

Quantum Field Theory 1

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Part I

Field Theory

Classical Field Theory

1.1 Continuous limit

1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x -axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a . Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n , one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2} \dot{\eta}_n^2 - \frac{\lambda}{2} (\eta_n - \eta_{n-1})^2 \right] \quad (1.1)$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda (\eta_{n+1} - 2\eta_n + \eta_{n-1}) \quad (1.2)$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \quad (1.3)$$

where the dispersion relation is:

$$\omega^2 = 2\lambda (1 - \cos k) \approx \lambda k^2 \quad (1.4)$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \quad \Rightarrow \quad k_m = \frac{2\pi m}{N}, \quad m = 0, 1, \dots, N-1 \quad (1.5)$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} [\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)}] \quad (1.6)$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjugated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \quad (1.7)$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*¹:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \quad (1.8)$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators*² $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} [e^{i(k_m n - \omega_m t)} \hat{a}_m + e^{-i(k_m n - \omega_m t)} \hat{a}_m^\dagger] \quad (1.9)$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \quad (1.10)$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2} \hat{\pi}_n^2 + \frac{\lambda}{2} (\hat{\eta}_n - \hat{\eta}_{n-1})^2 \right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) \quad (1.11)$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m |0\rangle \quad \forall m = 0, 1, \dots, N-1 \quad (1.12)$$

$$|n_0, n_1, \dots, n_{N-1}\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} |0\rangle \quad (1.13)$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2} \sum_{m=0}^{N-1} \hbar\omega_m \quad (1.14)$$

$$E_{n_0, n_1, \dots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \quad (1.15)$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as “particles” and n_m is the number of phonons in the m^{th} normal mode.

¹Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t} \hat{\mathcal{O}}(0) e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{d\hat{\mathcal{O}}}{dt} = \frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

²For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{d\hat{x}}{dt} = \hat{p}(t)$ and $\frac{d\hat{p}}{dt} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) + \hat{a}^\dagger(t)] \quad \hat{p}(t) = -i\omega \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) - \hat{a}^\dagger(t)]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega (\hat{a}^\dagger(t)\hat{a} + \frac{1}{2})$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.

1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \rightarrow 0$ and $N \rightarrow \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x, t)$ dependent on the continuous real coordinate $x \in [0, R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x} \right)^2 \quad \sum_n \longrightarrow \frac{1}{a} \int_0^R dx$$

Proposition 1.1.1. *In the continuous limit:*

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x - x') = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-x')}$$

Proof. By direct calculation:

$$a \sum_n f(an) \frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R dx f(x) \delta(x - y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \rightarrow kx$, symmetrizing $k_m \in [-\pi, \pi]$ (instead of $[0, 2\pi]$) one finds:

$$\delta(x - x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na} \sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-x')}$$

where integration limits are $\pm \frac{\pi}{a} \rightarrow \pm \infty$. □

Proof. The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R dx e^{i(k-k')x} = 2\pi \delta(k - k')$$

□

By these relations, it can be seen that $\frac{dk}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k + dk$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.

In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R dx \left[\frac{1}{2} \rho_0 (\partial_t \eta)^2 - \frac{\kappa}{2} (\partial_x \eta)^2 \right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x, t) \equiv \sqrt{\rho_0} \eta(x, t)$ and $\kappa = c^2 \rho_0$ and adding a pinning term $\propto \phi^2$, the Lagrangian can be written as:

$$L = \int_0^R dx \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{c^2}{2} (\partial_x \phi)^2 - \frac{m^2 c^4}{2} \phi^2 \right] \quad (1.16)$$

The classical equation of motion of this field yields:

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi - m^2 c^4 \phi \quad (1.17)$$

The solutions of this wave equation can be written as:

$$\phi(x, t) = e^{i(kx - \omega_k t)} \quad (1.18)$$

with dispersion relation:

$$\omega_k^2 = c^2 k^2 + m^2 c^4 \quad (1.19)$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x, t) := \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi(x, t) \quad (1.20)$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R dx \left[\frac{1}{2} \Pi^2 + \frac{c^2}{2} (\partial_x \phi)^2 + \frac{m^2 c^4}{2} \phi^2 \right] \quad (1.21)$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x, t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \sqrt{\frac{\hbar}{2\omega_k}} \left[e^{i(kx - \omega_k t)} \hat{a}_k + e^{-i(kx - \omega_k t)} \hat{a}_k^\dagger \right] \quad (1.22)$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi \delta(k - k') \quad (1.23)$$

$$[\hat{\phi}(x, t), \hat{\Pi}(x', t)] = i\hbar \delta(x - x') \quad (1.24)$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{2} \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) = E_0 + \int_{\mathbb{R}} \frac{dk}{2\pi} \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \quad (1.25)$$

