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Alternating Permutations and Binary Increasing Trees

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In [3], Foata and Schützenberger prove that the binary increasing trees are equinumerous with half of the alternating permutations considered by André [1].

In this paper we present a direct recursive proof of this fact, and then exhibit a natural bijection between the two families. In the process a second class of permutations, which forms a main concern of Foata and Schützenberger's paper, is encountered in a natural setting.

I. INTRODUCTION

1. The *alternating* permutations on $\{1, 2,..., n\}$ are those permutations with successive pairs of terms alternately in increasing and decreasing order. The alternating permutations on $\{1, 2, 3\}$, for example, are 132, 231, 213, and 312.

The number of alternating permutations on $\{1, 2,..., n\}$ is even for $n > 1$ since the complement of an alternating permutation, given by the mapping

$$k \to n + 1 - k, \qquad k = 1(1)\, n, \tag{1}$$

is again an alternating permutation. For $n = 3$, the permutation pairs produced by this mapping are 132, 312; 231, 213.

With A_n the number of complementary pairs of alternating permutations and $A_0 = 1$ by convention, it has been shown by André [1], by a direct combinatorial argument, that

$$2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k}, \qquad n = 1, 2,...$$

and hence, writing

$$A(t) = \sum_{n=0}^{\infty} A_n t^n/n! \tag{2}$$

for the exponential generation function of A_n,

$$2(dA(t)/dt) = 1 + A^2(t), \tag{3}$$

a relation satisfied by

$$A(t) = \tan t + \sec t. \tag{4}$$

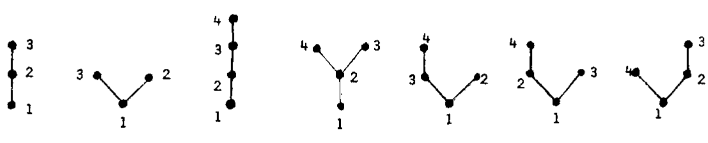
Thus for n odd, A_n is a tangent number, and for n even a signless Euler number. The first few numbers, for $n = 1, 2, \ldots$ are 1, 1, 2, 5, 16, 61, 272,... [5].[1]

The derivative of (3) supplies a second well-known recurrence which is more convenient for present purposes, namely

$$A_{n+2} = \sum_{k=0}^{n} \binom{n}{k} A_{k+1}A_{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{5}$$

This recurrence can be derived by a direct combinatorial argument as follows. A_{n+2} enumerates complementary pairs of permutations, so first select the permutation from each pair in which the element $n + 2$ precedes 1. Then classify these permutations by the position of element 1, whose range is positions 2 to $n + 2$. If 1 is in position $k + 2$, the k elements besides $n + 2$ to 1's left may be chosen in $\binom{n}{k}$ ways, and arranged in A_{k+1} orders; the $n - k$ elements to the right of 1, fixed by the first choice, may be arranged in A_{n-k} ways.

2. The *binary increasing* trees are vertex labeled rooted trees with at most two (upward) branchings at any vertex and with vertex labels in increasing order on any (upward) path from the root. For $n = 3$ and 4 the trees are



(6)

The recurrence (5) derived for the alternating permutations also holds for the binary increasing trees, and can be derived for them by an equally direct combinatorial argument. Every tree has at most two edges up from the root, and removing these edges partitions the remaining $n + 1$ vertices of an $n + 2$ vertex tree into two trees.

Classify these trees by the size of the tree containing vertex $n + 2$, whose range is size 1 to $n + 1$. If the size is $k + 1$, the k vertex labels other

[1] The numbers are designated here as André numbers, contrary to what seems an unfortunate current usage of calling them Euler numbers.

than $n+2$ may be chosen in $\binom{n}{k}$ ways, and arranged into a tree in A_{k+1} ways; the second tree can be arranged in A_{n-k} ways.

Hence the binary increasing trees are equinumerous with half of the alternating permutations. It is reasonable to expect a natural two-to-one correspondence between the alternating permutations and these trees; this follows below.

2. The Correspondence

1. The bijection presented here is formed by intertwining two operations, called nestling and relative complementing.

The complement of a permutation on $\{1, 2,..., n\}$ has been defined above in (1). Given a permutation P on a subset $\{a_1, a_2,..., a_k\}$ of $\{1, 2,..., n\}$, we define its *relative complement*, P^c, by the mapping

$$a_i \to a_{k+1-i}, \qquad i = 1(1)\,k.$$

For example, $(1284)^c = 8412$, and $(31528)^c = 38251$.

