Single-Assumption Systems in Proof-Theoretic Semantics

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Abstract

Proof-theoretic semantics is an inferentialist theory of meaning, usually developed in a multiple-assumption and single-conclusion framework. In that framework, this theory seems unable to justify classical logic, so some authors have proposed a multiple-conclusion reformulation to accomplish this goal. In the first part of this paper, the debate originated by this proposal is briefly exposed and used to defend the diverging opinion that proof-theoretic semantics should always endorse a single-assumption and single-conclusion framework. In order to adopt this approach some of its *criteria* of validity, especially separability, need to be weakened. This choice is evaluated and defended. The main argument in this direction is based on the circular dependences of meaning between multiple assumptions and conjunctions, and between multiple conclusions and disjunctions. In the second part of this paper, some systems that suit the new requirements are proposed for both intuitionistic and classical logic. A proof that they are valid, according to the weakened *criteria*, is sketched.

Keywords. Proof-Theoretic Semantics; Single Conclusions; Multiple Conclusion; Single Assumption; Multiple Assumption; Separability

1 Introduction

Proof-theoretic semantics (PTS) is an inferentialist theory of meaning for logical terms.[1] Since Natural Deduction is closer to actual inferential practice than any other proof systems, PTS heavily relies on it. According to this theory of meaning there is a mismatch between two main aspects of our standard argumentative practice:

1. Apparently we use arguments that derive a single conclusion from (possibly) more than one assumption;

2. Apparently we reason classically.

Indeed, there seems to be no formulation of classical logic in the standard multiple-assumption, single-conclusion Natural Deduction that suits all the *criteria* of PTS. The traditional solution in the context of PTS is to privilege intuitionistic logic, which, as opposed to classical, can be characterized by a well behaved formal system of Natural Deduction. In this paper, I will argue for two controversial positions:

[1] For a recent short introduction, see [Steinberger and Murzi, 2017]; for a more detailed presentation, see [Francez, 2015].

1. The multiple-assumption, single-conclusion framework should be changed with a *single*-assumption, single-conclusion framework in PTS, as well as in any investigation regarding the meaning of logical constants;
2. Classical logic can be justified in this reformulation of PTS, together with intuitionistic logic.

The structure of the paper is the following. In section 2, PTS is presented and some modifications of its standard version are proposed and defended. More in detail: in subsection 2.1, an unorthodox *criterion* of Harmony is presented and justified, while a standard notion of Separability is temporarily endorsed; in subsection 2.2, some attempts to justify classical logic in PTS using single-conclusion systems are evaluated and rejected; in subsection 2.3, the same thing is done with multiple-conclusion systems. At the end of the section, some general arguments against a multiple-conclusion approach in PTS are evaluated and endorsed, and it is argued that we should extend the ban to multiple-assumption systems as well. The adoption of a new *criterion* of Harmony and the rejection of multiple-assumption systems are the key results of this section. Then, in section 3 the single-assumption, single-conclusion version of PTS is developed formally. More in detail: in subsection 3.1 a pure[2] single-assumption, single-conclusion system is developed as a toy example; in subsection 3.2 purity is rejected, and the *criterion* of Separability is adapted for impure systems. In the last part of the section single-assumption, single-conclusion systems for intuitionistic and classical logics are developed, and it is proved that they suit both Harmony and Separability.

2 Reasons for single-assumption

2.1 Proof-theoretic semantics

2.1.1 Harmony

According to PTS the introduction and the elimination rules (I-rules and E-rules) correspond to the two main aspects of the inferential practice: I-rules specify what are the grounds for asserting a sentence and E-rules specify which conclusions we are entitled to derive from a sentence. Under this assumption, it is unsurprising that PTS imposes some restrictions on the acceptable pairs of I and E-rules. In the end, it would be very strange if the grounds for deriving a sentence were fully independent of its consequences. Indeed without any kind of restriction we are forced to accept even pairs of rules for pathological connectives like Prior's *tonk*:[3]

$$tonk\mathrm{I}\,\frac{A}{AtonkB} \qquad tonk\mathrm{E}\,\frac{AtonkB}{B}$$

As is well known, this pair of rules leads to triviality any transitive logical system. At first glance, PTS's diagnosis of a mismatch between the grounds for deriving $AtonkB$ and its consequences seems to suit the case: A and B are completely unrelated.

[2] In the sense of definition 2.4.

[3] [Prior, 1960]

PTS's *criterion* to discern between acceptable and unacceptable pairs of rules is generally called "Harmony". There are different proposals to formulate Harmony, but as far as we are concerned we can just use the possibility of circumventing *maximal formulae*:

Definition 2.1 (Maximal formulae)**.** Given a derivation $\mathfrak{D}$, a *maximal formula* in it is a formula that is the conclusion of an I-rule and the major premise of an E-rule.

If E-rules match I-rules, *maximal formulae* represent in some way detours in a derivation, so we should be able to avoid them. A derivation in which there are no *maximal formulae* is called "in normal form". As a consequence, our Harmony *criterion* is:[4]

Definition 2.2 (Harmony)**.** The rules for the logical constants of a system are in Harmony iff for every derivation in this system there is an equivalent derivation in normal form.[5]

It is usually required that there is an effective reduction procedure that leads from a non-normal derivation to its normal form.[6] I claim that we can reject this extra requirement, since it is essentially motivated by the intuitionistic skepticism about purely existential results, and we will gain a justification of *classical* logic by the end of this paper.[7] Before moving on, let us reply to some objections that could be raised against this "purely existential" formulation of Harmony:

- Someone could object that using purely existential results to prove the validity of classical logic is circular, and so unacceptable. I reply to this objection by pointing out that this argument is only *pragmatically* circular – logic is used in order to justify a logical system –, and that pragmatical circularity is inevitable in any justification of logic. Moreover, while standard circular arguments are trivial and uninformative, since we assume what we have to prove, pragmatically circular arguments can be demanding and informative.[8]

- Another easy objection could be that we should not rely on controversial classical principles, like purely existential results or excluded middle, in trying to persuade skeptics of classical logic, and that proof-theoretic semanticists are generally skeptical about classical logic. In other words, pragmatical circularity could be fine to explain why a logic is valid to someone who already endorses it, but not to convert logical "heretics", for which an impartial ground is needed.[9] This could seem a good objection as far as we consider only intuitionist and classical logic, but it has

[4]This *criterion* can be used as the main ingredient of an explicit definition of validity, following [Prawitz, 1973] (especially p. 236).

[5]While we will follow this definition across the entire paper, there are other, completely different approaches to Harmony. For a very brief overview of them, see [Steinberger, 2013].

[6][Prawitz, 1973], as an example.

[7]That the connections between Harmony and normalization have been exaggerated in Prawitz's papers of the Seventies is remarked in [Schroeder-Heister, 2006] as well, especially regarding the ask for strong normalization to obtain what Prawitz calls strong validity. Here we are just pushing the argument a little further.

[8][Dummett, 1991], p. 202.

[9]This is indeed the position that we find in [Dummett, 1991], p. 202. See [Wright, 2018], note 10 on p. 447 for another critical view on Dummett's position.

some obviously unacceptable consequences when more non-classical positions regarding logic are considered: McGee's skepticism regarding *modus ponens*, minimal logic, counter-classical systems like connexive logic and logical nihilism *inter alia*. Should we abandon our logical systems unless we are able to defend them using only those very few logical tools accepted by *all* those contenders? Accepting such a conclusion would mean giving up to justify any interesting logical system. Moreover, even taking for granted that a classical logician is unable to convert an intuitionist relying on purely existential results, the intuitionist is equally unable to convert the classical logician relying on its skepticism about purely existential results. So if we follow impartiality, as opposed to prudence, we conclude that the justification of classical logic has the same strength as the usual justification of intuitionistic logic.[10]

2.1.2 Separability

Another *criterion* imposed by PTS for the acceptability of a logical system is:[11]

Definition 2.3 (Separability)**.** To prove a logical consequence $\Gamma \vdash C$ we only need to use the rules for the logical constants that occur in Γ or in C.

That is, the only rules that are needed in order to prove a logical consequence are those for the logical terms that occur in it. There are two reasons to ask for this property:

Atomism PTS standardly assumes that the meanings of the logical terms are independent of each other;

Analyticity Logical consequences hold in virtue of the meaning of the logical terms that occur in them.

Since logical consequences are true in virtue of the meaning of the logical terms that occur in them and the meaning of those terms can be specified without reference to any other terms, every logical consequence should be provable using only the rules for the logical terms that occur in them.

Atomism asks for a further requirement too: only one logical term can occur in each meaning-conferring rule. I-rules are standardly (but not universally) considered as the only meaning-conferring rules for a term, so that Harmony justifies the E-rules based on the meaning given to logical terms by the I-rules. We will follow this tradition here, and so equate Atomism's requirement with:

Definition 2.4 (Purity)**.** Only one logical constant figures in each I-rule.

In order to endorse Atomism, we need to follow Purity, and if we endorse Atomism and we want logical consequences to be analytically valid, then we need to prove Separability. While there have been some attempts to identify Harmony

[10] An anonymous referee suggested that with the existence of normal form as the only requirement for Harmony, we cannot explain self-referential paradoxes in PTS as caused by loops in the reduction sequence, a well-known proposal made by Read ([Read, 2000], pp. 140-142.), but already sketched by Prawitz ([Prawitz, 1965] Appendix B). While this cannot be a decisive objection to my proposal, it is surely one of its worse downsides, and I will need to deal with it in further work.

[11] This requirement has been proposed in [Belnap, 1962] *outside PTS* and then integrated in this approach by [Dummett, 1991].

and Separability,[12] there are no uncontroversial results in this direction.[13] For this reason we will address them separately.

2.2 Single-conclusion classical logic?

The standard formulation **NJ** of intuitionistic logic suits both Harmony and Separability.[14] Unfortunately, it is much harder to find a good formulation of classical logic that suits both of them. Prawitz proposed a rule of classical *reductio* that preserves Harmony,[15] but the status of this rule is at least controversial: it is hard to consider it as an I-rule,[16] and considering it as an E-rule, it lacks a general justification, due to its disharmonious structure. Moreover, there is an even stronger problem for Separability, since the $\supset$-rules of Prawitz's classical system are the same of **NJ** but Peirce's law – that is, $((A \supset B) \supset A) \supset A$, in which only $\supset$ occurs – is classically but not intuitionistically valid. As a consequence it is provable only passing through classical *reductio*, which is a rule for $\perp$ and not for $\supset$.

Under these circumstances, the only solution for a classical system to suit Separability is to change the $\supset$-rules too.[17] There is general agreement that the adoption of something like Peirce's rule

$$\begin{array}{c} [A \supset B] \\ \vdots \\ \textit{Peirce}\ \dfrac{A}{A} \end{array}$$

could solve the issue of Separability without damaging Harmony, but there are different approaches to this rule. Both in [Zimmermann, 2002] and in [Pereira et al., 2010], Peirce's rule is considered as a *sui generis* E-rule: like *ex falso quodlibet* it cannot lead to maximal formulae, so the only requirement for accepting it in the system is that it does not prevent the normalization of the standard maximal formulae, rendering disharmonious the system.[18] On the other

[12] [Dummett, 1991]

[13] See for example [Read, 2000] (especially the first section) and [Steinberger, 2013] (especially section 3.1).

[14] Two clarifications: the negation $\neg A$ is considered as a shortenings for $A \supset \perp$; *ex falso quodlibet* is considered as the E-rule for $\perp$, which is in Harmony with its lack of I-rules (see [Read, 2000], p. 139).

[15] [Prawitz, 1965] proposed it for a $\vee\exists$-free fragment of classical logic, while [Andou, 1995] extended the result to cover the entire language.

[16] The thesis that classical *reductio* should be treated as an I-rule is evaluated in [Milne, 1994], pp. 58-60, but this alternative seems to give rise to more problems than it solves.

[17] Bilateral systems for classical logic, that is systems that assume *rejection* ($-$), together with assertion ($+$), as one of the basic speech acts, seems to behave much better; see [Rumfitt, 2000]. Nonetheless, in order to exclude paradoxical rules from our bilateral systems, we need to impose accordance between the rules of assertion and those of rejection for the same constant, as proposed in [Francez, 2018] (a reply to [Gabbay, 2017]). While this approach is innovative, so far there seems to be no solution to the apparently *ad hoc* status of both the Coordination Principles between $+$ and $-$, and the accordance restriction proposed by Francez. Moreover, to evaluate Separability it seems to me that we should consider the occurrences of $+$ and $-$ as well, but in this way Rumfitt's system for classical logic does not suit Separability anymore. As an example, $-$ is needed to prove $+(\neg\neg A) \vdash +A$. I thank Nissim Francez for stressing the point of bilateral systems.

