MATH 786: Cooperative Game Theory
HW02

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Abstract

Convex Games, Balanced Games, Shapley-Bondareva Theorem, Market Games.

1. Give an example of a 3-player balanced game which is not convex.

   Solution:
   Let $G = (N, V)$ where

   $$N = \{1, 2, 3\}$$

   and

   $$V(S) = \begin{cases} |S| & \text{if } |S| > 1 \\ 0 & \text{otherwise} \end{cases}$$

   Recall that for a game to be balanced, then

   $$\sum_{S \in T} \delta_S V(S) \leq V(N) \qquad \circledast$$

   must be true for all balanced families, T, of G. The following table displays all of the minimal balanced families of G. Non-minimal balanced families are not considered because they are trivially balanced using a weight of 0 for the "extraneous" coalitions and will meet the $\circledast$ criterion just as the minimal balanced families.

T	$\{\delta_S\}$	$\circledast$ Check
$\{\overline{1}, \overline{2}, \overline{3}\}$	$1, 1, 1$	$1 * 0 + 1 * 0 + 1 * 0 = 0 \leq 3 \implies \text{OK}$
$\{\overline{1, 2}, \overline{3}\}$	$1, 1$	$1 * 2 + 1 * 0 = 2 \leq 3 \implies \text{OK}$
$\{\overline{1, 3}, \overline{2}\}$	$1, 1$	$1 * 2 + 1 * 0 = 2 \leq 3 \implies \text{OK}$
$\{\overline{2, 3}, \overline{1}\}$	$1, 1$	$1 * 2 + 1 * 0 = 2 \leq 3 \implies \text{OK}$
$\{\overline{1, 2, 3}\}$	1	$1 * 3 = 3 \leq 3 \implies \text{OK}$
$\{\overline{1, 2}, \overline{1, 3}, \overline{2, 3}\}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{2} * 2 + \frac{1}{2} * 2 + \frac{1}{2} * 2 = 3 \leq 3 \implies \text{OK}$

   G is balanced.
   To show that a game is not convex, then

   $$V(S_1) + V(S_2) > V(S_1 \cup S_2) + V(S_1 \cap S_2)$$

   must be true for some pair S_1, S_2 in 2^N.

   Let $S_1 = \overline{1, 2}$ and $S_2 = \overline{2, 3}$. Then we have:

   $$\begin{gathered} V(\overline{1, 2}) + V(\overline{2, 3}) > V(\overline{1, 2, 3}) + V(\overline{2}) \\ 2 + 2 > 3 + 0 \\ 4 > 3 \end{gathered}$$

   Which is true.
   Thus, G is balanced and non-convex.

2. Recall the glove game from Problem Set #1. We showed that this game had a nonempty core by exhibiting a core vector for any value of m and p. [m and p were the numbers of left-hand and right-hand gloves in the game.] Now prove the core is nonempty again, this time by using the Shapley-Bondareva theorem. HINTS: A) You may assume without loss of generality that $m \leq p$. B) For each S_i in your arbitrary balanced family, define $\tilde{S}_i$ as a subset of S_i for which a) $\tilde{S}_i$ contains equal numbers of left- and right-hand glove players; b) $V(\tilde{S}_i) = V(S_i)$.

Solution:
If the glove game has a non-empty core, than any arbitrary balanced family will satisfy:

$$\sum_{S_i} \delta_{S_i} V(S_i) \leq V(N)$$

So we have:

$$\begin{aligned}
\sum_{S_i \in T} \delta_{S_i} V(S_i) &\leq V(N) \\
\sum_{S_i \in T} \delta_{S_i} V(\tilde{S}_i) &\leq V(N) && \text{by the assumption in the hints} \\
\sum_{S_i \in T} \delta_{S_i} * 5 * |\tilde{S}_i \cap M| &\leq V(N) && M \text{ is the set of left-hand glove holding players} \\
\sum_{S_i \in T} \delta_{S_i} * 5 * \sum_{j \in M} |\tilde{S}_i \cap \{j\}| &\leq V(N) \\
5 * \sum_{j \in M} \sum_{S_i \in T} \delta_{S_i} |\tilde{S}_i \cap \{j\}| &\leq V(N) \\
5 * \sum_{j \in M} \sum_{S_i \in T} \delta_{S_i} |S_i \cap \{j\}| &\leq V(N) && \text{because } |\tilde{S}_i \cap \{j\}| \leq |S_i \cap \{j\}| \\
5 * \sum_{j \in M} (1) &\leq V(N) && \text{by definition of a balanced game} \\
5m &\leq V(N) \\
5m &\leq 5m
\end{aligned}$$

The idea: because of the balancing weights, each value $V(S)$ is "scaled down" so that they will only sum to the value of the grand coalition, at most.

