Advanced Spacecraft Dynamics - Homework VI

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1 Introduction

This report analyses the Ballistic Reentry of the Soyuz capsule from a LEO at a specified altitude. We will analyse two cases which differ in the ΔV which is provided to the S/C at t_0. In particular we'll first analyse in detail Case 1 for which $\Delta V = 0.2\ km/s$ and subsequently report the results for Case 2 also providing a comment on the key differences between the two.

2 Initial Spacecraft Descent

The first step in the resolution of the problem is to define the initial conditions, which we are given under the form of

Quantity	*Units*	*Value*
h_0	km	400
$\theta_{G,0}$	deg	60
$\lambda_{g,0}$	deg	30

Table 1: Initial Conditions

Now, our aim will be that of propagating the position and velocity of the S/C in the inertial (ECI) reference frame, hence we use the provided initial conditions to define the initial conditions for the state $\underline{X}$.

$$\underline{r_0} = R_3^T(\theta_{G,0} + \lambda_{g,0}) \begin{bmatrix} r_E + h_0 \\ 0 \\ 0 \end{bmatrix} \tag{2.1}$$

$$\underline{v_0} = R_3^T(\theta_{G,0} + \lambda_{g,0}) \begin{bmatrix} 0 \\ \sqrt{\frac{\mu}{r_0}} - \Delta V \\ 0 \end{bmatrix} \tag{2.2}$$

where with $R_i(\alpha_j)$ we indicate a rotation about axis i of an angle α_j while with r_E and μ we indicate the equatorial earth radius and gravitational parameter respectively. Also notice that the velocity's second component is given by the circular velocity of the orbit at that altitude minus the ΔV applied for braking.

2.1 State Propagation

After having defined the initial conditions we can group them inside a state $\underline{X}$ as

$$\underline{X_0} = \begin{bmatrix} \underline{r_0} \\ \underline{v_0} \end{bmatrix} \quad , \quad \underline{\dot{X}} = \begin{bmatrix} \underline{v} \\ \frac{\mu}{r^3}\,\underline{r} \end{bmatrix} \tag{2.3}$$

and we can propagate this state using the well known Newton's Gravitational Law as reported above.
It is important to remark that we are required to stop the integration at the specified altitude for the Earth Interface $h_{EI} = 100\ km$. In the `Matlab` code this is obtained with the use of events inside the `ode113()` function.

The result of the propagation is the following trajectory,

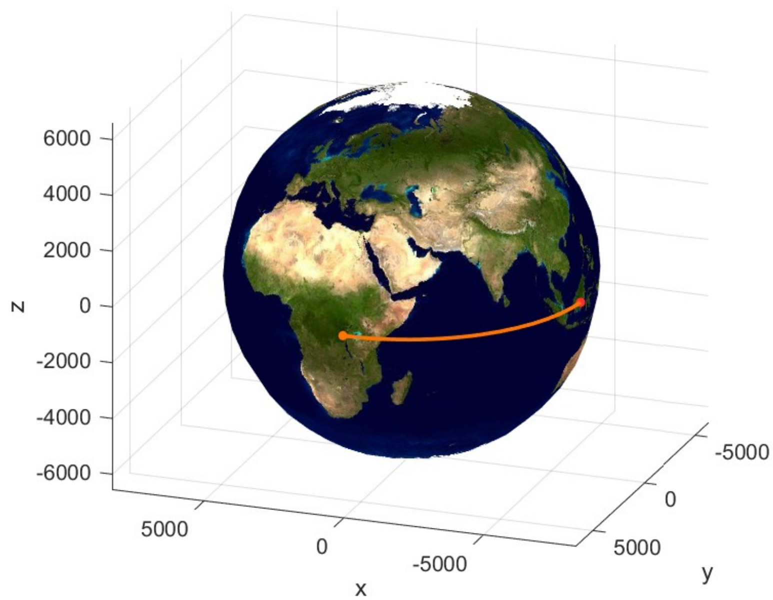

Figure 1: 3D Initial Descent Trajectory - Case 1

2.2 Inertial Velocity Analysis

Notice that we have considered all vectors as represented in the ECI reference frame while now, in order to compute the flight path angle γ, we need to rewrite the velocity in the Local Horizontal Plane reference frame LH. This is obtained through a series of rotations as follows,

$$\underline{v}^{(LH)^T} = \underline{v}^{(ECI)^T}\, R_3^T(\theta_G)\, R_3^T(\lambda_g)\, R_2^T(-L) \tag{2.4}$$

then, the radial and tangential components of velocity will be the respectively the first two components of $\underline{v}^{(LH)}$, therefore we'll be able to compute $\gamma_{I,ini}$ thanks to the `atan2()` function as

$$\gamma_{I,ini} = \arctan\left(\frac{v_r}{v_t}\right) \tag{2.5}$$

Finally, the results of this first part are reported in the table here below,

$v_{I,ini}$	$7.8247\ km/s$
$\gamma_{I,ini}$	$-2.9373°$

Table 2: Results of Part A - Case 1

2.3 Relative Velocity Analysis

The components of the relative velocity in the ECI reference frame can be computed from the following relation,

$$\underline{v_R} = \underline{v_I} - \underline{\omega_E} \times \underline{r} \tag{2.6}$$

where every vector has also been represented in the ECI reference frame and $\underline{\omega_E}$ is defined as

$$\underline{\omega_E} = \begin{bmatrix} 0 & 0 & \frac{2\pi}{D_{sid}} \end{bmatrix}^T \tag{2.7}$$

with D_{sid} representing the Sidereal Day of Earth.

Now, the procedure to find γ_R is the same as before and again, we can report the results in a table

$v_{R,ini}$	$7.3543\ km/s$
$\gamma_{R,ini}$	$-3.1254°$

Table 3: Results of Part B - Case 1

These will be the initial conditions for the Earth Reentry problem which will be investigated in the following sections.

3 Earth Reentry

In this section we'll analyse the Earth Reentry using a different mathematical model which we'll refer to as B1. It is important to notice that the Initial Conditions for this section will be the final results that we have obtained from the previous integration. Therefore we can introduce a new state vector $\underline{X}$ and define its initial conditions in $\underline{X0}$.

$$\underline{X} = \begin{bmatrix} r \\ \lambda_g \\ \gamma_R \\ v_R \end{bmatrix} \qquad , \qquad \underline{X_0} = \begin{bmatrix} r \\ \lambda_g \\ \gamma_R \\ v_R \end{bmatrix}\Bigg|_{EI} \tag{3.1}$$

3.1 State Propagation

Now, with the initial conditions defined, we'll propagate the state following the B1 mathematical model reported here below,

$$\dot{r} = v_R \sin\gamma_R \tag{3.2}$$

$$\dot{\lambda_g} = \frac{v_R \cos\gamma_R}{r} \tag{3.3}$$

$$\dot{\gamma_R} = -\frac{\mu}{r^2 v_R} + \frac{v_R}{r}\cos\gamma_R + \frac{\omega_E^2 r \cos\gamma_R}{v_R} + 2\omega_E \tag{3.4}$$

$$\dot{v_R} = -\frac{\mu}{r^2}\sin\gamma_R - \frac{D}{m} + \omega_E^2 r \sin\gamma_R \tag{3.5}$$

where

$$\rho = \rho_0 e^{-\beta_\rho h} \tag{3.6}$$

$$D = \frac{1}{2} C_d S \rho v_R^2 \tag{3.7}$$

with $\rho_0, \beta_\rho, C_d, S$ all being specified constant values.

