Advanced Spacecraft Dynamics - Homework V

Leonardo Russo

2015563

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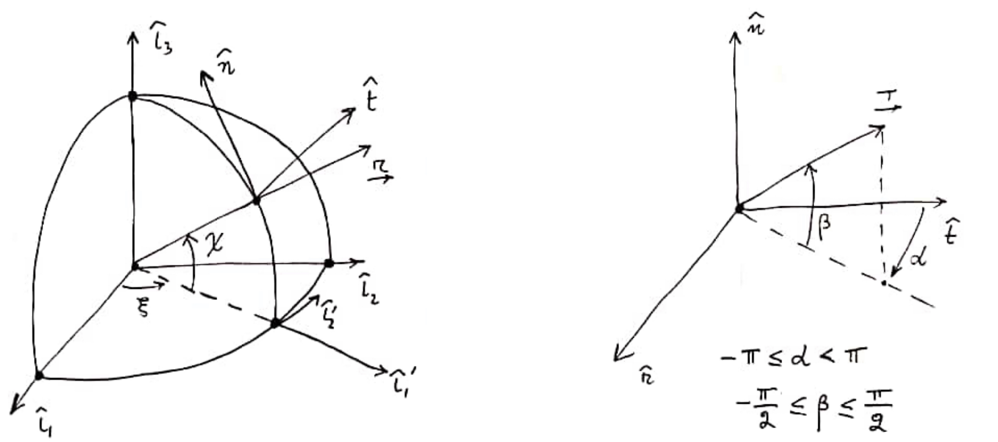
Contents

1 Introduction

In this report we will analyse the Minimum Time Continuous Thrust Transfer between two specified LEO and GEO orbits. First, we can assume that the two orbits will be coplanar, so that the problem becomes two-dimensional in which the state of the S/C can be represented inside the $\underline{x}$ vector which contains

$$\underline{x} = \begin{bmatrix} r & \xi & v_r & v_t \end{bmatrix}^T \tag{1.1}$$



(a) Inertial and Rotating Frames

(b) Generic Thrust Direction

Figure 1: Geometry of the Problem

Moreover, we will introduce the provided constants for these problem and store them together with other known quantities inside a parameter vector $\underline{p}$ as

$$c = 20 \quad km/s \tag{1.2}$$

$$u_T^{(max)} = 0.05 \cdot g_0 \quad m/s^2 \tag{1.3}$$

$$\underline{p} = \begin{bmatrix} c & u_T^{(max)} & \mu \end{bmatrix}^T \tag{1.4}$$

1.1 Initial Guess

Finally, for the solution of the problem we are given 4 initial values which will serve as our initial guess. These 4 values are the t_f and the initial costate vector $\underline{\lambda_0}$. Notice that from a simple analysis performed later we will see that λ_{2f} and $\dot{\lambda}_2$ are both null, meaning that $\lambda_2 = 0$ at each epoch. This is coherent with the data provided since, in fact, we are only given a initial guess of λ_{10}, λ_{30}, λ_{40} meaning that,

$$\underline{\lambda_0} = \begin{bmatrix}-1 & 0.2 & -1\end{bmatrix}^T \tag{1.5}$$

$$t_{f,0} = 10 \quad TU \tag{1.6}$$

Now, we can define the vector which will be the main input to `fmincon()` as $\underline{X}$, which will have initial conditions $\underline{X_0}$ as reported below,

$$\underline{X} = \begin{bmatrix}\underline{\lambda} \\ t_f\end{bmatrix} \qquad , \qquad \underline{X_0} = \begin{bmatrix}\underline{\lambda_0} \\ t_{f,0}\end{bmatrix} \tag{1.7}$$

2 Optimization

First of all, out goal will be that of minimizing the transfer time, therefore we will adopt an objective function

$$J = t_f \tag{2.1}$$

Then we can create the function `minTimeTransf` which will compute the value of the objective function from the state $\underline{X}$.

