Generating Uniformly Random Ranked Unlabeled Trees

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Plan

- Definitions
- Bijection between ranked unlabeled trees and alternating permutations.
- Generating uniformly random alternating permutations.

Alternating Permutations

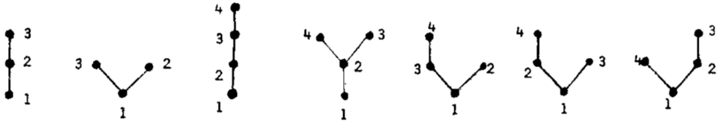
- An alternating permutation is a permutation such that successive terms alternate in increasing and decreasing order.
- Letting A_n denote the set of up-down $(\sigma(1) < \sigma(2))$ alternating permutations on $\{1, \ldots n\}$, then for $n > 0$,

$$2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k} \tag{1}$$

  The $n + 1$st Euler-Zigzag number.
- For $n = 3$, the up-down alternating permutations are $132, 231$ and the down-up alternating permutations are $312, 213$.

Binary Increasing Trees

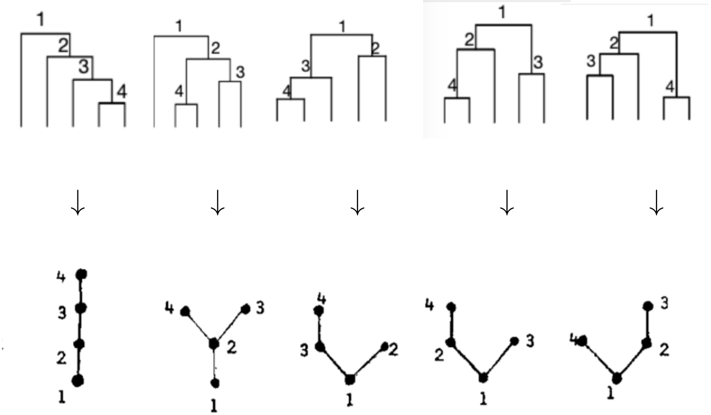
- Binary increasing trees are rooted, vertex labeled trees in increasing order along up-
  ward paths. The trees for $n = 3$ and $n = 4$ are below (from [1]).



- The cardinality of binary increasing trees with n vertices is also the *nth* Euler-Zigzag number A_n by a similar argument
- Bijection between binary increasing trees and ranked unlabeled trees as binary increasing trees with n vertices are the internal branches of ranked unlabeled trees with $n + 1$ leaves.

Binary Increasing Trees -> Ranked Unlabeled Trees

- The bijection between ranked unlabeled trees with 5 leaves and binary increasing trees with 4 vertices is presented below.



Converting Alternating Permutations to Binary Increasing Trees

- Define a bijection from alternating permutations to binary increasing trees in terms of relative complements and nestling.
- Let P be a permutation of a subset $\{a_1, \ldots a_k\}$ (with $a_i < a_{i+1}$) of $\{1, \ldots n\}$. Then we define the relative complement of P, denoted P^c by the mapping $a_i \to a_{k+1-i}$.
- $(1284)^c \to 8412$ as $1 \to 8$, $2 \to 4$, $8 \to 1$ and $4 \to 2$
- $(31528)^c = 38251$ as $3 \to 3$, $8 \to 1$, $5 \to 2$, $2 \to 5$ and $8 \to 1$
- If P is an up-down alternating permutation then its relative complement is a down-up alternating permutation and vice-versa. Also easy to see that $(P^c)^c = P$ and thus the complement operation is a bijection between down-up and up-down permutations.

Converting Alternating Permutations to Binary Increasing Trees

- Nestling, a recursive bracketing procedure made clear by the following example
- Let $P = 956273184$. Then,

$$956273184 \rightarrow (956273184) \rightarrow ((956273)\,1\,(84))$$

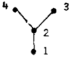
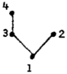
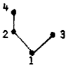
$$\rightarrow (((956)\,2\,(73))\,1\,((8)\,4)) \rightarrow ((((9)\,5\,(6))\,2\,((7)\,3))\,1\,((8)\,4))$$

- Recursively find the smallest element and draw parentheses around the block to its left and right until every number is directly surrounded by parentheses.
- For a permutation P let (P) denote its bracketed form.