Again, the propagation is interrupted when the value of r reaches r_E thanks to the landing event included in the `ode113()` options.
Finally, we can compute the evolution of the altitude as $h = r - r_E$.

3.2 Numerical Results Analysis

First, we can report the evolution in time of altitude h, flight path angle γ_R and the relative velocity v_R.

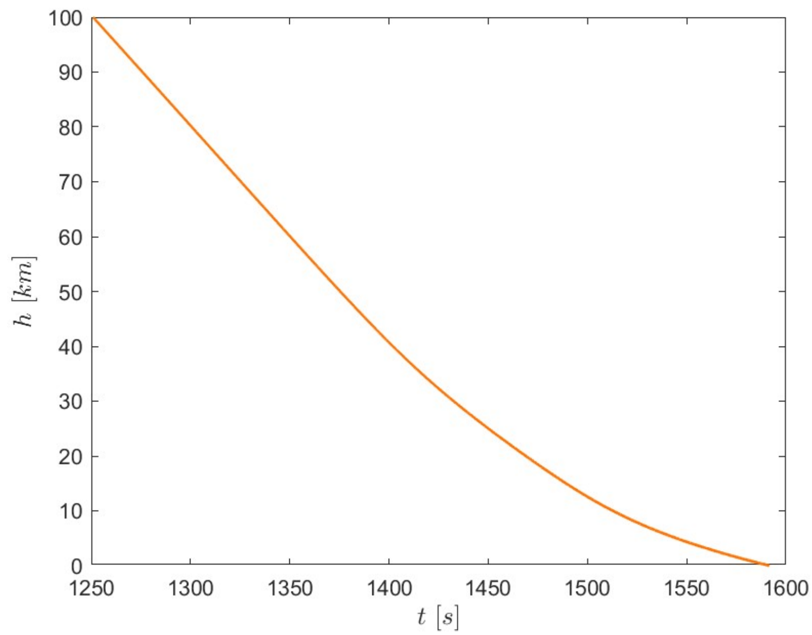

Figure 2: Evolution in time of the altitude of the S/C - Case 1

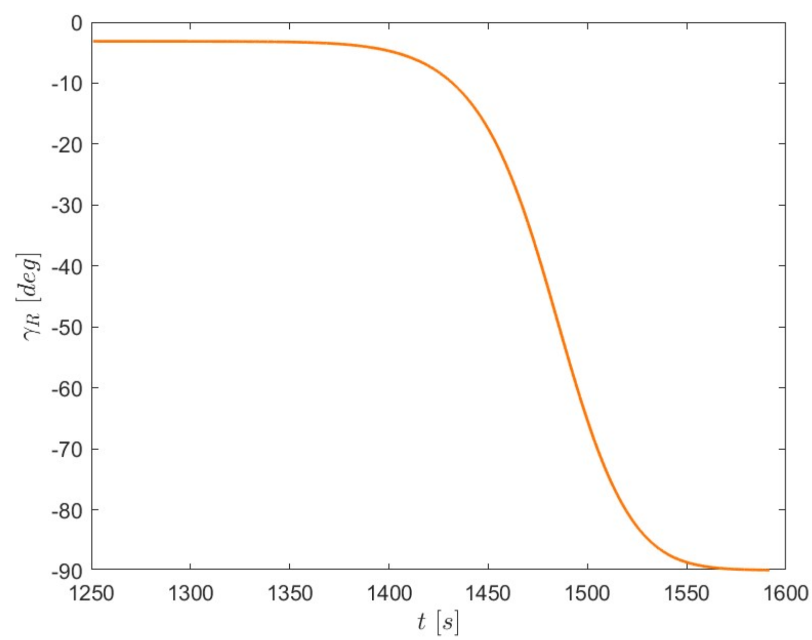

Figure 3: Evolution in time of the flight path angle - Case 1

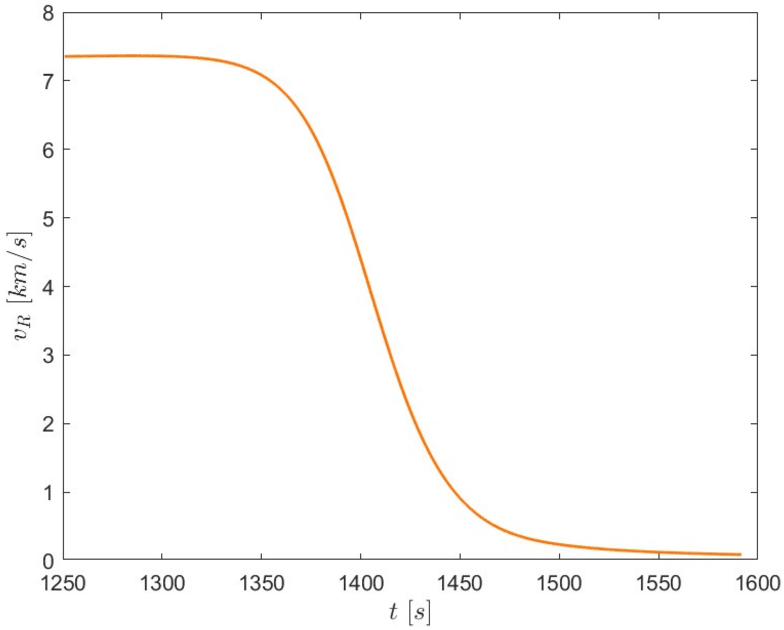

Figure 4: Evolution in time of relative velocity - Case 1

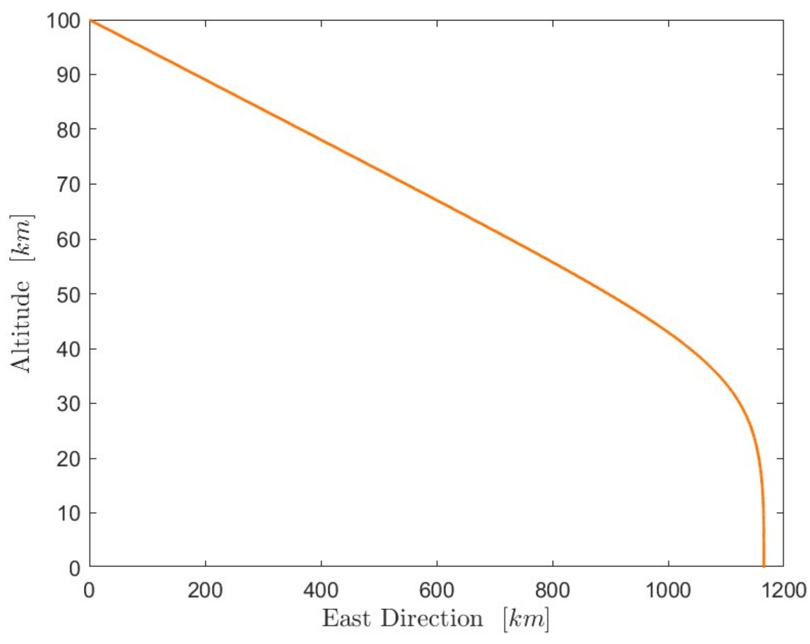

Figure 5: 2D Reentry Trajectory - Case 1

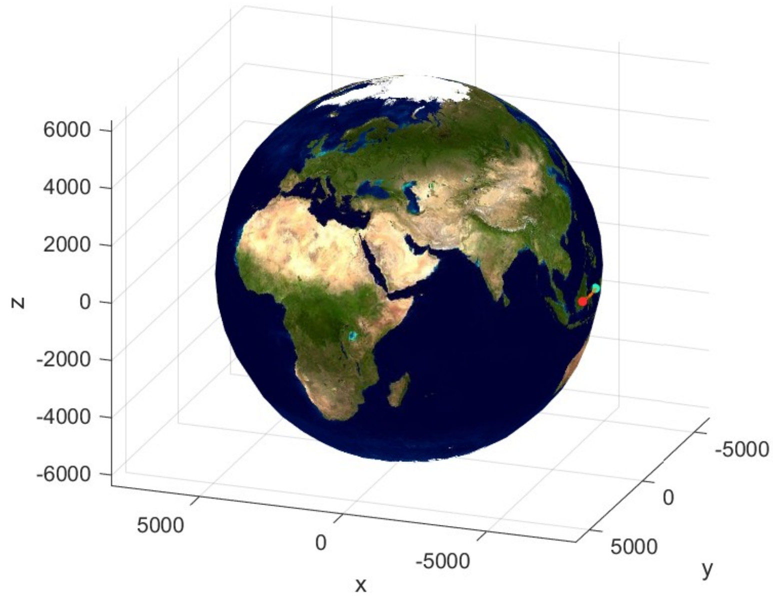

Figure 6: 3D Reentry Trajectory - Case 1

3.3 Spacecraft Attitude

It is relevant to investigate the Spacecraft Attitude as the quaternions associated to the rotation matrix from ECI to BA (Body Axes) reference frames $\underset{C\leftarrow N}{R}$.

In the more general case, the rotation matrix can be computed a succession of simple rotations as reported below,

$$\underset{C\leftarrow N}{R} = R_3(\theta_G)\, R_3(\lambda_g)\, R_2(-L)\, R_1(\xi_R)\, R_3(-\gamma_R)\, R_2(\sigma)\, R_1(\beta)\, R_3(-\alpha)\, R_A \tag{3.8}$$

where R_A is the rotation matrix from the auxiliary body axes frame to the body axes frame defined as

$$R_A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \tag{3.9}$$

In our particular case, we have that most angles are null

$$\alpha, \beta, \sigma, L, \xi_R = 0 \tag{3.10}$$

therefore, the rotation matrix reduces to

$$\underset{C \leftarrow N}{R} = R_A \, R_3(-\gamma_R) \, R_3(\lambda_g) \, R_3(\theta_G) \tag{3.11}$$

