Test II will cover all sections of the text from 3.3 through 5.2. You may bring a formula sheet to the test (please do not include worked examples). There will be a review Tuesday and a supplementary review session Wednesday morning if there is interest (DMS 315, 9:30 to 10:30 am). Go over homework and quizzes (solutions available at http://wolfweb.unr.edu/homepage/alex/330/). Here is a list of sample questions.

(1) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 2 \\ 1 & 3 & 5 & 11 & 9 \end{bmatrix}.$$

Use Gauss-Jordan elimination to find the reduced echelon form of A (indicate which operations you use). What is the complete solution for the system with augmented matrix A? (The system is $B\mathbf{x} = \mathbf{b}$ where $A = [B\,\mathbf{b}]$.) Indicate which of the variables are pivot variables and which are free.

(2) Determine the rank of the matrix A above. Find the nullspace of A. Under what conditions will a vector $\mathbf{b} = (b_1, b_2, b_3, b_4)$ be in the column space of $\mathbf{C}(A)$? $\text{rank}(A) = 3$, $11b_1 - 4b_2 - 2b_3 + b_4 = 0$

(3) Show that the columns of the matrix A above are linearly dependent. Find a maximal independent set of columns and express the other columns as linear combinations of these columns. Find a basis for the four fundamental subspaces of A: the nullspace, column space, row space and left null space; find the dimension of each of these subspaces. $2, 3, 3, 1$

(4) The matrix $[A\ \mathbf{b}]$ is the augmented matrix of a system of equations

$$[A\ \mathbf{b}] = \begin{bmatrix} 2 & 2 & 2 & 3 & b_1 \\ 1 & 1 & 1 & 1 & b_2 \\ 2 & 3 & 4 & 5 & b_3 \\ 1 & 3 & 5 & 11 & b_4 \\ 2 & 2 & 2 & 2 & b_5 \end{bmatrix}.$$

Find conditions on the components of $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$ so that the system $A\mathbf{x} = \mathbf{b}$ has a solution and find the complete solution in this case. Find a basis for the left null space of A. $-4a + 11b - 2c + d = 0$, $e - 2b = 0$.

(5) Find if possible a 3×4 matrix C so that $C\mathbf{x} = \mathbf{b}$ has a solution if and only if tbe components of $\mathbf{b} = (b_1, b_2, b_3)$ satisfy $b_1 - b_2 + 2b_3 = 0$. Choose your matrix so that each pair of columns is independent. Find three different bases for the column space of C. Find the dimension of the nullspace of C. $\dim(\mathbf{N}(C)) = 2.$

(6) Determine whether the following vectors are linearly independent:

$$(1, 2, 3, 0), (1, 0, 1, -1), (2, -1, 0, 3)\}.$$

Does this set form a basis for $\mathbb{R}^4$? If possible find a vector which is not in the span of these vectors. Yes, No.

(7) Explain why three independent vectors in $\mathbb{R}^3$ must span $\mathbb{R}^3$. Do the vectors $(1, 2, 3), (1, 0, 1), (2, -1, 0)$, form a basis for $\mathbb{R}^3$? Yes.

(8) Find if possible a matrix a 3×4 matrix B with rank 2 so that the vectors, $\mathbf{x}_1 = (1, 2, 1, 0)$ and $\mathbf{x}_2 = (2, -3, 0, 1)$ are in its nullspace. Do these vectors form a basis of $\mathbf{N}(B)$? Yes.

(9) Find a basis for the subspace $\mathcal{W}$ of $\mathbb{R}^4$ spanned by the vectors:

$$(1, 2, 0, 3), \quad (2, 1, 1, 1), \quad (1, 4, 3, -3), \quad (3, 15, 8, -4)$$

Express each of these vectors as a linear combination of your basis elements. Is $(3, 15, 8, 5)$ in $\mathcal{W}$? Find the dimension of $\mathcal{W}$. $\dim(\mathcal{W}) = 3.$

(10) Find the rank of the following matrix and the dimension of each of the four fundamental subspaces; find a basis for each of these subspaces.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(11) Find the projection matrix P onto the line through $\mathbf{a} = (2, -1, 1)$. Use P to find the projection $\mathbf{p}$ of the vector $\mathbf{b} = (1, 2, 3)$ onto this subspace. Is $I - P$ a projection? Show that $Q = 2P - I$ is a symmetric orthogonal matrix. $\frac{1}{2}\mathbf{a}$, yes

(12) Use the least squares method to fit a line $y = c + dt$ to the following data. How good is this approximation?

t	-1	1	2	4
y	1	4	6	7

$\frac{69}{26} + \frac{16}{13}t$, $\|\mathbf{e}\|^2 = \frac{17}{13}$

(13) Find the projection matrix P for the plane $x + 2y - z = 0$.

(14) Use the least squares method to fit a parabola $y = a + bx + cx^2$ to the following data:

x	0	1	2	3
y	5	2	1	3

How good is this approximation? Would a line fit the data well? $\frac{1}{20}(101 - 89t + 25t^2)$, $\|\mathbf{e}\|^2 = \frac{1}{20}$

(15) Find three orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ such that $\mathbf{q}_1$, $\mathbf{q}_2$ span the subspace of vectors (x, y, z) satisfying the condition $x + 2y + 3z = 0$. Is $Q = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3]$ invertible? If so find Q^{-1}. $\frac{1}{\sqrt{5}}(2, -1, 0)$, $\frac{1}{\sqrt{70}}(3, 6, -5)$, $\frac{1}{\sqrt{14}}(1, 2, 3)$

(16) Find the QR factorization of the matrix A:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ -2 & 0 & 5 \end{bmatrix}$$

(17) Suppose that A is a 3×3 matrix with $\det A = -5$. Find $\det(2A)$ and $\det(A^3)$. $-40, -125$

(18) For A given below compute $\det A$ in three ways: by row reduction, the "big formula" and cofactor expansion (using any row). Is A invertible? If so find $\det(A^{-1})$. -6

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

(19) Two $n \times n$ matrices A and B are said to be similar if there is an invertible matrix C so that $B = CAC^{-1}$. Show that if A and B are similar, then $\det A = \det B$.