IR-Safe NLO QCD with Massive Quarks

An extension of the NSC subtraction scheme

Bachelor Degree in Physics

Leonardo Cerasi (11410A)

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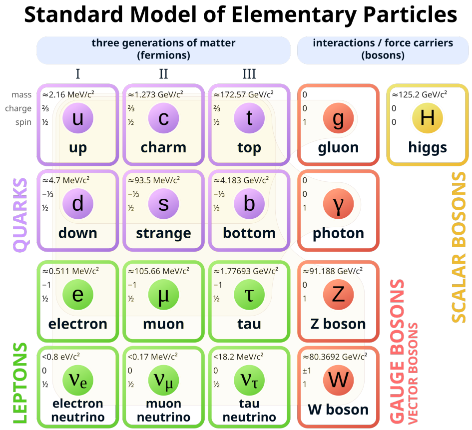
DEGLI STUDI
DI MILANO

Precision estimates at the LHC

Standard Model of fundamental interactions

Gauge theory: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$

- $\mathrm{SU}(3)$: strong interaction, with 8 gluons
- $\mathrm{SU}(2)$: weak interaction, with Z, $W^{\pm}$ bosons
- $\mathrm{U}(1)$: electromagnetic interaction, with the photon
- EW symmetry breaking: the Higgs boson



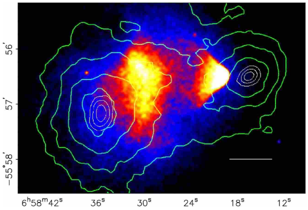


Figure from Clowe et al. 2006. Offset between the observed baryonic mass distribution and the gravitational potential in the Bullet Cluster (1E 0657-56).

Main BSM proposals

- supersymmetric models (MSSM, ...)
- dark matter models (WIMPs, axions, ...)
- extended gauge sectors ($SO(10)$, ...)
- SM Effective Field Theory (SMEFT)

Main BSM evidence

- dark matter and dark energy
- matter-antimatter asymmetry
- neutrino masses

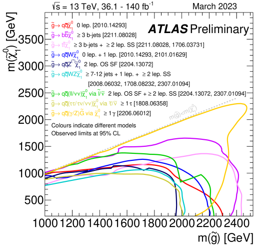
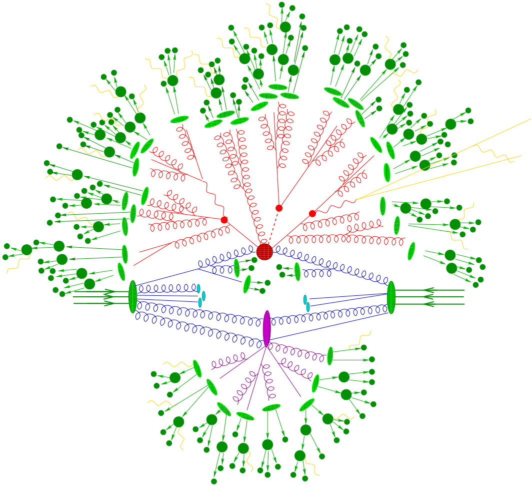


Figure from ATLAS PUB Note 2023-025. Exclusion limits in the $\tilde{g} - \tilde{\chi}_1^0$ mass plane for various models for the decay of the gluino to the lightest supersymmetric particle.

Precision estimates at the LHC

Precision studies of hadronic scatterings



Hard scattering processes

Characterized by a large momentum transfer, which allows for a perturbative description thanks to *asymptotic freedom*.
Individual partons treated as free particles: hadronic scattering cross-sections studied in terms of partonic scattering cross-section.

Figure from Höche 2015.
Hadronization of jets in hadronic scattering.

