On the Uniqueness of the Shapley Value by P. Dubey[1]

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Outline

Abstract

A Unique Value for Superadditive Games

Shapley showed that there is a unique value for the class of all superadditive games D that follows certain, intuitive axioms

Shapley raised the question of whether an axiomatic foundation could be found for such a value in the context of the subclass C or simple games alone.

This paper shows that this is possible.

Theorem 1: A proof of Shapley's Theorem for all games

The paper give a new, simple proof of Shapley's Theorem for the set of all games G, not just superadditive ones

Theorem 2: A unique value using an axiomatic variant

For the classes of games C' and C'', Shapley's axioms do not specify a unique value.

However, with some modification to one of the axioms (specifically the third), a unique value is obtained, and it happens to be the Shapley Value.

Notation

N-player games may be referred to by their characteristic function, since that is the differentiating feature.

$\mathbb{R}$ and $\mathbb{Z}$ are the sets of Real and Integer numbers, respectively

Lowercase s, t, and n might be used in place of $|S|, |T|$, and $|N|$ respectively. Hopefully it is intuitive where this has occurred.

For a vector v, v_i is the i^{th} component of v.

i is used as both a number and a name for the same player in N.

$\vee$ and $\wedge$ are the join and meet operations, respectively.

Introduction

A value for the power of players in n-Player Cooperative Games

An *n-person cooperative game in characteristic function form* G is defined as a pair

$$G = (N, v)$$

where N is a set of n players

$$N = (1, \ldots, n)$$

and v is a function

$$v : 2^N \to \mathbb{R} \text{ such that } v(\emptyset) = 0$$

- $v(S)$ represents the "worth"/"value"/"power" of coalition S
- Given a game G, or more specifically a characteristic function v, it is desirable to know the "value" of each player in the game.

Let the set of all n player games be G. We also refer to a game as its characteristic function v

Let ϕ be a function

$$\phi : G \to \mathbb{R}^n$$

where $\phi_i(v)$ is the value of the i^{th} player in the game.

Preliminary Concepts

1. A coalition S is called a *carrier* for the game if

$$v(T) = v(T \cap S) \qquad \forall \qquad T \subset N$$

2. If $\pi : N \to N$ is a permutation of N, then the game with characteristic function πv is defined by

$$\pi_v(T) = v(\pi(T)) \qquad \forall \qquad T \subset N$$

3. Given any two games v_1, v_2, $v_1 + v_2$ is defined by

$$(v_1 + v_2)(T) = v_1(T) + v_2(T)$$

Three Axioms

Shapley defined 3 axioms (in class we covered 4) that a ϕ should satisfy.

1. If S is a carrier for v, then $\sum_{i \in S} \phi_i(v) = v(S)$
2. For any permutation π and $i \in N$,

$$\phi_{\pi(i)}(\pi v) = \phi_i(v)$$

3. If v_1 and v_2 are any games, then

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$$

Theorem 1

Theorem (Shapley (1953))

There is a unique function ϕ, defined on G, which satisfies the axioms S1, S2, S3.

Proof of Theorem 1

Proof. For each coalition S and a constant c, define the game $v_{S,c}$ by

$$v_{S,c} = \begin{cases} 0 \text{ if } S \not\subset T \\ c \text{ if } S \subset T \end{cases}$$

Then S and its supersets must all be carriers for $v_{S,c}$ (since only S and its supersets can have an intersection with S that is S, see the definition of a carrier) $\square$

By the First Axiom, we have

$$\sum_{i \in S} \phi_i(v_{S,c}) = c$$

and

$$\sum_{i \in S \cup \{j\}} \phi_i(v_{S,c}) = c \text{ if } j \notin S$$

Together, these imply

$$\phi_j(v_{S,c}) = 0 \text{ if } j \notin S$$

By the Second Axiom, if π is a permutation that interchanges i and j $(i, j \in S)$ and leaves the other players fixed, then $\pi v_{S,c} = v_{S,c}$ (because the makeup of S is unchanged), thus

$$\phi_i(v_{S,c}) = \phi_j(v_{S,c}) \text{ for any } i, j \in S$$

Therefore, $\phi_{v_{S,c}}$ is unique (since every way to "make" $v_{S,c}$ is the same one) if ϕ exists, and is given by

$$\phi_i(v_{S,c}) = \begin{cases} c/|S| & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

because, the players of S are all interchangeable, in a sense, they should have similar value

The games $\{v_{S,c} | S \neq \emptyset, S \subset N, c \in \mathbb{R}\}$ form an additive basis for the vector space G (Shapley [1953] proves this).