Finally, one must notice that since we have adopted the initial conditions in Table 1 as the conditions at $t_0 = 0 \ s$, then the initial conditions in the analysis of the reentry will not be the same ones, but they will actually be the final conditions of the initial descent analysed in Section 2.

Finally, we have all the data to compute this matrix at each epoch inside the timespan and compute the associated quaternions, which we can report here below.

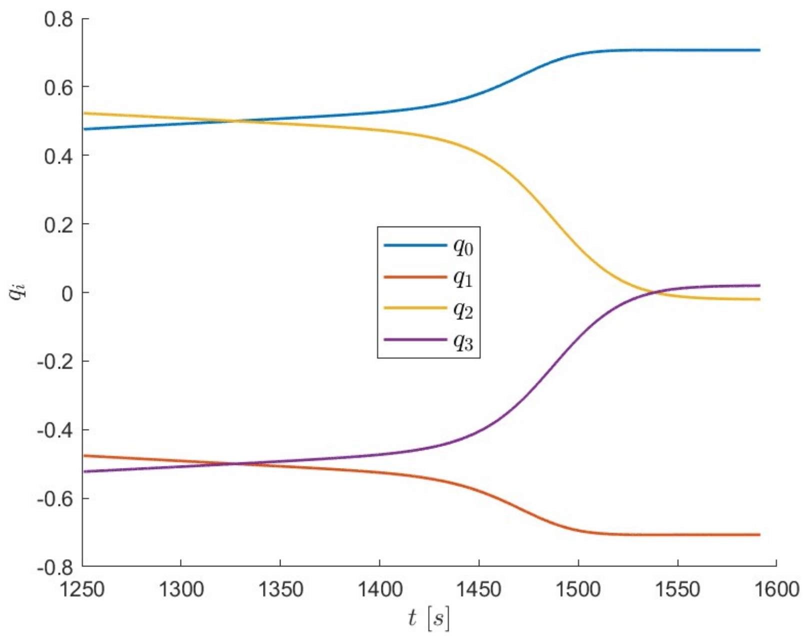

Figure 7: Spacecraft Attitude Quaternions - Case 1

It is easy to notice that the quaternions are symmetric as follows

$$\begin{cases} q_0 = -q_1 \\ q_2 = -q_3 \end{cases} \tag{3.12}$$

In order to understand the reason for this symmetry, we must convert to the associated Euler angles. This is quite useful since studying the attitude of the Body with respect to the Inertial frame may be useful in different applications.

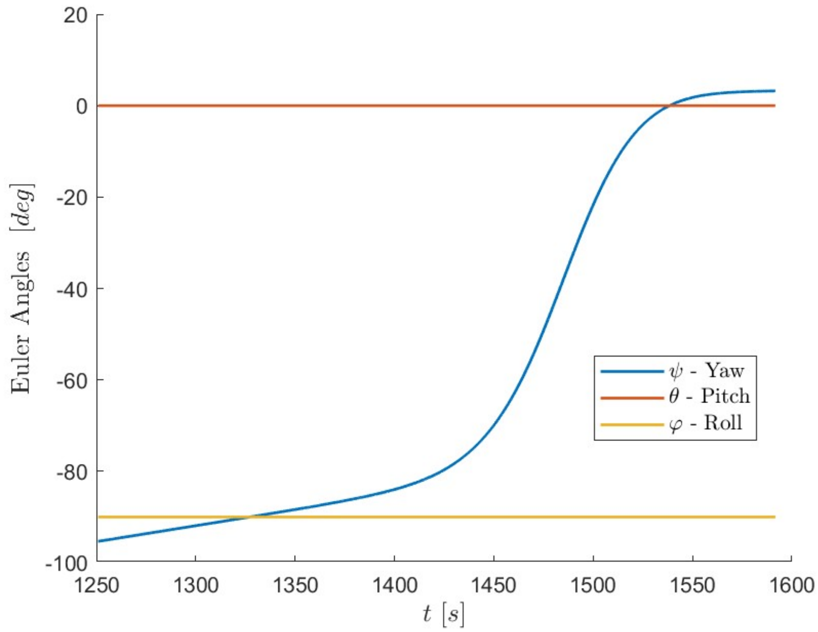

Figure 8: Spacecraft Attitude with Euler Angles - Case 1

From here we notice that while both pitch and roll are constant in time and respectively equal to 0 and $-\pi/2$, the yaw changes over time. By then looking at the general relation between quaternions and euler angles, we notice that in this particular case they can be reduced to

$$\begin{cases} q_0 = \cos\left(-\frac{\pi}{4}\right)\cos\frac{\psi}{2} \\ q_1 = \sin\left(-\frac{\pi}{4}\right)\cos\frac{\psi}{2} \\ q_2 = \sin\left(-\frac{\pi}{4}\right)\sin\frac{\psi}{2} \\ q_3 = \cos\left(-\frac{\pi}{4}\right)\sin\frac{\psi}{2} \end{cases} \tag{3.13}$$

Which proves the symmetry shown by the quaternions is no coincidence but it is instead a consequence of the particular conditions of the problem.

