MATH 786: Cooperative Game Theory
HW05

Terence Henriod

March 28, 2016

Abstract

Shapley Value, Shapley-Shubik Power Index, Banzhaf Index, Simple Games, Simple Majority Games.

1. Find the Shapley Value of the following games:

   (a) $N = \{1, 2, 3\}$, $V(\emptyset) = 0$, $V(\overline{1}) = 3$, $V(\overline{2}) = 2$, $V(\overline{3}) = 0$, $V(\overline{1,2}) = 4$, $V(\overline{1,3}) = 6$, $V(\overline{2,3}) = 8$, $V(\overline{N}) = 10$.

   Recall that the Shapley Value is computed by:

$$\varphi_i = \frac{1}{n!} * \sum_{\text{All Orderings } R} V(P_R(i) \cup \{i\}) - V(P_R(i))$$

   Solution:

   $\varphi = (\frac{6}{2}, \frac{7}{2}, \frac{7}{2})$

	$V(P_R(i) \cup \{i\}) - V(P_R(i))$		
Ordering	$i = 1$	$i = 2$	$i = 3$
$R_1 = \{1, 2, 3\}$	$3 - 0 = 3$	$4 - 3 = 1$	$10 - 4 = 6$
$R_2 = \{1, 3, 2\}$	$3 - 0 = 3$	$10 - 6 = 4$	$6 - 3 = 3$
$R_3 = \{2, 1, 3\}$	$4 - 2 = 2$	$2 - 0 = 2$	$10 - 4 = 6$
$R_4 = \{2, 3, 1\}$	$10 - 8 = 2$	$2 - 0 = 2$	$8 - 2 = 6$
$R_5 = \{3, 1, 2\}$	$6 - 0 = 6$	$10 - 6 = 4$	$0 - 0 = 0$
$R_6 = \{3, 2, 1\}$	$10 - 8 = 2$	$8 - 0 = 8$	$0 - 0 = 0$
$\sum$	18	21	21
φ	$18/6 = \frac{6}{2}$	$21/6 = \frac{7}{2}$	$21/6 = \frac{7}{2}$

   (b) The weighted majority game $[\frac{1}{2}; \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}]$.

   Solution:

   $\varphi = (\frac{168}{720}, \frac{168}{720}, \frac{168}{720}, \frac{72}{720}, \frac{72}{720}, \frac{72}{720}) = (\frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30})$

   The mathematically inclined will observe that there are only two types of players in the game, those with a weight of $\frac{1}{5}$ and those with a weight of $\frac{2}{15}$. Using the "substitute axiom", we can find the value for one player with a given weight and know the value for the rest of the players with the same weight.

   Consider that there are $6! = 720$ orderings. For a player with weight $\frac{1}{5}$, call them p, consider orderings with that player at a given position:

   With p in position 1 in the ordering, there are 120 such orderings. In none of these is p a swing player.

   With p in position 2 in the ordering, there are 120 such orderings. Again, in none of these is p a swing player.

   With p in position 3 in the ordering, there are 120 such orderings. There are 4 cases: the p can be preceded by players with weights $\frac{2}{15}$ and $\frac{2}{15}$; $\frac{2}{15}$ and $\frac{1}{5}$; $\frac{1}{5}$ and $\frac{2}{15}$; and $\frac{1}{5}$ and $\frac{1}{5}$. In only the first of these cases ($\frac{2}{15}$ and $\frac{2}{15}$) is p not a swing player, and there are $3 * 2 * 3 * 2 = 36$ of these orderings. This leaves $120 - 36 = 84$ orderings.

With p in position 4 in the ordering, there are 120 such orderings. In only the case where there are 2 $\frac{1}{5}$ players and 1 $\frac{2}{15}$ player preceding p is p not a swing player; there are $2 * 1 * 3$ of these orderings. Thus, in $120 - 36 = 84$ of these p is a swing player.

With p in position 5 in the ordering, there are 120 such orderings. In none of these is p a swing player.

With p in position 6 in the ordering, there are 120 such orderings. In none of these is p a swing player.

This gives us $84 + 84 = 168$ orderings where p is a swing player, so $\varphi_1 = \frac{168}{720}$, and by the "substitute axiom", $\varphi_2 = \varphi_3 = \varphi_1 = \frac{168}{720}$.

Now, since the Shapley value satisfies efficiency, we know that the remaining players will share a value of $\frac{720}{720} - 3 * (\frac{168}{720}) = \frac{216}{720}$. Since the remaining players are the $\frac{2}{15}$ type and there are 3 of them, we know that $\varphi_4 = \varphi_5 = \varphi_6 = frac216720/3 = frac72720$.