For our purposes, we can consider the games

$$\{v'_{S,c} | S \neq \emptyset, S \subset N, c \in \mathbb{R}\}$$

which are defined by

$$v'_{S,c} = \begin{cases} 0 \text{ if } S \neq T \\ c \text{ if } S = T \end{cases}$$

Any game v can be written as a finite sum of games of type $v'_{S,c}$.

The uniqueness of ϕ follows, using the Third Axiom, if we can show that each $\phi(v'_{S,c})$ is unique.

We can prove this by induction.

Assume that $\phi(v'_{S,c})$ is unique for $|S| = k+1, \ldots, n$.

If we show that this is true for $|S| = k \rightarrow$ mission accomplished.

Let $S_1, \ldots S_l$ be all of the proper supersets of S.

Since $|S_i| > k, 1 \leq i \leq l$, $\phi(v'_{S,c})$ is unique by the inductive assumption.

Since any game can be written as a sum of v' type games, We have

$$v_{S,c} = v'_{S,c} + v'_{S_1,c} + \cdots + v'_{S_l,c}$$

(Recall that the $v'_{S,c}$ corresponds to our k case.)

And so, by the Third Axiom, we have

$$\phi(v_{S,c}) = \phi(v'_{S,c}) + \phi(v'_{S_1,c}) + \cdots + \phi(v'_{S_l,c})$$

Since we know that all of the terms in the sum except for $\phi(v'_{S,c})$ are unique, then $\phi(v'_{S,c})$ must also be unique.

Thus, ϕ satisfies all three axioms and is unique for any game in G if it exists.

Implicit in the proof of uniqueness of ϕ is a recipe for constructing ϕ.

Suppose that for $s = |S| = k+1, \dots, n$

$$\phi_i(v_{S,c}) = \begin{cases} \frac{(s-1)!(n-s)!}{n!} * c & \text{if } i \in S \\ -\frac{s}{n-s} \frac{(s-1)!(n-s)!}{n!} * c & \text{if } i \notin S \end{cases}$$

Using

$$\phi(v_{S,c}) = \phi(v'_{S,c}) + \phi(v'_{S_1,c}) + \dots + \phi(v'_{S_l,c})$$

it follows that for $|S| = k$

$$\phi_i(v'_{S,c}) = \begin{cases} \frac{(s-1)!(n-s)!}{n!} * c & \text{if } i \in S \\ -\frac{s}{n-s} \frac{(s-1)!(n-s)!}{n!} * c & \text{if } i \notin S \end{cases}$$

Now, it is straightforward to obtain $\phi(v)$ for any v.

Since

$$v = \sum_{S \subset N : S \neq \emptyset} v'_{S,v(S)}$$

then by the Third Axiom

$$\phi v = \sum_{S \subset N : S \neq \emptyset} \phi(v'_{S,v(S)})$$

Simplifying the right hand side we have

$$\phi_i(v) = \sum_{i \in T, T \subset N} \frac{(t-1)!(n-t)!}{n!} [v(T) - v(T - \{i\})]$$

This is the Shapley value (perhaps written differently than we saw in class, but it is the same).

We can verify that this formula satisfies the three axioms (which we did do in class).

Uniqueness for Subclasses

(Major takeaways:

- Since the Shapley Value was show to exist on G, we just need to show its uniqueness.
- Games can be represented by linear combinations of games of type $v_{S,c}$

)

Subclasses of G

What if we want to apply the Three Axioms to subclasses of G, call them K.

Then the Second Axiom is only required to hold for $\pi v \in K$ if $v \in K$.

The Third Axiom is only required to hold if $v_1 + v_2 \in K$ if $v_1, v_2 \in K$

But can the Axioms, restricted in this way, specify a unique ϕ for K?

Clearly, by looking at the Shapley value on G, we can see that there is at least one ϕ for these subclasses, but is it the only one?

For any K, we can use a procedure similar to the previous proof, we can establish the uniqueness of ϕ on K.

It is not necessary to do so by starting with ϕ defined on G and restricting its domain to K.

Case: $K = D$

Consider the case $K = D$, where D is the subclass of G representing all superadditive games in G.

Recall that a superadditive game is one for which

$$v(S \cup T) \geq v(S) + v(T) \text{ for } S \cap T = \emptyset$$

Start by demonstrating that a basis for G is formed by

$$\{v_{S,c} | S \neq \emptyset, S \subset N, c \in \mathbb{R}\}$$

Suppose that $v'_{S,1}$ is in the linear span of

$$\{v_{T,1} | T \supset S\} \text{ when } |S| = k+1, \ldots, n$$

Let $|S^* = k|$.