2.1 Definition of the Constraints

Then, the initial and final position and velocities of the satellite are specified as the vectors $\underline{x_{ini}}$ and $\underline{x_{fin}}$, which will form the constraints vector $\underline{\Psi}$ as,

$$\underline{x_{ini}} = \begin{bmatrix} R_{LEO} \\ 0 \\ 0 \\ \sqrt{\frac{\mu}{R_{LEO}}} \end{bmatrix} \qquad , \qquad \underline{x_{fin}} = \begin{bmatrix} R_{GEO} \\ - \\ 0 \\ \sqrt{\frac{\mu}{R_{GEO}}} \end{bmatrix} \tag{2.2}$$

meaning that $\underline{\Psi}$ will be a 7x1 vector since ξ_{fin} is not specified.

$$\underline{\Psi} = \begin{bmatrix} \underline{x_0} - \underline{x_{ini}} \\ \underline{x_f} - \underline{x_{fin}} \end{bmatrix} \tag{2.3}$$

Then, we must compute the Hamiltonian for the problem but first we must recall the S/C state dynamics described inside vector $\underline{f}$. Note that we will report both quantities dropping the names of each variable and using just their index inside the $\underline{x}$ vector,

$$
\underline{\dot{x}} = \underline{f}(x) = \begin{bmatrix} x_3 \\ \frac{x_4}{x_1} \\ -\frac{\mu}{x_1^2} + \frac{x_4^2}{x_1} + a_{T,r} \\ -\frac{x_4 x_3}{x_1} + a_{T,t} \end{bmatrix} \tag{2.4}
$$

Then we can compute H as

$$
H = \underline{\lambda}^T \underline{f} \tag{2.5}
$$

$$
H = \lambda_1 x_3 + \lambda_2 \frac{x_4}{x_1} + \lambda_3 \left[-\frac{\mu}{x_1^2} + \frac{x_4^2}{x_1} + a_{T,r} \right] + \lambda_4 \left[-\frac{x_4 x_3}{x_1} + a_{T,t} \right] \tag{2.6}
$$

where $a_{T,r}$ and $a_{T,t}$ can be respectively rewritten as

$$
a_{T,r} = \frac{u_T}{x_7} \sin u \tag{2.7}
$$

$$
a_{T,t} = \frac{u_T}{x_7} \cos u \tag{2.8}
$$

The term x_7 can be considered as the last component of the $\underline{x}$ in the general 3D case of optimization. In this case this term can be removed from $\underline{x}$ because its dynamic allows a solution in closed from if we know a priori u_T. Hence, since we are in the case of minimum time continuous thrust transfers, we know that

$$
u_T = u_T^{(max)} = const. \tag{2.9}
$$

and therefore we will have

$$
\dot{x_7} = \frac{-u_T^{(max)}}{c} \qquad \rightarrow \qquad x_7(t) = 1 - \frac{u_T^{(max)}}{c} t \tag{2.10}
$$

This means that H can be finally rewritten as

$$
H = \lambda_1 x_3 + \lambda_2 \frac{x_4}{x_1} + \lambda_3 \left[-\frac{\mu}{x_1^2} + \frac{x_4^2}{x_1} + \frac{u_T^{(max)} c}{c - u_T^{(max)} t} \sin u \right] + \lambda_4 \left[-\frac{x_4 x_3}{x_1} + \frac{u_T^{(max)} c}{c - u_T^{(max)} t} \cos u \right] \tag{2.11}
$$

After we have computed H, we can solve the adjoint equations to find $\underline{\dot{\lambda}}$ as

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} \tag{2.12}$$

which leads us to

$$\dot{\lambda_1} = \frac{\lambda_2 x_4}{x_1^2} - \lambda_3 \left[\frac{2\mu}{x_1^3} - \frac{x_4^2}{x_1^2}\right] - \lambda_4 \left[\frac{x_3 x_4}{x_1^2}\right] \tag{2.13}$$

$$\dot{\lambda_2} = 0 \tag{2.14}$$

$$\dot{\lambda_3} = -\lambda_1 + \lambda_4 \frac{x_4}{x_1} \tag{2.15}$$

$$\dot{\lambda_4} = -\frac{\lambda_2}{x_1} - \lambda_3 \frac{2x_4}{x_1} + \lambda_4 \frac{x_3}{x_1} \tag{2.16}$$

Finally one can conclude that the constraints for this minimization problem regard 3 equality conditions on the final conditions of the S/C state $\underline{x}$ and, thanks to the ignorability of the transversality condition, a inequality condition on H_f as,

$$r_f - R_{GEO} = 0 \tag{2.17}$$

$$v_{r,f} = 0 \tag{2.18}$$

$$v_{t,f} - sqrt\frac{\mu}{R_{GEO}} = 0 \tag{2.19}$$

$$H_f \leq 0 \tag{2.20}$$

All 4 constraints will be generated by the properly defined `nonlinconstr()` function, which will take $\underline{X}$ as input along other quantities mainly needed for the computation of the final states.
In fact, it important to realize that in order to compute $\underline{x_f}$ and H_f it is necessary to propagate forward the initial states following the dynamics that we have seen above.

