

# Quantum Field Theory 1

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**Part I**

**Field Theory**

# Classical Field Theory

## 1.1 Continuous limit

### 1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass  $m \equiv 1$  lie at rest on the  $x$ -axis, with equilibrium positions labelled by  $n \in \mathbb{N}$  and equally spaced by a distance  $a$ . Assuming these atoms are free to vibrate only in the  $x$  direction (longitudinal waves), and denoting the displacement of the atom at position  $n$  as  $\eta_n$ , one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[ \frac{1}{2} \dot{\eta}_n^2 - \frac{\lambda}{2} (\eta_n - \eta_{n-1})^2 \right] \quad (1.1)$$

where  $\lambda$  is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda (\eta_{n+1} - 2\eta_n + \eta_{n-1}) \quad (1.2)$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \quad (1.3)$$

where the dispersion relation is:

$$\omega^2 = 2\lambda (1 - \cos k) \approx \lambda k^2 \quad (1.4)$$

Therefore, in the long-wavelength limit  $k \ll 1$  waves propagate with velocity  $c = \sqrt{\lambda}$ . To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \quad \Rightarrow \quad k_m = \frac{2\pi m}{N}, \quad m = 0, 1, \dots, N-1 \quad (1.5)$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} [\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)}] \quad (1.6)$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjugated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \quad (1.7)$$

In quantum mechanics,  $\eta_n$  and  $\Pi_n$  become operators with canonical commutator  $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$ . Implementing time evolution with the *Heisenberg picture*<sup>1</sup>:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \quad (1.8)$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators*  $\hat{a}(t)$  and  $\hat{a}^\dagger(t)$ , so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} [e^{i(k_m n - \omega_m t)} \hat{a}_m + e^{-i(k_m n - \omega_m t)} \hat{a}_m^\dagger] \quad (1.9)$$

where  $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$  and the  $N^{-1/2}$  ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \quad (1.10)$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[ \frac{1}{2} \hat{\pi}_n^2 + \frac{\lambda}{2} (\hat{\eta}_n - \hat{\eta}_{n-1})^2 \right] = \sum_{m=0}^{N-1} \hbar\omega_m \left( \hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) \quad (1.11)$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m |0\rangle \quad \forall m = 0, 1, \dots, N-1 \quad (1.12)$$

$$|n_0, n_1, \dots, n_{N-1}\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} |0\rangle \quad (1.13)$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2} \sum_{m=0}^{N-1} \hbar\omega_m \quad (1.14)$$

$$E_{n_0, n_1, \dots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \quad (1.15)$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as “particles” and  $n_m$  is the number of phonons in the  $m^{\text{th}}$  normal mode.

<sup>1</sup>Recall that  $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t} \hat{\mathcal{O}}(0) e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$  and  $\frac{d\hat{\mathcal{O}}}{dt} = \frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\mathcal{O}}]$ .

<sup>2</sup>For a harmonic oscillator  $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$ , so  $\frac{d\hat{x}}{dt} = \hat{p}(t)$  and  $\frac{d\hat{p}}{dt} = -\omega^2\hat{x}(t)$  and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) + \hat{a}^\dagger(t)] \quad \hat{p}(t) = -i\omega \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) - \hat{a}^\dagger(t)]$$

Inverting these expressions one finds  $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$  and  $\hat{\mathcal{H}} = \hbar\omega (\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2})$ . The time evolution  $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$  ensures that  $\hat{\mathcal{H}}$  is times-independent.

### 1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits  $a \rightarrow 0$  and  $N \rightarrow \infty$  while keeping the total length  $R \equiv Na$  fixed. In this context, the displacement becomes a field  $\eta(x, t)$  dependent on the continuous real coordinate  $x \in [0, R]$ , therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left( \frac{\partial \eta}{\partial x} \right)^2 \quad \sum_n \longrightarrow \frac{1}{a} \int_0^R dx$$

#### Proposition 1.1.1

In the continuous limit:

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x - x') = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-x')}$$

*Proof.* By direct calculation:

$$a \sum_n f(an) \frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R dx f(x) \delta(x - y)$$

Recalling Eq. 1.10, since  $k_m n = \frac{k_m}{a} na \rightarrow kx$ , symmetrizing  $k_m \in [-\pi, \pi]$  (instead of  $[0, 2\pi]$ ) one finds:

$$\delta(x - x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na} \sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-x')}$$

where integration limits are  $\pm \frac{\pi}{a} \rightarrow \pm \infty$ . □

#### Proposition 1.1.2

The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R dx e^{i(k-k')x} = 2\pi \delta(k - k')$$

By these relations, it can be seen that  $\frac{dk}{2\pi}$  has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between  $k$  and  $k + dk$ , while the interpretation of the divergent  $\delta(0)$  varies: in  $x$  space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in  $k$  space  $2\pi\delta(0)$  is the (hyper-)volume of the system.