Precision estimates at the LHC

Factorization theorem and perturbative QCD

Factorization theorem for hard hadronic scattering processes (Collins et al. 1989):

$$
\begin{aligned}
&\mathrm{d}\sigma_{h_1,h_2}(P_1,P_2) \\
&= \sum_{a,b} \int_{[0,1]^2} \frac{\mathrm{d}\xi_1}{\xi_1} \frac{\mathrm{d}\xi_2}{\xi_2} f_a^{(h_1)}(\xi_1,\mu_\mathrm{F}^2) f_b^{(h_2)}(\xi_2,\mu_\mathrm{F}^2) \times \\
&\times \mathrm{d}\hat{\sigma}_{a,b}(\xi_1 P_1, \xi_2 P_2, \alpha_\mathrm{s}, \mu_\mathrm{R}^2, \mu_\mathrm{F}^2) \left[1 + o\left(\frac{\Lambda_\mathrm{QCD}^n}{Q^n}\right)\right]
\end{aligned}
$$

Clear separation of energy scales:

- hard hadronic scattering: $Q \sim 100\,\mathrm{GeV} - 1\,\mathrm{TeV}$
- hadronization: $\Lambda_\mathrm{QCD} \approx 200\,\mathrm{MeV}$

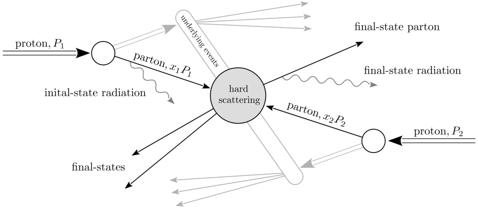


Figure from Asteriadis 2021.
Detail of the partonic scattering in a hard hadronic scattering.

Precision estimates at the LHC

Factorization theorem and perturbative QCD

Asymptotic freedom allows for a perturbative analysis of the underlying partonic scattering:

$$\mathrm{d}\hat{\sigma}_{a,b}(p_1, p_2) = \sum_{n\in\mathbb{N}_0} \mathrm{d}\hat{\sigma}^{(n)}_{a,b}(p_1, p_2)$$

where $\mathrm{d}\hat{\sigma}^{(n)} \sim \alpha_\mathrm{s}^{n_0+n}$ and $\alpha_\mathrm{s} \sim 0.1$ due to asymptotic freedom.

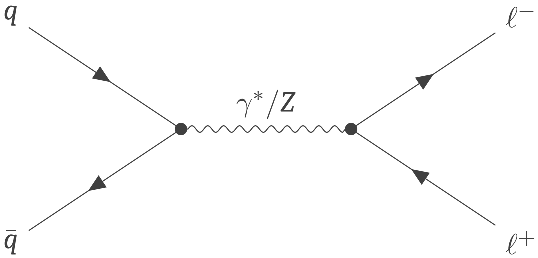
$n \geq 1$ are denoted by N^{n}LO QCD corrections:

- real corrections: additional initial- or final-state radiation
- virtual corrections: additional partonic loops

IR-pole structure of QCD

Radiative corrections to partonic processes

Our focus is on NLO QCD corrections. Consider e.g. the Drell-Yan process:



LO process

IR-pole structure of QCD

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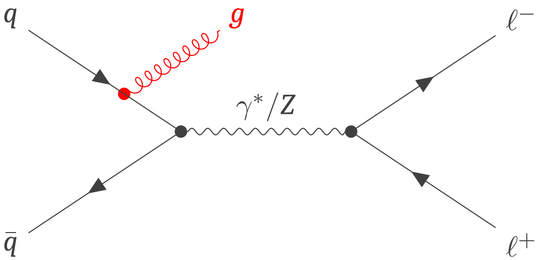

Real correction

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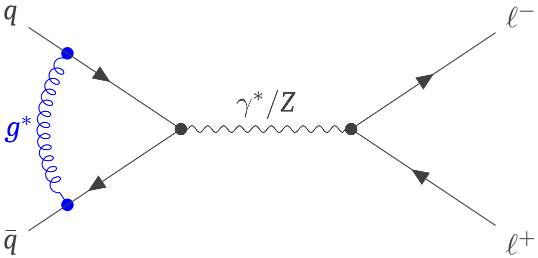
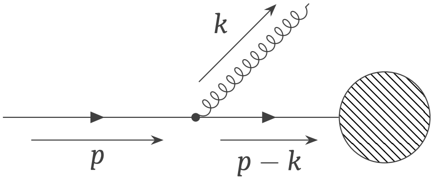


