Advanced Spacecraft Dynamics - Homework II

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Contents

1 Introduction

The aim of this report is to evaluate the attitude of a complex multibody structure over time by numerical integration of a dynamical model to be defined.

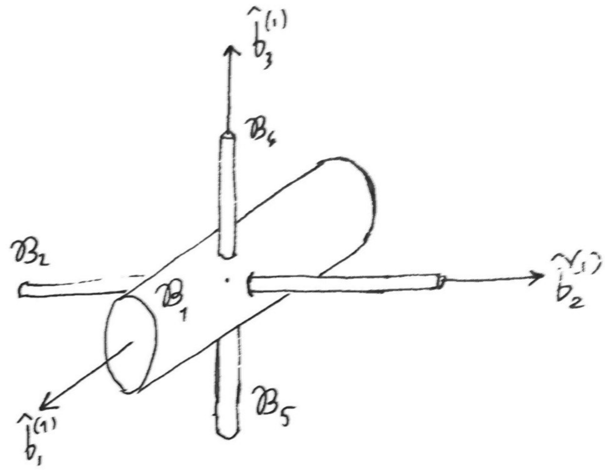

Figure 1: Description of the Body

The first step in the analysis of a multi-body structure is to define the Tree Diagram which shows the relative connection between all the bodies.

1	2	3	4	5
N	1	1	1	1
	N	N	N	N

Table 1: Tree-like Structure

At this point it is important to notice that the system is composed of one inner body and four outer bodies, all connected to the inner one. In particular, we'll have that $\mathcal{B}_1$ is our inner body with its own attitude with respect to the Inertial Frame, while $\mathcal{B}_j$ with $j \in [2, 5]$ are the outer bodies. Each outer body is connected to $\mathcal{B}_1$ via a rotary joint, meaning that each body will have only 1 degree of freedom represented by an angle θ_j.

In order to name efficiently each body and joint, we'll say that each outer body $\mathcal{B}_j$ is connected to $\mathcal{B}_1$ via the joint G_{j-1} which has joint partial $\Gamma_{G,j-1}$, angular position θ_{j-1} and angular rate σ_{j-1}.

Then, from the definition of the frames of reference solidal to each body, we can evaluate the rotation matrices from each outer body to the inner one.

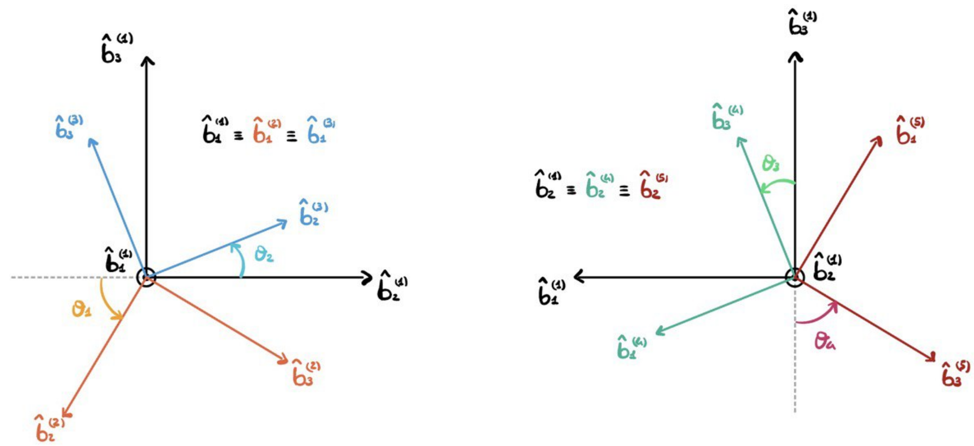

Figure 2: Body Reference Frames

$$\underset{1\leftarrow 2}{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & -\cos\theta_1 \end{bmatrix} \quad , \quad \underset{1\leftarrow 3}{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \tag{1.1}$$

$$\underset{1\leftarrow 4}{R} = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 \\ 0 & 1 & 0 \\ -\sin\theta_3 & 0 & \cos\theta_3 \end{bmatrix} \quad , \quad \underset{1\leftarrow 5}{R} = \begin{bmatrix} -\cos\theta_4 & 0 & -\sin\theta_4 \\ 0 & 1 & 0 \\ \sin\theta_4 & 0 & -\cos\theta_4 \end{bmatrix} \tag{1.2}$$

Also, since $\underline{\underline{B_1}}$ and $\underline{\underline{N}}$ are have the same orientation at t_0 (even though they different origin points), then we have that

$$\underset{N\leftarrow 1}{R}(t_0) = I_{3\text{x}3} \tag{1.3}$$

and this will be the initial condition that will be propagated in time using quaternions.