In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R dx \left[ \frac{1}{2} \rho_0 (\partial_t \eta)^2 - \frac{\kappa}{2} (\partial_x \eta)^2 \right]$$

where  $\rho_0$  is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting  $\phi(x, t) \equiv \sqrt{\rho_0} \eta(x, t)$  and  $\kappa = c^2 \rho_0$  and adding a pinning term  $\propto \varphi^2$ , the Lagrangian can be written as:

$$L = \int_0^R dx \left[ \frac{1}{2} (\partial_t \phi)^2 - \frac{c^2}{2} (\partial_x \phi)^2 - \frac{m^2 c^4}{2} \phi^2 \right] \quad (1.16)$$

The classical equation of motion of this field yields:

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi - m^2 c^4 \phi \quad (1.17)$$

The solutions of this wave equation can be written as:

$$\phi(x, t) = e^{i(kx - \omega_k t)} \quad (1.18)$$

with dispersion relation:

$$\omega_k^2 = c^2 k^2 + m^2 c^4 \quad (1.19)$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x, t) := \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi(x, t) \quad (1.20)$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R dx \left[ \frac{1}{2} \Pi^2 + \frac{c^2}{2} (\partial_x \phi)^2 + \frac{m^2 c^4}{2} \phi^2 \right] \quad (1.21)$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x, t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \sqrt{\frac{\hbar}{2\omega_k}} \left[ e^{i(kx - \omega_k t)} \hat{a}_k + e^{-i(kx - \omega_k t)} \hat{a}_k^\dagger \right] \quad (1.22)$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi \delta(k - k') \quad (1.23)$$

$$[\hat{\phi}(x, t), \hat{\Pi}(x', t)] = i\hbar \delta(x - x') \quad (1.24)$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{2} \hbar \omega_k \left( \hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) = E_0 + \int_{\mathbb{R}} \frac{dk}{2\pi} \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \quad (1.25)$$

The ground-state energy can be computed from Eq. 1.14, defining  $\text{Vol} := 2\pi \delta(k = 0)$ :

$$E_0 = \text{Vol} \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{2} \hbar \omega_k \quad (1.26)$$

For a strictly continuous system there is no cut-off in the  $k$  integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in  $E_0$  are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

## 1.2 Spacetime symmetries

### 1.2.1 Lie groups

#### Definition 1.2.1: Lie group

A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters  $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$ .

A Lie group can be seen both as a group and as a  $d$ -dimensional differentiable manifold (with coordinates  $\theta_a$ ). WLOG it is always possible to choose  $g(0, \dots, 0) = e$ .

#### Definition 1.2.2: Representation

Given a group  $G$  and a vector space  $V(\mathbb{K})$ , a *representation* of  $G$  on  $V$  is a homomorphism  $\rho : G \rightarrow \text{GL}(V)$ .

Given the isomorphism  $\text{GL}(V) \rightarrow \mathbb{K}^{n \times n}$ , with  $n \equiv \dim_{\mathbb{K}} V$ , it is usual to *de facto* represent  $G$  as matrices acting on elements of  $V$ , i.e.  $\rho : G \rightarrow \mathbb{K}^{n \times n}$ .

#### Theorem 1.2.1

Given a Lie group  $G$  and  $g \in G$  connected with the identity, a representation of degree  $n$  on  $V(\mathbb{C})$  as:

$$\rho(g(\theta)) = e^{i\theta_a T^a} \quad (1.27)$$

where  $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$  are the *generators* of  $G$  on  $V$ .

#### Definition 1.2.3: Lie algebra

Given a Lie group  $G$  with generators  $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$  on  $V(\mathbb{C})$ , its *Lie algebra* is:

$$[T^a, T^b] = i f^{ab}_c T^c \quad (1.28)$$

where  $f^{ab}_c$  are called the *structure constants*.

#### Proposition 1.2.1

The Lie algebra of a Lie group is independent of the representation.

#### Proposition 1.2.2

Any  $d$ -dimensional abelian Lie algebra is isomorphic to the direct sum of  $d$  one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree  $n = 1$ .



**Definition 1.2.4: Casimir operator**

Given a Lie group with generators  $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$  on  $V(\mathbb{C})$ , a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation, Casimir operators are operators proportional to  $\text{id}_V$ , and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

**Proposition 1.2.3**

A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

**1.2.2 Lorentz group**

Consider the group of linear transformations  $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$  on  $\mathbb{R}^{1,3}$  which leave invariant the quantity  $\eta_{\mu\nu} x^\mu x^\nu$ , i.e. the orthogonal group  $O(1,3)$  (with signature  $(+, -, -, -)$ ). The condition that  $\Lambda^\mu{}_\nu$  must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \quad (1.29)$$

This implies that  $\det \Lambda = \pm 1$ : a transformation with  $\det \Lambda = -1$  can always be written as the product of a transformation with  $\det \Lambda = 1$  and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines  $SO(1,3) := \{\Lambda \in O(1,3) : \det \Lambda = 1\}$ .

