Advanced Spacecraft Dynamics - Homework I

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Contents

1 Introduction

In this report we will analyse the stability of a spacecraft modelled as a rigid body containing three wheels and one damper inside of it.
Each subsystem $\mathcal{W}_i, \mathcal{D}$ naturally has its own reference system therefore, in order to avoid any confusion, all the spatial quantities such as vectors and dyads will be just represented by means of their components in the frame of reference of Body $\mathcal{B}$ with the help of auxiliary rotation matrices $\underset{B\leftarrow W_i}{R}$.

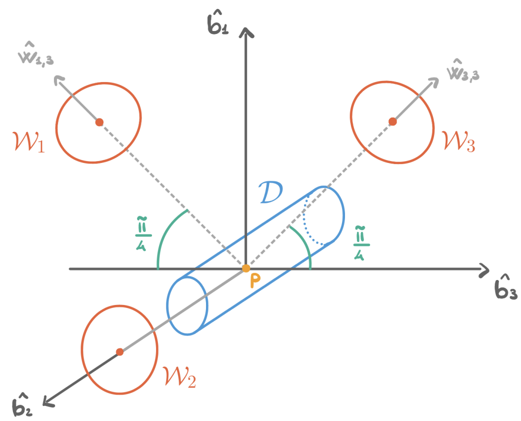

Figure 1: Sketch of the Spacecraft Model

The first steps involve the calculation of the characteristic quantities of the problem, which we will later need at each time step of the integration.
We start by computing the total mass of the systems, which remains constant in time as

$$M = m_b + \sum_{i}^{3} m_{w,i} + m_d \tag{1.1}$$

We can also compute now the Inertia Moments the wheels, which in this exercise are considered equal between them,

$$I_S = \frac{m_w R_w}{2} \tag{1.2}$$

$$I_T = \frac{m_w R_w}{4} \tag{1.3}$$

Then we'll procede by computing the static moment $\underline{S_p}^{(B)}$ as

$$S_p^{(B)} = S_{B,p}^{(B)} + \sum_{i}^{3} m_{w,i}\, r_{w,i}^{(B)} + m_d r_d^{(B)} \tag{1.4}$$

where

$$\underset{\rightarrow}{r_d} = \underset{\rightarrow}{b} + \xi \hat{n} \;\rightarrow\; \underline{r_d} = \begin{bmatrix} 0 & \xi & 0 \end{bmatrix} \tag{1.5}$$

Finally we compute the components of the Inertia Matrix $\underset{\rightarrow}{J_p}$ as

$$J_p^{(B)} = J_{B,p}^{(B)} + \sum_i^3 J_{Wi,p}^{(B)} + J_{D,p}^{(B)} \tag{1.6}$$

where $J_{B,p}^{(B)}$ is given and

$$\underset{\rightarrow}{J_{Wi,p}} = m_{w,i} \left[\left(\underset{\rightarrow}{r_{w,i}} \cdot \underset{\rightarrow}{r_{w,i}} \right) \underset{\rightarrow}{1} - \underset{\rightarrow}{r_{w,i}} \underset{\rightarrow}{r_{w,i}} \right] + \underset{\rightarrow}{J_{Wi,wi}} \tag{1.7}$$

$$\underset{\rightarrow}{J_{Wi,wi}} = I_T \underset{\rightarrow}{1} + (I_S - I_T)\hat{a}\hat{a} \tag{1.8}$$

$$\underset{\rightarrow}{J_{D,p}} = m_d \left[\left(\underset{\rightarrow}{r_d} \cdot \underset{\rightarrow}{r_d} \right) \underset{\rightarrow}{1} + \underset{\rightarrow}{r_d} \underset{\rightarrow}{r_d} \right] \tag{1.9}$$

With this informations, we can introduce the fundamental quantities for this problem, which are

$$\chi = \begin{Bmatrix} \underline{P} \\ \underline{H_p} \\ h_{a1} \\ h_{a2} \\ h_{a3} \\ P_n \end{Bmatrix} \qquad \eta = \begin{Bmatrix} \underline{v_p} \\ \underline{\omega} \\ \omega_{s1} \\ \omega_{s2} \\ \omega_{s3} \\ \dot{\xi} \end{Bmatrix} \tag{1.10}$$

The vector χ is composed of $\underline{P}$ which contains the components of the linear momentum of the spacecraft, $\underline{H_p}$ the components of the angular momentum while $\underline{h_{ai}}$ represent the angular momentum around the spin axis $\hat{a}$ of the $i-th$ wheel and $\underline{P_n}$ is the linear momentum of the damper mass along direction $\hat{n}$.
Instead, the vector η contains the velocity of the point P v_p, the components of angular velocity of the body $\mathcal{B}$ with respect to the inertial frame $\underline{\omega}$, the angular velocities of each wheel $\omega_{s,i}$ and the velocity of the mass inside the damper $\dot{x}i$.

