# Linear Algebra 04

时间：2022年4月28日

1. Find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

[解答]

Row reduce the system's augmented matrix:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \sim \sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

This echelon matrix shows that the system is inconsistent, because its rightmost column is a pivot column; the third row corresponds to the equation $0 = 5$. There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

1. 求解线性方程组的通解.

① 写出其系数矩阵或对应的增广矩阵.

② 化简增广矩阵. 根据是否有解考虑是否化简到简化阶梯形矩阵.

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

主元列个数为2且出现 $0x = b\,(b \neq 0)$ 的情况，显然不相容.

⇒ 复习教材第19页定理2和20页的应用行化简算法求解线性方程组

2. a. If a matrix **A** is diagonalizable, then the columns of **A** are linearly dependent. (T/F?)

如果一个矩阵A是可对角化的，则A的列线性相关。 (False)

2.a. A可对角化，A的各列线性无关 (反例如下). false.

5.3节定理 ① 可逆，但不可对角化. 5.3节 习题31.

② 可对角化但不可逆 5.3节 习题32.

b. Two eigenvectors corresponding to the distinct eigenvalues are always linearly independent. (T/F?)

对应于不同特征值的两个特征向量总是线性无关的。 (True)

2.b. 对应于两个不同的特征值的特征向量总是线性无关的 (True)

证明：设矩阵 $A^{(n\times n)}$，特征值为 $\lambda_1, \lambda_2$，$(\lambda_1 \neq \lambda_2)$ 对应的特征向量为 $\vec{x_1}, \vec{x_2}$.

由特征向量的定义.（教材5.1节定义 $P_{207}$）.

有 $A\vec{x_1} = \lambda_1 \vec{x_1}$ ， $A\vec{x_2} = \lambda_2 \vec{x_2}$

于是发现到这步，不好证明下去，（正难则反）. 我们采用反证法.

我们假设对应于不同特征值的特征向量线性相关.

故存在不全为零的实数 $k_1, k_2$. s.t: $k_1\vec{x_1} + k_2\vec{x_2} = 0$ ，故同时乘以A. 得

$A k_1\vec{x_1} + A k_2\vec{x_2} = \lambda_1 \vec{x_1}\cdot k_1 + \lambda_2\vec{x_2}\cdot k_2 = 0$ ，又 $k_1\vec{x_1} = -k_2\vec{x_2}$

代入 $k_1\lambda_1\vec{x_1} + k_2\lambda_2\vec{x_2} = 0$ 可得 $-\lambda_1\cdot k_2\vec{x_2} + \lambda_2\cdot k_2\vec{x_2} = 0$

即 $k_2\vec{x_2}(\lambda_2 - \lambda_1) = 0$. 又 $\lambda_1 \neq \lambda_2$ ，$\forall k_2, \vec{x_2}$ 不为零，故矛盾!

故对应于不同特征值的特征向量线性无关.

c. A 3x3 matrix **A** with 3 linearly dependent eigenvectors is invertible. (T/F?)

一个有3个线性相关特征向量的3x3的矩阵A是可逆的。(False)

2.c. 3个线性相关的特征向量对应的3x3的矩阵A是不可对角化的.

由2.a，与矩阵是否可逆无关.

总结：矩阵是否可以对角化，特征值和特征向量的情况如何与矩阵的秩没有直接关系!

3. 如果题2(b)是错误的，请修改为正确的并证明题2(b)；如果题2(b)是正确的，请证明之。

   [解答]

   [如何理解不同特征值对应的特征向量线性无关？ - 知乎 (zhihu.com)]