Writing explicitly the temporal component  $1 = (\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2$ , it is clear that  $(\Lambda^0_0)^2 \geq 1$ . Therefore,  $O(1,3)$  has two disconnected components: the orthochronous component with  $\Lambda^0_0 \geq 1$  and the non-orthochronous component with  $\Lambda^0_0 \leq -1$ . Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a discrete transformation which reverses the sign of the temporal component.

**Definition 1.2.5: Lorentz group**

The *Lorentz group*  $SO^+(1,3)$  is the orthochronous component of  $SO(1,3)$ .

The discrete transformations are factored out of the Lorentz group: these are *parity* and *time reversal*, which can be represented as  $\mathcal{P}^\mu{}_\nu = \text{diag}(+1, -1, -1, -1)$  and  $\mathcal{T}^\mu{}_\nu = \text{diag}(-1, +1, +1, +1)$ . Applying these discrete transformations in all combinations (id,  $\mathcal{P}$ ,  $\mathcal{T}$  and  $\mathcal{PT}$ ) one gets the four disconnected components of  $SO(1,3)$ , which are not simply connected. This means that Lorentz invariance does not include parity and time reversal invariance.

Considering the infinitesimal transformation and applying Eq. 1.29:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad \Rightarrow \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that  $\omega_{\mu\nu}$  has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the  $(x, y)$ ,  $(y, z)$  and  $(z, x)$  planes and the 3 rapidities of hyperbolic rotations in the  $(t, x)$ ,  $(t, y)$  and  $(t, z)$  planes.

**Theorem 1.2.2**

The Lorentz group is a non-compact Lie group.

*Proof.* Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in  $[0, 2\pi)$ , rapidities vary in  $\mathbb{R}$ , so the differentiable manifold associated to  $\text{SO}^+(1, 3)$  is not compact.  $\square$

**1.2.2.1 Lorentz algebra**

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as  $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$ , the generic element  $\Lambda \in \text{SO}^+(1, 3)$  can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \quad (1.30)$$

The  $\frac{1}{2}$  factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a  $n$ -dimensional representation of  $\text{SO}^+(1, 3)$ , both  $[J^{\mu\nu}]^i_j$  and  $[\Lambda]^i_j$  are  $\mathbb{C}^{n \times n}$  matrices ( $\Lambda$  is real): for example, the  $n = 1$  representation acts on *scalars*, which are invariant under Lorentz transformations, so  $J^{\mu\nu} \equiv 0 \forall \mu, \nu = 0, \dots, 3$ .

**4-vectors** The  $n = 4$  representation acts on *contravariant 4-vectors*  $v^\mu$ , which transform according to  $v^\mu \mapsto \Lambda^\mu_\nu v^\nu$ , and *covariant 4-vectors*  $v_\mu$ , which transform according to  $v_\mu \mapsto \Lambda_\mu^\nu v_\nu$ . In this representation, the generators are represented as  $\mathbb{C}^{4 \times 4}$  matrices:

$$[J^{\mu\nu}]^\rho_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma) \quad (1.31)$$

This is an irreducible representation, and the associated Lie algebra  $\mathfrak{so}^+(1, 3)$ , called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \quad (1.32)$$

It is convenient to rearrange the 6 components of  $J^{\mu\nu}$  into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \quad K^i := J^{i0} \quad (1.33)$$

The  $\mathfrak{so}^+(1, 3)$  can then be rewritten as:

$$[J^i, J^j] = i\epsilon^{ijk}J^k \quad [J^i, K^j] = i\epsilon^{ijk}K^k \quad [K^i, K^j] = -i\epsilon^{ijk}J^k \quad (1.34)$$

The first equation defines a  $\mathfrak{su}(2)$  sub-algebra, thus showing that  $J^i$  are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \quad \eta^i := \omega^{i0} \quad (1.35)$$

so that:

$$\Lambda = \exp[-i\boldsymbol{\theta} \cdot \mathbf{J} + i\boldsymbol{\eta} \cdot \mathbf{K}] \quad (1.36)$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts which increase velocities with respect to said frame.

### 1.2.2.2 Tensor Representations

A generic  $(p, q)$ -tensor transforms as:

$$T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mapsto \Lambda^{\mu_1}_{\alpha_1} \dots \Lambda^{\mu_p}_{\alpha_p} \Lambda_{\nu_1}^{\beta_1} \dots \Lambda_{\nu_q}^{\beta_q} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \quad (1.37)$$

The representation of the Lorentz group which acts on  $(p, q)$ -tensors is of degree  $n = 4^{p+q}$ , however it is reducible into the direct product of  $p + q$  4-dimensional representations as of Eq. 1.38.