Virtual correction

IR-pole structure of QCD
Infrared singularities of scattering amplitudes

One of the main difficulties: infrared singularities in particular kinematic regimes.
Example in real corrections:



$$\sim \quad \frac{1}{(p-k)^2} = -\frac{1}{2E_pE_k(1-\cos\theta)}$$

IR-pole structure of QCD
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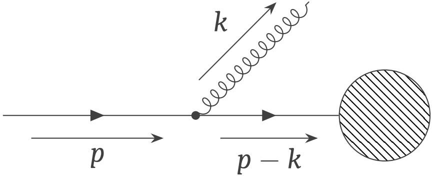

$$\sim \frac{1}{(p-k)^2} = -\frac{1}{2E_p\textcolor{red}{E_k}(1-\cos\theta)}$$

Soft singularity: $E_k \to 0$

IR-pole structure of QCD

Infrared singularities of scattering amplitudes

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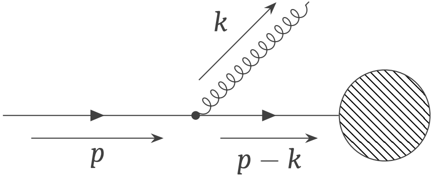
Example in real corrections:



$$\sim \quad \frac{1}{(p-k)^2} = -\frac{1}{2E_pE_k(1-\cos\theta)}$$

Collinear singularity: $\theta \to 0$

IR-pole structure of QCD
Dimensional regularization

The key idea to regularize IR divergences is dimensional regularization ('t Hooft et al. 1972):

$$d = 4 - 2\epsilon \quad , \quad \epsilon \in \mathbb{C} : \Re\epsilon < 0$$

Then, soft and collinear singularities are expressed as poles in ϵ:

$$\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k} \left|\mathcal{A}(p_k)\right|^2 \sim \int_0^{\mathcal{E}} \frac{\mathrm{d}E_k}{E_k^{5-d}} \int_0^{\pi} \mathrm{d}\theta \frac{\sin^{d-3}\theta}{1-\cos\theta} \left|\mathcal{A}_0\right|^2$$

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IR-pole structure of QCD
Subtraction schemes

ϵ-poles can be extracted from partonic cross-sections via subtraction methods.
General idea using a regular function $f(x)$:

$$I = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} f(x) = \underbrace{\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} \left[f(x) - f(0)\right]}_{\text{regular}} + \underbrace{f(0) \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}}_{\text{singular}}$$

- $\dfrac{f(x) - f(0)}{x^{1+\epsilon}}$ regular at $x = 0$, so it can be numerically integrated with $\epsilon \to 0$;
- $f(0) \displaystyle\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} = -\frac{f(0)}{\epsilon}$ contains the explicit ϵ-pole.

Our aim is finding the most general subtraction terms for partonic scattering.

NSC subtraction scheme

Sequential extraction of singularities

The Nested Soft-Collinear (NSC) Subtraction Scheme (SS) at NNLO is modular and local (Caola et al. 2017):

- singular limits extracted by operators: S_i soft limit $E_i \to 0$, C_{ij} collinear limit $\theta_{ij} \to 0$;

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- potentially-unresolved partons isolated using partitions of unity Δ^i and ω^{ij};
- first soft singularities removed applying $\overline{\mathsf{S}}_{\mathfrak{m}} \equiv \mathrm{id} - \mathsf{S}_{\mathfrak{m}}$, then collinear ones removed using $\overline{\mathsf{C}}_{i\mathfrak{m}} \equiv \mathrm{id} - \mathsf{C}_{i\mathfrak{m}}$, with $\mathfrak{m}$ unresolved parton;

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- IR-safe part of cross-section extracted by:

$$\mathrm{id} = \mathsf{S}_{\mathfrak{m}} + \sum_{i \in \mathcal{H}^n_{m,0}(\mathfrak{m})} \overline{\mathsf{S}}_{\mathfrak{m}} \mathsf{C}_{i\mathfrak{m}} + \mathsf{O}^{\mathfrak{m}}_{\mathsf{NLO}} \qquad \mathsf{O}^{\mathfrak{m}}_{\mathsf{NLO}} := \sum_{i \in \mathcal{H}^n_{m,0}} \overline{\mathsf{S}}_{\mathfrak{m}} \overline{\mathsf{C}}_{i\mathfrak{m}} \omega^{\mathfrak{m}i}$$

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- counterterms contain the IR divergences + a finite reminder.

