General Relativity

Prof. E. Castorina, a.a. 2024-25

Leonardo Cerasi[1]

GitHub repository: LeonardoCerasi/notes

[1] leo.cerasi@pm.me

Contents

Introduction

Nello studio delle interazioni a distanza si introducono le cosiddette teorie di campo: un campo è un'entità fisica che esiste in ogni punto dello spaziotempo (es: campo elettrico, magnetico, etc...) e che viene modificata dalla presenza di portatori della carica associata al campo.
Nel caso del di una teoria di campo per descrivere la gravità, è necessario un campo gravitazionale che sia influenzato dalla massa. Nel caso Newtoniano il campo gravitazionale $\Phi(\mathbf{r},t)$ è legato alla densità di massa $\rho(\mathbf{r},t)$ da un'equazione di Poisson:

$$\triangle\Phi = 4\pi G\rho \tag{1}$$

dove $G \approx 6.67 \cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ è la costante universale di Newton.
È banale ricavare il campo gravitazionale di una massa puntiforme M: in questo caso $\rho(\mathbf{r}) = M\delta^3(\mathbf{r})$, dunque $\Phi(\mathbf{r}) = -\frac{GM}{r}$. Il caso in cui invece ρ dipende dal tempo è non banale e per essere trattato necessita di un'equazione più generale di Eq. 1: l'equazione di campo di Einstein.

Analogie e differenze con l'Elettromagnetismo Superficialmente, il problema della generalizzazione relativistica della gravitazione potrebbe sembrare analogo a quello dell'elettromagnetismo: entrambe le forze, nel caso stazionario, sono governate da una legge proporzionale a r^{-2} ed entrambi i campi sono determinati da equazioni di Poisson in cui cambia solo la costante dimensionale.
La differenza tra le due teorie di campo, però, sta proprio nella descrizione matematica delle sorgenti che subentrano nelle equazioni di Poisson: nel caso dell'Elettromagnetismo, in regime stazionario il campo elettromagnetico è determinato dalla densità di carica ρ_e e dalla densità di corrente $\mathbf{J}$, e per avere una descrizione relativistica bisogna combinarle in una densità di corrente quadrivettoriale $j^\mu = (c\rho_e, \mathbf{J})$ (si può vedere che ρ_e trasforma come una componente temporale poiché $\rho_e \sim \mathrm{Vol}_3^{-1} \sim (\mathrm{Vol}_4/ct)^{-1} \sim ct$, dato che il quadrivolume è un invariante di Lorentz): ciò risulta naturalmente in un potenziale quadrivettoriale $A^\mu = (\phi/c, \mathbf{A})$.
D'altra parte, per quanto riguarda la gravitazione, bisogna ricordare l'uguaglianza relativistica tra massa energia; inoltre, a differenza della carica elettrica, l'energia non è un invariante relativistico, ma è la componente temporale del quadrivettore impulso: in particolare, a generare il campo gravitazionale sono la densità di energia ρ e la densità di momento ρ_p, alle quali sono associate una densità di corrente di energia $\mathbf{j}$ ed una densità di corrente di momento $\mathbf{T}^i$ per ciascuna componente ρ_p^i. Risulta evidente che l'equazione relativistica che descrive il campo gravitazionale sia notevolmente più complicata di quella del campo elettromagnetico, poiché le sorgenti non sono descritte da un quadrivettore, bensì da un tensore, il tensore energia-impulso:

$$T^{\mu\nu} \sim \begin{bmatrix} \rho c & \rho_p c \\ \mathbf{j} & \mathrm{T} \end{bmatrix} \tag{2}$$

Naturalmente, anche il potenziale gravitazionale sarà un tensore $h_{\mu\nu}$, ed il potenziale Newtoniano sarà $h_{00} \sim \Phi$.

Scala della Relatività Generale Tramite le costanti fondamentali G e c è possibile associare ad una massa M una sua lunghezza caratteristica, detta raggio di Schwarzschild:

$$R_s := \frac{2GM}{c^2} \tag{3}$$

Le correzioni relativistiche alla teoria della gravitazione sono determinate dal parametro R_s/r e, nella maggior parte delle situazioni, sono trascurabili: basti calcolare che per la Terra $R_s \approx 10^{-2}$ m, mentre il suo raggio è $R_T = 6 \cdot 10^6$ m, dunque sulla superficie terrestre le correzioni relativistiche alla gravità Newtoniana sono dell'ordine di 10^{-8}.
Gli effetti relativistici diventano importanti quando si considerano oggetti compatti come stelle di neutroni e buchi neri.

Part I

Il Principio d'Equivalenza

Chapter 1

Geodetiche

Nelle teorie classiche di campo vengono considerati due oggetti distinti: le particelle e i campi. I campi determinano il moto delle particelle, mentre le particelle determinano le oscillazioni dei campi.

1.1 Particelle non-relativistiche

Per descrivere il moto di una particella tra due punti fissati $\mathbf{x}(t_1) \equiv \mathbf{x}_1$ e $\mathbf{x}(t_2) \equiv \mathbf{x}_2$ si studia l'azione S associata alla traiettoria $\mathbf{x}(t)$, definita come:

$$S\left[\mathbf{x}(t)\right] := \int_{t_1}^{t_2} dt\, L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \tag{1.1}$$

dove L è la lagragiana che descrive la particella.
La traiettoria percorsa dalla particella, per il principio di minima azione, è quella che estremizza S, ovvero tale per cui $\delta S = 0\ \forall \delta\mathbf{x}(t) : \delta\mathbf{x}(t_1) = \delta\mathbf{x}(t_2) = 0$; esplicitando:

$$\begin{aligned}\delta S = \int_{t_1}^{t_2} dt\, \delta L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) &= \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial \dot{x}^i}\delta \dot{x}^i\right) \\ &= \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)\delta x^i + \left[\frac{\partial L}{\partial \dot{x}^i}\delta x^i\right]_{t_1}^{t_2}\end{aligned} \tag{1.2}$$

dove si è usata la convenzione di somma di Einstein.
Si vede subito che il termine di bordo è nullo, dunque estremizzare l'azione equivale alle equazioni di Eulero-Lagrange:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0 \tag{1.3}$$

1.1.1 Equazione geodetica

In generale, il moto di una particella libera su una generica varietà differenziale è descritto dalla lagrangiana $L = \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$; bisogna dunque tener conto della metrica della varietà considerata:

$$L = \frac{m}{2} g_{ij}(x)\dot{x}^i\dot{x}^j \tag{1.4}$$

dove x rappresenta collettivamente tutte le coordinate x^i sulla varietà. Si ricordi che, per una varietà reale n-dimensionale, $g_{ij} \in \mathbb{R}^{n\times n}$ è una matrice reale simmetrica.

Le equazioni di Eulero-Lagrange diventano dunque:

$$\frac{m}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j - \frac{d}{dt}\left(mg_{ik}\dot{x}^i\right) = 0 \tag{1.5}$$

Espandendo il secondo termine:

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j - \frac{\partial g_{ik}}{\partial x^j}\dot{x}^i\dot{x}^j - g_{ik}\ddot{x}^i = 0 \tag{1.6}$$

Del termine $g_{ik,j} - \frac{1}{2}g_{ij,k}$, essendo contratto con un fattore simmetrico $\dot{x}^i\dot{x}^j$, sopravvive solo la parte simmetrica rispetto agli indici i e j, ovvero:

$$g_{ik}\ddot{x}^i + \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j = 0 \tag{1.7}$$

A questo punto, si contrae per la metrica inversa g^{lk}, che per definizione soddisfa $g^{lk}g_{ik} = \delta^l_i$, così da ottenere (rinominando gli indici):

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \tag{1.8}$$

dove è stato definito il simbolo di Christoffel:

$$\Gamma^i_{jk} := \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \tag{1.9}$$

Questa equazione del moto è nota come equazione geodetica e le sue soluzioni sono dette geodetiche.

1.2 Particelle relativistiche

È possibile estendere la meccanica lagrangiana allo spaziotempo di Minkowski $\mathbb{R}^{1,3}$, descritto dalla metrica:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \tag{1.10}$$

Dato che questa metrica non è definita positiva, è possibile classificare due punti deparati da una distanza infinitesima $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ in base al segno di ds^2: se $ds^2 < 0$ si dicono timelike-separated, se $ds^2 > 0$ spacelike-separated e se $ds^2 = 0$ lightlike-separated (o null).
A differenza del caso classico, in cui l'orbita è parametrizzata dal tempo (che è assoluto), nel caso relativistico essa deve essere parametrizzata da un generico $\sigma \in \mathbb{R}$ monotono crescente lungo la traiettoria.
In ambito relativistico, il principio di minima azione ha un'interpretazione geometrica: la traiettoria deve estremizzare la distanza tra due punti dello spaziotempo. Di conseguenza, dato che una particella di massa m deve seguire una traiettoria timelike, si definisce l'azione come:

$$S = -mc\int_{x_1}^{x_2}\sqrt{-ds^2} = -mc\int_{\sigma_1}^{\sigma_2}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} \tag{1.11}$$

Il coefficiente è necessario per rendere l'azione dimensionalmente omogenea con $\hbar$.
L'azione così definita presenta due simemtrie:

1. invarianza di Lorentz: l'azione è invariante per $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$, con $\Lambda : \Lambda^\mu{}_\rho\eta_{\mu\nu}\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$;

2. invarianza per riparametrizzazioni: essendo σ un parametro arbitrario, è normale che l'azione non dipenda dalla sua scelta, infatti se si riparametrizza con una funzione monotona $\tilde{\sigma}(\sigma)$ si ha:

$$\tilde{S} = -mc \int_{\tilde{\sigma}_1}^{\tilde{\sigma}_2} d\tilde{\sigma} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tilde{\sigma}} \frac{dx^\nu}{d\tilde{\sigma}}} = -mc \int_{\sigma_1}^{\sigma_2} d\sigma \frac{d\tilde{\sigma}}{d\sigma} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \left(\frac{d\sigma}{d\tilde{\sigma}} \right)^2} = S \tag{1.12}$$

Grazie all'invarianza per riparametrizzazioni, il valore dell'azione tra due punti dello spaziotempo assume un significato ben preciso, il tempo proprio, ovvero il tempo misurato dalla particella in moto stessa:

$$\tau(\sigma) = \frac{1}{c} \int_0^\sigma d\sigma' \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma'} \frac{dx^\nu}{d\sigma'}} \tag{1.13}$$

Una conseguenza dell'identificazione tra azione e tempo proprio è che il principio di minima azione richiede che la traiettoria estremizzi il tempo proprio. È anche possibile riparametrizzare l'azione col tempo proprio, essendo questo una funzione monotona crescente lungo la traiettoria.

1.2.1 Equazione geodetica

Nel caso relativistico su una varietà differenziabile generica, la lagrangiana di una particella libera è:

$$L = \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} \tag{1.14}$$

Dunque, le equazioni di Eulero-Lagrange diventano:

$$-\frac{1}{2L} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \dot{x}^\mu \dot{x}^\nu - \frac{d}{d\sigma} \left(-\frac{1}{L} g_{\rho\nu} \dot{x}^\nu \right) = 0 \tag{1.15}$$

L'unica differenza con Eq. 1.5 è che $L = L(\sigma)$, dunque si trova un'equazione analoga all'Eq. 1.7 ma con un termine aggiuntivo:

$$g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\mu\nu}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\mu} \right) \dot{x}^\nu \dot{x}^\rho = \frac{1}{L} \frac{dL}{d\sigma} g_{\mu\rho} \dot{x}^\rho \tag{1.16}$$

È possibile annullare il termine $\frac{dL}{d\sigma}$ con un'opportuna scelta di parametrizzazione. Dall'Eq. 1.13 si vede che:

$$c \frac{d\tau}{d\sigma} = L(\sigma) \tag{1.17}$$

Dunque, riparametrizzando con $\tau(\sigma)$:

$$L(\tau) = \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} = \frac{d\sigma}{d\tau} L(\sigma) = c \tag{1.18}$$

In generale, qualsiasi riparametrizzazione con $\tilde{\tau} = a\tau + b$ (parametri affini della worldline) porta ad avere una lagrangiana costante.
Ricordando la definizione di connessione affine in Eq. 1.9, si trova l'*equazione geodetica*:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} \tag{1.19}$$

1.2.2 Momento coniugato

La differenza sostanziale tra lo spazio euclideo e lo spaziotempo di Minkowski è che, mentre nello spazio euclideo un corpo può rimanere fermo, nello spaziotempo nessun corpo può fermarsi nella direzione temporale. Questo fatto deve essere rispecchiato dal momento della particella:

$$p_\mu = \frac{dL}{dx^\mu} = \frac{d}{dx^\mu}\left(-mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\right) = mc\frac{\eta_{\mu\nu}\dot{x}^\nu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} = -\frac{m^2c^2}{L}\eta_{\mu\nu}\dot{x}^\nu \tag{1.20}$$

Non tutte le componenti del 4-momento sono indipendenti:

$$p^2 = p^\mu p_\mu = \frac{m^4c^4}{L^2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m^2c^2 \tag{1.21}$$

$$\left(p^0\right)^2 = \mathbf{p}^2 + m^2c^2 \tag{1.22}$$

Di conseguenza, si ha sempre $\left(p^0\right)^2 > 0$.
Si noti che riparametrizzando la worldline col tempo proprio, dato che $\frac{d\tau}{d\sigma} = -\frac{L}{mc^2}$:

$$p^\mu = m\frac{d\sigma}{d\tau}\frac{dx^\mu}{d\sigma} = m\frac{dx^\mu}{d\tau} \tag{1.23}$$

La non-indipendenza di una delle componenti del 4-momento è naturale: da una descrizione classica del sistema risultano tre gradi di libertà $x^i(t)$, dunque passando ad una descrizione relativistica non può risultare un grado di libertà in più. Ciò è legato all'invarianza per riparametrizzazione: risolvendo le equazioni del moto si trovano le componenti della traiettoria $x^\mu = x^\mu(\sigma)$, ma il parametro σ non può rappresentare dell'informazione sul sistema, dunque una delle quattro equazioni del moto va utilizzata per eliminare la dipendenza da σ, riducendo di nuovo a tre i gradi di libertà.

1.2.3 Interazioni

Dall'invarianza per riparametrizzazione, è possibile scegliere come parametro $\sigma = t$ il tempo misurato in un qualunque RF inerziale; considerando una particella libera nello spaziotempo di Minkowski, l'azione in Eq. 1.11 diventa:

$$S = -mc^2\int_{t_0}^{t_1} dt\,\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}} \tag{1.24}$$

In questa forma, è chiara la presenza di soli tre gradi di libertà dovuti a $\mathbf{x}(t)$.

1.2.3.1 Elettromagnetismo

Analogamente al caso classico, per rappresentare l'interazione elettromagnetica è necessario aggiungere un termine potenziale all'azione. Il problema è che la semplice aggiunta di $\int d\sigma V(\mathbf{x})$ non soddisfa l'invarianza per riparametrizzazione: per soddisfarre questo requisito, è necessario individuare un potenziale che cancelli il fattore Jacobiano derivante dalla trasformazione della misura $d\sigma$. Un'opzione è considerare un termine lineare in $\dot{x}^\mu$, dunque per l'invarianza di Lorentz è necessario che l'indice μ sia contratto:

$$S = \int_{\sigma_1}^{\sigma_2} d\sigma\left[-mc^2\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{x^\nu}{d\sigma}} - qA_\mu(x)\dot{x}^\mu\right] \tag{1.25}$$

dove q è la carica associata all'interazione e $A_\mu(x)$ è il suo potenziale quadrivettoriale.
Scrivendo $A_\mu(x) = (\phi(x)/c, \mathbf{A}(x))$, l'azione in Eq. 1.25 descrive l'interazione elettromagnetica; ciò diventa evidente riparametrizzando con $\sigma = t$:

$$S = \int_{t_0}^{t_1} dt \left[-mc^2 \sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}} - q\phi(x) - q\mathbf{A}(x) \cdot \mathbf{x} \right] \tag{1.26}$$

1.2.3.2 Gravitazione

Per descrivere l'interazione gravitazione è necessario considerare un'azione generalizzata del tipo:

$$S = \int_{t_0}^{t_1} dt \left[-mc^2 \sqrt{1 + \frac{2\Phi(\mathbf{x})}{c^2} - \frac{\dot{\mathbf{x}}^2}{c^2}} \right] \tag{1.27}$$

Nel limite non-relativistico $\dot{\mathbf{x}}^2 \ll c^2$ e $2\Phi(\mathbf{x}) \ll c^2$, dunque approssimando al prim'ordine:

$$S = \int_{t_0}^{t_1} dt \left[-mc^2 + \frac{m}{2}\dot{\mathbf{x}}^2 - m\Phi(\mathbf{x}) \right] \tag{1.28}$$

Il primo termine (l'energia a riposo della particella) non ha effetti sull'azione poiché è costante, mentre gli altri termini descrivono il moto non-relativistico di una particella in un campo gravitazionale $\Phi(\mathbf{x})$. Il termine $1 + 2\Phi(\mathbf{x})/c^2$ in Eq. 1.27 deriva dalla componente η_{00} della metrica, dunque si osserva che la metrica deve dipendere da x, ovvero la descrizione dell'interazione gravitazionale introduce uno spaziotempo curvo. La condizione che deve soddisfarre la metrica in un weak gravitational field è:

$$g_{00}(x) \approx -\left(1 + \frac{2\Phi(x)}{c^2}\right) \tag{1.29}$$

con $\Phi(x)$ il campo gravitazionale Newtoniano.

1.3 Principio di Equivalenza

Come si evince dall'Eq. 1.28, la "carica" dell'interazione gravitazionale è porprio la massa della particella: questo fatto viene definito *weak equivalence principle* (WEP) ed è solitamente espresso tramite l'uguaglianza tra la massa inerziale e la massa gravitazionale:

$$m_{\mathrm{i}} = m_{\mathrm{g}} \tag{1.30}$$

Questo è un fatto sperimentale misurato con una precisione dell'ordine di 10^{-13}.

1.3.1 Metrica di Kottler-Möller

Una conseguenza del WEP è l'indistinguibilità tra un'accellerazione costante ed un campo gravitazionale costante: ciò può essere visto in maniera analitica.
Considerando una particella di massa m con accellerazione costante $\mathbf{a} = a\hat{\mathbf{e}}_x$ in un RF inerziale $\mathcal{O}$, dalla relatività speciale si vede subito che la traiettoria non può essere $x(t) = \frac{1}{2}at^2$, poiché la velocità eccederebbe c; bisogna invece ricordare la composizione relativistica delle velocità:

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \tag{1.31}$$

È possibile definire la rapidità $\varphi : v = c \tanh \varphi$, così da poter riscrivere la composizione delle velocità come $\varphi = \varphi_1 + \varphi_2$.
Un'accelerazione costante significa che la rapidità della particella aumenta linearmente rispetto al tempo proprio, ovvero $\varphi(\tau) = a\tau/c$, quindi:

$$v(\tau) = \frac{dx}{d\tau} = c \operatorname{sech}\left(\frac{a\tau}{c}\right) \tag{1.32}$$

La relazione tra il tempo misurato nel RF inerziale ed il tempo proprio è:

$$\frac{dt}{d\tau} = \gamma(\tau) = \sqrt{\frac{1}{1 - v(\tau)^2/c^2}} = \cosh\left(\frac{a\tau}{c}\right) \quad \Longrightarrow \quad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right) \tag{1.33}$$

con costante d'integrazione tale per cui $\tau = 0$ corrisponda a $t = 0$. Per quanto riguarda la traiettoria:

$$x(\tau) = \frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a} \tag{1.34}$$

con costante d'integrazione tale per cui $x(0) = 0$. Si trova dunque un'iperbole nello spaziotempo:

$$\left(x + \frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2} \tag{1.35}$$

con asintoti $ct = \pm\left(x + c^2/a\right)$ per $\tau \to \pm\infty$. Si può mostrare che la trasformazione tra le coordinate (t, x) nel RF inerziale e quelle (τ, ρ) solidali alla particella è data da:

$$\begin{aligned} ct &= \left(\rho + \frac{c^2}{a}\right)\sinh\left(\frac{a\tau}{c}\right) \\ x &= \left(\rho + \frac{c^2}{a}\right)\cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a} \end{aligned} \tag{1.36}$$

Infatti, l'orbita della particella è giustamente descritta da $\rho = 0$. Inoltre, si vede che le coordinate (τ, ρ) non ricoprono tutto lo spaziotempo di Minkowski (t, x): questo dimostra che ci sono delle regioni dello spaziotempo causalmente disconnesse dalla particella.
Ricordando che $ds^2 = -c^2dt^2 + d\mathbf{r}^2$, sostituendo l'Eq. 1.36 si ottiene la *metrica di Kottler-Möller*:

$$ds^2 = -\left(1 + \frac{a\rho}{c^2}\right)^2 c^2 d\tau^2 + d\rho^2 + dy^2 + dz^2 \tag{1.37}$$

Mentre la parte spaziale rimane piatta, si vede che $g_{00} = g_{00}(x)$; inoltre, nel caso sub-relativistico:

$$g_{00} \approx -\left(1 + \frac{2a\rho}{c^2}\right) \tag{1.38}$$

Definendo $\Phi(\rho) = a\rho$, si trova proprio la condizione 1.29: questo è proprio l'assero del WEP, poiché un'accellerazione costante dà una metrica indistinguibile da quella di un campo gravitazionale costante (sub-relativistico).

1.3.1.1 Principio di equivalenza di Einstein

Dal WEP deriva il fatto che un campo gravitazionale costante può essere annullato dalla scelta di un particolare RF, il free-fall RF.
Il *principio di equivalenza di Einstein* è una generalizzazione del WEP: esso afferma che esiste sempre un local RF in cui gli effetti di un qualsiasi campa gravitazionali sono localmente annullati. Formalmente, ciò equivale a dire che la metrica $g_{\mu\nu}$ è sempre localmente approssimabile con la metrica di Minkowski $\eta_{\mu\nu}$.
Gli effetti di un campo gravitazionale non uniforme diventano evidenti quando è possibile svolgere misurazioni su una regione estesa di spazio. Si consideri ad esempio un osservatore confinato in un cubo chiuso in free-fall verso la Terra: non esiste alcun esperimento locale in grado di distinguere se l'osservatore stia fluttuando nello spazio oppure sia in free-fall, bensì è necessario un esperimento non-locale; un esempio di questo tipo di esperimenti consiste nel lasciare libere due masse test (quindi non influenzate vicendevolmente per interazione gravitazionale) separate da una certa distanza: se il cubo sta fluttuando nello spazio, le masse rimaranno nella loro posizione iniziale per inerzia, mentre se esso è in free-fall ciascuna massa sarà attratta verso il centro della Terra, dunque il loro spostamento avrà non solo una componente verticale (rispetto alla caduta), ma anche una orizzontale, per quanto piccola: le masse si sposteranno dunque l'una verso l'altra per effetto di una *tidal force*, uno dei principali fattori discriminanti dei campi gravitazionali non uniformi.

1.3.2 Gravitational time dilation

In condizioni di campo gravitazionale debole si ha $g_{00}(x) = 1 + \frac{2\Phi(x)}{c^2}$. Considerando il campo gravitazionale di un corpo sferico di massa M uniforme, $\Phi(r) = -\frac{GM}{r}$, si ha che un osservatore ad una distanza fissa r misurerà degli intervalli di tempo dati da:

$$d\tau^2 = g_{00}dt^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2 \tag{1.39}$$

Dunque, definendo t il tempo misurato da un osservatore a $r \to \infty$, l'osservatore a r misurerà:

$$T(r) = t\sqrt{1 - \frac{2GM}{rc^2}} \tag{1.40}$$

ovvero il tempo scorre più lentamente vicino ad un corpo massivo.
È anche possibile mettere in relazione i tempi misurati a due distanze finite r_1 ed $r_2 = r_1 + \Delta r$:

$$\begin{aligned} T_2 = t\sqrt{1 - \frac{2GM}{(r_1 + \Delta r)c^2}} &\approx t\sqrt{1 - \frac{2GM}{r_1 c^2} + \frac{2GM\Delta r}{r_1^2 c^2}} \\ &\approx t\sqrt{1 - \frac{2GM}{r_1 c^2}}\left(1 + \frac{GM\Delta r}{r_1^2 c^2}\right) = T_1\left(1 + \frac{GM\Delta r}{r_1^2 c^2}\right) \end{aligned} \tag{1.41}$$

dove si è assunto che $\Delta r \ll r_1$ e $2GM \ll r_1 c^2$. Ad esempio, con un per un dislivello di 10^3 m sul livello del mare ($\sim 6 \cdot 10^6$ m) si ha una differenza di 10^{-12} s, confermato sperimentalmente.

1.3.2.1 Gravitational redshift

Un'importante conseguenza della gravitational time dilation è il gravitational redshift.
Sempre in condizioni di campo debole, si consideri un segnale a distanza r_1 che si ripete ad intervalli

ΔT_1: un osservatore a r_2 misurerà:

$$\Delta T_2 = \sqrt{\frac{1 + 2\Phi(r_2)/c^2}{1 + 2\Phi(r_1)/c^2}} \Delta T_1 \approx \left(1 + \frac{\Phi(r_2) - \Phi(r_1)}{c^2}\right) \Delta T_1 \tag{1.42}$$

Dato che $\omega \sim T^{-1}$, si ha:

$$\omega_2 \approx \left(1 + \frac{\Phi(r_2) - \Phi(r_1)}{c^2}\right)^{-1} \omega_1 \tag{1.43}$$

Dato che $\Phi(r) \sim r^{-1}$ (nel caso considerato), se $r_2 > r_1$ si ha $\omega_2 < \omega_1$ (redshift), mentre se $r_2 < r_1$ si ha $\omega_2 > \omega_1$ (blueshift).

Part II

Differential Geometry

Chapter 2

Manifolds

2.1 Topological spaces

Definition 2.1.1. The *topology* $\mathcal{T}$ of a set X is a family of subsets of X, i.e. $\mathcal{T} \subseteq \mathcal{P}(X)$, defined as *open sets*, with the following properties:

1. $\emptyset, X \in \mathcal{T}$;

2. $O_\alpha, O_\beta \in \mathcal{T} \Rightarrow O_\alpha \cap O_\beta \in \mathcal{T}$;

3. $\{O_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ (I arbitrary index set) $\Rightarrow \bigcup_{\alpha \in I} O_\alpha \in \mathcal{T}$.

Definition 2.1.2. A *topological space* M is a set of points, endowed with a topology $\mathcal{T}$.

Definition 2.1.3. Given a topological space $(M, \mathcal{T})$, $O \in \mathcal{T}$ is a *neighbourhood* of a point $p \in M$ if $p \in O$.

Definition 2.1.4. A topological space $(M, \mathcal{T})$ is *Hausdorff* if $\forall p, q \in M\, \exists O_1, O_2 \in \mathcal{T}$ neighbourhoods of p and q respectively such that $O_1 \cap O_2 = \emptyset$.

Definition 2.1.5. A *homeomorphism* between two topological spaces $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ is a bijective map $f : M_1 \to M_2$ which is bicontinuous, i.e. both f and f^{-1} are continuous: f is continuous if $O \in \mathcal{T}_2 \Rightarrow f^{-1}(O) \in \mathcal{T}_1$.

2.2 Differentiable Manifolds

Definition 2.2.1. An n-dimensionale *differentiable manifold* $\mathcal{M}$ is a Hausdorff topological space such that:

1. $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall p \in \mathcal{M}\, \exists O \in \mathcal{T}(\mathcal{M}) : p \in O \wedge \exists \varphi : O \to U \in \mathcal{T}(\mathbb{R}^n)$ homeomorphism;

2. given $O_\alpha, O_\beta \in \mathcal{T}(\mathcal{M}) : O_\alpha \cap O_\beta \neq \emptyset$, the corresponding maps $\varphi_\alpha : O_\alpha \to U_\alpha, \varphi_\beta : O_\beta \to U_\beta$ must be *compatible*, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(O_\alpha \cap O_\beta) \to \varphi_\beta(O_\alpha \cap O_\beta)$ and its inverse must be smooth (of $\mathcal{C}^\infty$ class).

The maps φ_α are called *charts* and a collection of compatible charts is called an *atlas*: a *maximal atlas* $\mathcal{A}$ is an atlas such that $\bigcup_{\alpha \in I} O_\alpha = \mathcal{M}$. Two atlases are compatible if each chart of one atlas is compatible with every chart of the other: they define the same *differentiable structure* on the manifold.
Each chart φ_α provides a coordinate system on the region O_α: $\varphi_\alpha(p) = (x^1(p), \dots, x^\mu(p), \dots, x^n(p))$. The *transition functions* $\varphi_\beta \circ \varphi_\alpha^{-1}$ are therefore coordinate transformations on overlapping regions.

Example 2.2.1. The n-sphere $\mathbb{S}^n$ is a differentiable manifold.

Example 2.2.2. To define a differentiable structure on $\mathcal{S}^1$ an atlas of two charts is needed: the standard parametrization $\theta \in [0, 2\pi)$ is not a well-defined chart because $[0, 2\pi)$ is not an open set in the Euclidean topology of $\mathbb{R}$, therefore the elimination of a point is necessary; usually, the two charts of the atlas are defined by $\theta_1 \in (0, 2\pi)$, excluding $(1, 0)$ (in the embedding space $\mathbb{R}^2$), and $\theta_2 \in (-\pi, \pi)$, excluding $(-1, 0)$: they are evidently compatible, thus they form a maximal atlas.

2.2.1 Maps between manifolds

Locally mapping $\mathcal{M}$ to $\mathbb{R}^n$ allows to import concepts of Analysis from $\mathbb{R}^n$ to $\mathcal{M}$.

Definition 2.2.2. A function $f : \mathcal{M} \to \mathbb{R}$ on a differentiable manifold $(\mathcal{M}, \mathcal{A})$ is *smooth* if $f \circ \varphi_\alpha^{-1} : U_\alpha \to \mathbb{R}$ is smooth for all charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$.

Definition 2.2.3. A map $f : \mathcal{M} \to \mathcal{N}$ between two differentiable manifolds $(\mathcal{M}, \mathcal{A}_1), (\mathcal{N}, \mathcal{A}_2)$ is *smooth* if $\psi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1} : U_{\alpha_1} \to V_{\alpha_2}$ is smooth for all charts $(U_{\alpha_1}, \varphi_{\alpha_1}) \in \mathcal{A}_1, (V_{\alpha_2}, \varphi_{\alpha_2}) \in \mathcal{A}_2$.

Definition 2.2.4. A *diffeomorphism* between two differentiable manifolds $\mathcal{M}, \mathcal{N}$ is a smooth homeomorphism $f : \mathcal{M} \to \mathcal{N}$.

Proposition 2.2.1. *If $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic, then* $\dim_{\mathbb{R}} \mathcal{M} = \dim_{\mathbb{R}} \mathcal{N}$.

Example 2.2.3. $\mathbb{S}^7$ can be covered by multiple incompatible atlases: the resulting manifolds are homeomorphic but not diffeomorphic.

Example 2.2.4. $\mathbb{R}^n$ has a unique differentiable structure for all $n \in \mathbb{N}$, except for $n = 4$: $\mathbb{R}^4$ can be covered by infinitely-many incompatible atlases.

2.3 Tangent spaces

The notions of calculus can be defined on a differential manifold $(\mathcal{M}, \mathcal{A})$ via tangent spaces.

Definition 2.3.1. The derivative of a function $f : \mathcal{M} \to \mathbb{R}$ at a point $p \in \mathcal{M}$, covered by the chart (φ, U), is defined as:

$$\left. \frac{\partial f}{\partial x^\mu} \right|_p := \left. \frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu} \right|_{\varphi(p)} \tag{2.1}$$

Evidently, this definition depends on the choise of coordinates x^μ, thus it depends on the chart.

2.3.1 Tangent vectors

Definition 2.3.2. The set of all smooth functions on $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$.

Definition 2.3.3. A *tangent vector* to $\mathcal{M}$ in $p \in \mathcal{M}$ is an operator $X_p : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ such that:

1. $X_p(f+g) = X_p(f) + X_p(g)\,\forall f, g \in \mathcal{C}^\infty(\mathcal{M})$;
2. $X_p(f) = 0$ for all constant functions;
3. $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)\,\forall f, g \in \mathcal{C}^\infty(\mathcal{M})$.

Proposition 2.3.1. $X_p(\alpha f) = \alpha X_p(f)\,\forall \alpha \in \mathbb{R}$.

Proof. Trivial from conditions 2. and 3. of Def. 2.3.3. □

It is simple to check that $\partial_\mu|_p$ satisfies the conditions of Def. 2.3.3.

Theorem 2.3.1. *The set $T_p\mathcal{M}$ of all tangent vectors at a point $p \in \mathcal{M}$ forms an n-dimensional space, called tangent space, and $\{\partial_\mu|_p\}_{\mu=1,\dots,n}$ is a basis of such space.*

Proof. Defining $f \circ \varphi^{-1} \equiv F : U \subset \mathcal{M} \to \mathbb{R}$, with $f : \mathcal{M} \to \mathcal{M}$ and $(\varphi, U) \in \mathcal{A}$, it can be proved that, in some neighbourhood of p (not necessarily U), F cal always be written as:

$$F(x) = F(x^\mu(p)) + (x^\mu - x^\mu(p))\,F_\mu(x)$$

for some n functions F_μ (ex.: a Taylor series, or more generally $F(x) = F(0) + x\int_0^1 dt\,F(xt)$). Applying $\partial_\mu|_{x(p)}$:

$$\left.\frac{\partial F}{\partial x^\mu}\right|_{x(p)} = F_\mu(x(p))$$

Defining $f_\mu \equiv F_\mu \circ \varphi$, for any $q \in \mathcal{M}$ in an appropriate neighbourhood of p:

$$f(q) = f(p) + (x^\mu(q) - x^\mu(p))\,f_\mu(q)$$

Moreover, remembering Eq. 2.1:

$$f_\mu(p) = F_\mu \circ \varphi(p) = F_\mu(x(p)) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x(p)} = \left.\frac{\partial f}{\partial x^\mu}\right|_p$$

Using these facts, the action of a tangent vector can be written explicitly:

$$\begin{aligned}
X_p(f) &= X_p\left(f(p) + (x^\mu - x^\mu(p))\,f_\mu\right) \\
&= X_p\left(f(p)\right) + X_p\left((x^\mu - x^\mu(p))\right) f_\mu(p) + (x^\mu - x^\mu(p))\,(p)X_p\left(f_\mu\right) \\
&= X_p\left(x^\mu\right) f_\mu(p)
\end{aligned}$$

because $f(p)$ is a constant and $(x^\mu - x^\mu(p))\,(p) = x^\mu(p) - x^\mu(p) = 0$. Therefore, remembering the expression for $f_\mu(p)$:

$$X_p = X_p(x^\mu)\left.\frac{\partial}{\partial x^\mu}\right|_p \equiv X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p$$

Thus, $T_p\mathcal{M} = \mathrm{span}\{\partial_\mu|_p\}$. To check for linear independence, suppose $\alpha = \alpha^\mu\partial_\mu|_p \equiv 0$: acting on $f = x^\nu$, it gives $\alpha(f) = \alpha_\mu\partial_\mu(x^\nu)|_p = \alpha_\nu = 0$. This concludes the proof. □

2.3.1.1 Changing coordinates

Although $\partial_\mu|_p$ depends on the choice of coordinates (it is a *coordinate basis*), the existence of X_p is independent of that choice.
If two different charts $(\varphi, U), (\tilde{\varphi}, V)$ intersect in a neighbourhood of $p \in U \cap V$, the transition from x^μ to y^μ can be expressed as:

$$X_p(f) = X^\mu \frac{\partial f}{\partial x^\mu}\bigg|_p = X^\mu \frac{\partial y^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} \frac{\partial f}{\partial y^\nu}\bigg|_p \tag{2.2}$$

This equation can have two interpretations: the alibi interpretation:

$$\frac{\partial}{\partial x^\mu}\bigg|_p = \frac{\partial y^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} \frac{\partial}{\partial y^\nu}\bigg|_p \tag{2.3}$$

and the alias interpretation:

$$\tilde{X}^\nu = X^\mu \frac{\partial y^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} \tag{2.4}$$

Components of vectors which transform this way are called *contravariant*.

