Math 330 — Test II (60 pts)
Thursday, 15 November 2012

Name: Solutions

PLEASE SHOW ALL WORK

Nº	pts
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Σ	/60

Each problem is worth 10 points. For full credit please show work and clearly indicate which operations you use in row reduction. No wireless devices are permitted. A calculator may be used to check answers.

(1) Determine if the vectors $(1,-1,3)$, $(1,-2,5)$, $(2,0,2)$ are linearly independent. If possible, express one of these vectors as a linear combination of the others.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 0 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

No, the vectors are not linearly independent because A does not have full column rank.

Note $N(A) \neq 0$

$$x_1 + x_2 + 2x_3 = 0$$
$$-x_2 + 2x_3 = 0$$

Special solution
$x_3 = 1$
$x_2 = 2$
$x_1 = -x_2 - 2x_3 = -4$

So $\vec{s}_1 = (-4, 2, 1)$, $-4A_1 + 2A_2 + A_3 = 0$.

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = A_3 = 4A_1 - 2A_2 = 4\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

(2) With A and $\mathbf{b}$ as given, find the complete solution to $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}.$$

* Special solutions : free variables x_3, x_5

$$x_1 + 3x_3 - 4x_5 = 0$$
$$x_2 + 2x_3 + x_5 = 0$$
$$x_4 + x_5 = 0$$

$x_3 = 1, x_5 = 0$
$\rightarrow x_1 = -3$
$x_2 = -2$
$x_4 = 0$
$\vec{s}_1 = (-3, -2, 1, 0, 0)$

$x_3 = 0, x_5 = 1$
$x_1 = 4$
$x_2 = -1$
$x_4 = -1$
$\vec{s}_2 = (4, -1, 0, -1, 1)$

* particular solution: $x_3 = x_5 = 0$

$x_1 = 1$
$x_2 = -2$
$x_4 = 5$

so $\vec{x}_p = (1, -2, 0, 5, 0)$

* complete solution

$$\vec{x} = \vec{x}_p + x_3\vec{s}_1 + x_5\vec{s}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(3) The matrix A and its reduced echelon form R are given below. Find the rank of A and the dimension of each of the four fundamental subspaces of A. Find a basis for the column space and a basis for the row space.

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 1 & 1 & -5 & 0 & 1 \\ 2 & -1 & -1 & 0 & -4 \\ 2 & 0 & -4 & 3 & 1 \end{bmatrix} \xrightarrow{\text{reduced echelon form}} R = \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three pivots so $\operatorname{rank} A = 3$.

$\dim C(A) = \dim C(A^T) = 3$

$\dim N(A) = n - r = 5 - 3 = 2$

$\dim N(A^T) = m - r = 4 - 3 = 1$

Basis for column space: (pivot columns of A) $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

Basis for row space: (nonzero rows of R as columns) $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

(4) Compute the following determinant in two ways: First, using a cofactor expansion (clearly indicate which row or column you use) and, next, using row reduction to echelon form.

$$|A| = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{vmatrix}$$

* cofactor expansion:
First expand along 1^{st} row:

$$|A| = 1(-1)^{1+1}\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} + 2(-1)^{3+1}\begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

Next use 2nd row for both 3x3 dets.

$$= 1(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 2\left(2(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right) = 1(1-4) - 4(1-4) = 9$$

* row reduction

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - 2R_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} = U$$

$|A| = |U| = 1 \cdot 1 \cdot (-3)(-3) = 9$

(5) Use the least squares method to fit a line $y = c + dt$ to the given data.

t	-1	0	1	2
y	-1	1	2	4

$y = c + dt$
$-1 = c + d(-1)$
$1 = c + d(0)$
$2 = c + d(1)$
$4 = c + d(2)$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad \hat{x} = \begin{bmatrix} c \\ d \end{bmatrix}$

Solve $A^T A \hat{x} = A^T \vec{b}$

$\hat{x} = (A^T A)^{-1} A^T \vec{b}$

$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$ $\quad A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$

$(A^T A)^{-1} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

$\hat{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 7 \\ 16 \end{bmatrix}$

Least squares line

$y = \frac{7}{10} + \frac{8}{5} t$

(6) Find three orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ such that $\mathbf{q}_1$, $\mathbf{q}_2$ span the subspace of vectors (x, y, z) satisfying the condition $x - y - 2z = 0$.

First find basis for $x - y - 2z = 0$.

If $y = 1, z = 0$, $\quad \vec{a}_1 = (1, 1, 0)$

If $y = 0, z = 1$, $\quad \vec{a}_2 = (2, 0, 1)$

Apply Gram-Schmidt to $\vec{a}_1, \vec{a}_2$

take $\vec{v}_1 = \vec{a}_1 = (1, 1, 0)$

$\vec{v}_2 = \vec{a}_2 - \frac{\vec{v}_1^T \vec{a}_2}{\vec{v}_1^T \vec{v}_1} \vec{v}_1 = (2, 0, 1) - \frac{2}{2}(1, 1, 0) = (1, -1, 1)$

Normalize: $\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0)$

$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{3}}(1, -1, 1)$

Observe that $\vec{v}_3 = (1, -1, -2)$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$;

so we take $\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{6}}(1, -1, -2)$

Thus $\vec{q}_1 = \frac{1}{\sqrt{2}}(1, 1, 0)$, $\vec{q}_2 = \frac{1}{\sqrt{3}}(1, -1, 1)$, $\vec{q}_3 = \frac{1}{\sqrt{6}}(1, -1, -2)$