For those who would just *brute force* the problem, the following program written in the Rust programming language was used to produce the unreduced answer:

```
extern crate permutohedron;

fn main() {
    let weights = &[1.0/5.0, 1.0/5.0, 1.0/5.0, 2.0/15.0, 2.0/15.0, 2.0/15.0];
    let quota = 1.0 / 2.0;
    let (shapley_totals, n_orderings) = compute_shapley_totals(quota, weights);
    println!("The Shapley value is {:?} / {}", shapley_totals, n_orderings);
}

fn compute_shapley_totals(quota: f64, weights: &[f64]) -> (Vec<f64>, usize) {
let n = weights.len();
    let n_orderings = permutohedron::factorial(n);

    let players = (0..n).collect::<Vec<usize>>();

    let mut other = players.clone();
    let orderings = permutohedron::Heap::new(&mut other);

    let mut shapley_value = vec![0.0; n];

    for ordering in orderings {
        let mut predecessor_coalition_value = 0.0;

        for i in 1..(n + 1) {
            let (coalition, _) = ordering.split_at(i);
            let new_coalition_value = coalition_value(quota, coalition, weights);
            let difference = new_coalition_value - predecessor_coalition_value;

            let player = ordering[i - 1];
            shapley_value[player] += difference;

            predecessor_coalition_value = new_coalition_value;
        }
    }
```

```
    (shapley_value, n_orderings)
}

fn coalition_value(quota: f64, players: &[usize], weights: &[f64]) -> f64 {
    let mut total_weight = 0.0;
    for weight in weights {
        total_weight = total_weight + weight;
    }

    let mut weight_sum = 0.0;
    for player in players {
        weight_sum = weight_sum + weights[*player];
    }

    if weight_sum > quota {
        1.0
    } else {
        0.0
    }
}
```

2. The *Banzhaf Index* is another power index used to evaluate players' power in simple games. It is defined as the n-vector β where

$$\beta_i = \frac{\sum_{S:i \notin S} V(S \cup \{i\}) - V(S)}{\sum_{i=1}^{n} \left[\sum_{S:i \notin S} V(S \cup \{i\}) - V(S)\right]}$$

   (a) Compare the formula for the Banzhaf Index with that for the Shapley-Shubik power index. [Here I simply wish for you to compare the above definition with the "alternative formula for the Shapley Value" presented in class.]

   Solution:

   Recall the the "alternative formula for the Shapley Value":

$$\varphi_i = \frac{1}{n!} \sum_{S:i \notin S} |S|!\,(n - |S| - 1)!\,[V(S \cup \{i\}) - V(S)]$$

   Common to both formulae is the expression $[V(S \cup \{i\}) - V(S)]$, which counts whether or not a player i is a "swing player" in coalition $S \in 2^N$.

   In the Banzhaf Index, we count the number of times each player is a swing player for every unique coalition (but not ordering), and then divide that by the total number of times all players were swing players. It measures the value/power of a player normalized relative to the other players that have value/power.

   In the Shapley-Shubik power Index, we count the number of times a player is a swing player in every permutation of every coalition, and then divide that by the overall number of permutations of the grand coalition. It measures the value/power of a player normalized relative to all possible coalitions.

(b) Find the Banzhaf Index for the weighted majority game given in problem 1-b above.

Solution:

$\beta = (\frac{14}{60}, \frac{14}{60}, \frac{14}{60}, \frac{6}{60}, \frac{6}{60}, \frac{6}{60}) = (\frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30})$

Since the Banzhaf Index produces the same value for players with the same role, and there are only two roles in the game (a $\frac{1}{5}$ player or a $\frac{2}{15}$ player), we can evaluate the swing player instances for players 1 and 4, use those values for their respective substitutes, and then compute the index vector as a whole.

The following tables display the form of only the coalitions where there a player might be a swing player. The multiplicities indicate the number of ways to form a coalition of a given archetype (composition of players with the specified weights) from the players other than the "would-be swing player".

First, consider one of the players whose value is $\frac{1}{5}$, arbitrarily, we can use player 1.

S-archetype	Multiplicity	1 is swing	Count Towards Index
$\frac{1}{5}, \frac{2}{15}, [\frac{1}{5}]$	6	yes	6
$\frac{1}{5}, \frac{1}{5}, [\frac{1}{5}]$	1	yes	1
$\frac{2}{15}, \frac{2}{15}, \frac{2}{15}, [\frac{1}{5}]$	1	yes	1
$\frac{1}{5}, \frac{2}{15}, \frac{2}{15}, [\frac{1}{5}]$	6	yes	6

Now consider a player whose value is $\frac{2}{15}$, arbitrarily we use player 4.