Since

$$v'_{S^*,1} = v_{S^*,1} - v'_{S_1^*,1} - \cdots - v'_{S^*,1}$$

where $S_1^*, \ldots, S_j^*$ are all proper supersets of S^* and since by our inductive assumption each $v_{S_i^*,1}$ is in the linear span of $\{v_{T,1}|T \supset S_i^*\}$, it follows that $v'_{S^*,1}$ is in the linear span of $\{v_{T,1}|T \supset S_i^*\}$.

Knowing that $\{v'_{S,1}|S \neq \emptyset, S \subset N\}$ spans G, we can see that $\{v_{S,1}|S \neq \emptyset, S \subset N\}$ also spans G.

In fact, $\{v_{S,1}|S \neq \emptyset, S \subset N\}$ also spans G because

$$|\{v_{S,1}|S \neq \emptyset, S \subset N\}| = |\{v'_{S,1}|S \neq \emptyset, S \subset N\}|$$

which already was known to be a basis of G.

We can express a game in D uniquely as $v = \sum c_S v_{S,1}$.

Since some of the c_S terms could be negative, this would mean that the game is not in D, and we can't apply the Third Axiom.

But if we transpose the terms with a negative c_S coefficient to the left, we can see that the new equation contains only games in D. We can now apply the Third Axiom and prove the uniqueness of ϕ on D.

To find c_S explicitly, express each $v'_{S,1}$ in terms of the basis

$$\{v_{S,1} | S \neq \emptyset, S \subset N\}$$

using

$$v'_{S^*,1} = v_{S^*,1} - v'_{S_1^*,1} - \cdots - v'_{S^*,1}$$

and induction, then substitute into

$$v = \sum_{T:T \neq \emptyset, T \subset N} v'_{T,v(T)}$$

This way, it can be shown that $c_S = \sum_{T \subset S} (-1)^{s-t} v(T)$

This result allows us to write out an explicit formula for $\phi(v)$, as did Shapley with his Value.

This is not simple though, and it is easier to show by restricting a ϕ found on G to D

Case: $K \neq D$

The following examples (which are not exhaustive), we can prove the uniqueness of ϕ on that each class K the same way we did for G in Theorem 1, and construct phi recursively in a similar manner.

1. $K = \{v|v(S) = 0$ or $1, \quad \forall \quad S \subset N\}$, i.e. *simple games*.
2. $K = \{v|v(S) = 0$ if $|S| \leq k$ for some $k\}$, as well as all simple games with this restriction.
3. The subclass of games where certain players $i_1, \ldots, i_k$ are distinguished and $v(S) = 0$ if $\{i_1, \ldots, i_k\} \not\subset S$, as well as all simple games with this restriction.

Remarks

1. The convex cone generated by simple games with veto players (players i such that $v(S) = 0$ if $i \notin S \quad \forall \quad S \subset N$) is the subclass L of all games with non-empty cores. Therefore, case C shows that the axioms specify the Shapley value on L. In fact, this is true for convex cones generated by the class of games in any one of A, B, C or any of their unions.

2. For any $P \subset G, |P| < \inf$, we can determine a finite number of steps whether or not the axioms uniquely specify the Shapley value on P; and if they do not, we can construct different ϕs on P which do satisfy the axioms. This corresponds to checking if a system of linear equations has a unique solution. The size of the linear system can be reduced with a procedure mimicing the proof of Theorem 1 (not discussed).

Monotonic Simple Games

Definitions

Recall that C is the set of all simple games.

Let C' be the subclass of all *monotonic simple games* in G, i.e., games for which $v(S) = 1 \implies v(T) = 1, \quad \forall\, S \subset T$.

Let C'' be the subclass of all *superadditive simple games* in G.

Note that $C'' \subset C'$.

An Example of the Insufficiency of the Three Axioms

The Three Axioms *do not* uniquely specify the Shapley value on C' or C'' if $|N| > 2$.

There are games in C' and C'' for which the Shapley Value is determined by the First and Second Axioms, the games of type $v_{S,1}$.