2 Integration Procedure

First it is important to make one remark about the choice of reference frame used to write the components of each vector.
From now on we will express the angular velocities $\underset{\rightarrow}{\omega_j}$ in the frame of reference solidal to each $\mathcal{B}_j$, meaning that $\underline{\omega_j}$ is actually $\underline{\omega_j^{(B_j)}}$.

On the other hand the linear velocities $\underset{\rightarrow}{v_j}$ and the position vectors $\underset{\rightarrow}{r_j}$ will always be represented in the Inertial frame of reference $\underline{\underline{N}}$, meaning that $\underline{r_{p1}}$ is just $\underline{r_{p1}^{(N)}}$.

Finally, we will define the state which will be considered for the integration as follows,

$$X = \begin{bmatrix} \underline{\omega_1} \\ \underline{\sigma} \\ \underline{v_{p1}} \\ \underline{r_{p1}} \\ \underline{\theta} \\ \underline{x} \end{bmatrix} \quad , \quad x = \begin{bmatrix} q_0 \\ \underline{q} \end{bmatrix} \quad , \quad \underline{u} = \begin{bmatrix} \underline{\omega_1} \\ \underline{\sigma} \\ \underline{v_{p1}} \end{bmatrix} \tag{2.1}$$

where $\underline{x}$ contains the attitude of $\mathcal{B}_1$ with respect to the Inertial Frame $\underline{\underline{N}}$ under the form of quaternions q_0, $\underline{q}$. Also, we'll later see that we use a subset of the integrated state called $\underline{u}$, which contains all the velocities of the central body and the angular rates of each joint, to simplify some mathematical steps.

2.1 Initial Conditions

The initial conditions provided for the problem are the ones that follow,

$$\underline{\omega_1} = \begin{bmatrix} 1 & 0.1 & 0.1 \end{bmatrix} \qquad rad/s \tag{2.2}$$

$$\underline{\sigma}(t_0) = \begin{bmatrix} -0.2 & 0.2 & 0.1 & -0.3 \end{bmatrix} \qquad rad/s \tag{2.3}$$

$$\underline{v_{p1}}(t_0) = \begin{bmatrix} 0 & \sqrt{\frac{\mu}{R_0}} & 0 \end{bmatrix} \qquad m/s \tag{2.4}$$

$$\underline{r_{p1}} = \begin{bmatrix} R_0 & 0 & 0 \end{bmatrix} \qquad m \tag{2.5}$$

$$\underline{\theta} = \begin{bmatrix} -30 & 30 & -30 & 10 \end{bmatrix} \qquad deg \tag{2.6}$$

where $\underset{\rightarrow}{\omega_1}$ is expressed through its components in $\underline{\underline{B_1}}$ while $\underset{\rightarrow}{v_{p1}}$, $\underset{\rightarrow}{r_{p1}}$ are expressed in $\underline{\underline{N}}$.

2.2 Local Variables

At each step of the integration it is necessary to compute some quantities which will be used inside the Dynamical Model. These quantities are derived from analytical calculus and the steps used to obtain the relations will not be shown here for sake of brevity.

The mass matrix the masses of each body on its diagonal as follows,

$$\underline{\underline{M}} = \begin{bmatrix} M_{B1}\, I_{3\text{x}3} & 0_{3\text{x}12} \\ 0_{12\text{x}3} & m_{Bj}\, I_{12\text{x}12} \end{bmatrix} \tag{2.7}$$

The joint partials can be easily defined from the geometrical analysis of the multibody system.

$$\Gamma_{G1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \tag{2.8}$$
$$\Gamma_{G2} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \tag{2.9}$$
$$\Gamma_{G3} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \tag{2.10}$$
$$\Gamma_{G4} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \tag{2.11}$$

$$\dot{\Gamma}_{Gj} = 0 \qquad\qquad \forall\, j \in [\,1, 4\,] \tag{2.12}$$

Now, before computing the Inertia Momenta Matrix it is important to notice that each submatrix inside $\underline{\underline{J}}$ is given by the Inertia Moments of each body in its own reference frame. Hence we define $\underline{\underline{J}}$ as,

$$\underline{\underline{J}} = \begin{bmatrix} J_{p1}^{B_1} & 0 & 0 & 0 & 0 \\ 0 & J_{p2}^{B_2} & 0 & 0 & 0 \\ 0 & 0 & J_{p3}^{B_3} & 0 & 0 \\ 0 & 0 & 0 & J_{p4}^{B_4} & 0 \\ 0 & 0 & 0 & 0 & J_{p5}^{B_5} \end{bmatrix} \tag{2.13}$$

with

$$J_{p1}^{B_1} = \begin{bmatrix} I_{a1} & 0 & 0 \\ 0 & I_{t1} & 0 \\ 0 & 0 & I_{t1} \end{bmatrix} \tag{2.14}$$

$$J_{p2}^{B_2} = \begin{bmatrix} I_{t2} & 0 & 0 \\ 0 & I_{a2} & 0 \\ 0 & 0 & I_{t2} \end{bmatrix} = J_{p3}^{B_3} \tag{2.15}$$

$$J_{p4}^{B_4} = \begin{bmatrix} I_{t4} & 0 & 0 \\ 0 & I_{t4} & 0 \\ 0 & 0 & I_{a4} \end{bmatrix} = J_{p5}^{B_5} \tag{2.16}$$