S-archetype	Multiplicity	4 is swing	Count Towards Index
$\frac{1}{5}, \frac{2}{15}, [\frac{2}{15}]$	6	no	0
$\frac{1}{5}, \frac{1}{5}, [\frac{2}{15}]$	1	yes	3
$\frac{2}{15}, \frac{2}{15}, \frac{2}{15}, [\frac{2}{15}]$	0	no	0
$\frac{1}{5}, \frac{2}{15}, \frac{2}{15}, [\frac{2}{15}]$	3	yes	3

Thus, players 1, 2, 3 have $[V(S \cup \{i\}) - V(S)] = 14$. Players 4, 5, 6 have $[V(S \cup \{i\}) - V(S)] = 6$. Also, we have for the denominator of the formula:

$$\sum_{i=1}^{n} [\sum_{S:i \notin S} V(S \cup \{i\}) - V(S)] = 3 * (14) + 3 * (6) = 60$$

Finally, we have the Banzhaf Index: $\beta = (\frac{14}{60}, \frac{14}{60}, \frac{14}{60}, \frac{6}{60}, \frac{6}{60}, \frac{6}{60}) = (\frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30})$

3. Find an example of a monotonic simple game, with four players, which is not a weighted majority game for ANY of the values of $[q; w_1,\ w_2,\ w_3,\ w_4]$. [Note: a *monotonic* TU game is a game in which $S \subseteq T \to V(S) \le V(T)$.]

Solution:

The game is defined by its characteristic function:

S	$V(S)$	S	$V(S)$	S	$V(S)$	S	$V(S)$
$\emptyset$	0	$\overline{4}$	0	$\overline{2,3}$	0	$\overline{1,2,4}$	1
$\overline{1}$	0	$\overline{1,2}$	1	$\overline{2,4}$	0	$\overline{1,3,4}$	1
$\overline{2}$	0	$\overline{1,3}$	0	$\overline{3,4}$	1	$\overline{2,3,4}$	1
$\overline{3}$	0	$\overline{1,4}$	0	$\overline{1,2,3}$	1	$\overline{1,2,3,4}$	1

This game is simple because all payoffs are 0 or 1. This game is monotonic since $V(S) \le V(T)$ if $S \subseteq T$ for any S, T. The game cannot be a weighted majority game; if the game were a wighted majority game, a contradiction would arise.

Suppose the game is a weighted majority game with a non-negative quota q and non-negative weight w_i for each player i. Based on the characteristic function, then the following statements must hold (not a complete list):

$$\begin{aligned} w_1 + w_2 &\ge q \\ w_1 + w_3 &< q \\ w_2 + w_4 &< q \\ w_3 + w_4 &\ge q \end{aligned}$$

Further, we can deduce:

$$\begin{aligned} w_1 + w_2 \ge q,\ w_1 + w_3 < q \quad &\Longrightarrow \quad w_2 > w_3 \\ w_3 + w_4 \ge q,\ w_2 + w_4 < q \quad &\Longrightarrow \quad w_3 > w_2 \end{aligned}$$

However, $w_2 > w_3$ and $w_3 > w_2$ cannot both be true, thus we have a contradiction - the game cannot be a weighted majority game.

4. If one considers the concept of Shapley Value limited to the universe of simple games (i.e., the Shapley-Shubik power index), the 4 axioms - in particular the additivity axiom - characterizing it are somewhat unsatisfactory. This is because the sum of two simple games is not necessarily a simple game.

   Dubey (IJGT, 1975) suggested the following axiom as a replacement for additivity, in the case where $\mathcal{G}^n$ is replaced by $\mathcal{S}^n$ (the set of all n-player monotonic simple games). First, for any simple games $S_1 = (N, V_1)$ and $S_2 = (N, V_2)$ define

   $$S_1 \vee S_2 = (N, V^*) \text{ where } V^*(T) = \max(V_1(T), V_2(T)) \text{ and}$$

   $$S_1 \wedge S_2 = (N, V_*) \text{ where } V_*(T) = \min(V_1(T), V_2(T))$$

   Dubey's axiom is then

   $$F(S_1) + F(S_2) = F(S_1 \vee S_2) + F(S_1 \wedge S_2)$$

   Show that the Shapley-Shubik power index actually satisfies this axiom.