2.3.1.2 Curves

Consider a smooth curve on $\mathcal{M}$, i.e. a smooth map $\sigma : I \in \mathcal{T}(\mathbb{R}) \to \mathcal{M}$, parametrized as $\sigma(t) : \sigma(0) = p \in \mathcal{M}$; with a given chart (φ, U), this curve becomes $\varphi \circ \sigma : I \to \mathbb{R}^n$, parametrized by $x^\mu(t)$. The *tangent vector* to the curve in p is:

$$X_p = \frac{dx^\mu(t)}{dt}\bigg|_{t=0} \frac{\partial}{\partial x^\mu}\bigg|_p \tag{2.5}$$

This operator, applied to a function $f \in \mathcal{C}^\infty(\mathcal{M})$, calculates the directional derivative of f along the curve. It can be showed that every tangent vector can be written as in Eq. 2.5, therefore the tangent space is literally the space of all possible tangents to curves passing through p.
It must be noted that tangent spaces at different points are entirely different spaces: there's no way to directly compare vectors between them.

2.3.2 Vector fields

Definition 2.3.4. A *vector field* X is a smooth map $X : p \in \mathcal{M} \mapsto X_p \in T_p\mathcal{M}$. It can also be viewed as a smooth map $X : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, as $(X(f))(p) = X_p(f) \in \mathbb{R}$.

Definition 2.3.5. The space of all vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$.

Given a chart (φ, U), a vector field X can be expressed as:

$$X = X^\mu \frac{\partial}{\partial x^\mu} \tag{2.6}$$

with $X^\mu \in \mathcal{C}^\infty(\mathcal{M})$. This expression is only defined on U.

2.3.2.1 Lie brakets

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, their product is clearly not a vector field, as it does not satisfy Leibniz' rule:

$$XY(fg) = XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) \neq XY(f)g + fXY(g)$$

where $XY(f) \equiv X(Y(f))$.

Definition 2.3.6. Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, their *commutator* (or *Lie bracket*) is defined as:

$$[X, Y](f) = XY(f) - YX(f) \tag{2.7}$$

With a given chart:

$$\begin{aligned}
[X, Y](f) &= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial f}{\partial x^\nu}\right) - Y^\mu \frac{\partial}{\partial x^\mu}\left(X^\nu \frac{\partial f}{\partial x^\nu}\right) \\
&= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial f}{\partial x^\nu}
\end{aligned}$$

therefore:

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial x^\nu} \tag{2.8}$$

Theorem 2.3.2 (Jacobi)**.** *Given $X, Y, Z \in \mathfrak{X}(\mathcal{M})$, the Jacobi identity holds:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{2.9}$$

Proposition 2.3.2. *$\mathfrak{X}(\mathcal{M})$ is a Lie algebra.*

2.3.2.2 Integral curves

Definition 2.3.7. A *flow* on $\mathcal{M}$ is a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$, labelled by $t \in \mathbb{R}$, with group structure: $\sigma_0 = \mathrm{id}_{\mathcal{M}}$ and $\sigma_s \circ \sigma_t = \sigma_{s+t}$, thus $\sigma_{-t} = \sigma_t^{-1}$.

Such flows give rise to streamlines on the manifold: these streamlines are required to be smooth. Defining $x^\mu(\sigma_t) \equiv x^\mu(t)$, a vector field can be defined by the tangent to the streamlines at each point on the manifold:

$$X^\mu(x^\mu(t)) = \frac{dx^\mu(t)}{dt} \tag{2.10}$$

The inverse reasoning is also possible.

Definition 2.3.8. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, streamlines described by Eq. 2.10 are called *integral curves* generated by X.

Proposition 2.3.3. *The infinitesimal flow generated by $X \in \mathfrak{X}(\mathcal{M})$ is:*

$$x^\mu(t) = x^\mu(0) + tX^\mu(x(t)) + o(t) \tag{2.11}$$

Definition 2.3.9. A vector field which generates a flow defined for all $t \in \mathbb{R}$ is called *complete*.

Theorem 2.3.3. *If $\mathcal{M}$ is compact, then all $X \in \mathfrak{X}(\mathcal{M})$ are complete.*

Example 2.3.1. On $\mathbb{S}^2$, the flow generated by $X = \partial_\phi$ is described by $\dot\phi = 1, \dot\theta = 0$, thus $\theta(t) = \theta_0$ and $\phi(t) = \phi_0 + t$: the flow lines are lines of constant latitude.

2.3.3 Lie derivative

Defining calculus for vector fields requires a way to compare vectors of different tangent spaces.

Definition 2.3.10. Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a function $f : \mathcal{N} \to \mathbb{R}$, the *pull-back* of f is the function $\varphi^* f : \mathcal{M} \to \mathbb{R}$ such that $\varphi^* f(p) = f(\varphi(p))$.

Definition 2.3.11. Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$, the *push-forward* of X is the vector field $\varphi_* X \in \mathfrak{X}(\mathcal{N})$ such that $\varphi_* X(f) = X(\varphi^* f)$.

This last equality must be evaluated at the appropriate points: $[\varphi_* X(f)](\varphi(p)) = [X(\varphi^* f)](p)$.
With the appropriate charts on $\mathcal{M}$ and $\mathcal{N}$, the definitions above can be rewritten with coordinates:

$$\varphi^* f(x) = f(y(x)) \tag{2.12}$$

$$\varphi_* X(f) = X^\mu \frac{\partial f(y(x))}{\partial x^\mu} = X^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial f(y)}{\partial y^\alpha} \tag{2.13}$$

The notions of pull-back and push-forward allow to compare tangent vectors at neighbouring points and, in particular, to define the derivative along a vector field.

Definition 2.3.12. Given a function $f : \mathcal{M} \to \mathbb{R}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$, the derivative of f along X (called *Lie derivative*) is defined as:

$$\mathcal{L}_X f(x) \coloneqq \lim_{t \to 0} \frac{f(\sigma_t(x)) - f(x)}{t} = \left. \frac{df(\sigma_t(x))}{dt} \right|_{t=0} \tag{2.14}$$

where σ_t is the flow generated by X.

Proposition 2.3.4. $\mathcal{L}_X f = X(f)$.

Proof. $\mathcal{L}_X f = \frac{df(\sigma_t)}{dt} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu(t)}{dt} = X^\mu \frac{\partial f}{\partial x^\mu} = X(f)$. □

Definition 2.3.13. Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, the *Lie derivative* of Y along X is defined as:

$$\mathcal{L}_X Y_p \coloneqq \lim_{t \to 0} \frac{((\sigma_{-t})_* Y)_p - Y_p}{t} \tag{2.15}$$

where σ_t is the flow generated by X.

The use of the inverse flow σ_{-t} is necessary because to evaluate the vector field $\mathcal{L}_X Y$ at the point $p \in \mathcal{M}$, the tangent vector $Y_{\sigma_t(p)} \in T_{\sigma_t(p)}\mathcal{M}$ must be "pushed-back" to $T_p\mathcal{M} = T_{\sigma_0(p)}\mathcal{M}$.
With $t \to 0$, the infinitesimal flow σ_{-t} is, according to Eq. 2.11, $x^\mu(t) = x^\mu(0) - tX^\mu + o(t)$, therefore the Lie derivative of base tangent vectors can be expressed as:

$$(\sigma_{-t})_* \partial_\mu = \frac{\partial x^\nu(t)}{\partial x^\mu} \frac{\partial}{\partial x^\nu(t)} = \left(\delta^\nu_\mu - t \frac{\partial X^\nu}{\partial x^\mu} + o(t) \right) \partial_\nu(t) \quad \Longrightarrow \quad \mathcal{L}_X \partial_\mu = - \frac{\partial X^\nu}{\partial x^\mu} \partial_\nu \tag{2.16}$$

Proposition 2.3.5. $\mathcal{L}_X Y = [X, Y]$.

Proof. $\mathcal{L}_X Y = \mathcal{L}_X(Y^\mu \partial_\mu) = (\mathcal{L}_X Y^\mu)\, \partial_\mu + Y^\mu\, (\mathcal{L}_X \partial_\mu) = X^\nu \frac{\partial Y^\mu}{\partial x^\nu} \partial_\mu - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \partial_\nu = [X, Y]$. □

Proposition 2.3.6. $\mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z = \mathcal{L}_{[X,Y]} Z$.

Proof. Trivial with Jacobi identity. □

2.4 Tensors

2.4.1 Dual Spaces

Definition 2.4.1. Given a vector space V, its *dual* V^* is the space of all linear maps $f : V \to \mathbb{R}$.

Given a basis $\{\mathbf{e}_\mu\}_{\mu=1,\ldots,n}$ of V, its *dual basis* $\{\mathbf{f}^\mu\}_{\mu=1,\ldots,n}$ of V^* can be defined by:

$$\mathbf{f}^\nu(\mathbf{e}_\mu) = \delta^\nu_\mu \tag{2.17}$$

A general vector in V can be written as $X = X^\mu \mathbf{e}_\mu$, thus according to Eq. 2.17 $X^\mu = \mathbf{f}^\mu(X)$.

Proposition 2.4.1. *The map $f : \mathbf{e}_\mu \mapsto \mathbf{f}^\mu$ is an isomorphism between V and V^*.*

This isomorphism, however, is basis-dependent.

Proposition 2.4.2. $\dim_\mathbb{R} V = \dim_\mathbb{R} V^*$.

Proposition 2.4.3. $(V^*)^* = V$.

Proof. The natural isomorphism between $(V^*)^*$ and V is basis-independent: suppose $X \in V$ and $\omega \in V^*$, so that $\omega(X) \in \mathbb{R}$; X can be viewed as $X \in (V^*)^*$ by setting $V(\omega) \equiv \omega(V)$. □

2.4.2 Cotangent vectors

Definition 2.4.2. Given a differentiable manifold $(\mathcal{M}, \mathcal{A})$ and a point $p \in \mathcal{M}$, the *cotangent space* to $\mathcal{M}$ at p is defined as $T^*_p\mathcal{M} := (T_p\mathcal{M})^*$.

Elements of $T^*_p\mathcal{M}$ are called *cotangent vectors* (or *covectors*).

Definition 2.4.3. A *covector field* (or *1-form*) is a smooth map $\omega : p \in \mathcal{M} \mapsto \omega_p \in T^*_p\mathcal{M}$. It can also be viewed as a smooth map $\omega : \mathfrak{X}(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, as $(\omega(X))(p) = \omega_p(X_p) \in \mathbb{R}$.

Definition 2.4.4. The space of all 1-forms on $\mathcal{M}$ is denoted by $\Lambda^1(\mathcal{M})$.

Proposition 2.4.4. *$\{dx^\mu\}_{\mu=1,\ldots,n}$ is a basis of $\Lambda^1(\mathcal{M})$ dual to the basis $\{\partial_\mu\}_{\mu=1,\ldots,n}$ of $\mathfrak{X}(\mathcal{M})$.*

Proof. Consider $f \in \mathcal{C}^\infty(\mathcal{M})$ and define $df \in \Lambda^1(\mathcal{M})$ by $df(X) = X(f)$: taking $f = x^\mu$ and $X = \partial_\mu$, $df(X) = \partial_\nu(x^\mu) = \delta^\mu_\nu$, therefore $\{dx^\mu\}_{\mu=1,\ldots,n}$ is the dual basis of $\Lambda^1(\mathcal{M})$. □

This is also confirmed by $df = \frac{\partial f}{\partial x^\mu} dx^\mu$. These are coordinate basis: in fact, given two different charts $(\varphi, U), (\tilde{\varphi}, V)$:

$$dy^\mu = \frac{dy^\mu}{dx^\nu} dx^\nu \tag{2.18}$$

which is the inverse of Eq. 2.3 (not evaluated at a specific point). This ensures that:

$$dy^\mu \left(\frac{\partial}{\partial y^\nu} \right) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^\nu} dx^\alpha \left(\frac{\partial}{\partial x^\beta} \right) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^\nu} = \delta^\mu_\nu$$

A 1-form $\omega \in \Lambda^1(\mathcal{M})$ can thus be expressed both as $\omega = \omega_\mu dx^\mu = \tilde{\omega}_\mu dx^\mu$, with:

$$\tilde{\omega}_\omega = \frac{\partial x^\nu}{\partial y^\mu} \omega_\nu \tag{2.19}$$

Components of 1-forms which transform this way are called *covariant*.

Definition 2.4.5. Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a 1-form $\omega \in \Lambda^1(\mathcal{N})$, the *pull-back* of ω is the 1-form $\varphi^*\omega \in \Lambda^1(\mathcal{M})$ such that $\varphi^*\omega(X) = \omega(\varphi_* X)$.

With the appropriate charts on $\mathcal{M}$ and $\mathcal{N}$, the definition above can be rewritten with coordiantes:

$$\varphi^*\omega = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu} dx^\mu \tag{2.20}$$

Definition 2.4.6. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a 1-form $\omega \in \Lambda^1(\mathcal{M})$, the *Lie derivative* of ω along X is defined as:

$$\mathcal{L}_X \omega_p := \lim_{t\to 0} \frac{(\sigma_t^*\omega)_p - \omega_p}{t} \tag{2.21}$$

where σ_t is the flow generated by X.

In contranst with the Lie derivative of a vector field, which pushes forward with σ_{-t} (i.e. pushes back), the Lie derivative of a 1-form pulls back with σ_t: this results in the difference of a minus sign with respect to Eq. 2.16, giving:

$$\mathcal{L}_X dx^\mu = \frac{\partial X^\mu}{\partial x^\nu} dx^\nu \tag{2.22}$$

Therefore, on a general 1-form $\omega = \omega_\mu dx^\mu$:

$$\mathcal{L}_X \omega = \left(X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu\right) dx^\mu \tag{2.23}$$

2.4.3 Tensor fields

Definition 2.4.7. A *tensor of rank* (r, s) at a $p \in \mathcal{M}$ of a differentiable manifold $(\mathcal{M}, \mathcal{A})$ is a multi-linear map defined as:

$$T_p : \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r} \times \overbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}^{s} \to \mathbb{R} \tag{2.24}$$

Example 2.4.1. A cotangent vector $\omega_p \in T_p^*\mathcal{M}$ is a tensor of rank $(1, 0)$, while a tangent vector $X_p \in T_p\mathcal{M}$ is a tensor of rank $(0, 1)$.

Definition 2.4.8. A *tensor field* of rank (r, s) is a smooth map $T : p \in \mathcal{M} \mapsto T_p$ tensor of rank (r, s) at p. It can also be viewed as a smooth map $T : [\Lambda^1(\mathcal{M})]^r \times [\mathfrak{X}(\mathcal{M})]^s \to \mathcal{C}^\infty(\mathcal{M})$.

Given appropriate basis for vector fields $\{\mathbf{e}_\mu\}_{\mu=1,\dots,n}$ and 1-forms $\{\mathbf{f}^\mu\}_{\mu=1,\dots,n}$, the components of a tensor field are defined as:

$$T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} := T(\mathbf{f}^{\mu_1}, \dots, \mathbf{f}^{\mu_r}, \mathbf{e}_{\nu_1}, \dots, \mathbf{e}_{\nu_s}) \tag{2.25}$$

Proposition 2.4.5. *On an n-dimensional manifold, a (r, s) tensor field has n^{r+s} components, each being element of $\mathcal{C}^\infty(\mathcal{M})$.*

Consider two general basis transformations, for vector fields and 1-forms, described by invertible matrices A, B such that $\tilde{\mathbf{e}}_\mu = A^\nu{}_\mu \mathbf{e}_\nu$ and $\tilde{\mathbf{f}}^\mu = B^\mu{}_\nu \mathbf{f}^\nu$, with necessary condition $A^\mu{}_\nu B^\rho{}_\mu = \delta^\rho_\nu$ to ensure duality: this implies $B = A^{-1}$, i.e. covectors transform inversely with respect to vectors. Thus:

$$\tilde{T}^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = B^{\mu_1}{}_{\rho_1} \dots B^{\mu_r}{}_{\rho_r} A^{k_1}{}_{\nu_1} \dots A^{k_s}{}_{\nu_s} T^{\rho_1\dots\rho_r}{}_{k_1\dots k_s} \tag{2.26}$$

If the considered basis are coordinate basis, then $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}$ and $B^\mu{}_\nu = \frac{\partial y^\mu}{\partial x^\nu}$.

2.4.4 Operations on tensors

Algebric addition and multiplication by functions are trivially defined on tensors of the same rank.

Proposition 2.4.6. *The space of all (r, s) tensors at a point $p \in \mathcal{M}$ is denoted by $T_p^{(r,s)}\mathcal{M}$, and it is a vector space.*

Definition 2.4.9. Given two tensor fields S of rank (p, q) and T of rank (r, s), their *tensor product* is defined as:

$$\begin{aligned} S \otimes T(\omega_1, \dots, \omega_p, \eta_1, \dots, \eta_r, X_1, \dots, X_q, Y_1, \dots, Y_s) \\ = S(\omega_1, \dots, \omega_p, X_1, \dots, X_q)T(\eta_1, \dots, \eta_r, Y_1, \dots, Y_s) \end{aligned} \tag{2.27}$$

or, in components:

$$(S \otimes T)^{\mu_1 \dots \mu_p \nu_1 \dots \nu_r}{}_{\rho_1 \dots \rho_q \dots \sigma_1 \dots \sigma_s} = S^{\mu_1 \dots \mu_p}{}_{\rho_1 \dots \rho_q} T^{\nu_1 \dots \nu_r}{}_{\sigma_1 \dots \sigma_s} \tag{2.28}$$

It is also possible to contract tensor $((r, s) \mapsto (r-1, s-1))$: for example, given a rank $(2, 1)$ tensor, a rank $(1, 0)$ tensor can be defined as $S(\omega) = T(\omega, \mathbf{f}^\mu, \mathbf{e}_\mu)$, with components $S^\mu = T^{\nu\mu}{}_\mu$; it must be noted that, in general, $T^{\nu\mu}{}_\mu \neq T^{\mu\nu}{}_\mu$.

Definition 2.4.10. Given an object $T_{\mu_1 \dots \mu_n}$ dependent on some indices, its *symmetric* and *antisymmetric* parts are respectively defined as:

$$T_{(\mu_1 \dots \mu_n)} := \frac{1}{n!} \sum_{\sigma \in S^n} T_{\sigma(\mu_1) \dots \sigma(\mu_n)} \tag{2.29}$$

$$T_{[\mu_1 \dots \mu_n]} := \frac{1}{n!} \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) T_{\sigma(\mu_1) \dots \sigma(\mu_n)} \tag{2.30}$$

Conventionally, indices surrounded by || are not (anti-)symmetrized (ex: $T_{[\mu|\nu|\rho]} = \frac{1}{2}(T_{\mu\nu\rho} - T_{\rho\nu\mu})$). As previously seen, vector fields are pushed forward and 1-form are pulled back: tensors will thus behave in a mixed way.

Definition 2.4.11. Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a (r, s) tensor field T on $\mathcal{M}$, the *push-forward* of T is the (r, s) tensor field $\varphi_* T$ on $\mathcal{N}$ such that, for $\omega_j \in \Lambda^1(\mathcal{N})$ and $X_j \in \mathfrak{X}(\mathcal{N})$:

$$\varphi_* T(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = T(\varphi^* \omega_1, \dots, \varphi^* \omega_r, \varphi_*^{-1} X_1, \dots, \varphi_*^{-1} X_s) \tag{2.31}$$

Definition 2.4.12. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a (r, s) tensor field T on $\mathcal{M}$, the *Lie derivative* of T along X is defined as:

$$\mathcal{L}_X T_p := \lim_{t \to 0} \frac{((\sigma_{-t})_* T)_p - T_p}{t} \tag{2.32}$$

where σ_t is the flow generated by X.

2.5 Differential forms

Definition 2.5.1. A totally anti-symmetric $(0, p)$ tensor is defined as a *p-form.* The set of all p-forms over a manifold $\mathcal{M}$ is denoted as $\Lambda^p(\mathcal{M})$.

Proposition 2.5.1. *A p-form has* $\binom{n}{p}$ *independent components.*

Proposition 2.5.2. *The maximum degree of differential forms is* $p = n \equiv \dim_{\mathbb{R}} \mathcal{M}$*: forms in* $\Lambda^n(\mathcal{M})$ *are called top forms.*

Definition 2.5.2. Given $\omega \in \Lambda^p(\mathcal{M}), \eta \in \Lambda^q(\mathcal{M})$, their *wedge product* is a $(p+q)$-form defined as:

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\nu_1 \dots \nu_q]} \tag{2.33}$$

Example 2.5.1. Given $\omega, \eta \in \Lambda^2(\mathcal{M})$, their wedge product is $(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu$.

Proposition 2.5.3. *Given* $\omega \in \Lambda^p(\mathcal{M}), \eta \in \Lambda^q(\mathcal{M})$*:*

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \tag{2.34}$$

Corollary 2.5.3.1. $\omega \wedge \omega = 0 \ \forall \omega \in \Lambda^p(\mathcal{M}) : p$ *is odd.*

Proposition 2.5.4. *The wedge product is associative.*

Proposition 2.5.5. *If* $\{\mathbf{f}^\mu\}_{\mu=1,\dots,n}$ *is a basis of* $\Lambda^1(\mathcal{M})$*, then* $\{\mathbf{f}^{\mu_1} \wedge \dots \wedge \mathbf{f}^{\mu_p}\}_{\mu_1,\dots,\mu_p=1,\dots,n}$ *is a basis of* $\Lambda^p(\mathcal{M})$.

Locally $\{dx^\mu\}_{\mu=1,\dots,n}$ is a basis of $T_p^*\mathcal{M}$, thus a general p-form can be locally written as:

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{2.35}$$

Definition 2.5.3. The *exterior derivative* is a map $d : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$ defined as:

$$(d\omega)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]} \tag{2.36}$$

In local coordinates:

$$d\omega = \frac{1}{p!} \frac{\partial \omega_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{2.37}$$

Theorem 2.5.1 (Poincaré)**.** $d^2 = 0$.

Proof. Consequence of Schwarz lemma. □

Proposition 2.5.6. *Given* $\omega \in \Lambda^p(\mathcal{M}), \eta \in \Lambda^q(\mathcal{M})$*, then:*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \tag{2.38}$$

Proposition 2.5.7. *Given a diffeomorphism between two manifolds* $\varphi : \mathcal{M} \to \mathcal{N}$ *and* $\omega \in \Lambda^p(\mathcal{M})$*, then* $d(\varphi^*\omega) = \varphi^*(d\omega)$.

Corollary 2.5.7.1. *Given* $X \in \mathfrak{X}(\mathcal{M}), \omega \in \Lambda^p(\mathcal{M})$*, then* $d(\mathcal{L}_X \omega) = \mathcal{L}_X(d\omega)$.

Definition 2.5.4. $\omega \in \Lambda^p(\mathcal{M})$ is *closed* if $d\omega = 0$.

Definition 2.5.5. $\omega \in \Lambda^p(\mathcal{M})$ is *exact* if $\exists \eta \in \Lambda^{p-1}(\mathcal{M}) : \omega = d\eta$.

Theorem 2.5.2. *$\omega \in \Lambda^p(\mathcal{M})$ is exact $\Rightarrow$ it is closed.*

Definition 2.5.6. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, the *interior product* determined by X is a map $\iota_X : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ defined as:

$$\iota_X \omega(Y_1, \dots, Y_{p-1}) := \omega(X, Y_1, \dots, Y_{p-1}) \tag{2.39}$$

On 0-forms (i.e. scalar functions), it is defined as $\iota_X f \equiv 0$.

Proposition 2.5.8. *Given $X, Y \in \mathfrak{X}(\mathcal{M})$, then $\iota_X \iota_Y = -\iota_Y \iota_X$.*

Proof. Consequence of the total anti-symmetry of p-forms. □

Proposition 2.5.9. *Given $X \in \mathfrak{X}(\mathcal{M}), \omega \in \Lambda^p(\mathcal{M}), \eta \in \Lambda^q(\mathcal{M})$, then:*

$$\iota_X(\omega \wedge \eta) = \iota_X \omega \wedge \eta + (-1)^p \omega \wedge \iota_X \eta \tag{2.40}$$

Theorem 2.5.3 (Cartan)**.** *Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, then:*

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d \tag{2.41}$$

Proof. Consider $\omega \in \Lambda^1(\mathcal{M})$:

$$\iota_X(d\omega) = \iota_X \frac{1}{2} \left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu = X^\mu \partial_\mu \omega_\nu dx^\nu - X^\nu \partial_\mu \omega_\nu dx^\mu$$

$$d(\iota_X \omega) = d(\omega_\mu X^\mu) = X^\mu \partial_\nu \omega_\mu dx^\nu + \omega_\mu \partial_\nu X^\mu dx^\nu$$

Thus, adding these expressions and recalling Eq. 2.23:

$$(d\iota_X + \iota_X d)\omega = \left(X^\mu \partial_\mu \omega_\nu + \omega_\mu \partial_\nu X^\mu\right) dx^\nu = \mathcal{L}_X \omega$$

□

2.5.1 de Rham cohomology

While exact $\Rightarrow$ closed, the converse is not true, in general: it depends on the topological properties of the manifold.

Lemma 2.5.1 (Poincaré)**.** *If $\mathcal{M}$ is simply connected, then $\omega \in \Lambda^p(\mathcal{M})$ closed $\Rightarrow$ ω exact.*

In general, it is always possible to choose a simply connected neighbourhood of a point $p \in \mathcal{M}$, in which every closed form is exact, but that may not always be possible globally.
It is convenient to set the notation $d_p \equiv d : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$.

Definition 2.5.7. The set of all closed p-forms on $\mathcal{M}$ is denoted by $Z^p(\mathcal{M}) := \ker d_p$.

Definition 2.5.8. The set of all exact p-forms on $\mathcal{M}$ is denoted by $B^p(\mathcal{M}) := \operatorname{ran} d_{p-1}$.

Definition 2.5.9. Two closed p-forms $\omega, \omega' \in Z^P(\mathcal{M})$ are said to be *equivalent* if $\omega = \omega' + \eta$ for some $\eta \in B^p(\mathcal{M})$.

Definition 2.5.10. The p^{th} *de Rham cohomolgy group* of a manifold $\mathcal{M}$ is defined to be:

$$H^p(\mathcal{M}) \coloneqq Z^p(\mathcal{M})/B^p(\mathcal{M}) \tag{2.42}$$

Definition 2.5.11. The *Betti numbers* of a manifold $\mathcal{M}$ are defined as:

$$B_p \coloneqq \dim_{\mathbb{R}} H^p(\mathcal{M}) \tag{2.43}$$

Theorem 2.5.4. *Given a differentiable manifold, its Betti numbers are always finite.*

$B_0 = 1$ for any connected manifold: there exist constant functions, which are manifestly closed and not exact, due to the non-existence of "-1-forms". Higher Betti numbers are non-zero only if the manifold has some non-trivial topology.

Definition 2.5.12. The *Euler's character* of a manifold $\mathcal{M}$ is defined as:

$$\chi(\mathcal{M}) \coloneqq \sum_{p\in\mathbb{N}_0} (-1)^p B_p \tag{2.44}$$

Example 2.5.2. The n-sphere $\mathbb{S}^n$ has only $B_0 = B_n = 1$, thus $\chi(\mathbb{S}^n) = 1 + (-1)^n$.

Example 2.5.3. The n-torus $\mathbb{T}^n$ has $B_p = \binom{n}{p}$, thus $\chi(\mathbb{T}^n) = 0$.

2.5.2 Integration

Definition 2.5.13. A *volume form* on an n-dimensional differentiable manifold $\mathcal{M}$ is a nowhere-venishing top form v, i.e. locally $v = v(x)dx^1 \wedge \cdots \wedge dx^n : v(x) \neq 0$. If such a form exists, the manifold is said to be *orientable*.

Definition 2.5.14. Given an orientable manifold $\mathcal{M}$ with volume form v, the orientation is:

- right-handed if $v(x) > 0$ locally on every neighbourhood of $\mathcal{M}$;
- left-handed if $v(x) < 0$ locally on every neighbourhood of $\mathcal{M}$;

To ensure that the handedness of the manifold doesn't change on overlapping charts:

$$v = v(x)\frac{\partial x^1}{\partial y^{\mu_1}}dy^{\mu_1} \wedge \cdots \wedge \frac{\partial x^n}{\partial x^{\mu_n}}dx^{\mu_n} = v(x)\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)dy^1 \wedge \cdots \wedge dy^n$$

It is therefore necessary that the two sets of coordinates on the overlapping region satisfy:

$$\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) > 0 \tag{2.45}$$

Non-orientable manifolds cannot be covered by overlapping charts satisfying this condition.

Example 2.5.4. The real projective space $\mathbb{RP}^n$ is orientable for odd n and non-orientable for even n.

Example 2.5.5. The complex projective space $\mathbb{CP}^n$ is orientable for all $n \in \mathbb{N}$.

Definition 2.5.15. Given a function $f : \mathcal{M} \to \mathbb{R}$ on an orientable manifold $\mathcal{M}$ with volume form v and a chart (φ, U) on $\mathcal{M}$ with coordinates $\{x^\mu\}_{\mu=1,\dots,n}$, the *integral* of f on $O = \varphi^{-1}(U) \subset \mathcal{M}$ is defined as:

$$\int_O fv := \int_U dx_1 \dots dx_n \, f(x)v(x) \tag{2.46}$$

It is clear that the volume form acts like a measure on the manifold. To integrate over the whole manifold, it must be divided up into different regions, each covered by a single chart.

Definition 2.5.16. A k-dimensional manifold Σ is a *submanifold* of an n-dimensionale manifold $\mathcal{M}$, with $n > k$, if there exists an injective map $\varphi : \Sigma \to \mathcal{M}$ such that $\varphi_* : T_p(\Sigma) \to T_{\varphi(p)}(\mathcal{M})$ is injective.

Definition 2.5.17. Given a k-form $\omega \in \Lambda^k(\mathcal{M})$, its integral over a k-dimensional submanifold Σ of $\mathcal{M}$ is defined as:

$$\int_{\varphi(\Sigma)} \omega := \int_\Sigma \varphi^* \omega \tag{2.47}$$

Example 2.5.6. Consider a 1-form $\omega \in \Lambda^1(\mathcal{M})$ and a 1-dimensional submanifold γ of $\mathcal{M}$ described by a curve $\sigma : \gamma \to \mathcal{M} : x^\mu = \sigma^\mu(t)$: locally $\omega = \omega_\mu(x) dx^\mu$, thus the integral of ω on γ can be calculated as $\int_{\sigma(\gamma)} \omega = \int_\gamma \sigma^* \omega = \int_\gamma d\tau \, \omega_\mu(x) \frac{dx^\mu}{d\tau}$.

2.5.2.1 Stokes' theorem

Integration can be generalized beyond smooth (i.e. differentiable) manifolds.

Definition 2.5.18. An n-dimensional *manifold with boundary* is a Hausdorff topological space, equipped with a compatible maximal atlas, which is locally homeomorphic to $\mathbb{R}^{n-1} \times [a, \infty) : a \in \mathbb{R}$. The *boundary* $\partial\mathcal{M}$ is the 1-dimensional submanifold determined by $x^n = a$.

Theorem 2.5.5 (Stokes)**.** *Given an n-dimensional manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, then for any $\omega \in \Lambda^{n-1}(\mathcal{M})$:*

$$\int_\mathcal{M} d\omega = \int_{\partial\mathcal{M}} \omega \tag{2.48}$$

This important theorem unifies many different results.
Given the 1-dimensional manifold $I = [a, b] \subset \mathbb{R}$, then for any 0-form (i.e. scalar function) $\omega = \omega(x)$:

$$\int_I d\omega = \int_a^b \frac{d\omega}{dx} dx = \int_{\partial I} \omega = \omega(b) - \omega(a)$$

which is the fundamental theorem of calculus.
Given a 2-dimensional manifold with boundary $S \subset \mathbb{R}^2$ and a 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$, then $d\omega = (\partial_1\omega_2 - \partial_2\omega_1) dx^1 \wedge dx^2$ and:

$$\int_S d\omega = \int_S \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2} \right) dx^1 dx^2 = \int_{\partial S} \omega = \int_{\partial S} \omega_1 dx^1 + \omega_2 dx^2$$

which is Green's theorem.
GIven a 3-dimensional manifold with boundary $V \subset \mathbb{R}^3$ and a 2-form $\omega = \omega_1 dx^2 \wedge dx^3 + \omega_2 dx^3 \wedge dx^1 + \omega_3 dx^1 \wedge dx^2$, then $d\omega = (\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3) dx^1 \wedge dx^2 \wedge dx^3$ and:

$$\int_V d\omega = \int_V \left(\frac{\partial\omega_1}{\partial x^1} + \frac{\partial\omega_2}{\partial x^2} + \frac{\partial\omega_3}{\partial x^3} \right) = \int_{\partial V} \omega = \int_{\partial V} \omega_1 dx^2 dx^3 + \omega_2 dx^3 dx^1 + \omega_3 dx^1 dx^2$$

which is Gauss' theorem.

Chapter 3

Riemannian Geometry

3.1 Metric manifolds

Definition 3.1.1. A *metric* g is a (0,2) tensor field on a manifold $\mathcal{M}$ that is:

1. symmetric: $g(X, Y) = g(Y, X)$;
2. non-degenerate: $\exists p \in \mathcal{M} : g(X, Y)|_p = 0 \, \forall Y \in T_p\mathcal{M} \Rightarrow X_p = 0$.

Definition 3.1.2. A *metric manifold* $(\mathcal{M}, g)$ is a manifold equipped with a metric.

With a choice of coordinates, the metric can be written as:

$$g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu \tag{3.1}$$

where:

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \tag{3.2}$$

It is often written also as $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$. The matrix $g_{\mu\nu}(x) \in \mathbb{R}^{n\times n}$ is symmetric, and there's always a choice of basis on each tangent space such that this matrix is diagonal: the non-degeneracy condition implies that none of the diagonal elements vanish.

Proposition 3.1.1. *The signature of a metric, i.e. the number of negative entries when diagonalized, is independent on the choice of basis.*

Proof. From Sylvester's theorem of inertia. □

Riemannian manifolds

Definition 3.1.3. A *Riemannian manifold* $(\mathcal{M}, g)$ is a manifold equipped with a metric with totally-positive signature.

Example 3.1.1. The Euclidean space $\mathbb{R}^n$, equipped with the metric $g_{\mu\nu} = \delta_{\mu\nu}$ (in Cartesian coordinates), is a Riemannian manifold.

Definition 3.1.4. Given a Riemannian manifold $(\mathcal{M}, g)$ and $X \in \mathfrak{X}(\mathcal{M})$, the *length* of X at $p \in \mathcal{M}$ is:

$$|X_p| := \sqrt{g(X, X)|_p} \tag{3.3}$$

Given $Y \in \mathfrak{X}(\mathcal{M})$, the *angle* between X and Y at $p \in \mathcal{M}$ is:

$$\cos\theta := \frac{g(X, Y)|_p}{|X_p|\,|Y_p|} \tag{3.4}$$

This can be generalized to distances between points on a curve $\sigma : \mathbb{R} \to \mathcal{M}$:

$$d(p,q) = \int_a^b dt \sqrt{g(X,X)|_{\sigma(t)}} \tag{3.5}$$

where $\sigma(a) = p$, $\sigma(b) = q$ and X is the tangent vector field of the curve. With parametrization $x^\mu(t)$, the tangent vector has components $X^\mu = \frac{dx^\mu}{dt}$, thus:

$$d(p,q) = \int_a^b dt \sqrt{g_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \tag{3.6}$$

It is important to note that this distance is independent of the parametrization.

Lorentzian manifolds

Definition 3.1.5. A *Lorentzian manifold* $(\mathcal{M}, g)$ is a manifold equipped with a metric which has a signature with a single negative sign.

Example 3.1.2. The simplest Lorentzian manifold is $\mathbb{R}^n$ with the *Minkowski metric*:

$$\eta = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^{n-1} \otimes dx^{n-1} \tag{3.7}$$

Its components are $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$, thus this is a Lorentzian manifold.

On a general Lorentzian manifold, at any point $p \in \mathcal{M}$ it is always possible to choose an orthonormal basis $\{e_\mu\}_{\mu=0,\ldots,n-1}$ of $T_p\mathcal{M}$ such that $g_{\mu\nu}|_p = \eta_{\mu\nu}$: this fact is closely related to the equivalence principle. Consider a different basis $\tilde{e}_\mu = \Lambda^\nu{}_\mu e_\nu$: the condition for it to leave the Minkowski metric unchanged is:

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} \tag{3.8}$$

This is the defining equation of a Lorentz transformation: on a Lorentzian manifold, the basic features of special relativity are locally recovered. Thus, other ideas from special relativity can be imported.