Moreover, consider the action of the Lorentz group on  $(2, 0)$ -tensors: being  $T^{\mu\nu} \mapsto \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$ , if  $T^{\mu\nu}$  is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as  $T \equiv \eta_{\mu\nu} T^{\mu\nu} \mapsto \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta} = T$ , so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$4 \otimes 4 = 1 \oplus 6 \oplus 9 \quad (1.38)$$

These are irreducible representations which, given a generic tensor  $T^{\mu\nu}$ , act on  $S$ ,  $A^{\mu\nu}$  and  $S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S$  respectively, with  $A^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$  and  $S^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$ .

**Decomposition under rotations** Restricting the action to the  $\text{SO}(3)$  sub-group of  $\text{SO}^+(1, 3)$ , tensors can be decomposed according to irreducible representations of  $\text{SO}(3)$ , which are labelled by the angular momentum  $j \in \mathbb{N}_0$  and are of degree  $n = 2j + 1$ . Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \quad (1.39)$$

A Lorentz scalar  $\alpha$  is a scalar under rotations too, so  $\alpha \in \mathbf{0}$ . A 4-vector  $v^\mu$  is irreducible under the action of  $\text{SO}^+(1, 3)$ , but under  $\text{SO}(3)$  it is decomposed into  $v^0$  and  $\mathbf{v}$ , so  $v^\mu \in \mathbf{0} \oplus \mathbf{1}$ . A  $(2, 0)$ -tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.38: the trace is a scalar, so  $S \in \mathbf{0}$ , while the anti-symmetric part can be written as two spatial vectors  $A^{0i}$  and  $\frac{1}{2}\epsilon^{ijk}A^{jk}$ , so  $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$ . The traceless symmetric part then decomposes as  $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$  under spatial rotations.

Equivalently,  $T^{\mu\nu}$  can be decomposed into  $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$ ,  $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$ ,  $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$  and  $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$ : the formers are a scalar and two spatial vectors associated to  $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$ , while the latter can be decomposed into the trace, which is  $\mathbf{0}$ , the anti-symmetric part, which is  $\mathbf{1}$  ( $\epsilon^{ijk}A^{jk}$ ), and the traceless symmetric part, which is  $\mathbf{2}$ .

#### Example 1.2.1

Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have  $j = 2$  (spin of the graviton).

There are two *invariant tensors* under  $\text{SO}^+(1, 3)$ : the metric  $\eta_{\mu\nu}$ , by Eq. 1.29, and the Levi-Civita symbol  $\epsilon^{\mu\nu\sigma\rho}$ :

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\sigma_\gamma \Lambda^\rho_\delta \epsilon^{\alpha\beta\gamma\delta} = (\det \Lambda) \epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

### 1.2.2.3 Spinorial representations

The Lie algebras  $\mathfrak{su}(2)$  and  $\mathfrak{so}(3)$  are the same, which means that  $SU(2)$  and  $SO(3)$  are indistinguishable by infinitesimal transformations; however, they are globally different, as  $SO(3)$  rotations are periodic by  $2\pi$ , while  $SU(2)$  rotations are periodic by  $4\pi$ : in particular, it can be shown that  $SO(3) \cong SU(2)/\mathbb{Z}_2$ , i.e.  $SU(2)$  is the universal cover of  $SO(3)$ . This means that  $SU(2)$  representations can be labelled by  $j \in \frac{1}{2}\mathbb{N}_0$ , where half-integer spin representations are known as *spinorial representations*: they act on spinors, i.e. objects which change sign under rotations of  $2\pi$  (thus not suitable to represent  $SO(3)$ ).

#### Example 1.2.2

The  $\frac{1}{2}$  representation of  $SU(2)$  is a 2-dimensional representation where  $J^i = \frac{\sigma^i}{2}$ : Pauli matrices satisfy  $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$ , thus the  $\mathfrak{su}(2)$  algebra is satisfied. Denoting the  $m = \pm\frac{1}{2}$  states in the  $\frac{1}{2}$  representation as  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , the Clebsch-Gordan decomposition  $\frac{1}{2} \otimes \frac{1}{2} = \mathbf{0} \oplus \mathbf{1}$  yields the triplet ( $j = 1$ )  $|\uparrow\uparrow\rangle$ ,  $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ ,  $|\downarrow\downarrow\rangle$  and the singlet ( $j = 0$ )  $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ .

#### Proposition 1.2.4

The Lorentz algebra  $\mathfrak{so}^+(1, 3)$  can be decomposed as  $\mathfrak{su}(2) \times \mathfrak{su}(2)$ .