[18] [Zimmermann, 2002], pp. 565, 567-568 and [Pereira et al., 2010], p. 98. The two papers, which extend respectively Prawitz's and Seldin's strategies of normalization, interpret Peirce's rule essentially in the same way.

side, Peter Milne considers (a modified version of) Peirce's rule as a general-introduction rule (gI-rule), a new class of rules that in combination with general-elimination rules give a complete formulation for classical logic.[19] A $\oplus$gI rule has a conclusion that is fully general and has the same shape of all the minor premisses, it discharges a formula that has $\oplus$ as its outermost connective and eventually some of its subformulae, and all its major premises are subformulae of the $\oplus$-formula as well. As an example, the following two gI-rules for $\supset$ give a complete system for the purely implicational fragment of classical logic:

$$\text{Tarski}\ \frac{\begin{array}{c}[A \supset B]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array}}{C} \qquad \text{Weak}{\supset}\text{I}\ \frac{\begin{array}{c}[A \supset B]\\ \vdots \\ C\end{array} \qquad B}{C}$$

The first is a generalization of Peirce's rule, while the second is a weakening of standard $\supset$I, needed to avoid redundancy in the system containing also primitive rules for negation. Milne's interpretation makes Peirce's rule a *standard* gI-rule, while in [Zimmermann, 2002] and [Pereira et al., 2010] it is seen only as a *sui generis* E-rule,[20] so Milne's reading seems preferable.

Although Milne's solution suits both Harmony and Separability, it encounters some objections:

- As acknowledged by the author himself, this solution does not work for first-order logic;[21]
- General-introduction rules do not suit the standard definition of an I-rule.

Even though we do not consider first-order logic explicitly in this work, the first problem cannot be overlooked. About the second problem, Milne proposes a reinterpretation of Harmony in order to justify his rules.[22] As a consequence, the endorsement of this solution asks for a bigger departure from standard PTS than we are ready to accept, at least *prima facie*.[23]

2.3 Multiple-conclusion classical logic?

2.3.1 Harmony and Separability

It is very easy to find a well-behaved system of Natural Deduction for classical logic if we just adopt a multiple-conclusion approach.[24] Indeed the system **MCNK**, displayed in table 1, suits both Harmony and Separability:[25]

As an example, Peirce's law can be proved using only $\supset$-rules (and structural rules), and so its truth depends only on the meaning of this logical term:

[19] [Milne, 2015].

[20] First of all, as a gI-rule, it can give rise to maximal formulae, while as an E-rule it cannot.

[21] To be precise, it is not possible to have Separability without adding *ad hoc* restrictions to the rules for $\forall$; see [Milne, 2015], p. 217.

[22] [Milne, 2015], p. 202.

[23] Peter Milne acknowledged in a private conversation that his I-rules and standard I-rules have different purposes.

[24] Since there are very well-behaved sequent-calculus systems for classical logic that allow multiple formulae in the succedent, this is not very surprising.

[25] This system is Read's version of the original system of [Boričić, 1985]; see [Read, 2000], pp. 143-150. While Boričić discusses only normalization, Read deals also with Separability.

$$\frac{A,\Delta}{A,B,\Delta}\,\text{Weakening} \qquad\qquad \frac{A,A,\Delta}{A,\Delta}\,\text{Contraction}$$

$$\frac{A,\Gamma \qquad B,\Delta}{A\wedge B,\Gamma,\Delta}\,\wedge I \qquad\qquad \frac{A\wedge B,\Gamma \qquad \begin{array}{c}[A][B]\\ \vdots\\ \Delta\end{array}}{\Delta,\Gamma}\,\wedge E$$

$$\frac{A,\Gamma}{A\vee B,\Gamma}\,\vee I \qquad \frac{A\vee B,\Gamma \qquad \begin{array}{c}[A]\\ \vdots\\ \Delta\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ \Delta\end{array}}{\Delta,\Gamma}\,\vee E \qquad \frac{B,\Gamma}{A\vee B,\Gamma}\,\vee I$$

$$\frac{\begin{array}{c}[A]\\ \vdots\\ B,\Gamma\end{array}}{A\supset B,\Gamma}\supset I \qquad \frac{\bot,\Delta}{A,\Delta}\,\bot E \qquad \frac{A\supset B,\Gamma \qquad A,\Theta}{\Gamma,\Theta}\supset E$$

Table 1: **MCNK**

$$\supset\mathrm{I}_2\,\dfrac{Contr\,\dfrac{\supset\mathrm{E}\,\dfrac{[(A\supset B)\supset A]^2 \qquad \supset\mathrm{I}_1\,\dfrac{Weak\,\dfrac{[A]^1}{A,B}}{A,A\supset B}}{A,A}}{A}}{((A\supset B)\supset A)\supset A}$$

So, although the single-conclusion (SC) and the multiple-conclusion (MC) formulations of classical logic are expressively equivalent, they are evaluated differently by PTS: taking for granted that MC-formulations are acceptable in PTS, **MCNK** is a valid system according to PTS, while there seems to be no valid SC-system for this logic. This situation raises the following question: are MC-systems acceptable in PTS?

2.3.2 The problem with MC-systems

There are several objections to MC-systems in PTS but, for the scope of this work, we will deal only with the issue of circularity between the meanings of comma and disjunction.[26] According to this objection, since (1) the multiple conclusion A, B is equivalent to the single conclusion of disjunctive form $A \vee B$ and (2) we cannot explain what is the meaning of a multiple conclusion without using the disjunction, the meaning of the comma depends on that of the disjunction.[27]

As a consequence, **MCNK** suits Separability (and arguably Harmony) only apparently: if we look at multiple conclusions for what they really are, that is

[26] Another frequent objection against these systems is the rarity of MC-arguments in actual reasoning.

[27] The *locus classicus* for this argument is [Dummett, 1991], p. 187, for a more recent analysis, see [Steinberger, 2011].

disjunctions in disguise, it loses these good properties. Indeed, Peirce's law is provable in this system using only $\supset$-rules *and structural rules*, and we cannot consider innocent the appeal to these rules any more. So it seems that a MC-formulation of classical logic causes more problems than it solves, at least if we are interested in evaluating the dependence of meaning between logical terms.[28]

2.3.3 The problem with MA-systems

It is natural to wonder if the previous argument about circularity proves too much: does it reject multiple-assumption (MA) systems too, based on the circularity between multiple assumptions and conjunction?[29] The argument should be as follows: since (1) the multiple assumption A, B is equivalent to the single assumption of conjunctive form $A \wedge B$ and (2) we cannot explain what is the meaning of a multiple assumption without using the conjunction, the meaning of the comma depends on that of the conjunction.

In order to block this argument, the philosophers who are sceptic about MC-systems but endorse MA-systems reject the second premise; while the single assumption $A \wedge B$ and the multiple assumption A, B have the same meaning, we have a privileged access to A, B that do not pass through conjunction.[30] If this is right, then we can ground our grasp of conjunction on our previous understanding of multiple assumptions, but we do not need to do the opposite, so avoiding circularity. According to those philosophers, multiple assumptions are different from multiple conclusions because we naturally interpret multiple assertions in a conjunctive way. That is, asserting A and asserting B is tantamount to asserting $A \wedge B$, while asserting $A \vee B$ is neither tantamount to asserting A and asserting B, nor to asserting A or asserting B.

I grant that asserting $A \vee B$ is not tantamount to asserting A or asserting B,[31] but I am not so sure that the interpretation of multiple assertions is *naturally* conjunctive. Although it seems natural to interpret multiple assertions in this way, I see nothing more than a habit developed to simplify communication. There seems to be nothing that suggests that the conjunctive reading of multiple assertions is conceptually (and meaning-theoretically) prior to conjunction.[32] For this reason, we should not only reject the adoption of MC-systems but also adopt a single-assumption (SA) formalism. One would think that without multiple assumptions it is impossible to develop any theory of meaning, especially in PTS.[33] I will show that – against this defeatist position – a SASC-formalism is the best framework for investigating the meaning of logical terms.[34]

[28] While it is of course a wonderful formulation for other purposes, like proof search.

[29] We will endorse the generally but not universally accepted position that the commas between the assumptions of an entailment should be read as extensional conjunctions. In [Read, 2003], the author claims that they should instead be interpreted as *fusions*, in order to avoid the paradoxical situation in which a derivation that necessarily preserves the truth could nonetheless be relevantly invalid. Since the rules for fusion cannot be single-assumption, the cost of my framework seems to be at least the rejection of Read's solution to the apparently paradoxical situation of relevant logic. I thank Nissim Francez for stressing this point.

[30] [Dummett, 1991], p. 187.

[31] See p. 348 of [Steinberger, 2011], for an extensive argument.

[32] As far as I know, the most developed argument for the conjunctive nature of multiple assertions is given in [Steinberger, 2011], on pages 340 and 348, with an explicit reference to Gareth Evans. I believe that Evans arguments are purely naturalistic and so irrelevant for a purely meaning-theoretical investigation.

[33] [Steinberger, 2011], p. 347.

[34] As we already stressed in note 28 about the rejection of multiple-conclusion systems,

$$\wedge\mathrm{I}\ \frac{C \qquad \begin{array}{c}[C]\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[C]\\ \vdots\\ B\end{array}}{A \wedge B} \qquad \wedge\mathrm{E}\ \frac{A \wedge B \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array}}{C} \qquad \wedge\mathrm{E}\ \frac{A \wedge B \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}$$

$$\vee\mathrm{E}\ \frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C} \qquad \begin{array}{c}\vee\mathrm{I}\ \dfrac{A}{A \vee B}\\[2ex] \vee\mathrm{I}\ \dfrac{B}{A \vee B}\end{array}$$

$$\supset\mathrm{I}\ \frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \supset B} \qquad \supset\mathrm{E}\ \frac{A \supset B \qquad \begin{array}{c}\emptyset\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}$$

$$\neg\mathrm{I}\ \frac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A} \qquad Efq\ \frac{\bot}{C} \qquad \neg\mathrm{E}\ \frac{\neg A \qquad \begin{array}{c}\emptyset\\ \vdots\\ A\end{array}}{\bot}$$

Table 2: **SASCNJDJ**

3 SASC systems and their properties

3.1 A Pure SASC-system

With our standard requirements of Separability, Atomism, and Purity we cannot justify classical or intuitionistic logic in a SASC-system. However, it could be interesting to develop a (very weak) system that suits all these requirements, so let us consider the system **SASCNJDJ**, displayed in table 2.[35]

Every I-rule is Pure, and so this system suits Atomism. If the discharge of the assumptions is not optional, these rules give a SASC-system since we can have at most one (open) assumption and one conclusion at a time. The reason why we need the discharge to be compulsory is that otherwise we could have multiple assumptions in the following way

the restriction to single assumptions is needed only for the semantic investigation about the meaning of connectives and for defending the analytical status of logical truths. We could even conclude, adapting a statement of Shoesmith and Smiley, that we have been speaking single assumption all our life without knowing it; see [Shoesmith and Smiley, 1978], p. 4. Granted that, there is nonetheless nothing bad in using MASC or even MAMC systems for practical reasons, when we are not investigating the meaning of logical terms. In the same way as an Ardent Physicalist has no reason to ban everyday expressions like “chair” and “table”, or expressions of special sciences like “cell” and “mammal” when he is not describing the most fundamental ingredients of the universe. I thank an anonymous referee for addressing this issue.

[35] The second part of the acronym stands for **N**: natural (deduction) and **JDJ**: intuitionistic and dual-intuitionistic, since this system suits both these logics.

$$\vee\text{E}\ \frac{A \vee B \qquad \vee\text{I}\ \dfrac{A}{A \vee B} \qquad \vee\text{I}\ \dfrac{B}{A \vee B}}{A \vee B}$$

The proof of Separability and Harmony is just routine: we can obtain both properties using a translation from **SASCNJDJ** to an equivalent sequent system and then eliminating the Cut rule.[36] However, we have to address a possible objection. Someone might argue that, although there cannot be multiple *open* assumptions in our derivations, nonetheless there can be multiple (eventually closed) assumptions. Indeed the proof-trees of **SASCNJDJ** have branches on the top, so it seems not to be a real SA-system.

I will argue that the shape of the derivations in **SASCNJDJ** is due only to our habit of constructing proofs starting from the assumption and then deriving the conclusion. That is with our standard notation we can modify the conclusions of a derivation, but we cannot modify the assumptions. What we can at most do is to discharge them. This asymmetry can be shown analysing $\wedge$I and $\vee$E.

In a MC-formulation $\vee$E should be formalised using a downward branching, that is with

$$\vee\text{E}_{MC}\ \frac{A \vee B}{\begin{matrix} A & & B \\ \vdots & & \vdots \\ C & & C \end{matrix}}$$

In a SC-formulation, since we write down the derivations handling essentially the formula on the bottom, we can prevent the downward branching by formalizing $\vee$E in the usual way. With this formulation, we obtain a single conclusion and we also prevent the branching.