Definition 3.1.6. Given a Lorentzian manifold $(\mathcal{M}, g)$ and $X \in \mathfrak{X}(\mathcal{M})$, at $p \in \mathcal{M}$ the vector field is said to be:

- *timelike* if $g(X_p, X_p) < 0$;
- *null* if $g(X_p, X_p) = 0$;
- *spacelike* if $g(X_p, X_p) > 0$.

At each point $p \in \mathcal{M}$ it is possible to draw *lightcones*, i.e. the null tangent vectors at that point, which are past-directed or future-directed: these lightcones vary smoothly as the point is varied smoothly on the manifold, elucidating the causal structure of spacetime.

The distance between two points on a curve depends on the nature of the tangent vector field of the curve: a *timelike curve* is a curve whose tangent vector field is everywhere timelike, and analogously for the other cases. The distance on a spacelike curve is defined as in Eq. 3.5, while that on a timelike curve gets a negative sign in the square root. With parametrization $x^\mu(t)$, it is possible to define the *proper time* on a timelike curve as:

$$\tau = \int_a^b dt \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \tag{3.9}$$

This is precisely the action of a free particle moving in spacetime.

3.1.1 Metric properties

The metric defines a natural isomorphism between vectors and covectors.

Proposition 3.1.2. *Given a metric manifold* $(\mathcal{M}, g)$*, the metric defines for each* $p \in \mathcal{M}$ *a natural isomorphism* $g : X_p \in T_p\mathcal{M} \to \omega_p \in T_p^*\mathcal{M} : \omega_p(Y_p) = g(X_p, Y_p)\, \forall Y_p \in T_p\mathcal{M}$.

In a chosen coordinate basis, the vector $X = X^\mu \partial_\mu$ is mapped to the one-form $X = X_\mu dx^\mu$, thus the following identity holds:

$$X_\mu = g_{\mu\nu} X^\nu \tag{3.10}$$

Being g non-degenerate, the matrix $g_{\mu\nu}$ is invertible, with inverse $g^{\mu\nu}$ such that:

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \tag{3.11}$$

Its elements are the components of a (2,0) symmetric tensor $\hat{g} \coloneqq g^{\mu\nu} \partial_\mu \otimes \partial_\nu$, which defines the inverse of the natural isomorphism in Prop. 3.1.2:

$$X^\mu = g^{\mu\nu} X_\nu \tag{3.12}$$

The metric also defines a natural volume form on the manifold.

Definition 3.1.7. Given an n-dimensional metric manifold $(\mathcal{M}, g)$, the *volume form* is the top-form:

$$v \coloneqq \sqrt{\mathrm{g}}\, dx^1 \wedge \cdots \wedge dx^n \tag{3.13}$$

where $\mathrm{g} \coloneqq |\det g_{\mu\nu}|$.

Proposition 3.1.3. *The volume form is basis-independent.*

Proof. Consider a new set of coordinates y^μ such that $dx^\mu = A^\mu{}_\nu dy^\nu$, where $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}$. In general:

$$dx^1 \wedge \cdots \wedge dx^n = A^1{}_{\mu_1} \ldots A^n{}_{\mu_n} dy^{\mu_1} \wedge \cdots \wedge dy^{\mu_n}$$

Recalling the anti-symmetry of the wedge product and the definition of determinant, this can be rewritten as:

$$dx^1 \wedge \cdots \wedge dx^n = \sum_{\pi \in S^n} \operatorname{sgn} \pi\, A^1{}_{\pi(1)} \ldots A^n{}_{\pi(n)} dy^1 \wedge \cdots \wedge dy^n = \det A\, dy^1 \wedge \cdots \wedge dy^n$$

Note the Jacobian factor which arises when changing the measure. On the other hand:

$$g_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\nu} \tilde{g}_{\rho\sigma} = (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \tilde{g}_{\rho\sigma} \quad \Rightarrow \quad \det g_{\mu\nu} = \frac{\det \tilde{g}_{\mu\nu}}{(\det A)^2}$$

The factors $\det A$ and $(\det A)^{-1}$ cancel, thus yielding the thesis. □

The volume form can be rewritten as:

$$v = \frac{1}{n!} v_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \equiv \frac{1}{n!} \sqrt{\mathrm{g}}\, \epsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{3.14}$$

where $\epsilon_{\mu_1 \ldots \mu_n}$ is the totally-antisymmetric n-dimensional symbol (generalization of the Levi-Civita symbol). $\epsilon_{\mu_1 \ldots \mu_n}$ cannot be considered a proper tensor, as its components are always $+1, -1, 0$

indipendently if the indices are covariant or contravariant: it is, in fact, a *tensor density*, i.e. a tensor divided by $\sqrt{\mathrm{g}}$. It can be shown that:

$$v^{\mu_1\dots\mu_n} = g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n} v_{\mu_1\dots\mu_n} = \sigma\frac{1}{\sqrt{\mathrm{g}}}\epsilon^{\mu_1\dots\mu_n} \tag{3.15}$$

where σ is the sign of the signature (ex.: $\sigma = +1$ for Riemannian manifolds and $\sigma = -1$ for Lorentzian manifolds). As notation, the integral of a generic function f on $\mathcal{M}$ is denoted as:

$$\int_{\mathcal{M}} f v \equiv \int_{\mathcal{M}} d^n x\,\sqrt{\mathrm{g}} f \tag{3.16}$$

3.1.1.1 Hodge theory

Definition 3.1.8. Given an n-dimensional oriented metric manifold $(\mathcal{M}, g)$, the *Hodge dual* is defined as the map $\star : \Lambda^p(\mathcal{M}) \to \Lambda^{n-p}(\mathcal{M}) : \omega \mapsto \star\,\omega$ such that:

$$\star\,\omega_{\mu_1\dots\mu_{n-p}} := \frac{1}{(n-p)!}\sqrt{\mathrm{g}}\,\epsilon_{\mu_1\dots\mu_{n-p}\nu_1\dots\nu_p}\omega^{\nu_1\dots\nu_p} \tag{3.17}$$

In this section, the orientedness and n-dimensionality of the manifold are implied.

Proposition 3.1.4. *The Hodge dual is basis-independent.*

It is useful to state a lemma for future calculations.

Lemma 3.1.1. $v^{\mu_1\dots\mu_p\rho_1\dots\rho_{n-p}} v_{\nu_1\dots\nu_p\rho_1\dots\rho_{n-p}} = \sigma p!(n-p)!\delta^{\mu_1}_{[\nu_1}\dots\delta^{\mu_p}_{\nu_p]}$.

Proposition 3.1.5. $\star\,(\star\,\omega) = \sigma\,(-1)^{p(n-p)}\,\omega$.

The Hodge dual defines an inner product on each $\Lambda^p(\mathcal{M})$:

$$\langle\omega, \eta\rangle := \int_{\mathcal{M}} \omega\wedge\star\,\eta \tag{3.18}$$

This allows to define operators and their adjoints on the form spaces.

Proposition 3.1.6. *Given a metric manifold $(\mathcal{M}, g)$ and two forms $\omega \in \Lambda^p(\mathcal{M})$, $\alpha \in \Lambda^{p-1}(\mathcal{M})$, then:*

$$\langle d\alpha, \omega\rangle = \langle\alpha, d^\dagger\omega\rangle \tag{3.19}$$

where the adjoint of the exterior derivative $d^\dagger : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ is defined as:

$$d^\dagger := \sigma\,(-1)^{np+n-1}\star d\,\star \tag{3.20}$$

Proof. To simplify the proof, consider a closed manifold; then, from Stokes' theorem and Eq. 2.38:

$$0 = \int_{\mathcal{M}} d(\alpha\wedge\star\,\omega) = \langle d\alpha, \omega\rangle + \int_{\mathcal{M}} (-1)^{p-1}\,\alpha\wedge d\star\omega$$

The second term is proportional to $\langle\alpha, \star\, d\star\omega\rangle$: to determine the relative sign, note that $d\star\omega \in \Lambda^{n-p+1}(\mathcal{M})$, thus, from Prop. 3.1.5, $\star\star d\star\omega = \sigma\,(-1)^{(n-p+1)(p-1)}\,d\star\omega$. In conclusion:

$$\langle\alpha, \star\, d\star\omega\rangle = \sigma\,(-1)^{(n-p)(p-1)}\int_{\mathcal{M}} (-1)^{p-1}\,\alpha\wedge d\star\omega \quad\Rightarrow\quad \langle d\alpha, \omega\rangle = \sigma\,(-1)^{(n-p)(p-1)+1}\,\langle\alpha, \star\, d\star\omega\rangle$$

Noting that $(-1)^{(n-p)(p-1)+1} = (-1)^{np+n-1}$, as in general $(-1)^{-n} = (-1)^n$ and $(-1)^{-p^2+p+1} = (-1)^{-1}$ due to $p(p-1)$ being always even, concludes the proof. $\square$

Definition 3.1.9. Given a metric manifold $(\mathcal{M}, g)$, the *Laplacian* $\triangle : \Lambda^p(\mathcal{M}) \to \Lambda^p(\mathcal{M})$ is defined as the operator:

$$\triangle := (d + d^\dagger)^2 \tag{3.21}$$

Proposition 3.1.7. $\triangle = dd^\dagger + d^\dagger d = \{d, d^\dagger\}$.

Proof. Trivial, given $d^2 = d^{\dagger 2} = 0$. □

It is possible to calculate an explicit expression for the Laplacian of functions.

Lemma 3.1.2. *Given $f \in \mathcal{C}^\infty(\mathcal{M})$, then $d^\dagger f = 0$.*

Proof. Trivial noting that $\star f$ is a top-form. □

Proposition 3.1.8. *Given $f \in \mathcal{C}^\infty(\mathcal{M})$, then:*

$$\triangle f = -\frac{\sigma}{\sqrt{\mathrm{g}}}\partial_\nu \left(\sqrt{\mathrm{g}}\, g^{\mu\nu}\partial_\mu f\right) \tag{3.22}$$

Proof. Via direct calculation, using Lemma 3.1.2:

$$\begin{aligned}
\triangle f &= \sigma \left(-1\right)^{n^2+n-1} \star d \star \left(\partial_\mu f dx^\mu\right) = -\sigma \star d \left(\partial_\mu f \star dx^\mu\right) \\
&= -\frac{\sigma}{(n-1)!} \star d \left(\partial_\mu f g^{\mu\nu} \sqrt{\mathrm{g}}\, \epsilon_{\nu\rho_1 \dots \rho_{n-1}} dx^{\rho_1} \wedge \dots \wedge dx^{\rho_{n-1}}\right) \\
&= -\frac{\sigma}{(n-1)!} \star \partial_\alpha \left(\sqrt{\mathrm{g}}\, g^{\mu\nu}\partial_\mu f\right) \epsilon_{\nu\rho_1 \dots \rho_{n-1}} dx^\alpha \wedge dx^{\rho_1} \wedge \dots \wedge dx^{\rho_{n-1}} \\
&= -\sigma \star \partial_\nu \left(\sqrt{\mathrm{g}}\, g^{\mu\nu}\partial_\mu f\right) dx^1 \wedge \dots dx^n = -\frac{\sigma}{\sqrt{\mathrm{g}}}\partial_\nu \left(\sqrt{\mathrm{g}}\, g^{\mu\nu}\partial_\mu f\right)
\end{aligned}$$

□

The Laplacian operator is linked to the de Rham cohomology.

Definition 3.1.10. Given $\omega \in \Lambda^p(\mathcal{M})$, it is said to be *harmonic* if $\triangle\omega = 0$.

Definition 3.1.11. The space of harmonic p-forms on $(\mathcal{M}, g)$ is denoted as $\mathrm{Harm}^p(\mathcal{M})$.

Proposition 3.1.9. *A harmonic form is both closed and co-closed.*

Proof. $0 = \langle\omega, \triangle\omega\rangle = \langle d\omega, d\omega\rangle + \langle d^\dagger\omega, d^\dagger\omega\rangle$, thus $d\omega = 0$ and $d^\dagger\omega = 0$, for the inner product is positive-defined. □

Theorem 3.1.1. *Given a compact Riemannian manifold $(\mathcal{M}, g)$, any $\omega \in \Lambda^p(\mathcal{M})$ can be uniquely decomposed as $\omega = d\alpha + d^\dagger\beta + \gamma$, with $\alpha \in \Lambda^{p-1}(\mathcal{M})$, $\beta \in \Lambda^{p+1}(\mathcal{M})$ and $\gamma \in \mathrm{Harm}^p(\mathcal{M})$.*

Theorem 3.1.2 (Hodge)**.** *Given a compact Riemannian manifold $(\mathcal{M}, g)$, there is an isomorphism:*

$$\mathrm{Harm}^p(\mathcal{M}) \cong H^p(\mathcal{M}) \tag{3.23}$$

Proof. From Prop. 3.1.9 $\mathrm{Harm}^p(\mathcal{M}) \subset Z^p(\mathcal{M})$, but the uniqueness of decomposition in Th. 3.1.1 implies $\forall\gamma \in \mathrm{Harm}^p(\mathcal{M})\, \exists\eta_\gamma \in \Lambda^{p-1}(\mathcal{M}) : \gamma \neq d\eta_\gamma$, thus $\mathrm{Harm}^p(\mathcal{M}) \subset H^p(\mathcal{M})$.
WTS that any equivalence class $[\omega] \in H^p(\mathcal{M})$ can be represented by a harmonic form. By Th, 3.1.1 $\omega = d\alpha + d^\dagger\beta + \gamma$, but $\omega \in H^p(\mathcal{M})$ implies $d\omega = 0$ by definition, so:

$$0 = \langle d\omega, \beta\rangle = \langle\omega, d^\dagger\beta\rangle = \langle d\alpha + d^\dagger\beta + \gamma, d^\dagger\beta\rangle = \langle d^\dagger\beta, d^\dagger\beta\rangle$$

The inner product is positive-definite, thus $d^\dagger\beta = 0$, hence $\omega = \gamma + d\alpha$. By definition $H^p(\mathcal{M}) := Z^p(\mathcal{M})/B^p(\mathcal{M})$, so $[\omega] = \gamma$. □

Corollary 3.1.2.1. $B_p = \dim_{\mathbb{R}} \mathrm{Harm}^p(\mathcal{M})$.

3.2 Connections

There's a different way to differentiate tensor fields distinct from the Lie derivative, associated to a different way to map different vector spaces at different points: the covariant derivative.
From now on, $\mathcal{M}$ is implied to be an n-dimensional metric manifold with metric g.

3.2.1 Covariant derivative

Definition 3.2.1. The *connection* is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, usually written as $\nabla(X, Y) \equiv \nabla_X Y$, where ∇_X is called the *covariant derivative*, satisfying the following properties for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;
2. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z \,\forall f, g \in \mathcal{C}^\infty(\mathcal{M})$;
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y \,\forall f \in \mathcal{C}^\infty(\mathcal{M})$.

Usually $X(f) \equiv \nabla_X f$. The covariant derivative endows the manifold with more structure: in particular, given a basis $\{e_\mu\}$ of $\mathfrak{X}(\mathcal{M})$, its covariant derivative is expressed as:

$$\nabla_{e_\rho} e_\mu \equiv \Gamma^\mu_{\rho\nu} e_\mu \tag{3.24}$$

The $\Gamma^\mu_{\rho\nu}$ are the components of the connection on that basis. Usually $\nabla_{e_\mu} \equiv \nabla_\mu$, thus resembling a partial derivative. To elucidate how the covariant derivative acts on vector fields:

$$\begin{aligned}
\nabla_X Y &= \nabla_X(Y^\mu e_\mu) \\
&= X(Y^\mu)e_\mu + Y^\mu \nabla_X e_\mu \\
&= X^\nu e_\nu(Y^\mu)e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu \\
&= X^\nu \left[e_\nu(Y^\mu) + \Gamma^\mu_{\nu\rho} Y^\rho\right] e_\mu \\
&= X^\nu \nabla_\nu Y = X^\nu (\nabla_\nu Y)^\mu e_\mu
\end{aligned}$$

The dependency on X can therefore be eliminated, and in components:

$$(\nabla_\nu Y)^\mu = e_\nu(Y^\mu) + \Gamma^\mu_{\nu\rho} Y^\rho \tag{3.25}$$

A sloppy notation is often used: $(\nabla_\nu Y)^\mu \equiv \nabla_\nu Y^\mu$. This must not be confused as the covariant derivative of Y^μ. Moreover $\nabla_\nu Y^\mu \equiv Y^\mu_{\ ;\nu}$, while $\partial_\mu f \equiv f_{,\mu}$. On the coordinate basis $e_\mu = \partial_\mu$, then:

$$Y^\mu_{\ ;\nu} = Y^\mu_{\ ,\nu} + \Gamma^\mu_{\nu\rho} Y^\rho \tag{3.26}$$

Note that $Y^\mu_{\ ;\nu}$ is the μ^{th} component of $\nabla_\nu Y$, while $Y^\mu_{\ ,\nu}$ is the partial derivative of Y^μ along ∂_ν.
The covariant derivative coincides with other derivatives on $\mathcal{C}^\infty(\mathcal{M})$: it can be shown that $\nabla_X f = \mathcal{L}_X f = X(f)$ and $\nabla_\mu f = \partial_\mu f$. On $\mathfrak{X}(\mathcal{M})$, however, ∇_X and $\mathcal{L}_X$ are distinct: while $\nabla_X = X^\mu \nabla_\mu$, there's no way to write the same relation for $\mathcal{L}_X$, for it depends not only on X but on its first derivative too. The covariant derivative is thus the natural generalization of the partial derivative to curved manifolds.

Proposition 3.2.1. *$\Gamma^\mu_{\rho\nu}$ are not components of a tensor.*

Proof. Given the basis transformation $\tilde{e}_\nu = A^\mu{}_\nu e_\mu$, with A and invertible matrix (if they're both coordinate basis, then $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu}$), the components of a (1,2) tensor must transform as:

$$\tilde{T}^\mu{}_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu T^\tau{}_{\sigma\lambda}$$

In the new basis:

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\rho\nu}\tilde{e}_\mu &= \nabla_{\tilde{e}_\rho}\tilde{e}_\nu = \nabla_{A^\sigma{}_\rho e_\sigma}(A^\lambda{}_\nu e_\nu) = A^\sigma{}_\rho \nabla_{e_\sigma}(A^\lambda{}_\nu e_\lambda)\\
&= A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} e_\tau + A^\sigma{}_\rho e_\lambda \partial_\sigma A^\lambda{}_\nu = \left[A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} + A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu\right] e_\tau\\
&= \left[A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} + A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu\right](A^{-1})^\mu{}_\tau \tilde{e}_\mu
\end{aligned}$$

Thus, there's a second term proportional to ∂A which deviates from the transformation law:

$$\tilde{\Gamma}^\mu_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho \left[A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} + \partial_\sigma A^\tau{}_\nu\right]$$

□

3.2.2 Covariant derivative of tensors

First of all, it is necessary to elucidate how the covariant derivative acts on one-forms. Given a one-form ω, the one-form $\nabla_X\omega$ is defined by its action on vector fields. By Leibniz rule:

$$\nabla_X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)$$

Recalling that $\omega(Y)$ is a function, $\nabla_X(\omega(Y)) = X(\omega(Y))$, therefore:

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) \tag{3.27}$$

Expressing it in coordinates:

$$\begin{aligned}
X^\mu(\nabla_\mu\omega)_\nu Y^\nu &= X^\mu\partial_\mu(\omega_\nu Y^\nu) - \omega_\nu X^\mu\left[\partial_\mu Y^\nu + \Gamma^\nu_{\mu\rho}Y^\rho\right]\\
&= X^\mu\left[\partial_\mu\omega_\rho - \Gamma^\nu_{\mu\rho}\omega_\nu\right]Y^\rho
\end{aligned}$$

Crucially, the ∂Y terms cancel out, allowing to define $\nabla_X\omega$ without referencing Y:

$$(\nabla_\mu\omega)_\rho = \partial_\mu\omega_\rho - \Gamma^\nu_{\mu\rho}\omega_\nu \tag{3.28}$$

Using the same notation as for vector fields $(\nabla_\mu\omega)_\rho \equiv \nabla_\mu\omega_\rho \equiv \omega_{\rho;\mu}$:

$$\omega_{\rho;\mu} = \omega_{\rho,\mu} - \Gamma^\nu_{\mu\rho}\omega_\nu \tag{3.29}$$

This kind of argument can be extended to a general (p,q) tensor field:

$$\begin{aligned}
\nabla_\rho T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\mu_q} = \partial_\rho T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} &+ \Gamma^{\mu_1}_{\rho\sigma}T^{\sigma\mu_2\ldots\mu_p}{}_{\nu_1\ldots\nu_q} + \cdots + \Gamma^{\mu_p}_{\rho\sigma}T^{\mu_1\ldots\mu_{p-1}\sigma}{}_{\nu_1\ldots\nu_q}\\
&- \Gamma^\sigma_{\rho\nu_1}T^{\mu_1\ldots\mu_p}{}_{\sigma\nu_2\ldots\nu_q} - \cdots - \Gamma^\sigma_{\rho\nu_q}T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_{q-1}\sigma}
\end{aligned} \tag{3.30}$$

The pattern is clear: for each upper index μ there's a $+\Gamma^\mu_{\rho\sigma}T^\sigma$ term, while for each lower index ν there's $-\Gamma^\sigma_{\rho\nu}T_\sigma$ term. Furthermore, it is necessary to generalize the comma-notation: for example, $X^\mu{}_{;\nu\rho} \equiv \nabla_\rho\nabla_\nu X^\mu$, so the rightmost index is the one whose covariant derivative acts first.

3.2.2.1 Torsion and curvature

Even though the connection is not a tensor, it is used to construct two important tensors.

Definition 3.2.2. The *torsion* is a (1,2) tensor defined on $\Lambda^1(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$ as:

$$T(\omega, X, Y) \coloneqq \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \tag{3.31}$$

Alternatively, the torsion can be viewed as a map $T : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ sucht that:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{3.32}$$

Definition 3.2.3. The *curvature* is a (1,3) tensor defined on $\Lambda^1(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$ as:

$$R(\omega, X, Y, Z) \coloneqq \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{3.33}$$

Alternatively, the curvature can be viewed as a map from $\mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$ to the space of differential operators on $\mathfrak{X}(\mathcal{M})$ such that:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{3.34}$$

The fact that these are indeed tensors, i.e. they are linear in each argument, can be shown by direct calculation, recalling that $[fX, Y] = f[X, Y] - Y(f)X$.

Proposition 3.2.2. *On the coordinate basis $\{\partial_\mu\}$ and $\{dx^\mu\}$ the torsion components are:*

$$T^\rho{}_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{3.35}$$

Proof. By direct calculation:

$$\begin{aligned}
T^\rho{}_{\mu\nu} &= T(dx^\rho, \partial_\mu, \partial_\nu) = dx^\mu(\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu - [\partial_\mu, \partial_\nu]) \\
&= dx^\mu(\partial_\mu \partial_\nu - \Gamma^\sigma_{\mu\nu} \partial_\sigma - \partial_\nu \partial_\mu + \Gamma^\sigma_{\nu\mu} \partial_\sigma) \\
&= \left[\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right] \delta^\mu_\sigma = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}
\end{aligned}$$

□

Interestingly, even though $\Gamma^\rho_{\mu\nu}$ is not a tensor, its anti-symmetric part $\Gamma^\rho_{[\mu\nu]} = \frac{1}{2} T^\sigma{}_{\mu\nu}$ is. Clearly, the torsion tensor is anti-symmetric in its lower indices, thus for connections which are symmetric in their lower indices the torsion is null: such connections are said to be *torsion-free.*

Proposition 3.2.3. *On the coordiante basis $\{\partial_\mu\}$ and $\{dx^\mu\}$ the curvature components are:*

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda} \tag{3.36}$$

Proposition 3.2.4. *By direct calculation:*

$$\begin{aligned}
R(dx^\sigma, \partial_\mu, \partial_\nu, \partial_\rho) &= dx^\sigma(\nabla_\mu \nabla_\nu \partial_\rho - \nabla_\nu \nabla_\mu \partial_\rho - \nabla_{[\partial_\mu, \partial_\nu]} \partial_\rho) \\
&= dx^\sigma(\nabla_\mu \nabla_\nu \partial_\rho - \nabla_\nu \nabla_\mu \partial_\rho) = dx^\sigma(\nabla_\mu(\Gamma^\lambda_{\nu\rho} \partial_\lambda) - \nabla_\nu(\Gamma^\lambda_{\mu\rho} \partial_\lambda)) \\
&= dx^\sigma((\partial_\mu \Gamma^\lambda_{\nu\rho}) \partial_\lambda + \Gamma^\lambda_{\nu\rho} \Gamma^\tau_{\mu\lambda} \partial_\tau - (\partial_\nu \Gamma^\lambda_{\mu\rho}) \partial_\lambda - \Gamma^\lambda_{\mu\rho} \Gamma^\tau_{\nu\lambda} \partial_\tau) \\
&= \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda}
\end{aligned}$$

Clearly, the curvature tensor is anti-symmetric in its last two lower indices, i.e. $R^\sigma{}_{\rho\mu\nu} = R^\sigma{}_{\rho[\mu\nu]}$. It's also easy to show that:

$$R^\sigma{}_{\rho\mu\nu} = 2\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + 2\Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho} \tag{3.37}$$

Theorem 3.2.1. *The following idendity, known as the Ricci identity, holds:*

$$2\nabla_{[\mu}\nabla_{\nu]}Z^\sigma = R^\sigma{}_{\rho\mu\nu}Z^\rho - T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma \tag{3.38}$$

Proof. By direct calculation:

$$\begin{aligned}
\nabla_{[\mu}\nabla_{\nu]}Z^\sigma &= \partial_{[\mu}(\nabla_{\nu]}Z^\sigma) + \Gamma^\sigma_{[\mu|\lambda|}\nabla_{\nu]}Z^\lambda - \Gamma^\rho_{[\mu\nu]}\nabla_\rho Z^\sigma \\
&= \partial_{[\mu}\partial_{\nu]}Z^\sigma + (\partial_{[\mu}\Gamma^\sigma_{\nu]\rho})Z^\rho + (\partial_{[\mu}Z^\rho)\Gamma^\sigma_{\nu]\rho} + \Gamma^\sigma_{[\mu|\lambda|}\partial_{\nu]}Z\lambda + \Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho}Z^\rho - \frac{1}{2}T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma \\
&= \left(\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + \Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho}\right)Z^\rho - \frac{1}{2}T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma = \frac{1}{2}R^\sigma{}_{\rho\mu\nu}Z^\rho - \frac{1}{2}T^\sigma{}_{\mu\nu}\nabla_\rho Z^\sigma
\end{aligned}$$

□

3.2.2.2 Levi-Civita connection

The discussion on the connection has so far been independent of the metric. Starting to consider it, an important result is the *fundamental theorem of Riemannian geometry.*

Theorem 3.2.2 (Riemann)**.** *On a metric manifold $(\mathcal{M}, g)$, there exists a unique torsion-free connection that is compatible with the metric, i.e. for all $X \in \mathfrak{X}(\mathcal{M})$:*

$$\nabla_X g = 0 \tag{3.39}$$

This is called the Levi-Civita connection.

Proof. WTS uniqueness: suppose such a connection exists. Then, by Leibniz:

$$X(g(Y,Z)) = \nabla_X(g(Y,Z)) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X, Z)$$

Since $\nabla_X g = 0$, by cyclic permutations of X, Y and Z:

$$\begin{aligned}
X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\
Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\end{aligned}$$

Since the connection is torsion-free, $\nabla_X Y - \nabla_Y X = [X,Y]$, thus these equations become:

$$\begin{aligned}
X(g(Y,Z)) &= g(\nabla_Y X, Z) + g(\nabla_X Z, Y) + g([X,Y], Z) \\
Y(g(Z,X)) &= g(\nabla_Z Y, X) + g(\nabla_Y X, Z) + g([Y,Z], X) \\
Z(g(X,Y)) &= g(\nabla_X Z, Y) + g(\nabla_Z Y, X) + g([Z,X], y)
\end{aligned}$$

Adding the first two and subtracting the third:

$$\begin{aligned}
g(\nabla_Y X, Z) = \frac{1}{2}[&X(g(Y,Z)) + Y(g(Z,X)) + Z(g(X,Y)) \\
&- g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)]
\end{aligned}$$

The metric is non-degenerate, thus this uniquely specifies the connection. By direct calculation it can be shown that it indeed satisfies all the properties of a connection. □

Proposition 3.2.5. *On the coordinate basis $\{\partial_\mu\}$ and $\{dx^\mu\}$ the Levi-Civita connection's components, called Christoffel symbols, are:*

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{3.40}$$

Proof. Recalling that $[\partial_\mu, \partial_\nu] = 0$:

$$\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = g(\nabla_\mu \partial_\nu, \partial_\rho) = \frac{1}{2}(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu})$$

□

Example 3.2.1. In flat space $\mathbb{R}^n$, endowed with either Euclidean or Minkowski metric, it is always possibile to choose Cartesian coordinates, in which case the Christoffel symbols vanish. Being the Riemann tensor a genuine tensor, it therefore will vanish in all possibile coordinate systems on $\mathbb{R}^n$, even in those with $\Gamma^\rho_{\mu\nu} \neq 0$: this expresses the flatness of $\mathbb{R}^n$.

3.2.2.3 Gauss' theorem

The divergence theorem (or Gauss' theorem) states that the integral of a total derivative is a boundary term. It is possible to express this theorem on curved manifolds in a convenient way.

Lemma 3.2.1. $\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{\mathrm{g}}} \partial_\nu \sqrt{\mathrm{g}}$.

Proof. A useful identity for invertible matrices: $\operatorname{tr} \log A = \log \det A$. Thus (WLOG $\det g > 0$):

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\rho} \partial_\nu g_{\mu\rho} = \frac{1}{2} \operatorname{tr}(g^{-1} \partial_\nu g) = \frac{1}{2} \operatorname{tr}(\partial_\nu \log g) = \frac{1}{2} \partial_\nu \log \det g = \frac{1}{\sqrt{\det g}} \partial_\nu \sqrt{\det g}$$

□

Theorem 3.2.3 (Gauss)**.** *Given a Riemannian manifold $(\mathcal{M}, g)$, consider a region $M \subseteq \mathcal{M}$ with boundary ∂M and let n^μ be an outward-pointing unit vector orthogonal to ∂M. Then, for any vector field X^μ on M:*

$$\int_M d^n x \sqrt{\mathrm{g}}\, \nabla_\mu X^\mu = \int_{\partial M} d^{n-1} x \sqrt{\gamma}\, n_\mu X^\mu \tag{3.41}$$

where γ_{ij} is the pull-back of the metric to ∂M and $\gamma \equiv \det \gamma_{ij}$.

Proof. From Lemma 3.2.1:

$$\sqrt{\mathrm{g}}\, \nabla_\mu X^\mu = \sqrt{\mathrm{g}} \left(\partial_\mu X^\mu + \Gamma^\mu_{\mu\nu} X^\nu\right) = \sqrt{\mathrm{g}} \left(\partial_\mu X^\mu + X^\nu \frac{1}{\sqrt{\mathrm{g}}} \partial_\nu \sqrt{\mathrm{g}}\right) = \partial_\mu(\sqrt{\mathrm{g}} X^\mu)$$

The integral becomes:

$$\int_M d^n x \sqrt{\mathrm{g}}\, \nabla_\mu X^\mu = \int_M d^n x\, \partial_\mu(\sqrt{\mathrm{g}} X^\mu)$$

This is the integral of an ordinary partial derivative, so the ordinary divergence theorem applies. To evaluate the integral on the boundary, it is convenient to pick coordinates so that ∂M is a surface at constant x^n. Moreover, to simplify the proof, the possible metrics will be restricted to $g_{\mu\nu} = \mathrm{diag}\,(\gamma_{ij}, N^2)$. By usual integration rules:

$$\int_M d^n x\, \partial_\mu(\sqrt{\mathrm{g}} X^\mu) = \int_{\partial M} d^{n-1}x \sqrt{\gamma N^2} X^n$$

The unit normal vector is $n^\mu = (0, \ldots, 0, \frac{1}{N})$, so that $g_{\mu\nu} n^\mu n^\nu = 1$, therefore $n_\mu = g_{\mu\nu} n^\nu = (0, \ldots, 0, N)$. The proof is then concluded because:

$$\int_{\partial M} d^{n-1}x \sqrt{\gamma N^2} X^n = \int_{\partial M} d^{n-1}x \sqrt{\gamma} n_\mu X^\mu$$

$\square$

Note that this theorem holds on Lorentzian manifolds too, with the condition that ∂M must be purely timelike or purely spacelike, ensuring that $\gamma \neq 0$ at any point.

3.2.2.4 Maxwell action

Consider spacetime as a manifold $\mathcal{M}$. The electromagnetic field can be described by a form on this manifold: indeed, the electromagnetic gauge field $A_\mu = (\phi, \mathbf{A})$ is to be thought as the components of a one-form $A = A_\mu(x) dx^\mu$. The exterior derivative of this form is a 2-form $F = dA$:

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu$$

The components $F_{\mu\nu}$ are in reality the components of a tensor, the *Faraday tensor*. By construction, a useful identity holds, sometimes called the *Bianchi identity*:

$$dF = 0 \tag{3.42}$$

From this identity derive two Maxwell equations: $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$. Moreover, note that the gauge field is not unique: the gauge transformation $A \mapsto A + d\alpha$, which equals $A_\mu \mapsto A_\mu + \partial_\mu \alpha$, leaves F unchanged.
To study the dynamics of these fields, an action is needed: Differential Geometry allows very few actions to be written down.
For example, suppose that on the considered manifold no metric is defined. To integrate over $\mathcal{M}$ a 4-form is needed, but F is a 2-form, thus the only possible action is:

$$\mathcal{S}_{\text{top}} = -\frac{1}{2} \int F \wedge F \tag{3.43}$$

The integrand become $dx^0 dx^1 dx^2 dx^3 \mathbf{E} \cdot \mathbf{B}$. Actions of this kind, independent of the metric, are called *topological actions* and are of no interest in classical physics: in fact, $F \wedge F = d(A \wedge F)$, so the action is a total derivative and doesn0't affect the equations of motion.
To construct an action of classical interest, a metric is needed. This allows to introduce a second 2-form, $\star F$, so to construct the *Maxwell action*:

$$\mathcal{S}_{\text{M}} = -\frac{1}{2} \int F \wedge \star F \tag{3.44}$$

The integrand can then be expanded as:

$$\mathcal{S}_{\mathrm{M}} = -\frac{1}{4}\int d^4x\sqrt{\mathrm{g}}\, g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} = -\frac{1}{4}\int d^4x\sqrt{\mathrm{g}}F^{\mu\nu}F_{\mu\nu}$$

In flat spacetime $F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$. In a general curved spacetime, the equation of motion resulting from the variation of the Maxwell action is $d\star F = 0$.
To complete the theory, consider a gauge field coupled to a current, described by a one-form J. The Maxwell action then becomes:

$$\mathcal{S}_{\mathrm{M}} = \int -\frac{1}{2}F\wedge\star F + A\wedge\star J \tag{3.45}$$

This action must retain its gauge invariance, but under $A \mapsto A + d\alpha$ it transforms as $S_{\mathrm{M}} \mapsto S_{\mathrm{M}} + \int d\alpha\wedge\star J$, therefore, after integrating by parts, the condition of gauge invariance translates to:

$$d\star J = 0 \tag{3.46}$$

This is current conservation in the language of forms. Varying the action in Eq. 3.44 now leads to the Maxwell equations with source terms:

$$d\star F = \star J \tag{3.47}$$

To define electric and magnetic charges, integrate over submanifolds. Consider a three-dimensional spatial submanifold Σ: the electric charge in Σ is defined as:

$$Q_e(\Sigma) := \int_\Sigma \star J \tag{3.48}$$

This agrees with the usual definition in flat spacetime $Q_e = \int_\Sigma d^3xJ^0$. Using the equations of motion and Stokes' theorem, a general form of Gauss' law is obtained:

$$Q_e(\Sigma) = \int_{\partial\Sigma} \star F \tag{3.49}$$

Simiilarly, the magnetic charge in Σ is defined as:

$$Q_m(\Sigma) := \int_{\partial\Sigma} F \tag{3.50}$$

The non-existence of magnetic charges, following from Bianchi identity, can be evaded in topologically interesting manifolds.
From charge conservation in Eq. 3.46, it follows that the electric charge in a region cannot change, unless current flows in or out of that region. Consider a cylindrical region of spacetime V, ending in two spatial hypersurfaces Σ_1 and Σ_2: its boundary is $\partial V = \Sigma_1 \cup \Sigma_2 \cup B$, where B is a cylindrical timelike hypersurface. The statement that no current flows in or out of V means that $J|_B = 0$. Then:

$$Q_e(\Sigma_1) - Q_e(\Sigma_2) = \int_{\Sigma_1}\star J - \int_{\Sigma_2}\star J = \int_{\partial V}\star J - \int_B \star J = \int_{\partial V}\star J = \int_V d\star J = 0$$

Thus, electric charge in remains constant in time.