However, we want to define the Shapley Value for all of these games

Take a game that is not of type $v_{S,1}$ in C'' (and thus C' also). An example is:

$$v(N - \{i\}) = v(N - \{j\}) = v(N) = 1$$

$$v(S) = 0 \text{ for all other } S$$

$$i, j \in N, i \neq j, |N| > 2$$

Let ϕ be the following:

$$\phi_i(v) = \phi_j(v) = p \text{ for } p \in \mathbb{R}$$

$$\phi_k(v) = \frac{1 - 2p}{|N - \{i, j\}|} \text{ for } k \neq i, j$$

The game satisfies the First Axiom because

$$\sum_{i \in N} \phi_i(v) = v(N)$$

The game satisfies the Second Axiom because players maintain the same value, even if the game is permuted.

This ϕ also satisfies the Third Axiom vacuously. Suppose

$$v + v' = v'' \text{ for } v', v'' \in C'$$

But

$$v(N) = 1 \implies v'' = 1 \implies v'(N) = 0 \implies v' = 0 \text{ (because it is monotonic)}$$

Also, if

$$v - v' = v'' \text{ for } v', v'' \in C'$$

then two cases arise...

1. $v''(N) = 1$
   $v'(N)$ must $= 0 \implies v' = 0$

2. $v''(N) = 1$
   Therefore $v'' = 0$ and $v'' = v'$

The Third Axiom is satisfied for any value of p, so ϕ is not uniquely specified on C' or C'' by the Three Axioms.

A Variant of the Third Axiom

We can replace the Third Axiom with a variant, and ϕ will be uniquely specified on C' or C'', and it will be the Shapley Value (given later).

Note that as we construct the variant, we will do so for C''. The construction is the same for C'.

Let us start with some definitions. Consider the games $v, v' \in C''$:

Define the join of the two games $(v \vee v')$ to be

$$(v \vee v')(S) = \begin{cases} 1 \text{ if either } v(S) = 1 \text{ or } v'(S) = 1 \\ 0 \text{ if } v(S) = 0 \text{ and } v'(S) = 0 \end{cases}$$

Define the join of the two games $(v \wedge v')$ to be

$$(v \wedge v')(S) = \begin{cases} 1 \text{ if } v(S) = 1 \text{ and } v'(S) = 1 \\ 0 \text{ if either } v(S) = 0 \text{ or } v'(S) = 0 \end{cases}$$

Note that $v \vee v'$ may not always be in C'', but that $v \wedge v'$ is, even for $v, v' \in C'$.

We can verify that $v \wedge v' \in C''$ for $v, v' \in C''$.

(By Contradiction) Suppose that $v \wedge v' \notin C''$.

Then there are coalitions S, T s.t. $S \cap T = \emptyset$ and

$$(v \wedge v')(S \cup T) < (v \wedge v')(S) + (v \wedge v')(T)$$

But by the definition of $v \wedge v'$ this means that either

$$v(S \cup T) < v(S) + v(T) \text{ or}$$

$$v'(S \cup T) < v'(S) + v'(T) \text{ or}$$

Which is a contradiction. (Consider the possible values for the various terms.)

The Variant of the Third Axiom can be stated as:

If $v \vee v' \in C''$ whenever $v, v' \in C''$ then

$$\phi(v \vee v') + \phi(v \wedge v') = \phi(v) + \phi(v')$$

(For the C' version, we may drop the if because $v \vee v' \in C'$ always.)

Theorem 2: A Variant of the Third Axiom (Finally)

Theorem (Dubey 1975)
There is a unique function ϕ, defined on C'', which satisfies the Three Axioms (the third one being our variant). Moreover, ϕ is the Shapley Value.

Proof of Theorem 2

Every $v \in C''$ has a finite number of minimal winning coalitions $S_1, \dots S_k$, i.e., coalitions S_i such that $v(T) = 1$ if $S_i \subset T$ for some i and $v(T) = 0$ if $S_i \not\subset T \quad \forall\, i$.

Clearly $v = v_{S_1,1} \vee \cdots \vee v_{S_k,1}$, where the right side is defined associatively.

Let

$$n^1(v) = min\{p \in \mathbb{Z}^+ | \exists \text{ a minimal winning coalition } T \text{ with } |T| = p\} \text{ and}$$

$$n^2(v) = \text{the number of winning coalitions } T \text{ such that } |T| = n^1(v)$$

The proof of uniqueness of ϕ will be on induction of $n^1(v)$ and $n^2(v)$.

For the case where $n^1(v) = n, v = v_{N,1}$, ϕ is obviously unique.