2.2 Propagation Procedure

A propagation will be therefore necessary to compute the final values of the state S, which is be defined as

$$\underline{S} = \begin{bmatrix} \underline{x} \\ \underline{\lambda} \\ \underline{p} \end{bmatrix} \tag{2.21}$$

where we need to provide the necessary parameters inside $\underline{p}$ even though these will not change in time.

2.2.1 Initial Conditions

The initial conditions that we will use for each propagation will be

$$\underline{S_0} = \begin{bmatrix} \underline{x_0} \\ \underline{\lambda_0} \\ \underline{p} \end{bmatrix} \tag{2.22}$$

where $\underline{\lambda_0}$ are the initial values of the costate computed in each iteration of the `fmincon()` function, while $\underline{x_0}$ is always the same and equal to $\underline{x_{ini}}$.

2.2.2 State Update

Now, we already have the dynamics for the $\underline{\lambda}$ of the $\underline{S}$ vector, but we still need to compute the control in order to integrate the dynamics of the S/C state $\underline{x}$.
Following the Pontryagin's principle, it is possible to prove that the control u that minimizes H is the one that aligns $\underset{\rightarrow}{T}$ with the Primer vector $\underset{\rightarrow}{\tilde{p}}$, which leads to

$$\sin u = -\frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}} \tag{2.23}$$

$$\cos u = -\frac{\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}} \tag{2.24}$$

Finally, after performing the integration we will have all the necessary quantities to test the constraints in an iterative process which is carried out by the `fmincon()` function.