3.4 Sensible Design Parameters

In the study of a reentry trajectory it is important to compute certain quantities since they will bear a heavy importance in the design of the S/C. In particular, we will focus on the deceleration a_D, the heat in the stagnation point q_s and the dynamical pressure P_d and we will aim to compute their maximum values in order to dimension the S/C accordingly.

Before analysing these parameters it is important to report the time of flight, which is the time required by the S/C to complete the reentry.

$$ToF = 340\ s \tag{3.14}$$

3.4.1 Spacecraft Deceleration

We can consider the spacecraft deceleration defined as

$$a_D = -\dot{v_I} = -\frac{d}{dt}\sqrt{v_R^2 + \omega_E^2 r^2 + 2\omega_E r v_R \cos\gamma_R} \tag{3.15}$$

In the previous equation we notice that each parameter varies in time except for ω_E which can be treated as a constant. Therefore, by applying the chain rule, the previous equation can be rewritten into

$$a_D = -\frac{1}{2}\frac{2v_R\dot{v_R} + 2r\omega_E^2\dot{r} + 2\omega_E\left[r\cos\gamma_R\dot{v_R} + v_R\cos\gamma_R\dot{r} - rv_R\sin\gamma_R\dot{\gamma_R}\right]}{\sqrt{v_R^2 + \omega_E^2 r^2 + 2\omega_E v_R r\cos\gamma_R}} \tag{3.16}$$

3.4.2 Stagnation Point Heat

Again, we start from the definition of the Stagnation Point Heat which is provided as

$$q_s = \frac{K}{\sqrt{R_c}}\sqrt{\rho}\, v_R^3 \tag{3.17}$$

where R_c is the curvature radius at the stagnation point and K is a constant which depends on the planet.

3.4.3 Dynamical Pressure

Finally, the Dynamical Pressure is defined as

$$P_d = \frac{1}{2}\rho\, v_R^2 \tag{3.18}$$

Once we have evaluated these parameters at each value of the descent altitude, we can numerically find their maximum value and the report the time instant for which it verifies.

Quantity	*Units*	*Value*	*Time* [s]
$a_{D,max}$	g_0	9.7036	153.8906
$P_{d,max}$	bar	0.45184	153.9906
$q_{s,max}$	MW/m^2	1.0385	125.6906

Table 4: Results of Part E - Case 1

3.5 Analytical-Numerical Solutions Comparison

Finally, it is possible to compare the results obtained from the numerical propagation performed above with the results of a simplified version of the problem which allows an analytical solution called Allen-Eggers solution which uses the following assumptions,

$$\begin{cases} \frac{D}{m} >> \frac{\mu}{r^2}\sin\gamma_R \\ \frac{v_r}{r} - \frac{mu}{r^2 v_r} \approx 0 \end{cases} \tag{3.19}$$

It is then possible to show that these hypothesis lead to a analytical solution of the problem reported here below,

$$v_{R,AE} = v_{R,EI}\exp\left[Ce^{-\beta h}\right] \tag{3.20}$$

$$a_{D,AE} = \frac{B\rho_0}{2}v_{R,EI}^2 e^{-\beta h}\exp\left[2Ce^{-\beta h}\right] \tag{3.21}$$

$$q_{s,AE} = \frac{K}{\sqrt{R_c}}\sqrt{\rho}\,v_{R,AE}^3 \tag{3.22}$$

$$P_{d,AE} = \frac{1}{2}\rho\, v_{R,AE}^2 \tag{3.23}$$

where

$$B = \frac{C_d S}{m} \tag{3.24}$$

$$C = \frac{B\rho_0}{2\beta\sin\gamma_{R,EI}} \tag{3.25}$$

Finally, we can compare the solutions by analysing their evolutions with respect to the altitude h.