On the contrary, with $\wedge$I we cannot handle the open assumptions in order to avoid an upward branching, because the order in which we write the proof is from the assumption down to the conclusion. So what we can at most have is the rule of **SASCNJDJ**, in which all the assumptions apart from one are closed.

Things are reversed, if we construct the derivations starting from the bottom (that is the conclusion) and going upward.[37] Indeed, the system $\mathbf{SASCNJDJ}_T$, displayed in table 3, is equivalent to **SASCNJDJ** but its derivations have downward (instead of upward) branching. You can see that with this formulation all derivations have a single assumption and a single open conclusion, but eventually multiple *closed conclusions*. In particular, $\wedge$I now has only one (open or closed) assumption and $\vee$E has multiple closed conclusions. So neither upward branching is a symptom of endorsing an MA-formulation, nor downward branching is a symptom of endorsing an MC-formulation.

3.2 Weak Separability

While **SASCNJDJ** suits all the requirements we imposed on a logical system, it is very weak. If this were the only valid system according to our SASC-version

[36] Following the example of Gentzen in his seminal work [Gentzen, 1969]. I proved this result in my Ph.D. thesis [Ceragioli, 2020].

[37] [Tranchini, 2012] proposes this as the most natural formulation of dual-intuitionistic logic.

$$\wedge\mathrm{I}\ \dfrac{C}{A \wedge B \qquad \begin{array}{c} C \\ \vdots \\ [A] \end{array} \qquad \begin{array}{c} C \\ \vdots \\ [B] \end{array}} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B}{A} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B}{B}$$

$$\vee\mathrm{I}\ \dfrac{C}{A \vee B \qquad \begin{array}{c} C \\ \vdots \\ [A] \end{array}} \qquad \vee\mathrm{I}\ \dfrac{C}{A \vee B \qquad \begin{array}{c} C \\ \vdots \\ [B] \end{array}} \qquad \vee\mathrm{E}\ \dfrac{A \vee B}{C \qquad \begin{array}{c} A \\ \vdots \\ [C] \end{array} \qquad \begin{array}{c} B \\ \vdots \\ [C] \end{array}}$$

$$\supset\mathrm{I}\ \dfrac{\top}{A \supset B \qquad \begin{array}{c} A \\ \vdots \\ [B] \end{array}} \qquad \supset\mathrm{E}\ \dfrac{A \supset B}{\begin{array}{c} \top \\ \vdots \\ [A] \end{array} \qquad B} \qquad \top\mathrm{I}\ \dfrac{C}{\top}$$

Table 3: $\mathbf{SASCNJDJ}_T$

of PTS, it would be a Pyrrhic victory against the sceptics of single-assumption systems. Following a suggestion from Peter Milne, we can justify some more interesting logical systems.[38] Milne points out that, although Atomism is almost universally accepted in PTS, we do not have really strong reasons for endorsing it. As a consequence, we should not reject I-rules just because they are not Pure.[39]

The idea is to substitute Atomism with

Molecularity The dependence relation between the meanings of the logical constants is not circular.

and Purity with

Definition 3.1 (Non-circularity)**.** For every pair of distinct logical constants $\oplus$ and $\ominus$, it does not hold that both $\ominus \prec \oplus$ and $\oplus \prec \ominus$.

Where $\ominus \prec \oplus$ (the meaning of $\oplus$ depends on the meaning of $\ominus$) iff there is a sequence of logical terms $\circ_1, \dots, \circ_n$ such that $\circ_1 = \ominus$, $\circ_n = \oplus$ and for every $1 \leq i < n$, $\circ_i$ occurs in the premisses or in the discharged assumptions of an I-rule for $\circ_{i+1}$.

Since we still endorse Analyticity, we cannot do without some kind of Separability requirement. However, standard Separability is obviously too strong: why should we require that a logical consequence about $\oplus$ is provable using only $\oplus$-rules, given that the meaning of $\oplus$ can depend on that of some other terms like $\ominus$? Indeed Analyticity asks that the logical consequences hold in virtue of the meaning of the logical terms that occur in them and, when $\oplus \prec \ominus$, $\ominus$I is constitutive of the meaning of $\oplus$,[40]. As a consequence, under these circumstances $\ominus$-rules can be used to derive logical consequences regarding $\oplus$ without violating Analyticity. In summary, we should substitute standard Separability with

[38] [Milne, 2002].

[39] Indeed even Dummett pointed out that it was an "exorbitant" demand; see [Dummett, 1991], p. 257.

[40] And $\ominus$E is justified by $\ominus$I.

$$\wedge\mathrm{I}\ \dfrac{C \qquad \begin{matrix}[C]\\ \vdots\\ A\end{matrix} \qquad \begin{matrix}[C]\\ \vdots\\ B\end{matrix}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{C} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$$

$$\vee\mathrm{E}\ \dfrac{(A \vee B)\{\wedge D\} \qquad \begin{matrix}[A\{\wedge D\}]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B\{\wedge D\}]\\ \vdots\\ C\end{matrix}}{C} \qquad \vee\mathrm{I}\ \dfrac{A}{A \vee B} \qquad \vee\mathrm{I}\ \dfrac{B}{A \vee B}$$

$$\supset\mathrm{I}\ \dfrac{\{C\} \qquad \begin{matrix}[A\{\wedge C\}]\\ \vdots\\ B\end{matrix}}{A \supset B} \qquad \supset\mathrm{E}\ \dfrac{(A \supset B)\{\wedge(D \wedge E)\} \qquad \begin{matrix}[\{E\}]\\ \vdots\\ A\end{matrix} \qquad \begin{matrix}[B\{\wedge D\}]\\ \vdots\\ C\end{matrix}}{C}$$

$$\bot\mathrm{E}\ \dfrac{\bot}{C}$$

Table 4: **SASCNJ**

Definition 3.2 (Weak Separability)**.** To prove a logical consequence $\Gamma \vdash C$ we only need to use the rules for the logical constants that occur in Γ or C, together with the rules for the constants on which those depend. That is, in order to prove a logical consequence $\Gamma \vdash C$, it is enough to use the rules for the constants $\circ_1, \dots, \circ_n$ such that for every $1 \leq i \leq n$:

- $\circ_i$ occurs in Γ or C; or
- for some $j \neq i$ such that $1 \leq j \leq n$, $\circ_j$ occurs in Γ or C and $\circ_i \prec \circ_j$.

While we managed easily to adapt Separability to this SASC-framework, it is not so clear how to interpret Harmony in this liberalized approach. Indeed, the standard notion of *maximal formula* relies heavily on the Purity of the I-rules. We will deal with this problem in the concrete examples of intuitionistic and classical logics, and we will see that as soon as we have a clear adaptation of the notion of *maximal formula* in these cases, the proof of the existence of the normal form is just a purely technical issue.

3.2.1 Intuitionistic system

Let us now consider the system **SASCNJ** for intuitionistic logic, displayed in table 4. The curly brackets in '$\{A\}$' mean that an application of the rule is correct whether the formula A occurs in it or not. For example $\supset$I can be used to prove $A \supset B$ from a derivation of B from A, or to return a derivation of $A \supset B$ from C, given a derivation of B from $A \wedge C$.[41]

According to this formulation, the particularity of intuitionistic logic is that it allows the elimination of connectives that occur in a subordinate position with

[41] I take this usage of curly brackets from [Milne, 2002].

respect to conjunction. Looking at the I-rules, we can reconstruct the following dependence clauses: $\wedge$, $\vee$ and $\bot$ do not depend on any connective, while $\supset$ depends on $\wedge$ ($\wedge \prec\supset$). As a consequence, Weak Separability allows the usage of $\wedge$-rules in the derivation of purely implicational logical consequences. Given this, we have

Theorem 3.1 (Weak Separability for **SASCNJ**)**.** *If $A \vdash_{SASCNJ} B$, then we can derive B from A using only rules for the connectives that occur in A or B, together with $\wedge$ if $\supset$ occurs in A or B.*

Proof. Since **SASCNJ**$^{\wedge}$ (the subsystem of **SASCNJ** with only the $\wedge$-rules) can prove all the consequences of **NJ**$^{\wedge}$ and **SASCNJ** is equivalent to **NJ**, **SASCNJ** conservatively extends **SASCNJ**$^{\wedge}$.[42] Analogously, the $\vee$-rules of **SASCNJ** are complete for the purely disjunctive fragment of intuitionistic logic, so **SASCNJ** conservatively extends **SASCNJ**$^{\vee}$.[43] In a similar way we can prove that: **SASCNJ** conservatively extends **SASCNJ**$^{\wedge\vee}$, so neither $\supset$ nor $\bot$ are needed to derive consequences about $\wedge$ and $\vee$; **SASCNJ** conservatively extends **SASCNJ**$^{\wedge\supset}$; **SASCNJ** conservatively extends **SASCNJ**$^{\vee\supset}$; **SASCNJ** conservatively extends **SASCNJ**$^{\wedge\vee\supset}$; **SASCNJ** conservatively extends **SASCNJ**$^{\bot}$; **SASCNJ** conservatively extends **SASCNJ**$^{\wedge\bot}$; **SASCNJ** conservatively extends **SASCNJ**$^{\vee\bot}$; **SASCNJ** conservatively extends **SASCNJ**$^{\supset\bot}$; **SASCNJ** conservatively extends **SASCNJ**$^{\wedge\vee\bot}$; **SASCNJ** conservatively extends **SASCNJ**$^{\wedge\supset\bot}$.[44] $\square$

Let us now consider the problem with Harmony by evaluating the following application of $\supset$E in a derivation:

$$
\supset\mathrm{E}_1\ \dfrac{\begin{array}{c}\mathtt{d}\\ (A\supset B)\wedge(D\wedge E)\end{array}\qquad \begin{array}{c}[E]^1\\ \vdots\\ A\end{array}\qquad \begin{array}{c}[B\wedge D]^1\\ \vdots\\ C\end{array}}{C}
$$

The major premise $(A \supset B)\wedge(D\wedge E)$ cannot be derived directly using $\supset$I, since $\supset$ is not its principal connective. Does this mean that these rules never give rise to maximality? This cannot be right. Indeed, the following pair of rules violate Separability due to their respective disharmonious nature:

$$
tonk\mathrm{I}\ \dfrac{A}{A tonk B}\qquad\qquad tonk\mathrm{E}_1\ \dfrac{(A tonk B)\{\wedge C\}\qquad \begin{array}{c}[B\{\wedge C\}]^1\\ \vdots\\ D\end{array}}{D}
$$

[42] To be precise, since **NJ** allows derivations with more than one open assumption, we need to specify how these are translated using conjunctions of formulae. This imposes a choice about how to associate formulae in the conjunctions: $A, B, C \vdash D$ should be translated with $(A\wedge B)\wedge C \vdash D$ or $A\wedge(B\wedge C) \vdash D$? Technicalities apart, we associate to every **NJ**-derivation all its possible translation with different associativity, so that for every **NJ**-derivation there are several translations in **SASCNJ**.

[43] Even the $\vee$-rules of **SASCNJDJ** are complete for this fragment of intuitionistic logic, so the conjunction in the elimination rule for $\vee$ adopted in **SASCNJ** is needed only to prove consequences in which $\vee$ does occur alongside with $\wedge$, like distribution of conjunction over disjunction.

[44] The complete proof is included in my Ph.D. thesis Ceragioli [2020].

This is evidently displayed by the derivation

$$\dfrac{\dfrac{A \qquad \dfrac{[A]^1}{A tonk B}\,tonk\mathrm{I} \qquad [A]^1}{(A tonk B) \wedge A}\,\wedge\mathrm{I}_1 \qquad [B \wedge A]^2}{B \wedge A}\,tonk\mathrm{E}_2$$

The key idea of Harmony is that, since the meaning of every formula is given completely by I-rules, when the major premise of an elimination rule is derived using an I-rule, the E-rule will just exploit this ground to derive its conclusion. According to this picture, the I-rules give canonical grounds to derive a sentence, while the E-rules exploit the meaning of connectives to give non-canonical grounds for the conclusion. Harmony requires that we can always avoid introducing canonically a sentence only to give a non-canonical ground for one of its premises, by an immediately successive application of an E-rule. This requirement is usually equated with the ask for normalization, while as we already stressed in section 2.1.1 we require only the existence of a normal form for any valid derivation.