Then, we must introduce the matrices Ω and V, which respectively relate the angular velocities and linear velocities of each body with the elements of $\underline{u}$. Again, recall that each row of Ω is built using the vector components in $\underline{\underline{B_j}}$ while $\underline{\underline{V}}$ is built using vectors components in $\underline{\underline{N}}$.

$$\Omega = \begin{bmatrix} I_{3x3} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x3} \\ \underset{2\leftarrow 1}{R} & \Gamma_{G1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x3} \\ \underset{3\leftarrow 1}{R} & 0_{3x1} & \Gamma_{G2} & 0_{3x1} & 0_{3x1} & 0_{3x3} \\ \underset{4\leftarrow 1}{R} & 0_{3x1} & 0_{3x1} & \Gamma_{G3} & 0_{3x1} & 0_{3x3} \\ \underset{5\leftarrow 1}{R} & 0_{3x1} & 0_{3x1} & 0_{3x1} & \Gamma_{G4} & 0_{3x3} \end{bmatrix} \tag{2.17}$$

$$V = \begin{bmatrix} 0_{3x3} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & I_{3x3} \\ \underline{\tilde{r_{21}}}\underset{N\leftarrow 1}{R} & \underline{\tilde{r_{2,G1}}}\underset{N\leftarrow 2}{R}\Gamma_{G1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & I_{3x3} \\ \underline{\tilde{r_{31}}}\underset{N\leftarrow 1}{R} & 0_{3x1} & \underline{\tilde{r_{3,G2}}}\underset{N\leftarrow 3}{R}\Gamma_{G2} & 0_{3x1} & 0_{3x1} & I_{3x3} \\ \underline{\tilde{r_{41}}}\underset{N\leftarrow 1}{R} & 0_{3x1} & 0_{3x1} & \underline{\tilde{r_{4,G3}}}\underset{N\leftarrow 4}{R}\Gamma_{G3} & 0_{3x1} & I_{3x3} \\ \underline{\tilde{r_{51}}}\underset{N\leftarrow 1}{R} & 0_{3x1} & 0_{3x1} & 0_{3x1} & \underline{\tilde{r_{5,G4}}}\underset{N\leftarrow 5}{R}\Gamma_{G4} & I_{3x3} \end{bmatrix} \tag{2.18}$$

Then we need to compute the residual accelerations from their analytical definition, which after some mathematical steps and following the nomenclature used until now, leads to

$$\alpha_j^{(R)} = \dot{\Gamma}_{G,j-1}\,\sigma_{j-1} + \underline{\tilde{\omega_j}}\,\Gamma_{G,j-1}\,\sigma_{j-1} \qquad , \qquad \forall\, j \in [\,2,5\,] \tag{2.19}$$

$$\begin{aligned} a_{pj}^{(R)} = & \left(\underset{N\leftarrow 1}{R}\,\underline{\omega_1}\right)^{\sim}\left(\underset{N\leftarrow 1}{R}\,\underline{\omega_1}\right)^{\sim}\underline{r_{1,Gj-1}} + \\ & - \left(\underset{N\leftarrow 1}{R}\,\underline{\omega_1} + \underset{N\leftarrow j}{R}\,\Gamma_{G,j-1}\sigma_{j-1}\right)^{\sim} \cdot \left[\left(\underset{N\leftarrow 1}{R}\,\underline{\omega_1} + \underset{N\leftarrow j}{R}\,\Gamma_{G,j-1}\sigma_{j-1}\right)^{\sim}\underline{r_{j,Gj-1}}\right] + \\ & - \left[\underset{N\leftarrow j}{R}\left(\dot{\Gamma}_{G,j-1}\sigma_{j-1} + \underline{\tilde{\omega_j}}\Gamma_{G,j-1}\sigma_{j-1}\right)\right]^{\sim} \qquad , \qquad \forall\, j \in [\,2,5\,] \end{aligned} \tag{2.20}$$

with

$$\alpha_1^{(R)} = 0 \tag{2.21}$$

$$a_{p1}^{(R)} = 0 \tag{2.22}$$

The last intermediate step is to compute the Torques applied on each body in its solidal reference frame and all the Forces applied on each body in the inertial frame, which in this case is only the gravitational force.

$$\underline{T_j} = (-K_1\,\theta_{j-1} - K_2\,\sigma_{j-1})\,\Gamma_{G,j-1} \qquad , \qquad \forall\, j \in [\,2,5\,] \tag{2.23}$$

$$\underline{T_1} = -\left(\underset{1\leftarrow 2}{R}\,\underline{T_2} + \underset{1\leftarrow 3}{R}\,\underline{T_3} + \underset{1\leftarrow 4}{R}\,\underline{T_4} + \underset{1\leftarrow 5}{R}\,\underline{T_5}\right) \tag{2.24}$$

$$\underline{F_j} = \frac{\mu\, m_{Bj}}{r_j^3}\,\underline{r_j} \qquad , \qquad \forall\, j \in [\,1,5\,] \tag{2.25}$$

where $\underline{r_j}$ are the vectors connecting the center of mass of each body with the center of the Inertial reference frame, which are obtained via summation of the path vectors.

