HW 1

1| Show that $Var(X) = E[X^2] - E[X]^2$

The expected value of a random variable X ($E[X]$, μ) is defined as the sum of all possible outcomes of X multiplied by the respective likelihood of those outcomes:

$E[X] = \sum_{i=1}^{\infty} p_i x_i$

The variance of random variable X ($Var(X)$, σ^2), is defined as the sum of the likelihood of all outcomes of X multiplied the square of the difference of the expected value of X and the respective outcomes of X:

$Var[X] = \sum_{i=1}^{\infty} p_i \cdot (x_i - E[X])^2$

If we expand the $Var[X]$ expression, we get:

$Var[X] = \sum p \cdot (x^2 - 2x E[X] + E[X]^2)$

$= \sum px^2 - 2px E[X] + p E[X]^2$

Break the single sum into 3 separate ones:

$Var[X] = \sum px^2 - \sum 2px E[X] + \sum p E[X]^2$

$E[X]$ is a constant, so $\sum p E[X]^2$ can be rearranged to become $E[X]^2 \sum p$.

$\sum p = 1$, so the third sum becomes simply $E[X]^2$:

$Var[X] = \sum px^2 - \sum 2px E[X] + E[X]^2$

Again, 2 is a constant, as is $E[X]$, so the second term becomes $2E[X] \sum px$.

$\sum px = E[X]$, so the second term is $2E[X] \cdot E[X] = 2E[X]^2$:

$Var[X] = \sum px^2 - 2E[X]^2 + E[X]$

$= \sum px^2 - E[X]^2$

Finally, if we were to "assume" x^2 was an outcome of X^2 (which, of course, it would be), then it would follow that $\sum px^2 = E[X^2]$, giving us our sought result:

$Var[X] = E[X^2] - E[X]^2$

2| Show that $\sum_k f(k) = 1$, where $f(k)$ is the pmf of the following distributions:

! a) Bernoulli $\rightarrow f(k) = \begin{cases} p & \text{for } k=1 \\ 1-p & \text{for } k=0 \end{cases}$

Therefore $\sum$ Bernoulli $= f(0) + f(1) = (1-p) + (p) = 1$

b) Binomial $X \sim Bin(n, p)$

$$f(k) = \begin{cases} \frac{n!}{k!(n-k)!} p^k q^{n-k} & \text{for } k = 0, 1, \dots n \\ 0 & \text{otherwise} \end{cases} \quad \text{Note: } q = 1-p$$

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

The above expression is the binomial expansion of $(p+q)^n$, therefore:

$$\sum \frac{n!}{k!(n-k)!} p^k q^{n-k} = (p+q)^n = (p+(1-p))^n = 1^n = 1$$

c) Geometric $X \sim Geometric(p)$

$$f(k) = \begin{cases} p(1-p)^{k-1} & \text{for } k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{k=0} p(1-p)^{k-1}$$

This is a geometric series, which is in the form $\sum ar^n$, which we know will converge if $|r| < 1$ and $|1-p|$ is < 1.

Whe can find where such a series converges by finding the limit as $n \to \infty$. Through some manipulation of the sum (not shown) we can evaluate the sum of the series using:

$$\sum p(1-p)^{k-1} = \frac{p(1-(1-p)^{k-1})}{1-(1-p)},$$

so, remembering that $n = k-1$,

$$\lim_{n \to \infty} \frac{p(1-(1-p)^{k-1})}{1-(1-p)} = \frac{p(1-(1-p)^{(\infty)})}{(1-(1-p))} = \frac{p(1-0)}{1-(1-p)} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

– d) Poisson

$$f(k) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \text{for } x = \text{any non-negative integer} \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$e^{-\lambda}$ is a constant and can be pulled out to the front of the series as a multiplier:

$$e^{-\lambda} \sum \frac{\lambda^k}{k!}$$

Further, $\sum \frac{\lambda^k}{k!}$ is a power series that sums to e^{λ}. Therefore:

$$e^{-\lambda} \sum \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1$$

! 3| Show that for a Binomial r.v. Y with parameters (n, p):

$$P(Y=k) = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

Such a variable is considered to be the number of "successes" found when conducting n number of successive, independent Bernoully trials with the same probability of success. Thus, following simple probability theory, we multiply the successive likelihoods of the outcomes in a sample. Consider the example where the number of successes $k = 3$, the number of trials in the sample $n = 3$, and the probability of succes $p = .4$ (and conversely, the probability of failure is $1-p = 1-.4 = .6$):

$$P(\text{Success, Success, Failure}) \longrightarrow (.4)(.4)(.6) = (.4)^2 (.6)^1$$

Notice that the exponent for the failures is the remaining number of trials, or $n-k$. With all this information, we can generalize to:

$$P(SSF) \Rightarrow p^k (1-p)^{n-k}$$

Further, this event can occur in different combinations of F and S, i.e. FSS or SFS, therefore we must add the likelihoods of these combinations as well, which can be done using by multiplying the number of combinations by this generalized likelihood formula like so:

$$\binom{n}{k} p^k (1-p)^{n-k} \quad \text{or} \quad \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k}$$

