Space Missions and Systems - Homework I

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April 4, 2022

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Contents

1 Estimation of State Vector components and associated Uncertainties

1.1 Assignment of known Data

First of all, we start by reporting here the given parameters such as X_0,

$$X_0 = \begin{bmatrix} x_0 \\ y_0 \\ u_0 \\ v_0 \\ \mu_{v,0} \\ C_{D,0} \end{bmatrix} = \begin{bmatrix} -0.8 \\ 6419.4 \\ -7.11389 \\ -0.24912 \\ 324860.3 \\ 2.2 \end{bmatrix} \begin{bmatrix} km \\ km \\ km/s \\ km/s \\ km^3/s^2 \\ \ \end{bmatrix} \tag{1.1}$$

and also the other main known quantities such as,

$$R_\oplus = 6378\, km \tag{1.2}$$
$$R_v = 6052\, km \tag{1.3}$$
$$x_\oplus = -38.2 \cdot 10^6\, km \tag{1.4}$$
$$y_\oplus = 0\, km \tag{1.5}$$
$$S = 40 \cdot 10^-6\, km^2 \tag{1.6}$$
$$m = 2000\, kg \tag{1.7}$$

from which we can evaluate the coordinate of the Ground Station in our reference system

$$x_s = D + R_\oplus \cdot cos(30°) \tag{1.8}$$
$$y_s = R_\oplus \cdot sin(30°) \tag{1.9}$$

Then, the first step to integrate the trajectory is to correctly define the equation of motion, which we do with the help of some new parameters,

$$r = \sqrt{x^2 + y^2} \tag{1.10}$$
$$V = \sqrt{u^2 + v^2} \tag{1.11}$$
$$z = r - R_v \tag{1.12}$$

Then, from a simple analysis we obtain the equation of motion along x and y as,

$$\begin{cases} \ddot{x} = -\frac{\mu_v}{r^3}x - \frac{1}{2}\rho(z)\, C_D\, \frac{S}{m}\, V \cdot u \\ \ddot{y} = -\frac{\mu_v}{r^3}y - \frac{1}{2}\rho(z)\, C_D\, \frac{S}{m}\, V \cdot v \end{cases} \tag{1.13}$$

where $\rho(z)$ is the function that describes the atmospheric density from the altitude z with the following equation,

$$\rho(z) = \rho_0\, e^{-\frac{z-z_0}{H}} \tag{1.14}$$

with ρ_0, z_0 and H all constant known parameters.

We can now rewrite the equations as first order system as,

$$F(X,t) = \dot{X} = \begin{bmatrix} y \\ v \\ -\frac{\mu_v}{r^3}x - \frac{1}{2}\rho(z)\, C_D\, \frac{S}{m}\, V \cdot u \\ -\frac{\mu_v}{r^3}y - \frac{1}{2}\rho(z)\, C_D\, \frac{S}{m}\, V \cdot v \\ 0 \\ 0 \end{bmatrix} \tag{1.15}$$

and, by defining a dynamical model based on this linearised equations, we can integrate from $X0$ through MATLAB's ode113() function.

We can then evaluate the computed observables following their definition,

$$G(X,t) = \begin{cases} \rho & = \sqrt{(x-x_s)^2 + (y-y_s)^2} \\ \dot{\rho} & = \frac{1}{\rho}[(x-x_s)u + (y-y_s)v] \end{cases} \tag{1.16}$$

and, finally, we can put together the data from the observed observables and the computed observables to obtain a *pre-fit* plot of the residuals