The relations between each of the terms contained in the two auxiliary vectors is such that the two vectors can be related in a compact way by defining the Momenta Matrix M_T as follows,

$$
M_T = \begin{bmatrix} M\, I_{3x3} & -\tilde{S}_p & 0 & 0 & 0 & m_d\, \underline{n} \\ \tilde{S}_p & J_p^{(B)} & I_S\, \underline{a_1} & I_S\, \underline{a_2} & I_S\, \underline{a_3} & 0 \\ 0 & I_S\, \underline{a_1^T} & I_S & 0 & 0 & 0 \\ 0 & I_S\, \underline{a_2^T} & 0 & I_S & 0 & 0 \\ 0 & I_S\, \underline{a_3^T} & 0 & 0 & I_S & 0 \\ m_d\, \underline{n^T} & 0 & 0 & 0 & 0 & m_d \end{bmatrix} \tag{1.11}
$$

such that

$$
\chi = M_T\ \eta \tag{1.12}
$$

Now, recalling that our aim is to propagate in time the quantities inside the η vector, we notice that can be easily done by following the procedure here below.

$$
\dot{\eta} = [M_T - B]^{-1} \left\{ \left(A\, M_T - \dot{M}_T \right) \eta + c_0 \right\} \tag{1.13}
$$

Note that this procedure comes from the introduction of new parameters A, B, c_0 to describe the dynamics equations, which are defined as follows,

$$
A \ :\ \dot{\chi} = A(\eta)\, \chi + c \tag{1.14}
$$

$$
B \ :\ c = B(\eta)\, \dot{\eta} + c_0(\eta) \tag{1.15}
$$

This means that A is a matrix that is based on the terms of the dynamics equation which depend on χ, while B and c_0 represent the terms that depend on $\dot{\eta}$ and the external terms respectively.

Also $\dot{M}_T$ is the variation in time of the components of the M_T matrix, where the only parameter changing is ξ.

We will now analyse the two proposed cases and see how, depending on the conditions, these final matrices will be marginally different.

2 Case 1

In this case we suppose that the wheels $\mathcal{W}_1$ and $\mathcal{W}_3$ rotate with fixed angular velocity each thanks to a provided torque

$$g_{a,i} = Is\, \underline{a_i^T}\, \underline{\dot{\omega}} \tag{2.1}$$

while the second wheel $\mathcal{W}_2$ is subject to viscous damping modelled by a factor
$K_w = 0.1\ kg\, m^2/s$.

Then we'll have

$$A = \begin{bmatrix} -\tilde{\omega} & 0_{3x3} & 0_{3x1} & 0_{3x1} \\ -\tilde{v_p} & -\tilde{\omega} & 0_{3x1} & 0_{3x1} \\ 0_{1x3} & 0_{1x3} & 0 & 0 \\ 0_{1x3} & 0_{1x3} & 0 & 0 \end{bmatrix} \tag{2.2}$$

$$B = \begin{bmatrix} 0_{6x3} & 0_{6x3} & 0_{6x3} & 0 \\ 0_{1x3} & I_S\, \underline{a_1^T} & 0_{1x3} & 0 \\ 0_{1x3} & 0_{1x3} & 0_{1x4} & 0 \\ 0_{1x3} & I_S\, \underline{a_3^T} & 0_{1x3} & 0 \\ 0_{1x3} & 0_{1x3} & 0_{1x3} & 0 \end{bmatrix} \tag{2.3}$$

$$c_0 = \begin{bmatrix} F_{ext} \\ M_p \\ 0 \\ -K_w\, \omega_{s2} \\ 0 \\ m_d\, \underline{\omega}^T\, \underline{\tilde{n}}\, (\underline{v_p} - \underline{\tilde{r_d}}\, \underline{\omega}) - C_d\, \dot{\xi} - K_d\, \xi \end{bmatrix} \tag{2.4}$$

