Random Doubly Stochastic Tridiagonal Matrices

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Plan

- Definitions
- Using a random doubly stochastic tridiagonal matrix to generate an alternating permutation
- Sampling a uniformly random doubly stochastic tridiagonal matrix (multiple ways)

Goal

- Want to understand where the motivation for the algorithm presented in [2] came from.
- Want to sample uniformly an alternating permutation by sampling uniformly from an alternating sequence of reals and recovering a uniform permutation with a sorting algorithm.

Tridiagonal Doubly Stochastic Matrices

- Let τ_N denote the set of tridiagonal doubly stochastic matrices. That is, matrices of the form

$$\begin{pmatrix} 1-c_1 & c_1 & & & & 0 \\ c_1 & 1-c_1-c_2 & c_2 & & & \\ & c_2 & 1-c_2-c_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{n-1} & 1-c_{n-1}-c_n & c_n \\ 0 & & & & c_n & 1-c_n \end{pmatrix}$$

  where all elements are positive and the rows sum to 1.
- A tridiagonal doubly stochastic matrix is uniquely determined by its superdiagonal, namely the elements $c_1, \ldots, c_n$
- These matrices represent are transition probability matrices for birth and death chains, the original motivation for this paper being to study these chains by examining the distribution of the eigenvalues of these matrices.

Polytopes

- We can view τ_N as a polytope by considering only the elements on the superdiagonal. In this sense τ_N is $n-dimensional$ and determined by the following linear inequalities

$$c_i \geq 0 \text{ and } c_i + c_{i+1} \leq 1 \ \forall i \in \{0, \ldots, n\}$$

where $c_0 = c_{n+1} = 0$

- As shown in [3], with the below argument, $Vol\,(\tau_N) = \frac{E_n}{n!}$, where E_n is the *nth* Euler-Zigzag number. This is equivalent to the fraction of permutations of $\{1, \ldots n\}$, that are alternating.

Proof of Volume of τ_N (from [3])

$$f_n(t) = \int_{x_1=0}^{t} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n. \tag{3.3}$$

Clearly $f(1) = \mathrm{vol}(\mathcal{C}_n)$. Differentiating equation (3.3) yields $f_n'(t) = f_{n-1}(1-t)$. There are various ways to solve this recurrence for $f_n(t)$ (with the initial conditions $f_0(t) = 1$ and $f_n(0) = 0$ for $n > 0$), yielding

$$\sum_{n\geq 0} f_n(t)x^n = (\sec x)(\cos(t-1)x + \sin tx).$$

Putting $t = 1$ gives

$$\sum_{n\geq 0} f_n(1)x^n = \sec x + \tan x,$$

so we conclude that $\mathrm{vol}(\mathcal{C}_n) = E_n/n!$.

Volume of Alternating Permutation Polytope

- Let P_n be the polytope determined by by $d_1 \leq d_2 \geq d_3 \leq d_4 \dots d_n$. We would like to get a uniform sample from this polytope, from which we can recover a uniform alternating permutation.
- There
  is a volume-preserving bijection between P_n and τ_N given below.

  Proof of algorithm. In [30, Theorem 2.3], Richard Stanley shows that the polytope $\{(x_1, x_2, \dots, x_n) : 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq n \text{ and } x_i + x_{i+1} \leq 1, \text{ for } 1 \leq i \leq n-1\}$ is affinely equivalent to the polytope $\{(y_1, y_2, \dots, y_n) : 0 \leq y_i \leq 1 \text{ for } 1 \leq i \leq n \text{ and } y_1 \leq y_2 \geq y_3 \leq y_4 \geq \cdots y_n\}$. In particular, the map $\phi(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ defined by

  $$y_i = \begin{cases} x_i & \text{if } i \text{ is odd} \\ 1 - x_i & \text{if } i \text{ is even.} \end{cases}$$

- Because of this fact, to sample from P_n, we need only sample from τ_N. Thus to sample a uniform alternating permutation, it is sufficient to sample from τ_n.

Sampling from τ_n

- One way to sample a tridiagonal doubly stochastic matrix with superdiagonal $c_1, \ldots c_n$ is given below.

  1. Choose an alternating permutation σ uniformly at random (see below).
  2. Choose n points uniformly in $[0, 1]$ and order them from smallest to largest, calling them $0 < x_1 < x_2 < \cdots < x_n < 1$.
  3. Define the c_i as follows:

  $$c_i := \begin{cases} x_{\sigma_i} & \text{if } i \text{ is odd, and} \\ 1 - x_{\sigma_i} & \text{if } i \text{ is even.} \end{cases}$$

  This step uses the map given by Richard Stanley in [30, Theorem 2.3].