$$y_\rho = \rho_{obs} - \rho_{computed} \tag{1.17}$$

$$y_{\dot{\rho}} = \dot{\rho_{obs}} - \dot{\rho_{computed}} \tag{1.18}$$

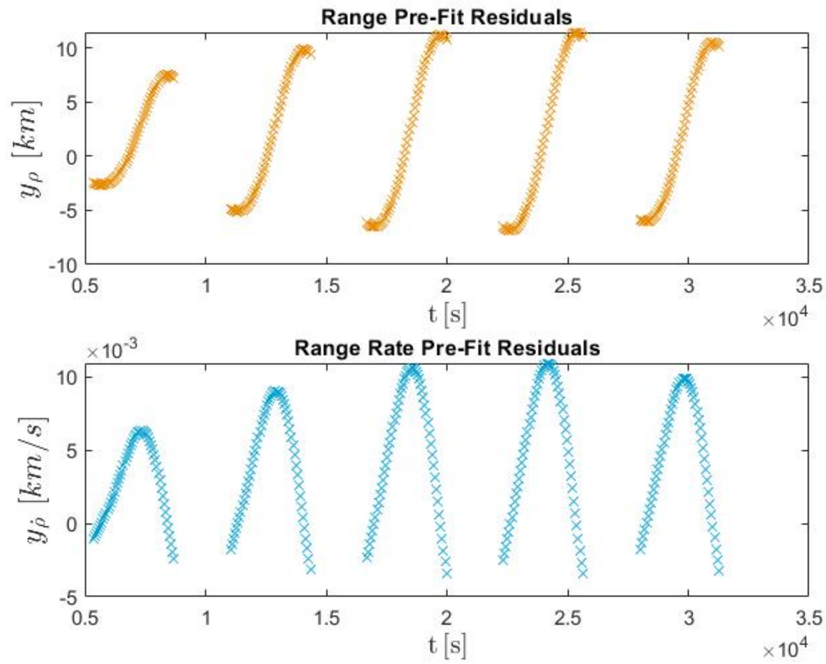

Figure 1: Pre-Fit Residuals

1.2 Model and Transition Definition

Since the linearised equations for the state deviation and state transition matrices are

$$\dot{X} = A\,X \tag{1.19}$$

$$\dot{\Phi} = A\,\Phi \tag{1.20}$$

we can compute analytically the A matrix as $\frac{\partial F}{\partial X}$,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.21}$$

where the elements on the third and fourth rows are too long to be reported here, hence they will be reported at the end of this report in the Additional Information Section 6.

On the other hand, it is impossible to compute Φ analytically, so we must integrate equation 1.20, again by using the ode113() function.

The last step before starting the iterative procedure is to define a function that returns the matrix $\tilde{H}$ from a state X and the coordinates X_s. Since $\tilde{H}$ is defined as $\frac{\partial G}{\partial X}$, we can write,

$$\tilde{H} = \begin{bmatrix} \frac{x-x_s}{\rho} & \frac{y-y_x}{\rho} & 0 & 0 & 0 & 0 \\ \tilde{H}_{2,1} & \tilde{H}_{2,1} & \frac{x-x_s}{\rho} & \frac{y-y_x}{\rho} & 0 & 0 \end{bmatrix} \tag{1.22}$$

where,

$$\tilde{H}_{2,1} = \frac{u}{\rho} - \frac{(x-x_s)\cdot[u(x-x_s)+v(y-y_s)]}{\rho^3} \tag{1.23}$$

$$\tilde{H}_{2,2} = \frac{v}{\rho} - \frac{(y-y_s)\cdot[u(x-x_s)+v(y-y_s)]}{\rho^3} \tag{1.24}$$

1.3 Weighted Least Square (WLS) filter with a-priori information

The least square method is an iterative process which, at each step, computes a new initial state estimation based both on the the weights of the observables and on an a priori information.
In particular, the a-priori weight matrix W_{apr} is built as

$$\begin{bmatrix} \sigma_{x_0}^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{y_0}^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{u_0}^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{v_0}^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\mu_{v,0}}^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{C_D}^{-2} \end{bmatrix} \tag{1.25}$$

where every σ represents the standard deviation of each variable of the state vector.
On the other hand, the weight matrix for the observables W_{obs} is a square

matrix which has two elements $\left(\sigma_{\rho}^{-2}, \sigma_{\dot{\rho}}^{-2}\right)$ alternately repeated on the diagonal.

Eventually, during every iteration we are able to compute $\tilde{H}$ for every couple of observables and, thanks to the knowledge of Φ, we are able to build the H matrix.
Then, we can simply compute the correction step $\hat{x}$ as

$$\hat{x} = (H^T W_{obs} H + W_{apr})^{-1}(H^T W_{obs} y + W_{apr}\bar{x}) \tag{1.26}$$

and we can update the initial state as

$$X_0 = X_0 + \hat{x} \tag{1.27}$$

To understand weather the estimation converges or diverges, we resort to some useful indicators such as SOS, $MEAN$ and RMS.
In our case to achieve convergence 10 iterations are needed, at this point in fact we observe that noise can be described as white and that all SOS, $MEAN$ and RMS satisfy the required conditions 6.13.