This traditional picture is adequate for our single-assumption system too, but we have to acknowledge that some E-rules exploit the meaning of more than one connective, and so we will be able to normalize the derivation only if we have more than one I-rule in the direct derivation of the major premise. As a first approximation, we can claim that the connectives used by an E-rule are those that occur in its schema.[45] In case an E-rule uses the meaning of both $\wedge$ and $\supset$ to derive the conclusion, it is natural to ask that all these connectives are introduced by their respective canonical grounds, in order to have normalizability.[46] So it is not strange to ask for an application of a $\wedge$-rule in order to use $\supset$. In other words, the applications of $\supset$E that we have to normalize have the form:

$$\dfrac{\dfrac{F \qquad \dfrac{[F]^3 \qquad \begin{matrix}[A \wedge F]^1 \\ \vdots \\ B\end{matrix}}{A \supset B}\,\supset\mathrm{I}_1 \qquad \dfrac{[F]^3 \qquad \begin{matrix}[F]^2 \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}[F]^2 \\ \vdots \\ E\end{matrix}}{D \wedge E}\,\wedge\mathrm{I}_2}{(A \supset B) \wedge (D \wedge E)}\,\wedge\mathrm{I}_3 \qquad \begin{matrix}[E]^4 \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[B \wedge D]^4 \\ \vdots \\ C\end{matrix}}{C}\,\supset\mathrm{E}_4$$

The same argument holds for $\vee$E too, and so even

[45] We will see in definition 3.3 a more refined definition of active occurrences of connectives in an *application* of a rule. This will be the key notion to define maximality. However, for heuristic reasons, a rule-centered approach is best suited to give a first sketch of how to generalize maximality.

[46] We could be tempted to say that $\supset$E is an elimination rule both for $\supset$ and $\wedge$, but we must remember our separability results: the meaning of $\supset$E depends on that of $\wedge$, but the opposite does not hold.

$$\vee\mathrm{E}_2\,\dfrac{\wedge\mathrm{I}_1\,\dfrac{E \qquad \vee\mathrm{I}\,\dfrac{\begin{matrix}[E]^1\\ \vdots\\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}[E]^1\\ \vdots\\ C\end{matrix}}{(A\vee B)\wedge C} \qquad \begin{matrix}[A\wedge C]^2\\ \vdots\\ D\end{matrix} \qquad \begin{matrix}[B\wedge C]^2\\ \vdots\\ D\end{matrix}}{D}$$

is not in normal form and need to be normalized.

To be more precise, let us define the notion of *active occurrence of a connective in an inference*:

Definition 3.3 (Active Connective in an Inference)**.** An occurrence of the connective $\otimes$ in a formula A is active in an inference iff the inference is an exemplification of a rule in which, in the formula exemplified by A, $\otimes$ already has the same occurrence.

This notion is a useful tool to evaluate the consequences of allowing the occurrence of more than one connective in the rules of the system. As an example, in the derivation

$$\vee\mathrm{E}_2\,\dfrac{(A\vee B)\wedge(C\wedge D) \qquad \wedge\mathrm{E}_1\,\dfrac{[A\wedge(C\wedge D)]^2 \qquad [C\wedge D]^1}{C\wedge D} \qquad \wedge\mathrm{E}_1\,\dfrac{[B\wedge(C\wedge D)]^2 \qquad [C\wedge D]^1}{C\wedge D}}{C\wedge D}$$

the outermost $\wedge$ in $(A\vee B)\wedge(C\wedge D)$ is active in the instantiation of $\vee$E, since $\wedge$ already occurs in this role in the schema of $\vee$E. On the contrary, the second occurrence of the same connective, which ties together C and D is not active, since conjunction does not occur in this role in the schema of $\vee$E.

With this definition at hand, we can give the following characterization of *maximal formulae* in **SASCNJ**:

Definition 3.4 (Maximal formulae (**SASCNJ**))**.** Given a derivation $\mathfrak{D}$, a formula A that occurs in it is a *maximal formula* iff:

- A is the major premise of an application of a $\oplus$E rule and the last rules applied in its immediate subderivation are I-rules for all the connectives which occur actively in it in that application of $\oplus$E.

When the major premise of an E-rule can have only one active connective, we have the usual definition of *maximal formulae*; in **SASCNJ**, this happens for $\wedge$E. When more than one connective can occur actively in the major premise of an application of an E-rule, we can have the new form of maximality that we observed for $\vee$E and $\supset$E, alongside the usual ones. Since these are the only E-rules of **SASCNJ** that allow for more than one active connective in their major premises, these are all the new kinds of maximal formulae possible in this system. Arguably, this definition of maximality can be generalized for all the systems that allow more than one connective in the major premise of the E-rules.

Now that we have a clear idea of what a normal derivation is in **SASCNJ**, we can state the following theorem.

Theorem 3.2 (Existence of normal form for **SASCNJ**)**.** *If $A \vdash_{SASCNJ} B$, then there is a normal derivation of B from A in **SASCNJ**.*

Axioms

$$A \Rightarrow A$$

Structural rules

$$Weak \Rightarrow \frac{\Rightarrow \ C}{A \ \Rightarrow \ C} \qquad \Rightarrow Weak \frac{C \ \Rightarrow}{C \ \Rightarrow \ A}$$

$$Cut \ \frac{C \ \Rightarrow \ A \qquad \{G\wedge\}A\{\wedge H\} \ \Rightarrow \ D}{\{G\wedge\}C\{\wedge H\} \ \Rightarrow \ D}$$

Operational rules

$$\wedge \Rightarrow \frac{A \ \Rightarrow \ C}{A \wedge B \ \Rightarrow \ C} \qquad \wedge \Rightarrow \frac{B \ \Rightarrow \ C}{A \wedge B \ \Rightarrow \ C}$$

$$\Rightarrow \wedge \ \frac{C \ \Rightarrow \ A \qquad C \ \Rightarrow \ B}{C \ \Rightarrow \ A \wedge B}$$

$$\vee \Rightarrow \frac{A\{\wedge D\} \ \Rightarrow \ C \qquad B\{\wedge D\} \ \Rightarrow \ C}{(A \vee B)\{\wedge D\} \ \Rightarrow \ C}$$

$$\Rightarrow \vee \ \frac{C \ \Rightarrow \ B}{C \ \Rightarrow \ A \vee B} \qquad \Rightarrow \vee \ \frac{C \ \Rightarrow \ A}{C \ \Rightarrow \ A \vee B}$$

$$\supset\Rightarrow \frac{\{E\} \ \Rightarrow \ A \qquad B\{\wedge D\} \ \Rightarrow \ C}{(A \supset B)\{\wedge(D \wedge E)\} \ \Rightarrow \ C} \qquad \Rightarrow\supset \frac{A\{\wedge C\} \ \Rightarrow \ B}{\{C\} \ \Rightarrow \ A \supset B}$$

$$\Rightarrow \perp \ \frac{A \ \Rightarrow \ \perp}{A \ \Rightarrow}$$

Table 5: **SASCLJ**

In order to prove this theorem, we need some definitions and lemmas. We also need to introduce the single-antecedent and single-succedent sequent calculus system **SASCLJ** for intuitionistic logic, displayed in table 5.[47]

Definition 3.5 ($As \Rightarrow$, $Idem \Rightarrow$ and $Comm \Rightarrow$)**.** Associativity, commutativity and idempotence are derivable properties of conjunction in **SASCLJ**. We abbreviate their derivation in the following way:

$As \Rightarrow_1$ stands for

$$
{\scriptstyle Cut}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{A \Rightarrow A}{A\wedge B \Rightarrow A}}{(A\wedge B)\wedge C \Rightarrow A} \qquad {\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{B \Rightarrow B}{A\wedge B \Rightarrow B}}{(A\wedge B)\wedge C \Rightarrow B} \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{C \Rightarrow C}{(A\wedge B)\wedge C \Rightarrow C}}{(A\wedge B)\wedge C \Rightarrow B\wedge C}}{(A\wedge B)\wedge C \Rightarrow A\wedge(B\wedge C)} \qquad (E\wedge(A\wedge(B\wedge C)))\wedge F \Rightarrow D}{(E\wedge(A\wedge B)\wedge C)\wedge F \Rightarrow D}
$$

$As \Rightarrow_2$ stands for

$$
{\scriptstyle Cut}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{A \Rightarrow A}{A\wedge(B\wedge C) \Rightarrow A} \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{B \Rightarrow B}{B\wedge C \Rightarrow B}}{A\wedge(B\wedge C) \Rightarrow B}}{A\wedge(B\wedge C) \Rightarrow A\wedge B} \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{C \Rightarrow C}{B\wedge C \Rightarrow C}}{A\wedge(B\wedge C) \Rightarrow C}}{A\wedge(B\wedge C) \Rightarrow (A\wedge B)\wedge C} \qquad (E\wedge((A\wedge B)\wedge C))\wedge F \Rightarrow D}{(E\wedge(A\wedge(B\wedge C)))\wedge F \Rightarrow D}
$$

$Idem \Rightarrow$ stands for

$$
{\scriptstyle Cut}\,\dfrac{{\Rightarrow\wedge}\,\dfrac{A \Rightarrow A \qquad A \Rightarrow A}{A \Rightarrow A\wedge A} \qquad C\wedge(A\wedge A) \Rightarrow D}{C\wedge A \Rightarrow D}
$$

$Comm \Rightarrow$ stands for

$$
{\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{A \Rightarrow A}{B\wedge A \Rightarrow A} \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{B \Rightarrow B}{B\wedge A \Rightarrow A}}{B\wedge A \Rightarrow A\wedge B} \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{C \Rightarrow C}{(B\wedge A)\wedge C \Rightarrow C}}{(B\wedge A)\wedge C \Rightarrow (A\wedge B)\wedge C}}{D\wedge((B\wedge A)\wedge C) \Rightarrow (A\wedge B)\wedge C}
$$

$$
\vdots
$$

$$
{\scriptstyle Cut}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{D \Rightarrow D}{D\wedge((B\wedge A)\wedge C) \Rightarrow (B\wedge A)\wedge C} \qquad \begin{matrix}\vdots \\ D\wedge((B\wedge A)\wedge C) \Rightarrow (A\wedge B)\wedge C\end{matrix}}{D\wedge((B\wedge A)\wedge C) \Rightarrow D\wedge((A\wedge B)\wedge C)} \qquad D\wedge((A\wedge B)\wedge C) \Rightarrow E}{D\wedge((B\wedge A)\wedge C) \Rightarrow E}
$$

Definition 3.6 (Semi-Cut-free form)**.** A **SASCLJ**-derivation is in semi-Cut-free form iff all its applications of Cut occur in exemplifications of $As \Rightarrow$, $Idem \Rightarrow$ or $Comm \Rightarrow$.

Lemma 3.1 (Existence of semi-Cut-free form for **SASCLJ**)**.** *If* $\vdash_{SASCLJ} A \Rightarrow B$*, then there is a semi-Cut-free derivation of* $A \Rightarrow B$ *in* ***SASCLJ****.*

[47] This system is just a useful technical device, devoid of real philosophical interest. The curly brackets have the same usage that we already saw for natural deduction. Following Gentzen, the '**L**' in the acronym specifies that it is a sequent calculus.

Proof. **LJ** formulated without Cut is complete for intuitionistic logic. We can translate every Cut-free **LJ**-derivation with semi-Cut-free **SASCLJ**-derivations in the following way:[48]

- The axiom is obvious;
- The applications of $\Rightarrow \otimes$ and $\Rightarrow Weakening$ are translated by an application of the corresponding rule of **SASCLJ**;
- An application of $\otimes \Rightarrow$ is translated by an application of the corresponding rule of **SASCLJ**, eventually together with several applications of $As \Rightarrow$;
- An application of $Weakening \Rightarrow$ is translated by an application of the corresponding rule of **SASCLJ** when the antecedent of the premise is void, and by $\wedge \Rightarrow$ otherwise, eventually together with several applications of $As \Rightarrow$;
- An application of $Contraction \Rightarrow$ is translated by an application of $Idem \Rightarrow$, eventually together with several applications of $As \Rightarrow$;
- An application of $Permutation \Rightarrow$ is translated by an application of $Comm \Rightarrow$, eventually together with several applications of $As \Rightarrow$;

Since in this translation, Cut occurs only in the sub-derivations $As \Rightarrow$, $Idem \Rightarrow$ and $Comm \Rightarrow$, we obtain a semi-Cut-free derivation. ☐

Lemma 3.2 (Translation from **SASCLJ** to **SASCNJ**)**.** *We can translate every **SASCLJ**-derivation of $A \Rightarrow B$ with an **SASCNJ**-derivation of B from A. In the translation, we need to resort to derived major premises of E-rules only for applications of the rule of Cut. In all other cases, major premises are open or closed assumptions.*

Proof. The translation goes as follows:[49]

- The axiom is translated by an open assumption;
- An application of $\Rightarrow \otimes$ is translated by an application of $\otimes$I;
- An application of $\otimes \Rightarrow$ is translated by an application of $\otimes$E in which the major premise is an open assumption,[50] e.g.:

$$\supset\Rightarrow \dfrac{E \Rightarrow A \qquad B \wedge D \Rightarrow C}{(A \supset B) \wedge (D \wedge E) \Rightarrow C} \quad \rightsquigarrow \quad \supset\mathrm{E}_{1,2} \dfrac{(A \supset B) \wedge (D \wedge E) \qquad \begin{matrix}[E]^1 \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[B \wedge D]^2 \\ \vdots \\ C\end{matrix}}{C}$$

[48] Since in **LJ** antecedents can contain more than one formula, we need to specify how multiple-antecedents are translated using conjunctions of formulae. We already saw this problem in note 42, and here we will apply the same solution: we associate to every **LJ**-sequent all its possible translation with different associativity, so that for every **LJ**-derivation there are several translations in **SASCLJ**. This is the reason for the eventual several applications of $As \Rightarrow$ used in the translation.

