MATH 786: Cooperative Game Theory
HW06

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Abstract

Marriage Game, Hasse Diagram, NTU Games.

1. Consider the marriage game below:

M_1: W_2, W_3, W_5, W_1, s.s., W_4, W_6, W_7
M_2: W_2, W_6, W_7, W_3, s.s., W_1, W_4, W_5
M_3: W_2, W_3, W_1, W_4, W_5, s.s., W_6, W_7
M_4: W_5, W_1, W_3, W_4, W_2, s.s., W_6, W_7
M_5: W_6, W_7, W_4, W_2, W_1, s.s., W_3, W_5
M_6: W_7, W_6, W_2, W_1, W_3, s.s., W_4, W_5

W_1: M_1, M_3, M_2, M_4, s.s., M_5, M_6
W_2: M_2, M_3, M_6, M_5, M_4, M_1, s.s.
W_3: M_2, M_5, M_4, M_1, W_3, s.s., M_6
W_4: M_3, M_2, M_1, M_6, W_4, s.s., M_5
W_5: M_1, M_2, M_3, M_4, M_5, M_6, s.s.
W_6: M_6, M_2, M_3, M_1, M_4, M_5, s.s.
W_7: M_5, M_4, M_1, M_3, M_2, M_6, s.s.

(a) Find μ_m and μ_w for this game.

Solution:
$\mu_m =$
$\{(M_1 \to W_3), (M_2 \to W_2), (M_3 \to W_1), (M_4 \to W_5), (M_5 \to W_6), (M_6 \to W_7), (\text{S.S.} \to W_4)\}$

$\mu_w =$
$\{(W_1 \to M_3), (W_2 \to M_2), (W_3 \to M_4), (W_4 \to \text{S.S.}), (W_5 \to M_1), (W_6 \to M_6), (W_7 \to M_5)\}$

The Deferred Acceptance Procedure proceeds as follows when run with the men proposing:

$M_1 : R$	$M_2 : R$	$M_3 : R$	$M_4 : R$	$M_5 : R$	$M_6 : R$	
$W_1 : SS$	$W_2 : SS$	$W_3 : SS$	$W_4 : SS$	$W_5 : SS$	$W_6 : SS$	$W_7 : SS$
$M_1 :$	propose $\rightarrow W_2$					
	W_2 tentatively accepts; rejects S.S.					
$M_2 :$	propose $\rightarrow W_2$					
	W_2 tentatively accepts; rejects M_1					
$M_3 :$	propose $\rightarrow W_2$					
	W_2 rejects					
$M_4 :$	propose $\rightarrow W_5$					
	W_5 tentatively accepts; rejects S.S.					
$M_5 :$	propose $\rightarrow W_6$					
	W_6 tentatively accepts; rejects S.S.					
$M_6 :$	propose $\rightarrow W_7$					
	W_7 tentatively accepts; rejects S.S.					
$M_1 : R$	$M_2 : W_2$	$M_3 : R$	$M_4 : W_5$	$M_5 : W_6$	$M_6 : W_7$	
$W_1 : SS$	$W_2 : M_2$	$W_3 : SS$	$W_4 : SS$	$W_5 : M_4$	$W_6 : M_5$	$W_7 : M_6$
$M_1 :$	propose $\rightarrow W_3$					
	W_3 tentatively accepts; rejects S.S.					
$M_2 :$	(tentatively accepted)					
$M_3 :$	propose $\rightarrow W_3$					
	W_3 rejects					
$M_4 :$	(tentatively accepted)					
$M_5 :$	(tentatively accepted)					
$M_6 :$	(tentatively accepted)					
$M_1 : W_3$	$M_2 : W_2$	$M_3 : R$	$M_4 : W_5$	$M_5 : W_6$	$M_6 : W_7$	
$W_1 : SS$	$W_2 : M_2$	$W_3 : M_1$	$W_4 : SS$	$W_5 : M_4$	$W_6 : M_5$	$W_7 : M_6$
$M_1 :$	(tentatively accepted)					
$M_2 :$	(tentatively accepted)					
$M_3 :$	propose $\rightarrow W_1$					
	W_1 tentatively accepts; rejects S.S.					
$M_4 :$	(tentatively accepted)					
$M_5 :$	(tentatively accepted)					
$M_6 :$	(tentatively accepted)					
$M_1 : W_3$	$M_2 : W_2$	$M_3 : W_1$	$M_4 : W_5$	$M_5 : W_6$	$M_6 : W_7$	
$W_1 : M_4$	$W_2 : M_2$	$W_3 : M_1$	$W_4 : SS$	$W_5 : M_4$	$W_6 : M_5$	$W_7 : M_6$

...and as follows with the women proposing:

$W_1 : R$	$W_2 : R$	$W_3 : R$	$W_4 : R$	$W_5 : R$	$W_6 : R$	$W_7 : R$
$M_1 : SS$	$M_2 : SS$	$M_3 : SS$	$M_4 : SS$	$M_5 : SS$	$M_6 : SS$	
$W_1 :$	propose $\to M_1$					
	M_1 tentatively accepts; rejects S.S.					
$W_2 :$	propose $\to M_2$					
	M_2 tentatively accepts; rejects S.S					
$W_3 :$	propose $\to M_2$					
	M_2 rejects					
$W_4 :$	propose $\to M_3$					
	M_3 tentatively accepts; rejects S.S.					
$W_5 :$	propose $\to M_1$					
	M_1 tentatively accepts; rejects W_1					
$W_6 :$	propose $\to M_6$					
	M_6 tentatively accepts; rejects S.S.					
$W_7 :$	propose $\to M_5$					
	M_5 tentatively accepts; rejects S.S.					
$W_1 : R$	$W_2 : W_2$	$W_3 : R$	$W_4 : M_3$	$W_5 : M_1$	$W_6 : W_6$	$W_7 : M_5$
$M_1 : W_5$	$M_2 : W_2$	$M_3 : W_4$	$M_4 : SS$	$M_5 : W_7$	$M_6 : W_6$	
$W_1 :$	propose $\to M_3$					
	M_3 tentatively accepts; rejects M_4					
$W_2 :$	(tentatively accepted)					
$W_3 :$	propose $\to M_5$					
	M_5 rejects					
$W_4 :$	propose $\to M_2$					
	M_2 rejects					
$W_5 :$	(tentatively accepted)					
$W_6 :$	(tentatively accepted)					
$W_7 :$	(tentatively accepted)					
$W_1 : M_3$	$W_2 : W_2$	$W_3 : R$	$W_4 : R$	$W_5 : M_1$	$W_6 : W_6$	$W_7 : M_5$
$M_1 : W_5$	$M_2 : W_2$	$M_3 : W_1$	$M_4 : SS$	$M_5 : W_7$	$M_6 : W_6$	
$W_1 :$	(tentatively accepted)					
$W_2 :$	(tentatively accepted)					
$W_3 :$	propose $\to M_4$					
	M_4 tentatively accepts; rejects S.S.					
$W_4 :$	propose $\to M_1$					
	M_1 rejects					
$W_5 :$	(tentatively accepted)					
$W_6 :$	(tentatively accepted)					
$W_7 :$	(tentatively accepted)					
$W_1 : M_3$	$W_2 : W_2$	$W_3 : M_4$	$W_4 : R$	$W_5 : M_1$	$W_6 : W_6$	$W_7 : M_5$
$M_1 : W_5$	$M_2 : W_2$	$M_3 : W_1$	$M_4 : W_3$	$M_5 : W_7$	$M_6 : W_6$	
$W_4 :$	propose $\to M_6$					
	M_6 rejects					
$W_4 :$	propose $\to M_4$					
	M_4 rejects					
$W_4 :$	propose $\to S.S.$					
	SS accepts					
$W_1 : M_3$	$W_2 : W_2$	$W_3 : M_4$	$W_4 : SS$	$W_5 : M_1$	$W_6 : W_6$	$W_7 : M_5$
$M_1 : W_5$	$M_2 : W_2$	$M_3 : W_1$	$M_4 : W_3$	$M_5 : W_7$	$M_6 : W_6$	

(b) Find two other core matchings by trial and error.

Solution:

$\mu_1 =$
$\{(M_1 \to W_5), (M_2 \to W_2), (M_3 \to W_1), (M_4 \to W_3), (M_5 \to W_6), (M_6 \to W_7), (\text{S.S.} \to W_4)\}$

$\mu_2 =$
$\{(M_1 \to W_3), (M_2 \to W_2), (M_3 \to W_1), (M_4 \to W_5), (M_5 \to W_7), (M_6 \to W_6), (\text{S.S.} \to W_4)\}$

Recall that the core of a marriage game is described as the set of matchings such that

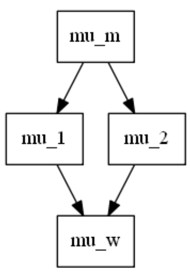
i. $\mu(i) \succeq^{M_i}$ s.s. $\quad \forall i \in \{1, \ldots, |M|\}$
ii. $\mu^{-1}(j) \succeq^{W_j}$ s.s. $\quad \forall j \in \{1, \ldots, |W|\}$
iii. $\nexists\, i, j$ for which $W_j \succeq^{M_i} W_{\mu(i)}$ AND $M_i \succeq^{W_j} W_{\mu^{-1}(j)}$

Both of these pairs meet the first two, the "individual rationality", conditions, and neither has any "eloping" pairs (the third condition).

Further, we might remark that we know which pairs in the matching cannot change: those where a player was matched to staying single, and those that do not change between the man-optimal and woman-optimal matchings. Next, by examining the preference lists, we can determine a subset of preferences that are allowable for a core matching. After taking these steps, the number of matchings to try becomes relatively small. (In this example, there are only two pairs of matchings that can "trade" their partners after this.)