Maxwell equations from connections First note that, given the gauge field $A \in \Lambda^1(\mathcal{M})$, the field strength can be expressed via covariant derivatives:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

The Christoffel symbols cancel out due to anti-symmetry: this is what allows to define the exterior derivative without introducing connections first.

Proposition 3.2.6. *Current conservation can be written as:* $d \star J = 0 \Leftrightarrow \nabla_\mu J^\mu = 0$.

Proof. Recalling Lemma 3.2.1:

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\rho} J^\rho = \partial_\mu J^\mu + \partial_\rho(\log\sqrt{\mathrm{g}}) J^\rho = \frac{1}{\sqrt{\mathrm{g}}}\partial_\mu(\sqrt{\mathrm{g}} J^\mu) \propto d \star J$$

□

As an aside, in general the divergence in different coordinate systems can be computed using the formula $\nabla_\mu J^\mu = \frac{1}{\sqrt{\mathrm{g}}}\partial_\mu(\sqrt{\mathrm{g}} J^\mu)$.

Proposition 3.2.7. $d \star F = \star J \Leftrightarrow \nabla_\mu F^{\mu\nu} = J^\nu$.

Proof. Recalling Lemma 3.2.1:

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} = \frac{1}{\sqrt{\mathrm{g}}}\partial_\mu(\sqrt{\mathrm{g}} F^{\mu\nu})$$

where $\Gamma^\nu_{\mu\rho} F^{\mu\rho} = 0$ because $\Gamma^\nu_{\mu\rho}$ is symmetric in μ and ρ, while $F^{\mu\rho}$ is anti-symmetric. The proof follows recalling the definition of the Hodge dual in Eq. 3.17. □

3.3 Parallel transport

The connection connects tangent spaces, or more generally any tensor vector space, at different points of the manifold: this map is called *parallel transport* and it's necessary for the definition of differentiation.

Definition 3.3.1. Consider a vector field X and some associated integral curve γ, with coordinates $x^\mu(\tau)$ such that:

$$X^\mu\big|_\gamma = \frac{dx^\mu(\tau)}{d\tau}$$

A tensor field T is said to be *parallely transported* along γ if:

$$\nabla_X T = 0 \tag{3.51}$$

Suppose that γ connects two points $p, q \in \mathcal{M}$: Eq. 3.51 provides a map from the tensor vector space defined at p to that defined at q. To illustrate this, consider the parallel transport of a vector field Y:

$$X^\nu(\partial_\nu Y^\mu + \Gamma^\mu_{\nu\rho} Y^\rho) = 0$$

Evaluating this equation on γ, considering $Y^\mu = Y^\mu(x(\tau))$:

$$\frac{dY^\mu}{d\tau} + X^\nu \Gamma^\mu_{\nu\rho} Y^\rho = 0 \tag{3.52}$$

These are a set of coupled ODEs, thus, given an initial condition (ex.: at $\tau = 0$, i.e. at p), these equations can be solved to find a unique vector at each point along the curve.
Note that the parallel transport depends both on the path (characterized by the vector field X) and on the connection.

Definition 3.3.2. Given a vector field X, a *geodesic* is a curve tangent to X such that:

$$\nabla_X X = 0 \tag{3.53}$$

Proposition 3.3.1. *A geodesic is described by the geodesic equation:*

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{3.54}$$

Proof. From the above calculations, along γ:

$$0 = \frac{dX^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} X^\nu X^\rho = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

□

For the Levi-Civita connection $\nabla_X g = 0$. If $Y \in \mathfrak{X}(\mathcal{M})$ is parallely transported along a geodesic associated to $X \in \mathfrak{X}(\mathcal{M})$, then $\nabla_X Y = \nabla_X X = 0$, therefore $\frac{d}{d\tau} g(X, Y) = 0$: this ensures that the two tangent vectors always make the same angles along the geodesic.

3.3.1 Normal coordinates

Theorem 3.3.1. *Given a Riemannian manifold $(\mathcal{M}, g)$ and $p \in \mathcal{M}$,in a neighbourhood of p it's always possible to find coordinates, called normal coordinates, such that $g_{\mu\nu}(p) = \delta_{\mu\nu}$ and $g_{\mu\nu,\rho}(p) = 0$.*

Proof. By brute force, consider initial coordinates y^μ and find a change of coordinates x^μ which satisfy the requirements:

$$\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma} = g_{\mu\nu}$$

WLOG p is the origin of both sets of coordinates, so:

$$y^\rho = \left.\frac{\partial y^\rho}{\partial x^\mu}\right|_{x=0} x^\mu + \frac{1}{2}\left.\frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu}\right|_{x=0} x^\mu x^\nu + \dots$$

This, together with the Taylor expansion og $\tilde{g}_{\rho\sigma}$, can be inserted in the transformation equation of the metric, thus finding a set of PDEs for each power of x, which can be solved to characterize the normal coordinates. For example, the first condition is:

$$\left.\frac{\partial y^\rho}{\partial x^\mu}\right|_{x=0}\left.\frac{\partial y^\sigma}{\partial x^\nu}\right|_{x=0}\tilde{g}_{\rho\sigma}(p) = \delta_{\mu\nu}$$

Given any $\tilde{g}_{\rho\sigma}(p)$, it's always possible to find $\frac{\partial y}{\partial x}$ that satisfies this condition. In fact, if $\dim_{\mathbb{R}} \mathcal{M} = n$, the Jacobian of the transformation has n^2 independent elements and the equation above puts $\frac{1}{2}n(n+1)$ constraints: the remaining free parameters are $\frac{1}{2}n(n-1)$, which is precisely the dimension of $\mathrm{SO}(n)$, the symmetry group of the flat metric. A similar counting shows that $g_{\mu\nu,\rho}(p) = 0$ puts $\frac{1}{2}n^2(n+1)$ constraints, precisely the number of independent elements of the Hessian of the transformation. □

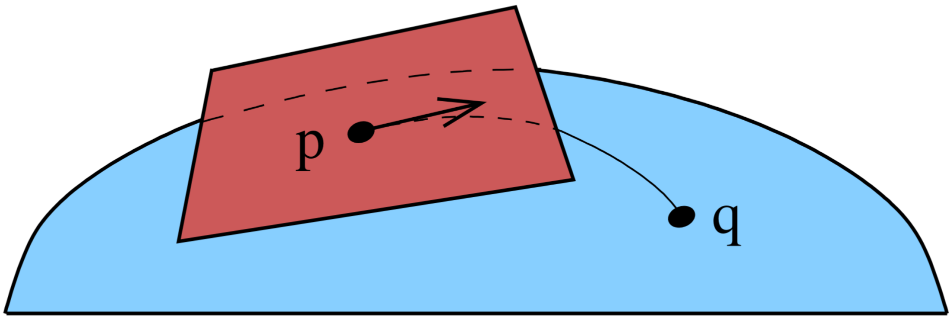


Figure 3.1: Visualization of the exponential map.

This theorem holds for Lorentzian manifolds too, but the flat metric is now $\eta_{\mu\nu}$ and its symmetry group is $\mathrm{SO}(1, n-1)$. The condition $g_{\mu\nu,\rho}(p) = 0$ implies that $\Gamma^\mu_{\nu\rho}(p) = 0$, but generally the Christoffel symbols won't vanish away from p. Note, however, that it's not generally possible to ensure the vanishing of second derivatives too: indeed, $g_{\mu\nu,\rho\sigma}(p) = 0$ would put $\frac{1}{4}n^2(n+1)^2$ constraints, but the independent $\frac{\partial^3 y}{\partial^3 x}$ terms are $\frac{1}{6}n^2(n+1)(n+2)$, thus leaving $\frac{1}{12}n^2(n^2-1)$ free terms: this is precisely the number of indepedent components of the Riemann tensor, therefore in general it's not possible to pick coordinates as to make the Riemann tensor vanish too.

3.3.1.1 Exponential map

A simple way to construct normal coordinates is the following: given a tangent vector $X_p \in T_p\mathcal{M}$, there is a unique affinely parametrized geodesic through p with tangent vector X_p at p; then, any point q in the neighbourhood of p is labelled by the coordinates of the geodesic that takes from p to q a fixed amount of time.

Analytically, introducing a coordinate system (not necessarily normal) $\tilde{x}^\mu$ in the neighbourhood of p, an affinely parametrized geodesic solves Eq. 3.54, with initial conditions $\frac{\partial \tilde{x}^\mu}{\partial \tau}|_{\tau=0} = \tilde{X}^\mu_p$ and $\tilde{x}^\mu(\tau = 0) = 0$ that make the solution unique. The uniqueness of the solution allows to define a map $\mathrm{Exp} : T_p\mathcal{M} \to \mathcal{M}$, called the *exponential map*, which acts as follows: given $X_p \in T_p\mathcal{M}$, construct the appropriate geodesic as above and follow it for a fixed affine distance, conventionally $\tau = 1$, to get a new point $q \in \mathcal{M}$. See Fig. 3.1 for visual aid. Obviously, there may be points which cannot be reached from p by geodesics, or there may be tangent vectors X_p for which Exp is ill-defined: in General Relativity, this occurs when spacetime has singularities, but these are not relevant issues.

Pick a basis $\{e_\mu\}$ of $T_p\mathcal{M}$. Then $\mathrm{Exp} : T_p\mathcal{M} \ni X^\mu e_\mu \mapsto q \in \mathcal{M}$, thus it is possible to assign coordinates in the neighbourhood of p such that $x^\mu(q) = X^\mu$: these are the normal coordinates. To show this, note that if $\{e_\mu\}$ is orthonormal, then the geodesics will point in orthogonal directions, ensuring that $g_{\mu\nu}(p) = \delta_{\mu\nu}$. Now, fix a point q associated to a given tangent vector $X_p \in T_p\mathcal{M}$: this means that q is at distance $\tau = 1$ from p along the given geodesic. Note that the geodesic equation is homogeneous in τ, thus in general $\mathrm{Exp} : \tau X_p \mapsto x^\mu(\tau) = \tau X^\mu$, which means that geodesics take a simple form in these coordinates:

$$x^\mu(\tau) = \tau X^\mu$$

Being these geodesics, they must solve Eq. 3.54, that is:

$$\Gamma^\mu_{\nu\rho}(x(\tau)) X^\nu X^\rho = 0$$

which holds at any point along the geodesic, i.e. at any $\tau \in \mathbb{R}^+$. At most points $x(\tau)$, this equation only holds for those choices of X^μ tangent to the geodesics. However, at $x(0) = 0$, i.e. at p, it must hold for any tangent vector: this means that $\Gamma^{\mu}_{(\nu\rho)}(p) = 0$ which, for a torsion-free connection, ensures that $\Gamma^{\mu}_{\nu\rho}(p) = 0$. But vanishing Christoffel symbols imply a vanishing first derivative of the metric: for the Levi-Civita connection $2g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} = g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}$, thus symmetrizing $(\mu\nu)$ cancels the last two terms, leaving an identity that, evaluated at p, gives $g_{\mu\nu,\rho}(p) = 0$. Hence, these are indeed normal coordinates.

3.3.1.2 Equivalence principle

Normal coordinates are conceptually important in General Relativity: an observer at point p who parametrizes their immediate surroundings using coordinates constructed by geodesics will experience a locally flat metric. This is precisely Einstein's equivalence principle: any free-falling observer, performing local experiments, will not experience a gravitational field. The formal definition of free-falling observer is an observer which follows geodesics, while the local lack of gravitational field means $g_{\mu\mu}(p) = \eta_{\mu\nu}$. In this context, normal coordinates are called *local inertial frame*.
To understand what "local" means, note that there is a way to distinguish whether a gravitational field is present at p: a non-vanishing Riemann tensor. This depends on the second derivatives of the metric, which in general will be non-vanishing. However, to measure the effects of the Riemann tensor, one needs to compare the results of experiments at p and at a nearby point q: this is a non-local observation.

3.3.2 Curvature and torsion

With reference to Fig. 3.2, consider a tangent vector $Z_p \in T_p\mathcal{M}$ and two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$: $[X, Y] = 0$, i.e. they are linearly independent. Construct two curved γ, γ' as in figure, both leading to a point $r \in \mathcal{M}$ which, for simplicity, is close to p. It is possible to impose normal coordinates centered at p such that $x^\mu = (\tau, \sigma, \dots)$, so that $X = \frac{\partial}{\partial\tau}$ and $Y = \frac{\partial}{\partial\sigma}$: then $x^\mu(p) = (0, 0, 0, \dots)$, $X^\mu(q) = (\delta\tau, 0, 0, \dots)$, $x^\mu(s) = (0, \delta\sigma, 0, \dots)$ and $x^\mu(r) = (\delta\tau, \delta\sigma, 0, \dots)$, with $\delta\tau, \delta\sigma$ small.

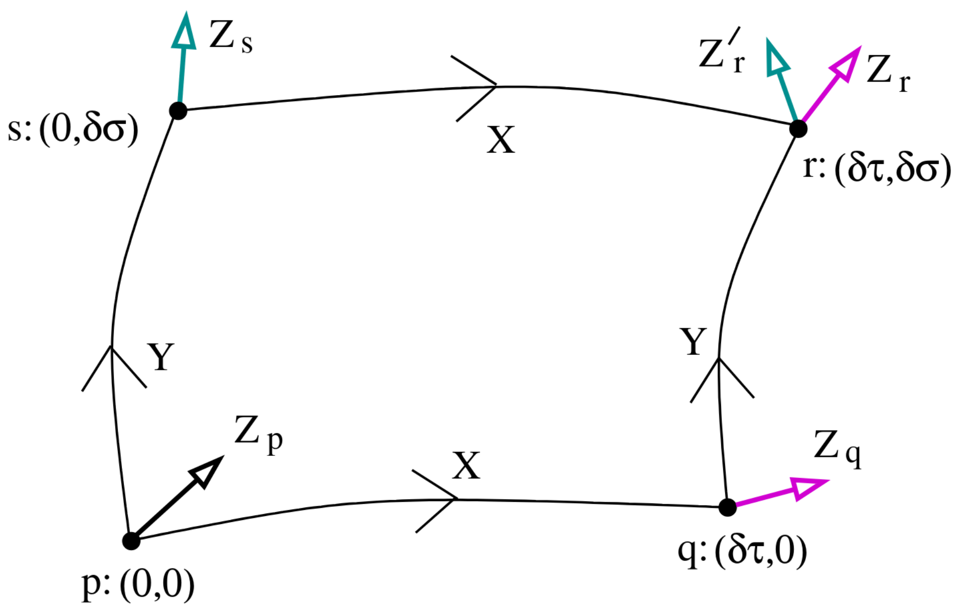


Figure 3.2: Parallel transport along different paths.

First, parallel transport Z_p along X to Z_q, so that Z^μ solves:

$$\frac{dZ^\mu}{d\tau} + X^\nu \Gamma^\mu_{\nu\rho} Z^\rho = 0$$

In normal coordinates $\Gamma^\mu_{\nu\rho}(p) = 0$, thus $\frac{dZ^\mu}{d\tau}\Big|_{\tau=0} = 0$ and the Taylor expansion is:

$$\begin{aligned}
Z^\mu_q &= Z^\mu_p + \frac{\delta\tau^2}{2} \frac{d^2 Z^\mu}{d\tau^2}\bigg|_{\tau=0} + o(\delta\tau^3) \\
&= Z^\mu_p - \frac{\delta\tau^2}{s} \left[X^\nu Z^\rho \frac{d\Gamma^\mu_{\nu\rho}}{d\tau} + \frac{dX^\nu}{d\tau} Z^\rho \Gamma^\mu_{\nu\rho} + X^\nu \frac{dZ^\rho}{d\tau} \Gamma^\mu_{\nu\rho} \right]_{\tau=0} + o(\delta\tau^3) \\
&= Z^\mu_p - \frac{\delta\tau^2}{2} X^\nu Z^\rho \frac{d\Gamma^\mu_{\nu\rho}}{d\tau}\bigg|_{\tau=0} + o(\delta\tau^3) = Z^\mu_p - \frac{\delta\tau^2}{2} \left[X^\nu X^\sigma Z^\rho \Gamma^\mu_{\nu\rho,\sigma} \right]_p + o(\delta\tau^3)
\end{aligned}$$

where $\frac{d}{d\tau} = X^\sigma \partial_\sigma$. Now, Z_q needs to be parallely transported along Y to Z_r, but this time $\frac{dZ^\mu}{d\sigma}\Big|_{\sigma=0}$ doesn't vanish, in general. From the parallel transport equation:

$$\frac{dZ^\mu}{d\sigma}\bigg|_{\sigma=0} = -\left[Y^\nu Z^\rho \Gamma^\mu_{\nu\rho} \right]_q = -\left[Y^\nu Z^\rho X^\sigma \Gamma^\mu_{\nu\rho,\sigma} \right]_p \delta\tau + o(\delta\tau^2)$$

The expansions of Y^ν and Z^ρ at leading order multiply $\Gamma^\mu_{\nu\rho}(p) = 0$, thus only contribute to higher order terms. Next order in $\delta\sigma$:

$$\frac{d^2 Z^\mu}{d\sigma^2}\bigg|_{\sigma=0} = -\left[\left(\frac{dY^\nu}{d\sigma} Z^\rho + Y^\nu \frac{dZ^\rho}{d\sigma} \right) \Gamma^\mu_{\nu\rho} + Y^\nu Z^\rho \frac{d\Gamma^\mu_{\nu\rho}}{d\sigma} \right]_q = -\left[Y^\nu Y^\sigma Z^\rho \Gamma^\mu_{\nu\rho,\sigma} \right]_p + o(\delta\tau)$$

The complete expansion thus is:

$$\begin{aligned}
Z^\mu_r &= Z^\mu_q - \left[Y^\nu Z^\rho X^\sigma \Gamma^\mu_{\nu\rho,\sigma} \right]_p \delta\tau\delta\sigma - \frac{1}{2} \left[Y^\nu Y^\sigma Z^\rho \Gamma^\mu_{\nu\rho,\sigma} \right]_p \delta\sigma^2 + o(\delta^3) \\
&= Z^\mu_p - \frac{1}{2} \Gamma^\mu_{\nu\rho,\sigma}(p) \left[X^\nu X^\sigma Z^\rho \delta\tau^2 + 2 Y^\nu Z^\rho X^\sigma \delta\tau\delta\sigma + Y^\nu Y^\sigma Z^\rho \delta\sigma^2 \right] + o(\delta^3)
\end{aligned}$$

Parallel transport along γ' leads to a similar expression (exchange of X and Y):

$$Z'^\mu_r = Z^\mu_p - \frac{1}{2} \Gamma^\mu_{\nu\rho,\sigma}(p) \left[Y^\nu Y^\sigma Z^\rho \delta\sigma^2 + 2 X^\nu Z^\rho Y^\sigma \delta\sigma\delta\tau + X^\nu X^\sigma Z^\rho \delta\tau^2 \right] + o(\delta^3)$$

The difference between the parallely transported tangent vectors to leading order is:

$$\Delta Z^\mu_r = Z^\mu_r - Z'^\mu_r = -\left[\Gamma^\mu_{\nu\rho,\sigma} - \Gamma^\mu_{\sigma\rho,\nu} \right]_p \left[Y^\nu Z^\rho X^\sigma \right]_p \delta\tau\delta\sigma + o(\delta^3)$$

Recalling that $\Gamma^\mu_{\nu\rho}(p) = 0$, it is possible to write:

$$\Delta Z^\mu_r = -\left[R^\mu{}_{\rho\sigma\nu} Y^\nu Z^\rho X^\sigma \right]_p \delta\sigma\delta\tau + o(\delta^3) \tag{3.55}$$

It would be possible to evaluate the expression at r too, as it would differ only by higher order terms. Although the calculation was carried in a particular choice of coordinates, Eq. 3.55 is a tensor relation, therefore it must hold in all coordinate systems: the Riemann tensor thus determines the path dependence of parallel transport.

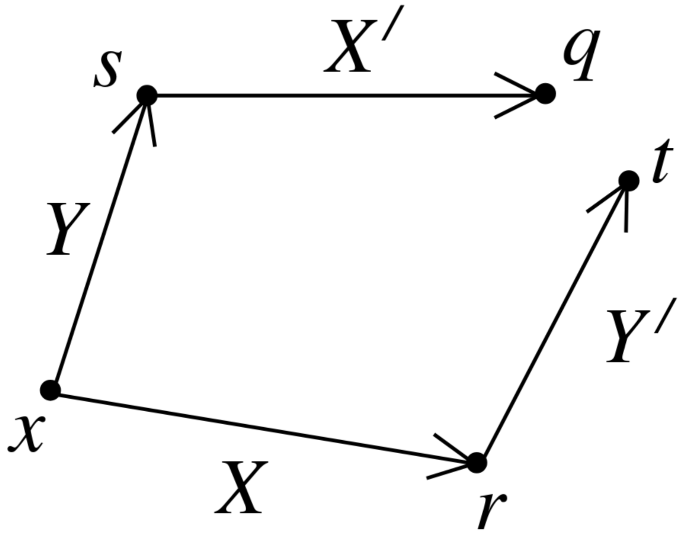


Figure 3.3: Visualization of torsion.

3.3.2.1 Torsion

Consider two tangent vectors $X_p, Y_p \in T_p\mathcal{M}$ and a coordinate system x^μ such that $X_p = X^\mu \partial_\mu$ and $Y_p = Y^\mu \partial_\mu$. If $p : x^\mu$, as in Fig. 3.3 construct $r, s \in \mathcal{M}$ such that $r : x^\mu + \varepsilon X^\mu$ and $s : x^\mu + \varepsilon Y^\mu$, with ε an infinitesimal parameter. Now, parallel transport $X_p \in T_p\mathcal{M}$ along the direction of Y_p to $X'_s \in T_s\mathcal{M}$ and $Y_p \in T_p\mathcal{M}$ along X_p to $Y'_r \in T_r\mathcal{M}$; their components will be:

$$X'_s = \left(X^\mu - \varepsilon \Gamma^\mu_{\nu\rho} Y^\nu X^\rho\right) \qquad\qquad Y'_r = \left(Y^\mu - \varepsilon \Gamma^\mu_{\nu\rho} X^\nu Y^\rho\right)$$

Repeating this process, starting from point s and moving along the direction of X'_s, a new point $q \in \mathcal{M}$ is determined, with coordinates:

$$q : x^\mu + \varepsilon\left(X^\mu + Y^\mu\right) - \varepsilon^2 \Gamma^\mu_{\nu\rho} Y^\nu X^\rho$$

Analogously, starting at point r and moving along the direction of Y'_r, a new point $t \in \mathcal{M}$ is determined, with coordinates:

$$t : x^\mu + \varepsilon\left(X^\mu + Y^\mu\right) - \varepsilon^2 \Gamma^\mu_{\nu\rho} X^\nu Y^\rho$$

If the connection is torsion-free, then $q \equiv t$. On the other hand, if $T^\mu_{\nu\rho} \neq 0$, the parallelogram fails to close, as in Fig. 3.3.

3.3.3 Geodesic deviation

Definition 3.3.3. Given a one-parameter family of geodesics $\{x^\mu(\tau; s)\}_{s\in\mathbb{R}}$ on a manifold $\mathcal{M}$, the tangent vector field and the *deviation vector* field are defined as:

$$X^\mu := \left.\frac{\partial x^\mu}{\partial \tau}\right|_s \qquad\qquad S^\mu := \left.\frac{\partial x^\mu}{\partial s}\right|_\tau \tag{3.56}$$

The meaning of these vector fields is evident: the tangent vector field fixes a particular geodesics (i.e. a particular s) and assigns at each point of the geodesic its tangent vector, while the deviation

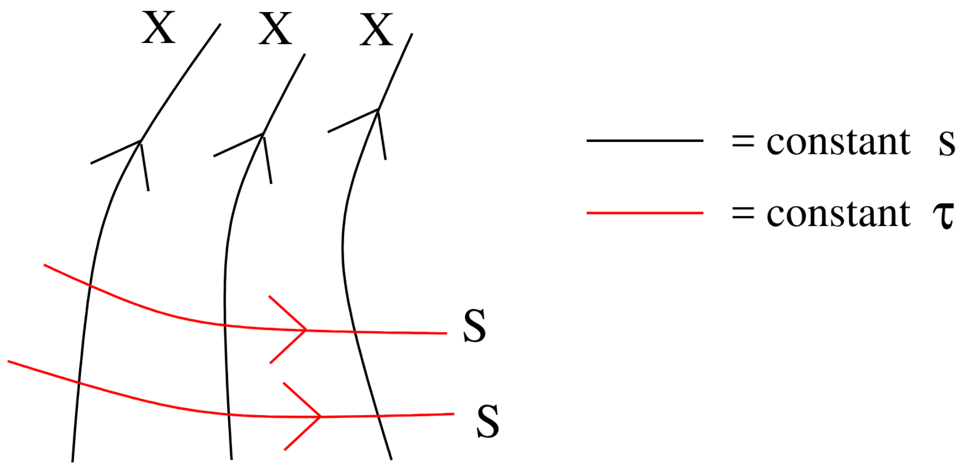


Figure 3.4: A one-parameter family of geodesics generated by X.

vector field fixes a particular value of the affine parameter τ and assigns at each point with this value a vector which takes to a nearby geodesic (at the same τ).
The family of geodesics sweeps a surface embedded in the manifold, so there is freedom in the choice of coordinates s and τ. In particular, it's always possible to pick them so that $X = \frac{\partial}{\partial \tau}$ and $S = \frac{\partial}{\partial s}$, in order for them to be linearly independent: $[X, S] = 0$, as in Fig. 3.4. Consider a torsion-free connection, so that:

$$\nabla_X S - \nabla_S X = [X, S] = 0 \qquad \Rightarrow \qquad \nabla_X \nabla_X S = \nabla_X \nabla_S X = \nabla_S \nabla_X X + R(X, S)X$$

But X is tangent to geodesics, so from Eq. 3.53 $\nabla_X X = 0$ and:

$$\nabla_X \nabla_X S = R(X, S)X \tag{3.57}$$

Restricting to an integral curve γ of the vector field X, i.e. $X^\mu|_\gamma = \frac{dx^\mu}{d\tau}$, the covariant derivative along γ becomes:

$$\nabla_X\big|_\gamma = X^\mu\big|_\gamma \nabla_\mu = \frac{dx^\mu}{d\tau}\nabla_\mu \equiv \frac{D}{D\tau}$$

Hence, in index notation, the change of the deviation vector along the geodesic is expressed as:

$$\frac{D^2 S^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} X^\rho S^\sigma X^\nu \tag{3.58}$$

This can be interpreted as the relative acceleration of neighbouring geodesics, and it is determined by the Riemann tensor. Experimentally, such geodesic deviations are called *tidal forces*.

3.4 Riemann tensor

Recall Eq. 3.36 for the components of the Riemann tensor $R^\sigma{}_{\rho\mu\nu}$: it is manifestly anti-symmetric in its last two indices, but there are also other subtle symmetries when using the Levi-Civita connection.

Proposition 3.4.1. *On a metric manifold with a Levi-Civita connection:*

$$R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu} = -R_{\rho\sigma\mu\nu} = R_{\mu\nu\sigma\rho} \tag{3.59}$$

$$R_{\sigma[\rho\mu\nu]} = 0 \tag{3.60}$$

Proof. Set normal coordinates centered at a point p: then, $\Gamma^{\mu}_{\nu\rho} = 0$ and $\partial_\mu g^{\lambda\sigma} = 0$ at that point. At p, the Riemann tensor can be written as:

$$\begin{aligned} R_{\sigma\rho\mu\nu} &= g_{\sigma\lambda} R^{\lambda}{}_{\rho\mu\nu} = g_{\sigma\lambda}\left[\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho}\right] \\ &= \frac{1}{2}\left[\partial_\mu\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right) - \partial_\nu\left(\partial_\mu g_{\sigma\rho} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho}\right)\right] \\ &= \frac{1}{2}\left[\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\mu\rho}\right] \end{aligned}$$

The symmetries are then manifest, and being these tensor equations they are valid in all coordinate systems. □

An important computation tool is the *Bianchi identity.*

Theorem 3.4.1 (Bianchi)**.** *On a metric manifold with a Levi-Civita connection:*

$$\nabla_{[\lambda} R_{\sigma\rho]\mu\nu} = 0 \qquad \Leftrightarrow \qquad R^{\sigma}{}_{\rho[\mu\nu;\lambda]} = 0 \tag{3.61}$$

Proof. The two equations are equivalent, so the proof is of the first one. In normal coordinates $\nabla_\mu = \partial_\mu$ at p, so schematically: $R = \partial\Gamma + \Gamma\Gamma$, thus $\nabla R = \partial R = \partial^2\Gamma + \Gamma\partial\Gamma = \partial^2\Gamma$. Explicitly:

$$\partial_\lambda R_{\sigma\rho\mu\nu} = \frac{1}{2}\partial_\lambda\left[\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\mu\rho}\right]$$

Anti-symmetrizing the appropriate indices yields the result. □

Note that Eq. 3.60-3.61 do not require that the connection is a Levi-Civita connection, but are valid for general torsion-free connections.

3.4.1 Ricci and Einstein tensors

Definition 3.4.1. On a metric manifold, the *Ricci tensor* is defined as:

$$R_{\mu\nu} := R^{\rho}{}_{\mu\rho\nu} \tag{3.62}$$

The *Ricci scalar* is defined as:

$$R := g^{\mu\nu} R_{\mu\nu} \tag{3.63}$$

Proposition 3.4.2. *On a metric manifold with a Levi-Civita connection:*

$$R_{\mu\nu} = R_{\nu\mu} \tag{3.64}$$

Proof. Using Eq. 3.59: $R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu} = g^{\rho\sigma} R_{\rho\nu\sigma\mu} = R_{\nu\mu}$. □

Proposition 3.4.3. *On a metric manifold:*

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \tag{3.65}$$

Proof. Writing explicitly Bianchi identity:

$$\nabla_\lambda R_{\sigma\rho\mu\nu} + \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\rho R_{\lambda\sigma\mu\nu} = 0$$

Contracting with $g^{\mu\lambda} g^{\rho\nu}$:

$$\nabla^\mu R_{\mu\sigma} - \nabla_\sigma R + \nabla^\nu R_{\nu\sigma} = 0$$

which yields the thesis. □

Definition 3.4.2. On a metric manifold, the *Einstein tensor* is defined as:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3.66}$$

Proposition 3.4.4. *On a metric manifold, the Einstein tensor is covariantly constant:*

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{3.67}$$

Proof. Trivial from Eq. 3.65. □

3.4.2 Connection and curvature forms

This section is based on a Lorentzian manifold, but the discussion is equivalent on a Riemannian one: it's enough to swap η_{ab} with δ_{ab} as the flat metric.

3.4.2.1 Vielbeins

Although typically calculations are carried on a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, there are possible basis without such an interpretation. For example, a linear combination of a coordinate basis won't in general be a coordinate basis itself:

$$\hat{e}_a = e_a{}^{\mu} \partial_\mu \tag{3.68}$$

A particularly useful non-coordinate basis is one such that:

$$g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu} = \eta_{ab} \tag{3.69}$$

The components $e_a{}^{\mu}$ are called *vielbeins* or *tetrads*. In this non-coordinate system, the manifold looks flat (or, at least, its patch covered by the given chart). In the following computations, greek indices are raised/lowered by the metric $g_{\mu\nu}$, while latin indices by the metric η_{ab}.
The vielbeins aren't unique, for given a set of vielbeins $e_a{}^{\mu}$ it is always possible to find a new one:

$$\tilde{e}_a{}^{\mu} = e_b{}^{\mu} (\Lambda^{-1})^b{}_a \tag{3.70}$$

The transformation matrix must satisfy the condition imposed by Eq. 3.69, i.e.:

$$\Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab} \tag{3.71}$$

These are *local Lorentz transformations*, because the condition is that of Lorentz transformation, but Λ is now allowed to vary over the manifold. The dual basis of one-forms $\{\hat{\theta}^a\}$ is defined by $\hat{\theta}^a(\hat{e}_b) = \delta^a_b$. The relation to the coordinate basis is:

$$\hat{\theta}^a = e^a{}_{\mu} dx^{\mu} \tag{3.72}$$

where the coefficients satisfy:

$$e^a{}_{\mu} e_b{}^{\mu} = \delta^a_b \tag{3.73}$$

$$e^a{}_{\mu} e_a{}^{\nu} = \delta^{\nu}_{\mu} \tag{3.74}$$

The metric is a tensor, so $g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \hat{\theta}^a \otimes \hat{\theta}^b$, thus it is related to the vielbeins by:

$$g_{\mu\nu} = e^a{}_{\mu} e^b{}_{\nu} \eta_{ab} \tag{3.75}$$

3.4.2.2 Connection 1-form

On a non-coordinate basis $\{\hat{e}_a\}$, connection components are computed in the usual way:

$$\nabla_{\hat{e}_c}\hat{e}_b = \Gamma^a_{cb}\hat{e}_a \tag{3.76}$$

However, these are not the same components $\Gamma^\mu_{\nu\rho}$ as in a coordinate basis.

Definition 3.4.3. On a metric manifold with vielbeins, the *connection 1-form* is defined as:

$$\omega^a{}_b := \Gamma^a_{cb}\hat{\theta}^c \tag{3.77}$$

Note that these are really n^2 1-forms, according to values of $a, b = 1, \ldots, n \equiv \dim_{\mathbb{R}} \mathcal{M}$. This is also known as the *spin connection*, due to its relationship to spinors in curved spacetime.

Proposition 3.4.5. *Given a local Lorentzian transformation Λ:*

$$\tilde{\omega}^a{}_b = \Lambda^a{}_c\omega^c{}_d(\Lambda^{-1})^d{}_b + \Lambda^a{}_c(d\Lambda^{-1})^c{}_b \tag{3.78}$$

The second term reflects the second term in Prop. 3.2.1, which involves the derivative of the coordinate transformation. Important results for connection 1-forms is the *Cartan structure equation.*

Theorem 3.4.2 (Cartan)**.** *On a metric manifold with a torsion-free connection:*

$$d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0 \tag{3.79}$$

Proof. From Eq. 3.76 $\Gamma^a_{cb} = e^a{}_\rho e_c{}^\mu \nabla_\mu e_b{}^\rho$, thus, remembering $\Gamma^\rho_{[\mu\nu]} = 0$ (torsion-free):

$$\begin{aligned}
\omega^a{}_b \wedge \hat{\theta}^b &= \Gamma^a_{cb}\left(e^c{}_\mu dx^\mu\right) \wedge \left(e^b{}_\nu dx^\nu\right) \\
&= e^a{}_\rho e_c{}^\mu \left(\partial_\mu e_b{}^\rho + e_b{}^\nu \Gamma^\rho_{\mu\nu}\right)\left(e^c{}_\mu dx^\mu\right) \wedge \left(e^b{}_\nu dx^\nu\right) \\
&= e^a{}_\rho \underbrace{e_c{}^\lambda e^c{}_\mu}_{\delta^\lambda_\mu} e^b{}_\nu \left(\partial_\lambda e_b{}^\rho + e_b{}^\sigma \Gamma^\rho_{\lambda\sigma}\right) dx^\mu \wedge dx^\nu = e^a{}_\rho e^b{}_\nu \partial_\mu e_b{}^\rho dx^\mu \wedge dx^\nu
\end{aligned}$$

But $e^b{}_\nu e_b{}^\rho = \delta^\rho_\nu$, so $e^b{}_\nu \partial_\mu e_b{}^\rho = -e_b{}^\rho \partial_\mu e^n{}_\nu$, hence:

$$\omega^a{}_b \wedge \hat{\theta}^b = -e^a{}_\rho e_b{}^\rho \partial_\mu e^b{}_\nu dx^\mu \wedge dx^\nu = -\partial_\mu e^a{}_\nu dx^\mu \wedge dx^\nu = -d\hat{\theta}^a$$

□

For a Levi-Civita connection, a stronger result holds.