[49] Keep in mind that the curly brackets have the same meaning both in natural deduction and sequent calculus.

[50] Subsequent applications of rules could close it, but only translations of Cut could cause a recombination of the proof that makes it the conclusion of a derivation.

- The applications of $\Rightarrow \bot$ and $Weak \Rightarrow$ are translated without modifying the **SASCNJ**-derivation;[51]
- An application of $\Rightarrow Weak$ is translated by an application of *Efq*;
- An application of Cut

$$Cut\ \dfrac{\begin{array}{c}\vdots \mathtt{d}_1\\ C \ \Rightarrow\ A\end{array}\qquad \begin{array}{c}\vdots \mathtt{d}_1\\ (F \wedge A) \wedge G \ \Rightarrow\ H\end{array}}{(F \wedge C) \wedge G \ \Rightarrow\ H}$$

is translated by

$$\begin{array}{c}\wedge I_2\ \dfrac{(F \wedge C) \wedge G \qquad \wedge I_1\ \dfrac{[(F \wedge C) \wedge G]^2 \qquad \wedge E\ \dfrac{[(F \wedge C) \wedge G]^1}{F} \qquad \begin{array}{c}\wedge E\ \dfrac{[(F \wedge C) \wedge G]^1}{C}\\ \vdots \mathtt{d}_1^*\\ A\end{array}}{F \wedge A} \qquad \wedge E\ \dfrac{[(F \wedge C) \wedge G]^2}{G}}{(F \wedge A) \wedge G}\\ \vdots \mathtt{d}_2^*\\ H\end{array}$$

Even though these do not correspond to primitive rules of **SASCLJ**, Associativity, Commutativity and Idempotence of conjunction have a key role in the proof of theorem 3.2. So let us see how they are translated in **SASCNJ**.

$$\begin{array}{c}\dfrac{\alpha \qquad \dfrac{[\alpha]^4 \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^3}{E} \qquad \dfrac{[\alpha]^3 \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^2}{A} \qquad \wedge\mathrm{I}_1\ \dfrac{[\alpha]^2 \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^1}{B} \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^1}{C}}{B \wedge C}}{A \wedge (B \wedge C)}\ \wedge\mathrm{I}_2}{E \wedge (A \wedge (B \wedge C))}\ \wedge\mathrm{I}_3 \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^4}{F}}{\beta}\ \wedge\mathrm{I}_4\\ \vdots\\ D\end{array}$$

$$\begin{array}{c}\dfrac{\beta \qquad \dfrac{[\beta]^4 \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^3}{E} \qquad \dfrac{[\beta]^3 \qquad \wedge\mathrm{I}_1\ \dfrac{[\beta]^2 \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^1}{A} \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^1}{B}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^2}{C}}{(A \wedge B) \wedge C}\ \wedge\mathrm{I}_2}{E \wedge ((A \wedge B) \wedge C)}\ \wedge\mathrm{I}_3 \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^4}{F}}{\alpha}\ \wedge\mathrm{I}_4\\ \vdots\\ D\end{array}$$

Where $\alpha = (E \wedge ((A \wedge B) \wedge C)) \wedge F$ and $\beta = (E \wedge (A \wedge (B \wedge C))) \wedge F$. We will call the first derivation $As_1\wedge$ and the second $As_2\wedge$.

[51] The first, because we can apply *Efq* directly to $\bot$; the second because the kind of monotonicity that it exemplifies is metatheoretical in natural deduction.

$$
\dfrac{\gamma \qquad \wedge\mathrm{E}\,\dfrac{[\gamma]^3}{C} \qquad \dfrac{[\gamma]^3 \qquad \wedge\mathrm{I}_1\,\dfrac{[\gamma]^2 \qquad \wedge\mathrm{E}\,\dfrac{[\gamma]^1}{A} \qquad \wedge\mathrm{E}\,\dfrac{[\gamma]^1}{B}}{A \wedge B} \qquad \wedge\mathrm{E}\,\dfrac{[\gamma]^2}{D}}{(A \wedge B) \wedge D}\,\wedge\mathrm{I}_2}{\delta}\,\wedge\mathrm{I}_3
$$

$$
\begin{array}{c} \delta \\ \vdots \\ D \end{array}
$$

Where $\gamma = C \wedge ((B \wedge A) \wedge D)$ and $\delta = C \wedge ((A \wedge B) \wedge D)$. We call this derivation $Comm\wedge$.

$$
\wedge\mathrm{I}_2\,\dfrac{A \wedge B \qquad \wedge\mathrm{E}\,\dfrac{[A \wedge B]^2}{A} \qquad \wedge\mathrm{I}_1\,\dfrac{\wedge\mathrm{E}\,\dfrac{[A \wedge B]^2}{B} \qquad [A \wedge B]^1 \qquad [A \wedge B]^1}{B \wedge B}}{A \wedge (B \wedge B)}
$$

$$
\begin{array}{c} A \wedge (B \wedge B) \\ \vdots \\ D \end{array}
$$

We call this derivation $Idem\wedge$.

□

We can now prove theorem 3.2.[52]

Proof. If $A \vdash_{SASCNJ} B$ then, since **SASCNJ** and **SASCLJ** are equivalent each other, $\vdash_{SASCLJ} A \Rightarrow B$. Given lemma 3.1 we know that there is a semi-Cut-free derivation `d` for $A \Rightarrow B$. As a consequence, the only non-assumed major premises in the derivation $\mathtt{d}^*$, obtained from `d` by the translation displayed in lemma 3.2, are due to $Idem \Rightarrow$, $Comm \Rightarrow$ and $As \Rightarrow$, that are the only Cut in the **SASCLJ**-derivation. We call the **SASCNJ**-sub-derivations obtained by translating these **SASCLJ**-sub-derivations $Con\wedge$, $Per\wedge$ and $As\wedge$.

To end the proof, we will show that:

1. the only situation in which this translation gives rise to maximal formulae is when the end-formula of a chain of $Con\wedge$, $Per\wedge$ and $As\wedge$ is the major premise of an application of $\wedge$E;

2. in this case we can normalize the derivation

Let us start with the first point. $Con\wedge$, $Per\wedge$ and $As\wedge$ allow to compose deductions on the bottom, that is the bottom-formula of $X\wedge$ (that is obviously derived) can be the major premise of an E-rule.[53] Of course the open assumption of $X\wedge$ can itself be derived by $X\wedge$, thanks to composition. So we have to deal with chains of $X\wedge$ that derive major premises of E-rules.

It follows from definition 3.4 that maximal formulae can occur only when the end-formula of the chain of $X\wedge$ is the major premise of $\wedge E$. Indeed, each $X\wedge$ derive its conclusion with $\wedge I$ as last rule and applies only $\wedge$-rules. It is important also to notice that there are no maximal formulae in $X\wedge$, and that

[52]The complete proof is included in my Ph.D. thesis Ceragioli [2020].
[53]I will use $X\wedge$ to indicate each of $Con\wedge$, $Per\wedge$ and $As\wedge$.

their open assumption is always a major premise of $\wedge I$. So there are no maximal formulae inside the chains of $X\wedge$ (neither in a single $X\wedge$, nor in the assumption of an $X\wedge$ derived by the preceding $X\wedge$), only the end-formula of the entire chain can give rise to a maximal formula with the next rule.

Now, let us prove that we can normalize these non-normal derivations that we eventually obtain by this translation. Let us consider the chain that derives a maximal formula, and in particular its last $X\wedge$. Let us now see a procedure that removes completely the maximal formula or removes the last $X\wedge$, so that we can obtain the normalization by induction on the length of the chain. The proof is by cases on the last $X\wedge$. We will see the case in which $As_1\wedge$ is the last rule of the chain. The other cases are similar.

$As_1\wedge$

$$\dfrac{\begin{matrix}\dfrac{\begin{matrix}\vdots_{X\wedge}\\ \alpha\end{matrix} \quad \dfrac{[\alpha]^4 \quad \wedge\mathrm{E}\dfrac{[\alpha]^3}{E} \quad \dfrac{[\alpha]^3 \quad \wedge\mathrm{E}\dfrac{[\alpha]^2}{A} \quad \wedge\mathrm{I}_1\dfrac{[\alpha]^2 \quad \wedge\mathrm{E}\dfrac{[\alpha]^1}{B} \quad \wedge\mathrm{E}\dfrac{[\alpha]^1}{C}}{B\wedge C}}{A\wedge(B\wedge C)}\wedge\mathrm{I}_2}{E\wedge(A\wedge(B\wedge C))}\wedge\mathrm{I}_3 \quad \wedge\mathrm{E}\dfrac{[\alpha]^4}{F}}{\beta}\wedge\mathrm{I}_4 \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}[\gamma]^5\\ \vdots\\ G\end{matrix}}{G}\wedge\mathrm{E}_5$$

Where $\alpha = (E\wedge((A\wedge B)\wedge C))\wedge F$ and $\beta = (E\wedge(A\wedge(B\wedge C)))\wedge F$.

If $\gamma = F$, then we reduce the length of the chain by: $\begin{matrix}\wedge\mathrm{E}\dfrac{\begin{matrix}\vdots_{X\wedge}\\ (E\wedge((A\wedge B)\wedge C))\wedge F\end{matrix}}{F}\\ \vdots\\ G\end{matrix}$

If $\gamma = E\wedge((A\wedge B)\wedge C)$, then we reduce the length of the chain by:

$$\begin{matrix}\dfrac{\begin{matrix}\vdots_{X\wedge}\\ \alpha\end{matrix} \quad \wedge\mathrm{E}\dfrac{[\alpha]^3}{E} \quad \dfrac{[\alpha]^3 \quad \wedge\mathrm{E}\dfrac{[\alpha]^2}{A} \quad \wedge\mathrm{I}_1\dfrac{[\alpha]^2 \quad \wedge\mathrm{E}\dfrac{[\alpha]^1}{B} \quad \wedge\mathrm{E}\dfrac{[\alpha]^1}{C}}{B\wedge C}}{A\wedge(B\wedge C)}\wedge\mathrm{I}_2}{E\wedge(A\wedge(B\wedge C))}\wedge\mathrm{I}_3\\ \vdots\\ G\end{matrix}$$

If $E\wedge(A\wedge(B\wedge C))$ is maximal, then it is the major premise of an $\wedge E$, let us call δ its discharged assumption.

If $\delta = E$, then we reduce to $\begin{matrix}\wedge\mathrm{E}\dfrac{\begin{matrix}\vdots_{X\wedge}\\ (E\wedge((A\wedge B)\wedge C))\wedge F\end{matrix}}{E}\\ \vdots\\ G\end{matrix}$

If $\delta = A \wedge (B \wedge C)$, then we reduce to

$$\dfrac{\begin{array}{c}\vdots_{X\wedge}\\ \alpha\end{array} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^2}{A} \qquad \wedge\mathrm{I}_1\,\dfrac{[\alpha]^2 \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{B} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{C}}{B \wedge C}}{\begin{array}{c}A \wedge (B \wedge C)\\ \vdots\\ G\end{array}}\,\wedge\mathrm{I}_2$$

If $A \wedge (B \wedge C)$ is maximal, then it is the major premise of an application of $\wedge E$. Let us call ε its discharged assumption.

If $\varepsilon = A$, then we reduce to $$\begin{array}{c}\vdots_{X\wedge}\\ \wedge\mathrm{E}\,\dfrac{(E \wedge ((A \wedge B) \wedge C)) \wedge F}{A}\\ \vdots\\ G\end{array}$$

If $\varepsilon = B \wedge C$, then we reduce to

$$\wedge\mathrm{I}_1\,\dfrac{\begin{array}{c}\vdots_{X\wedge}\\ \alpha\end{array} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{B} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{C}}{\begin{array}{c}B \wedge C\\ \vdots\\ G\end{array}}$$

If $B \wedge C$ is maximal, then it is major premise of $\wedge E$, and its discharged premise can only be B or C. We then reduce to

$$\begin{array}{c}\vdots_{X\wedge}\\ \wedge\mathrm{E}\,\dfrac{(E \wedge ((A \wedge B) \wedge C)) \wedge F}{B}\\ \vdots\\ G\end{array} \quad \text{or} \quad \begin{array}{c}\vdots_{X\wedge}\\ \wedge\mathrm{E}\,\dfrac{(E \wedge ((A \wedge B) \wedge C)) \wedge F}{C}\\ \vdots\\ G\end{array}$$

In both cases we have reduced the length of the chain of $X\wedge$ or normalized the derivation if there is no chain.