Nestling is a multistep double bracketing procedure that the following example should make clear:

$$\begin{aligned} 956273184 &\to ((956273)\,1(84)) \\ &\to (((956)\,2(73))\,1((8)\,4)) \\ &\to ((((9)\,5(6))\,2((7)\,3))\,1((8)\,4)). \end{aligned} \tag{7}$$

Thus after placing the entire permutation within brackets, each segment within brackets in turn is double bracketed by placing brackets around the elements to the left and to the right of the least element. This process is continued until every two elements are separated by left or right brackets.

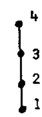
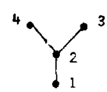
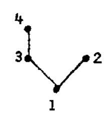
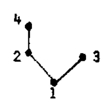
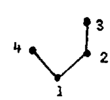
2. Our bijection uses a modified nestling in which each segment in turn, before it is double bracketed, is relative complemented if its least term is not preceded by its greatest term. Calling this the nestled transform, "(P^T)", of P, we illustrate with an alternating permutation:

$$\begin{aligned} 845173926 &\to (845173926) \\ \xrightarrow{c} (265937184) &\to ((265937)\,1(84)) \\ \xrightarrow{c} ((956273)\,1(84)) &\to (((956)\,2(73))\,1((8)\,4)) \\ &\to ((((9)\,5(6))\,2((7)\,3))\,1((8)\,4)), \end{aligned}$$

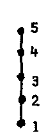
which is the same as the nestled permutation of (7). Note that the nestling illustrated in (7) is also a nestled transform since complementing is never required there.

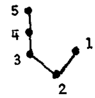
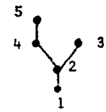
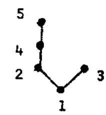
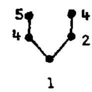
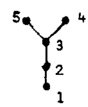
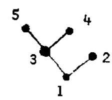
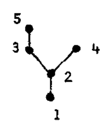
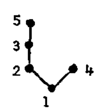
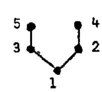
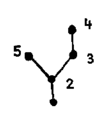
By design the nestled transform of a permutation is the same as that of its complement $((P^T) = ((P^c)^T))$, and each transform (P^T) defines a tree as follows: the elements within each pair of brackets are all above the least element of that bracket, and in particular the smallest elements to the left and right of this element within the facing brackets are connected to it by edges in the tree.

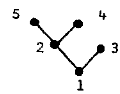
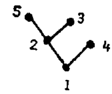
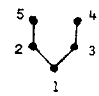
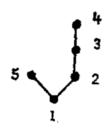
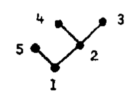
To prove that this map from P to (P^T) to T is well defined requires only a simple induction. The first double bracketing in the map $P \to (P^T)$ changes P to $((P_1)\ 1(P_2))$, with $P_1 1 P_2 = P$ or P^c. So if the maps $P_1 \to T_1$ and $P_2 \to T_2$ are well defined, then so is $P \to T$. The map $P \to T$ is trivially well defined when $n = 1$ or 2, so the result is immediate. When $n = 4$, the map is:

P	(P^T)	T
3241 & 2314	((((4) 3) 2) 1)	
4231 & 1324	(((4) 2(3)) 1)	
3412 & 2143	(((4) 3) 1(2))	
2413 & 3142	(((4) 2) 1(3))	
4132 & 1423	((4) 1((3) 2))	

For $n = 5$, the map is:

P	(P^T)	T
34251 & 32415	(((((5) 4) 3) 2) 1)	

P	(P^T)	T
43512 & 23154	((((5) 4) 3) 1(2))	
45231 & 21435	((((5) 4) 2(3)) 1)	
42513 & 24153	((((5) 4) 2) 1(3))	
45132 & 21534	(((5) 4) 1((3) 2))	
24351 & 42315	((((5) 3(4)) 2) 1)	
53412 & 13254	(((5) 3(4)) 1(2))	
35241 & 31425	((((5) 3) 2(4)) 1)	
32514 & 34152	((((5) 3) 2) 1(4))	
35142 & 31524	(((5) 3) 1((4) 2))	
25341 & 41325	(((5) 2((4) 3)) 1)	

P	(P^T)	T
52413 & 14253	(((5) 2(4)) 1(3))	
52314 & 14352	(((5) 2(3)) 1(4))	
25143 & 41523	(((5) 2) 1((4) 3))	
51324 & 15342	((5) 1(((4) 3) 2))	
51423 & 15243	((5) 1((4) 2(3)))	

3. The Foata–Schützenberger Permutations

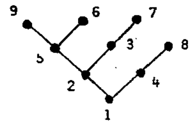
1. A corollary result of the map $P \to (P^T) \to T$ just described is that the transforms P^T, that is the nestled transforms (P^T) with the brackets removed, are equinumerous with the binary increasing trees. The map $P^T \to (P^T) \to T$ has just been given. The inverse map $T \to P^T$ can be stated iteratively by the set of instructions:

(1) Starting at the tip (end point) of T with the highest label, read down the vertices to and including the first branch point encountered.