Proposition 3.4.6. *On a metric manifold with a Levi-Civita connection:*

$$\omega_{ab} = -\omega_{ba} \tag{3.80}$$

Proof. Being the Levi-Civita connection compatible with the metric:

$$\begin{aligned}
\Gamma_{abc} &= \eta_{ad} e^d{}_\rho e_b{}^\mu \nabla_\mu e_c{}^\rho = -\eta_{ad} e_c{}^\rho e_b{}^\mu \nabla_\mu e^d{}_\rho = -\eta_{cf} e^f{}_\sigma e_b{}^\mu \nabla_\mu (\eta_{ad} g^{\rho\sigma} e^d{}_\rho) \\
&= -\eta_{cf} e^f{}_\rho e_b{}^\mu \nabla_\mu e_a{}^\rho = -\Gamma_{cba}
\end{aligned}$$

From Eq. 3.77 $\omega_{ab} = \Gamma_{acb}\hat{\theta}^c$, thus completing the proof. □

Eq. 3.79-3.80 allow to quickly compute the spin connection, as they uniquely define it. Indeed, ω_{ab} being anti-symmetric means that there are $\frac{1}{2}n(n-1)$ independent 1-forms, i.e. $\frac{1}{2}n^2(n-1)$ independent components. The Cartan structure equation relates two 2-forms, each with $\frac{1}{2}n(n-1)$ independent components, thus posing $\frac{1}{2}n^2(n-1)$ constraints (as there are n equations) and uniquely fixing the spin connection.

3.4.2.3 Curvature 2-form

Recall Eq. 3.59, which holds for Levi-Civita connections. Computing the Riemann tensor in the non-coordinate basis ${R^a}_{bcd} = R(\hat{\theta}^a, \hat{e}_b, \hat{e}_c, \hat{e}_d)$, the anti-symmetry of the last two indices persists: ${R^a}_{bcd} = -{R^a}_{bdc}$.

Definition 3.4.4. On a metric manifold with a Levi-Civita connection, the *curvature 2-form* is defined as:

$$ {\mathcal{R}^a}_b := \frac{1}{2} {R^a}_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d \tag{3.81} $$

Again, these are really n^2 2-forms. A second *Cartan structure relation* holds.

Theorem 3.4.3 (Cartan)**.** *On a metric manifold with a Levi-Civita connection:*

$$ {\mathcal{R}^a}_b = d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b \tag{3.82} $$

Connection and curvature forms make computing the Riemann tensor less tedious, as exterior derivatives take significant less effort than covariant derivatives.

Part III

General Relativity

Chapter 4

Einstein Field Equations

The force of gravity is mediated by a gravitational field, which is identified with a metric $g_{\mu\nu}(x)$ on a 4d Lorentzian manifold called spacetime. This metric is a dynamical object, as all other fields in Nature, thus the laws governing its dynamics must be studied.

4.1 Einstein-Hilbert action

Differential Geometry places strict limits to the possible actions that can be formulated, ensuring it is something intrinsic to the metric and independent on the particular choice of coordinates.
Spacetime is a Lorentzian manifold $\mathcal{M}$, thus, recalling the canonical volume form in Eq. 3.13, there need to be metric-dependent scalar function on $\mathcal{M}$: the obvious non-trivial choice is the Ricci scalar. The resulting action is the *Einstein-Hilbert action*:

$$\mathcal{S} = \int d^4x\sqrt{-\mathrm{g}}R \tag{4.1}$$

where the negative sign makes it manifest that the metric has signature $(-,+,+,+)$. Schematically, the Riemann tensor is $R \sim \partial\Gamma + \Gamma\Gamma$, while $\Gamma \sim \partial g$, thus the Einstein-Hilbert action is second order in derivatives of the metric, like most other actions in physics.

4.1.1 Equations of motion

To determine the Euler-Lagrange equations of the Einstein-Hilbert action, consider a fixed initial metric $g_{\mu\nu}(x)$ and a perturbation $g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$. For the inverse metric:

$$g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu \quad \Rightarrow \quad \delta g_{\mu\nu}g^{\nu\rho} + g_{\mu\nu}\delta g^{\nu\rho} = 0 \quad \Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$$

Lemma 4.1.1. *The variation of* $\sqrt{-\mathrm{g}}$ *is:*

$$\delta\sqrt{-\mathrm{g}} = -\frac{1}{2}\sqrt{-\mathrm{g}}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{4.2}$$

Proof. Recalling that for a diagonalizable matrix $\log\det A = \operatorname{tr}\log A$, then $(\det A)^{-1}\delta(\det A) = \operatorname{tr}(A^{-1}\delta A)$. Applying this to the metric:

$$\delta\sqrt{-\mathrm{g}} = \frac{1}{2}\frac{1}{\sqrt{-\mathrm{g}}}\delta(-\mathrm{g}) = \frac{1}{2}\frac{1}{\sqrt{-\mathrm{g}}}(-\mathrm{g})g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-\mathrm{g}}\, g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-\mathrm{g}}\, g_{\mu\nu}g^{\mu\nu}$$

□

Lemma 4.1.2. *The variation of the Christoffel symbols is:*

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\nabla_{\mu}\delta g_{\sigma\nu} + \nabla_{\nu}\delta g_{\sigma\mu} - \nabla_{\sigma}\delta g_{\mu\nu}) \tag{4.3}$$

Proof. First note that, although $\Gamma^{\rho}_{\mu\nu}$ is not a tensor, $\delta\Gamma^{\rho}_{\mu\nu}$ is, for it is the difference of two Christoffel symbols, one computed with $g_{\mu\nu}$ and the other with $g_{\mu\nu} + \delta g_{\mu\nu}$, but the extra term in the transformation law of $\Gamma^{\rho}_{\mu\nu}$ is independent of the metric, thus cancels out.
This osservation allows to work in normal coordinates at $p \in \mathcal{M}$, so that $\partial_{\rho}g_{\mu\nu} = 0$ and $\Gamma^{\rho}_{\mu\nu} = 0$ at that point. Then, to linear order in the variation, at p:

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_{\mu}\delta g_{\sigma\nu} + \partial_{\nu}\delta g_{\sigma\mu} - \partial_{\sigma}\delta g_{\mu\nu}) = \frac{1}{2}g^{\rho\sigma}(\nabla_{\mu}\delta g_{\sigma\nu} + \nabla_{\nu}\delta g_{\sigma\mu} - \nabla_{\sigma}\delta g_{\mu\nu})$$

This is a tensor equation, hence valid in all coordinate system and, being p arbitrary, on all $\mathcal{M}$. □

Lemma 4.1.3. *The variation of the Ricci tensor is:*

$$\delta R_{\mu\nu} = \nabla_{\rho}\delta\Gamma^{\rho}_{\rho\nu} - \nabla_{\nu}\delta\Gamma^{\rho}_{\rho\mu} \tag{4.4}$$

Proof. Working in normal coordinates, the Riemann tensor becomes $R^{\sigma}{}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\sigma}_{\nu\mu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\mu}$, so:

$$\delta R^{\sigma}{}_{\mu\rho\nu} = \partial_{\rho}\delta\Gamma^{\sigma}_{\nu\mu} - \partial_{\nu}\delta\Gamma^{\sigma}_{\rho\mu} = \nabla_{\rho}\delta\Gamma^{\sigma}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\sigma}_{\rho\mu}$$

This is a tensor equation, hence valid in all coordinates systems and on all $\mathcal{M}$. Contracting indices σ, ρ and working to leading order yields the result. □

Proposition 4.1.1. *The Euler-Lagrange equations of the Einstein-Hilbert action are:*

$$G_{\mu\nu} = 0 \tag{4.5}$$

Proof. Varying the Einstein-Hilbert action:

$$\begin{aligned}
\delta\mathcal{S} &= \delta\int d^4x\sqrt{-\mathrm{g}}\, g^{\mu\nu}R_{\mu\nu} \\
&= \int d^4x\,\left[\delta(\sqrt{-\mathrm{g}})g^{\mu\nu}R_{\mu\nu} + \sqrt{-\mathrm{g}}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-\mathrm{g}}\, g^{\mu\nu}(\delta R_{\mu\nu})\right] \\
&= \int d^4x\sqrt{-\mathrm{g}}\left[\left(-\frac{1}{2}Rg_{\mu\nu} + R_{\mu\nu}\right)\delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_{\rho}\delta\Gamma^{\rho}_{\mu\nu} - \nabla_{\nu}\Gamma^{\rho}_{\rho\mu}\right)\right] \\
&= \int d^4x\sqrt{-\mathrm{g}}\left[\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \nabla_{\mu}\left(g^{\rho\nu}\nu\Gamma^{\mu}_{\rho\nu} - g^{\mu\nu}\delta\Gamma^{\rho}_{\rho\nu}\right)\right]
\end{aligned}$$

The last term is a total derivative, hence by the divergence theorem Eq. 3.41 it yields a boundary term which can be ignored (Gibbons-Hawking boundary term). The Euler-Lagrange equations are then found imposing $\delta\mathcal{S}$ for arbitrary $\delta g_{\mu\nu}$, so recalling the definition of the Einstein tensor Eq. 3.66 the proof is completed. □

These equations are called *Einstein field equations* in the absence of any matter. For this reason, they can be further simplified: by contracting with $g^{\mu\nu}$ one finds $R = 0$, thus:

$$R_{\mu\nu} = 0 \tag{4.6}$$

4.1.1.1 Dimensional analysis

The Einstein-Hilbert action in Eq. 4.1 doesn't have the right dimensions, which will be necessary when considering the presence of matter.
If x^μ has dimension of length, then $g_{\mu\nu}$ is dimensionless and the Ricci scalar is $[R] = L^{-2}$. Including the integration measure $[\mathcal{S}] = L^2$, but it must have the same dimension of $[\hbar] = ML^2T^{-1}$ (i.e. energy $\times$ times). Thus, the action with the right dimension is:

$$\mathcal{S} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}R \tag{4.7}$$

In the following, natural units are adopted: $c = \hbar = 1$.

4.1.1.2 Cosmological constant

It reality, Eq. 4.1 is not the simplest action allowed by Differential Geometry: in fact, a constant term could be added to the Ricci scalar. The action then becomes:

$$\mathcal{S} = \frac{1}{16\alpha G}\int d^4x\sqrt{-\mathrm{g}}(R - 2\Lambda) \tag{4.8}$$

Λ is called the *cosmological constant* and has dimension $[\Lambda] = L^{-2}$. The resulting field equations are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{4.9}$$

Contracting with $g^{\mu\nu}$ yields $R = 4\Lambda$, thus in the absence of matter:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{4.10}$$

4.1.2 Diffeomorphisms

Being the metric a symmetric $\mathbb{R}^{4,4}$ matrix, it should have $\frac{1}{2} \times 4 \times 5 = 10$ degrees of freedom. However, not all the components of $g_{\mu\nu}$ are physical. Indeed, two metrics related by a change of coordinates $x^\mu \mapsto \tilde{x}^\mu(x)$ describe the same physical spacetime, thus there is a redundancy in any given representation of the metric, which removes precisely 4 degrees of freedom, leaving with only 6 remaining degrees of freedom.
Mathematically, given a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, it maps all fields on $\mathcal{M}$ to a new set of fields on $\mathcal{M}$: the result is physically indistinguishable from the original, describing the same spacetime but in different coordinates. Such diffeomorphisms are analogous to gauge symmetries in Yang-Mills theory.
Consider a diffeomorphism $x^\mu \mapsto \tilde{x}^\mu = x^\mu + \delta x^\mu$: this can be viewed as an active change, where points in spacetime are mapped from one another, or as a passive change, in which only the coordinates of each point are affected, but the two are equivalent. This change of coordinates can be thought as generated by an infinitesimal vector field $X^\mu : \delta x^\mu = -X^\mu(x)$, so that the metric transforms as:

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}g_{\rho\sigma}(x) = \left(\delta^\rho_\mu + \partial_\mu X^\rho\right)\left(\delta^\sigma_\nu + \partial_\nu X^\sigma\right)g_{\rho\sigma}(x) \\ &= g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g_{\nu\rho}(x)\partial_\mu X^\rho\end{aligned}$$

Meanwhile, the Taylor expansion around $\tilde{x} = x + \delta x$ is:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x + \delta x) = \tilde{g}_{\mu\nu}(x) - X^\lambda \partial_\lambda \tilde{g}_{\mu\nu}(x)$$

Comparing the metrics at the same point, it is understood that it undergoes an infinitesimal change:

$$\delta g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = X^\lambda \partial_\lambda g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g_{\nu\rho}(x)\partial_\mu X^\rho$$

By Eq. 2.23, this is the Lie derivative of the metric:

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu} \tag{4.11}$$

Thus, an infinitesimal diffeomorphism along $X \in \mathfrak{X}(\mathcal{M})$ makes the metric change by an infinitesimal amount given by its Lie derivative along X. This can be viewed as the leading term in the Taylor expansion along X. These equations can be rewritten in a simpler form:

$$\delta g_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \partial_\nu X_\mu - X^\rho \partial_\nu g_{\mu\rho} + \partial_\mu X_\nu - X^\rho \partial_\mu g_{\nu\rho} = \partial_\mu X_\nu + \partial_\nu X_\mu + 2 g_{\rho\sigma} \Gamma^\sigma_{\mu\nu} X^\rho$$

Therefore:

$$\delta g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu \tag{4.12}$$

This equation can be used in the path integral. In fact, insisting that $\delta\mathcal{S} = 0$ for any $\delta g_{\mu\nu}$ gives the equations of motion; on the other hand, those variations $\delta g_{\mu\nu}$ for which $\delta\mathcal{S} = 0$ for any metric are the *symmetries* of the action. From Prop. 4.1.1:

$$\delta\mathcal{S} = \int d^4x \sqrt{-\mathrm{g}}\, G^{\mu\nu} \delta g_{\mu\nu} = 2 \int d^4x \sqrt{-\mathrm{g}}\, G^{\mu\nu} \nabla_\mu X_\nu$$

Invariance for $x^\mu \mapsto x^\mu - X^\mu$ means that $\delta\mathcal{S} = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$, hence, integrating by parts:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{4.13}$$

This is Bianchi identity: although it is a result from Differential Geometry, is follows from diffeomorphism invariance of the Einstein-Hilbert action. Bianchi identity is comprised of 4 equations, which make the 10 Einstein equations not completely independent: in fact, only 6 of then are independent, the same number of independent components of the metric, thus ensuring that the field equations are not overdetermined.

4.2 Simple solutions

The Einstein equations in the absence of matter are:

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

with $\Lambda \in \mathbb{R}$.

4.2.1 Minkowski space

The simplest case is $\Lambda = 0$. The Einstein equations then reduce to $R_{\mu\nu} = 0$, with the condition of $g_{\mu\nu}$ being non-degenerate, for it is a metric and the field equations require the existence of the inverse metric $g^{\mu\nu}$. This restriction is physically unusual: it is not a holonomic constraint on the physical degrees of freedom, but an inequality $\det g < 0$ (with required signature $(-,+,+,+)$) which is not found for other fields of the Standard Model.
The simplest Ricci flat metric is Minkowski space $ds^2 = -dt^2 + d\mathbf{x}^2$, but it is not the only metric obeying $R_{\mu\nu} = 0$. Another example is Schwarzschild metric.

4.2.2 de Sitter space

Consider $\Lambda > 0$. A possible ansatz is:

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

4.2.2.1 Riemann tensor

First, one needs to compute the Ricci tensor for this metric: a simple way is using the curvature form. The non-coordinate 1-forms satisfy $d^2 = \eta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$, thus:

$$\hat{\theta}^0 = f dt \qquad \hat{\theta}^1 = f^{-1} dr \qquad \hat{\theta}^2 = r d\vartheta \qquad \hat{\theta}^3 = r\sin\vartheta d\varphi$$

$$d\hat{\theta}^0 = f' dr \wedge dt \qquad d\hat{\theta}^1 = 0 \qquad d\hat{\theta}^2 = dr \wedge d\vartheta \qquad d\hat{\theta}^3 = \sin\vartheta dr \wedge d\varphi + r\cos\vartheta d\vartheta \wedge d\varphi$$

The first Cartan structure relation Eq. 3.79, together with Eq. 3.80, allow to determine the connection 1-form. For example, the first equation is ${\omega^0}_1 = f' f dt = f' d\hat{\theta}^0$, and then ${\omega^1}_0 = \omega_{10} = -\omega_{01} = {\omega^0}_1$. Using the other structure equation, the only non-vanishing components of the connection 1-form are:

$${\omega^0}_1 = {\omega^1}_0 = f'\hat{\theta}^0 \qquad {\omega^2}_1 = -{\omega^1}_2 = \frac{f}{r}\hat{\theta}^2 \qquad {\omega^3}_1 = -{\omega^1}_3 = \frac{f}{r}\hat{\theta}^3 \qquad {\omega^3}_2 = -{\omega^2}_3 = \frac{\cot\theta}{r}\hat{\theta}^3$$

The curvature 2-form can be computed from the second Cartan structure relation Eq. 3.82. For example, ${\mathcal{R}^0}_1 = d{\omega^0}_1 + {\omega^0}_c \wedge {\omega^c}_1$, but ${\omega^0}_c \wedge {\omega^c}_1 = {\omega^0}_1 \wedge {\omega^1}_1 = 0$, thus ${\mathcal{R}^0}_1 = d{\omega^0}_1 = ((f')^2 + f'' f) dr \wedge dt$. From the curvature 2-form, the Riemann tensor can be calculated via Eq. 3.81 (recall the anti-symmetries of the Riemann tensor), finding the only non-vanishing components:

$$R_{0101} = f f'' + (f')^2 \qquad R_{0202} = R_{0303} = \frac{f f'}{r} \qquad R_{1212} = R_{1313} = -\frac{f f'}{r} \qquad R_{2323} = \frac{1-f^2}{r^2}$$

To convert back to $x^\mu = (t, r, \vartheta, \varphi)$, use $R_{\mu\nu\rho\sigma} = {e^a}_\mu {e^b}_\nu {e^c}_\rho {e^d}_\sigma R_{abcd}$, which is particularly easy given that the matrices ${e_a}^\mu$ which define the non-coordiante 1-forms are diagonal:

$$R_{trtr} = f(r)f''(r) + f'(r)^2 \qquad R_{t\vartheta t\vartheta} = f(r)^3 f'(r) r \qquad R_{t\varphi t\varphi} = f(r)^3 f'(r) r \sin^2\vartheta$$

$$R_{r\vartheta r\vartheta} = -\frac{f'(r) r}{f(r)} \qquad R_{r\varphi r\varphi} = -\frac{f'(r) r}{f(r)}\sin^2\vartheta \qquad R_{\vartheta\varphi\vartheta\varphi} = (1 - f(r)^2) r^2 \sin^2\vartheta$$

4.2.2.2 Ricci tensor

Given the Riemann tensor, it is easy to check that the Ricci tensor is diagonal with components:

$$R_{tt} = -f(r)^4 R_{rr} = f(r)^3 \left[f''(r) + \frac{2f'(r)}{r} + \frac{f'(r)^2}{f(r)} \right]$$

$$R_{\varphi\varphi} = \sin^2\vartheta R_{\vartheta\vartheta} = (1 - f(r)^2 - 2f(r)f'(r) r)\sin^2\vartheta$$

Imposing $R_{\mu\nu} = \Lambda g_{\mu\nu}$ determines two constraints on $f(r)$:

$$f''(r) + \frac{2f'(r)}{r} + \frac{f'(r)^2}{f(r)} = -\frac{\Lambda}{f(r)} \qquad\qquad 1 - 2f(r)f'(r) r - f(r)^2 = \Lambda r^2$$

A solution is:

$$f(r) = \sqrt{1 - \frac{r^2}{R^2}} \qquad R^2 \equiv \frac{3}{\Lambda}$$

This determines the metric of *de Sitter space*:

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{4.14}$$

More precisely, this is the *static patch* of de Sitter space.

4.2.2.3 de Sitter geodesics

To interpret the metric, it's useful to study its geodesics. First, note that a non-trivial $g_{tt}(r)$ term means that a particle won't remain at rest at $r \neq 0$, but it will be pushed to smaller values of $g_{tt}(r)$, i.e. larger values of r. The action of a particle in the general $f(r)$ metric, parametrized by its proper time σ, is:

$$\mathcal{S} = \int d\sigma \left[-f(r)^2\dot{t}^2 + f(r)^{-2}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\varphi^2)\right] \tag{4.15}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\sigma}$. This Lagrangian has two ignorable degrees of freedom which lead to conserved quantities: $t(\sigma)$ and $\varphi(\sigma)$, as they appear only with time derivatives. The first one has energy as the conserved quantity, while the second has angular momentum:

$$E = -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} = f(r)^2\dot{t} \tag{4.16}$$

$$\ell = \frac{1}{2}\frac{\partial L}{\partial \dot{\varphi}} = r^2\sin^2\vartheta\dot{\varphi} \tag{4.17}$$

The $\frac{1}{2}$ are due to the absence of the usual factor in the kinetic term. The equations of motion from the action Eq. 4.15 should be supplemented with a constraint to distinguish whether a particle is massive or massless. For a massive particle, the trajectory must be timelike, so:

$$-f(r)^2\dot{t}^2 + f(r)^{-2}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2) = -1$$

WLOG consider geodeiscs that lie in the $\vartheta = \frac{\pi}{2}$ plane, so that $\dot{\theta} = 0$ and $\sin^2\vartheta = 1$. Replacing (t, φ) with (E, ℓ), the constraint becomes:

$$\dot{r}^2 + V_{\text{eff}}(r) = E^2 \qquad V_{\text{eff}}(r) = \left(1 + \frac{\ell^2}{r^2}\right) f(r)^2 \tag{4.18}$$

For de Sitter space:

$$V_{\text{eff}}(r) = \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{r^2}{R^2}\right)$$

This effective potential is plotted in Fig. 4.1: immediately, on sees that the potential pushes the particle to larger values of r. Focusing on geodesics with $\ell = 0$, the potential is a harmonic repulsor: a particle stationary at $r = 0$ is an unstable geodesic, for if it has non-zero initial velocity it will follow the trajectory:

$$r(\sigma) = R\sqrt{E^2 - 1}\sinh\frac{\sigma}{R} \tag{4.19}$$

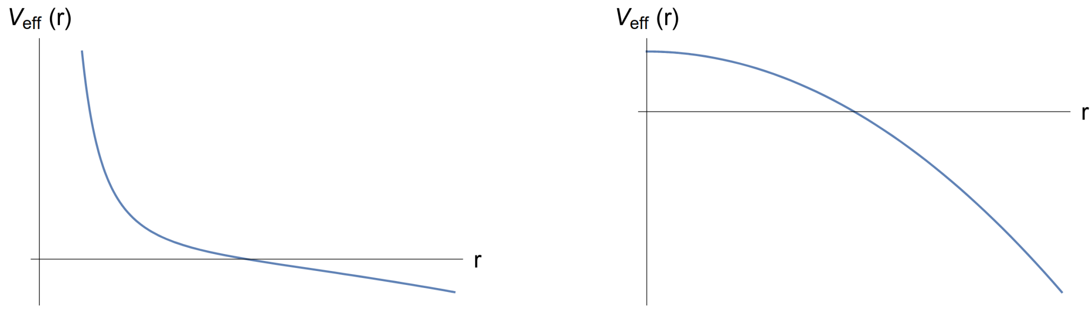


Figure 4.1: Effective potential for de Sitter geodesics, with $\ell \neq 0$ and $\ell = 0$ respectively.

Note that the metric is singular at $r = R$, a fact not manifest in the geodesic Eq. 4.19 which shows that any observer reaches $r = R$ in finite proper time. Problems arise when studying the coordinate time t, which has the interpretation of the time experienced by someone stationary at $r = 0$; from Eq. 4.16:

$$\frac{dt}{d\sigma} = E\left(1 - \frac{r^2}{R^2}\right)^{-1}$$

This shows that $t(\sigma) \to \infty$ as $r(\sigma) \to R$: in fact, if $r(\sigma^*) = R$, then for $\sigma = \sigma^* - \varepsilon$ one has $\frac{dt}{d\sigma} = -\frac{\alpha}{\varepsilon}$, i.e. $t(\varepsilon) = -\log(\varepsilon/R)$, so $t(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This means that a particle moving along the geodesic Eq. 4.19 will reach $r = R$ in finite proper time, but a stationary observer at $r = 0$ will measure an infinite amount of time.

This strange behaviour is similar to what happens at the horizon of a black hole ($r = 2GM$): however, the Schwarzschild metric has a singularity at $r = 0$, while de Sitter metric looks flat at $r = 0$. de Sitter space seems like an inverted black hole in which particles are pushed outwars to $r = R$.

4.2.2.4 de Sitter embeddings

de Sitter space can be nicely embedded as a submanifold of $\mathbb{R}^{1,4}$ with metric:

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{4}(dX^i)^2 \tag{4.20}$$

In particular, de Sitter metric Eq. 4.14 is a metric on the submanifold of $\mathbb{R}^{1,4}$ defined by the timelike hyperboloid:

$$-(X^0)^2 + \sum_{i=1}^{4}(X^i)^2 = R^2 \tag{4.21}$$

A way of parametrizing this constraint is by imposing that $r^2 = (X^1)^2 + (X^2)^2 + (X^3)^2$, so that:

$$R^2 - r^2 = (X^4)^2 - (X^0)^2$$

The solutions are parametrized as:

$$X^0 = \sqrt{R^2 - r^2}\sinh\frac{t}{R} \qquad X^4 = \sqrt{R^2 - r^2}\cosh\frac{t}{R}$$

Computing the respective variations, along with $\sum_{i=1}^{3}(dX^i)^2 = dr^2 + r^2 d\Omega_2^2$, where $d\Omega_n^2$ is the metric element on $\mathbb{S}^n$, allows to show that the pull-back of the Minkowski metric Eq. 4.20 on the hyperboloid

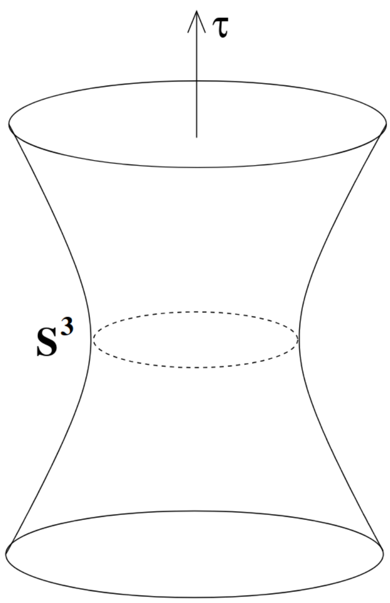


Figure 4.2: de Sitter space visualization with global coordinates.

Eq. 4.21 so parametrized gives the de Sitter metric Eq. 4.14 in the static patch coordinates.
The coordinates $\{X^i\}_{i=0,\ldots,4}$ so defined are not the most intuitive: they single out X^4 as special, when the constraint does no such thing, and they do not cover the whole hyperboloid, as they are limited to $X^4 \geq 0$. A better choice of parametrization is:

$$X^0 = R\sinh\frac{\tau}{T} \qquad\qquad X^i = y^i R\cosh\frac{\tau}{R}\ :\ \sum_{i=1}^{4}(y^i)^2 = 1$$

Given this constraint, $\{y^i\}_{i=1,,2,3,4}$ parametrize $\mathbb{S}^3$. These coordinates retain more of the symmetry of de Sitter space and cover the whole space, thus are a better parametrization. The pull-back of Minkowski metric, however, gives a rather different metric on de Sitter space:

$$ds^2 = -d\tau^2 + R^2\cosh^2\frac{\tau}{R}\,d\Omega_3^2 \tag{4.22}$$

These are known as *global coordinates*, as they cover the whole space (except for singularities at the poles for any choice of coordinates on $\mathbb{S}^3$). Since this metric is related to that in Eq. 4.14 by a change of coordinates, it too must obey the Einstein equations. Global coordiantes also show that the singularity which happens at X^4, i.e. at $r = R$, is nothing but a coordinate singularity.
These coordinates provide a clearer intuition for the physics of de Sitter space: it is a time-dependent solution of the field equations in which a spatial $\mathbb{S}^3$ first shrinks to a minimal radius R and then expands, as shown in Fig. 4.2. The expansionary phase is a good approximation to our current universe on large scales.

4.2.3 Anti-de Sitter space

Consider $\Lambda < 0$. With the same ansatz as for de Sitter space, it's easy to find the metric for *anti-de Sitter space* (AdS):

$$ds^2 = -\left(1+\frac{r^2}{R^2}\right)dt^2 + \left(1+\frac{r^2}{R^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2 \qquad R^2 \equiv -\frac{3}{\Lambda} \tag{4.23}$$

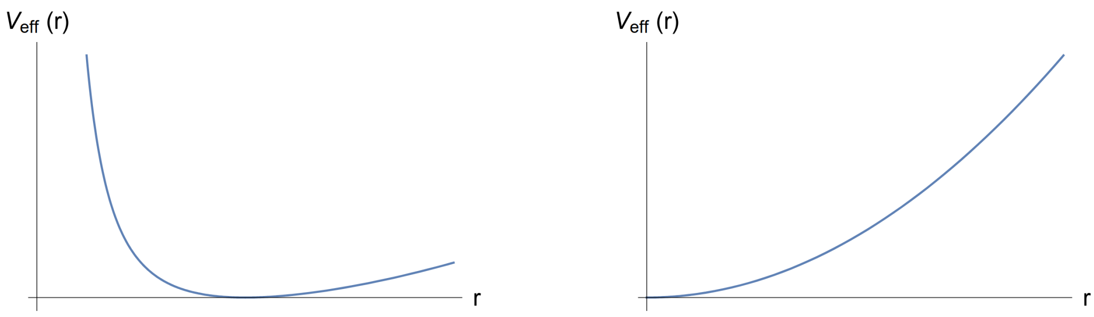


Figure 4.3: Effective potential for anti-de Sitter geodesics, with $\ell \neq 0$ and $\ell = 0$ respectively.

Equivalent coordinates are found setting $r = R\sinh\rho$:

$$ds^2 = -\cosh^2\rho\, dt^2 + R^2 d\rho^2 + R^2\sinh^2\rho\, d\Omega_2^2 \tag{4.24}$$

In AdS there's no coordinate singularity in r and indeed now coordinates cover the whole space.

4.2.3.1 Anti-de Sitter geodesics

AdS has the same action as in Eq. 4.15, thus the radial trajectory of a massive particle moving along a geodesic in the $\vartheta = \frac{\pi}{2}$ plane is $\dot{r}^2 + V_{\text{eff}}(r) = E^2$, with:

$$V_{\text{eff}}(r) = \left(1 + \frac{\ell^2}{R^2}\right)\left(1 + \frac{r^2}{R^2}\right) \tag{4.25}$$

Plotting it in Fig. 4.3, the geodesics' behavior is clear: with no angular momentum, anti-de Sitter space acts like a harmonic oscillator, pulling the particle towards the origin and making it oscillate around $r = 0$. If the particle has non-zero angular momentum, then the potential has a minimum at $r_*^2 = R\ell$, thus particles oscillate around r_*: importantly, particles with finite energy cannot escape to $r \to \infty$, but are constrained by the spacetime within some finite distance from the origin.
This picture of AdS as a harmonic trap which pulls particle to the origin clashes with the fact that AdS is a homogeneous space (roughly, all points are the same). To understand how these two facts are compatible, consider a stationary observer at $r = 0$: this is a geodesic and, from its perspective, other observers (with $\ell = 0$) will oscillate around $r = 0$ along geodesics. However, since these observers move along geodesics, in their reference frame they are stationary at $r = 0$, with all other observers oscillating. Thus, while in dS each observer views themselves at the center of the universe, with other observers moving away from them, in AdS each observer views themselves as the center of the universe, with other observer oscillating around them.
Its possible to study geodesics for a massless particle too. This time, the constraint to the action is:

$$-f(r)^2\dot{t}^2 + f(r)^{-2}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2) = 0$$

This means that the particle follows a null geodesic. Its equation of motion is:

$$\dot{r}^2 + V_{\text{null}}(r) = E^2 \qquad V_{\text{null}}(r) = \frac{\ell^2}{r^2}\left(1 + \frac{r^2}{R^2}\right)$$

Its plot in Fig. 4.4 makes it clear that massless particle can escape to $r \to \infty$, as the null potential is asymptotically constant, and it will only experience the usual gravitational redshift. AdS only

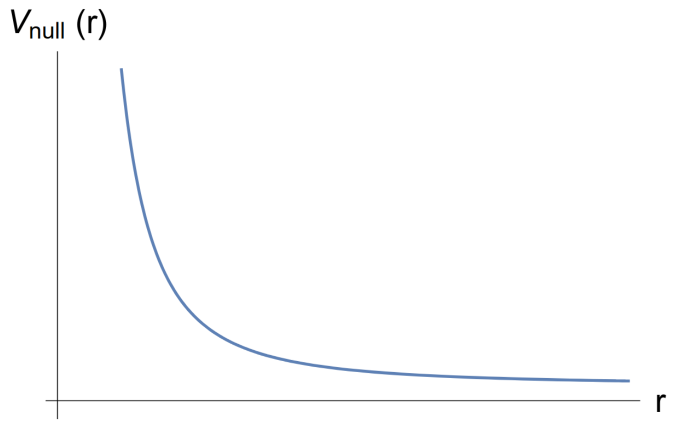


Figure 4.4: Potential experienced by massless particles with $\ell \neq 0$ in AdS.

confines massive particles. To solve for the trajectory, it's easier to work with $r = R\sinh\rho$ and $\ell = 0$, so that:

$$R\dot{\rho} = \pm\frac{E}{\cosh\rho} \quad \Rightarrow \quad R\sinh\rho(\sigma) = E(\sigma - \sigma_0)$$

One sees that $\rho \to \infty$ only in infinite affine time (i.e. $\sigma \to \infty$). However, in coordinate time, recalling Eq. 4.16, $E = \cosh^2\rho\,\dot{t}$, so:

$$R\tan\frac{t}{R} = E(\sigma - \sigma_0)$$

Hence, as affine time $\sigma \to \infty$, coordinate time $t \to \frac{\pi}{2}R$. This means that light rays escape to infinity in a finite amount of coordinate time: to make sense of dynamics in AdS, one must specify some boundary conditions at infinity to dictate how massless particles and field behave.
AdS does not appear to be of any cosmological interest.

4.2.3.2 Anti-de Sitter embeddings

AdS too has a natural embedding in a 5d spacetime: it is a submanifold of $\mathbb{R}^{2,3}$ with metric:

$$ds^2 = -(dX^0)^2 - (dx^4)^2 + \sum_{i=1}^{3}(dX^i)^2 \tag{4.26}$$

In particular, anti-de Sitter metric Eq. 4.23 is a metric on the hyperboloid:

$$-(X^0)^2 - (X^4)^2 + \sum_{i=1}^{3}(X^i)^2 = -R^2 \tag{4.27}$$

This constraint can be solved via the parametrization:

$$X^0 = R\cosh\rho\sin\frac{t}{R} \qquad X^4 = R\cosh\rho\cos\frac{t}{R} \qquad X^i = y^i R\sinh\rho \;:\; \sum_{i=1}^{3}(y^i)^2 = 1$$

Given this constraint, $\{y^i\}_{i=1,2,3}$ parametrize $\mathbb{S}^2$. The pull-back of the metric Eq. 4.26 on the hyperboloid so parametrized gives anti-de Sitter metric Eq. 4.24.