(c) Draw a Hasse diagram for your set of core matchings.

Solution:



2. Suppose in a marriage game that $\mu_m = \mu_w$. How many core matchings must there be in this game? Justify your answer.

Solution:

There can only be one (1) core matching. We know that the set of core matchings of the marriage game is a lattice, and we also know that, from the men's perspective, μ_m is the greatest point and μ_w is the least point (in the lattice sense), i.e. $\mu_m \geq \mu_w$. However, since $\mu_m = \mu_w$, for a matching to be greater than μ_w, that point would have to also be greater than μ_m, which, of course, is not possible if that point/matching is to still be in the core. By similar argument, there cannot be another matching that is less than μ_m.

3. A lattice L is *distributive* if, for every $x, y, z \in L$, we have

$$\text{A) } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ AND}$$
$$\text{B) } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

It has been proven that the set of core matchings is always a distributive finite lattice (the converse of this statement, although it has to be more carefully stated, is also true).

In the "ten" core matchings example from class, verify A) and B) in the case where x $= \mu_5$, $y = \mu_6$, and $z = \mu_9$.

Solution:

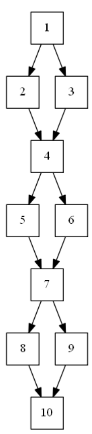
Recall that the Hasse diagram ranks the matchings, and from it we can easily determine the $\vee$ and $\wedge$ rlationship between two matchings, where $a \vee b$ is the matching that is the lowest common element at or above a, b, and $a \wedge b$ is the matching that is the highest common element at or below a, b.

A)

$$\begin{aligned} \mu_5 \wedge (\mu_6 \vee \mu_9) &= (\mu_5 \wedge \mu_6) \vee (\mu_5 \wedge \mu 9) \\ \mu_5 \wedge \mu_6 &= \mu_7 \vee \mu_9 \\ \mu_7 &= \mu_7 \qquad \surd \end{aligned}$$

B)

$$\begin{aligned} \mu_5 \vee (\mu_6 \wedge \mu_9) &= (\mu_5 \vee \mu_6) \wedge (\mu_5 \vee \mu_9) \\ \mu_5 \vee \mu_9 &= \mu_4 \wedge \mu_5 \\ \mu_5 &= \mu_5 \qquad \surd \end{aligned}$$



4. Suppose a given marriage game in which $|M| = |W| = m$ (m odd). For simplicity, also assume that every player's last choice is "s.s.". Show that μ_m must give the players on average at least their $\frac{m+1}{2}^{th}$ choice. HINT: Consider the "total ranking" for the men under μ_m (by "total ranking" we mean the sum of all the men's individual ranking fo the women they get under μ_m), and consider the ramifications of this quantity for the DAP.

Solution:

Start by considering a "worst case" (or perhaps "most extreme case") man-optimal matching, where each man gets his first choice, and each woman gets her last choice (above s.s. since she did get matched). In this case, the total ranking of the men's choices is m, and the total ranking of the women's choices is $m * m$. In this scenario, since there are $2m$ players, for the average player we have:

$$\frac{m + m^2}{2m} = \frac{1 + m}{2}$$

Now, suppose we alter things to make the outcome less extreme. If we alter the game so that any man does worse, than at there is a woman who must do better. This is because for a man to do worse, he mst get rejected, and each time a woman rejects him, that implies that she has at least one prospect who is better that proposed to her. So even as we alter the game, we see that the net total ranking cannot decrease. This argument covers all of the cases that are less extreme than our original one.

Since we considered the most extreme case, and deviation from it, we can say that in all other cases, players will actually do better in terms of the rankings of their partners on average than $\frac{1+m}{2}$.

5. Consider the modeling of NTU games in which the characteristic function values $V(S)$ are sets in $\mathbb{R}^S$ (and not $\mathbb{R}^N$). State the definition of superadditivity in this case.

Solution:

If an NTU game is superadditive then it must satisfy $V(A) \times V(B) \subset V(A \cup B) \quad \forall\ A \in \mathbb{R}^{S_1}, B \in \mathbb{R}^{S_2}, A \cup B = \emptyset$.

Superadditivity, in any sense, means that the total value of coalitions working separately must not exceed the value of those coalitions working together. In the NTU-sense, this means that the region specified by the $V(S)$s must not be outside the regions defined by the $V(S)$ for the coalitions together. If we consider the one dimensional (one player) case for each S_1, S_2, then we might say that A represents all of the payoffs that player 1 can achieve (call the best one a), and B represents the payoffs 2 can achieve (call the best one b). Together, the two players can achieve any combination of payofs inside the region bounded by $(0,0)$, $(0,b)$, $(a,0)$, and (a,b). Similar logic can be used in higher dimensions (with coalitions of more players).