(2) Then beginning at the highest labeled tip of T above this branch point not yet encountered, read down the vertices to the first branch point, not counting and skipping over any branch points previously encountered.

(3) Iterate step 2 until all vertices have been read.

For example, the permutation $P^T = 956273184$ of (13) is read in this manner from the tree

$T =$ 

$(95 - 6_5 2 - 73_2 1 - 84)$

2. This map can be used to determine several properties of the permutations P^T, as follows.

LEMMA 1. *Given $P^T = p_1 p_2 \cdots p_n$, if $p_i < p_{i+1}$, then $p_{i+1} > p_{i+2}$.*

Proof. Let $P^T \leftrightarrow T$. Then p_i is a branch point of T and p_{i+1} is a tip of T.

LEMMA 2. *The nestled transforms (P^T) are completely characterized by the property: the maximal element within each pair of facing brackets is the extreme leftmost element.*

Proof. $T \leftrightarrow (P^T) = ((P_1^T)\, 1 (P_2^T))$, $(P_1^T) \to T_1$ and $(P_2^T) \to T_2$. The first term read from a tree is its maximal vertex so the result follows from induction on the size of P^T.

LEMMA 3. *The transforms P^T are completely characterized by the two criteria (the second due to Foata and Schützenberger* [3]):

(1) *$p_1 = n$, the maximal term of P^T.*

(2) *If $p_i p_{i+1}$ and $p_j p_{j+1}$ are both in increasing order with $i < j$, $p_i < p_j$, and $p_{i+1} < p_{j+1}$, then there is a third pair $p_k p_{k+1}$ in increasing order with $i < k < j$, $p_k < p_i < p_{i+1}$, and $p_{i+1} < p_{j+1} < p_{k+1}$.*

Proof. $P^T = P_1^T 1 P_2^T$, so if $p_i p_{i+1}$ and $p_j p_{j+1}$ satisfy criterion 2, then either both pairs are in one of P_1^T, P_2^T and the result follows by induction, or else $p_i p_{i+1}$ is in P_1^T and $p_j p_{j+1}$ is in P_2^T. In this latter case the elements 1 and the first (maximal) element of P_2^T meet the requirements of $p_k p_{k+1}$.

Complementing the transforms P^T (Eq. (2)) after deleting $p_1 = n$ yield the permutations studied by Foata and Schützenberger and which they have named the André permutations, type one.

4. REMARKS

1. In a binary increasing tree each vertex above the root is connected by an edge down to a unique vertex of lower label. We use this to define the functional code [2][2] for T by setting c_i to be the label of the vertex directly below vertex $i + 1$, for $i + 1 = 2(1)\, n$.

The functional codes for the trees on $\{1, 2, \ldots, n\}$ satisfy the two criteria:

(1) $1 \leqslant c_i \leqslant i,\ i = 1(1)\,n-1$.

(2) Each code number appears at most twice.

The codes for the trees for $n = 3$ and 4 listed in (6) are, respectively:

$$12, \quad 11, \quad 123, \quad 122, \quad 113, \quad 112, \quad 121.$$

Since the pairs $(c_i, i+1)$ define the edges of T, the set of functional codes is enumerated by (2) and (4). Interestingly, there does not appear to be a direct approach to enumerating these sequences without matching them to the binary increasing trees.

2. The trees also have Prüfer codes [4], of size $n-2$, obtained by removing largest end points in succession and recording the adjacent vertices. However this code, applied to the binary increasing trees, is just the functional code in reverse order with the term $c_1 = 1$ deleted.

3. Every binary increasing tree determines a partial ordering $\prec$ on $\{1, 2, \ldots, n\}$ characterized by the three criteria:

(1) Every pair (i, j) has a meet $i \wedge j$ but their join exists only if $i \prec j$ or $j \prec i$ (i.e., only if the two vertices are linearly ordered in the tree).

(2) There are no three pairwise unordered numbers i, j and k with meets $i \wedge j = i \wedge k = j \wedge k$.

(3) $i \prec j \Rightarrow i \leqslant j$ (by convention $i \prec i$).

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[a] These codes were communicated to J. Riordan by H. W. Becker in 1949 as being equinumerous with the alternating permutations, but without association with the trees (and without any proofs). Riordan subsequently observed that they were the functional codes for the binary increasing trees.