NSC subtraction scheme

IR-safe cross-section

The total ϵ-finite NLO cross-section can be written as (Devoto et al. 2025):

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{(1)}_{a,b} = \mathrm{d}\hat{\sigma}^{\text{NLO,reg}}_{a,b} &+ \frac{1}{2\hat{s}} \sum_{n \in \mathcal{G}_m} \left[\frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}^2)}{2\pi} \left\langle I^{(0)}_{\mathrm{T}} \mathcal{F}_{a,b}[\mathcal{X}^n_m] \right\rangle + \left\langle \mathcal{F}^{\text{fin}}_{a,b}[\mathcal{X}^n_m] \right\rangle \right] \\
&+ \frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}^2)}{2\pi} \sum_c \left[P^{\text{gen}}_{f_c,f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\text{gen}}_{f_c,f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]
\end{aligned}
$$

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$$+ \frac{\alpha_s(\mu_R^2)}{2\pi} \sum_c \left[P^{\text{gen}}_{f_c,f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\text{gen}}_{f_c,f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]$$

- NLO final reminder, defined using $\mathrm{O}^{\mathfrak{m}}_{\text{NLO}}$;

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- NLO final reminder, defined using $\mathrm{O}^{\mathfrak{m}}_{\mathrm{NLO}}$;
- finite counterterm from virtual corrections;

NSC subtraction scheme

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$$+ \frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}^2)}{2\pi} \sum_{c} \left[P^{\mathrm{gen}}_{f_c,f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\mathrm{gen}}_{f_c,f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]$$

- NLO final reminder, defined using $\mathrm{O}^{\mathfrak{m}}_{\mathrm{NLO}}$;
- finite counterterm from virtual corrections;
- finite counterterm from PDF renormalization;

NSC subtraction scheme

IR-safe cross-section

The total ϵ-finite NLO cross-section can be written as (Devoto et al. 2025):

$$\mathrm{d}\hat{\sigma}^{(1)}_{a,b} = \mathrm{d}\hat{\sigma}^{\text{NLO,reg}}_{a,b} + \frac{1}{2\hat{s}} \sum_{n\in\mathcal{G}_m} \left[\frac{\alpha_s(\mu_R^2)}{2\pi} \left\langle I_T^{(0)} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \right\rangle + \left\langle \mathcal{F}^{\text{fin}}_{a,b}[\mathcal{X}_m^n] \right\rangle \right]$$
$$+ \frac{\alpha_s(\mu_R^2)}{2\pi} \sum_c \left[P^{\text{gen}}_{f_c,f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\text{gen}}_{f_c,f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]$$

- NLO final reminder, defined using $\mathrm{O}^{\mathfrak{m}}_{\text{NLO}}$;
- finite counterterm from virtual corrections;
- finite counterterm from PDF renormalization;
- counterterm we are interested in.

NSC subtraction scheme

Pole extraction through operators

IR singularities are extracted as operators acting on the LO matrix element:

$$I_{\mathrm{S}}(\epsilon) := -\frac{1}{\epsilon^2}\left(\frac{2\mathcal{E}}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\sum_{i\neq j\in\mathcal{X}^n_{m,0}}\eta_{ij}(\mathbf{T}_i\cdot\mathbf{T}_j)\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij})$$

$$I_{\mathrm{C}}(\epsilon) := \sum_{i\in\mathcal{H}^n_{m,0}}\frac{\Gamma_{i,f_i}}{\epsilon} = \frac{1}{\epsilon}\sum_{i\in\mathcal{H}^n_{m,0}}\left[\gamma_i + 2\mathbf{T}_i^2\log\frac{\mathcal{E}}{E_i}\right] + o(\epsilon^0)$$

$$I_{\mathrm{V}}(\epsilon) \equiv \sum_{i\neq j\in\mathcal{X}^n_{m,0}}(\mathbf{T}_i\cdot\mathbf{T}_j)\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon}\cos\left(\lambda_{ij}\pi\epsilon\right)\left[\frac{1}{\epsilon^2}+\frac{1}{\mathbf{T}_i^2}\frac{\gamma_i}{\epsilon}\right]$$

