Linear Algebra 02

时间：2022年04月09日

1. If an augmented matrix $[A \quad \mathbf{b}]$ is transformed into $[C \quad \mathbf{d}]$ by elementary row operations, then the equations $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ have exactly the same solution sets. (T/F?)

[解答]

True. See the box after the definition of elementary row operations, in Section 1.1. If $[A \quad \mathbf{b}]$ is transformed into $[C \quad \mathbf{d}] \text{ by elementary row operations, then the two augmented matrices are row }$ equivalent.

1. Find a basis for the set of all vectors of the form $\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix}$.(Be careful.)

   [解答]

   The set is Span S, where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \right\}$. Note that S is a linearly dependent set, but each pair of vectors in S forms a linearly independent set. Thus any two of the three vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}$ will be a basis for Span S.

2. $$T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

   Show that $\det T = (b-a)(c-a)(c-b)$.

[解答]

$$\det T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$
$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$$

3. 设有实矩阵 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 1 & a \end{bmatrix}$，$B = \begin{bmatrix} 1 & b & c \\ 2 & b^2 & c+1 \end{bmatrix}$，其中 a, b, c 为实常数. 已知齐次线性方程组 $AX = 0$ 和 $BX = 0$ 同解，试求 a, b, c 的值.

   [解答]

   解:由题意，矩阵 A 和 B 必然有相同的秩，从而 A 必然不满秩，于是

$$\det A = \det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & a \end{bmatrix} = 0$$

得 $a=2$.
设 $X=(x_1,x_2,x_3)^T$，解方程组 $AX=0$ 得 $X=t(1,1,-1)^T$，其中 t 为任意实数. 这也说明两个方程组的解空间维数都是 1，从而矩阵 A 和 B 的秩都是 2 .
令 $B\begin{bmatrix}1\\1\\-1\end{bmatrix}=0$，得 $\begin{cases}b-c+1=0\\b^2-c+1=0\end{cases}$，解得 $\begin{cases}b=0\\c=1\end{cases}$ 或 $\begin{cases}b=1\\c=2\end{cases}$.
进一步计算可得，当 $b=0,c=1$ 时， B 的两行线性相关，此时 B 的秩为 1；而当 $b=1,c=2$ 时， B 的两行线性无关，此时 B 的秩为 2 .
综上所述， $a=2,b=1,c=2.$

4. 证明替换定理：设向量组 $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_s\}$ 线性无关，$\boldsymbol{\beta}=b_1\boldsymbol{\alpha}_1+b_2\boldsymbol{\alpha}_2+\cdots+b_s\boldsymbol{\alpha}_s$. 如果 $b_i\neq 0$，那么用 $\boldsymbol{\beta}$ 替换 $\boldsymbol{\alpha}_i$ 后得到的向量组 $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_{i-1},\boldsymbol{\beta},\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_s\}$ 也线性无关.
[解答]
证明：设存在标量 $c_1,c_2,\ldots,c_s$ 使得

$$c_1\boldsymbol{\alpha}_1+c_2\boldsymbol{\alpha}_2+\cdots+c_{i-1}\boldsymbol{\alpha}_{i-1}+c_i\boldsymbol{\beta}+c_{i+1}\boldsymbol{\alpha}_{i+1}+\cdots+c_s\boldsymbol{\alpha}_s=0$$

即

$$(c_ib_1+c_1)\boldsymbol{\alpha}_1+(c_ib_2+c_2)\boldsymbol{\alpha}_2+\cdots+(c_ib_{i-1}+c_{i-1})\boldsymbol{\alpha}_{i-1}+b_i\boldsymbol{\alpha}_i+(c_ib_{i+1}+c_{i+1})\boldsymbol{\alpha}_{i+1}+\cdots+(c_ib_s+c_s)\boldsymbol{\alpha}_s=0$$

因为向量组 $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_s\}$ 线性无关，所以

$$c_ib_1+c_1=c_ib_2+c_2=\cdots=c_ib_{i-1}+c_{i-1}=b_ic_i=c_ib_{i+1}+c_{i+1}=\cdots=c_ib_s+c_s=0$$

因为 $b_i\neq 0$，所以 $c_i=0$，所以 $c_1=c_2=\cdots=c_n=0.$
这说明向量组 $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_{i-1},\boldsymbol{\beta},\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_s\}$ 是线性无关的.

5. 问题解答：

31. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

[解答]