   问题：为什么不同特征值对应的特征向量线性无关?
   解答: 首先我们对于矩阵 $A$ 的特征值 $\lambda_1$ ，有等式满足 $Ax_1 = \lambda_1 x_1$ ，特征向量为 $x_1$.
   对于特征值 $\lambda_2$ ，有等式满足 $Ax_2 = \lambda_2 x_2$ 。
   下面用反证法进行证明! 首先假设不同特征值对应的特征向量线性相关
   如果对于不同的 $\lambda_1$，$\lambda_2$ 所对应的特征向量线性相关的话，那么满足下面等式:
   $k_1x_1 + k_2x_2 = 0$，那么等式两边同时乘以矩阵 A，得到 $Ak_1x_1 + Ak_2x_2 = 0$，化简为:
   $\lambda_1 k_1 x_1 + \lambda_2 k_2 x_2 = 0$， 又因为根据等式 $k_1x_1 + k_2x_2 = 0$ 可以得到,
   $k_2x_2 = -k_1x_1$ ，带入到 $\lambda_1 k_1 x_1 + \lambda_2 k_2 x_2 = 0$ ，得到 $k_1x_1(\lambda_1 - \lambda_2) = 0$, 又因为 $\lambda_1, \lambda_2$ 不相同，则造成矛盾。
   所以不同特征值对应的特征向量线性相关是错误的。
   所以：不同特征值对应的特征向量是线性无关的

4. 已知

$$
\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}
$$

   求可逆矩阵 $\boldsymbol{P}$ 和对角矩阵 $\boldsymbol{D}$, 使 $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$

[解答]

4. ① 求解特征值

$$\det(A-\lambda I)=0 \Rightarrow \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = (A-\lambda I)$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ 0 & -2-\lambda & -2-\lambda \\ 3 & 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 0 & \lambda+2 \\ 0 & 1 & 1 \\ 3 & 3 & 1-\lambda \end{vmatrix} -(2+\lambda)$$

$$= -(2+\lambda)(2+\lambda)\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= -(2+\lambda)^2\begin{vmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (2+\lambda)^2\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (2+\lambda)^2 \left| (1-\lambda)+3-3 \right| = -(\lambda-1)(2+\lambda)^2.$$

特征值 $\lambda_1=1$, $\lambda_2=-2$, $\lambda_3=-2$. 二重根.

② 求解特征向量, $\lambda_1=1$ 时, $A-I=\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1+x_2=0 \\ x_2+x_3=0 \end{matrix}$$ 取 $\vec{x}=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ 即可

$\lambda_2=\lambda_3=-2$ 时, $[A+2I]=\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow x_1+x_2+x_3=0$ $\Rightarrow x_1=-x_2-x_3$

$$\vec{x}=\begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ 线性无关

③ 构造矩阵 P 和 D.

$$P=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

④ 验证 AP=PD, 自行验证!

There are four steps to implement the description in Theorem 5 .

Step 1. Find the eigenvalues of A. As mentioned in Section $5.2$, the mechanics of this step are appropriate for a computer when the matrix is larger than $2 \times 2$. To avoid unnecessary distractions, the text will usually supply information needed for this step. In the present case, the characteristic equation turns out to involve a cubic polynomial that can be factored:

$$\begin{aligned} 0=\det(A-\lambda I) &= -\lambda^3-3\lambda^2+4 \\ &= -(\lambda-1)(\lambda+2)^2 \end{aligned}$$

The eigenvalues are $\lambda=1$ and $\lambda=-2$.

Step 2. Find three linearly independent eigenvectors of $A$. Three vectors are needed because $A$ is a $3 \times 3$ matrix. This is the critical step. If it fails, then Theorem 5 says that $A$ cannot be diagonalized. The method in Section $5.1$ produces a basis for each eigenspace:

Basis for $\lambda=1$ : $\quad \mathbf{v}_1=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda=-2$ : $\quad \mathbf{v}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\quad \mathbf{v}_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

You can check that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a linearly independent set.