3. Determine if the following game has a nonempty core. If the core is nonempty, state a core vector; if not, present an argument using the Shapley-Bondareva Theorem.

$$N = \{1, 2, 3, 4\}$$

$$V(N) = 10$$

$$V(\overline{1,4}) = V(\overline{1,2,4}) = V(\overline{1,3,4}) = 8$$

$$V(\overline{1,2,3}) = V(\overline{2,3,4}) = 7$$

$$V(S) = 0 \text{ for all other coalitions } S$$

Solution:
The game G has an empty core.

The Shapley-Bondareva Theorem states that a game's core must be non-empty if that game is balanced; that is if:

$$\sum_{S \in 2^N,\, S \neq \emptyset} \delta_S V(S) \leq V(N)$$

for every balanced family T, where:

$$\sum_{S \in 2^N \,:\, i \in S} \delta_S = 1,\ 0 \leq \delta_S \leq 1$$

Consider the balanced family $T = \{\overline{1,4},\ \overline{1,2,3},\ \overline{2,3,4}\}$. T can be balanced with all weights $\delta_S = \frac{1}{2}$. Using the Shapley-Bondareva criterion, we have:

$$\begin{aligned}
\frac{1}{2}V(\overline{1,4}) + \frac{1}{2}V(\overline{1,2,3}) + \frac{1}{2}V(\overline{2,3,4}) &\leq V(\overline{1,2,3,4}) \\
\frac{1}{2}(8) + \frac{1}{2}(7) + \frac{1}{2}(7) &\leq (10) \\
\frac{8}{2} + \frac{7}{2} + \frac{7}{2} &\leq 10 \\
\frac{22}{2} &\leq 10 \\
11 &\not\leq 10
\end{aligned}$$

Since T violates the condition, the core of G must be empty.

4. Are all convex games market games? Give an argument or a counterexample.

   Solution:
   Yes, all convex games are market games.

   When we talk about convex games, by definition, all sub-games of a convex game are also convex. If a game is convex that means that it has a non-empty core. Having a non-empty core is equivalent to being a balanced game. Thus, all sub-games of a convex game are balanced. Since all sub-games of a convex game are balanced, the original convex game is totally balanced. All totally balanced games are market games. Thus, all convex games are market games.

5. Suppose given a market game in which $n = 3$ and $m = 2$. The initial endowments are $a^1 = (1,0)$, $a^2 = (0,3)$, and $a^3 = (1,1)$. The utility functions are $u^1(w_1, w_2) = 2w_1 + 2w_2$, $u^2(y_1, y_2) = 4y_1 + y_2$, and $u^3(z_1, z_2) = 4\sqrt{z_1} + 4\sqrt{z_2}$.

   (a) Find the characteristic function of the game.

   Solution:

$V(\emptyset)$	$= 0$	$V(\overline{1,2})$	$= 10$
$V(\overline{1})$	$= 2$	$V(\overline{1,3})$	$= 10$
$V(\overline{2})$	$= 3$	$V(\overline{2,3})$	$= 12$
$V(\overline{3})$	$= 8$	$V(\overline{1,2,3})$	$= 18$

$V(\emptyset) = 0$
True by definition.

$V(\overline{1})$
$a^{\overline{1}} = (1, 0)$
Solves maximally with $w_1 = 1$, $w_2 = 0$.
$u^1(1, 0) = 2(1) + 2(0) = 2$
$V(\overline{1}) = 2$

$V(\overline{2})$
$a^{\overline{2}} = (0, 3)$
Solves maximally with $y_1 = 0$, $y_2 = 3$.
$u^2(0, 3) = 4(0) + (3) = 3$
$V(\overline{2}) = 3$

$V(\overline{3})$
$a^{\overline{3}} = (1, 1)$
Solves maximally with $z_1 = 1$, $z_2 = 1$.
$u^3(1, 1) = 4\sqrt{(1)} + 4\sqrt{(1)} = 8$
$V(\overline{3}) = 8$

$V(\overline{1, 2})$
$a^{\overline{1,2}} = (1, 0) + (0, 3) = (1, 3)$
Solves maximally with $w_1 = 0$, $w_2 = 3$, $y_1 = 4$, $y_2 = 0$.
$u^1(1, 0) = 2(0) + 2(3) = 6$
$u^2(0, 3) = 4(1) + (0) = 4$
$V(\overline{1, 2}) = 6 + 4 = 10$