2.3 State Update

Finally, once we have computed all the necessary quantities it is possible to update the State using the following relations,

$$\underline{\dot{u}} = \left\{\Omega^T \underline{\underline{J}}\Omega + V^T \underline{\underline{M}} V\right\}^{-1} \left\{\Omega^T \left[\underline{T} - \underline{\underline{J}}\,\underline{\alpha}^{(R)} - \underline{\underline{\tilde{\omega}}}\,\underline{\underline{J}}\,\underline{\omega}\right] + V^T \left[\underline{F} - \underline{\underline{M}}\,\underline{a}^{(R)}\right]\right\} \tag{2.26}$$

$$\underline{\dot{r_{p1}}} = \underline{v_{p1}} \tag{2.27}$$

$$\underline{\dot{\theta}} = \underline{\sigma} \tag{2.28}$$

$$\dot{q_0} = -\frac{1}{2}\underline{q} \cdot \omega_1 \tag{2.29}$$

$$\underline{\dot{q}} = \frac{1}{2}\left(q_0\,\underline{\omega_1} + \underline{q} \times \underline{\omega_1}\right) \tag{2.30}$$

where equation 2.26 is obtained by developing Kane Equation from its initial form,

$$F_{GR} + F_{GR}^{*} = 0 \tag{2.31}$$

and the remaining relations are developed trivially.

3 Results

We will study two different cases which differ in only two quantities: the length of bodies $\mathcal{B}_4$, $\mathcal{B}_5$ and the final propagation time t_f.

3.1 Case 1

In the first case we adopt the following values for the variable quantities,

$$l_B = 9\, m \tag{3.1}$$

$$t_f = 1800\, sec \tag{3.2}$$

Already from the evolution in time of $\underline{\omega_1}$ one can recognize that the attitude of the satellite undergoes a transient period which eventually leads to a steady state attitude.
In particular we see that we move from the initial rotation about axis $b_1^{(1)}$, which is the axis of minimum inertia, towards a rotation about the axis of maxima inertia due to the presence of dissipation in the joints.