*Proof.* Given the  $\mathfrak{so}^+(1, 3)$  algebra in Eq. 1.33, it is possible to define:

$$\mathbf{J}_\pm := \frac{1}{2} (\mathbf{J} \pm i\mathbf{K})$$

The Lie algebra then becomes:

$$[\mathbf{J}_\pm^i, \mathbf{J}_\pm^j] = i\epsilon^{ijk} \mathbf{J}_\pm^k \quad [\mathbf{J}_\pm^i, \mathbf{J}_\mp^j] = 0$$

These are two commuting  $\mathfrak{so}(2)$  algebras, thus proving the thesis.  $\square$

As observed before, this does not imply that  $SO^+(1, 3)$  is globally equivalent to  $SU(2) \times SU(2)$ : in fact,  $SU(2) \times SU(2)/\mathbb{Z}_2 \cong SO(4)$ , while the universal cover of  $SO^+(1, 3)$  is  $SL(2, \mathbb{C})$ , as it can be shown that  $SO^+(1, 3) \cong SL(2, \mathbb{C})/\mathbb{Z}_2$ .

By Prop. 1.2.4, representations of  $SO^+(1, 3)$  can be labelled by  $(j_-, j_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$ , with each index labelling a representation of  $SU(2)$ : as  $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$ , the  $(j_-, j_+)$  representation contains states with all possible spins  $|j_+ - j_-| \leq j \leq j_+ + j_-$ , and it is a representation of degree  $n = (2j_- + 1)(2j_+ + 1)$ .  $(\mathbf{0}, \mathbf{0})$  is the trivial (scalar) representation, as both  $\mathbf{J}_\pm = \mathbf{0}$  and  $\mathbf{J} = \mathbf{K} = \mathbf{0}$ .

$(\frac{1}{2}, \mathbf{0})$  and  $(\mathbf{0}, \frac{1}{2})$  are 2-dimensional spinorial representations. These representations act on different spinors  $(\psi_L)_\alpha \in (\frac{1}{2}, \mathbf{0})$  and  $(\psi_R)_\alpha \in (\mathbf{0}, \frac{1}{2})$ , with  $\alpha = 1, 2$ , which are called *left-* and *right-handed Weyl spinors*. In  $(\frac{1}{2}, \mathbf{0})$  the generators are  $\mathbf{J}_- = \frac{\sigma}{2}$  and  $\mathbf{J}_+ = \mathbf{0}$ , while in  $(\mathbf{0}, \frac{1}{2})$  they are  $\mathbf{J}_- = \mathbf{0}$  and  $\mathbf{J}_+ = \frac{\sigma}{2}$ , thus one finds  $\mathbf{J}_L = \mathbf{J}_R = \frac{\sigma}{2}$  and  $\mathbf{K}_L = -\mathbf{K}_R = i\frac{\sigma}{2}$ , so that:

$$\psi_L \mapsto \Lambda_L \psi_L = \exp \left[ (-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2} \right] \psi_L \quad (1.40)$$

$$\psi_R \mapsto \Lambda_R \psi_R = \exp \left[ (-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2} \right] \psi_R \quad (1.41)$$

Note that the generators  $K^i$  are not hermitian, as expected from Prop. 1.2.3. Furthermore, note that  $\Lambda_{L,L} \in \mathbb{C}^{2 \times 2}$ , therefore  $\psi_{L,R} \in \mathbb{C}^2$ .

### Proposition 1.2.5

Given  $\psi_L \in (\frac{1}{2}, \mathbf{0})$  and  $\psi_R \in (\mathbf{0}, \frac{1}{2})$ , then  $\sigma^2 \psi_L^* \in (\mathbf{0}, \frac{1}{2})$  and  $\sigma^2 \psi_R^* \in (\frac{1}{2}, \mathbf{0})$ .

*Proof.* Recall that for Pauli matrices  $\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$ , so  $\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$  and:

$$\sigma^2 \psi_L^* \mapsto \sigma^2 (\Lambda_L \psi_L)^* = (\sigma^2 \Lambda_L^* \sigma^2) \sigma^2 \psi_L^* = \Lambda_R \sigma^2 \psi_L^* \Rightarrow \sigma^2 \psi_L^* \in (\mathbf{0}, \frac{1}{2})$$

where  $\sigma^2 \sigma^2 = I_2$  was used. The proof for  $\sigma^2 \psi_R^*$  is analogous.  $\square$

### Definition 1.2.6: Charge conjugation

On Weyl spinors, the *charge conjugation operator* is defined as:

$$\psi_L^c := i\sigma^2 \psi_L^* \quad \psi_R^c := -i\sigma^2 \psi_R^* \quad (1.42)$$

By Prop. 1.2.5, charge conjugation changes transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the  $i$  factor ensures that applying this operator twice yields the identity operator.