$\square$

As a consequence of theorem 3.1, **SASCNJ** suits Harmony.

$$
\wedge\mathrm{I}\ \frac{C \qquad \begin{array}{c}[C]\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[C]\\ \vdots\\ B\end{array}}{A \wedge B}
\qquad
\wedge\mathrm{E}\ \frac{A \wedge B \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array}}{C}
\qquad
\wedge\mathrm{E}\ \frac{A \wedge B \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}
$$

$$
\vee\mathrm{E}\ \frac{(A \vee B)\{\wedge D\} \qquad \begin{array}{c}[A\{\wedge D\}]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B\{\wedge D\}]\\ \vdots\\ C\end{array}}{C}
\qquad
\begin{array}{c}
\vee\mathrm{I}\ \dfrac{A}{A \vee B}\\[2ex]
\vee\mathrm{I}\ \dfrac{B}{A \vee B}
\end{array}
$$

$$
\supset\mathrm{I}\ \frac{\{C\} \qquad \begin{array}{c}[A\{\wedge C\}]\\ \vdots\\ B\{\vee D\}\end{array}}{(A \supset B)\{\vee D\}}
\qquad
\supset\mathrm{E}\ \frac{(A \supset B)\{\wedge(D \wedge E)\} \qquad \begin{array}{c}[\{E\}]\\ \vdots\\ A\{\vee F\}\end{array} \qquad \begin{array}{c}[B\{\wedge D\}]\\ \vdots\\ C\end{array}}{C\{\vee F\}}
$$

$$
\bot\mathrm{E}\ \frac{\bot}{C}
$$

Table 6: **SASCNK**

3.2.2 Classical system

Let us now move on to classical logic and consider the system **SASCNK**, displayed in table 6. Classical logic allows both the introduction of connectives in a subordinate position with respect to disjunction,[54] and the elimination of connectives that occur in a subordinate position with respect to conjunction. The following dependence clauses hold between the meaning of classical connectives: $\wedge$, $\vee$ and $\bot$ do not depend on any connective, while $\supset$ depends on both $\wedge$ and $\vee$ ($\wedge \prec\supset$ and $\vee \prec\supset$). Given this, we have

Theorem 3.3 (Weak Separability for **SASCNK**)**.** *If $A \vdash_{SASCNK} B$, then we can derive B from A using only rules for the connectives that occur in A or B, together with $\wedge$ and $\vee$ if $\supset$ occurs in A or B.*

Proof. We can prove this theorem using conservative extension, as we did for theorem 3.1. The only difference is that in this case $\vee$-rules can extend $\mathbf{SASCNK}^{\wedge\supset}$ *non*-conservatively.[55] □

Let us see the consequences of allowing the introduction of connectives in a subordinate position for the identification of *maximal formulae*. This point is controversial and very important since the possibility of proving Harmony depends on it. Steinberger[56] observes that in the derivation

[54] This was the main point of [Milne, 2002].

[55] The complete proof is included in my Ph.D. thesis Ceragioli [2020].

[56] This worry is extensively exposed in [Steinberger, 2008], and remarked in [Steinberger, 2011], p. 345. It is directly used against Milne's system, but it can be used against my reformulation too.

$$
\vee\mathrm{E}_2\,\dfrac{\supset\mathrm{I}_1\,\dfrac{C \qquad \begin{array}{c}[A\wedge C]^1\\ \vdots\\ B\vee D\end{array}}{(A\supset B)\vee D} \qquad \begin{array}{c}[A\supset B]^2\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[D]^2\\ \vdots\\ E\end{array}}{E}
$$

the major premise $(A \supset B) \vee D$ of $\vee$E is derived by an I-rule, so it is derived canonically. Steinberger argues that for this reason, it should be possible to normalize this derivation, while this cannot be done in general.

We already saw that the main intuition about canonicity that is behind Harmony has to be adapted for SASC-systems in order to be applicable in this framework. Given Steinberger's observation, our classical system can suit Harmony only if the previous derivation does not display a genuine violation of normality, that is if it does not contain a *maximal formula*. This position is made plausible by the fact that $\supset$I does not influence the meaning of $\vee$, since $\mathbf{SASCNK}^{\vee\supset}$ is a conservative extension of $\mathbf{SASCNK}^{\vee}$. In other words, $\supset$I is not another I-rule for $\vee$. So, we do not have a real *maximal formula* in the derivation under analysis, because the meaning of $\vee$ is not introduced canonically and then immediately removed.

Steinberger's counterexample is just obtained applying the standard definition of *maximal formula* to Milne's system without questioning its suitability. On the contrary, in order to obtain a *maximal formula*, we have to use completely the meaning of $\supset$, that is we have to apply its E-rule. Comprehensibly, since the meaning of $\supset$ depends on that of $\vee$ (and $\wedge$), eventually we need to use $\vee$E as a precondition to apply $\supset$E. As a consequence, following a suggestion of Milne, we maintain that maximality arises only in the following sub-case of the derivation proposed by Steinberger:[57]

$$
\vee\mathrm{E}_3\,\dfrac{\supset\mathrm{I}_1\,\dfrac{C \qquad \begin{array}{c}[A\wedge C]^1\\ \vdots\\ B\vee D\end{array}}{(A\supset B)\vee D} \qquad \begin{array}{c}\supset\mathrm{E}_2\,\dfrac{[A\supset B]^3 \qquad \begin{array}{c}\emptyset\\ \vdots\\ A\vee F\end{array} \qquad \begin{array}{c}[B]^2\\ \vdots\\ G\end{array}}{G\vee F}\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[D]^3\\ \vdots\\ E\end{array}}{E}
$$

It follows from theorem 3.4 that in this case both applications of the $\supset$-rules can be removed (possibly without affecting the applications of the $\vee$-rules), so avoiding an unnecessary detour.

In order to argue further that this choice of maximality is not *ad hoc*, let us consider the following generalization of *tonk*:

$$
tonk\mathrm{I}\,\dfrac{A\{\vee C\}}{(A tonk B)\{\vee C\}} \qquad tonk\mathrm{E}_1\,\dfrac{(A tonk B)\{\wedge C\} \qquad \begin{array}{c}[B\{\wedge C\}]^1\\ \vdots\\ D\end{array}}{D}
$$

[57][Milne, 2002], pp. 518-519.

This pair of rules is unacceptable, since the following derivation cannot be regarded as valid:

$$\vee\mathrm{E}_2\,\dfrac{tonk\mathrm{I}\,\dfrac{A \vee B}{(AtonkB) \vee B} \qquad tonk\mathrm{E}_1\,\dfrac{[AtonkB]^2 \qquad [B]^1}{B} \qquad [B]^2}{B}$$

According to our analysis, the diagnosis is clear: *tonk*I and *tonk*E do not suit Harmony, since $(AtonkB) \vee B$ is a *maximal formula* that cannot be removed.

We can sum up our previous considerations in a general characterization of maximal formulae for **SASCNK**, but in order to do so, we need the following definition:

Definition 3.7 (Elimination Path (E-path))**.** Given a derivation $\mathfrak{D}$, a list of sentences $A_1, \ldots, A_n$ is an E-path iff for every m such that $1 \leq m \leq n$, A_m is the major premise of an E-rule and A_{m+1} is one of the discharged assumption of that rule.

As an example, in the previous derivation $(AtonkB) \vee B, AtonkB$ is an E-path. With these tools at hand, we can state the general definition of *maximal formulae* for **SASCNK**:

Definition 3.8 (Maximal formulae (**SASCNK**))**.** Given a derivation $\mathfrak{D}$, a formula that occurs in it is a *maximal formula* iff:

1. A is the major premise of an application of a $\oplus$E rule and the last rules applied in its immediate subderivation are I-rules for all the connectives which occur actively in it in that application of $\oplus$E; or

2. it is the conclusion of an application of $\oplus$I and the first formula of an E-path such that:

   (a) the last rule of the E-path is $\oplus$E;

   (b) each rule in the E-path eliminates occurrences of connectives that are active in the conclusion of the application of $\oplus$I.

Point 1 of the definition captures the general phenomenon behind the extension of the notion of *maximal formula* in **SASCNJ**: since more than one connective can be active in the major premise of an E-rule, all of them should be introduced by an I-rule in order to have maximality. Point 2 captures the generalization to the notion of maximality unlocked by the occurrence of more connectives in the conclusions of the I-rules of **SASCNK**. Arguably, this definition of *maximal formulae* is well suited for every impure system, since the main point of this generalization - at least regarding maximality - is the possibility of working with more connectives in the conclusion of an I-rule and in the major premise of an E-rule.

With this definition, we can state the theorem:

Theorem 3.4 (Existence of normal form for **SASCNK**)**.** *If* $A \vdash_{SASCNK} B$*, then there is a normal derivation of* B *from* A *in* ***SASCNK****.*

Axioms

$$A \Rightarrow A$$

Structural rules

$$Weak \Rightarrow \frac{\Rightarrow\ C}{A\ \Rightarrow\ C} \qquad \Rightarrow Weak\ \frac{C\ \Rightarrow}{C\ \Rightarrow\ A}$$

$$Cut\ \frac{C\ \Rightarrow\ \{E\vee\}A\{\vee F\} \qquad \{G\wedge\}A\{\wedge H\}\ \Rightarrow\ D}{\{G\wedge\}C\{\wedge H\}\ \Rightarrow\ \{E\vee\}D\{\vee F\}}$$

Operational rules

$$\wedge\Rightarrow \frac{A\ \Rightarrow\ C}{A\wedge B\ \Rightarrow\ C} \qquad \wedge\Rightarrow \frac{B\ \Rightarrow\ C}{A\wedge B\ \Rightarrow\ C}$$

$$\Rightarrow\wedge\ \frac{C\ \Rightarrow\ A \qquad C\ \Rightarrow\ B}{C\ \Rightarrow\ A\wedge B}$$

$$\vee\Rightarrow \frac{A\{\wedge D\}\ \Rightarrow\ C \qquad B\{\wedge D\}\ \Rightarrow\ C}{(A\vee B)\{\wedge D\}\ \Rightarrow\ C}$$

$$\Rightarrow\vee\ \frac{C\ \Rightarrow\ B}{C\ \Rightarrow\ A\vee B} \qquad \Rightarrow\vee\ \frac{C\ \Rightarrow\ A}{C\ \Rightarrow\ A\vee B}$$

$$\supset\Rightarrow \frac{\{E\}\ \Rightarrow\ A\{\vee F\} \qquad B\{\wedge D\}\ \Rightarrow\ C}{(A\supset B)\{\wedge(D\wedge E)\}\ \Rightarrow\ C\{\vee F\}} \qquad \Rightarrow\supset \frac{A\{\wedge C\}\ \Rightarrow\ B\{\vee F\}}{\{C\}\ \Rightarrow\ (A\supset B)\{\vee F\}}$$

$$\Rightarrow\bot\ \frac{A\ \Rightarrow\ \bot}{A\ \Rightarrow}$$

Table 7: **SASCLK**

Like for theorem 3.1, in order to prove this theorem we need some definitions and lemmas. We also need to introduce the single-antecedent and single-succedent sequent calculus system **SASCLK** for classical logic, displayed in table 7.