There's a small subtlety: the embedding hyperboloid has topology $\mathbb{S}^1 \times \mathbb{R}^3$, not $\mathbb{R}^4$: this corresponds to a compact time direction, as $t \in [0, 2\pi R)$. However, AdS metric does not have such restriction, with $t \in \mathbb{R}$: this is a universal covering of the hyperboloid.
Another useful parametrization of the hyperboloid is the following:

$$X^4 - X^3 = r \qquad X^4 + X^3 = \frac{R^2}{r} + \frac{r}{R^2}\eta_{ij}dx^i dx^j \qquad X^i = \frac{r}{R}x^i \, , \, i = 0, 1, 2$$

with $r \in [0, \infty)$. With these coordinates, the metric takes the form:

$$ds^2 = R^2 \frac{dr^2}{r^2} + \frac{r^2}{R^2}\eta_{ij}dx^i dx^j$$

These coordinates don't cover the whole AdS, but only one-half of the hyperboloid, restricted to $X^4 - X^3 > 0$: this is known as the *Poincaré patch* of AdS. Moreover, x^0 cannot be further extended beyond $x^0 \in (-\infty, +\infty)$, thus in global coordinates the restriction $t \in [0, 2\pi R)$ remains. In this case, massive particles fall towards $r = 0$.
Finally, there are two more possible coordinate systems on the Poincaré patch, obtained by setting $z = \frac{R^2}{r}$ and $r = Re^\rho$:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \eta_{ij}dx^i dx^j\right) \qquad ds^2 = R^2 d\rho^2 + e^{2\rho}\eta_{ij}dx^i dx^j$$

In each case, massive particles fall towards $z = \infty$ or $\rho = -\infty$.

4.3 Symmetries

What makes Minkowski, dS and AdS special solutions to the Einstein equations are their symmetries. The symmetries of Minkowski spacetime are familiar: translations and rotations of spacetime, with the latter splitting between proper rotations and Lorentz boosts. These symmetries are responsible for the existence of energy, momentum and angular momentum on a fixed Minkowski background. It is important to characterize the symmetries of a general metric.

4.3.1 Isometries

Recall Def. 2.3.7 of flow on a manifold: by Eq. 2.10, it is possible to identify a flow with a vector field $K \in \mathfrak{X}(\mathcal{M})$ such that it is tangent to the flow at each point of the manifold: $K^\mu = \frac{dx^\mu}{dt}$, with $x^\mu(t) \equiv x^\mu(\sigma_t)$.

Definition 4.3.1. Given a flow associated to $K \in \mathfrak{X}(\mathcal{M})$, it is said to be an *isometry* if:

$$\mathcal{L}_K g = 0 \quad \Leftrightarrow \quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{4.28}$$

Recall Eqq. 4.11-4.12 for the equivalence. This condition means that the metric doesn't change along flow lines: this is called *Killing equation*, and a vector field which satisfied it is a *Killing vector field*. Prop. 2.3.6 can be generalized to hold for the Lie derivative of arbitrary tensor fields, thus Killing vectors too form a Lie algebra: it is the Lie algebra of the isometry group of the manifold.

Example 4.3.1. Given a metric $g_{\mu\nu}(x)$, if it doesn't depend on $x^1 \equiv y$, then $X = \partial_y$ is a Killing vector field, for $(\mathcal{L}_{\partial_y} g)_{\mu\nu} = \partial_y g_{\mu\nu} = 0$. This is the case of ignorable degrees of freedom, as in Eqq. 4.16-4.17.

4.3.1.1 Minkowski spacetime

Consider Minkowski spacetime $\mathbb{R}^{1,3}$ with $g_{\mu\nu} = \eta_{\mu\nu}$. Killing equation is:

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 0$$

There two forms of solution. The first one is:

$$K_\mu = c_\mu$$

for any constant vector c_μ. These generate translations. Alternatively:

$$K_\mu = \omega_{\mu\nu} x^\nu \, : \, \omega_{\mu\nu} = -\omega_{\nu\mu}$$

These generate rotations and Lorentz boosts. The emergence of the Lie algebra structure can be elucidated defining the Killing vectors:

$$P_\mu := \partial_\mu \qquad M_{\mu\nu} := \eta_{\mu\rho} x^\rho \partial_\nu - \eta_{\nu\rho} x^\rho \partial_\mu$$

These are 10 Killing vectors: 4 for translations and 6 for rotations and boosts.

Proposition 4.3.1. *Killing vectors of Minkowski spacetime obbey:*

$$[P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\sigma] = -\eta_{\mu\sigma} P_\nu + \eta_{\sigma\nu} P_\mu$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}$$

Proof. By direct calculation. □

This is precisely the Lie algebra of the Poincaré group $\mathbb{R}^4 \times \mathrm{SO}(1,3)$, i.e. the isometry group of Minkowski spacetime.

4.3.1.2 de Sitter and anti-de Sitter space

The isometries of dS and AdS are best seen from their embeddings. The constraint Eq. 4.21 which defined de Sitter space is invariant under the rotations of $\mathbb{R}^{1,4}$, thus dS inherits the $\mathrm{SO}(1,4)$ isometry group. Similarly, the constraint Eq. 4.27 which defines anti-de Sitter space is invariant under rotations of $\mathbb{R}^{2,3}$, so AdS inherits the $\mathrm{SO}(2,3)$ isometry group. Note that both these groups are 10-dimensional, as $\mathbb{R}^4 \times \mathrm{SO}(1,3)$.
It's simple to determine the 10 Killing spinors of 5d spacetime:

$$M_{AB} = \eta_{AC} X^C \partial_B - \eta_{BC} X^C \partial_A$$

where $X^A, A = 0,1,2,3,4$ are 5d coordinates and η_{AB} is the appropriate Minkowski metric, with signature $(-,+,+,+,+)$ for dS and $(-,-,+,+,+)$ for AdS. In either case, the Lie algebra is that of the appropriate Lorentz group:

$$[M_{AB}, M_{CD}] = \eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}$$

The embedding hyperbolae are both invariant under these Killing vectors: flows generated by them map points on the hyperbolae to other points on the hyperbolae. Therefore, these Killing vectors are inherited by dS and AdS respectively.

Energy Energy is defined by timelike Killing vectors. In anti-de Sitter space there's no problem finding a timelike Killing vector, as the metric in global coordinates Eq. 4.24 is time-independent, so $K = \partial_t$. However, this changes in de Sitter space.
Considering dS in the static patch with coordinates $r^2 = (X^1)^2 + (X^2)^2 + (X^3)^2$, $X^0 = \sqrt{R^2 - r^2}\sinh\frac{t}{R}$ and $X^4 = \sqrt{R^2 - r^2}\cosh\frac{t}{R}$, the metric Eq. 4.14 is time-independent, thus $K = \partial_t$ is a Killing vector; pushing it forward to the 5d space:

$$\partial_t = \frac{\partial X^A}{\partial t}\partial_A = \frac{1}{R}(X^4\partial_0 + X^0\partial_4)$$

On the static patch, this Killing vector is timelike and the energy Eq. 4.16 follows from it. The problem is that the static patch is only one-half of the hyperboloid: when considering the whole AdS, one must account for the case $X^0 = 0, X^4 < 0$, in which the Killing vector points in the past direction, or $X^0 \neq 0, X^4 = 0$, in which it is spacelike. Therefore, the Killing vector can be both timelike (future-directed and past-directed) or spacelike when spanning the whole manifold: for this reason, it's not possible to define a global positive conserved energy on the total de Sitter space. The same conclusion follows by noting that the metric in global coordinates Eq. 4.22 is time-dependent.

4.3.2 Conserved quantities

It is possible to reframe Noether's theorem in the context of Killing vectors.

Theorem 4.3.1 (Noether)**.** *Given a massive particle moving on a geodesic $x^\mu(\tau)$ in a spacetime with metric g which admits a Killing vector field K, then the Noether charge is conserved along the geodesic:*

$$Q := K_\mu \frac{dx^\mu}{d\tau} \tag{4.29}$$

Proof. First, show that Q is indeed conserved along the geodesic, recalling Eq. 4.28:

$$\frac{dQ}{d\tau} = \partial_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + K_\mu \frac{d^2x^\mu}{d\tau^2} = \partial_\nu K_\mu \frac{dx^\nu}{d\tau} - K_\mu \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \nabla_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

To show that Q follows from Noether's theorem, consider the action of the massive particle and introduce an infinitesimal transformation $\delta x^\mu = K^\mu(x)$:

$$\begin{aligned}
\delta\mathcal{S} = \delta\int d\tau\, g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= \int d\tau \left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho + 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\right] \\
&= \int d\tau \left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}K^\rho + 2\frac{dx^\mu}{d\tau}\left(\frac{dK_\mu}{d\tau} - K^\nu \frac{dg_{\mu\nu}}{d\tau}\right)\right] \\
&= \int d\tau \left[\partial_\rho g_{\mu\nu}K^\rho - 2K^\rho\partial_\nu g_{\mu\rho} + 2\partial_\nu K_\mu\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \int d\tau\, 2\nabla_\nu K_\mu \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}
\end{aligned}$$

The transformation is a symmetry if $\delta\mathcal{S} = 0$, thus, by the symmetry of $\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$, the resulting equation is $\nabla_{(\nu}K_{\mu)} = 0$, i.e. Killing equation. □

Example 4.3.2. Energy and angular momentum defined on de Sitter geodesics by Eqq. 4.16-4.17 are Noether charges associated to Killing vectors ∂_t and ∂_φ respectively.

4.3.3 Komar integrals

Definition 4.3.2. Given a Killing vector $K = K^\mu \partial_\mu$, defined the 1-form $K \equiv K_\mu dx^\mu$, the associated *Komar form* is the 2-form defined as:

$$F := dK \tag{4.30}$$

Proposition 4.3.2. *Given a Killing vector $K = K^\mu \partial_\mu$, the associated Komar form is:*

$$F_{\mu\nu} = \nabla_\mu K_\nu - \nabla_\nu K_\mu \tag{4.31}$$

Proof. $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dK = \nabla_\mu K_\nu dx^\mu \wedge dx^\nu$. □

Theorem 4.3.2. *If the vacuum Einstein equations $R_{\mu\nu} = 0$ hold, then a Komar form obeys the vacuum Maxwell equations:*

$$d \star F = 0 \quad \Leftrightarrow \quad \nabla^\mu F_{\mu\nu} = 0 \tag{4.32}$$

Proof. Recall Prop. 3.2.7 for the equivalence. From Ricci identity Eq. 3.38:

$$(\nabla_\mu \nu_\nu - \nabla_\nu \nabla_\mu) K^\sigma = R^\sigma{}_{\rho\mu\nu} K^\rho \quad \Rightarrow \quad (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) K^\mu = R_{\rho\nu} K^\rho$$

From Killing equation $\nabla_{(\mu} K_{\nu)} = 0 \Rightarrow \nabla_\mu K^\mu = 0$, thus $\nabla_\mu \nabla_\nu K^\mu = R_{\rho\nu} K^\rho$ and:

$$\nabla^\mu F_{\mu\nu} = \nabla^\mu \nabla_\mu K_\nu - \nabla^\mu \nabla_\nu K_\mu = -2 \nabla^\mu \nabla_\nu K_\mu = -2 R_{\rho\nu} K^\rho$$

If $R_{\rho\nu} = 0$, then $\nabla^\mu F_{\mu\nu} = 0$. □

Definition 4.3.3. Given a Komar form F associated to a Killing vector K, the associated *Komar charge* (or Komar integral) on a spatial submanifold Σ is defined as:

$$Q_{\text{Komar}} := -\frac{1}{8\pi G} \int_\Sigma d \star F = -\frac{1}{8\pi G} \int_{\partial\Sigma} \star F = -\frac{1}{8\pi G} \int_{\partial\Sigma} \star dK \tag{4.33}$$

Proposition 4.3.3. *If Eq. 4.32 holds, then Q_{Komar} is conserved.*

Proof. Recall Eq. 3.49: the proof is identical. □

As for Noether integrals of a particle, Komar integrals of spacetime are interpreted based on the defining Killing vector. For example, if K^μ is everywhere timelike, i.e. $g_{\mu\nu} K^\mu K^\nu < 0$, then its Komar integral can be identified with the energy (or the mass) of spacetime; if the Killing vector is related to rotations, instead, the conserved charge is the angular momentum of spacetime.

4.4 Asymptotics of spacetime

The three special solution (Minkowski, dS, AdS) not only have different spacetime curvature and symmetries, but, more fundamentally, they have different behavior at infinity. Their importance lie in the fact that, however complicated the metric might be, if fields are suitably localized, then they will asymptote to one of these three symmetric spaces.

4.4.1 Conformal transformations

Definition 4.4.1. Given a metric manifold $(\mathcal{M}, g)$ and a non-vanishing $\Omega \in \mathcal{C}^\infty(\mathcal{M})$, a *conformal transformation* is defined as:

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{4.34}$$

Typically, g and $\tilde{g}$ describe different spacetime with considerably warped distances. However, conformal transformations preserve angles: in Lorentzian spacetime, the two metrics have the same causal structure, i.e. null/timelike/spacelike vector fields in one metric remain null/timelike/spacelike in the other too.

Proposition 4.4.1. *Only conformal transformations of the metric preserve its causal structure.*

Although timelike particle trajectories necessarily remain timelike under a conformal transformation, the same needs not be true for geodesics, as distances get warped.

Proposition 4.4.2. *Null geodesics are mapped to null geodesics under a conformal transformation.*

Proof. First compute the Christoffel symbols in the new metric:

$$\begin{aligned}
\Gamma^\mu_{\rho\sigma}[\tilde{g}] &= \frac{1}{2}\tilde{g}^{\mu\nu}(\partial_\rho \tilde{g}_{\nu\sigma} + \partial_\sigma \tilde{g}_{\rho\nu} - \partial_\nu \tilde{g}_{\rho\sigma}) \\
&= \frac{1}{2}\Omega^{-2} g^{\mu\nu}(\partial_\rho(\Omega^2 g_{\nu\sigma}) + \partial_\sigma(\Omega^2 g_{\rho\nu}) - \partial_\nu(\Omega^2 g_{\rho\sigma})) \\
&= \Gamma^\mu_{\rho\sigma}[g] + \Omega^{-1}(\delta^\mu_\sigma \nabla_\rho \Omega + \delta^\mu_\rho \nabla_\sigma \Omega - g_{\rho\sigma}\nabla^\mu \Omega)
\end{aligned}$$

In the last line, recall that $\nabla = \partial$ on scalar functions. Suppose an affinely parametrized geodesic in the metric g:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}[g]\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad \Rightarrow \quad \frac{dx^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}[\tilde{g}]\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \Omega^{-1}(\delta^\mu_\sigma \nabla_\rho \Omega + \delta^\mu_\rho \nabla_\sigma \Omega - g_{\rho\sigma}\nabla^\nu \Omega)\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$$

For a null geodesic $g_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$, thus:

$$\frac{dx^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}[\tilde{g}]\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 2\frac{dx^\mu}{d\tau}\frac{1}{\Omega}\frac{d\Omega}{d\tau}$$

This is the equation of non-affinely parametrized geodesic (like Eq. 1.16), hence the thesis. □

Usual curvature tensors are not invariant under conformal transformations. A curvature tensor that is indeed invariant is the *Weyl tensor*:

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{2}{n-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(n-1)(n-2)} R g_{\mu[\rho}g_{\sigma]\nu} \tag{4.35}$$

where $n \equiv \dim_{\mathbb{R}} \mathcal{M}$. The Weyl tensor has all the symmetries of the Riemann tensor, with the additional property that it vanishes when contracting any pair of indices with the metric: it can be viewed as the trace-free part of the Riemann tensor.

4.4.2 Penrose diagrams

To study the asymptotic behavior of spacetime, one needs to perform a conformal transformation that maps infinity to a finite distance: the resulting causal structure can then be drawn on a finite area and the resulting picture is called a *Penrose diagram*.

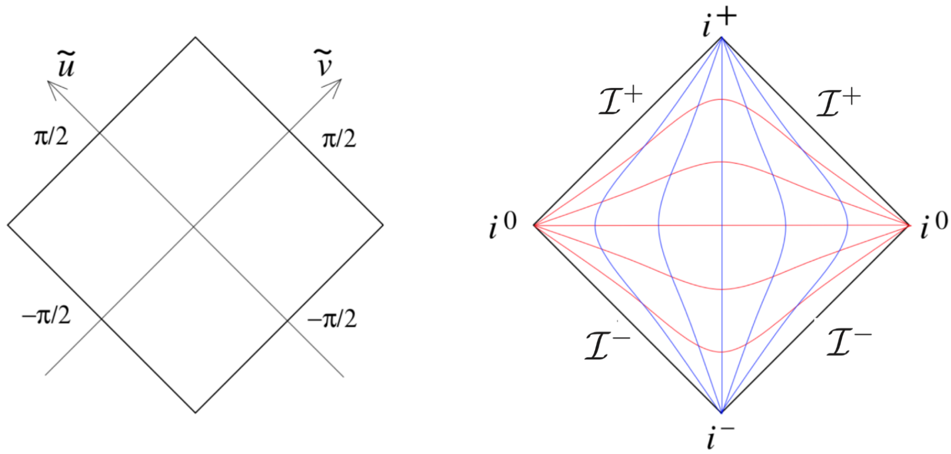


Figure 4.5: Penrose diagram for $d = 1 + 1$ Minkowski spacetime.

4.4.2.1 Minkowski spacetime

It is simplest to study $\mathbb{R}^{1,1}$, where the Minkowski metric is $ds^2 = -dt^2 + dx^2$. First, introduce *lightcone coordinates*:

$$u = t - x \qquad v = t + x \qquad \Rightarrow \qquad ds^2 = -du\,dv$$

These coordinates range in $u, v \in \mathbb{R}$. To work with finite quantities, define:

$$u = \tan\tilde{u} \qquad v = \tan\tilde{v} \qquad \Rightarrow \qquad ds^2 = -\frac{1}{\cos^2\tilde{u}\,\cos^2\tilde{v}}d\tilde{u}\,d\tilde{v}$$

where now $\tilde{u}, \tilde{v} \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$. Note that the metric diverges when approaching the boundary of Minkowski spacetime. However, a conformal transformation is possible with $\Omega(\tilde{u}, \tilde{v}) \equiv \cos\tilde{u}\,\cos\tilde{v}$:

$$d\tilde{s}^2 = \left(\cos^2\tilde{u}\,\cos^2\tilde{v}\right) ds^2 = -d\tilde{u}\,d\tilde{v}$$

It is now possible to add $\tilde{u} = \pm\frac{\pi}{2}$ and $\tilde{v} = \pm\frac{\pi}{2}$: this operation is called *conformal compactification.* Penrose diagrams, as other relativistic diagrams, present light-rays travelling at 45°, time in the vertical direction and space in the horizontal one. Fig. 4.5 shows the Penrose diagram for Minkowski spacetime $\mathbb{R}^{1,1}$ with $\tilde{u}, \tilde{v}$ coordinates, both with coordinate axis and geodesics. Moreover, the various infinities of Minkowski space are shown:

- timelike geodesics (blue) start at $i^-(\tilde{u}, \tilde{v}) = \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ and end at $i^+(\tilde{u}, \tilde{v}) = \left(+\frac{\pi}{2}, +\frac{\pi}{2}\right)$, called respectively *past* and *future timelike infinity*;
- spacelike geodesics (red) start and end at $i^0(\tilde{u}, \tilde{v}) = \left(\mp\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, both called *spacelike infinity*;
- all null curves (not shown) start at the boundary $\mathcal{I}^- \equiv \{\tilde{u} = -\frac{\pi}{2}\} \cup \{\tilde{v} = -\frac{\pi}{2}\}$ and end at the boundary $\mathcal{I}^+ \equiv \{\tilde{u} = +\frac{\pi}{2}\} \cup \{\tilde{v} = +\frac{\pi}{2}\}$, called respectively *past* and *future null infinity.*

It's clear that in Minkowski spacetime there are more ways reach infinity along a null direction than in a timelike or spacelike direction.
Penrose diagrams allow to immediately visualize the causal structure of spacetime. As shown in Fig. 4.6, given a particle moving along a timelike curve, as it approaches i^+ its past lightcone encompasses progressively more of spacetime: thus, an observer in Minkowski spacetime can in

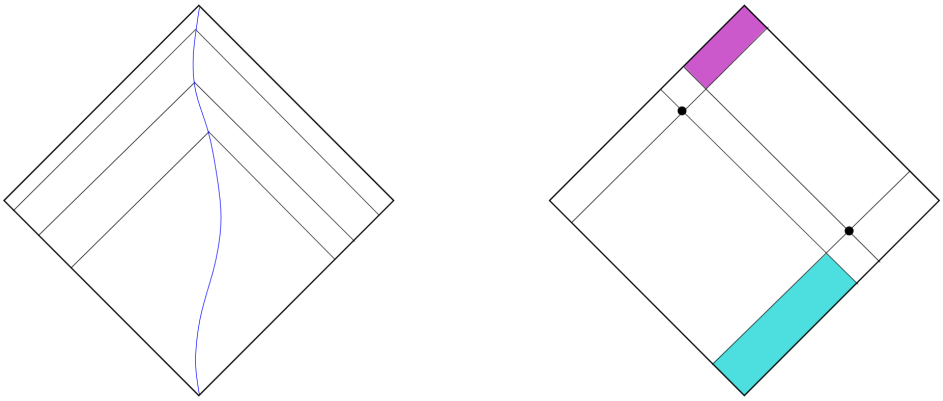


Figure 4.6: Lightcones in $d = 1 + 1$ Minkowski spacetime.

principle see everything, as long as they wait long enough. Relatedly, given any two points in Minkowski spacetime, they are causally connected both in the past and in the future, as both cones intersect (see Fig. 4.6): thus, there was always an event in the past that could have influenced both, and there will always be an event in the future that can be influenced by both.

4d Minkowski spacetime The analysis can be repeated for $\mathbb{R}^{1,3}$ with $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$. Lightcone coordinates are again:

$$u = t - r \qquad v = t + r \quad \Rightarrow \quad ds^2 = -du\,dv + \frac{1}{4}\left(u - v^2\right) d\Omega_2^2$$

Finite range coordinates are again:

$$u = \tan\tilde{u} \qquad v = \tan\tilde{v} \quad \Rightarrow \quad ds^2 = \frac{1}{4\cos^2\tilde{u}\,\cos^2\tilde{v}}\left(-4d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{u} - \tilde{v})d\Omega_2^2\right)$$

Finally, the conformal transformation with $\Omega(\tilde{u}, \tilde{v}) = 2\cos\tilde{u}\,\cos\tilde{v}$ leads to:

$$d\tilde{s}^2 = -4d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{u} - \tilde{v})d\Omega_2^2$$

In 4d Minkowski spacetime there's an additional constraint: $r \geq 0$, so $v \geq v$. Conformal compactification leads to:

$$-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq +\frac{\pi}{2}$$

The corresponding Penrose diagram is drawn in Fig. 4.7: the spatial $\mathbb{S}^2$ is not shows for simplicity, but every point on the diagram corresponds to an $\mathbb{S}^2$ of radius $|\sin(\tilde{u} - \tilde{v})|$. The line $\tilde{u} = \tilde{v}$ is not a boundary of Minkowski spacetime, but it's simply the origin $r = 0$ (at which $\mathcal{S}^2$ shrinks to a point): to illustrate this, a null geodesic is drawn.
In general, Penrose diagrams are only useful for spacetimes which contain an obvious $\mathcal{S}^2$, i.e. those with SO(3) isometry: however, these are the simplest and most important in physics.

4.4.2.2 de Sitter space

Recall global coordinates on dS: from Eq. 4.22, $ds^2 = -d\tau^2 + R^2\cosh^2\frac{\tau}{R}d\Omega_3^2$. To draw the Penrose diagram, define *conformal time* as:

$$\frac{d\eta}{d\tau} = \frac{1}{R\cosh(\tau/R)} \quad \Rightarrow \quad \cos\eta = \frac{1}{\cosh(\tau/R)}$$

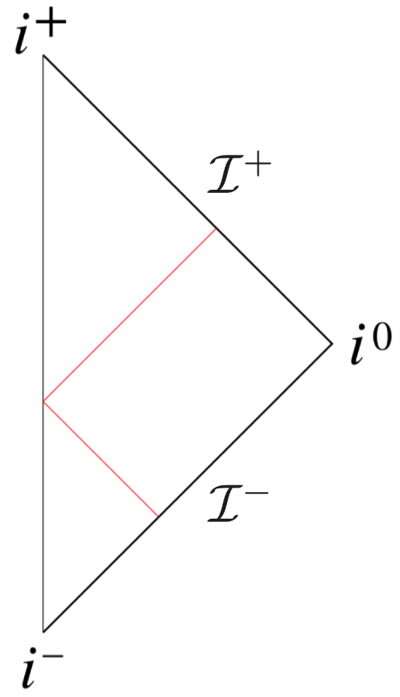


Figure 4.7: Penrose diagram for $d = 1 + 3$ Minkowski spacetime.

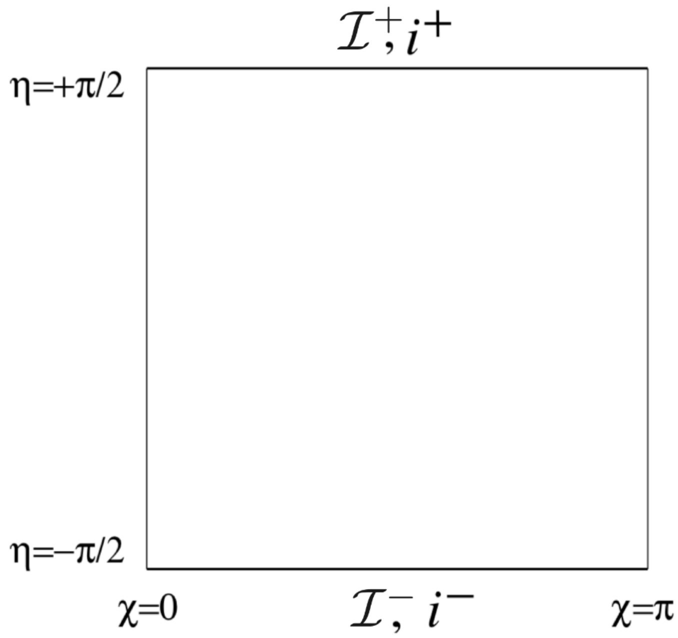


Figure 4.8: Penrose diagram for de Sitter space.

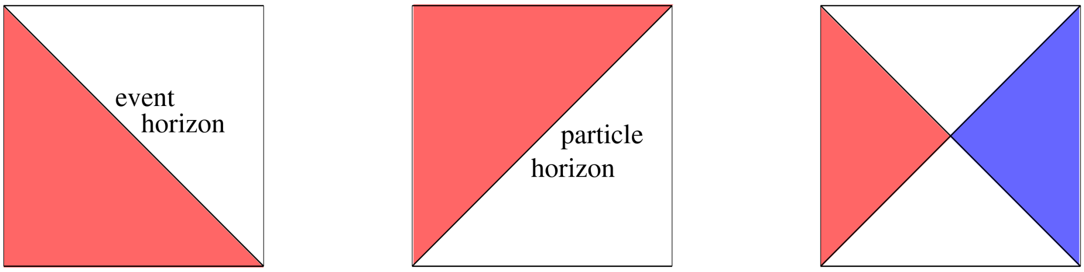


Figure 4.9: Event and particle horizons for an observer at the north pole in dS, and the causal diamonds for an observer at the north and south pole.

Given that $\tau \in \mathbb{R}$, then $\eta \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$. In conformal time, de Sitter space has the metric:

$$ds^2 = \frac{R^2}{\cos^2 \eta} \left(-d\eta^2 + d\Omega_3^2\right)$$

Writing $d\Omega_3^2 = d\chi^2 + \sin^2\chi \, d\Omega_2^2$, with $\chi \in [0, \pi]$, de Sitter metric is conformally equivalent to:

$$d\tilde{s}^2 = -d\eta^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2$$

After conformal compactification, $\eta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ and $\chi \in [0, \pi]$. The Penrose diagram is drawn in Fig. 4.8: the two vertical lines are not boundaries of dS, but simply the north and south poles of $\mathbb{S}^3$. The boundaries of the spacetime are the top and bottom lines, labelled both $i^\pm$ and $\mathcal{I}^\pm$ as they are where both timelike and null geodesics originate and terminate.

Proposition 4.4.3. *de Sitter spacetime has a spacelike $\mathbb{S}^3$ boundary with timelike normal vector.*

The causal structure of dS is very different from that of Minkowski spacetime: it is no longer true that any observer can see everything by waiting long enough. For example, as shown in Fig. 4.9, an observer at the north pole will eventually see only exactly half the spacetime: the boundary of this half-space is the observer's *event horizon*, in the sense that signals from beyond the horizon cannot reach them. It is also clear that this event horizon is observer-dependent: in this context, these are referred to as *cosmological horizons*.
Furthermore, an observer at the north pole will only be able to communicate with another half of the spacetime, as shown in Fig. 4.9: the boundary of this region of influence is known as the *particle horizon* and it represents the furthest distance light can travel since the beginning of time. Its intersection with the event horizon determines the (nothern) *causal diamond*: usually, nothern and southern causal diamonds are causally disconnected.
Penrose diagrams can also be used to explain the divergence at $r = R$ of the metric Eq. 4.14 on the static patch of dS. Recalling the embedding of the static patch in $\mathbb{R}^{1,4}$, parametrized as $X^0 = \sqrt{R^2 - r^2} \sinh \frac{t}{R}$ and $X^4 = \sqrt{R^2 - r^2} \cosh \frac{t}{R}$. Naively the surface $r = R$ corresponds to $X^0 = X^4 = 0$, but writing $r = R(1 - \varepsilon^2/2)$ yields that $X^0 \sim R\varepsilon \sinh \frac{t}{R}$ and $X^4 \sim R\varepsilon \cosh \frac{t}{R}$, so $\varepsilon \to 0$ can be obtained keeping $X^0, X^4 \neq 0$ and finite, provided that $t \to \pm\infty$: to do this, $\varepsilon \exp(\pm t/R)$ must be kept finite, thus the surface $r = R$ is identified with the lines $X^0 = \pm X^4$. Translation to polar coordinates is done by $X^0 = R \sinh \frac{\tau}{R}$ and $X^4 = R \cosh \frac{\tau}{R} \cos\chi$, with χ the polar angle on $\mathbb{S}^3$; after mapping to conformal time, one finds that:

$$X^0 = \pm X^4 \quad \Leftrightarrow \quad \sin\eta = \pm\cos\chi \quad \Leftrightarrow \quad \chi = \pm\left(\eta - \frac{\pi}{2}\right)$$

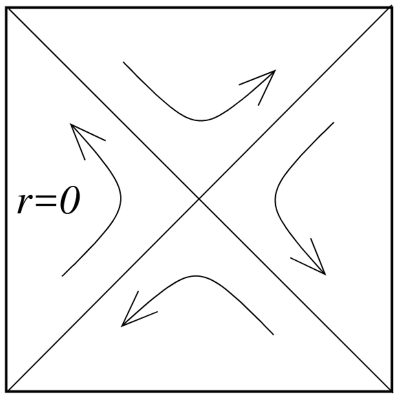


Figure 4.10: $K = \partial_t$ Killing vector field in de Sitter spacetime.

These are precisely the lines determining the polar causal diamonds. It can also be checked that $r = R$ on the static patch corresponds to the north pole $\chi = 0$ in global coordinates and that $t = \tau$ along this line. Therefore, the static patch of dS provides coordinates that cover only the nothern causal diamond of dS spacetime, with the coordinate singularity at $r = R$ corresponding to the past and future observer-dependent horizons.
Finally, Penrose diagrams help to understand the nature of the $K = \partial_t$ Killing vector field exhibited by the static patch metric. As shown in Fig. 4.10, there's no global timelike Killing vector field in dS spacetime: extending the Killing vector beyond the static patch, i.e. the nothern causal diamond, it is timelike but past-oriented on the souther causal diamond and spacelike on the upper and lower quadrants.

4.4.2.3 Anti-de Sitter space

The global coordinates on AdS are, from Eq. 4.24, $ds^2 = -\cosh^2\rho\, dt^2 + R^2 d\rho^2 + R^2\sinh^2\rho\, d\Omega_2^2$, with $\rho \in [0, \infty)$. Introduce a conformal radial coordinate:

$$\frac{d\psi}{d\rho} = \frac{1}{\cosh\rho} \quad \Rightarrow \quad \cos\psi = \frac{1}{\cosh\rho}$$

with $\psi \in [0, \frac{\pi}{2})$. With a dimensionless coordiante time $\tilde{t} = \frac{t}{R}$, the metric becomes:

$$ds^2 = \frac{R^2}{\cos^2\psi}\left(-d\tilde{t}^2 + d\psi^2 + \sin^2\psi\, d\Omega_2^2\right) = \frac{R^2}{\cos^2\psi}\left(-d\tilde{t}^2 + d\Omega_3^2\right)$$

AdS metric is conformally equivalent to:

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\psi^2 + \sin^2\psi\, d\Omega_2^2$$

After conformal compactification, $\tilde{t} \in \mathbb{R}$ and $\psi \in \left[0, \frac{\pi}{2}\right]$ and the resulting Penrose diagram is the infinite strip in Fig. 4.11. The edge at $\psi = 0$ is not a boundary, but the spatial origin where $\mathcal{S}^2$ shrinks to a point; in contrast, $\psi = \frac{\pi}{2}$ is a boundary of the spacetime, labelled $\mathcal{I}$, which should be viewed as a combination of $\mathcal{I}^-$, $\mathcal{I}^+$ and i^0, since null and spacelike geodesics begin and end there.

Proposition 4.4.4. *AdS spacetime has a timelike $\mathbb{R} \times \mathbb{S}^2$ boundary with spacelike normal vector.*

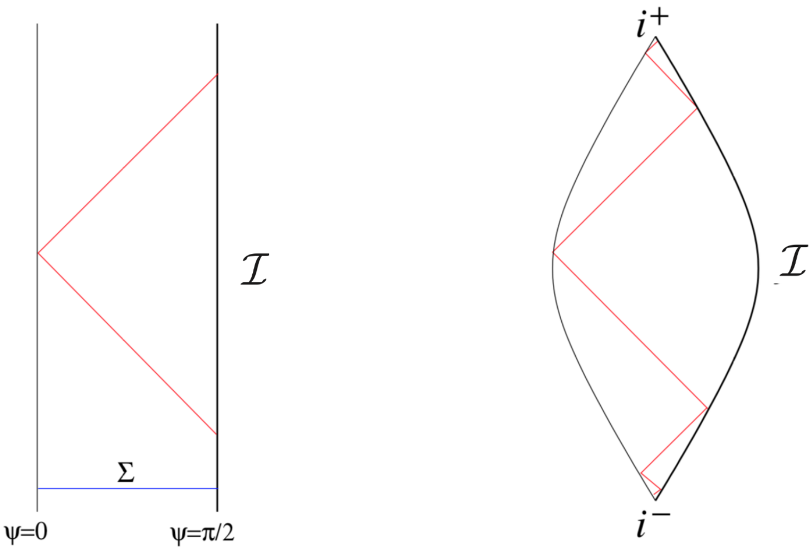


Figure 4.11: Penrose diagrams for AdS, without and with conformally compactified time coordinate.

Note that $\mathbb{R}$ is the time factor.
The Penrose diagram clearly shows that light rays reach the boundary in finite conformal time. To study physics in AdS, one needs to specify boundary conditions at $\mathcal{I}$: for example, reflecting boundary conditions make light rays bounce back and forth forever, rendering AdS a "box" spacetime in which massive particle are confined in the interior and massless particles bounce off the boundary.
Another characteristic of AdS space is that it isn't *globally hyperbolic*: there exists no Cauchy surface on which initial data can be specified. Consider for example the 3d spacelike hypersurface Σ in Fig. 4.11. Specifying initial data on Σ it's not sufficient to solve for their time evolution: in AdS, there exist points in the future of Σ which are in causal contact with the boundary, thus the time evolution depends on boundary conditions too.
To make the Penrose diagram for AdS not stretch to infinity, the time coordinate can be restricted to finite values by:

$$\tilde{t} = \tan\tau \quad \Rightarrow \quad ds^2 = \frac{R^2}{\cos^2\psi\,\cos^4\tau}\left(-d\tau^2 + \cos^4\tau\, d\Omega_3^2\right)$$

This metric is conformally equivalent to:

$$d\tilde{s}^2 = -d\tau^2 + \cos^4\tau\left(d\psi^2 + \sin^2\psi\, d\Omega_2^2\right)$$

with conformally compactified $\tau \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$. Ignoring the spatial $\mathbb{S}^2$, the resulting Penrose diagram is drawn in Eq. 4.11: the spatial $\mathbb{S}^3$ grows and shrinks in time, the timelike boundary $\mathcal{I}$ is still present and now the past and future timelike infinities $i^\pm$ are also shown. This diagram makes it clear that a lightray bounces back and forth off teh boundary of AdS an infinite number of times.