$$\dot{M}_T = \begin{bmatrix} 0_{3x3} & -\dot{\tilde{S}}_p & 0_{1x4} \\ \dot{\tilde{S}}_p & \dot{J}_p^{(B)} & 0_{1x4} \\ 0_{4x3} & 0_{4x3} & 0_{4x4} \end{bmatrix} \tag{2.5}$$

where

$$\dot{S}_p = \begin{bmatrix} 0 & m_d\, \dot{\xi} & 0 \end{bmatrix} \tag{2.6}$$

$$
\dot{J_p^{(B)}} = m_d\,\dot{\xi}\begin{bmatrix} 2\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\xi \end{bmatrix} \tag{2.7}
$$

Note that in order to solve the system as proposed we must define a new state of the system which includes η and the variable ξ, meaning that our state will have 11 variables.

Once we have propagated the state forward, we can store the results at each time step inside vectors and compute the Mechanical Energy of the system at each instant as

$$
E = \frac{1}{2}\eta^T\,M_T\,\eta + \frac{1}{2}K_d\,\xi^2 \tag{2.8}
$$

2.1 Plot of the Results

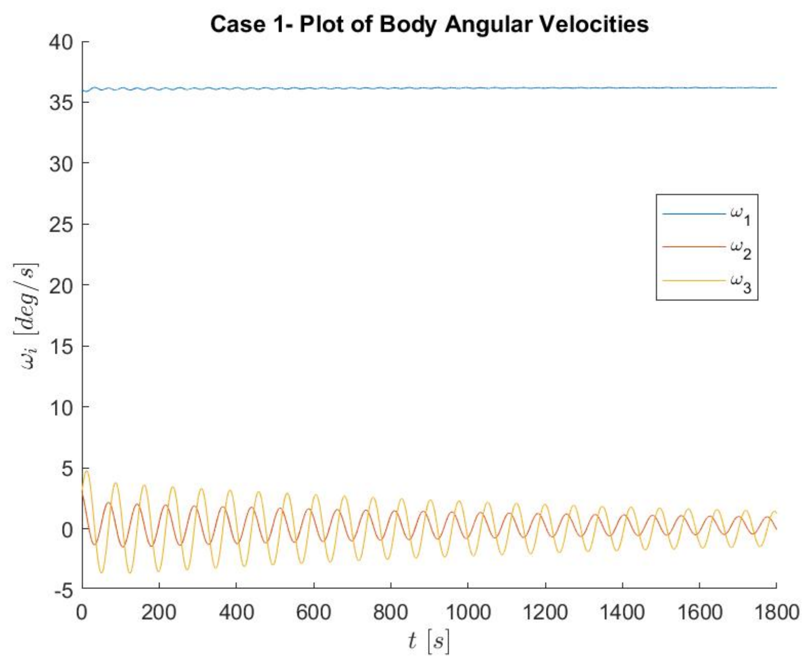

Figure 2: Body Angular Velocities

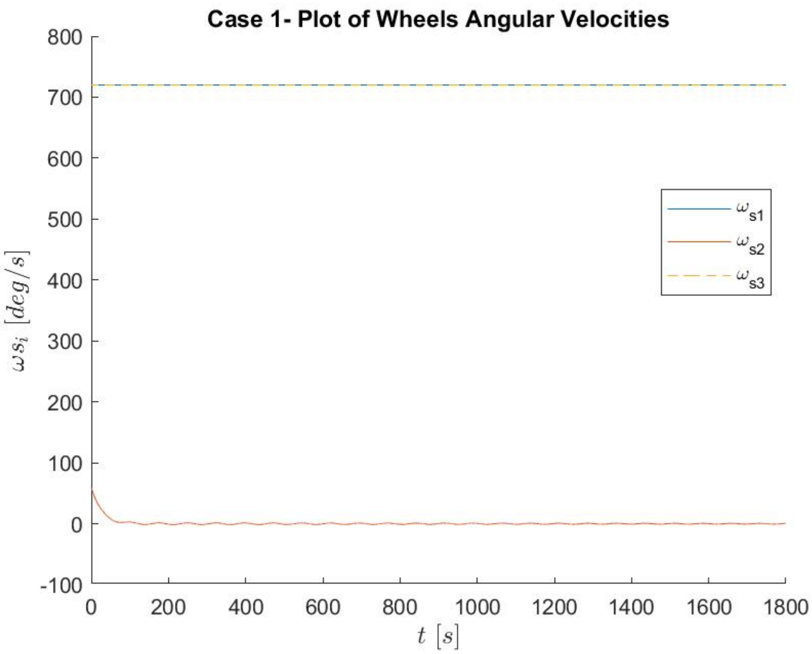

Figure 3: Wheels Angular Velocities

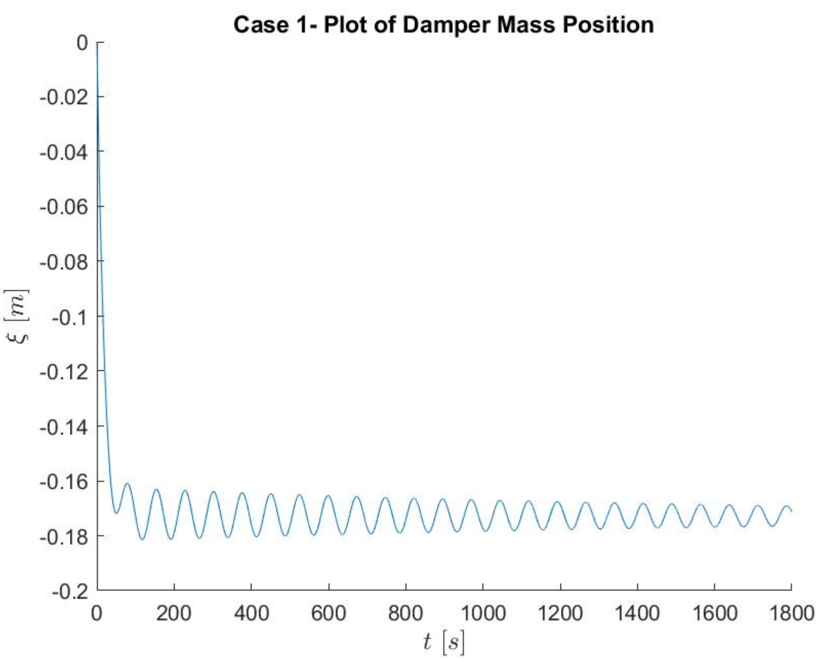

Figure 4: Damper Mass Position

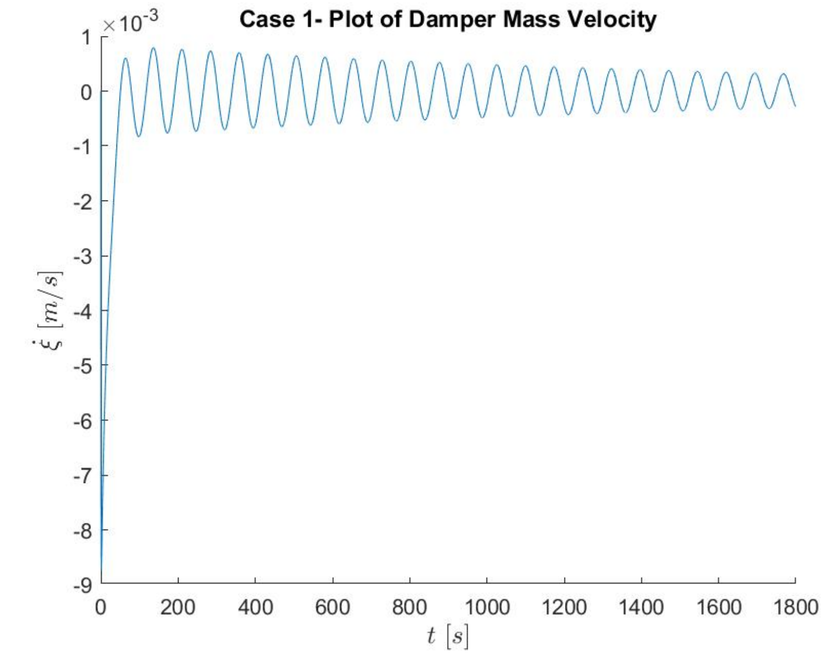

Figure 5: Damper Mass Velocity

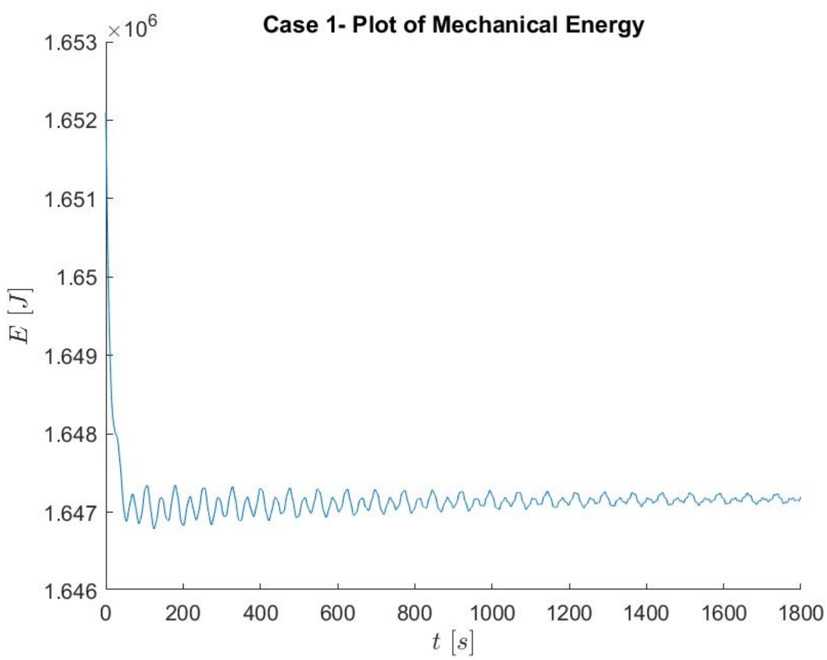

Figure 6: Mechanical Energy

2.2 Comments

As we can see, by keeping two out of the three wheels with fixed angular velocity, they behave as momentum wheels and provide the spacecraft with a certain rotation stiffness which does not allow it to rotate about the major axis of inertia.

Moreover, this stiffness does not allow the body to reach its lowest mechanical energy value and we can in fact see from the graph that it tends to an asymptotic value.

Eventually we can see how after some time the body reaches a steady configuration with very small oscillations from it.

3 Case 2

In the second case the only thing changing is the behaviour of the wheels, which this time only shows viscous dissipation for all three wheels allowing ω_{s1} and ω_{s3} to decrease in time.

In practical term this means that we won't have any term dependent from $\dot{\eta}$, meaning that B will be a null matrix and the new c_0 will be

$$B = \begin{bmatrix} 0_{10x10} \end{bmatrix} \tag{3.1}$$

$$c_0 = \begin{bmatrix} F_{ext} \\ M_p \\ -K_w\,\omega_{s1} \\ -K_w\,\omega_{s2} \\ -K_w\,\omega_{s3} \\ m_d\,\underline{\omega}^T\,\underline{\tilde{n}}\,(\underline{v_p} - \underline{\tilde{r_d}}\,\underline{\omega}) - C_d\,\dot{\xi} - K_d\,\xi \end{bmatrix} \tag{3.2}$$