The ground-state energy can be computed from Eq. 1.14, defining $\text{Vol} := 2\pi \delta(k = 0)$:

$$E_0 = \text{Vol} \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{2} \hbar \omega_k \quad (1.26)$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

1.2 Spacetime symmetries

1.2.1 Lie groups

Definition 1.2.1. A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d -dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0, \dots, 0) = e$.

Definition 1.2.2. Given a group G and a vector space $V(\mathbb{K})$, a *representation* of G on V is a homomorphism $\rho : G \rightarrow \text{GL}(V)$.

Given the isomorphism $\text{GL}(V) \rightarrow \mathbb{K}^{n \times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to *de facto* represent G as matrices acting on elements of V , i.e. $\rho : G \rightarrow \mathbb{K}^{n \times n}$.

Proposition 1.2.1. Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ as:

$$\rho(g(\theta)) = e^{i\theta_a T^a} \quad (1.27)$$

where $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ are the generators of G on V .

Definition 1.2.3. Given a Lie group G with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ on $V(\mathbb{C})$, its *Lie algebra* is:

$$[T^a, T^b] = i f_c^{ab} T^c \quad (1.28)$$

where f_c^{ab} are called the *structure constants*.

Proposition 1.2.2. The Lie algebra of a Lie group is independent of the representation.

Proposition 1.2.3. Any d -dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

Definition 1.2.4. Given a Lie group with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ on $V(\mathbb{C})$, a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation, Casimir operators are operators proportional to id_V , and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

Proposition 1.2.4. A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

1.2.2 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu} x^\mu x^\nu$, i.e. the orthogonal group $\text{O}(1,3)$ (with signature $(+, -, -, -)$). The condition that Λ^μ_ν must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \quad (1.29)$$

This implies that $\det \Lambda = \pm 1$: a transformation with $\det \Lambda = -1$ can always be written as the product of a transformation with $\det \Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines $\text{SO}(1,3) := \{\Lambda \in \text{O}(1,3) : \det \Lambda = 1\}$.

Writing explicitly the temporal component $1 = (\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2$, it is clear that $(\Lambda^0_0)^2 \geq 1$. Therefore, $\text{O}(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0_0 \geq 1$ and the non-orthochronous component with $\Lambda^0_0 \leq -1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a discrete transformation which reverses the sign of the temporal component.

Definition 1.2.5. The *Lorentz group* is the orthochronous component of $\text{SO}(1, 3)$.

The discrete transformations are factored out of the Lorentz group. Considering the infinitesimal transformation and applying Eq. 1.29:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \quad \Rightarrow \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x, y) , (y, z) and (z, x) planes and the 3 rapidities of hyperbolic rotations in the (t, x) , (t, y) and (t, z) planes.

Proposition 1.2.5. The Lorentz group is a non-compact Lie group.

Proof. Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in $[0, 2\pi)$, rapidities vary in \mathbb{R} , so the differentiable manifold associated to $\text{SO}(1, 3)$ is not compact. \square

1.2.2.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \text{SO}(1, 3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \quad (1.30)$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a n -dimensional representation of $\text{SO}(1, 3)$, both $[J^{\mu\nu}]^i{}_j$ and $[\Lambda]^i{}_j$ are $\mathbb{C}^{n \times n}$ matrices (Λ is real): for example, the $n = 1$ representation acts on *scalars*, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \forall \mu, \nu = 0, \dots, 3$.