$V(\overline{1, 3})$
$a^{\overline{1,3}} = (1, 0) + (1, 1) = (2, 1)$
Solves maximally with $w_1 = 1$, $w_2 = 0$, $z_1 = 1$, $z_2 = 0$.
$u^1(1, 0) = 2(1) + 2(0) = 2$
$u^3(1, 1) = 4\sqrt{(1)} + 4\sqrt{(1)} = 8$
$V(\overline{1, 3}) = 8 + 2 = 10$

$V(\overline{2, 3})$
$a^{\overline{2,3}} = (0, 3) + (1, 1) = (1, 4)$
Solves maximally with $y_1 = 1$, $y_2 = 0$, $z_1 = 0$, $z_2 = 4$.
$u^2(1, 0) = 4(1) + (0) = 4$
$u^3(0, 4) = 4\sqrt{(0)} + 4\sqrt{(4)} = 8$
$V(\overline{2, 3}) = 4 + 8 = 12$

$V(\overline{1, 2, 3})$
$a^{\overline{1,2,3}} = (1, 0) + (0, 3) + (1, 1) = (2, 4)$
Solves maximally with $w_1 = 0$, $w_2 = 3$, $y_1 = 1$, $y_2 = 0$, $z_1 = 1$, $z_2 = 1$.
$u^1(1, 0) = 2(0) + 2(3) = 6$
$u^2(1, 0) = 4(1) + (0) = 4$
$u^3(0, 4) = 4\sqrt{(1)} + 4\sqrt{(1)} = 8$
$V(\overline{1, 2, 3}) = 6 + 4 + 8 = 18$

(b) Graph the core of the game.

Solution:

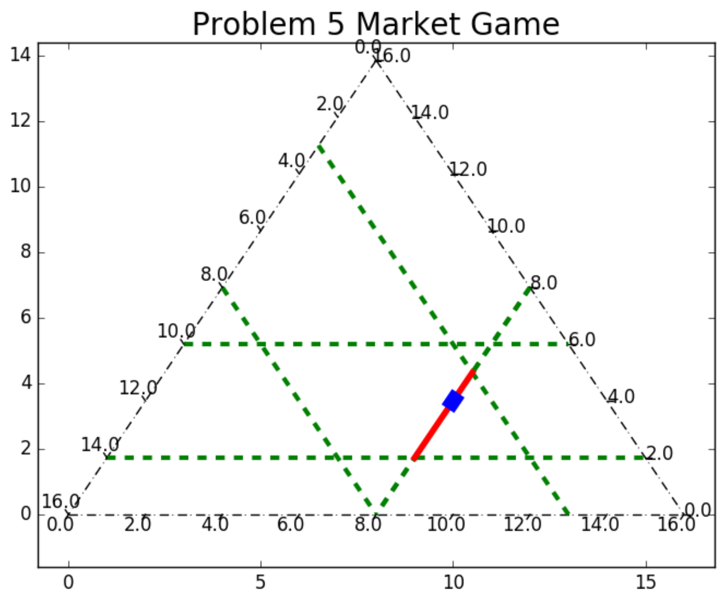

Figure 1: A graph of the core of the game on the 3-D simplex (solid red line) and the competitve solution (blue point) for the game.

(c) Verify that the vector $\Pi = (4, 2)$ is a price equilibrium vector for this game, by showing optimal consumptions (given Π) constitute an N-allocation.

Solution:
To determine the lower bounds of allowable price vectors, we can examine the optimal allocations:

$$\begin{aligned}
(w_1^*, w_2^*) &= \operatorname{argmax} u^1(w_1, w_2) - \Pi * (w_1 - a_1^1, w_2 - a_2^1) \\
&= \operatorname{argmax} 2w_1 + 2w_2 - \Pi_1 * (w_1 - 1) - \Pi_2(w_2 - 0) \\
&= \operatorname{argmax} w_1(2 - \Pi_1) + w_2(2 - \Pi_2) + \Pi_1 \\
&\implies \Pi_1 \geq 2,\ \Pi_2 \geq 2
\end{aligned}$$

$$\begin{aligned}
(y_1^*, y_2^*) &= \operatorname{argmax} u^2(y_1, y_2) - \Pi * (y_1 - a_1^2, y_2 - a_2^2) \\
&= \operatorname{argmax} 4y_1 + y_2 - \Pi_1 * (y_1 - 0) - \Pi_2(y_2 - 3) \\
&= \operatorname{argmax} y_1(4 - \Pi_1) + y_2(1 - \Pi_2) + 3\Pi_2 \\
&\implies \Pi_1 \geq 4,\ \Pi_2 \geq 1
\end{aligned}$$