Converting Alternating Permutations to Binary Increasing Trees

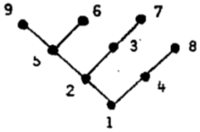
- For the final bijection, we use a process where we interchange nestling and coupling operations. The process works by at each stage before bracketing a block, relative complementing it if its least term is not preceeded by its greatest term.
- $$(845173926) \to^{c} (265937184) \to ((265937)\,1\,(84))$$
  $$\to^{c} ((956273)\,1\,(84)) \to (((956)\,2\,(73))\,1\,(84))$$
  $$\to ((((9)\,5\,(6))\,2\,(73))\,1\,(84)) \to ((((9)\,5\,(6))\,2\,((7)\,3))\,1\,((8)\,4))$$
- This transform is called the nestled transform of a permutation and for a permutation P, it's nestled transform is denoted $\left(P^{T}\right)$. By definition $\left((P^{c})^{T}\right) = \left(P^{T}\right)$.
- We can define a tree from the nestled transform by having that within each pair of brackets, the elements within those brackets are all above the least element in those brackets and the smallest elements to the left and right of the least element (if there are any), are connected to it by an edge.

- Below we present the operation in the case $n = 4$ from [1].

P	(P^T)	T
3241 & 2314	((((4) 3) 2) 1)	
4231 & 1324	(((4) 2(3)) 1)	
3412 & 2143	(((4) 3) 1(2))	
2413 & 3142	(((4) 2) 1(3))	
4132 & 1423	((4) 1((3) 2))	

- The inverse map (from a tree T to a nestled transformed permutation P^T (also has a natural recursive definition.
- For a tree T, start at the tip of T with the highest value and list out the vertices up to and including the first branch point. Then starting from the highest tip above this branch point not yet listed, list the vertices up to and including the first branch point.
- The second step is iterated until every label has been read off. A simple example is presented below.

$T =$



$(95 - 6_5 2 - 73_2 1 - 84)$

- With this, we have shown a bijection between alternating permutations and binary increasing trees (and hence ranked unlabeled trees).

Generating Uniformly Random Alternating Permutation

- Due to the bijection above, to sample uniformly from ranked unlabeled trees (with $n+1$ leaves) it is sufficient to sample uniformly from the space of alternating permutations (of $\{1,\ldots n\}$).
- Fortunately, [2] presents an algorithm to do this in $O\left(n \log n\right)$ time presented below.

- Idea: generate a random sequence of alternating reals and recover an alternating permutation from this using a sorting algorithm.
- Let $N \in \mathbb{N}$ and $U_1, \dots U_N$ be $i.i.d$ uniform on $[0, 1]$.
- Define a random sequence $(X_1, \dots X_N)$ by
- $X_1 = U_1$ and
- $x_{i+1} = 1 - \frac{2}{\pi} \arcsin \left(U_{n+1} \sin \left(\frac{\pi}{2} X_i \right) \right)$ for $i \in \{, \dots n - 1\}$

- Let $\alpha_N = \frac{\sin\left(\frac{\pi}{2} X_N\right)}{\sin\left(\frac{\pi}{2} X_1\right)}$
- We now define a random sequence Y with
- $Y = (X_1, 1 - X_2, X_3, \ldots)$ with probability $\frac{1}{\alpha_N + \frac{1}{\alpha_N}}$
- $Y = (1 - X_1, X_2, 1 - X_3, \ldots)$ with probability $\frac{1}{\alpha_N + \frac{1}{\alpha_N}}$
- Start over from the beginning with probability $1 - \frac{2}{\alpha_N + \frac{1}{\alpha_N}}$

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- We can attain an upper bound on the rejection probability as for all
$$E[1 - \frac{2}{\alpha_N + \frac{1}{\alpha_N}}] \leq 1 - \frac{2}{3\pi} \approx 0.788$$
- Thus once we have a target sequence Y, we can recover an alternating permutation (and hence a ranked unlabeled tree) in $O(n \log n)$ time using a sorting algorithm.

References

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