4-vectors The $n = 4$ representation acts on *contravariant 4-vectors* v^μ , which transform according to $v^\mu \mapsto \Lambda^\mu{}_\nu v^\nu$, and *covariant 4-vectors* v_μ , which transform according to $v_\mu \mapsto \Lambda_\mu{}^\nu v_\nu$. In this representation, the generators are represented as $\mathbb{C}^{4 \times 4}$ matrices:

$$[J^{\mu\nu}]^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma) \quad (1.31)$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}(1, 3)$, called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \quad (1.32)$$

It is convenient to rearrange the 6 components of $J^{\mu\nu}$ into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \quad K^i := J^{i0} \quad (1.33)$$

The $\mathfrak{so}(1, 3)$ can then be rewritten as:

$$[J^i, J^j] = i\epsilon^{ijk}J^k \quad [J^i, K^j] = i\epsilon^{ijk}K^k \quad [K^i, K^j] = -i\epsilon^{ijk}J^k \quad (1.34)$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^i are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \quad \eta^i := \omega^{i0} \quad (1.35)$$

so that:

$$\Lambda = \exp [-i\boldsymbol{\theta} \cdot \mathbf{J} + i\boldsymbol{\eta} \cdot \mathbf{K}] \quad (1.36)$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts wch increase velocities with respect to said frame.

1.2.2.2 Tensor Representations

A generic (p, q) -tensor transforms as:

$$T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mapsto \Lambda^{\mu_1}_{\alpha_1} \dots \Lambda^{\mu_p}_{\alpha_p} \Lambda^{\beta_1}_{\nu_1} \dots \Lambda^{\beta_q}_{\nu_q} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \quad (1.37)$$

The representation of the Lorentz group which acts on (p, q) -tensors is of degree $n = 4^{p+q}$, however it is reducible into the direct product of $p + q$ 4-dimensional representations as of Eq. 1.38.

Moreover, consider the action of the Lorentz group on $(2, 0)$ -tensors: being $T^{\mu\nu} \mapsto \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu} T^{\mu\nu} \mapsto \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$4 \otimes 4 = 1 \oplus 6 \oplus 9 \quad (1.38)$$

These are irreducible representations which, given a generic tensor $T^{\mu\nu}$, act on S , $A^{\mu\nu}$ and $S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$ and $S^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$.

Decomposition under rotations Restricting the action to the $\text{SO}(3)$ sub-group of $\text{SO}(1, 3)$, tensors can be decomposed according to irreducible representations of $\text{SO}(3)$, which are labelled by the angular momentum j and are of degree $n = 2j + 1$. Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \quad (1.39)$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$. A 4-vector v^μ is irreducible under the action of $\text{SO}(1, 3)$, but under $\text{SO}(3)$ it is decomposed into v^0 and \mathbf{v} , so $v^\mu \in \mathbf{0} \oplus \mathbf{1}$. A $(2, 0)$ -tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.38: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2}\epsilon^{ijk}A^{jk}$, so $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ under spatial rotations.

Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$, $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$, $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$ and $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk}A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

Example 1.2.1. Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two *invariant tensors* under $\text{SO}(1, 3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.29, and the Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\sigma{}_\gamma \Lambda^\rho{}_\delta \epsilon^{\alpha\beta\gamma\delta} = (\det \Lambda) \epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

1.2.2.3 Spinor representations

1.2.2.4 Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields A scalar field transforms as:

$$\phi'(x') = \phi(x) \tag{1.40}$$

Consider an infinitesimal transformation $x'^\rho = x^\rho + \delta x^\rho$, with $\delta x^\rho = -\frac{i}{2} \omega_{\mu\nu} [J^{\mu\nu}]^\rho{}_\sigma x^\sigma$ as of Eq. 1.30. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar representation of $\text{SO}(1, 3)$ is the trivial one ($J^{\mu\nu} = 0$).

However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$.

1.3 Classical equations of motion

Consider a *local field theory* of fields $\{\phi_i(x)\}_{i \in \mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkowski spacetime. Its Lagrangian takes the form:

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.41)$$

where \mathcal{L} is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int dt L = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.42)$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

Proposition 1.3.1. *The stationary action principle $\delta\mathcal{S} = 0$ determines the classical equations of motion:*

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} = 0 \quad (1.43)$$

Proof. Varying Eq. 1.42:

$$\delta\mathcal{S} = \int d^4x \sum_{i \in \mathcal{I}} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \delta (\partial_\mu \phi_i) \right] = \int d^4x \sum_{i \in \mathcal{I}} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right] \delta \phi_i = 0$$

□

Proposition 1.3.2. *Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.*

Proof. This is a consequence of Stokes theorem:

$$\int_\Sigma d^4x \partial_\mu K^\mu = \int_{\partial\Sigma} dA n_\mu K^\mu$$

□

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_i)} \quad (1.44)$$

$$\mathcal{H} = \sum_{i \in \mathcal{I}} \Pi_i(x) \partial_0 \phi_i(x) - \mathcal{L} \quad (1.45)$$