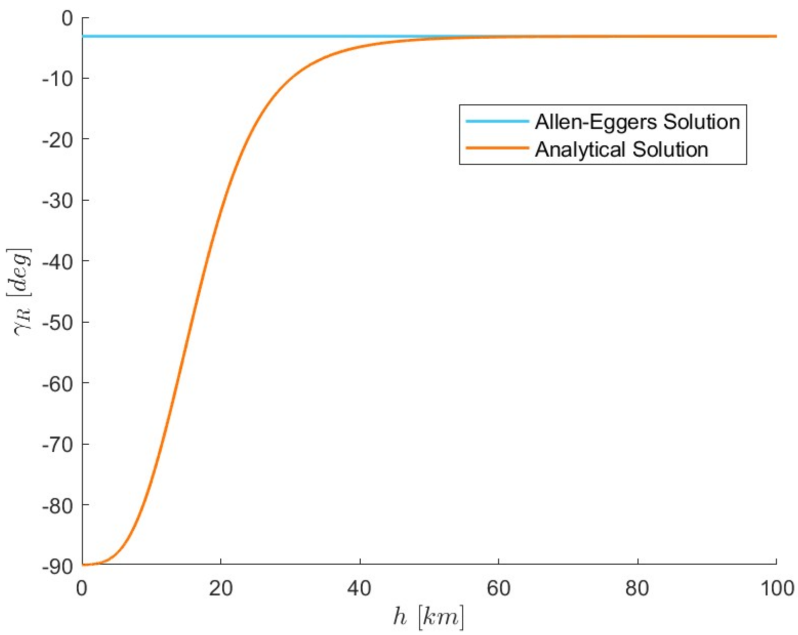

Figure 9: Evolution of the flight path angle along trajectory - Case 1

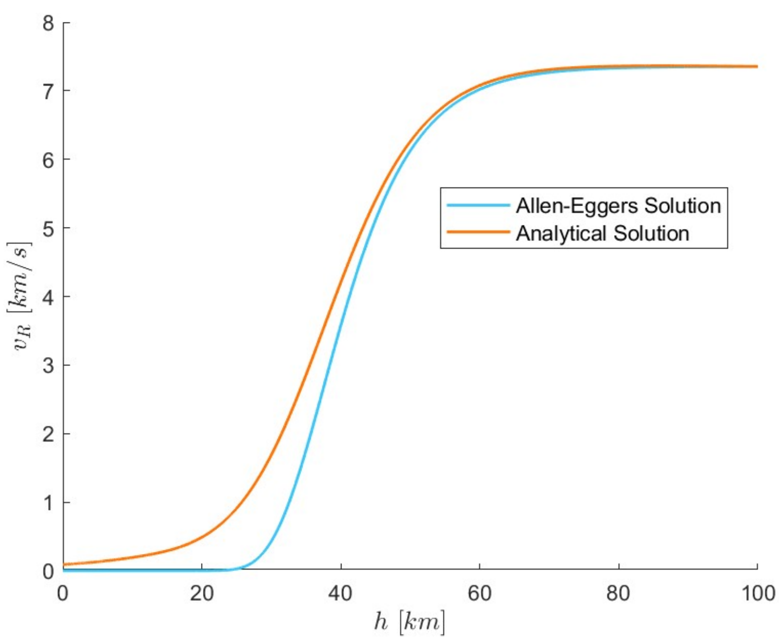

Figure 10: Evolution of the Relative Velocity along trajectory - Case 1

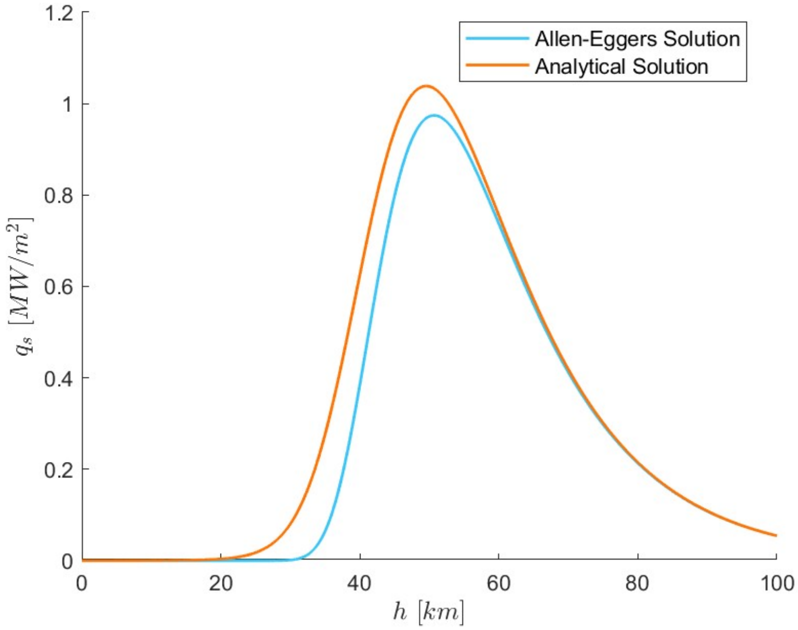

Figure 11: Evolution of the Stagnation Heat along trajectory - Case 1

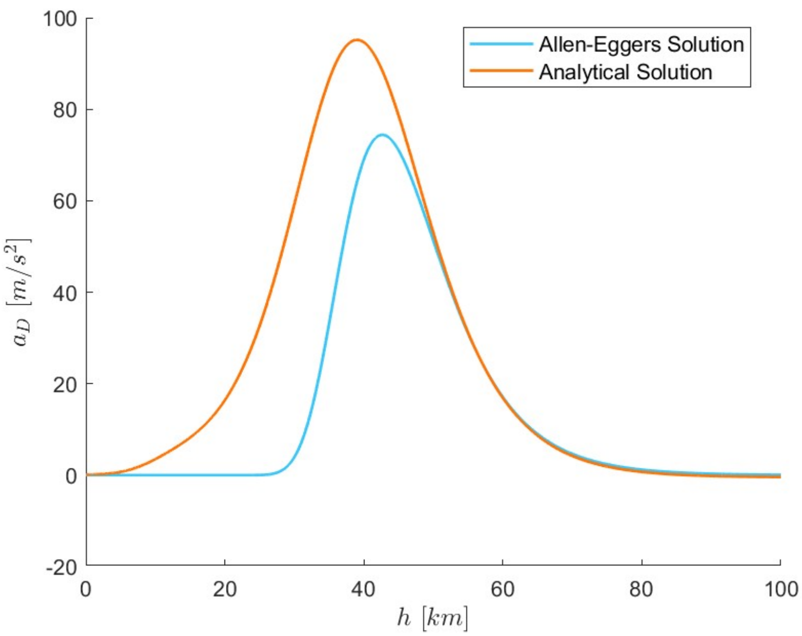

Figure 12: Evolution of the Spacecraft Deceleration along trajectory - Case 1

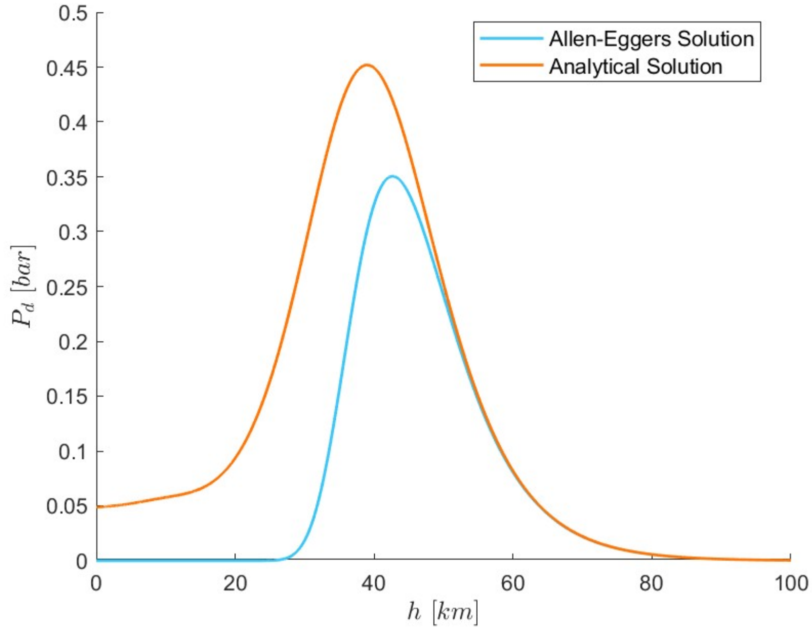

Figure 13: Evolution of the Dynamical Pressure along trajectory - Case 1

4 Second Case Analysis

The procedure followed for Case 2 in exactly the same as the one previously studied but naturally a different value of the initial ΔV will alter most of the results.
In fact, we notice that a higher initial braking will cause the earth reentry trajectory to be steeper, meaning that the S/C will land in a smaller total time of flight. This causes both a variation of the landing location as shown by the computed trajectories below, and also an increase in all sensible design parameters.
This means that a faster reentry will cause larger stress on the S/C under any aspect, hence we may need to improve its structural properties in order for this reentry approach to be viable.

4.1 Results

Again, the results for the analysis will be provided here below via tables and graphical representations.