   HINT: Consider any ordering R, and any player i. Then consider the nine cases

   i) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 1, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 1$

   ii) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 1, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 0$

   ...

   ix) $V_1(P_R(i) \cup \{i\}) = 0, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 0, \quad V_2(P_R(i)) = 0$

   In each case, show that

   $$\begin{aligned} &V_1(P_R(i) \cup \{i\}) - V_1(P_R(i)) + V_2(P_R(i) \cup \{i\}) - V_2(P_R(i)) = \\ &\qquad V^*(P_R(i) \cup \{i\}) - V^*(P_R(i)) + V_*(P_R(i) \cup \{i\}) - V_*(P_R(i)) \end{aligned}$$

   Solution:

   Note that

   $$\begin{aligned} &V_1(P_R(i) \cup \{i\}) - V_1(P_R(i)) + V_2(P_R(i) \cup \{i\}) - V_2(P_R(i)) = \\ &\qquad V^*(P_R(i) \cup \{i\}) - V^*(P_R(i)) + V_*(P_R(i) \cup \{i\}) - V_*(P_R(i)) \end{aligned}$$

   is equivalent to

   $$\begin{aligned} &V_1(P_R(i) \cup \{i\}) - V_1(P_R(i)) + V_2(P_R(i) \cup \{i\}) - V_2(P_R(i)) = \\ &\qquad \max(V_1(P_R(i) \cup \{i\}), V_2(P_R(i) \cup \{i\})) - \max(V_1(P_R(i)), V_2(P_R(i))) + \\ &\qquad \min(V_1(P_R(i) \cup \{i\}), V_2(P_R(i) \cup \{i\})) - \min(V_1(P_R(i)), V_2(P_R(i))) \end{aligned}$$

   i) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 1, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 1$

   $$\begin{aligned} (1) - (1) + (1) - (1) &= \max((1), (1)) - \max((1), (1)) + \min((1), (1)) - \min((1), (1)) \\ 0 &= 1 - 1 + 1 - 1 \\ 0 &= 0 \end{aligned}$$

ii) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 1, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 0$

$$\begin{aligned}(1) - (1) + (1) - (0) &= \max((1),(1)) - \max((1),(0)) + \min((1),(1)) - \min((1),(0)) \\ 1 &= 1 - 1 + 1 - 0 \\ 1 &= 1\end{aligned}$$

iii) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 1$

$$\begin{aligned}(1) - (0) + (1) - (1) &= \max((1),(1)) - \max((0),(1)) + \min((1),(1)) - \min((0),(1)) \\ 1 &= 1 - 1 + 1 - 0 \\ 1 &= 1\end{aligned}$$

iv) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 1, \quad V_2(P_R(i) \cup \{i\}) = 0, \quad V_2(P_R(i)) = 0$

$$\begin{aligned}(1) - (1) + (0) - (0) &= \max((1),(0)) - \max((1),(0)) + \min((1),(0)) - \min((1),(0)) \\ 0 &= 1 - 1 + 0 - 0 \\ 0 &= 0\end{aligned}$$

v) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 0$

$$\begin{aligned}(1) - (0) + (1) - (0) &= \max((1),(1)) - \max((0),(0)) + \min((1),(1)) - \min((0),(0)) \\ 2 &= 1 - 0 + 1 - 0 \\ 2 &= 2\end{aligned}$$

vi) $V_1(P_R(i) \cup \{i\}) = 0, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 1$

$$\begin{aligned}(0) - (0) + (1) - (1) &= \max((0),(1)) - \max((0),(1)) + \min((0),(1)) - \min((1),(1)) \\ 0 &= 1 - 1 + 0 - 0 \\ 0 &= 0\end{aligned}$$

vii) $V_1(P_R(i) \cup \{i\}) = 1, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 0, \quad V_2(P_R(i)) = 0$

$$\begin{aligned}(1) - (0) + (0) - (0) &= \max((1),(0)) - \max((0),(0)) + \min((1),(0)) - \min((0),(0)) \\ 1 &= 1 - 0 + 0 - 0 \\ 1 &= 1\end{aligned}$$

viii) $V_1(P_R(i) \cup \{i\}) = 0, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 1, \quad V_2(P_R(i)) = 0$

$$\begin{aligned}(0) - (0) + (1) - (0) &= \max((0),(1)) - \max((0),(0)) + \min((0),(1)) - \min((0),(0)) \\ 1 &= 1 - 0 + 0 - 0 \\ 1 &= 1\end{aligned}$$

ix) $V_1(P_R(i) \cup \{i\}) = 0, \quad V_1(P_R(i)) = 0, \quad V_2(P_R(i) \cup \{i\}) = 0, \quad V_2(P_R(i)) = 0$

$$\begin{aligned}(0) - (0) + (0) - (0) &= \max((0),(0)) - \max((0),(0)) + \min((0),(0)) - \min((0),(0)) \\ 0 &= 0 - 0 + 0 - 0 \\ 0 &= 0\end{aligned}$$