- This is the reverse of the algorithm to generate an alternating permutation that was presented in [2].

Gibbs Sampling from τ_N

- Another way to generate uniform sample from τ_n is to use gibbs sampling. Namely, start by letting $c_1 = c_2 \dots c_n = 0$.
- Let $U_1, U_2, \dots$ be a sequence of i.i.d $[0, 1]$ uniform random variables.
- At each time step k, select uniformly i from $\{1, \dots n\}$ and let $c_i = U_k$ if the inequality constraints are still satisfied. Namely if $c_{i-1} + U_k \leq 1$ and $c_{i+1} + U_k \leq 1$.
- This chain is reversible with respect to the uniform distribution on τ_N. The authors state that $10 \cdot n \log n$ steps are sufficient for the chain to mix and show experimentally that the distribution at this time step for small n is almost exactly uniform, but no proof is presented.

Using Markov Chain to Sample an Alternating Permutation

- In the paper, the authors determine, for a uniform element of τ_N, the joint distribution
  of the c_i's. They show that the elements c_i form a markov chain.

Theorem 4.1. *For any $1 \le i \le n-1$, for any real constants $a_1, \ldots, a_i$ in the interval $[0,1]$, and for any $0 \le t \le 1$,*

$$\Pr(c_{i+1} \le t | c_1 = a_1, c_2 = a_2, \ldots, c_i = a_i) = \Pr(c_{i+1} \le t | c_i = a_i). \tag{4.1}$$

- Then, in the limiting case, we get the following result

Corollary 4.4. *In the limit as $n \to \infty$,*

$$\lim_{n\to\infty} \Pr(c_1^{(n)} \le x) = \sin(x\pi/2) \quad \text{and}$$

$$\lim_{n\to\infty} \Pr(c_{i+1}^{(n)} \le x | c_i^{(n)} = a_i)) = \frac{\sin(\frac{\pi}{2}\min\{x, 1-a_i\})}{\sin(\frac{\pi}{2}(1-a_i))}.$$

Using Markov Chain to Sample an Alternating Permutation

- This looks very similar to the chain presented in [2], and it is.
- The stochastic representation of this chain, prove in [1] (apart from the initial value) is given by the following.

$$c_{i+1} = \frac{2}{\pi} \arcsin \left(U_{i+1} \cos \left(\frac{\pi}{2} \right) c_i \right)$$

where the U_i are i.i.d uniform on $[0, 1]$ and $c_1 = U_1$.
- For this chain, the density of $C = (c_1, \ldots c_n)$ is given by

$$1_{\{c_1+c_2 \leq 1, c_2+c_3 \leq 1 \ldots c_{n-1}+c_n \leq 1\}} \left(\frac{\pi}{2} \right)^{n-1} \alpha_n$$

where $\alpha_n = \frac{\cos\left(\frac{\pi}{2} c_n\right)}{\cos\left(\frac{\pi}{2} c_1\right)}$

Using Markov Chain to Sample an Alternating Permutation

- Then let $Z = C$ with probability $\frac{1}{\alpha_N + \alpha_N^{-1}}$, let $Z = C^{-1} = (c_n, \ldots, c_1)$ with probability $\frac{1}{\alpha_N + \alpha_N^{-1}}$ and with probability $1 - \frac{2}{\alpha_N + \alpha_N^{-1}}$, repeat the process (running a new markov chain) to find Z.
- Then Z has density,

$$\frac{1}{Vol\left(\tau_n\right)} 1_{\{c_1+c_2 \leq 1, c_2+c_3 \leq 1 \ldots c_{n-1}+c_n \leq 1\}}$$

$$= \frac{1}{Q_n}\left(\frac{\pi}{2}\right)^{n-1} 1_{\{c_1+c_2 \leq 1, c_2+c_3 \leq 1 \ldots c_{n-1}+c_n \leq 1\}}$$

where Q_n is the expected value of the rejection probability. Using the asymptotic expansion for E_n, we can thus deduce that $Q_n \approx \frac{8}{\pi^2} \approx 0.78$ for large n.

Using Markov Chain to Sample an Alternating Permutation

- Z is a uniform sample from τ_N so we can use the bijection presented earlier to recover from Z a uniform element P of P_n and from that, using a sorting algorithm, recover a uniform alternating permutation σ.

References

- P. Diaconis and P. M. Wood. *Random double stochastic tridiagonal matrices*. Random Structures and Algorithms, 2012.
- P. Marchal. *Generating random alternating permutations in time $n \log n$*. http://hal.archives-ouvertes.fr/docs/00/76/54/32/PDF/altperm.pdf, 2012.
- Richard Stanley. *A Survey of Alternating Permutations*. https://arxiv.org/abs/0912.4240, 2009.