$v_{I,ini}$	$7.6340\ km/s$
$\gamma_{I,ini}$	$-4.9852°$

Table 5: Results of Part A - Case 2

$v_{R,ini}$	$7.1648\ km/s$
$\gamma_{R,ini}$	$-5.3126°$

Table 6: Results of Part B - Case 2

$$ToF = 273.52\ s \tag{4.1}$$

Quantity	*Units*	*Value*	*Time [s]*
$a_{D,max}$	g_0	14.2200	96.1508
$P_{d,max}$	bar	0.6616	96.1508
$q_{s,max}$	MW/m^2	1.2541	80.2508

Table 7: Results of Part E - Case 2

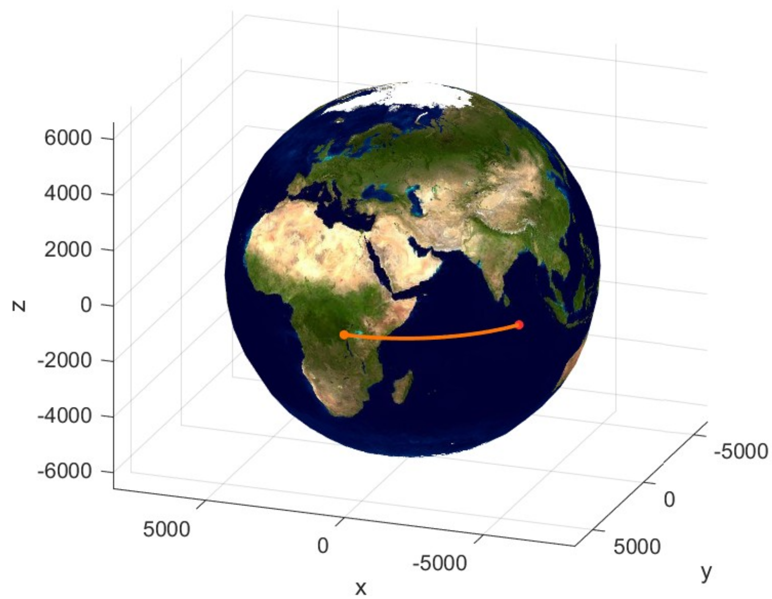

Figure 14: 3D Initial Descent Trajectory - Case 2

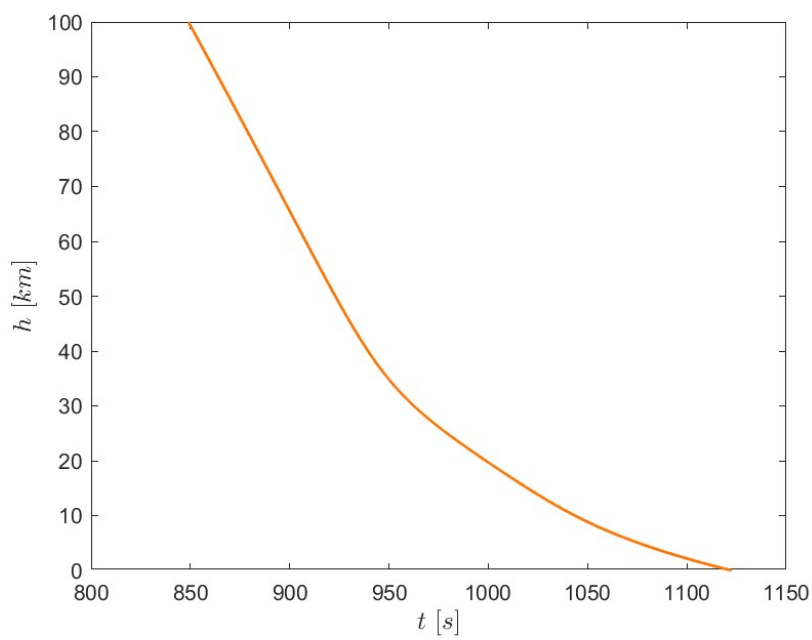

Figure 15: Evolution in time of the altitude of the S/C - Case 2