NSC subtraction scheme

Pole cancellation: $I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon)$

$$\equiv \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij}\, {}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) = 1 - \epsilon \log \eta_{ij} + o(\epsilon^2)$$

$$I_{\mathrm{S+V}} = \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[-\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} K_{ij} + \left(\frac{\mu^2}{2 p_i \cdot p_j} \right)^{\epsilon} \cos(\lambda_{ij} \pi \epsilon) \left(\frac{1}{\epsilon^2} + \frac{1}{\mathbf{T}_i^2} \frac{\gamma_i}{\epsilon} \right) \right]$$

$$= \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \left[-1 + \epsilon \log \eta_{ij} + \left(\frac{\mathcal{E}^2}{E_i E_j} \right)^{\epsilon} \eta_{ij}^{-\epsilon} \cos(\lambda_{ij} \pi \epsilon) \left(1 + \epsilon \frac{\gamma_i}{\mathbf{T}_i^2} \right) \right] + o(\epsilon^0)$$

$$= \frac{1}{\epsilon} \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(L_i + L_j + \frac{\gamma_i}{\mathbf{T}_i^2} \right) + o(\epsilon^0) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) + o(\epsilon^0)$$

NSC subtraction scheme

Pole cancellation: $I_{\mathsf{T}}(\epsilon)$

Compare:

$$I_{\mathsf{S+V}}(\epsilon) = -\frac{1}{\epsilon}\sum_{i\in\mathcal{H}^n_{m,0}}\left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{S+V}} + o(\epsilon)$$

$$I_{\mathsf{C}}(\epsilon) = +\frac{1}{\epsilon}\sum_{i\in\mathcal{H}^n_{m,0}}\left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{C}} + o(\epsilon)$$

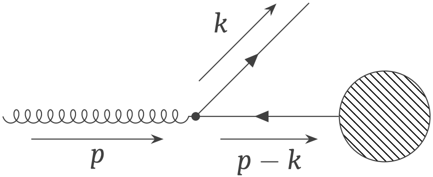
Hence (Devoto et al. 2025):

$$I_{\mathsf{T}}(\epsilon) \equiv I_{\mathsf{S}}(\epsilon) + I_{\mathsf{C}}(\epsilon) + I_{\mathsf{V}}(\epsilon) = I^{(0)}_{\mathsf{T}} + o(\epsilon)$$

NSC SS with massive quarks

Mass-regualtion of soft and collinear limits

Massive partons do not determine soft or collinear singularities:



$$\sim \quad \frac{1}{(p-k)^2 - m^2} = -\frac{1}{2E_p\left(E_k - |\mathbf{k}|\cos\theta\right)}$$

Clearly, no more collinear singularities as $E_k \neq |\mathbf{k}|$ for $m \neq 0$.
The soft limit is non-singular even in the massless case.

NSC SS with massive quarks
Generalized soft operator

- In presence of final-state massive partons, $I_{\mathrm{S}}(\epsilon)$ becomes:

$$I_{\mathrm{S}}(\epsilon) = \frac{1}{2\epsilon}\left(\frac{2\mathcal{E}}{\mu}\right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i,j\in\mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) I_{i,j}(\epsilon)$$

NSC SS with massive quarks

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- $I_{i,j}(\epsilon)$ depends on the nature (massive or massless) of partons i and j;

NSC SS with massive quarks

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- $I_{i,j}(\epsilon)$ depends on the nature (massive or massless) of partons i and j;
- $I_{i,j}(\epsilon)$ is singular as $\sim \epsilon^{-1}$ if at least one parton is massless, otherwise it is ϵ-finite;

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- $I_{i,j}(\epsilon)$ depends on the nature (massive or massless) of partons i and j;
- $I_{i,j}(\epsilon)$ is singular as $\sim\epsilon^{-1}$ if at least one parton is massless, otherwise it is ϵ-finite;
- ϵ^{-2}-poles of $I_{\mathrm{S}}(\epsilon)$ are determined only by massless partons.

NSC SS with massive quarks

Generalized virtual operator

- In presence of final-state massive partons, $I_{\mathrm{V}}(\epsilon)$ becomes (Catani et al. 2001):

$$I_{\mathrm{V}}(\epsilon) := \Re I_1(\epsilon) = \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{|s_{ij}|} \right)^{\epsilon} \left[\mathcal{V}_{i,j}(\epsilon) - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) \right] - \sum_{i \in \mathcal{X}^n_{m,0}} \Gamma_i(\epsilon)$$

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- $\Gamma_i(\epsilon)$ is ϵ^{-1}-singular;
- ϵ^{-2}-poles of $I_{\mathrm{V}}(\epsilon)$ are determined only by massless partons.