3.1 Plot of the Results

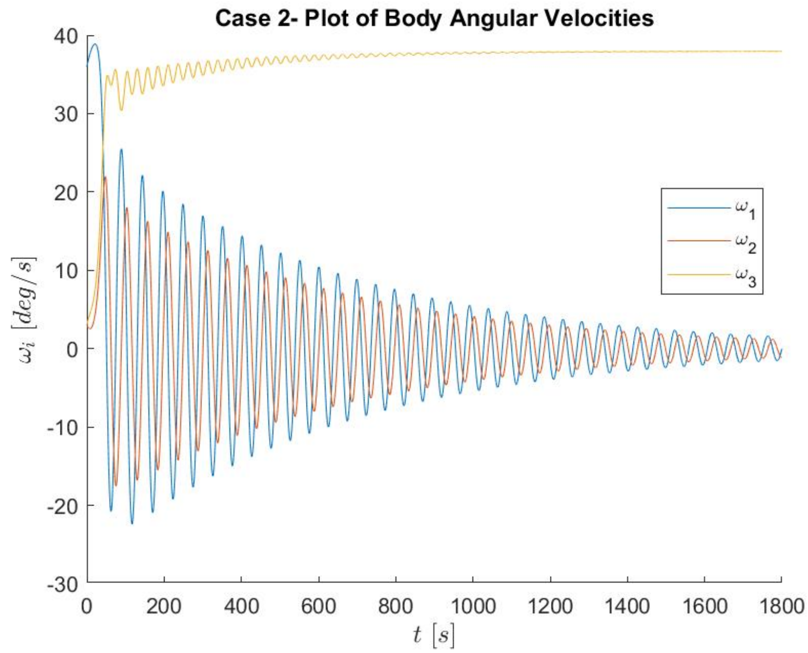

Figure 7: Body Angular Velocities

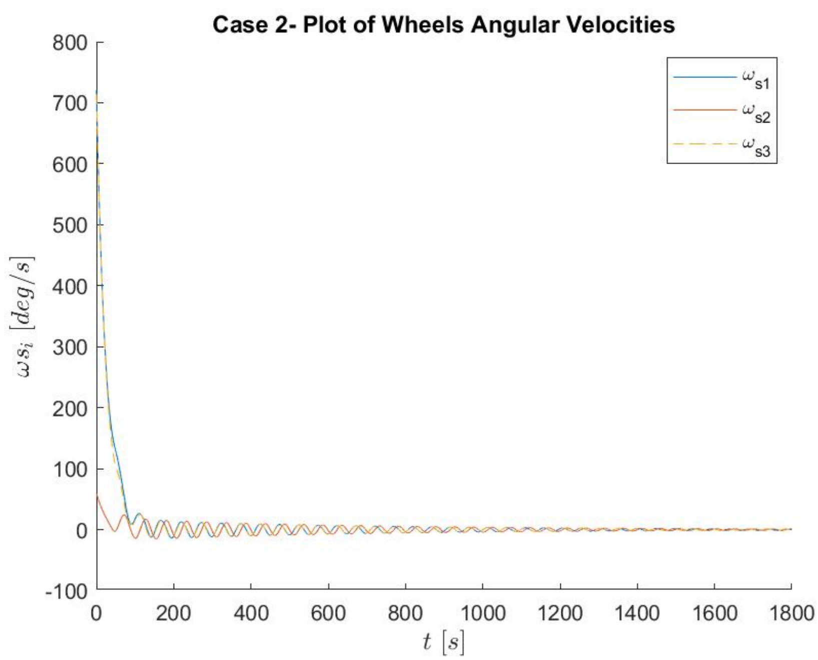

Figure 8: Wheels Angular Velocities

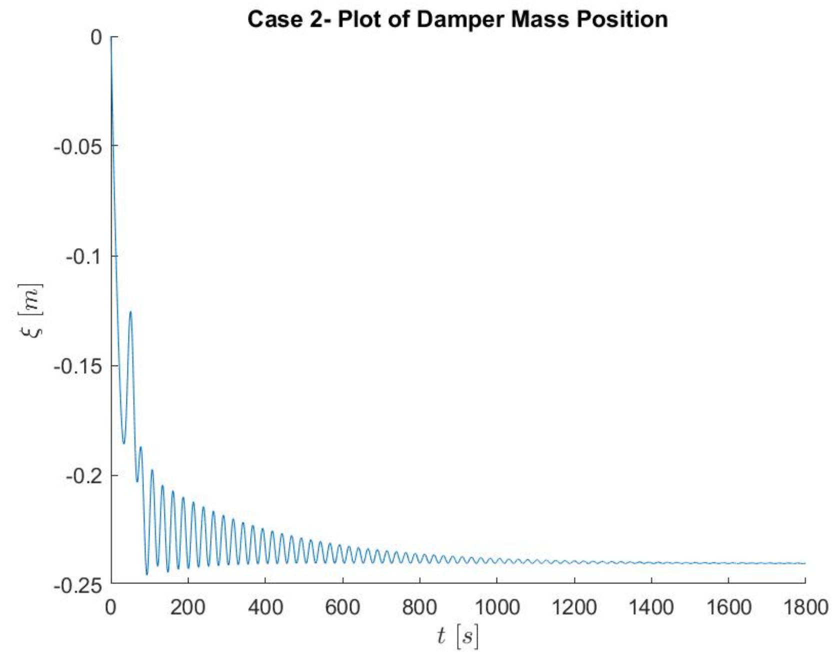

Figure 9: Damper Mass Position

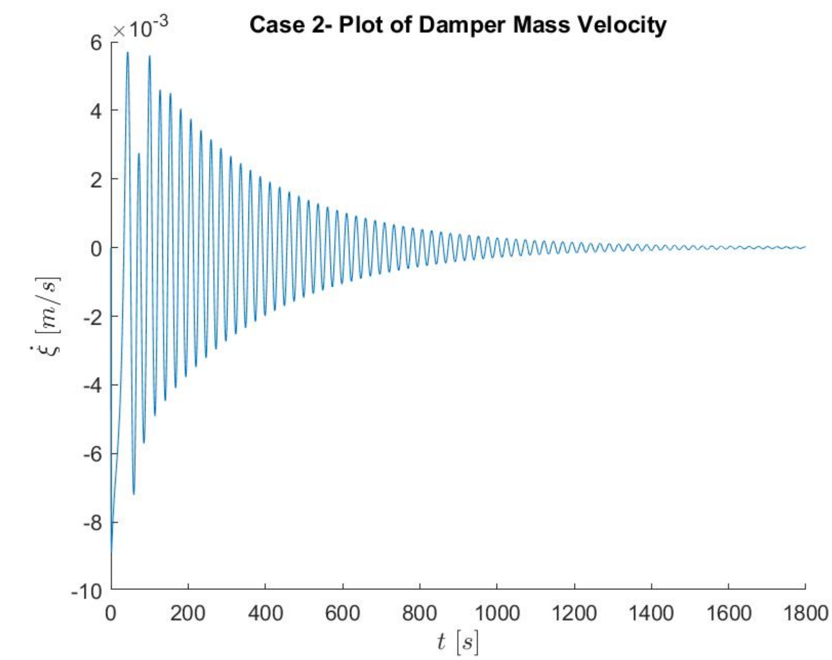

Figure 10: Damper Mass Velocity

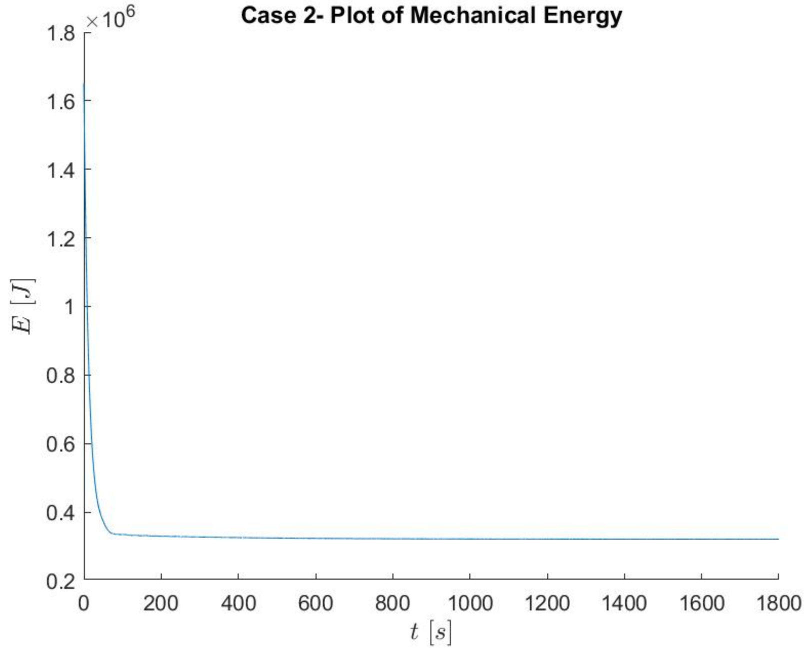

Figure 11: Mechanical Energy

3.2 Comments

In this second case all the wheels are subject to viscous damping, which explains why their angular velocity decreases in amplitude over time. In particular as the wheels slow down the body finds its equilibrium state with the damper mass no longer fixed in point P but about $25\,cm$ away from it.

Once this transient period is finished, then the body is very close to a rigid body and in fact it shows an almost constant rotation about its major axis of inertia as expected from the rigid body theory.

A great energy dissipation is also observed in the graph of Mechanical Energy which rapidly decreases up to the point where the body has reached the new rotation equilibrium.