$$\begin{aligned}
(z_1^*, z_2^*) &= \operatorname{argmax} u^3(z_1, z_2) - \Pi * (z_1 - a_1^3, z_2 - a_2^3) \\
&= \operatorname{argmax} 4\sqrt{z_1} + 4\sqrt{z_2} - \Pi_1 * (z_1 - 1) - \Pi_2(z_2 - 1) \\
&= \operatorname{argmax} \sqrt{z_1}(4 - \Pi_1\sqrt{z_1}) + \sqrt{z_2}(4 - \Pi_1\sqrt{z_2}) + \Pi_1 + \Pi_2 \\
&\text{(assuming that we allocate the maximum available goods } (2, 4) \text{ to z)} \\
&\implies \Pi_1 \geq \frac{4}{\sqrt{2}},\ \Pi_2 \geq 2
\end{aligned}$$

Taken together, this $\implies \Pi_1 \geq 4$, $\Pi_2 \geq 2$. (We take the most restrictive lower bound to ensure that no player is motivated to demand an infinite amount of any good)

To determine the upper bounds of possible price vectors, we can examine the utility functions to see at what price a good would be profitable to purchase:

$$\begin{aligned}
u^1(w_1, w_2) &= 2w_1 + 2w_2 \\
&\implies \Pi_1 w_1 \leq 2w_1 \implies \Pi_1 \leq 2 \\
&\implies \Pi_2 w_2 \leq 2w_2 \implies \Pi_2 \leq 2
\end{aligned}$$

$$\begin{aligned}
u^2(y_1, y_2) &= 4y_1 + y_2 \\
&\implies \Pi_1 y_1 \leq 4y_1 \implies \Pi_1 \leq 4 \\
&\implies \Pi_2 y_2 \leq y_2 \implies \Pi_2 \leq 1
\end{aligned}$$

$$\begin{aligned}
u^3(z_1, z_2) &= 4\sqrt{z_1} + 4\sqrt{z_2} \\
&\text{(assuming that we allocate the maximum available goods } (2,4) \text{ to z)} \\
&\implies \Pi_1 z_1 \leq 4\sqrt{z_1} \implies \Pi_1 \leq \frac{4}{\sqrt{z_1}} \implies \Pi_1 \leq \frac{4}{\sqrt{2}} \\
&\implies \Pi_2 z_2 \leq 4\sqrt{z_2} \implies \Pi_2 \leq \frac{4}{\sqrt{z_2}} \implies \Pi_2 \leq 2
\end{aligned}$$

Taken together, this $\implies \Pi_1 \leq 4$, $\Pi_2 \leq 2$. (We take the least restrictive upper bound to allow at least one person to purchase the goods and others to sell their goods for a good price)

If $\Pi = (4, 2)$, then we have $4 \leq \Pi_1 = 4 \leq 4$ and $2 \leq \Pi_2 = 2 \leq 2$, thus Π is an equilibrium vector.

(d) Find the competitive solution to the game which corresponds with Π.

Solution:

$$\vec{\beta} = (4, 6, 8)$$

Using the optimal consumptions found when finding $V(N)$ and the price equilibrium vector from the previous exercise, we can compute a competitive solution $\vec{\beta}$:

$$\begin{aligned}
\beta_1 &= u^1(w_1^*, w_2^*) + \Pi * (\vec{a}^1 - \vec{w}^*) \\
&= 2(0) + 2(3) + (4,2) * ((1,0) - (0,3)) \\
&= 6 + (4,2) * (1,-3) \\
&= 6 + -2 \\
&= 4
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= u^2(y_1^*, y_2^*) + \Pi * (\vec{a}^2 - \vec{y}^*) \\
&= 4(1) + (0) + (4,2) * ((0,3) - (1,0)) \\
&= 4 + (4,2) * (-1,3) \\
&= 4 + 2 \\
&= 6
\end{aligned}$$

$$\begin{aligned}
\beta_3 &= u^2(z_1^*, z_2^*) + \Pi * (\vec{a}^3 - \vec{z}^*) \\
&= 4\sqrt{(1)} + 4\sqrt{(1)} + (4,2) * ((1,1) - (1,1)) \\
&= 0 + 8 + (4,2) * (0,0) \\
&= 8 - 0 \\
&= 8
\end{aligned}$$

$\vec{\beta}$ is in the core of the game, so it should be acceptable.