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Pole cancellation: generalized pole terms in $I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon)$

It is convenient to group summations as follows:

$$
\begin{aligned}
I_{\mathrm{S+V}}(\epsilon) &= \sum_{i,j \in \mathcal{X}^n_{m,0}} I^{i,j}_{\mathrm{S}}(\epsilon) + \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \tilde{I}^{i,j}_{\mathrm{V}}(\epsilon) - \sum_{i \in \mathcal{X}^n_{m,0}} \Gamma_i(\epsilon) \\
&= \sum_{i \neq j \in \color{blue}{\mathcal{X}^n_{m,0}}} \left[I^{i,j}_{\mathrm{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathrm{V}}(\epsilon) \right] + \sum_{i \in \color{red}{\mathcal{X}^n_{m,\mathrm{m}}}} \left[I^{i,i}_{\mathrm{S}}(\epsilon) - \Gamma_i(\epsilon) \right] - \sum_{i \in \color{blue}{\mathcal{H}^n_{m,0}}} \Gamma_i(\epsilon)
\end{aligned}
$$

all partons — final-state massive partons — all massless partons

$$
\color{blue}{\mathcal{X}^n_{m,0}} = \color{red}{\mathcal{X}^n_{m,\mathrm{m}}} \cup \color{blue}{\mathcal{H}^n_{m,0}}
$$

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&= \underline{\sum_{i \neq j \in \mathcal{X}^n_{m,0}} \left[I^{i,j}_{\mathsf{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon) \right]} + \sum_{i \in \mathcal{X}^n_{m,\mathrm{m}}} \left[I^{i,i}_{\mathsf{S}}(\epsilon) - \Gamma_i(\epsilon) \right] - \sum_{i \in \mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)
\end{aligned}
$$

Then, ϵ^{-2}-singularities are only present in the <u>first sum</u>:

$$
I^{i,j}_{\mathsf{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon) = \chi_{i,j} + \text{Ж}_{i,j} + o(\epsilon)
$$

Laurent expansion: pole terms and ϵ-finite finite reminder.

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Pole cancellation: quadratic pole terms

Quadratic pole terms have a complex general form:

$$\chi_{i,j} \equiv \frac{1}{\epsilon^2}\left(\frac{1}{2}I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)}\right) + \frac{1}{\epsilon}\left(\frac{1}{2}I_{i,j}^{(0)} + \mathcal{V}_{i,j}^{(-1)} - \mathscr{L}_{\mathrm{m}} I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)} \log\frac{\mu^2}{|s_{ij}|}\right)$$

However, with further manipulation, the explicit structure is simple:

$$\chi_{i,j} = \begin{cases} 0 & i,j \text{ massive} \\ \frac{1}{\epsilon}L_j & i \text{ massive}, j \text{ massless} \\ \frac{1}{\epsilon}L_i & i \text{ massless}, j \text{ massive} \\ \frac{1}{\epsilon}\left(L_i + L_j\right) & i,j \text{ massless} \end{cases} \qquad L_k \equiv \log\frac{\mathcal{E}}{E_k}$$

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Pole cancellation: generalized pole terms in $I_{\mathsf{S}}(\epsilon) + I_{\mathsf{V}}(\epsilon)$

It is convenient to group summations as follows:

$$\begin{aligned} I_{\mathsf{S+V}}(\epsilon) &= \sum_{i,j\in\mathcal{X}^n_{m,0}} I_{\mathsf{S}}^{i,j}(\epsilon) + \sum_{i\neq j\in\mathcal{X}^n_{m,0}} \tilde{I}_{\mathsf{V}}^{i,j}(\epsilon) - \sum_{i\in\mathcal{X}^n_{m,0}} \Gamma_i(\epsilon) \\ &= \sum_{i\neq j\in\mathcal{X}^n_{m,0}} \left[I_{\mathsf{S}}^{i,j}(\epsilon) + \tilde{I}_{\mathsf{V}}^{i,j}(\epsilon)\right] + \underline{\sum_{i\in\mathcal{X}^n_{m,\mathrm{m}}} \left[I_{\mathsf{S}}^{i,i}(\epsilon) - \Gamma_i(\epsilon)\right]} - \sum_{i\in\mathcal{H}^n_{m,0}} \Gamma_i(\epsilon) \end{aligned}$$