4.5 Matter coupling

Spacetime is not merely the background on which matter exists, but it is dynamically influenced by the matter distribution on it. It is therefore necessary to study how matter couples to the spacetime metric.

4.5.1 Field theories in curved spacetime

The simplest way to describe matter is by fields governed by a Lagrangian. Consider a scalar field $\phi(x)$. In flat Minkowski spacetime, its action is:

$$\mathcal{S}_{\text{scalar}} := \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{4.36}$$

The negative sign of the kinetic term follows from the signature choice $(-,+,+,+)$. The generalization to curved spacetime is straightforward:

$$\mathcal{S}_{\text{scalar}} = \int d^4x\sqrt{-\mathrm{g}} \left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right]$$

Note the (useful) redundancy $\nabla_\mu\phi = \partial_\mu\phi$. Curved spacetime also introduces the possibility to add new terms to the Lagrangian. For example:

$$\mathcal{S}_{\text{scalar}} = \int d^4x\sqrt{-\mathrm{g}} \left[-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) - \frac{1}{2}\xi R\phi^2\right] \tag{4.37}$$

for some $\xi \in \mathbb{R}$. This theory correctly reduces to Eq. 4.36 on flat spacetime, as $R = 0$. To derive the equation of motion, vary the action keeping the metric fixed:

$$\begin{aligned}\delta\mathcal{S}_{\text{scalar}} &= \int d^4x\sqrt{-\mathrm{g}} \left[-g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi - \frac{\partial V}{\partial\phi}\delta\phi - \xi R\phi\delta\phi\right] \\ &= \int d^4x\sqrt{-\mathrm{g}} \left[\left(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\partial V}{\partial\phi} - \xi R\phi\right)\delta\phi - \nabla_\mu(\delta\phi\nabla^\mu\phi)\right]\end{aligned}$$

where integration by parts was possible due to $\nabla_\mu g_{\rho\sigma} = 0$ for the Levi-Civita connection. The last term is, by divergence theorem, a boundary term, thus the equation of motion for a scalar field theory in curved spacetime is:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\partial V}{\partial\phi} - \xi R\phi = 0 \tag{4.38}$$

Now covariant derivatives are necessary, as $\nabla_\mu\nabla_\nu \neq \partial_\mu\partial_\nu$.

4.5.2 Einstein equations with matter

To understand how matter fields back-react on spacetime, consider the combined action:

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}\,(R - 2\Lambda) + \mathcal{S}_M \tag{4.39}$$

where $\mathcal{S}_M$ is the action for matter fields, which in general depends on both the fields and the metric.

Definition 4.5.1. Given a field theory for matter described by the action $\mathcal{S}_M$, the *energy-momentum tensor* is defined as:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}} \tag{4.40}$$

Proposition 4.5.1. *The energy-momentum tensor is symmetric.*

Proof. It inherits the symmetry of the metric. □

Proposition 4.5.2. *The equations of motion derived from the action Eq. 4.39 are:*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{4.41}$$

Proof. Varying the full metric, by Def. 4.5.1:

$$\delta\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}\left[G_{\mu\nu} + \Lambda g_{\mu\nu}\right]\delta g^{\mu\nu} - \frac{1}{2}\int d^4x\sqrt{-\mathrm{g}}T_{\mu\nu}\delta g^{\mu\nu} = 0$$

□

These are the full *Einstein field equations*, describing gravity coupled to matter. It is possible to rewrite them by observing that the cosmological constant can be absorbed in the energy-momentum tensor as an additive component:

$$(T_\Lambda)_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}$$

This is justified by the fact that matter fields often mimic a cosmological constant. Contracting the remaining equation with $g^{\mu\nu}$ (i.e. taking the trace) then gives $-R = 8\pi GT$, where $T \equiv g^{\mu\nu}T_{\mu\nu}$, hence:

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{4.42}$$

Remember that the cosmological constant is present inside the energy-momentum tensor.

4.5.3 Energy-momentum tensor

The action $\mathcal{S}_M$ is, by hypothesis, diffeomorphism-invariant, thus, recalling the argument which lead to Bianchi identity Eq. 4.13, given $\delta g_{\mu\nu} = (\mathcal{L}_X g_{\mu\nu}) = 2\nabla_{(\mu}X_{\nu)}$:

$$\delta\mathcal{S}_M = -\frac{1}{2}\int d^4x\sqrt{-\mathrm{g}}T_{\mu\nu}\delta g^{\mu\nu} = -2\int d^4x\sqrt{-\mathrm{g}}T_{\mu\nu}\nabla^\mu X^\nu$$

Diffeomorphism invariance means that $\delta\mathcal{S}_M = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$, hence, integrating by parts:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{4.43}$$

Of course, this was necessary to make Einstein equations consistent, as $\nabla_\mu G^{\mu\nu} = 0$. Anyway, this equation hints to the more profound nature of the energy-momentum tensor, which has nothing to do with gravity: it can be shown that the energy-momentum tensor is linked to Noether currents associated to translational invariance in space and time. Trivially, Eq. 4.43 reduces in flat spacetime to $\partial_\mu T^{\mu\nu} = 0$, which is the usual conservation law enjoyed by Noether currents.
Consider a translation $x^\mu \mapsto x^\mu + \delta x^\mu$, with $\delta x^\mu = X^\mu(x)$. The action restricted to flat spacetime is not invariant under such shift, but one which is invariant can be constructed coupling the matter fields to a background metric and allowing this to vary. The change of the action in flat space, where the metric is fixed, must be equal and opposite to the change of the action where the metric can vary but x^μ is fixed, thus:

$$\begin{aligned}\delta\mathcal{S}_{\text{flat}} &= -\int d^4x\left.\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\delta g^{\mu\nu} = -\int d^4x\left.\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\partial^{(\mu}X^{\nu)}\\ &= -2\int d^4x\left.\frac{\partial\mathcal{S}_M}{\partial g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\partial^\mu X^\nu = -2\int d^4x\,\partial^\mu\left[\frac{\partial\mathcal{S}_M}{\partial g^{\mu\nu}}\right]_{g_{\mu\nu}=\eta_{\mu\nu}}X^\nu\end{aligned}$$

But $\delta\mathcal{S}_{\text{flat}} = 0$ for all constant X^μ, as this is the definition of a translationally-invariant theory, hence the conserved Noether current in flat space is:

$$T_{\mu\nu} = -2\frac{\partial \mathcal{S}_M}{\partial \delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

i.e. the flat version of Eq. 4.40.

4.5.3.1 Field theories

It is straightforward to compute $T_{\mu\nu}$ for a scalar field $\phi(x)$. Recall Eq. 4.37 (with $\xi = 0$) and Lemma 4.1.1:

$$\delta\mathcal{S}_{\text{scalar}} = \int d^4x\sqrt{-\text{g}}\left[\frac{1}{4}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi + \frac{1}{2}g_{\mu\nu}V(\phi) - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right]\delta g^{\mu\nu}$$

This gives the energy-momentum tensor:

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\rho\phi\nabla_\rho\phi + V(\phi)\right) \tag{4.44}$$

Restricting to flat Minkowski spacetime:

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

which is the energy density of a scalar field.

Maxwell theory Varying Maxwell action Eq. 3.44:

$$\delta\mathcal{S}_{\text{Maxwell}} = -\frac{1}{4}\int dx^4\sqrt{-\text{g}}\left[-\frac{1}{2}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + 2g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right]\delta g^{\mu\nu}$$

So the energy-momentum tensor for Maxwell theory is:

$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{4.45}$$

In flat Minkowski spacetime:

$$T_{00} = \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{B}^2$$

which is the energy density of the electromagnetic field.

4.5.3.2 Perfect fluids

A perfect fluid is described by its *energy density* $\rho(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$ and velocity 4-vector field $u^\mu(\mathbf{x}, t) : u^\mu u_\mu = -1$. Pressure and energy density are related by an *equation of state* $p = p(\rho)$.

Example 4.5.1. Dust is a fluid of massive particles floating around very slowly, so that the equation of state is $p = 0$.

Example 4.5.2. Radiation is a fluid of photons with $p = \rho/3$.

The energy-momentum tensor of a perfect fluid is:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \tag{4.46}$$

A fluid at rest ($u^\mu = \delta_\mu, 0$) in flat Minkowski spacetime has $T^{\mu\nu} = \mathrm{diag}\,(\rho, p, p, p)$, thus T_{00} is yet again the energy density, as expected. Generally, $\rho = T_{\mu\nu}u^\mu u^\nu$ is the energy density measured by an observer co-moving with the fluid.
Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$ determines two constraints. The first is:

$$u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu = 0 \tag{4.47}$$

which is a generalization of mass conservation (where mass is identified with ρ). The first term calculates how fast ρ changes along u^μ, while the second expresses it depending on the rate of flow out of the region $\nabla_\mu u^\mu$. The second constraint is:

$$(\rho + p)u^\mu \nabla_\mu u^\nu = -(g^{\mu\nu} + u^\mu u^\nu)\nabla_\mu p \tag{4.48}$$

which is a generalization of Euler equation, i.e. the fluid equivalent of $F = ma$ (or rather $ma = F$).

4.5.4 Energy conservation

There's a difference between the energy-momentum tensor and the current which arises from a global symmetry. Consider a conserved current $J^\mu : \nabla_\mu J^\mu = 0$. Invoking the divergence theorem Eq. 3.41:

$$0 = \int_V d^4x\sqrt{-\mathrm{g}}\,\nabla_\mu J^\mu = \int_{\partial V} d^3x\sqrt{\gamma}\,n_\mu J^\mu$$

where V is a spatial volume with boundary $\partial V = \Sigma_1 \cup \Sigma_2 \cup B$, with Σ_1, Σ_2 past and future spacelike boundarires and B timelike (lateral) boundary. If no current flows out of the region, i.e. $n_\mu J^\mu|_B = 0$, then this expression becomes the conservation $Q(\Sigma_1) = Q(\Sigma)_2$ of the charge associated to the current:

$$Q(\Sigma) \equiv \int_\Sigma d^3x\sqrt{\gamma}\,n_\mu J^\mu$$

Thus, for a vector field, covariant conservation is equivalent to actual conservation.
The same argument doesn't apply to the energy-momentum tensor: the problem arises from generalizing $\nabla_\mu J^\mu = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}J^\mu)$ to a higher-order tensor field, which is necessary in order to have a divergence theorem like Eq. 3.41. Indeed:

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho}T^{\rho\nu} + \Gamma^\nu_{\mu\rho}T^{\mu\rho} = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}\,T^{\mu\nu}) + \Gamma^\nu_{\mu\rho}T^{\mu\rho}$$

The last term doesn't allow to convert the integral of $\nabla_\mu T^{\mu\nu}$ to a boundary term. Instead:

$$\partial_\mu(\sqrt{-\mathrm{g}}\,T^{\mu\nu}) = -\sqrt{-\mathrm{g}}\,\Gamma^\nu_{\mu\rho}T^{\mu\rho} \tag{4.49}$$

Therefore, for higher-order tensors, covariant conservation is not equivalent to actual conservation.

4.5.4.1 Conserved energy from Killing vectors

Given a Killing vector K, it's possible to define a conserved current associated to the energy-momentum tensor as:

$$J_T^\nu := -K_\mu T^{\mu\nu} \tag{4.50}$$

This current is covariantly conserved, as:

$$\nabla_\nu J_T^\nu = -\left(T^{\mu\nu}\nabla_\nu K_\mu + K_\mu \nabla_\nu T^{\mu\nu}\right) = -T^{\mu\nu}\nabla_{(\nu}K_{\mu)} = 0$$

Its associated conserved charge on a spatial hypersurface Σ is defined as:

$$Q_T(\Sigma) := \int_\Sigma d^3x \sqrt{\gamma}\, n_\mu J_T^\mu \tag{4.51}$$

The interpretation of this charge depends on the properties of the Killing vector: if K is globally timelike, then the charge is the energy of matter $E = Q_T(\Sigma)$, meanwhile if K is globally spacelike, then it is the momentum of matter.

Absence of Killing vectors The problem of energy conservation becomes subtle when dealing with spacetimes which do not have any globally timelike Killing vector.
For example, a system comprised of two orbiting stars is modelled by a spacetime which doesn't have such a Killing vector: however, the problem of matter energy conservation does not arise in this case, as the stars, while orbiting each other, emit gravitational waves, thus losing energy and eventually spiraling towards each other. Nonetheless, a meaningful question is that of energy conservation of the total system, i.e. the two stars and the gravitational field. A guess would be to consider a total energy-momentum tensor defined similarly to Eq. 4.40, but:

$$T^{\text{total}}_{\mu\nu} = -\frac{2}{\sqrt{-\mathrm{g}}}\left[\frac{1}{16\pi G}\frac{\delta\mathcal{S}_{\text{EH}}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right] = -\frac{1}{8\pi G}G_{\mu\nu} + T_{\mu\nu} = 0$$

by Einstein field equations. This equation has no physical significance other than expressing the subtlety of energy conservation in Genera Relativity.
Clearly, one could try to understand the energy carried by the gravitational field alone. Unfortunately, there are compelling arguments that there exists no tensor which can be thought as the local energy density of the gravitational field: roughly speaking, the energy density of the Newtonian gravitational field Φ is proportional to $(\nabla\Phi)^2$, so the relativistic equivalent should be proportional to the first derivatives of the metric, which can be made locally vanishing by normal coordinates due to the equivalence principle, and a tensor which vanishes in one coordinate system does so in all of them.

4.5.5 Energy conditions

To study the general properties of a spacetime without explicitly referencing the specific characteristics of matter, one needs to place certain restrictions on the kinds of energy-momentum tensor which are considered physical: these are the *energy conditions* and express the idea that energy should be positive.

4.5.5.1 Weak energy condition

This condition states that, for any timelike vector field X:

$$T_{\mu\nu}X^\mu X^\nu \geq 0$$

This quantity is the energy measured by an observer moving along the timelike integral curves of X, and it should be non-negative. A timelike curve can be arbitrarily close to a null curve, thus by continuity:

$$T_{\mu\nu}X^\mu X^\nu \geq 0 \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}) : X_\mu X^\mu \leq 0 \tag{4.52}$$

Fluids Recall the energy-momentum tensor for a fluid Eq. 4.46 and impose the weak energy condition (WLOG $X \cdot X = -1$):

$$(\rho + p)(u \cdot X)^2 - p \geq 0$$

In the rest-frame $u^\mu = (1, 0, 0, 0)$, so considering a constant $X^\mu = (\cosh\varphi, \sinh\varphi, 0, 0)$, whose integral curves are world-lines of observers boosted with rapidity φ with respect to the fluid, then:

$$(\rho + p)\cosh^2\varphi - p \geq 0 \quad \Rightarrow \quad \begin{cases} \rho \geq 0 & \varphi = 0 \\ p \geq -\rho & \varphi \to \infty \end{cases}$$

The first condition ensures that the energy density is positive, the second allows for negative pressure limited from below.
Note that there are situations in which negative energy density makes physical sense: viewing the cosmological constant as part of the energy-momentum density, then any $\Lambda < 0$ violates the weak energy condition. In this sense, AdS space violates the weak energy condition.

Scalar fields The weak energy condition for the energy-momentum tensor of a scalar field theory Eq. 4.44 reads:

$$(X^\mu \partial_\mu \phi)^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) \geq 0$$

The sum of the first two terms is always positive: define $Y_\mu = \partial_\mu\phi + X_\mu X^\nu \partial_\nu \phi : X_\mu Y^\mu = 0$, i.e. orthogonal to X_μ, so it must be spacelike or null, hence $Y_\mu Y^\mu \geq 0$. Rewriting the above condition:

$$\frac{1}{2}(X^\mu\partial_\mu\phi)^2 + \frac{1}{2}Y_\mu Y^\mu + V(\phi) \geq 0 \quad \Rightarrow \quad V(\phi) \geq 0$$

The weak energy condition is clearly violated by any classical field theory with $V(\phi) \leq 0$.

4.5.5.2 Strong energy condition

This condition states that:

$$R_{\mu\nu}X^\mu X^\nu \geq 0 \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}) : X_\mu X^\mu \leq 0 \tag{4.53}$$

The strong energy condition ensures that timelike geodesics converge, i.e. that gravity is attractive. Using Eq. 4.42, this condition can be rewritten as:

$$\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) X^\mu X^\nu \geq 0$$

Taking yet again $X \cdot X = -1$, the strong energy condition for the energy-momentum tensor of a fluid Eq. 4.46 in the rest-frame reads:

$$(\rho+p)(u{\cdot}X)^2-p+\frac{1}{2}(3p-\rho) \geq 0 \quad \Rightarrow \quad (\rho+p)\cosh\varphi^2-p+\frac{1}{2}(3p-\rho) \geq 0 \quad \Rightarrow \quad \begin{cases} p \geq -\rho/3 & \varphi = 0 \\ p \geq -\rho & \varphi \to \infty \end{cases}$$

It's not difficult to show that $\Lambda > 0$ violates the strong energy condition: dS space is incompatible with it, as timelike geodesics are pulled apart by the expansion of space.
Moreover, any classical scalar field theory with $V(\phi) \geq 0$ violates the strong energy condition.

4.5.5.3 Null energy condition

This condition states that:

$$T_{\mu\nu}X^\mu X^\nu \geq 0 \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}) : X_\mu X^\mu = 0 \tag{4.54}$$

This condition is implied by both the weak and strong energy condition, but the converse is not true: it is weaker than both conditions. However, it is satisfied by any classical field theory and any perfect fluid with $\rho + p \geq 0$.

4.5.5.4 Dominant energy condition

Given a future-directed timelike vector field X, it's possible to define the energy density current measured by an observer moving along integral curves of X as:

$$J^\mu \equiv -T^{\mu\nu}X_\nu$$

The dominant energy condition requires that, in addition to the weak energy condition Eq. 4.52:

$$J_\mu J^\mu \leq 0 \tag{4.55}$$

This means that the energy density current is either timelike or null, so requiring that energy doesn't flow faster than time.
It's possible to check that this condition is always satisfied by a classical scalar field theory, while for a perfect fluid it imposes $\rho^2 \geq p^2$.

4.6 Cosmology

Of the few situations in which Einstein field equations sourced by matter Eq. 4.41 need to be solved directly, the one where $T_{\mu\nu}$ plays a crucial role is Cosmology, the study of the universe as a whole.

4.6.1 FLRW metric

The key assumption of cosmology is that the universe is spatially homogeneous and isotropic. These conditions restric the possible spatial geometries to only three:

- Euclidean space $\mathbb{R}^3$, with vanishing curvature and metric:

$$ds^2 = dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

- sphere $\mathbb{S}^3$, with uniform positive curvature and metric (implicit unitary radius):

$$ds^2 = \frac{1}{1-r^2}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

- hyperboloid $\mathbb{H}^3$, with uniform negative curvature and metric:

$$ds^2 = \frac{1}{1+r^2}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

The existence of these three symmetric spaces is analogous to the existance of three symmetric spacetimes as solutions of the vacuum field equations: dS and AdS have constant *spacetime curvature*, supplied by the cosmological constant, while $\mathbb{S}^3$ and $\mathbb{H}^3$ have constant *spatial curvature*. Indeed, the metric on $\mathcal{S}^3$ corresponds to the spatial part of the de Sitter metric Eq. 4.14, while the metric on $\mathcal{H}^3$ corresponds to the spatial part of the anti-de Sitter metric Eq. 4.23.
These spatial metrics are written in unified form as:

$$ds^2 = \gamma_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \tag{4.56}$$

with $k = +1, 0, -1$ on $\mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3$ respectively. Cosmology studies spacetimes in which space expands as the universe evolves, thus the metric takes the form:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \tag{4.57}$$

This is the *Friedmann-Lemaître-Robertson-Walker metric* and the dimensionless factor $a(t)$ can be viewed as the size of the spatial dimensions.

Example 4.6.1. de Sitter metric in global coordinates Eq. 4.22 is a FLRW metric with $k = +1$.

4.6.1.1 Curvature tensors

To solve the field equations for FLRW metrics, first compute the Ricci tensor. Christoffel symbols are straightforward:

$$\Gamma^\mu_{00} = \Gamma^0_{i0} = 0 \qquad \Gamma^0_{ij} = a\dot{a}\gamma_{ij} \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \qquad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right)$$

Proposition 4.6.1. *The Ricci tensor for a FLRW metric has non-vanishing components:*

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right)g_{ij} \tag{4.58}$$

Proof. First, $R_{0i} = 0$ because there's no covariant 3-vector that it could possibly equal to. Then, contracting Eq. 3.36 and recalling the only non-vanishing Christoffel symbols:

$$R_{00} = -\partial_0\Gamma^i_{i0} - \Gamma^j_{i0}\Gamma^i_{j0} = -3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

For the spatial components, consider the spatial metric in Cartesian coordinates:

$$\gamma_{ij} = \delta_{ij} + \frac{kx_i x_j}{1 - k\mathbf{x}\cdot\mathbf{x}}$$

The Christoffel symbols depend on $\partial\gamma$ and the Ricci tensor on $\partial^2\gamma$, thus to evaluate the latter at $\mathbf{x} = 0$ one only needs the metric up to quadratic order:

$$\gamma_{ij} = \delta_{ij} + kx_i x_j + o(x^4) \quad \Rightarrow \quad \gamma^{ij} = \delta^{ij} - kx^i x^j + o(x^4) \quad \Rightarrow \quad \Gamma^i_{jk} = kx^i \delta_{jk} + o(x^3)$$

where i, j indices are raised or lowered by δ^{ij}. The Ricci tensor is then computed as:

$$\begin{aligned} R_{ij} &= \partial_\rho \Gamma^\rho_{ij} - \partial_j \Gamma^\rho_{\rho i} + \Gamma^\lambda_{ij}\Gamma^\rho_{\rho\lambda} - \Gamma^\Gamma_{\rho i}\Gamma^\rho_{j\lambda} \\ &= (\partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij}) - \partial_j \Gamma^k_{ki} + (\Gamma^0_{ij}\Gamma^k_{k0} + \Gamma^k_{ij}\Gamma^l_{lk}) - (\Gamma^0_{ki}\Gamma^k_{j0} + \Gamma^k_{0i}\Gamma^0_{jk} + \Gamma^k_{li}\Gamma^l_{jk}) \end{aligned}$$

Evaluating this expression at $\mathbf{x} = \mathbf{0}$ allows to drop the $\Gamma^k_{ij}\Gamma^l_{lk}$ term and to replace any undifferentiated γ_{ij} with δ_{ij}, so that:

$$\begin{aligned} R_{ij}(\mathbf{x} = \mathbf{0}) &= \left(\partial_0(a\dot{a}) + 3k - k + 3\dot{a}^2 - \dot{a}^2 - \dot{a}^2\right)\delta_{ij} + o(x^2) \\ &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\delta_{ij} + o(x^2) \end{aligned}$$

Covariance implies that $R_{ij} \sim \gamma_{ij}$, thus the general result is:

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\gamma_{ij} = \frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) g_{ij}$$

□

The Ricci scalar is then easily computed:

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \tag{4.59}$$

Finally, the non-vanishing components of the Einstein tensor are:

$$G_{00} = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \qquad G_{ij} = -\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) g_{ij} \tag{4.60}$$

4.6.2 Friedmann equations

There only remains to specify the matter content of the universe. By hypothesis, it is filled by a perfect fluid, so that the energy-momentum tensor is that in Eq. 4.46. Assuming that the fluid is at rest in the preferred frame of the universe, i.e. $u^\mu = (1, 0, 0, 0)$ in the FLRW coordinates, the Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$ reads:

$$u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu = 0$$

Recalling that $\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\rho} u^\rho = \Gamma^i_{i0} u^0 = \dot{a}/a$, the *continuity equation* is obtained:

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0 \tag{4.61}$$

This equation expresses the conservation of energy in an expanding universe. The second constraint Eq. 4.48 is trivial for homogeneous isotropic fluids. There remains the equation of state, which for fluids of cosmological interest is simply:

$$p = w\rho \tag{4.62}$$

In particular, $w = 0$ describes pressureless dust, while $w = \frac{1}{3}$ radiation. The continuity equation thus becomes:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

This means that the energy density dillutes as the universe expands, for:

$$\rho = \frac{\rho_0}{a^{3(1+w)}} \tag{4.63}$$

For pressureless dust $\rho \sim a^{-3}$, which is the expected scaling of energy density with volume, while for radiation $\rho \sim a^{-4}$, accounting for an extra a^{-1} factor due to redshift.
With this setting, the temporal component of the Einstein field equations becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{4.64}$$

This is the *Friedmann equation*, and with Eq. 4.63 it describe how the universe expands. The spatial component of the field equations, computed using the Friedmann equation too, reads:

$$\frac{\ddot{a}}{a} - \frac{\Lambda}{3} = -\frac{4\pi G}{3}(\rho + 3p) \tag{4.65}$$

This is the *Raychaudhuri equation*, and it describes the acceleration of the expansion of the universe: it isn't independent of the Friedmann equation, as a time derivative of Eq. 4.64 yields Eq. 4.65.
A particularly simple solution can be found by setting $k = \Lambda = 0$, i.e. considering a universe dominated only by a single homogeneous isotropic fluid with energy density Eq. 4.63:

$$\left(\frac{\dot{a}}{a}\right)^2 \sim \frac{1}{a^{3(1+w)}} \quad \Rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$$

The solution $w = \frac{1}{3}$, i.e. $a(t) \sim t^{1/2}$, describes a radiation-dominated universe, which is a model for roughly the first 50'000 years of our Universe, while $w = 0$, i.e. $a(t) \sim t^{2/3}$, describes a matter-dominated universe, which is a model for roughly the following 10 billion years of our Universe.

Chapter 5

Weak Gravity

Although Einstein field equations are extremely difficult to solve, a possible ansatz is to consider an almost-flat metric with $\Lambda = 0$, which in the so called *almos-inertial coordinates* x^μ takes the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.1}$$

where the perturbation of the metric is assumed to be small: $h_{\mu\nu} \ll 1$.

5.1 Linerarized gravity

The aim is to expand the field equations to linear order in $h_{\mu\nu}$: at this order, gravity can be thought as a symmetric spin 2 field $\eta_{\mu\nu}$ propagating through flat Minkowski spacetime. Therefore, indices are raised and lowered by Minkowski metric $\eta_{\mu\nu} = \text{diag}\,(-1, +1, +1, +1)$, and the field theory inherits Lorentz invariance:

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu \quad \Rightarrow \quad h^{\mu\nu} \mapsto \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma h^{\rho\sigma}(\Lambda^{-1}x)$$

where $\eta^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. To leading order, the inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, thus:

$$\Gamma^\sigma_{\nu\rho} = \frac{1}{2}\eta^{\sigma\lambda}\left(\partial_\nu h_{\lambda\rho} + \partial_\rho h_{\nu\lambda} - \partial_\lambda h_{\nu\rho}\right) \tag{5.2}$$

Recalling Eq. 3.36, the $\Gamma\Gamma \sim h^2$ terms of the Riemann tensor are negligible to first order, so:

$$R^\sigma{}_{\rho\mu\nu} = \frac{1}{2}\eta^{\sigma\lambda}\left(\partial_\mu\partial_\rho h_{\nu\lambda} - \partial_\mu\partial_\lambda h_{\nu\rho} - \partial_\nu\partial_\rho h_{\mu\lambda} + \partial_\nu\partial_\lambda h_{\mu\rho}\right) \tag{5.3}$$

Contracting (σ, ρ), the Ricci tensor is:

$$R_{\mu\nu} = \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \tag{5.4}$$

where $\Box \coloneqq \partial^\mu\partial_\mu$ and $h = h^\mu{}_\mu$ is the trace of $h_{\mu\nu}$. The Ricci scalar is:

$$R = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h \tag{5.5}$$

Finally, the Einstein tensor can be expressed as:

$$G_{\mu\nu} = \frac{1}{2}\left[\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h\right)\eta_{\mu\nu}\right] \tag{5.6}$$

For linearized gravity, Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ becomes $\partial^\mu G_{\mu\nu} = 0$, which is indeed obeyed by Eq. 5.6. Einstein field equations with a source $T_{\mu\nu}$, which, for consistency, must be suitably small, are then a set of linear PDEs:

$$\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \left(\partial^\rho \partial^\sigma h_{\rho\sigma} - \Box h\right) \eta_{\mu\nu} = 16\pi G T_{\mu\nu} \tag{5.7}$$

This can be thought as $\mathfrak{L}(h_{\mu\nu}) = 16\pi G T_{\mu\nu}$, where $\mathfrak{L}$ is a linear differential operator known as *Lichnerowicz operator*.

Proposition 5.1.1. *Eq. 5.7 are the equations of motion derived from the Fierz-Pauli action:*

$$\mathcal{S}_{\text{FP}} = \frac{1}{8\pi G} \int d^4x \left[-\frac{1}{4} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\nu h^{\rho\mu} + \frac{1}{4} \partial_\mu h \partial^\mu h - \frac{1}{2} \partial_\nu h^{\mu\nu} \partial_\mu h \right] \tag{5.8}$$

Proof. Varying the action:

$$\begin{aligned} \delta \mathcal{S}_{\text{FP}} &= \frac{1}{8\pi G} \int d^4x \left[\frac{1}{2} \partial_\rho \partial^\rho h_{\mu\nu} - \partial^\rho \partial_\nu h_{\rho\mu} - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial_\rho h + \frac{1}{2} \partial_\nu \partial_\mu h + \frac{1}{2} \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} \right] \delta h^{\mu\nu} \\ &= \frac{1}{8\pi G} \int d^4x \left[-G_{\mu\nu} \delta h^{\mu\nu} \right] \end{aligned}$$

Hence, $G_{\mu\nu} = 0$. To get the matter coupling, add $T_{\mu\nu} h^{\mu\nu}$ to the action. □

5.1.1 Gauge symmetry

Linearized gravity inherits a useful gauge symmetry from the diffeomorphism invariance of the full theory. Under a change of coordinates $x^\mu \mapsto x^\mu - \xi^\mu(x)$, where $\xi(x)$ is assumed to be small, the metric changes by Eq. 4.11 as $\delta g_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$; for the linearized metric Eq. 5.1, being both h and ξ small, the covariant derivatives take the vanishing connection of Minkowski spacetime, thus:

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{5.9}$$

This is similar to the gauge transformation of Maxwell theory $A_\mu \mapsto A_\mu + \partial_\mu \alpha$: just like $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ is gauge-invariant, so is the linearized Riemann tensor Eq. 5.3.

Proposition 5.1.2. *The Fierz-Pauli action is invariant under the gauge symmetry Eq. 5.9.*

Proof. Recalling the linearized Bianchi identity $\partial^\mu G_{\mu\nu} = 0$:

$$\delta \mathcal{S}_{\text{FP}} = -\frac{1}{8\pi G} \int d^4x \, 2 G_{\mu\nu} \partial^\mu \xi^\nu = \frac{1}{4\pi G} d^4x \, \partial^\mu G_{\mu\nu} \xi^\nu = 0$$

□

As in Electromagnetism, it's useful to impose a gauge fixing condition.

Proposition 5.1.3. *It's always possible to pick de Donder gauge:*

$$\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h = 0 \tag{5.10}$$

Proof. Suppose that the doesn't obey de Donder condition, but WLOG $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = f_\nu$ for some functions f_ν. After the gauge transformation Eq. 5.9, this becomes $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h + \Box\xi_\nu = f_\nu$, thus one only needs to find $\xi_\nu : \Box\xi_\nu = f_\nu$, which always has a solution. $\Box$

In de Donder gauge, the field equations Eq. 5.7 are greatly simplified:

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T_{\mu\nu} \tag{5.11}$$

To simplify these equations even more, it's useful to define:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \quad \Rightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

as $\bar{h} = -h$. With this choice, the linearized Einstein equations in de Donder gauge reduce to a set of wave equations:

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{5.12}$$

Non-linear theory de Donder gauge can be extended to the full non-linear theory as the condition:

$$g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0 \tag{5.13}$$

Note that this isn't a tensor equation, as $\Gamma^\rho_{\mu\nu}$ isn't a tensor, and indeed the point of gauge fixing is to set a preferred choice of coordinates. This gauge condition simplifies the expression of the d'Alembertian $\Box \coloneqq \nabla^\mu\nabla_\mu = g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)$, which simply becomes $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$; moreover, the same applies to 1-forms: $\nabla^\mu\omega_\mu = g^{\mu\nu}\nabla_\mu\omega_\nu = g^{\mu\nu}(\partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho) = \partial^\mu\omega_\mu$.

5.1.2 Newtonian limit

In the presence of a low-density, slowly-moving distribution of matter, the linearized field equations reduce to the Newtonian theory of gravity. For a stationary matter configuration, the only non-vanishing component of the energy-momentum tensor is $T_{00} = \rho(\mathbf{x})$; moreover, since there's no time-dependence, $\Box = -\partial_t^2 + \nabla^2 = \nabla^2$. The Einstein equations become:

$$\nabla^2\bar{h}_{00} = -16\pi G\rho(\mathbf{x}) \qquad \nabla^2\bar{h}_{0i} = \nabla^2\bar{h}_{ij} = 0$$

With suitable boundary conditions, the solutions to these equations are:

$$\bar{h}_{00} = -4\Phi(\mathbf{x}) \qquad \bar{h}_{0i} = \bar{h}_{ij} = 0$$

where $\Phi : \nabla^2\Phi = 4\pi G\rho$ is the Newtonian gravitational potential. Then $\bar{h} = 4\Phi$ and:

$$h_{\mu\nu} = -2\Phi(\mathbf{x})\delta_{\mu\nu}$$

The full metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ it thus expressed as:

$$ds^2 = -\left(1 + 2\Phi(\mathbf{x})\right)dt^2 + \left(1 - 2\Phi(\mathbf{x})\right)d\mathbf{x}^2$$

This is exactly teh condition Eq. 1.29. Interestingly, for a point particle $\Phi(\mathbf{x}) = -\frac{GM}{r}$ and the metric coincides with the leading expansion of Schwarzschild metric (the g_{00} term is exact).

5.2 Gravitational waves

To study the propagation of gravitational waves in vacuum and in the absence of sources, one needs to solve the linearized wave equation:

$$\Box \bar{h}_{\mu\nu} = 0 \tag{5.14}$$

A possible solution is the gravitational wave:

$$\bar{h}_{\mu\nu}(x) = \Re\{H_{\mu\nu} e^{ik_\rho x^\rho}\} \tag{5.15}$$

where $H_{\mu\nu} \in \mathbb{C}^{4\times 4}$ is a symmetric polarization matrix and the wave-vector k^μ is a real 4-vector. For simplicity, the $\Re$ is made implicit in the following calculations. This plane wave ansatz solves Eq. 5.14 if the wave-vector is null:

$$k_\mu k^\mu = 0 \tag{5.16}$$

Therefore, gravitational waves propagate at the speed of light. Writing $k^\mu = (\omega, \mathbf{k})$, this condition becomes $\omega = \pm |\mathbf{k}|$. Moreover, being the vacuum wave equation linear, the general solution is just a linear combination of plane waves.
The polarization matrix has 10 components, but gauge conditions are yet to be imposed. The plane wave ansatz satisfies de Donder gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ only if:

$$k^\mu H_{\mu\nu} = 0 \tag{5.17}$$

The polarization is then transverse to the direction of propagation. Furthermore, de Donder gauge still allows for gauge transformations $h_{\mu\nu} \mapsto h_{\mu\nu}\partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, i.e. $\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial^\rho \xi_\rho$, so the solution is still in de Donder gauge only if:

$$\Box \xi_\mu = 0 \quad \Rightarrow \quad \xi_\mu(x) = \lambda_\mu e^{ik_\rho x^\rho}$$

This gauge transformation shifts the polarization matrix as:

$$H_{\mu\nu} \mapsto H_{\mu\nu} + i\left(k_\mu \lambda_\nu + k_\nu \lambda_\mu - \eta_{\mu\nu} k^\rho \lambda_\rho\right) \tag{5.18}$$

Polarization matrices which differ by this term thus describe the same gravitational wave. Hence, it is possible to choose λ_μ in order to have:

$$H_{0\mu} = 0 \quad \wedge \quad H^\mu{}_\mu = 0 \tag{5.19}$$

These condition, together with Eq. 5.17, are called *transverse traceless gauge*. Being $H_{\mu\nu}$ traceless, in this gauge $\bar{h}_{\mu\nu} = h_{\mu\nu}$, as it is traceless too. Of the 10 components of the polarization matrix, only 2 are independent: de Donder condition Eq. 5.17 poses 4 constraints and the gauge transformation Eq. 5.18 poses 4 more of them. Therefore, there are only two independent polarizations in $H_{\mu\nu}$.