$(\frac{1}{2}, \frac{1}{2})$  is a 4-dimensional complex representation: as  $j = 0, 1$ , this representation acts on complex 4-vectors of the form  $((\psi_L)_\alpha, (\xi_R)_\beta) \in \mathbb{C}^4$ , called *Dirac spinors*, and  $\Lambda = \text{diag}(\Lambda_L, \Lambda_R) \in \mathbb{C}^{4 \times 4}$ . To explicit this relation, set  $\psi_R \equiv i\sigma^2 \psi_L^*$ ,  $\xi_L \equiv -i\sigma^2 \xi_R^*$  and  $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ ,  $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$ : it can be shown, then, that  $\xi_R^\dagger \sigma^\mu \psi_R$  and  $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$  are contravariant 4-vectors. Although these 4-vectors are complex by construction, being the matrix  $\Lambda^\mu{}_\nu$  which represents the Lorentz transformation of a 4-vector real, a reality condition  $v_\mu^* = v_\mu$  is Lorentz invariant.

**Parity** Note that  $\mathcal{P}\mathbf{K} = -\mathbf{K}$ , as the velocity of the boost gets reversed, while  $\mathcal{P}\mathbf{J} = \mathbf{J}$ : this means that  $\mathcal{P}\mathbf{J}_\pm = \mathbf{J}_\mp$ , i.e. parity exchanges a  $(\mathbf{j}_-, \mathbf{j}_+)$  representation into a  $(\mathbf{j}_+, \mathbf{j}_-)$  representation. Therefore, a  $(\mathbf{j}_-, \mathbf{j}_+)$  representation of  $SO^+(1, 3)$  is a basis for the representation of the parity transformation iff  $j_- = j_+$ .

### Example 1.2.3

Weyl spinors (separately) are not a basis for a representation of the parity transformation, but Dirac spinors are.

#### 1.2.2.4 Field representations

Given a field  $\phi(x)$ , under a Lorentz transformation  $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$  it transforms as  $\phi(x) \mapsto \phi'(x')$ .

**Scalar fields** A scalar field transforms as:

$$\phi'(x') = \phi(x) \quad (1.43)$$

Consider an infinitesimal transformation  $x'^\rho = x^\rho + \delta x^\rho$ , with  $\delta x^\rho = -\frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}]^\rho_\sigma x^\sigma$  as of Eq. 1.30. Then, by definition,  $\delta\phi \equiv \phi'(x') - \phi(x) = 0$ , which corresponds to the fact that the scalar representation of  $SO^+(1,3)$  is the trivial one ( $J^{\mu\nu} = 0$ ).

However, one can consider the variation at fixed coordinate  $\delta_0\phi \equiv \phi'(x) - \phi(x)$ : while  $\delta\phi$  studies only a single degree of freedom, as the point  $P \in \mathbb{R}^{1,3}$  is kept constant and only  $\phi(P)$  can vary (i.e. the base space is one-dimensional),  $\delta_0\phi$  studies  $\phi(P)$  with  $P$  varying over  $\mathbb{R}^{1,3}$ , thus the base space is now a space of functions, which is infinite-dimensional. Therefore,  $\delta\phi$  provides a finite-dimensional representation of the generators, while  $\delta_0\phi$  an infinite-dimensional one.

To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho \partial_\rho \phi = \frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}]^\rho_\sigma x^\sigma \partial_\rho \phi \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi$$

Recalling Eq. 1.31, the generators can be expressed as:

$$L^{\mu\nu} := i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (1.44)$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As  $p^\mu = i\partial^\mu$  (with signature  $(+, -, -, -)$ ), it is clear that  $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$  is the orbital angular momentum.

**Weyl fields** A left-handed Weyl field transforms as:

$$\psi'_L(x') = \Lambda_L \psi_L(x) \quad (1.45)$$

with  $\Lambda_L$  defined in Eq. 1.40, and similarly for right-handed Weyl fields. The infinite-dimensional representation of the Lorentz generators determined by Weyl spinors can be found as:

$$\begin{aligned} \delta_0\psi_L &\equiv \psi'_L(x) - \psi_L(x) = \psi'_L(x' - \delta x) - \psi_L(x) \\ &= \psi'_L(x') - \delta x^\rho \partial_\rho \psi_L(x) - \psi_L(x) = (\Lambda_L - I_2) \psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x) \end{aligned}$$

The second term yields  $L^{\mu\nu}$ , while the first can be further elaborated by writing:

$$\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \quad (1.46)$$

Thus:

$$\delta_0\psi_L = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_L$$

where the angular momentum separates into the orbital and the spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \quad (1.47)$$

This separation is general:  $L^{\mu\nu}$  is always expressed as in Eq. 1.44, while  $S^{\mu\nu}$  depends on the specific representation. In the scalar representation  $S^{\mu\nu} = 0$ , while in the left/right-handed Weyl representation  $S^{i0} = \pm i\frac{\sigma^i}{2}$ .