The <u>second sum</u> is manifestly ϵ-finite:

$$\sum_{i\in\mathcal{X}^n_{m,\mathrm{m}}} \left[C_{\mathrm{F}}\left(\frac{1}{\epsilon} - \frac{1}{\kappa_i}\log\frac{1-\kappa_i}{1+\kappa_i} - 2\mathscr{L}_{\mathrm{m}}\right) - C_{\mathrm{F}}\left(\frac{1}{\epsilon} + \frac{1}{2}\log\frac{m_{Q_i}^2}{\mu^2} - 2\right)\right] \equiv \sum_{i\in\mathcal{X}^n_{m,\mathrm{m}}} Д_i$$

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Pole cancellation: generalized pole terms in $I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon)$

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&= \sum_{i\neq j\in\mathcal{X}^n_{m,0}} \left[I_{\mathrm{S}}^{i,j}(\epsilon) + \tilde{I}_{\mathrm{V}}^{i,j}(\epsilon)\right] + \sum_{i\in\mathcal{X}^n_{m,\mathrm{m}}} \left[I_{\mathrm{S}}^{i,i}(\epsilon) - \Gamma_i(\epsilon)\right] - \underline{\sum_{i\in\mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)}
\end{aligned}
$$

The <u>third sum</u> has isolated ϵ-poles by definition:

$$
\sum_{i\in\mathcal{H}^n_{m,0}} \Gamma_i(\epsilon) = \sum_{i\in\mathcal{H}^n_{m,0}} \left(\frac{1}{\epsilon}\gamma_i + Ш_i\right) \qquad Ш_i \equiv -\delta_{f_i,g}\frac{2}{3}T_{\mathrm{R}}\sum_{\rho=1}^{n_F}\log\frac{m^2_{Q_\rho}}{\mu^2}
$$

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Pole cancellation: generalized integrated counterterms

Puttin geverything together, using colour-conservation we find:

$$I_{\mathsf{S+V}}(\epsilon) = -\frac{1}{\epsilon} \sum_{i\in\mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{S+V}} + o(\epsilon)$$

$$I_{\mathsf{C}}(\epsilon) = +\frac{1}{\epsilon} \sum_{i\in\mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{C}} + o(\epsilon)$$

The total operator $I_{\mathsf{T}} \equiv I_{\mathsf{S}} + I_{\mathsf{C}} + I_{\mathsf{V}}$ is the ϵ-finite, and the integrated counterterms read:

$$\begin{aligned} I^{(0)}_{\mathsf{T}} = &\sum_{i\neq j\in\mathcal{X}^n_{m,0}} (\mathbf{T}_i\cdot\mathbf{T}_j)\text{Ж}_{i,j} + \sum_{i\in\mathcal{X}^n_{m,\mathrm{m}}} \text{Д}_i - \sum_{i\in\mathcal{H}^n_{m,0}} \text{Ц}_i - \sum_{i\in\mathcal{H}^n_{m,0}} \left[2\mathscr{L}_i\left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + 2\mathbf{T}_i^2 L_i^2\right] \\ &+ (N_q + N_{\bar{q}})\,\frac{C_{\mathrm{F}}}{6}\left(39 - 4\pi^2\right) + N_g\left[\frac{C_{\mathrm{A}}}{9}\left(67 - 6\pi^2\right) - \frac{23}{9} T_{\mathrm{R}} n_f\right] \end{aligned}$$

Conclusions

Results and future developments

Main results

- proof of IR-poles cancellation even with generic massive final-state partons
- computation of generalized integrated counterterms

Future developments

- extension to NNLO in the NSC SS
- inclusion of initial-state massive partons
- resummation of massive logarithms
- application to heavy-quark processes

Figure from CMS Report CERN–EP 2025–025. Example Feynman diagram of non-resonant mono-t production at tree-level mediated by a spin-1 boson M, which decays directly into a dark-matter pair $\chi\bar{\chi}$.