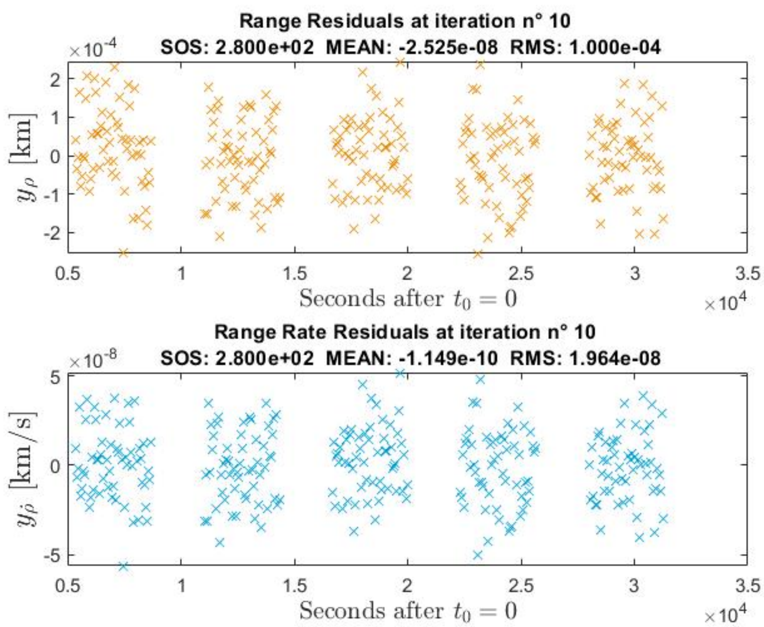

Figure 2: Range and Range Rate Residuals after iteration n° 10

Moreover, we report here the final value of X_0 with the associated uncertainties

	value	*uncertainty*
x_0	$-1.9318 \cdot 10^{-5}$	$8.0775 \cdot 10^{-6}$
y_0	$6.4192 \cdot 10^{3}$	$2.3110 \cdot 10^{-6}$
u_0	-7.1139	$2.5000 \cdot 10^{-9}$
v_0	$-0.2490 \cdot 10^{-5}$	$8.9794 \cdot 10^{-9}$
$\mu_{v,0}$	$3.2846 \cdot 10^{5}$	$2.3730 \cdot 10^{-3}$
C_D	2.0000	$3.7052 \cdot 10^{-7}$

2 Observable Analysis

In order to determine which observable is more important for our studies, we look at the information matrix for each one.
At this point, we remember that the elements on the diagonal of such matrix are the sum of the squares of the derivative of every observation with respect to each element in the state vector. Therefore, the bigger the value, the more important and useful that observable will be for that element of the state vector.

As shown in the code, we can definitely see that the values of the Range Rate information matrix are always bigger than the ones for the Range, hence indicating that the more useful observables between the two is the Range Rate.

3 Semimajor Axis and Eccentricity Evolution

In order to obtain an evolution of the semimajor axis a and eccentricity e over $24\,hours$ the first step is to define a vector containing time stamps from $1:86400$ seconds. Then we can just integrate the equation of motion previously described over that time interval and, from the values of the state vector X, we can compute both parameters.

$$e = ||\vec{e}\,|| = \left|\left|\frac{\vec{V} \times \vec{h}}{\mu_v} - \frac{\vec{r}}{r}\right|\right| \tag{3.1}$$

$$a = \frac{h^2}{\mu_v} \cdot \frac{1}{1 - e^2} \tag{3.2}$$

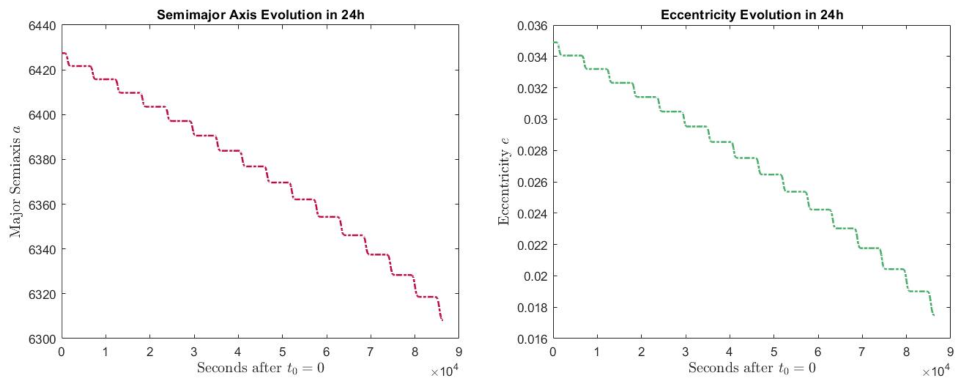

Figure 3: Evolution of semimajor axis and eccentricity in 24 hours

4 Altitude Evolution

Similarly to the study of semimajor axis and eccentricity, we can evaluate the altitude as

