Math 330 — Test I (60 pts) Name: Solutions

Thursday, 4 October 2012

PLEASE SHOW ALL WORK

Nº	pts
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Σ	/60

Each problem is worth 10 points. For full credit please show work and clearly indicate which operations you use in row reduction. No wireless devices are permitted. A calculator may be used to check answers.

(1) Find the augmented matrix for the following system. Use elimination and back substitution to solve it. Indicate which elimination steps you use.

$$x + 2y - 3z = 5$$
$$2x + 3y - 2z = 6$$
$$4x + 6y + z = 7$$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 3 & -2 & 6 \\ 4 & 6 & 1 & 7 \end{bmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -1 & 4 & -4 \\ 0 & -2 & 13 & -13 \end{bmatrix} \xrightarrow[R_3-2R_2]{} \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -1 & 4 & -4 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

$$x + 2y - 3z = 5$$
$$-y + 4z = -4$$
$$5z = -5$$

So $5z = -5$, gives $z = -1$, $-y + 4(-1) = -4$ so $y = 0$,

and $x + 2(0) - 3(-1) = 5$, so $x = 5-3 = 2$.

Solution: $x = 2$, $y = 0$, $z = -1$ or $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

(2) Apply elimination to the following matrix. How may pivots are there in the resulting upper triangular matrix? Find all special solutions and then find the complete solution to $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 2 & 1 & 5 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 2 & 1 & 5 \\ 1 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \xrightarrow[R_3-2R_1]{} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivots (1^{st} & 3^{rd} columns)

Free variables x_2, x_4

$$x_1 + x_2 + 3x_4 = 0$$
$$x_3 + x_4 = 0$$

if $x_2 = 1, x_4 = 0$

$x_1 + 1 + 0 = 0$, $x_1 = -1$

$x_3 + 0 = 0$, $x_3 = 0$

$\vec{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$;

if $x_2 = 0, x_4 = 1$

$x_1 + 0 + 3 = 0$

$x_3 - 1 = 0$

$x_1 = -3$, $x_3 = 1$

$\vec{s}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

complete solution $\vec{x} = x_2\vec{s}_1 + x_4\vec{s}_2 = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix}$$

(3) Find an LU factorization for the following matrix A. Verify that this factorization works.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & 12 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 11 \end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

Check $$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & 12 \end{bmatrix} = A$$

(4) With A and $\mathbf{b}$ as given, use *Gauss-Jordan elimination* to find A^{-1} and then use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)*R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = [I \,|\, A^{-1}]$$

So $$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Solve $A\vec{x} = \vec{b}$:

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

(5) With A as below, find conditions on the components of $\mathbf{b} = (b_1, b_2, b_3)$ so that the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Find a vector $\mathbf{b}$ which is *not* in the column space $\mathbf{C}(A)$. How do you know $\mathbf{b}$ is not in $\mathbf{C}(A)$?

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 1 & 1 & 2 & b_2 \\ 2 & 5 & 1 & b_3 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -1 & 1 & b_2-b_1 \\ 0 & 1 & -1 & b_3-2b_1 \end{array}\right] \xrightarrow{R_3+R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -1 & 1 & b_2-b_1 \\ 0 & 0 & 0 & b_3+b_2-3b_1 \end{array}\right]$$

So $A\vec{x} = \vec{b}$ has a solution exactly when $b_3+b_2-3b_1 = 0$. ✓

$\vec{b} = (1,0,0)$ is not in $C(A)$ because $b_3+b_2-3b_1 = -3 \neq 0$.

(6) Which of the following subsets of $\mathbb{R}^3$ are subspaces? Give a brief reason.

a. All vectors $\mathbf{b} = (b_1, b_2, b_3)$ satisfying $b_1 - 2b_2 - b_3 = 4$.

Not a subspace because it does not contain $\vec{0}$.

b. All linear combinations of $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (1, -1, 0)$.

This is a subspace since the set of linear combinations of any 2 vectors is a subspace.

c. All vectors $\mathbf{b} = (b_1, b_2, b_3)$ satisfying $b_1^2 + b_2^2 = b_3^2$.

The vectors $\vec{b} = (1,0,1)$ and $\vec{c} = (0,1,1)$ are both in the set since $1^2+0^2=1^2$, $0^2+1^2=1^2$; but $\vec{b}+\vec{c} = (1,1,2)$ is not since $1^2+1^2=2 \neq 4 = 2^2$. So this not a subspace.