Assume that $\phi(v)$ is unique for all v such that $n^1(v) = k+1, \ldots, n$. Then ϕ is unique when $n^1(v) = k$ and $n^2(v) = 1$

Let S be the unique minimal winning coalition with k players. If S is the only minimal winning coalition of v, then $v = v_{S,1}$ and $\phi(v)$ is unique. Otherwise, let $S_1, \ldots S_m$ denote all of the minimal winning coalitions that aren't S.

Note: $|S_i| > k$ for $1 \leq i \leq m$ since $n^2(v) = 1$.

Now,

$$v = (v_{S_1,1} \vee \cdots \vee v_{S_m,1}) \vee v_{S,1}$$

We restate this as

$$v = v' \vee v_{S,1}$$

It follows that $n^1(v') > k$. Therefore $\phi(v')$ is unique by the inductive assumption

Further, $n^1(v_{S,1} \wedge v') > k$. This is apparent from the definition of $\wedge$. Therefore $\phi(v \vee v')$ is also unique by the inductive assumption. Invoke our Third Axiom Variant, then

$$\phi(v) = \phi(v' \vee v_{S,1}) = \phi(v') + \phi(v_{S,1}) - \phi(v_{S,1} \wedge v')$$

Since all three vectors on the right side are unique, so is $\phi(v)$.

Next, suppose that $\phi(v)$ has been shown to be unique for all v such that either

$$n^1(v) = k+1, \dots n \text{ or}$$

$$n^1(v) = k \text{ and } n^2(v) = 1, \dots, j$$

Then $\phi(v)$ is unique if we can show that it is so for $n^1(v) = k$ and $n^2(v) = j+1$.

Let $S_1, \ldots, S_{j+1}$ be the minimal winning coalitions of v with k players each.

Let $T_1, \ldots, T_m$ be all the other minimal winning coalitions of v. By the conditions on $n^1(v)$ and $n^2(v)$ it is clear that $|T_i| > k$ for $1 \leq i \leq m$.

Now,

$$v = (v_{T_1,1} \vee \cdots \vee v_{T_m,1} \vee v_{S_1,1} \vee \cdots \vee v_{S_j,1}) \vee v_{S_{j+1,1}}$$

We abbreviate this as

$$v = v'' \vee v_{S_{j+1},1}$$

We can now see that v'' satisfies $n^1(v) = k$ and $n^2(v) = 1, \ldots, j$

and $v'' \wedge v_{S_{j+1,1}}$ satisfies $n^1(v) = k+1, \ldots n$.

Therefore, both $\phi(v'')$ and $\phi(v'' \wedge v_{S_{j+1},1})$ are unique by the inductive assumption.

By our Third Axiom Variant, we have

$$\phi(v) = \phi(v'' \vee v_{S_{j+1},1}) = \phi(v_{S_{j+1},1}) - \phi(v'' \wedge v_{S_{j+1},1})$$

which proves the uniqueness of ϕ.

Combining the two results, we see that $\phi(v)$ is unique fr any feasible numbers $n^1(v)$ and $n^2(v)$, i.e., for all $v \in C''$.

It is clear that the Shapley Value ϕ on G satisfies the Three Axioms (with our variant) when it is restricted to C''.

Indeed, $v + v' = (v \vee v') + (v \wedge v')$ when we consider the $+$ operation to occur in the space of G. Hence, by the Third Axiom, $\phi(v) + \phi(v') = \phi(v \vee v') + \phi(v \wedge v')$.

Thus, the Shapley Value is *the* unique ϕ on C'' which satisfies the Three Axioms (with our variant).

Note that we don't need to depend on the ϕ defined on G to establish the existence of ϕ on C''. The implicit proof of uniqueness of ϕ on C'' is a straightforward recursive construction.

Remarks

1. Theorem 2 holds when we replace c'' with C'. The proofs are similar and involve stopping the induction at appropriate stages, considering games that take on values in $\{c, 0\}$ instead of $\{1, 0\}$.

   Two examples are sub-classes of C' (or C'') for which:

   a) $v(S) = 0$ if $|S| \leq k$ (and 1 otherwise)

   b) $v(S) = 0$ if $\{i_1, \ldots, i_k\} \not\subset S$ (and 1 otherwise)

2. By changing the First Axiom, but retaining the Second and our variant of the Third, we can obtain an axiomatic value for the *Banzhaff Value* in its unnormalized form [Lucas, 1973] when it is restricted to C' or C''. This proof is similar to the proof of Theorem 2, and appears in a later paper.

References

[1] P. Dubey. On the uniqueness of the shapley value. *International Journal of Game Theory*, 4(3):131–139.

Thank you!