$$z = \sqrt{x^2 + y^2} - R_v \tag{4.1}$$

for every instant in 24 hours.
Then, we can just plot the result as here below

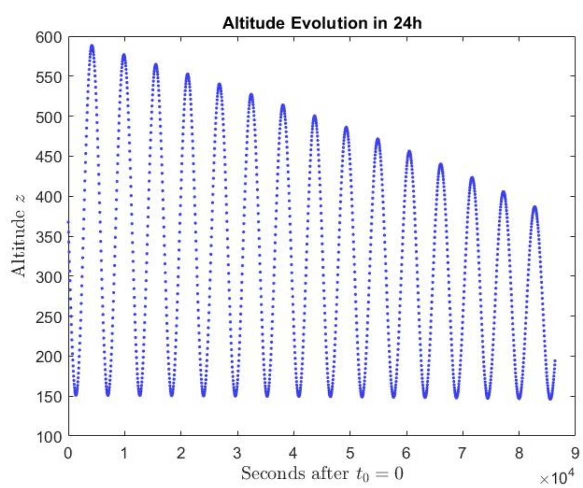

Figure 4: Altitude evolution in 24 hours

5 Correction Manoeuvre

As required, we must find the time at which, since we drop below $400, km$ in apocenter altitude, we must perform a correction manoeuvre.
We can evaluate the radius of the apocenter at each instant in the timespan as

$$R_{apo} = \frac{h^2}{\mu_v} \cdot \frac{1}{1-e} \tag{5.1}$$

and, when we find that $R_{apo} - R_v < 400\,km$ we can store the timestamp at which that occurred.
In particular, in our case the manoeuvre must occur after 22 hours, 10 minutes and 15 seconds.

6 Additional Information

$$A_{3,1} = -\frac{\mu}{r^3} + 3\frac{\mu x^2}{r^5} + \frac{1}{2} C_D \, \frac{S}{m} \, V \, u \, \frac{x}{Hr} \, \rho(z) \tag{6.1}$$

$$A_{3,2} = 3\frac{\mu xy}{r^5} + \frac{1}{2} C_D \frac{S}{m} V u \frac{y}{Hr} \rho(z) \tag{6.2}$$

$$A_{3,3} = -\frac{1}{2} C_D \frac{S}{m} \rho(z) \left(V + \frac{u^2}{V} \right) \tag{6.3}$$

$$A_{3,4} = -\frac{1}{2} C_D \frac{S}{m} u \rho(z) \frac{v}{V} \tag{6.4}$$

$$A_{3,5} = -\frac{x}{r^3} \tag{6.5}$$

$$A_{3,6} = -\frac{1}{2} \rho(z) \frac{S}{m} V u \tag{6.6}$$

$$A_{4,1} = 3\frac{\mu xy}{r^5} + \frac{1}{2} C_D \frac{S}{m} V v \frac{x}{Hr} \rho(z) \tag{6.7}$$

$$A_{4,2} = -\frac{\mu}{r^3} + 3\frac{\mu y^2}{r^5} + \frac{1}{2} C_D \, \frac{S}{m} \, V \, v \, \frac{y}{Hr} \, \rho(z) \tag{6.8}$$

$$A_{4,3} = -\frac{1}{2} C_D \frac{S}{m} v \frac{u}{V} \tag{6.9}$$

$$A_{4,4} = -\frac{1}{2} C_D \frac{S}{m} \rho(z) \left(V + \frac{v^2}{V} \right) \tag{6.10}$$

$$A_{4,5} = -\frac{y}{r^3} \tag{6.11}$$

$$A_{4,6} = -\frac{1}{2} \rho(z) \frac{S}{m} V v \tag{6.12}$$

$$\begin{cases} SOS \approx \text{n° of observations} \\ MEAN \approx 0 \\ RMS \approx \sigma_{obs} \end{cases} \tag{6.13}$$