Definition 3.9 ($\Rightarrow As$, $\Rightarrow Idem$, $\Rightarrow Comm$ and $\Rightarrow Distri^{\vee\wedge}$)**.** Associativity, commutativity and idempotence are derivable properties of disjunction in **SASCLK**. In the same way, distributivity of conjunction over disjunction is a derivable property. We abbreviate their derivation in the following way:

$\Rightarrow As_1$ stands for

$$
{\scriptstyle Cut}\,\dfrac{A \Rightarrow (E \vee ((B \vee C) \vee D)) \vee F \qquad {\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{B \Rightarrow B}{B \Rightarrow B \vee (C \vee D)} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{B \Rightarrow B \vee (C \vee D)}}{B \vee C \Rightarrow B \vee (C \vee D)} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{D \Rightarrow D}{D \Rightarrow B \vee (C \vee D)}}{(B \vee C) \vee D \Rightarrow B \vee (C \vee D)}}{A \Rightarrow (E \vee (B \vee (C \vee D))) \vee F}
$$

$\Rightarrow As_2$ stands for

$$
{\scriptstyle Cut}\,\dfrac{A \Rightarrow (E \vee (B \vee (C \vee D))) \vee F \qquad {\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{B \Rightarrow B}{B \Rightarrow (B \vee C) \vee D} \qquad {\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{B \Rightarrow (B \vee C) \vee D} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{D \Rightarrow D}{D \Rightarrow (B \vee C) \vee D}}{C \vee D \Rightarrow (B \vee C) \vee D}}{B \vee (C \vee D) \Rightarrow (B \vee C) \vee D}}{A \Rightarrow (E \vee ((B \vee C) \vee D)) \vee F}
$$

$\Rightarrow Idem$ stands for

$$
{\scriptstyle Cut}\,\dfrac{C \Rightarrow (A \vee A) \vee D \qquad {\scriptstyle \vee\Rightarrow}\,\dfrac{A \Rightarrow A \qquad A \Rightarrow A}{A \vee A \Rightarrow A}}{C \Rightarrow A \vee D}
$$

$\Rightarrow Comm$ stands for

$$
{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{D \Rightarrow D}{D \Rightarrow D \vee ((B \vee A) \vee C)} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{A \Rightarrow A}{A \Rightarrow B \vee A} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{B \Rightarrow B}{B \Rightarrow B \vee A}}{A \vee B \Rightarrow B \vee A}}{A \vee B \Rightarrow (B \vee A) \vee C} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{C \Rightarrow (B \vee A) \vee C}}{(A \vee B) \vee C \Rightarrow (B \vee A) \vee C}}{(A \vee B) \vee C \Rightarrow D \vee ((B \vee A) \vee C)}}{D \vee ((A \vee B) \vee C) \Rightarrow D \vee ((B \vee A) \vee C)}
$$

$$\vdots$$

$$
{\scriptstyle Cut}\,\dfrac{E \Rightarrow D \vee ((A \vee B) \vee C) \qquad \begin{matrix}\vdots \\ D \vee ((A \vee B) \vee C) \Rightarrow D \vee ((B \vee A) \vee C)\end{matrix}}{E \Rightarrow D \vee ((B \vee A) \vee C)}
$$

$\Rightarrow Distri^{\vee\wedge}$ stands for

$$
{\scriptstyle \vee\Rightarrow}\dfrac{{\scriptstyle \wedge\Rightarrow}\dfrac{{\scriptstyle \Rightarrow\vee}\dfrac{C \;\Rightarrow\; C}{C \;\Rightarrow\; (A\wedge B)\vee C}}{A\wedge C \;\Rightarrow\; (A\wedge B)\vee C} \qquad {\scriptstyle \wedge\Rightarrow}\dfrac{{\scriptstyle \Rightarrow\vee}\dfrac{C \;\Rightarrow\; C}{C \;\Rightarrow\; (A\wedge B)\vee C}}{C\wedge C \;\Rightarrow\; (A\wedge B)\vee C}}{{\scriptstyle Comm\Rightarrow}\dfrac{(A\vee C)\wedge C \;\Rightarrow\; (A\wedge B)\vee C}{C\wedge (A\vee C) \;\Rightarrow\; (A\wedge B)\vee C}}
$$

$$\vdots$$

$$
{\scriptstyle \vee\Rightarrow}\dfrac{{\scriptstyle Comm\Rightarrow}\dfrac{\dfrac{A\wedge B \;\Rightarrow\; A\wedge B \qquad {\scriptstyle \wedge\Rightarrow}\dfrac{{\scriptstyle \Rightarrow\vee}\dfrac{C \;\Rightarrow\; C}{C \;\Rightarrow\; (A\wedge B)\vee C}}{C\wedge B \;\Rightarrow\; (A\wedge B)\vee C}}{(A\vee C)\wedge B \;\Rightarrow\; (A\wedge B)\vee C}{\scriptstyle \vee\Rightarrow}}{B\wedge (A\vee C) \;\Rightarrow\; (A\wedge B)\vee C} \qquad \begin{array}{c}\vdots\\ C\wedge (A\vee C) \;\Rightarrow\; (A\wedge B)\vee C\end{array}}{{\scriptstyle Comm\Rightarrow}\dfrac{(B\vee C)\wedge (A\vee C) \;\Rightarrow\; (A\wedge B)\vee C}{(A\vee C)\wedge (B\vee C) \;\Rightarrow\; (A\wedge B)\vee C}}
$$

$$\vdots$$

$$
{\scriptstyle Cut}\dfrac{D \;\Rightarrow\; (A\vee C)\wedge(B\vee C) \qquad \begin{array}{c}\vdots\\ (A\vee C)\wedge(B\vee C) \;\Rightarrow\; (A\wedge B)\vee C\end{array}}{D \;\Rightarrow\; (A\wedge B)\vee C}
$$

Definition 3.10 (Semi-Cut-free form)**.** A **SASCLK**-derivation is in semi-Cut-free form iff all its applications of Cut occur in exemplifications of $As \Rightarrow$, $Idem \Rightarrow$, $Comm \Rightarrow$, $\Rightarrow As$, $\Rightarrow Idem$, $\Rightarrow Comm$ or $\Rightarrow Distri^{\vee\wedge}$.

Lemma 3.3 (Existence of semi-Cut-free form for **SASCLK**)**.** *If $\vdash_{SASCLK} A \Rightarrow B$, then there is a semi-Cut-free derivation of $A \Rightarrow B$ in* ***SASCLK****.*

Proof. **LK** formulated without Cut is complete for classical logic. We can translate every Cut-free **LK**-derivation with semi-Cut-free **SASCLK**-derivations in the following way:[58]

- The axiom is obvious;
- An application of $\Rightarrow \otimes$ are translated by an application of the corresponding rule of **SASCLK**, apart from $\Rightarrow \wedge$ that needs an extra application of $\Rightarrow Distri^{\vee\wedge}$ in the following way: ${\scriptstyle \Rightarrow Distri^{\vee\wedge}}\dfrac{{\scriptstyle \Rightarrow\wedge}\dfrac{\begin{array}{c}\mathtt{d}_1^*\\ C \Rightarrow_{SASCLK} A\vee D\end{array} \quad \begin{array}{c}\mathtt{d}_2^*\\ C \Rightarrow_{SASCLK} B\vee D\end{array}}{C \Rightarrow_{SASCLK} (A\vee D)\wedge(B\vee D)}}{C \Rightarrow_{SASCLK} (A\wedge B)\vee D}$
- An application of $\otimes \Rightarrow$ is translated by an application of the corresponding rule of **SASCLK**, eventually together with several applications of $As \Rightarrow$ and $\Rightarrow As$;

[58]To be precise, since in **LK** antecedent and succedent can contain more than one formula, we need to specify how multiple-antecedents and multiple-succeddents are translated using conjunctions and disjunctions of formulae. As we saw for **LJ**, we associate to every **LK**-sequent all its possible translation with different associativity, so that for every **LK**-derivation there are several translations in **SASCLK**. This is the reason of the eventual several applications of $As \Rightarrow$ and $\Rightarrow As$ used in the translation.

- An application of $Weakening \Rightarrow$ is translated by an application of the corresponding rule of **SASCLK** when the antecedent of the premise is void, and by $\wedge \Rightarrow$ otherwise, eventually together with several applications of $As \Rightarrow$;
- An application of $\Rightarrow Weakening$ is translated by an application of the corresponding rule of **SASCLK** when the succedent of the premise is void, and by $\Rightarrow \vee$ otherwise, eventually together with several applications of $\Rightarrow As$;
- An application of $Contraction \Rightarrow$ is translated by an application of $Idem \Rightarrow$, eventually together with several applications of $As \Rightarrow$;
- An application of $\Rightarrow Contraction$ is translated by an application of $\Rightarrow Idem$, eventually together with several applications of $\Rightarrow As$;
- An application of $Permutation \Rightarrow$ is translated by an application of $Comm \Rightarrow$, eventually together with several applications of $As \Rightarrow$;
- An application of $\Rightarrow Permutation$ is translated by an application of $\Rightarrow Comm$, eventually together with several applications of $\Rightarrow As$;

Since in this translation, Cut occurs only in the sub-derivations $As \Rightarrow$, $Idem \Rightarrow$ and $Comm \Rightarrow$, we obtain a semi-Cut-free derivation. $\square$

Lemma 3.4 (Translation from **SASCLK** to **SASCNK**). *We can translate every **SASCLK**-derivation of $A \Rightarrow B$ with an **SASCNK**-derivation of B from A. In the translation, we need to resort to derived major premises of E-rules only for applications of the rule of Cut. In all other cases, major premises are open or closed assumptions.*

Proof. The translation is like in lemma 3.2. The only change is in the translation of Cut

From
$$Cut\ \dfrac{\begin{matrix}\vdots \mathtt{d}_1 \\ C \Rightarrow (D \vee A) \vee E\end{matrix} \qquad \begin{matrix}\vdots \mathtt{d}_2 \\ (F \wedge A) \wedge G \Rightarrow H\end{matrix}}{(F \wedge C) \wedge G \Rightarrow (D \vee H) \vee E}$$
we obtain:

$$\wedge I_2\ \dfrac{[A \wedge ((F \wedge C) \wedge G)]^4 \qquad \wedge I_1\ \dfrac{[A \wedge ((F \wedge C) \wedge G)]^2 \qquad \wedge E\ \dfrac{[A \wedge ((F \wedge C) \wedge G)]^1}{F} \qquad \wedge E\ \dfrac{[A \wedge ((F \wedge C) \wedge G)]^1}{A}}{F \wedge A} \qquad \wedge E\ \dfrac{[A \wedge ((F \wedge C) \wedge G)]^2}{G}}{(F \wedge A) \wedge G}$$

$$\vdots \mathtt{d}_2^*$$

$$\vee E_4\ \dfrac{[(D \vee A) \wedge ((F \wedge C) \wedge G)]^5 \qquad \vee I\ \dfrac{\wedge E\ \dfrac{[D \wedge ((F \wedge C) \wedge G)]^4}{D}}{(D \vee H) \vee E} \qquad \vee I\ \dfrac{\begin{matrix}\vdots \mathtt{d}_2^* \\ H\end{matrix}}{(D \vee H) \vee E}}{(D \vee H) \vee E}$$

$$\vdots$$

$$\vee E_5\ \dfrac{\wedge I_3\ \dfrac{(F \wedge C) \wedge G \qquad \begin{matrix}\wedge E\ \dfrac{[(F \wedge C) \wedge G]^3}{C} \\ \vdots \mathtt{d}_1^* \\ (D \vee A) \vee E\end{matrix} \qquad [(F \wedge C) \wedge G]^3}{((D \vee A) \vee E) \wedge ((F \wedge C) \wedge G)} \qquad \begin{matrix}\vdots \\ (D \vee H) \vee E\end{matrix} \qquad \vee I\ \dfrac{\wedge E\ \dfrac{[E \wedge ((F \wedge C) \wedge G)]^5}{E}}{(D \vee H) \vee E}}{(D \vee H) \vee E}$$

The other cases of Cut are easy variations of this or identical with that of **SASCLJ**.

Associativity, Commutativity and Idempotence of disjunction are translated in **SASCNK** in the following way.

$$\dfrac{\begin{matrix}A\\ \vdots\mathtt{d}_1^*\\ \beta\end{matrix} \quad \dfrac{[E\vee((B\vee C)\vee D)]^4 \quad \vee\mathrm{I}\,\dfrac{[E]^3}{\alpha} \quad \vee\mathrm{E}_2\,\dfrac{[(B\vee C)\vee D]^3 \quad \vee\mathrm{E}_1\,\dfrac{[B\vee C]^2 \quad \vee\mathrm{I}\,\dfrac{[B]^1}{\alpha} \quad \vee\mathrm{I}\,\dfrac{[C]^1}{\alpha}}{\alpha} \quad \vee\mathrm{I}\,\dfrac{[D]^2}{\alpha}}{\alpha}}{\alpha}\,\vee\mathrm{E}_3 \quad \dfrac{[F]^4}{\alpha}\,\vee\mathrm{I}}{\begin{matrix}\alpha\\ \vdots\end{matrix}}\,\vee\mathrm{E}_4$$

$$\dfrac{\begin{matrix}A\\ \vdots\mathtt{d}_1^*\\ \alpha\end{matrix} \quad \dfrac{[E\vee(B\vee(C\vee D))]^4 \quad \vee\mathrm{I}\,\dfrac{[E]^3}{\beta} \quad \vee\mathrm{E}_2\,\dfrac{[B\vee(C\vee D)]^3 \quad \vee\mathrm{I}\,\dfrac{[B]^2}{\beta} \quad \vee\mathrm{E}_1\,\dfrac{[B\vee C]^2 \quad \vee\mathrm{I}\,\dfrac{[C]^1}{\beta} \quad \vee\mathrm{I}\,\dfrac{[D]^1}{\beta}}{\beta}}{\beta}}{\beta}\,\vee\mathrm{E}_3 \quad \dfrac{[F]^4}{\beta}\,\vee\mathrm{I}}{\begin{matrix}\beta\\ \vdots\end{matrix}}\,\vee\mathrm{E}_4$$

Where $\alpha = (E \vee (B \vee (C \vee D))) \vee F$ and $\beta = (E \vee ((B \vee C) \vee D)) \vee F$. We will call the first derivation $\vee As_1$ and the second $\vee As_2$.

$$\vee\mathrm{E}_3\,\dfrac{\begin{matrix}E\\ \vdots\mathtt{d}_1^*\\ \gamma\end{matrix} \quad \vee\mathrm{I}\,\dfrac{[C]^3}{\delta} \quad \vee\mathrm{E}_2\,\dfrac{[(A\vee B)\vee D]^3 \quad \vee\mathrm{E}_1\,\dfrac{[A\vee B]^2 \quad \vee\mathrm{I}\,\dfrac{[A]^1}{\delta} \quad \vee\mathrm{I}\,\dfrac{[B]^1}{\delta}}{\delta} \quad \vee\mathrm{I}\,\dfrac{[D]^2}{\delta}}{\delta}}{\begin{matrix}\delta\\ \vdots\end{matrix}}$$

Where $\gamma = C \vee ((A \vee B) \vee D)$ and $\delta = C \vee ((B \vee A) \vee D)$. We will call this derivation $\vee Comm$.

$$\vee\mathrm{E}_2\,\dfrac{\begin{matrix}E\\ \vdots\mathtt{d}_1^*\\ (A\vee A)\vee B\end{matrix} \quad \vee\mathrm{I}\,\dfrac{\vee\mathrm{E}_1\,\dfrac{[A\vee A]^2 \quad [A]^1 \quad [A]^1}{A}}{A\vee B} \quad \vee\mathrm{I}\,\dfrac{[B]^2}{A\vee B}}{\begin{matrix}A\vee B\\ \vdots\end{matrix}}$$

We will call this derivation $\vee Idem$.