4) Find the average number of roulette spins necessary to get the number 13. (Assume American roulette with 36 numbers and 2 zeros)

$P(x=13)=\frac{1}{38}$ $\quad X \sim Bin(n,p)$ $\quad \mu = np = 1$ "success"

$$np=1 \rightarrow n=\frac{1}{p}=\frac{1}{\left(\frac{1}{38}\right)} \boxed{=38 \text{ spins of the wheel}}$$

5) Find the average loss in a roulette game if one bets on a number, and on twelve numbers (equal bet on each). The answer should be given in % to the bet amount. (Again, assume American roulette)

payout $=\frac{1}{n}(36-n)=\frac{36}{n}-1$ where n is the number of squares a bet is placed on and assuming the casino does not pay back the original bet for a when. However, it is customary to pay back the original bet, so payout $=\frac{36}{n}$

a) Bet on one number:

$$\text{Average loss} = 1-\frac{\text{payout}}{\text{average number of spins (bets) to win}}$$

$$=1-\frac{\left[\frac{36}{(1)}\right]}{(38)}=.0526 \boxed{=5.26\%}$$

b) Bet on 12 numbers

$p=\frac{12}{38}$

$$\text{average number of spins for a win} = \frac{1 \text{ win}}{p} = \frac{1}{\frac{12}{38}} = 3.1\overline{6}$$

$$\text{Average loss} = 1-\frac{\left[\frac{36}{(12)}\right]}{(3.1\overline{6})} = .0526 \boxed{=5.26\%}$$

! 6| The probability for a complication after a particular type of surgery is $p = .005$

$X \sim Bin(n, .005)$

a) Find the probability that there will be at least one complication

$n = 100 \longrightarrow X \sim Bin(100, .005)$

$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$P(X=0) = \frac{100!}{0!(100-0)!} .005^0 (.995)^{100-0} = .60577...$

$1 - (.60577) \boxed{= .39423}$

b) Find the average number of complications after 100 surgeries.

$\mu = np = (100)(.005) \boxed{= .5}$

c) Find the probability that there will be at least one complication after 1000 surgeries.

$n = 1000, p = .005 \rightarrow$ I would say n is large and p is small, so I will use a Poisson approximation.

$X \sim Bin(n, p) \leftrightarrow X \sim Poisson(\lambda) \rightarrow \lambda = np = (1000)(.005) = 5$

$X \sim Poisson(5)$

$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$P(X=0) = e^{-(5)} \cdot \frac{(5)^{(0)}}{(0)!} = e^{-5} \cdot \frac{1}{1} = e^{-5}$

$1 - (e^{-5}) \boxed{= .99326}$

Not Required - I'm an undergrad. I did them to see if I could.

7] Use the definition of the mean to find the mean of a Poisson random variable with parameter λ

$E[X] = \sum_{x=0}^{\infty} xp$, p in a Poisson distribution is $e^{-\lambda} \cdot \frac{\lambda^x}{x!}$, so:

$E[X] = \sum_{x=0}^{\infty} x \cdot e^{-\lambda} \cdot \frac{1}{x!} \cdot \lambda^x$

At some point, we will have a series that is $\Sigma p = 1$, so we need a way to pull a λ out so we can have $\lambda \Sigma p = \lambda(1) = \lambda$. We can manipulate the series to get rid of the first term (which is 0):

$= 0 + \sum_{x=1} x \cdot e^{-\lambda} \cdot \frac{1}{x!} \cdot \lambda^x$

and we can get a $\lambda^{y+1} = \lambda \cdot \lambda^y$ by replacing $(x-1)$ with y:

$= \sum_{y=0} (y+1) \cdot e^{-\lambda} \cdot \frac{1}{(y+1)!} \cdot \lambda^{y+1} = \Sigma (y+1) \cdot e^{-\lambda} \cdot \frac{1}{(y+1) \cdot y!} \cdot \lambda \cdot \lambda^y = \lambda \Sigma e^{-y} \cdot \frac{\lambda^y}{y!}$

(using $(x-1)$ sets the starting term of the sum back to zero)

Now we can move back to using p with $p = e^{-y} \cdot \frac{\lambda^y}{y!}$ as this represents the p.m.f. of a Poisson r.v. with parameter λ. We also know that $\Sigma p = 1$ based on accepted probability theory.

$= \lambda \Sigma e^{-y} \cdot \frac{\lambda^y}{y!} = \lambda \cdot \Sigma p = \lambda(1) = \lambda$. Therefore, $E[X] = \lambda$

8] Prove the linear expectation: if $Y = aX + b$, where X is a r.v. and a, b are constants then $E(Y) = aE(X) + b$

If $E(X) = \frac{\Sigma x}{n}$, then $E(Y) = \frac{\Sigma y}{n} = \frac{\Sigma (ax+b)}{n}$

which is to say: $E(Y) = \frac{(ax_1 + b) + (ax_2 + b) \dots + (ax_n + b)}{n}$

Which can be manipulated to become: $E(Y) = \frac{a \Sigma x + n \cdot b}{n} = a \cdot \frac{\Sigma x}{n} + \frac{n \cdot b}{n} = a \cdot E(X) + b$