Linear Algebra

时间:2022年04月04日

一、判断正误

1. Every matrix is row equivalent to a unique matrix in echelon form.(F)

   [解析]教材13页定理一

   每个矩阵行等价于唯一的简化阶梯型矩阵。

   THEOREM 1 **Uniqueness of the Reduced Echelon Form**

   Each matrix is row equivalent to one and only one reduced echelon matrix.

2. If A is an invertible $n \times n$ matrix, then the equation $Ax = b$ is consistent for each b in $\mathbb{R}^n$.(T)

   [解析]教材104页定理五

   d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

   THEOREM 5 If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

二、计算

1. Find bases for $NulA$ and $ColA$.

   a.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

[解析]

1. 化为阶梯型矩阵，含有主元列的构成列空间。
2. 求解矩阵的零空间--->解$Ax = 0$
3. **13.** $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

13. Since B is a row echelon form of A, we see that the first and second columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix},\begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables: $x_1=-6x_3-5x_4$, $x_2=(-5/2)x_3-(3/2)x_4$, with x_3 and x_4 free. So

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}\right\}.$$

b.

$$A=\begin{bmatrix}1&2&0&4&5\\0&0&5&-7&8\\0&0&0&0&-9\\0&0&0&0&0\end{bmatrix}$$

14. $$A=\begin{bmatrix}1&2&-5&11&-3\\2&4&-5&15&2\\1&2&0&4&5\\3&6&-5&19&-2\end{bmatrix},$$

$$B=\begin{bmatrix}1&2&0&4&5\\0&0&5&-7&8\\0&0&0&0&-9\\0&0&0&0&0\end{bmatrix}$$

14. Since B is a row echelon form of A, we see that the first, third, and fifth columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}-5\\-5\\0\\-5\end{bmatrix},\begin{bmatrix}-3\\2\\5\\-2\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables, mentally completing the row reduction of B to get: $x_1=-2x_2-4x_4$, $x_3=(7/5)x_4$, $x_5=0$, with x_2 and x_4 free. So

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix}\right\}.$$

2.

Let $\mathbf{b}_1=\begin{bmatrix}1\\-3\end{bmatrix}$, $\mathbf{b}_2=\begin{bmatrix}-2\\4\end{bmatrix}$, $\mathbf{c}_1=\begin{bmatrix}-7\\9\end{bmatrix}$, $\mathbf{c}_2=\begin{bmatrix}-5\\7\end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B}=\{\mathbf{b}_1,\mathbf{b}_2\}$ and $\mathcal{C}=\{\mathbf{c}_1,\mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

[解答]

这里我们知道

a. Notice that ${}_{\mathcal{B}\leftarrow\mathcal{C}}^{\ P}$ is needed rather than ${}_{\mathcal{C}\leftarrow\mathcal{B}}^{\ P}$, and compute

$$\left[\begin{array}{cc:cc}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2\end{array}\right]=\left[\begin{array}{cc:cc}1 & -2 & -7 & -5\\ -3 & 4 & 9 & 7\end{array}\right]\sim\left[\begin{array}{cc:cc}1 & 0 & 5 & 3\\ 0 & 1 & 6 & 4\end{array}\right]$$

So

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}=\begin{bmatrix}5 & 3\\ 6 & 4\end{bmatrix}$$

b. By part (a) and property (6) above (with $\mathcal{B}$ and $\mathcal{C}$ interchanged),

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}=(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P})^{-1}=\frac{1}{2}\begin{bmatrix}4 & -3\\ -6 & 5\end{bmatrix}=\begin{bmatrix}2 & -3/2\\ -3 & 5/2\end{bmatrix}\quad\blacksquare$$

Another description of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ uses the change-of-coordinate matrices $P_{\mathcal{B}}$ and $P_{\mathcal{C}}$ that convert $\mathcal{B}$-coordinates and $\mathcal{C}$-coordinates, respectively, into standard coordinates. Recall that for each $\mathbf{x}$ in $\mathbb{R}^n$,

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}=\mathbf{x},\quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}=\mathbf{x},\quad\text{and}\quad[\mathbf{x}]_{\mathcal{C}}=P_{\mathcal{C}}^{-1}\mathbf{x}$$

Thus

$$[\mathbf{x}]_{\mathcal{C}}=P_{\mathcal{C}}^{-1}\mathbf{x}=P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

In $\mathbb{R}^n$, the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ may be computed as $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. Actually, for matrices larger than 2×2, an algorithm analogous to the one in Example 3 is faster than computing $P_{\mathcal{C}}^{-1}$ and then $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. See Exercise 12 in Section 2.2.

三、证明

1. 设A是n阶矩阵，若$A^2=A$，证明$A+E$可逆.

[解答]

证明:由于$A^2-A=0$,我们想要证明$(A+E)(?)=E$,那么如何求解?处的表达式呢？我们知道这样的因式分解是可以配凑的，于是$(A+E)(A-2E)=-2E$,这里的配凑我们显然需要利用$A^2-A=0$这个现成的表达式来证明。那么我们可以找到$(?)=\frac{A-2E}{-2}$,说明$A+E$可逆，证毕。

大家可以思考本题是否还有其他解法，采用特征值证明??

2. 设A是$m\times n$矩阵，B是$n\times s$矩阵，若$AB=O$，证明$rank(A)+rank(B)\leq n$.

[解答]

分析：总体思路：采用矩阵分块的方式求解.题目具有一定难度.

证明:
对矩阵B按照列分块，我们记$B=[\beta_1\quad\beta_2\quad\beta_3\quad\cdots\quad\beta_s]$
那么有

$$\begin{aligned}AB&=A\,[\beta_1\quad\beta_2\quad\beta_3\quad\cdots\quad\beta_s]\\&=[A\beta_1\quad A\beta_2\quad A\beta_3\quad\cdots\quad A\beta_s]\\&=[\vec{0}\quad\vec{0}\quad\vec{0}\quad\cdots\quad\vec{0}]\end{aligned}$$

于是我们有$A\beta_j=\vec{0},j=1,2,3,\cdots,s$,
所以B的列向量都是齐次方程组$A\vec{x}=\vec{0}$的解，由于方程组$A\vec{x}=\vec{0}$的解向量的
$rank(\vec{x})=n-rank(A)$,这里的$\vec{x}$与$\vec{\beta}$含义等价，所以

$$rank(\beta_1, \beta_2, \beta_3, \cdots, \beta_s) \leq n - rank(A)$$

我们又知道

$$rank(\beta_1, \beta_2, \beta_3, \cdots, \beta_s) = rank(B)$$

所以：$rank(A) + rank(B) \leq n.$