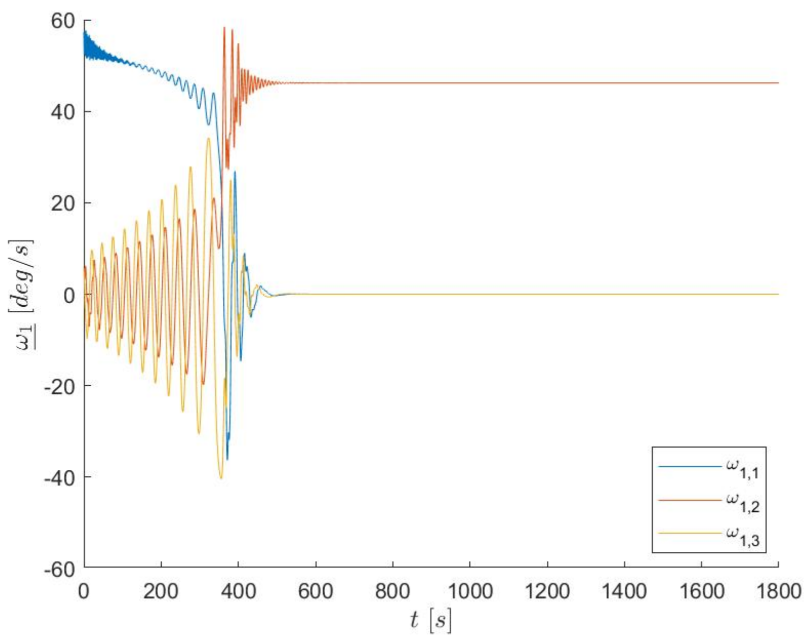

Figure 3: Angular Velocity of Body 1

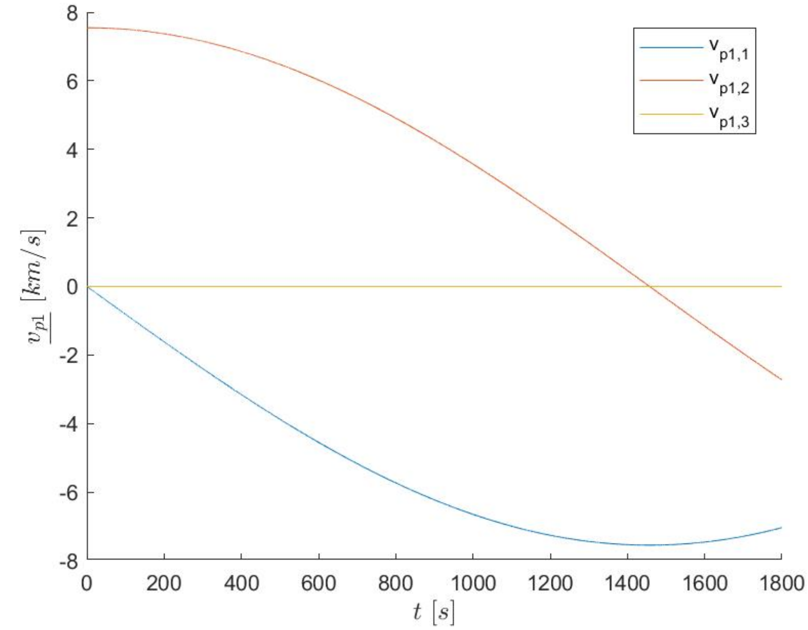

Figure 4: Linear Velocity of Body 1

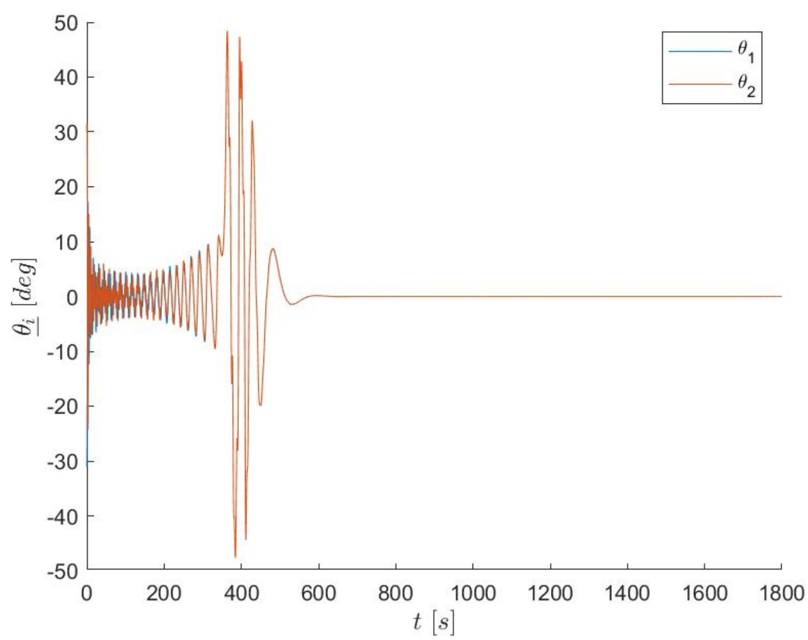

Figure 5: Angular Position of Joints 1 and 2

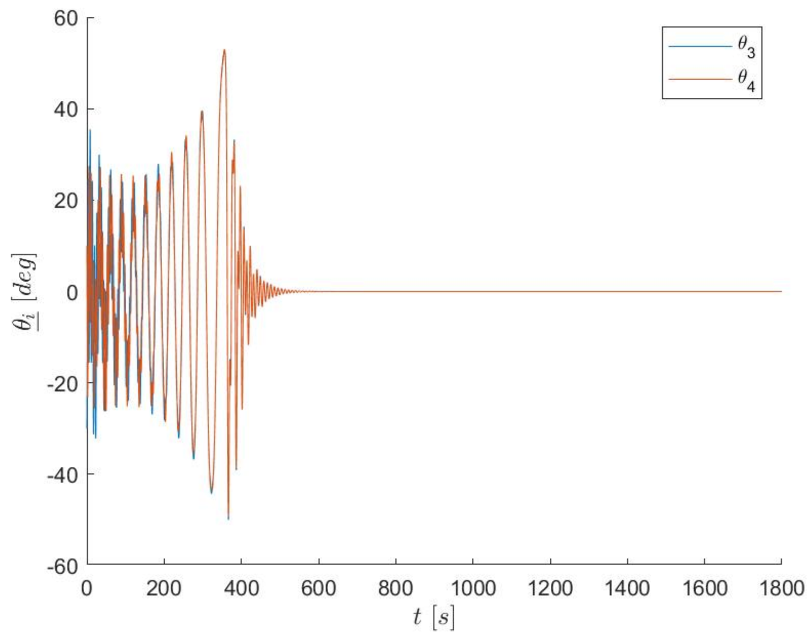

Figure 6: Angular Position of Joints 3 and 4

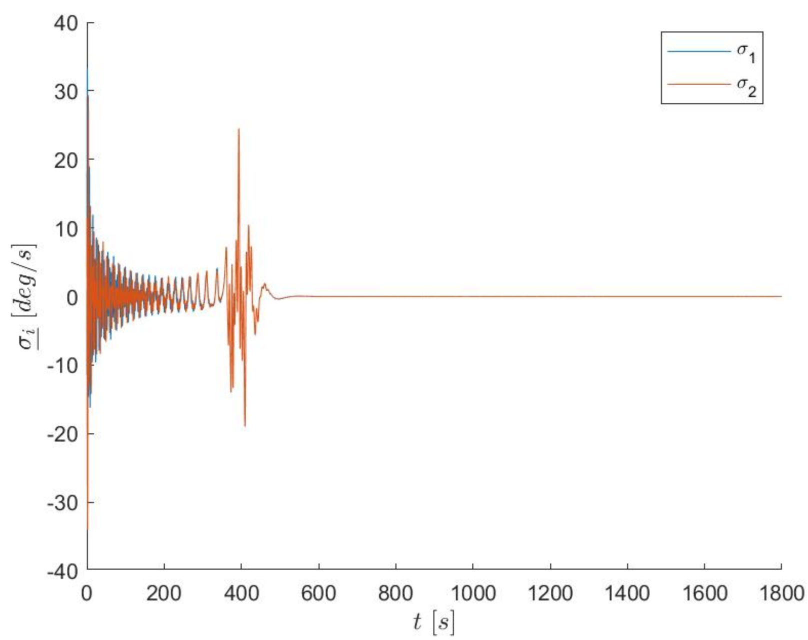

Figure 7: Angular Rate of Joints 1 and 2

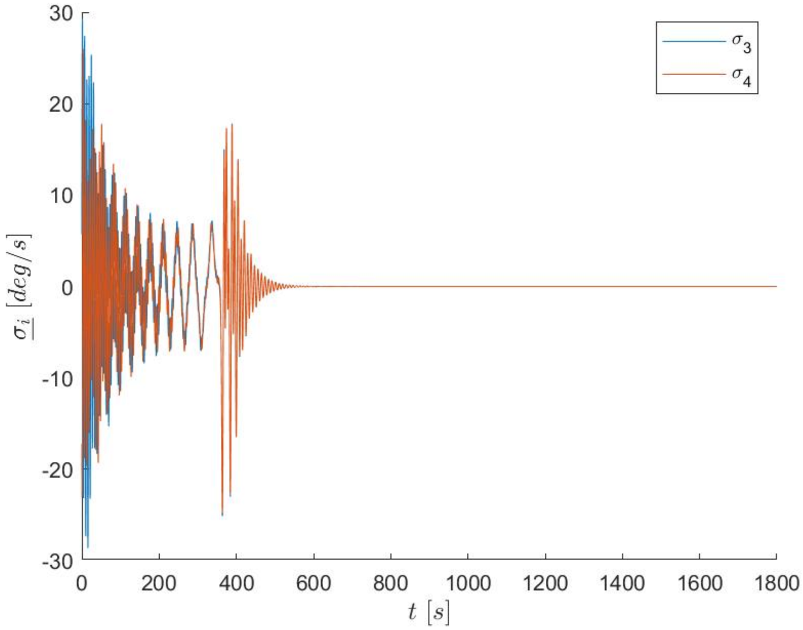

Figure 8: Angular Rate of Joints 3 and 4

Immediately one can recognize a particular behaviour of the appendixes. In fact, as time goes on, the opposite appendixes tend to align themselves with each other and find a fixed position with respect to body $\mathcal{B}_1$. This is evident both from the graphs of θ_j and from the ones of σ_j in which we observe that this apparently rigid behaviour is obtained for $t > 600\, sec$.

This behaviour is coherent with what we expected and it is explained by the presence of a dissipative torque that eventually slows down each joint.

3.2 Case 2

In the second case we instead adopt the following values,

$$
\begin{aligned}
l_B &= 3\,m && (3.3)\\
t_f &= 7200\,sec && (3.4)
\end{aligned}
$$

The key difference between the two studied case is immediately apparent in every graph depicted here below regarding the angular motion.

Having smaller appendices $\mathcal{B}_4$, $\mathcal{B}_5$ generates less dissipation inside the system and this slows down the evolution of the state. In fact we can observe that the transient period is much longer in this case since we reach a steady state only for $t > 2500\,sec$.

Moreover it is important to note that in this case the axis of maximum inertia of the complete body is not close to any of the axes of body 1, hence why we observe two non-null components of the steady state angular velocity $\underline{\omega_1}$.