**Vector fields** A (contravariant) vector field transforms as:

$$V'^\mu(x') = \Lambda^\mu_\nu V^\nu(x) \quad (1.48)$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the  $(\frac{1}{2}, \frac{1}{2})$  representation.

### 1.2.3 Poincaré group

#### Definition 1.2.7: Poincaré group

The *Poincaré group* is defined as  $\text{ISO}^+(1, 3) := \text{T}^{1,3} \times \text{SO}^+(1, 3)$ , where  $\text{T}^{1,3} \cong \mathbb{R}^{1,3}$  is the group of translations of  $\mathbb{R}^{1,3}$ .

Given a translation  $x^\mu \mapsto x^\mu + a^\mu$ , the associated group element can be written as:

$$T = e^{-ia_\mu P^\mu} \quad (1.49)$$

where  $P^\mu$  is the 4-momentum operator. Clearly translations commute, and so do their generators; on the other hand, as  $\mathbf{P}$  is a vector under rotations, while  $P^0$  (energy) a scalar, one has:

$$[J^i, P^j] = ie^{ijk}P^k \quad [J^i, P^0] = 0$$

These equations uniquely determine the *Poincaré algebra*  $\mathfrak{iso}^+(1, 3)$ :

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [J^{\mu\nu}, J^{\sigma\rho}] &= i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \\ [P^\mu, J^{\rho\sigma}] &= i(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho) \end{aligned} \quad (1.50)$$

It's easy to check that  $[K^i, P^0] = iP^i$ , while  $[J^i, P^0] = [P^i, P^0] = 0$ : given that  $P^0$  generates time translations, linear and angular momentum are conserved quantities, while  $\mathbf{K}$  is not.

#### 1.2.3.1 Field representations

Fields provide an infinite-dimensional representation of the Lorentz group as  $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$ , where  $S^{\mu\nu}$  does not depend on  $x^\mu$ , but only on the spin of the field.

To represent  $P^\mu$  on fields, their transformation law must be specified: all fields are required to be scalars under translations, independently of their spin. This means that, given a generic field  $\phi(x)$ , under a translation  $x' = x + a$  it transforms as  $\phi'(x') = \phi(x)$ , so, under an infinitesimal translation  $x' = x + \varepsilon$ :

$$\begin{aligned} \delta_0\phi &\equiv \phi'(x) - \phi(x) = \phi'(x' - \varepsilon) - \phi(x) = -\varepsilon^\mu \partial_\mu \phi(x) \\ &= e^{-i(-\varepsilon_\mu)P^\mu} \phi'(x') - \phi(x) = (e^{i\varepsilon_\mu P^\mu} - \mathbf{I}) \phi(x) = i\varepsilon_\mu P^\mu \phi(x) \end{aligned}$$

It is clear then that:

$$P^\mu = +i\partial^\mu \quad (1.51)$$

Explicitly,  $P^0 = i\partial_t$  and  $\mathbf{P} = -i\nabla$ . It is trivial to check that these generators obey the Poincaré algebra.

#### 1.2.3.2 Particle representations

The Poincaré group can also be represented using the Hilbert space  $\mathcal{H}$  of a free particle as a basis. Denoting a generic state as  $|\mathbf{p}, s\rangle \in \mathcal{H}$ , where  $\mathbf{p}$  is the particle's momentum and  $s$  collectively labels all other quantum numbers, it is clear that  $\mathcal{H}$  is infinite-dimensional, as  $\mathbf{p}$  is a continuous unbounded variable.

**Theorem 1.2.3: Wigner's theorem**

On the Hilbert space of physical states, any symmetry transformation can be represented by a linear and unitary or anti-linear and anti-unitary operator.

By this theorem, Poincaré transformations can be represented by unitary matrices, i.e.  $\mathbf{J}$ ,  $\mathbf{K}$ ,  $\mathbf{P}$  and  $P^0$  can be represented by hermitian operators. These representations can be labeled by Casimir operators, which for  $\text{ISO}^+(1, 3)$  are easily found as  $P_\mu P^\mu$  and  $W_\mu W^\mu$ , where  $W^\mu$  is the *Pauli-Lubanski operator*:

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}J_{\nu\sigma}P_\rho \quad (1.52)$$

On single-particle states  $P_\mu P^\mu = m^2$ , while  $W_\mu W^\mu$  can be conveniently computed in a particular frame (due to Lorentz invariance). If  $m \neq 0$ , this frame is the rest-frame of the particle:

$$W^\mu = -\frac{m}{2}\epsilon^{\mu\nu\sigma 0}J_{\nu\sigma} = \frac{m}{2}\delta^{\mu i}\epsilon^{ijk}J^{jk} = \delta^{\mu i}mJ^i$$

Therefore, on single-particle states of mass  $m$  and spin  $j$ , the Casimir operator takes the form:

$$W_\mu W^\mu = -m^2 j(j+1) \quad (1.53)$$

If  $m = 0$ , the rest-frame does not exist, but it is possible to choose a frame where  $P^\mu = (\omega, 0, 0, \omega)$ , where  $W^0 = W^3 = \omega J^3$ ,  $W^1 = \omega(J^1 - K^2)$  and  $W^2 = \omega(J^2 + K^1)$ , so that:

$$W_\mu W^\mu = -\omega^2 [(K^2 - J^1)^2 + (K^1 + J^2)^2] \quad (1.54)$$

It is clear that the  $m \rightarrow 0$  limit is not trivial, and massive and massless representation need to be studied separately.