3 Results

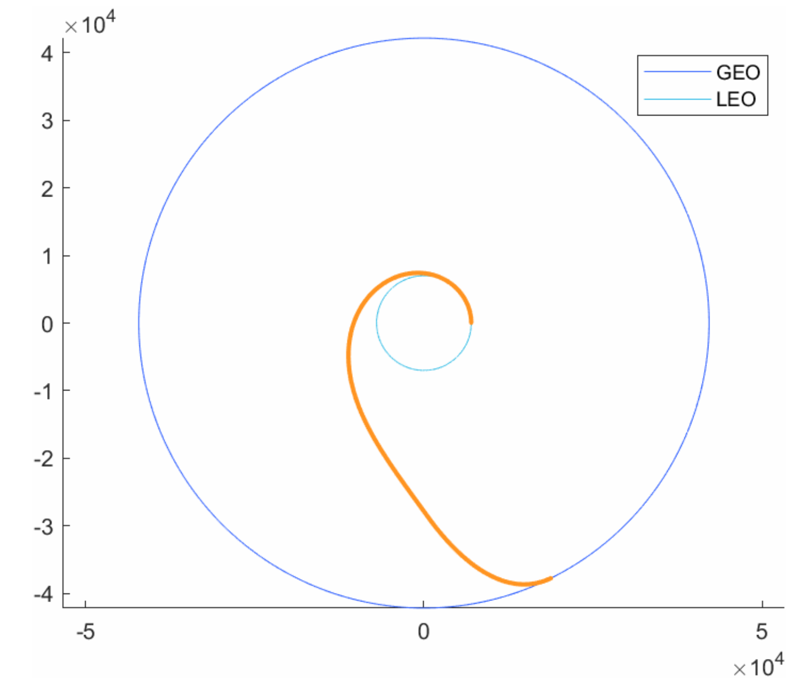


Figure 2: Animation of the 2D Orbit

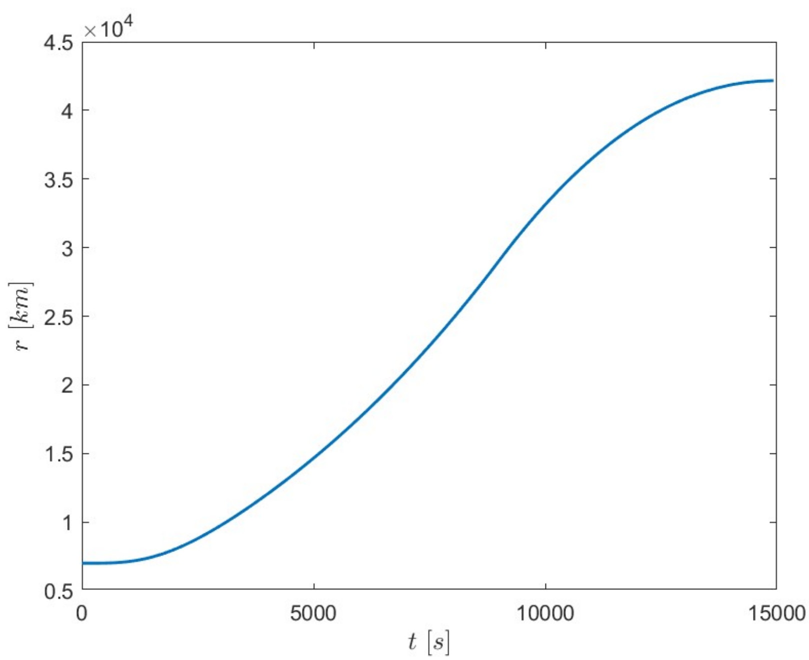


Figure 3: Plot of the radial position of the S/C

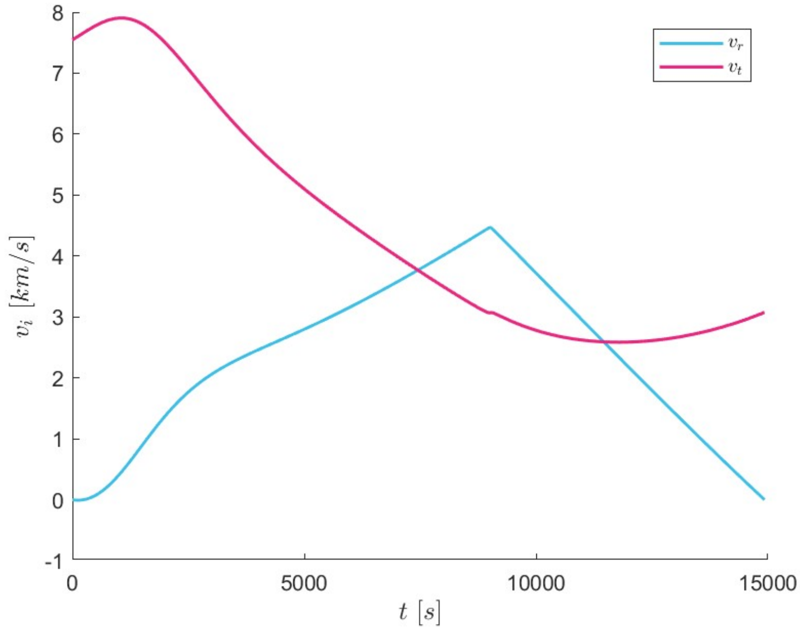


Figure 4: Plot of the radial position of the S/C

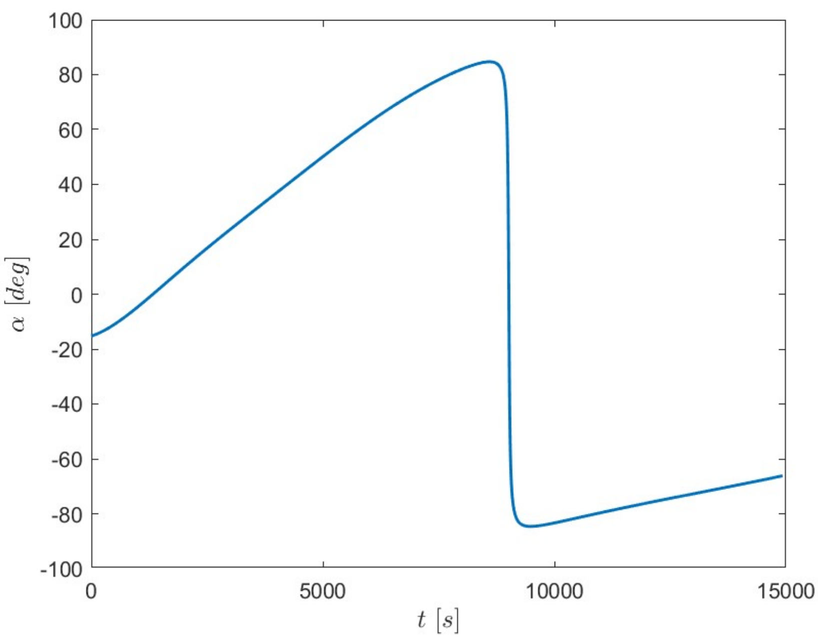


Figure 5: Plot of the radial position of the S/C