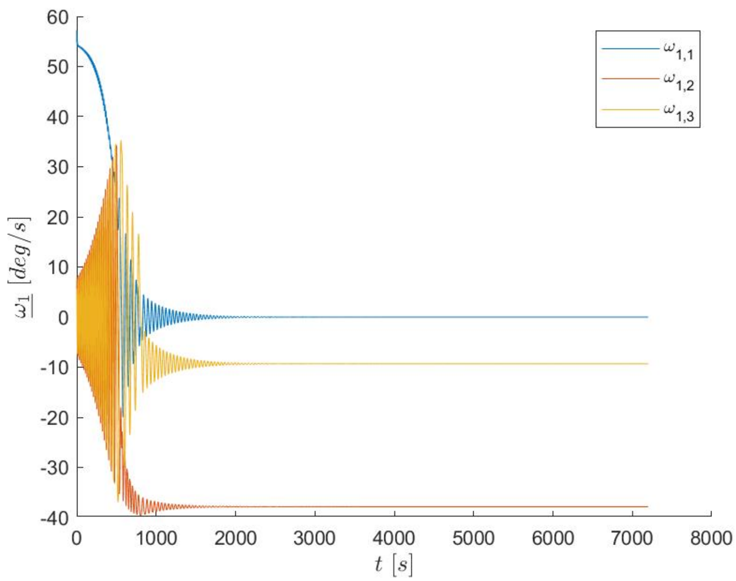

Figure 9: Angular Velocity of Body 1

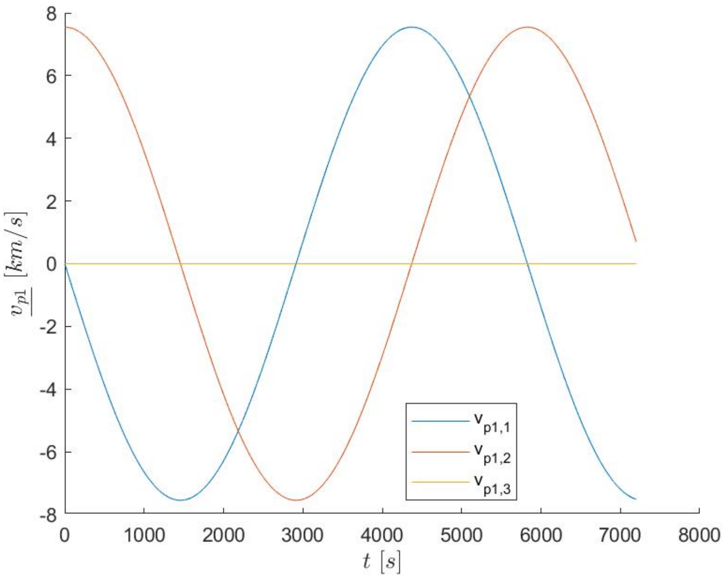

Figure 10: Linear Velocity of Body 1

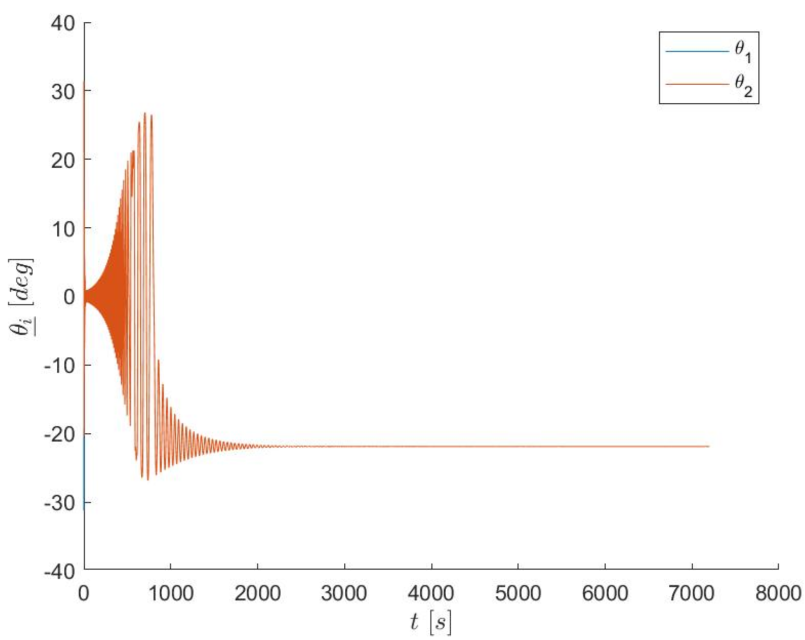

Figure 11: Angular Position of Joints 1 and 2