Step 3. Construct P from the vectors in step 2. The order of the vectors is unimportant. Using the order chosen in step 2 , form

$$P=[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]=\begin{bmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{bmatrix}$$

Step 4. Construct D from the corresponding eigenvalues. In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of $P$. Use the eigenvalue $\lambda=-2$ twice, once for each of the eigenvectors corresponding to $\lambda=-2$ :

$$D=\begin{bmatrix}1&0&0\\0&-2&0\\0&0&-2\end{bmatrix}$$

It is a good idea to check that $P$ and $D$ really work. To avoid computing $P^{-1}$, simply verify that $AP=PD$. This is equivalent to $A=PDP^{-1}$ when $P$ is invertible. (However, be sure that $P$ is invertible!) Compute

$$AP=\begin{bmatrix}1&3&3\\-3&-5&-3\\3&3&1\end{bmatrix}\begin{bmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{bmatrix}=\begin{bmatrix}1&2&2\\-1&-2&0\\1&0&-2\end{bmatrix}$$
$$PD=\begin{bmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-2&0\\0&0&-2\end{bmatrix}=\begin{bmatrix}1&2&2\\-1&-2&0\\1&0&-2\end{bmatrix}$$

5. (选做)请仔细阅读下面的证明过程：

$$T=\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix}$$

Show that $\det T=(b-a)(c-a)(c-b)$.

[证明]

$$\det T=\begin{vmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{vmatrix}=\begin{vmatrix}1&a&a^2\\0&b-a&b^2-a^2\\0&c-a&c^2-a^2\end{vmatrix}=\begin{vmatrix}1&a&a^2\\0&b-a&(b-a)(b+a)\\0&c-a&(c-a)(c+a)\end{vmatrix}$$
$$=(b-a)(c-a)\begin{vmatrix}1&a&a^2\\0&1&b+a\\0&1&c+a\end{vmatrix}=(b-a)(c-a)\begin{vmatrix}1&a&a^2\\0&1&b+a\\0&0&c-b\end{vmatrix}=(b-a)(c-a)(c-b)$$

请仿照上述的证明思路及证明过程，证明如下结论：

$$
\boldsymbol{A}=\begin{bmatrix}1 & a_1 & a_1^2 & \cdots & a_1^{n-1}\\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1}\end{bmatrix}
$$

Show that $\det A=\prod_{0\le i<j\le n}(x_j-x_i)$.

[证明]

[范德蒙行列式 - 知乎 (zhihu.com)]

现在我们证明 $A=\begin{bmatrix}1 & a_1 & \cdots & a_1^{n-1}\\ 1 & a_2 & \cdots & a_2^{n-1}\\ \vdots & \vdots & \cdots & \\ 1 & a_n & \cdots & a_n^{n-1}\end{bmatrix}$ 中 $\det A=|A|\neq 0$

简证：为了表达方便，

我们直接对A进行转置 令 $B=A^T=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & \cdots & 1\\ a_1 & a_2 & & & & & a_n\\ a_1^2 & a_2^2 & \cdots & & & & a_n^2\\ \vdots & & & & & & \\ a_1^{n-1} & a_2^{n-1} & & & & & a_n^{n-1}\end{bmatrix}$

拿第3行减去第2行的$a_1$倍，后面同理

$$
=\begin{bmatrix}1 & 1 & 1 & 1 & \cdots & 1\\ a_1-a_1 & a_2-a_1 & a_3-a_1 & a_4-a_1 & \cdots & a_n-a_1\\ a_1^2-a_1^2 & a_2^2-a_1^2 & a_3^2-a_1 & \cdots & \cdots & a_n^2-a_1^2\\ a_1^3-a_1\cdot a_1^2 & a_2^3-a_1\cdot a_2^2 & \cdots & \cdots & \cdots & a_n^3-a_1\cdot a_n^2\\ \vdots & & & & & \\ a_1^{n-1}-a_1\cdot a_1^{n-2} & a_2^{n-1}-a_1\cdot a_2^{n-2} & \cdots & & \cdots & a_n^{n-1}-a_1\cdot a_n^{n-2}\end{bmatrix}
\rightarrow
\begin{bmatrix}1 & 1 & \cdots & 1\\ a_1-a_1 & a_2-a_1 & & a_n-a_1\\ a_1^2-a_1^2 & a_2^2-a_1\cdot a_2 & \cdots & a_n^2-a_1\cdot a_n\\ a_1^3-a_1^3 & a_2^3-a_1\cdot a_2^2 & \cdots & a_n^3-a_1\cdot a_n^2\\ \vdots & \vdots & & \vdots\\ a_1^{n-1}-a_1^{n-1} & a_2^{n-1}-a_1\cdot a_2^{n-2} & & a_n^{n-1}-a_1\cdot a_n^{n-2}\end{bmatrix}
$$

