Name: Solutions

Please show your work
(10 pts.)

Math 330 – Quiz 12
Tuesday, 4 Dec 2012

1. Find the matrix exponential e^{tA} given that A is diagonalizable with $A = S\Lambda S^{-1}$ where A, Λ and S are given below. Use it to solve the differential equation $\mathbf{u}' = A\mathbf{u}$ subject to the initial condition $\mathbf{u}(0) = (1, -1)$.

$$A = \begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

$$e^{tA} = S e^{t\Lambda} S^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^{2t} \\ 3e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2e^t + 3e^{2t} & e^t - e^{2t} \\ -6e^t + 6e^{2t} & 3e^t - 2e^{2t} \end{bmatrix}$$

Solution to $\vec{u}' = A\vec{u}$, $\vec{u}(0) = (1,-1)$:

$$\vec{u}(t) = e^{tA}\vec{u}(0) = \begin{bmatrix} -2e^t + 3e^{2t} & e^t - e^{2t} \\ -6e^t + 6e^{2t} & 3e^t - 2e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3e^t + 4e^{2t} \\ -9e^t + 8e^{2t} \end{bmatrix}$$

2. Find the eigenvalues of the symmetric matrix A given below. Find an orthogonal matrix Q which diagonalizes it (that is, find an orthogonal matrix Q so that $A = Q\Lambda Q^T$ for some diagonal matrix Λ).

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$$

if $\lambda = 4, -1$. Thus $\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

$\lambda = 4$, $A - 4I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ $\quad 2x - y = 0$, take $x = 1, y = 2$ $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$, $A + I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad x + 2y = 0$, take $x = -2, y = 1$ $\quad \vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Note $\vec{x}_1^T \vec{x}_2 = 0$, $\|\vec{x}_1\| = \sqrt{5}$, $\|\vec{x}_2\| = \sqrt{5}$

Then $\vec{q}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{q}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

Then $A = Q\Lambda Q^T$.