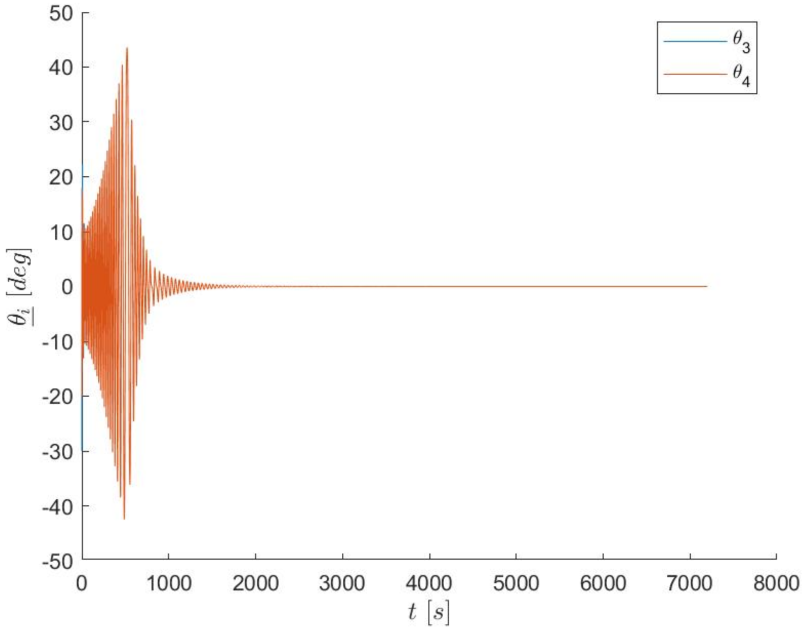

Figure 12: Angular Position of Joints 3 and 4

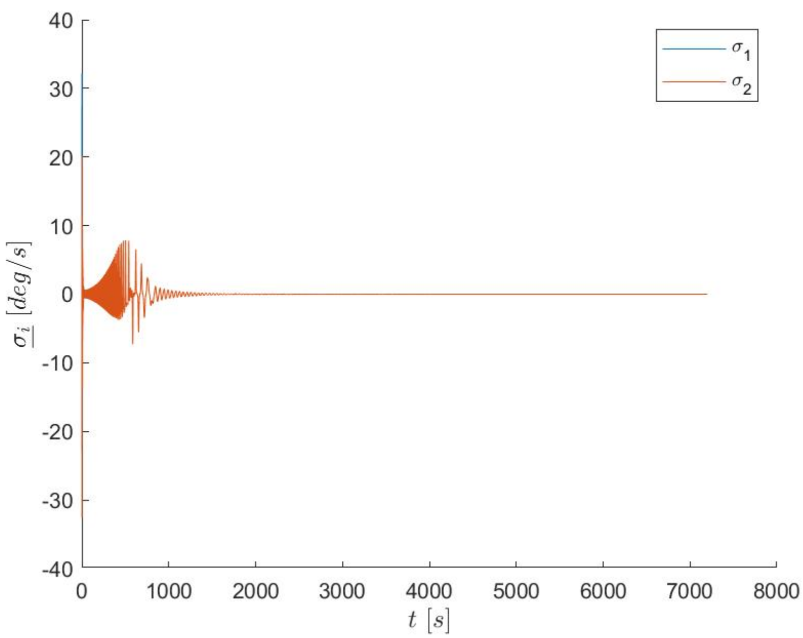

Figure 13: Angular Rate of Joints 1 and 2

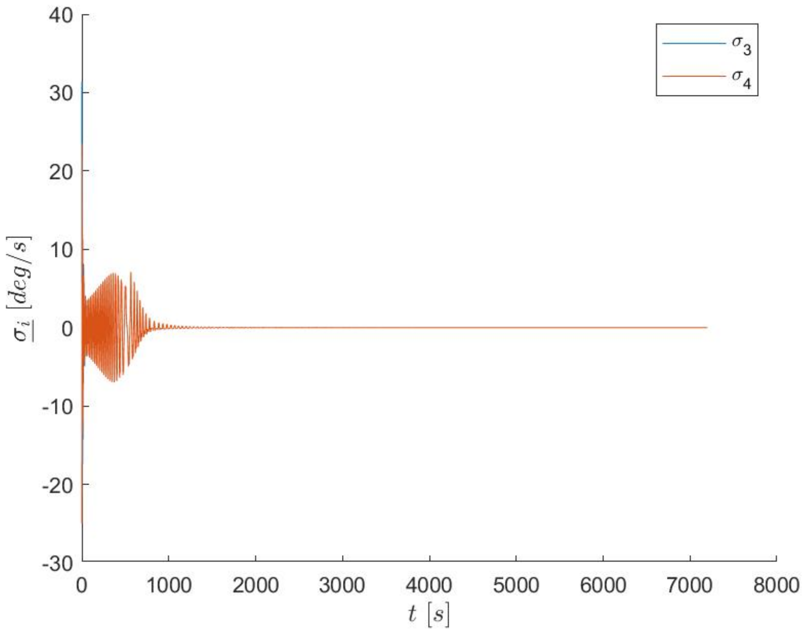

Figure 14: Angular Rate of Joints 3 and 4