In conclusion, let us see the translation of $\Rightarrow Distri^{\vee\wedge}$:

$$\vee\mathrm{E}_4\,\dfrac{\vee Comm\,\dfrac{\begin{matrix}D\\ \vdots\mathtt{d}_1^*\\ (A\vee C)\wedge(B\vee C)\end{matrix}}{(B\vee C)\wedge(A\vee C)} \quad \vee\mathrm{E}_2\,\dfrac{\vee Comm\,\dfrac{[B\wedge(A\vee C)]^4}{\epsilon} \quad \vee\mathrm{I}\,\dfrac{[A\wedge B]^2}{\epsilon} \quad \wedge\mathrm{E}_1\,\dfrac{[C\wedge B]^2 \quad \vee\mathrm{I}\,\dfrac{[C]^1}{\epsilon}}{\epsilon}}{\epsilon} \quad \wedge\mathrm{E}_3\,\dfrac{[C\wedge(A\vee C)]^4 \quad \vee\mathrm{I}\,\dfrac{[C]^3}{\epsilon}}{\epsilon}}{\epsilon}$$

Where $\epsilon = (A \wedge B) \vee C$ and the applications of $\vee Comm$ have form:

$$\wedge\mathrm{I}_3\ \dfrac{A \wedge B \qquad \wedge\mathrm{E}_1\ \dfrac{[A \wedge B]^3 \qquad [B]^1}{B} \qquad \wedge\mathrm{E}_2\ \dfrac{[A \wedge B]^3 \qquad [A]^2}{A}}{B \wedge A}$$

We will call this derivation $Distri^{\vee\wedge}$.

□

We can now prove theorem 3.4.[59]

Proof. If $A \vdash_{SASCNK} B$ then, since **SASCNK** and **SASCLK** are equivalent each other, $\vdash_{SASCLK} A \Rightarrow B$. Given lemma 3.3 we know that there is a semi-Cut-free derivation $\mathtt{d}$ for $A \Rightarrow B$. As a consequence, the only non-assumed major premises in the derivation $\mathtt{d}^*$, obtained from $\mathtt{d}$ by the translation displayed in lemma 3.4, are due to $Idem \Rightarrow$, $Comm \Rightarrow$, $As \Rightarrow$, $\Rightarrow Idem$, $\Rightarrow Comm$, $\Rightarrow As$ and $\Rightarrow Distri^{\vee\wedge}$ that are the only Cut in the **SASCLK**-derivation. We call the **SASCNK**-sub-derivations obtained by translating these **SASCLK**-sub-derivations $Con\wedge$, $Per\wedge$, $As\wedge$, $Con\vee$, $Per\vee$, $As\vee$ and $Distri^{\vee\wedge}$.

To end the proof, we will show that:

1. the only situations in which this translation gives rise to maximal formulae is when:
   (a) the end-formula of a chain of $Con\wedge$, $Per\wedge$ and $As\wedge$ is the major premise of an application of $\wedge$E; or
   (b) the first-formula of the chain of $Con\vee$, $Per\vee$ and $As\vee$ is a conclusion of $\vee$I, so in particular, $Distri^{\vee\wedge}$ does not cause any maximality;
2. in these cases we can normalize the derivation.

Let us start with the first point. About $Con\wedge$, $Per\wedge$ and $As\wedge$ we behave like for **SASCNJ**. Before passing to $Con\vee$, $Per\vee$ and $As\vee$, let us consider $Distri^{\vee\wedge}$.

First of all, let us check that there are no maximal formulae inside $Distri^{\vee\wedge}$ itself. It contains two major premises of E-rules that are derived, that correspond to the two applications of $Comm\wedge$. The first on the right could create a maximal formula of the kind $\wedge$I-$\vee$E-$\wedge$E. However, only one of the assumptions discharged by $\vee$E is major premise of $\wedge$E, while in order to have a maximal formula, both should be so (definition 3.8). The second has the same structure, and can not create a maximal formula, because the discharged assumption on the left is major premise of $\wedge$I and not of $\wedge$E. So there is no maximal formula inside $Distri^{\vee\wedge}$.

Let us now consider the possibility that $Distri^{\vee\wedge}$ creates maximal formulae by composition, that is by the rule used to derive $(A \vee C) \wedge (B \vee C)$ as last step of $\mathtt{d}_1^*$, together with the rules of $Distri^{\vee\wedge}$. The only rule that could create a maximal formula in this way is $\vee$I (in this way we could have a maximal formula of the kind $\vee$I-$\wedge$I-$\vee$E). Nonetheless this possibility is excluded by the form of $(A \vee C) \wedge (B \vee C)$ itself, since the only I-rule that could derive it is $\wedge$I. So $Distri^{\vee\wedge}$ neither contains maximal formulae inside itself, nor it creates them by composition.

[59] The complete proof is included in my Ph.D. thesis Ceragioli [2020].

Now, let us consider chains of $Con\vee$, $Per\vee$ and $As\vee$.[60] These steps of derivation allow to compose deductions on the top, that is the top-formula of $X\vee$ (that is a major premise of $\vee E$) can be derived by an I-rule. Of course the conclusion of $X\vee$ can itself be a top-formula of $X\vee$, thanks to composition. So we have to deal with chains of $X\vee$ that eliminate conclusions of I-rules.

The only situation in which we obtain a maximal formula is when the top-formula of the chain of $X\vee$ is the conclusion of $\vee I$. Indeed, each $X\vee$ derives its conclusion by $\vee E$ with its open assumption as major premise, and applies only $\vee$-rules.

It is important also to notice that there are no maximal formulae in $X\vee$, and that their conclusion is always a conclusion of $\vee E$. So there are no maximal formulae in the chains of $X\vee$ (neither in a single $X\vee$, nor in the conclusion of an $X\vee$ being the major premise of the next $X\vee$): only its first formula can be maximal, depending on the rule that derives it. In other words, the chains of $X\vee$ can create a maximal formula only at the point of application at their top.

Now, we just have to show that we are able to normalize these maximal formulae. We can deal with maximal formulae caused by chains of $X\wedge$ in the same way as we did regarding **SASCNJ** in preivous section.

Let us consider a chain of $X\vee$ that eliminates a maximal formula, and in particular the first $X\vee$, that directly eliminates it. We will normalize completely the derivation or we will remove the first $X\vee$, so by induction on the length of the chain we obtain the normalization. By cases on the first $X\vee$. We will see the case in which $As_1\vee$ is the first rule of the chain. The other cases are similar.

$$\vee As_1$$

$$\begin{array}{c}\vee E_4\,\dfrac{\vee I\,\dfrac{\begin{array}{c}\vdots\\ \alpha\end{array}}{\gamma} \quad \vee E_3\,\dfrac{[E\vee((B\vee C)\vee D)]^4 \quad \vee I\,\dfrac{[E]^3}{\delta} \quad \vee E_2\,\dfrac{[(B\vee C)\vee D]^3 \quad \vee E_1\,\dfrac{[B\vee C]^2 \quad \vee I\,\dfrac{[B]^1}{\delta} \quad \vee I\,\dfrac{[C]^1}{\delta}}{\delta} \quad \vee I\,\dfrac{[D]^2}{\delta}}{\delta}}{\delta} \quad \vee I\,\dfrac{[F]^4}{\delta}}{\delta}\\ \vdots_{X\vee}\end{array}$$

Where $\gamma = (E\vee((B\vee C)\vee D))\vee F$ and $\delta = (E\vee(B\vee(C\vee D)))\vee F$.

If $\alpha = F$, then we reduce the length of the chain by: $$\begin{array}{c}\vee I\,\dfrac{\begin{array}{c}\vdots\\ F\end{array}}{(E\vee(B\vee(C\vee D)))\vee F}\\ \vdots_{X\vee}\end{array}$$

If $\alpha = E\vee((B\vee C)\vee D)$, then we reduce the length of the chain by:

$$\begin{array}{c}\vee E_3\,\dfrac{\begin{array}{c}\vdots\\ E\vee((B\vee C)\vee D)\end{array} \quad \vee I\,\dfrac{[E]^3}{\delta} \quad \vee E_2\,\dfrac{[(B\vee C)\vee D]^3 \quad \vee E_1\,\dfrac{[B\vee C]^2 \quad \vee I\,\dfrac{[B]^1}{\delta} \quad \vee I\,\dfrac{[C]^1}{\delta}}{\delta} \quad \vee I\,\dfrac{[D]^2}{\delta}}{\delta}}{\delta}\\ \vdots_{X\vee}\end{array}$$

[60] I will use $X\vee$ to indicate each of them.

If it creates anormality, then $E \vee ((B \vee C) \vee D)$ is the conclusion of an $\vee I$, let us call β its premise.

If $\beta = E$, then we reduce to

$$\begin{array}{c}\vdots\\ \vee\mathrm{I}\ \dfrac{E}{(E \vee (B \vee (C \vee D))) \vee F}\\ \vdots_{X\vee}\end{array}$$

If $\beta = (B \vee C) \vee D$, then we reduce to

$$\begin{array}{c}\vee\mathrm{E}_2\ \dfrac{\begin{array}{c}\vdots\\ (B \vee C) \vee D\end{array} \qquad \vee\mathrm{E}_1\ \dfrac{[B \vee C]^2 \qquad \vee\mathrm{I}\ \dfrac{[B]^1}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[C]^1}{\delta}}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[D]^2}{\delta}}{\delta}\\ \vdots_{X\vee}\end{array}$$

If $(B \vee C) \vee D$ is maximal, then it is conclusion of $\vee I$, let us call ε its premise.

If $\varepsilon = D$, then we reduce to

$$\begin{array}{c}\vdots\\ \vee\mathrm{I}\ \dfrac{D}{(E \vee (B \vee (C \vee D))) \vee F}\\ \vdots_{X\vee}\end{array}$$

If $\varepsilon = B \vee C$, then we reduce to

$$\begin{array}{c}\vee\mathrm{E}_1\ \dfrac{\begin{array}{c}\vdots\\ B \vee C\end{array} \qquad \vee\mathrm{I}\ \dfrac{[B]^1}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[C]^1}{\delta}}{\delta}\\ \vdots_{X\vee}\end{array}$$

If $B \vee C$ is maximal, then it is conclusion of $\vee I$, and its premise can only be B or C. We then reduce to

$$\begin{array}{c}\vdots\\ \vee\mathrm{I}\ \dfrac{B}{(E \vee (B \vee (C \vee D))) \vee F}\\ \vdots_{X\vee}\end{array} \quad \text{or} \quad \begin{array}{c}\vdots\\ \vee\mathrm{I}\ \dfrac{C}{(E \vee (B \vee (C \vee D))) \vee F}\\ \vdots_{X\vee}\end{array}$$

In both cases we have reduced the length of the chain of $\vee X$ or normalized the derivation if there is no such chain.

□

As a consequence, **SASCNK** suits Harmony too.

4 Conclusion

We saw that PTS evaluates differently equivalent formulations of intuitionistic and classical logic, based on their notational aspects. Standard PTS adopts a MASC-framework and claims to justify intuitionistic logic, while a MAMC-framework has been proposed to justify classical logic. I argued that neither of these approaches is satisfying, since there is a circular dependence of meaning both between multiple conclusions and disjunctions and between multiple assumptions and conjunctions. In conclusion, we displayed a SASC-framework for PTS and managed to justify both intuitionistic and classical logics in it, by both a reinterpretation of the notion of *maximal formula* and a weakening of the requirements of Harmony and Separability.

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