**Massive representations** Restricting to  $m \in \mathbb{R}^+$  ( $m^2 < 0$  states, called tachyons, are excluded), the massive representations are labeled by mass  $m$  and spin  $j$ : in fact, after a Lorentz transformation such that  $P^\mu = (m, \mathbf{0})$ , spatial rotations can still be performed, i.e. the subspace of single-particle states with momentum  $P^\mu = (m, \mathbf{0})$  is still a basis for the representation of  $\text{SU}(2)$  (as spinors must be included too). The group of transformations which leaves invariant a certain choice of  $P^\mu$  is called the *little group*, so  $\text{SU}(2)$  is the little group of massive single-particle states: massive representations are labelled by  $m$  and  $j$ , which means that massive particles of spin  $j$  have  $2j+1$  degrees of freedom.

**Massless representations** The little group for  $P^\mu = (\omega, 0, 0, \omega)$  clearly is  $\text{SO}(2)$ , the group of rotations in the  $(x, y)$  plane generated by  $J^3$ : as for any abelian group, its irreducible representations are one-dimensional, and they are labeled by the eigenvalue  $h$  of  $J^3$ , which represents the angular momentum in the direction of propagation of the particle and is called *helicity*. Helicity can be shown to be quantized as  $h \in \frac{1}{2}\mathbb{Z}_0$  (by topologic considerations on  $\text{ISO}^+(1, 3) \equiv \mathbb{R}^4 \times \text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ ).

As a consequence, massless particles only have one degree of freedom and are characterized by their helicity  $h$ . As  $\text{SO}(2) \equiv \text{U}(1)$ , on a state of helicity  $h$  the little group is represented as:

$$U(\theta) = e^{-ih\theta} \quad (1.55)$$

Although massless particles with opposite helicities are logically two different species of particles, it can be written as  $h = \hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$  (unit vectors), so  $h$  is a pseudoscalar such that  $h \mapsto -h$  under parity: this means that, if the interaction conserves parity,  $h$  and  $-h$  must be symmetric.



**Example 1.2.4**

The electromagnetic and gravitational interactions conserve parity, thus photons and gravitons must be a basis for the representation of both  $\text{ISO}^+(1, 3)$  and parity: photons can have  $h = \pm 1$  (left- and right-handed), while gravitons have  $h = \pm 2$ .

**Example 1.2.5**

Neutrinos only interact via the weak interaction, which does not conserve parity, and in fact the two states  $h = \pm \frac{1}{2}$  are different particles: neutrinos have  $h = -\frac{1}{2}$ , while antineutrinos have  $h = +\frac{1}{2}$ .

## 1.3 Classical equations of motion

Consider a *local field theory* of fields  $\{\phi_i(x)\}_{i \in \mathcal{I}} \equiv \phi(x)$ , where  $x \in \mathbb{R}^{1,3}$  is a point in Minkowski spacetime. Its Lagrangian takes the form:

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.56)$$

where  $\mathcal{L}$  is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int dt L = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.57)$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

**Theorem 1.3.1**

The *stationary action principle*  $\delta \mathcal{S} = 0$  determines the classical equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} = 0 \quad (1.58)$$

*Proof.* Varying Eq. 1.57:

$$\delta \mathcal{S} = \int d^4x \sum_{i \in \mathcal{I}} \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \delta (\partial_\mu \phi_i) \right] = \int d^4x \sum_{i \in \mathcal{I}} \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right] \delta \phi_i = 0$$

□

**Corollary 1.3.1.1**

Two Lagrangians which differ by a total divergence  $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$  yield the same equations of motion.

*Proof.* This is a consequence of Stokes theorem:

$$\int_{\Sigma} d^4x \partial_{\mu} K^{\mu} = \int_{\partial\Sigma} dA n_{\mu} K^{\mu}$$

□

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \quad (1.59)$$

$$\mathcal{H} = \sum_{i \in \mathcal{I}} \Pi_i(x) \partial_0 \phi(x) - \mathcal{L} \quad (1.60)$$

Unlike the Hamiltonian formalism, the Lagrangian formalism keeps Lorentz covariance explicit.