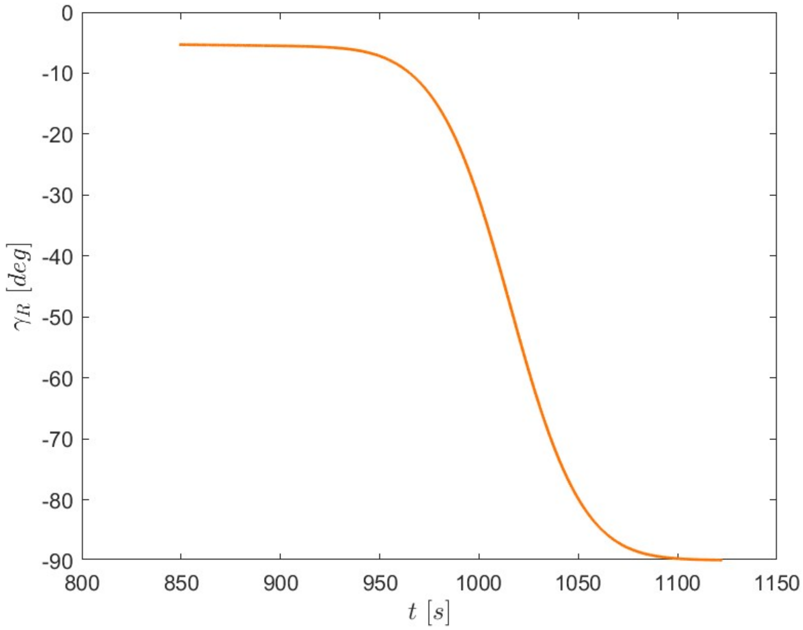

Figure 16: Evolution in time of the flight path angle - Case 2

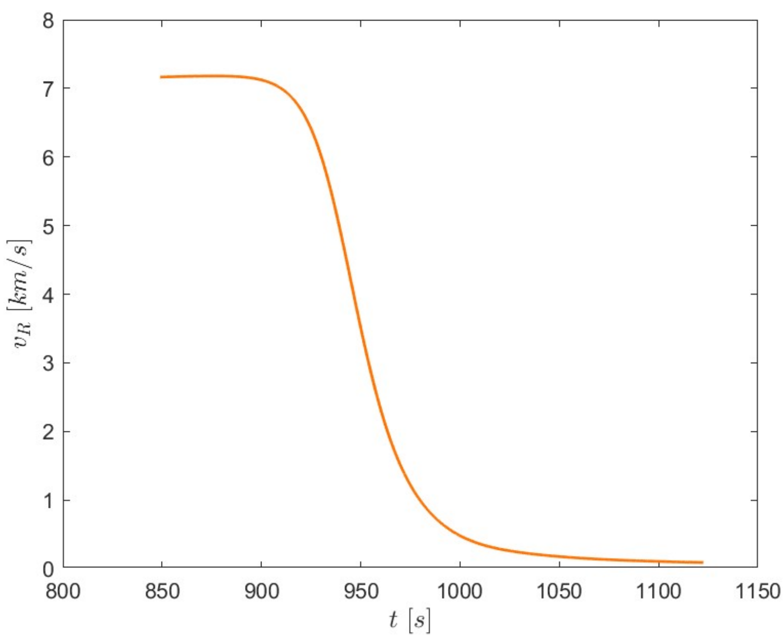

Figure 17: Evolution in time of relative velocity - Case 2

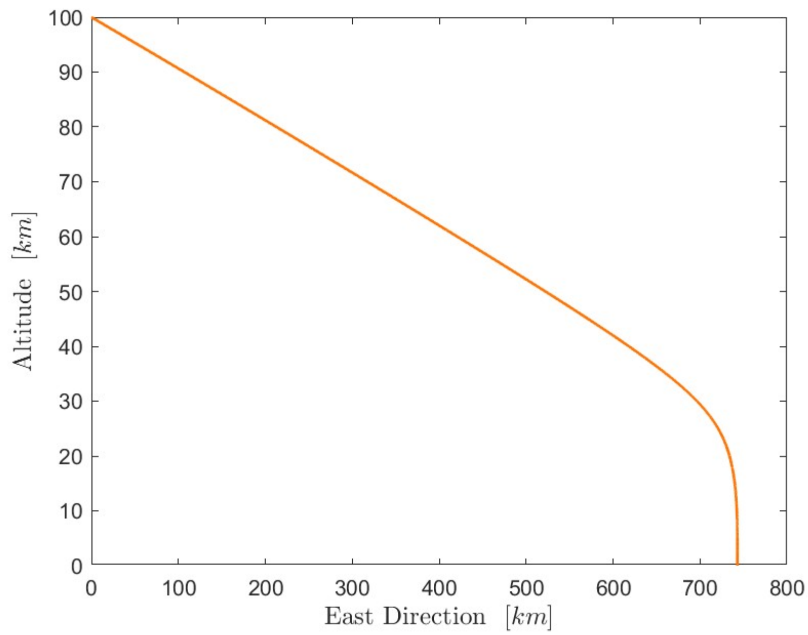

Figure 18: 2D Reentry Trajectory - Case 2

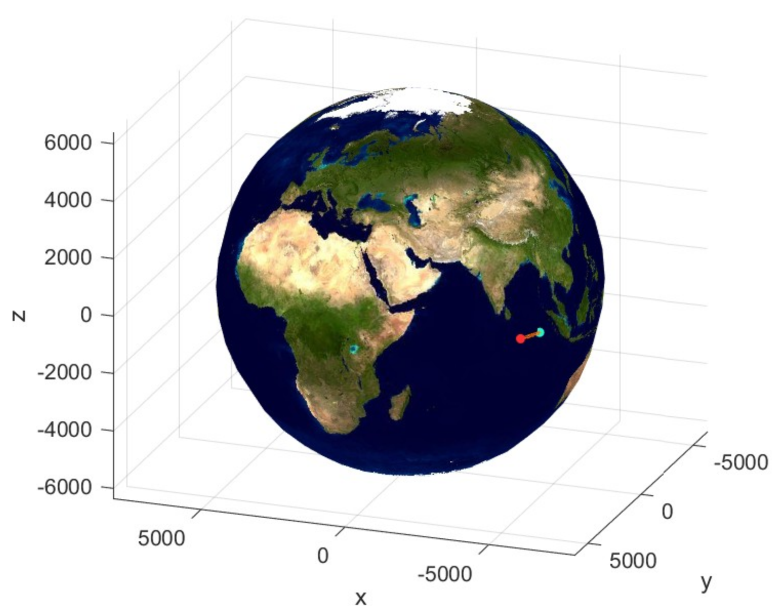

Figure 19: 3D Reentry Trajectory - Case 2

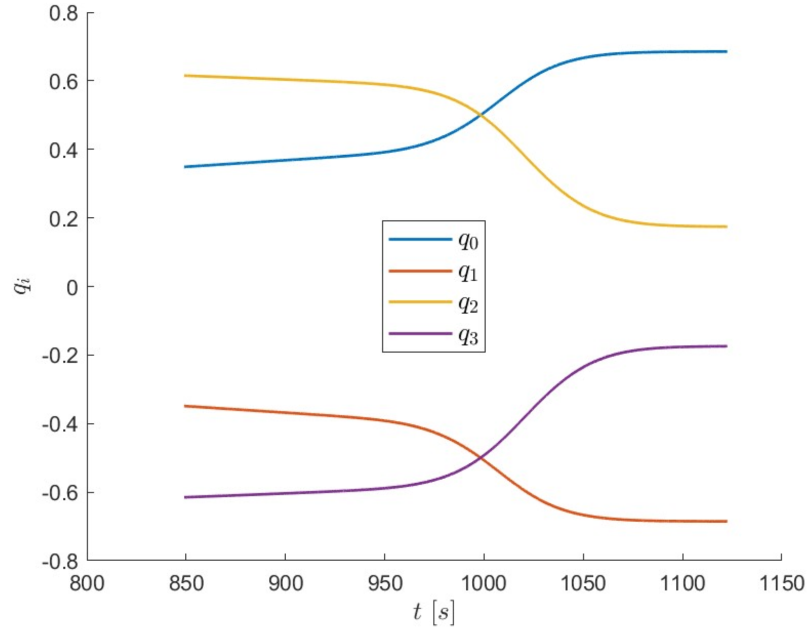

Figure 20: Spacecraft Attitude Quaternions - Case 2

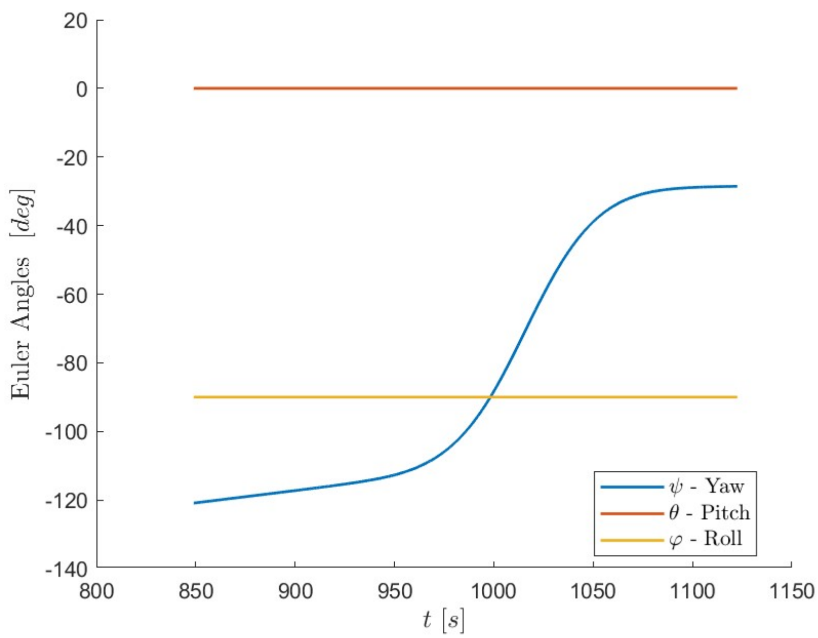

Figure 21: Spacecraft Attitude with Euler Angles - Case 2