$$
\rightarrow\begin{bmatrix}1 & 1 & \cdots & 1\\ 0 & (a_2-a_1)\cdot 1 & & (a_n-a_1)\cdot 1\\ 0 & (a_2-a_1)a_2 & & (a_n-a_1)\cdot a_n\\ 0 & (a_2-a_1)a_2^2 & & (a_n-a_1)\cdot a_n^2\\ \vdots & \vdots & & \vdots\\ 0 & (a_2-a_1)a_2^{n-2} & & (a_n-a_1)a_n^{n-2}\end{bmatrix}
$$

$$= (a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \begin{bmatrix} 1 & & 1 \\ a_2 & & a_n \\ a_2^2 & & a_n^2 \\ \vdots & & \vdots \\ a_2^{n-2} & & a_n^{n-2} \end{bmatrix} \\ 0 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

我们不妨继续考察 $A_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{bmatrix}$

采取同样的方式，后一行减去前一行的$a_2$倍

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_2-a_2 & a_3-a_2 & & a_n-a_2 \\ a_2^2-a_2\cdot a_2 & a_3^2-a_2\cdot a_3 & & a_n^2-a_2\cdot a_n \\ a_2^3-a_2\cdot a_2^2 & a_3^3-a_2\cdot a_3^2 & & a_n^3-a_2\cdot a_n^2 \\ \vdots & \vdots & & \vdots \\ a_2^{n-2}-a_2\cdot a_2^{n-3} & a_3^{n-2}-a_2\cdot a_3^{n-3} & & a_n^{n-2}-a_2\cdot a_n^{n-3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & (a_3-a_2)\cdot 1 & & (a_n-a_2)\cdot 1 \\ 0 & (a_3-a_2)\,a_3 & & (a_n-a_2)\cdot a_n \\ \vdots & (a_3-a_2)\,a_3^2 & & (a_n-a_2)\cdot a_n^2 \\ \vdots & \vdots & & \vdots \\ 0 & (a_3-a_2)\,a_3^{n-3} & & (a_n-a_2)\,a_n^{n-3} \end{bmatrix}$$

$$= (a_3-a_2)(a_4-a_2)\cdots(a_n-a_2)\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \boxed{\begin{matrix} 1 & 1 & & 1 \\ a_3 & a_4 & & a_n \\ a_3^2 & a_4^2 & & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_3^{n-3} & a_4^{n-3} & & a_n^{n-3} \end{matrix}} \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

递归执行n-5次后，我们得到

$$A_{n-3} = \begin{bmatrix} 1 & 1 & 1 \\ a_{n-2} & a_{n-1} & a_n \\ a_{n-2}^2 & a_{n-1}^2 & a_n^2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ a_{n-2}-a_{n-2} & a_{n-1}-a_{n-2} & a_n-a_{n-2} \\ a_{n-2}^2-a_{n-2}a_{n-2} & a_{n-1}^2-a_{n-2}a_{n-1} & a_n^2-a_{n-2}a_n \end{bmatrix}$$

$$= (a_{n-1}-a_{n-2})(a_n-a_{n-2})\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & a_{n-1} & a_n \end{bmatrix} = (a_{n-1}-a_{n-2})(a_n-a_{n-2})\cdot(a_n-a_{n-1})$$

所以 $|A| =$

$$(a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)$$
$$\cdot(a_3-a_2)(a_4-a_2)\cdots(a_n-{}_2)$$
$$\cdot(a_4-a_3)(a_5-a_3)\cdots(a_n-a_3)$$
$$\vdots$$
$$(a_{n-1}-a_{n-2})(a_n-a_{n-2})\cdot(a_n-a_{n-1}) = \prod_{1\le i<j<n}(x_j-x_i)$$