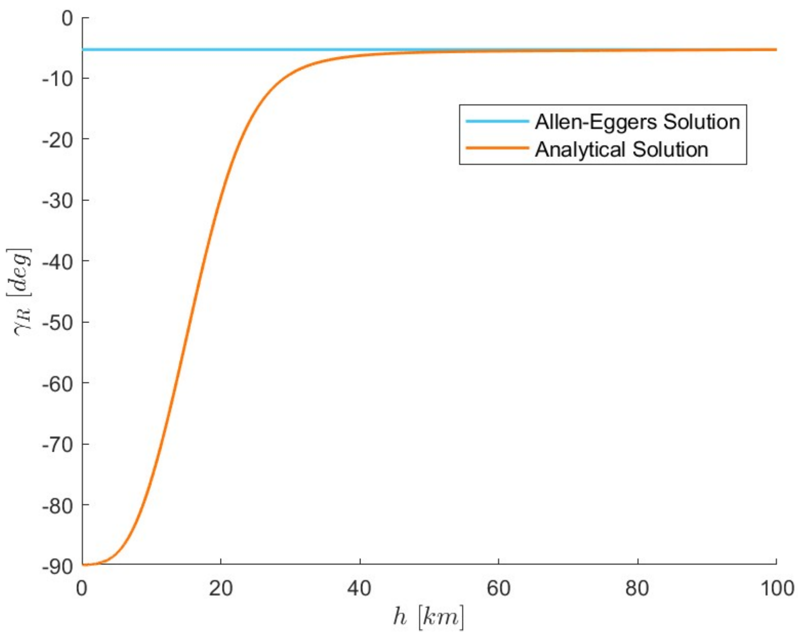

Figure 22: Evolution of the flight path angle along trajectory - Case 2

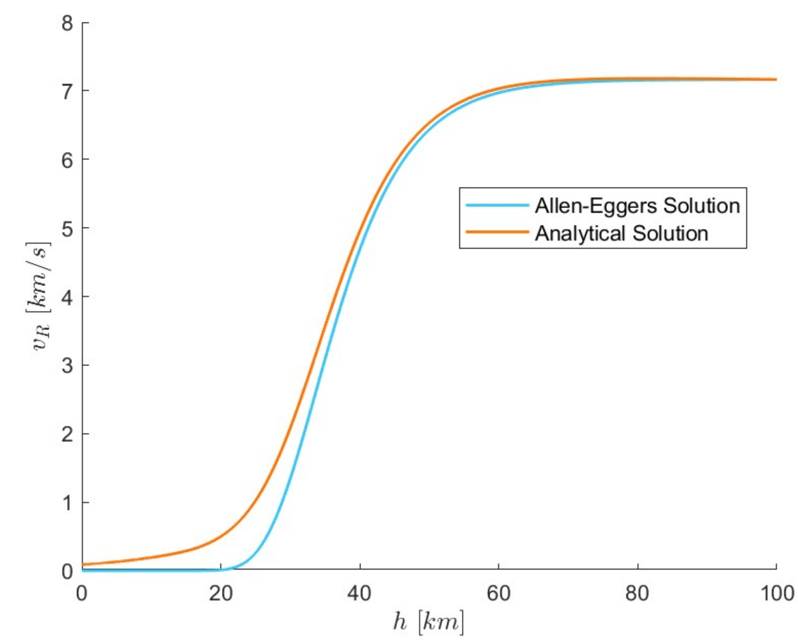

Figure 23: Evolution of the Relative Velocity along trajectory - Case 2

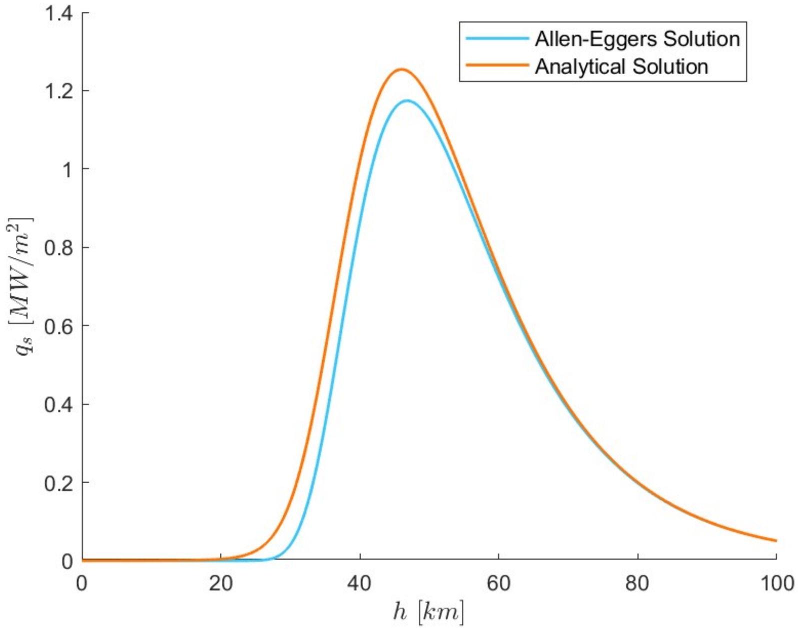

Figure 24: Evolution of the Stagnation Heat along trajectory - Case 2

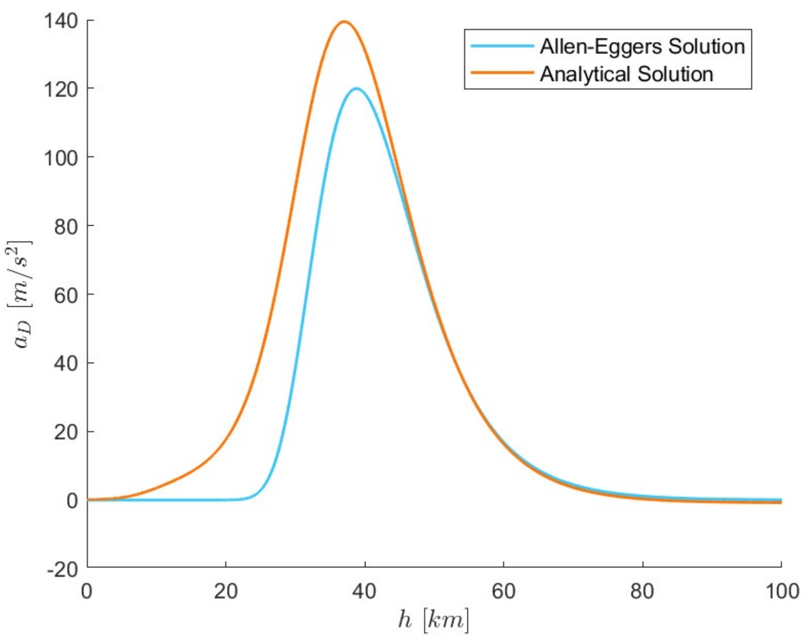

Figure 25: Evolution of the Spacecraft Deceleration along trajectory - Case 2

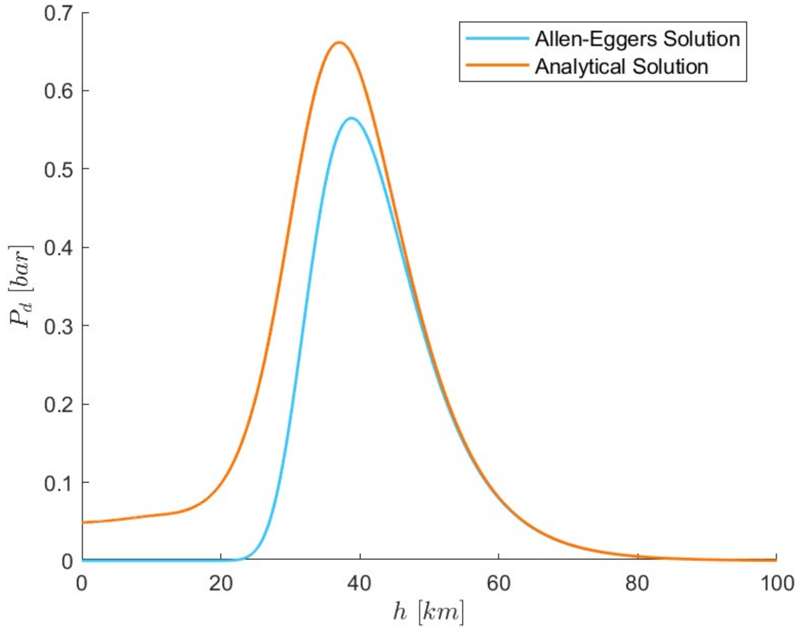

Figure 26: Evolution of the Dynamical Pressure along trajectory - Case 2