Example 5.2.1. Consider a gravitational wave propagating in the z direction: $k^\mu = (\omega, 0, 0, \omega)$, thus $H_{0\nu} + H_{3\nu} = 0$ by Eq. 5.17. By Eq. 5.19, the polarization matrix is then restricted to be:

$$H_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.20}$$

where in general $H_+, H_\times \in \mathbb{C}$. The two polarizations are seen explicitly.

5.2.1 Polarizations

Point particles moving along geodesics are not affected by gravitational waves due to the equivalence principle: to measure their passage, one needs to study how the relative distance between two observers changes, which is done using the geodesic deviation (recall Sec. 3.3.3).
Consider a family of geodesics $x^\mu(\tau, s)$ with tangent vector field $u^\mu = \partial_\tau|_s x^\mu$ and deviation vector field $S^\mu = \partial_s|_\tau x^\mu$. The geodesic deviation equation (Eq. 3.58) reads:

$$\frac{D^2 S^\mu}{D\tau^2} = R^\mu{}_{\rho\sigma\nu} u^\rho u^\sigma S^\nu$$

Suppose that, in the absence of gravitational waves, the geodesics are in a rest-frame such that $u^\mu = (1, 0, 0, 0)$: as the gravitational wave passes $u^\mu = (1, 0, 0, 0) + o(h)$, so the aim is to compute the geodesic deviation to leading order in h. The Riemann tensor is already $o(h)$, thus other correction terms can be neglected; similarly, proper time τ can be replaced with coordinate time t, so that the geodesic deviation equation becomes:

$$\frac{d^2 S^\mu}{dt^2} = R^\mu{}_{00\nu} S^\nu$$

In the linearized regime, the Riemann tensor is given by Eq. 5.3, hence, using $h_{\mu 0} = 0$, the needed components are $R^\mu{}_{00\nu} = \frac{1}{2}\partial_0^2 h^\mu{}_\nu$ and:

$$\frac{d^2 S^\mu}{dt^2} = \frac{1}{2}\frac{d^2 h^\mu{}_\nu}{dt^2} S^\nu \tag{5.21}$$

For simplicity, consider a wave propagating in the z direction with polarization matrix Eq. 5.20 and solve the geodesic deviation equation in the $z = 0$ plane only, as S^0 and S^3 are not affected by the gravitational wave.

5.2.1.1 + polarization

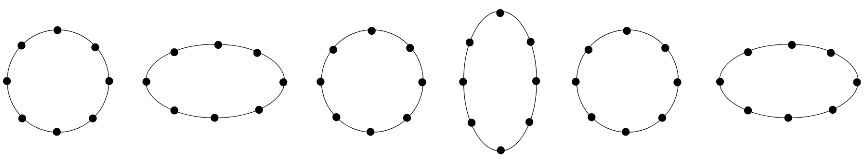
Setting $H_\times = 0$, Eq. 5.21 becomes:

$$\frac{d^2 S^1}{dt^2} = -\frac{\omega^2}{2} H_+ e^{i\omega t} S^1 \qquad \frac{d^2 S^2}{dt^2} = +\frac{\omega^2}{2} H_+ e^{i\omega t} S^2$$

Solving perturbatively in H_+, at leading order:

$$S^1(t) = S^1(0)\left[1 + \frac{1}{2} H_+ e^{i\omega t} + \dots\right] \qquad S^2(t) = S^2(t)\left[1 - \frac{1}{2} H_+ e^{i\omega t} + \dots\right] \tag{5.22}$$

where, again, the $\Re$ is implicit (recall that in general $H_+ \in \mathbb{C}$). To visualize these solutions, consider a family of neighbouring geodesics which, at $t = 0$, are arranged around a circle of radius R: the initial conditions then satisfy $S^1(0)^2 + S^2(0)^2 = R^2$. The relative negative sign determines:



5.2.1.2 × polarization

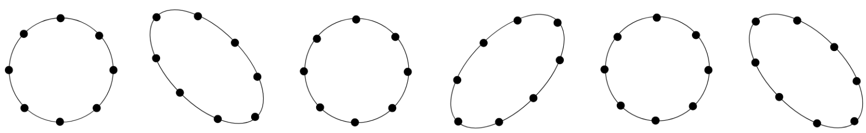
Setting $H_\times = 0$, Eq. 5.21 becomes:

$$\frac{d^2S^1}{dt^2} = -\frac{\omega^2}{2}H_\times e^{i\omega t}S^2 \qquad\qquad \frac{d^2S^2}{dt^2} = -\frac{\omega^2}{2}H_\times e^{i\omega t}S^1$$

Solving perturbatively in $H_\times$, at leading order:

$$S^1(t) = S^1(0) + \frac{1}{2}S^2(0)H_\times e^{i\omega t} + \dots \qquad S^2(t) = S^2(t) + \frac{1}{2}S^1(0)H_\times e^{i\omega t} + \dots \tag{5.23}$$

This is the same displacement as Eq. 5.22, but rotated by 45°: to see this, note that $S^1(t) \pm S^2(t)$ have the same functional expression as Eq. 5.22. Thus:



The general polarization is a linear combination of both: the result is yet an elliptic displacement whose axes rotate, analogously to the circular polarization of light. Interestingly, note that displacements due to gravitational waves are invariant under π rotations, while light polarization, being described by a vector, is invariant under 2π rotations: this reflects the fact that the graviton has spin 2 and the photon has spin 1.

5.2.1.3 Gravitational wave detection

Gravitational wave detectors are interferometers, which bounce light back and forth between two arms. If the gravitational wave propagates perpendicular to the plane of detection, it will shorten one arm and lengthen the other: assuming the arms are aligned with x and y axes, the maximum change in length by Eq. 5.22 is $L' = L(1 \pm H_+/2)$, i.e. $\delta L/L = H_+/2$. For a typical astrophysical source $H_+ \sim 10^{-21}$, while for LIGO $L \sim 3\,\mathrm{km}$, thus $\delta L \sim 10^{-18}\,\mathrm{m}$: this is smaller than the radius of the proton and extremely difficult to detect, however the first direct measurement of gravitational waves was performed in 2015 and now LIGO and VIRGO detectors have observed a large number of mergers involving black holes and neutron stars.

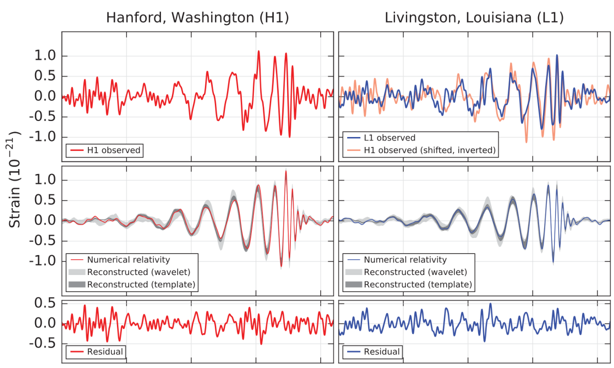


Figure 5.1: First detection of gravitational waves by LIGO.

5.2.2 Exact solutions

Given the found wave-like solution to the linearized field equations, the metric of a wave moving in the positive z direction takes the form:

$$ds^2 = -dt^2 + (\delta_{ab} + h_{ab}(z-t))\, dx^a dx^b + dz^2 \tag{5.24}$$

with $a, b = 1, 2$. Being the wave equation linear, any function $h_{ab}(z-t)$ is a solution: Eq. 5.15 is simply the Fourier decomposition of the general solution.
The extreme weakness of gravitational waves makes the linearized metric Eq. 5.1 suitable to describe their properties. However, the wave solution has an extension to the general non-linear field equations. Consider a wave propagating in the positive z direction and introduce lightcone coordinates:

$$u = t - z \qquad v = t + z$$

Then, consider the plane wave ansatz, also called *Brinkmann metric*:

$$ds^2 = -du\, dv + dx^a dx^a + H_{ab}(u) x^a x^b du^2$$

Note that to obtain the linearized metric Eq. 5.24 from Brinkmann metric, one needs some change of coordinates. It is possible to show that Brinkmann metric is Ricci flat for any traceless matrix $H_{ab}(u)$, hence solving the vacuum Einstein equations:

$$R_{\mu\nu} = 0 \quad \Leftrightarrow \quad H^a{}_a(u) = 0 \quad \Leftrightarrow \quad H_{ab}(u) = \begin{bmatrix} H_{11}(u) & H_{12}(u) \\ H_{12}(u) & -H_{11}(u) \end{bmatrix}$$

The general metric has again two independent polarization states.

5.3 Perturbing spacetime

The gravitational plane wave solution Eq. 5.15 is not realistic, as it is comes from infinity and goes to infinity: in reality, gravitational waves are produced at some point and radiate out. The analogy with Electromagnetism persists: as electromagnetic waves are produced by oscillating charges, gravitational waves are produced by oscillating masses.

5.3.1 Green's function

Recall the linearized Einstein field equations Eq. 5.12:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

which assumes that both $h_{\mu\nu}$ and $T_{\mu\nu}$ are small. These are a set of decoupled wave equations.
Consider a matter field localized in a spatial region Σ, in which a time-dependent energy-momentum source $T_{\mu\nu}(\mathbf{x}, t)$ is present (ex.: two orbiting black holes): the localization condition then is simply $T_{\mu\nu}(\mathbf{x}, t) = 0\, \forall \mathbf{x} \notin \Sigma$. The question is what the metric $h_{\mu\nu}$ looks like far away from Σ. The solution to Eq. 5.12 outside of Σ can be expressed using the retarded Green's function:

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = 4G \int_\Sigma d^3x' \, \frac{T_{\mu\nu}(\mathbf{x}', t_r)}{|\mathbf{x} - \mathbf{x}'|} \qquad t_r \equiv t - |\mathbf{x} - \mathbf{x}'| \tag{5.25}$$

Retarded time expresses the causality of the wave equation. This solution satisfies de Donder condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ only if $\partial^\mu T_{\mu\nu} = 0$, i.e. the energy-momentum tensor is conserved; however, it does not automatically satisfy conditions Eq. 5.19.
Denoting the size of Σ as d and $r \equiv |x|$, then approximating:

$$|\mathbf{x} - \mathbf{x}'| \gg d \,\forall \mathbf{x}' \in \Sigma \quad \Rightarrow \quad |\mathbf{x} - \mathbf{x}'| = r - \frac{\mathbf{x} \cdot \mathbf{x}'}{r} + \dots \quad \Rightarrow \quad \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x} \cdot \mathbf{x}'}{r^3} + \dots$$

To approximate the energy-momentum tensor, assume that the motion of matter is non-relativistic, so that $T_{\mu\nu}$ does not change much over the time $\tau \sim d$ needed to light to cross Σ. For example, in the case of two objects orbiting each other with characteristic frequency ω, then $T_{\mu\nu} \sim e^{-i\omega t}$ and the non-relativistic condition reads $d \ll 1/\omega$. Taylor expanding $T_{\mu\nu}$:

$$T_{\mu\nu}(\mathbf{x}', t_r) = T_{\mu\nu}(\mathbf{x}', t - r + \mathbf{x} \cdot \mathbf{x}'/r + \dots) = T_{\mu\nu}(\mathbf{x}', t - r) + \dot{T}_{\mu\nu}(\mathbf{x}', t - r)\frac{\mathbf{x} \cdot \mathbf{x}'}{r} + \dots$$

At leading order in d/r, then, the solution becomes:

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) \approx \frac{4G}{r} \int_\Sigma d^3x' \, T_{\mu\nu}(\mathbf{x}', t - r)$$

The temporal component is:

$$\bar{h}_{\mu\nu}(\mathbf{x}) \approx \frac{4G}{r} E \qquad\qquad E \equiv \int_\Sigma d^3x' \, T_{00}(\mathbf{x}', t - r)$$

This is simply the Newtonian limit; note that there's no time-dependency, as $\partial^\mu T_{\mu\nu} = 0$ ensures that the energy E inside Σ is conserved. Similarly:

$$\bar{h}_{0i}(\mathbf{x}) \approx -\frac{4G}{r} P_i \qquad\qquad P_i \equiv -\int_\Sigma d^3x' \, T_{0i}(\mathbf{x}', t - r)$$

Again, the total momentum of matter inside Σ is conserved, hence the absence of time-dependence. Note that it's always possible to choose a rest-frame where matter is stationary, i.e. $P_i = 0$ and $h_{0i} = 0$. The motion of matter inside Σ is instead described by $\bar{h}_{ij}$.

Proposition 5.3.1. *Far away from the source, the spatial metric takes the form:*

$$\bar{h}_{ij}(\mathbf{x}, t) \approx \frac{2G}{r} \ddot{I}_{ij}(t - r) \qquad\qquad I_{ij}(t) := \int_\Sigma d^3x \, T^{00}(\mathbf{x}, t) x_i x_j \tag{5.26}$$

where $I_{ij}(t)$ is the quadrupole moment for the energy.

Proof. The thesis is equivalent to:

$$\int_\Sigma d^3x' \, T_{ij}(\mathbf{x}, t) = \frac{1}{2} \ddot{I}_{ij}(t)$$

Recalling current conservation $\partial_\mu T^{\mu\nu} = 0$:

$$T^{ij} = \partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j = \partial_k(T^{ik}x^j) + \partial_0 T^{0i}x^j$$

$$T^{0i} = \partial_k(T^{0k}x^i) - (\partial_k T^{0k})x^i = \partial_k(T^{0k}x^i) + \partial_0 T^{00}x^i \quad \Rightarrow \quad T^{0(i}x^{j)} = \frac{1}{2}\partial_k(T^{0k}x^i x^j) + \frac{1}{2}\partial_0 T^{00}x^i x^j$$

Integrating the first over Σ, recalling that $T^{ij} = T^{(ij)}$, and dropping the total spatial derivatives yields the result. □

The physical meaning of Eq. 5.26 is that if a matter distribution is shaken then, after the time needed for signal to propagate, is will affect the metric. Given the linearity of these equations, if matter oscillates at a frequency ω, then spacetime will create waves at parametrically same frequency.
The gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ means that $\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}$ and $\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{i0}$. The first equation gives:

$$\partial_0 \bar{h}_{0i} = -\frac{2G\hat{x}_j}{r^2}\ddot{I}_{ij}(t-r) - \frac{2G\hat{x}_j}{r}\dddot{I}_{ij}(t-r)$$

where $\partial_j r = x_j/r \equiv \hat{x}_j$. Note that, though the first term is $\sim 1/r^2$ and the second $\sim 1/r$, the latter has an additional time derivative, i.e. an extra factor of the characteristic source frequency ω: therefore, the second term dominates for $r \gg 1/\omega$, i.e. $r \gg \lambda$ (emitted gravitational waves' wavelength). This is the so-called *far-field zone* or *radiation zone*, and in this regime:

$$\bar{h}_{0i} \approx -\frac{4G}{r}P_i - \frac{2G\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \tag{5.27}$$

Note that this expression, found integrating the preceding equation, contains an integration constant given by the aforementioned P_i, but it can be set to zero choosing coordinates in which the center of mass of the system is stationary. By analogous reasoning:

$$\bar{h}_{00} \approx \frac{4G}{r}E + \frac{2G\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \tag{5.28}$$

Example 5.3.1. Consider two objects of mass R and separated by a distance R, orbiting in the (x, y) plane: by Newtonian gravity, the orbit frequency is $\omega^2 = 2GM/R^3$. Treating these objects as point particles, their energy density is:

$$T^{00}(\mathbf{x}, t) = M\delta(z)\left[\delta\left(x - \frac{R}{2}\cos\omega t\right)\delta\left(y - \frac{R}{2}\sin\omega t\right) + \delta\left(x + \frac{R}{2}\cos\omega t\right)\delta\left(y + \frac{R}{2}\sin\omega t\right)\right]$$

The quadrupole moment then is:

$$I_{ij}(t) = \frac{MR^2}{2}\begin{bmatrix} \cos^2\omega t & \cos\omega t\sin\omega t & 0 \\ \cos\omega t\sin\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{MR^2}{4}\begin{bmatrix} 1+\cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & 1-\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The resulting metric perturbation is:

$$\bar{h}_{ij}(\mathbf{x}, t) \approx -\frac{2GMR^2\omega^2}{r}\begin{bmatrix} \cos 2\omega t_r & \sin 2\omega t_r & 0 \\ \sin 2\omega t_r & -\cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $t_r = t - r$. This gravitational wave propagates approximately radially. To estimate its expected strength:

$$|h_{ij}| \sim \frac{2GMR^2\omega^2}{r} \sim \frac{G^2M^2}{Rr}$$

Thus, the signal is largest for large masses orbiting very tightly. The most dense objects are black holes, for which $R_s = 2GM$, so considering two black holes orbiting at $R \approx R_s$ then $|h_{ij}| \sim GM/r \sim R_s/r$: black holes of a few solar masses have $R_s \sim 10\,\text{km}$, so if they are situated in a galaxy a billion light-years away from Earth $r \sim 10^{21}\,\text{km}$, hence $|h| \sim 10^{-20}$.

5.3.1.1 Multipole expansion

Comparing to Electromagnetism, recall the first multipoles of the charge distribution $\rho(\mathbf{x})$:

$$Q = \int_\Sigma d^3x\, \rho(\mathbf{x}) \qquad \mathbf{p} = \int_\Sigma d^3x\, \rho(\mathbf{x})\mathbf{x} \qquad \mathcal{Q}_{ij} = \int_\Sigma d^3x\, \rho(\mathbf{x}) \left(3x_i x_j - \delta_{ij}x^2\right)$$

Charge conservation $\dot{Q} = 0$ nullifies the monopole contribution to electromagnetic waves. The leading order contribution is that of the dipole:

$$\mathbf{A}(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi}\dot{\mathbf{p}}(t - r)$$

For gravity, the first multipoles of the energy distribution $T_{00}(\mathbf{x})$ are:

$$E = \int_\Sigma d^3x\, T_{00}(\mathbf{x}) \qquad \mathbf{X} = \frac{1}{E}\int_\Sigma d^3x\, T_{00}(\mathbf{x})\mathbf{x} \qquad I_{ij} = \int_\Sigma d^3x\, T_{00}(\mathbf{x}, t)x_i x_j$$

i.e. the total energy, the center of mass (energy) of the distribution and the quadrupole. Energy conservation $\dot{E} = 0$ is responsible for the lack of monopole contribution to gravitational radiation. But in contrast to Electromagnetism, the dipole contribution also vanishes due to momentum conservation $\dot{\mathbf{P}} = \mathbf{0}$:

$$E\dot{X}_i = \int_\Sigma d^3x\, (\partial_0 T_{00})x_i = \int_\Sigma d^3x\, (\partial_j T_{j0})x_i = -\int_\Sigma d^3x\, T_{i0} = P_i \quad \Rightarrow \quad E\ddot{\mathbf{X}} = \dot{\mathbf{P}} = \mathbf{0}$$

In Electromagnetism, another dipole contribution to the gauge potential is possible:

$$\mathbf{A}(\mathbf{x}, t) = -\frac{\mu_0}{4\pi r}\hat{\mathbf{x}} \times \dot{\mathbf{m}}(t - r) \qquad\qquad \mathbf{m} := \frac{1}{2}\int_\Sigma d^3x\, \mathbf{x} \times \mathbf{J}(\mathbf{x})$$

For gravity, there's an equivalent contribution with the analogue of magnetic dipole being:

$$J_i = \int_\Sigma d^3x\, \epsilon_{ijk}x_j T_{0k}$$

This is nothing other than the total angular momentum of the system, thus the dipole contribution to gravitational radiation is nullified by angular momentum conservation $\dot{\mathbf{J}} = \mathbf{0}$ too. The leading order effect for gravitational waves is then the quadrupole.

5.3.2 Radiated power

In the context of Electromagnetism, the power radiated as electromagnetic waves are emitted is easily found defining the Poynting vector $\mathbf{S} := \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$ and, using the dipole approximation, finding *Larmor formula*:

$$\mathcal{P} = \int_{\mathbb{S}^2} d^2\mathbf{r} \cdot \mathbf{S} = \frac{\mu_0}{6\pi c}\, |\ddot{p}|^2$$

Analogously, a source emitting gravitational waves loses energy which is carried by them. The problem of finding an equivalent to Larmor formula is that there's no local energy-momentum tensor for gravitational fields, so there's no analogue to the Poynting vector. A possible way forward is to define an energy-momentum tensor $t_{\mu\nu}$ for gravitatonal waves which, in the linearized theory, obeys $\partial^\mu t_{\mu\nu} = 0$: Eq. 4.49 does not allow to do so in a diffeomorphism-invariant way, i.e. in the full non-linear theory $t_{\mu\nu}$ is not a tensor and in the linearized theory it is not invariant under gauge transformations Eq. 5.9. It turns out that there are a number of different ways to define such a tensor.

5.3.2.1 Fierz-Pauli action

Recall the Fierz-Pauli action Eq. 5.8 for linearized gravity: viewed as an action describing a spin 2 field propagating through Minkowski spacetime, it can be treated as any classical field theory and used to compute an energy-momentum tensor. In the transverse traceless gauge, i.e. with $h = 0$ and $\partial^\mu h_{\mu\nu} = 0$, after integration by parts the action becomes:

$$\mathcal{S}_{\text{FP}} = -\frac{1}{8\pi G}\int d^4x\,\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}$$

This looks like the action for a set of massless scalar fields, hence the energy dentity takes the schematic form:

$$t^{00} \sim \frac{1}{G}\dot{h}_{\mu\nu}\dot{h}^{\mu\nu}$$

There are gradient terms too, but, due to the wave equation, they contribute as time derivatives. Note that t^{0i} scales in the same way. In the transverse traceless gauge, Eq. 5.26 becomes:

$$h_{ij}(\mathbf{x},t) \sim \frac{G}{r}\ddot{\mathcal{Q}}(t-r) \qquad\qquad \mathcal{Q}_{ij} \equiv I_{ij} - \frac{1}{3}I_{kk}\delta_{ij}$$

where $\mathcal{Q}_{ij}$ is the traceless part of the quadrupole moment I_{ij}. The energy density carried by gravitational waves should then be:

$$t^{00} \sim \frac{G}{r^2}\dddot{\mathcal{Q}}_{ij}^2$$

Integrating over a sphere at large distance suggests that $\mathcal{P} \sim G\,\dddot{\mathcal{Q}}_{ij}^2$, and this is indeed correct, as it can be shown that:

$$\mathcal{P}(t) = \frac{G}{5}\dddot{\mathcal{Q}}_{ij}(t_r)\,\dddot{\mathcal{Q}}^{ij}(t_r) \tag{5.29}$$

This is the *quadrupole formula* and it is analogous to Larmor formula. Note that r is the distance at which the gravitational waves are observed.

5.3.2.2 Conceptual issues

Although Eq. 5.29 is indeed correct, as the 1993 Nobel prize was awarded for data in agreement with it, there remains some conceptual issues in the definition of $t_{\mu\nu}$, which can be improved.
First, note that the definition of $t_{\mu\nu}$ from the Fierz-Pauli action suffers a number of ambiguities: if one attempts to compute it as the Noether currents associated to spacetime translations, the result is neither symmetric nor gauge invariant, but this is also true for Maxwell theory. An idea could be to add a term: $t_{\mu\nu} \mapsto t_{\mu\nu} + \partial^\rho\Theta_{\rho\mu\nu}$, with $\Theta_{\rho\mu\nu} = -\Theta_{\mu\rho\nu}$ as to not violate current conservation; such a term would make $t_{\mu\nu}$ symmetric, but still not gauge invariant.
A different approach is to interpret the lack of energy conservation for matter fields as energy transferred to the gravitational field. Although covariant conservation is not actual conservation for $T_{\mu\nu}$, it can be rewritten as:

$$0 = \nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}\,T^\mu{}_\nu) - \Gamma^\rho_{\mu\nu}T^\mu{}_\rho = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}\,T^\mu{}_\nu) - \frac{1}{2}\partial_\nu g_{\mu\rho}T^{\mu\rho}$$

where the symmetry of $T_{\mu\nu}$ was used. Note that $\Gamma^\rho_{\mu\nu}$ reduces to $g_{\mu\rho,\nu}$ only when ν is down: this reflects that this equation is non-covariant. Using the field equations:

$$\partial_\mu(\sqrt{-\mathrm{g}}\,T^\mu{}_\nu) = \frac{1}{16\pi G}\sqrt{-\mathrm{g}}\,\partial_\nu g_{\mu\rho}\left(R^{\mu\rho} - \frac{1}{2}Rg^{\mu\rho}\right) = \frac{1}{16\pi G}\sqrt{-\mathrm{g}}\,\partial_\nu g_{\mu\rho}R^{\mu\rho}$$

The idea is to express this equation as $\partial_\mu(\sqrt{-\mathrm{g}}\,T^\mu{}_\nu) = -\partial_\mu(\sqrt{-\mathrm{g}}\,t^\mu{}_\rho)$, for some $t^\mu{}_\nu$ referred to as *Landau-Lifshitz pseudotensor*: this suggests that the sum of matter energy $T^\mu{}_\nu$ and gravitational energy $t^\mu{}_\nu$ is conserved, however it would be coordinate-dependent as $t^\mu{}_\nu$ is not a real tensor.
The final approach assumes that $t_{\mu\nu}$ is quadratic in $h_{\mu\nu}$, so, keeping $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, expand the Einstein field equations to second order:

$$\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]^{(1)} + \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]^{(2)} = 8\pi G T_{\mu\nu} \quad \Leftrightarrow \quad \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]^{(1)} = 8\pi G(T_{\mu\nu} + t_{\mu\nu})$$

where superscripts (n) denote a restriction to terms of order h^n. The gravitational energy-momentum non-tensor thus is:

$$t_{\mu\nu} = -\frac{1}{8\pi G}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]^{(2)} = -\frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}R^{(2)}\eta_{\mu\nu} - \frac{1}{2}R^{(1)}h_{\mu\nu}\right]$$

Far away from the source $R^{(1)}$ can be neglected, since it vanishes by the equations of motion (at linear order), so:

$$t_{\mu\nu} = -\frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}R^{(2)}\eta_{\mu\nu}\right] \tag{5.30}$$

The linearized Bianchi identity is $\partial^\mu\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]^{(1)} = 0$, hence, far away from sources, i.e. $T_{\mu\nu} = 0$, necessarily $\partial^\mu t_{\mu\nu} = 0$ as befits a conserved current. However, $t_{\mu\nu}$ is still not gauge invariant.

5.3.2.3 Gauge invariance

Although none of the defined $t_{\mu\nu}$ is gauge invariant, it is still possible to extract a gauge-invariant quantity from it, which has physical meaning.
First, if the spacetime is asymptotically Minkowski, it could be possible to integrate t^{00} on an infinite spatial hypersurface, obtaining the so-called *ADM energy*, which is shown to be constant in time and gauge.invariant. Alternatively, one could integrate t^{0i} on a sphere at $\mathcal{I}^+$, yielding the so-called *Bondi energy*, which is gauge-invariant and time-dependent, and it can be defined in the full non-lnear theory too.
Moreover, consider that, like any other waves, gravitational waves vary over some typical length λ. Thus, averaging over these oscillations is possible by:

$$\langle t_{\mu\nu}\rangle := \int_V d^4y\, W(x-y)t_{\mu\nu}(y)$$

where V is a spacetime region of typical size a and $W \in \mathcal{C}^\infty(V) : \int_V d^4x\, W(x) = 1 \wedge W|_{\partial V} = 0$. This integral averages total derivatives as $\langle \partial X\rangle \sim 1/a$, so $\langle X\partial Y\rangle = -\langle Y\partial X\rangle + o(1/a)$. Then:

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi G}\langle \partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma}\rangle \tag{5.31}$$

This is indeed a conserved quantity:

$$\partial^\mu\langle t_{\mu\nu}\rangle = \frac{1}{32\pi G}\langle (\Box h_{\rho\sigma})\partial_\nu h^{\rho\sigma} + \frac{1}{2}\partial_\nu(\partial_\mu h_{\rho\sigma}\partial^\mu h^{\gamma\sigma})\rangle = 0$$

as the first term vanishes by equations of motion and the second yields a negligible total derivative. Under a gauge transformation like Eq. 5.9:

$$\delta \langle t_{\mu\nu} \rangle = \frac{1}{16\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu (\partial^\rho \xi^\sigma + \partial^\sigma \xi^\rho) \rangle = \frac{1}{16\pi G} \langle \partial_\mu \partial^\rho h_{\rho\sigma} \partial_\nu \xi^\sigma + \partial_\mu \partial^\sigma h_{\rho\sigma} \partial_\nu \xi^\rho \rangle + o(a^{-1}) = o(a^{-1})$$

where, after integration by parts, de Donder gauge condition $\partial^\rho h_{\rho\sigma} = 0$ was invoked. Hence, $t_{\mu\nu}$ is almost gauge-invariant, and it is properly gauge-invariant if $a \to \infty$, i.e. if the averaging is over all of spacetime. The power emitted by a gravitational wave at infinity is:

$$\mathcal{P} = \int_{\mathbb{S}^2_\infty} d^2x \, \hat{n}_i \, \langle t^{0i} \rangle$$

with $\hat{n}_i$ a normal versor to $\mathbb{S}^2_\infty$. This indeed gives Eq. 5.29.

5.3.2.4 Orders of magnitude

Black holes Consider two masses M, separated by a distance R and orbiting each other with frequency ω. Newtonian gravity approximation gives:

$$\omega^2 R \sim \frac{GM}{R^2}$$

The quadrupole is $\mathcal{Q} \sim MR^2$, hence $\dddot{\mathcal{Q}} \sim \omega^3 MR^2$, and the emitted power scales as:

$$\mathcal{P} \sim G \dddot{\mathcal{Q}}^2 \sim \frac{G^4 M^5}{R^5}$$

Returning to SI units, $[G] = M^{-1} L^3 T^{-2}$ and, for a black hole, $R_s = 2GM/c^2$, thus:

$$\mathcal{P} = \left(\frac{R_s}{R} \right)^5 L_\mathrm{p}$$

where the *Plank luminosity* is defined as:

$$L_\mathrm{p} \equiv \frac{c^5}{G} \approx 3.6 \cdot 10^{52} \, \mathrm{J\,s^{-1}}$$

To get a sense of scale, the Sun emits $L_\odot \approx 10^{-26} L_\mathrm{p}$ and the Milky Way, with 10^{11} stars, emits $L \approx 10^{-15} L_\mathrm{p}$. Moreover, in the visible Universe there are roughly 10^{10} galaxies, so all the stars of the visible Universe shine with $L \approx 10^{-5} L_\mathrm{p}$. Yet, when two black holes spiral towards each other, when their separation is comparable to their Schwarzschild radius, they emit in gravitational waves 10^5 times the energy emitted by all the stars of the visible Universe.

Solar System If two objects with masses $M_1 \gg M_2$ orbit each other, then their gravitationally-radiated power is:

$$\mathcal{P} \sim \frac{G^4 M_1^3 M_2^2}{R^5}$$

Considering Jupiter, which has $M \sim 10^{-3} M_\odot$ and orbits at $R \approx 10^9\,\mathrm{km}$ from the Sun, which has $R_s \approx 3\,\mathrm{km}$:

$$\mathcal{P} \approx 10^{-50} L_\mathrm{p} \approx 10^{-24} L_\odot$$

This is completely negligible.

Human body Consider a human being shaking its arms around. The mass of a human arm is a few kg and moves a distance of around $1\,\mathrm{m}$ with a frequency $\omega \sim 1\,\mathrm{Hz}$, so $\mathcal{Q} \approx 1\,\mathrm{kg\,m^2}$ and $\dddot{\mathcal{Q}} \approx 1\,\mathrm{kg\,m^2\,s^{-3}}$, hence the gravitationally-radiated power is:

$$\mathcal{P} \sim \frac{G\dddot{\mathcal{Q}}^2}{c^5} \approx 10^{-52}\,\mathrm{J\,s}^{-1\approx 10^{-52}\,\mathrm{J\,s^{-1}}}$$

Suppose the existence of gravitons with $E = \hbar\omega$: to produce a single graviton with $\omega = 1\,\mathrm{Hz}$, i.e. $E \approx 10^{-34}\,\mathrm{J}$, a human needs to shake its arms around for $t \approx 10^{18}\,\mathrm{s}$: this is approximately the age of the Universe.

Chapter 6

Black Holes

6.1 Schwarzschild black holes

The simplest black hole solution is the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right) \tag{6.1}$$

This is a special case of the general metric in Sec. 4.2 with $f(r)^2 = 1 - 2GM/r$: it solves the vacuum Einstein equations $R_{\mu\nu} = 0$.
This solution depends on a single parameter M, which is identified as the mass of the black hole: from the Newtonian approximation $g_{00} = -(1 + 2\Phi)$, so $\Phi = -\frac{GM}{r}$, which is precisely the potential for a point mass M at the origin. The same can be shown via Komar integrals: note that the Schwarzschild metric admits a timelike Killing vector field $K = \partial_t$, with dual 1-form $K = g_{00}dt$ and 2-form:

$$F = dK = -\frac{2GM}{r^2} dr \wedge dt$$

The associated Komar charge is:

$$M_{\text{Komar}} = -\frac{1}{8\pi G} \int_{\mathbb{S}^2_R} \star F = M$$

for all $R > 2GM$ (radius of the horizon). Although the 2-form F obeys the vacuum Maxwell equations $d \star F = 0$, thus one would expect the associated Komar charge to vanish as there's no current, $M_{\text{Komar}} = M \neq 0$: this is possible because the mass of the black hole is localized at the origin, where F diverges. Moreover, the Schwarzschild solution is mathematically valid for $M \in \mathbb{R}$: the solution $M = 0$ is just Minkowski spacetime, while $M < 0$ will be shown to be unphysical.

6.1.1 Birkhoff theorem

Theorem 6.1.1 (Birkhoff). *The Schwarzschild solution is the unique spherically-symmetric and asimptotically-flat solution to the vacuum Einstein field equations.*

Proof. First, spherical symmetry means that the metric has an SO(3) isometry. It can be shown that any metric with such isometry can be written as:

$$ds^2 = g_{\tau\tau}(\tau, \rho) d\tau^2 + 2g_{\tau\rho}(\tau, \rho) d\tau \, d\rho + g_{\rho\rho}(\tau, \rho) d\rho^2 + r^2(\tau, \rho) d\Omega_2^2$$

These coordinates make the SO(3) isometry manifest, as it acts on $\mathcal{S}^2$ leaving τ and ρ unchanged: this is a *foliation* of the space by $\mathbb{S}^2$. Being $r(\tau,\rho)$ the size of the sphere, it is useful to change coordinates to τ and r:

$$ds^2 = g_{\tau\tau}(\tau,r)d\tau^2 + 2g_{\tau r}(\tau,r)d\tau\, dr + g_{rr}(\tau,r)dr^2 + r^2 d\Omega_2^2$$

Note a subtlety: for some functions $r(\tau,\rho)$ it's not possible to exchange r and ρ (ex.: $r=\tau$), but these counterexamples are ruled out by the assumption that the spacetime is asymptotically the Minkowski spacetime. Now, to get rid of the cross-term, consider $\tilde{t}(\tau,r)$:

$$d\tilde{t}^2 = \left(\frac{\partial\tilde{t}}{\partial\tau}\right)^2 d\tau^2 + 2\frac{\partial\tilde{t}}{\partial\tau}\frac{\partial\tilde{t}}{\partial r}d\tau\, dr + \left(\frac{\partial\tilde{t}}{\partial r}\right)^2 dr^2$$

It's always ppossible to pick $\tilde{t}(\tau,r)$ such that the cross-term vanishes, so that:

$$ds^2 = -f(\tilde{t},r)d\tilde{t}^2 + g(\tilde{t},r)dr^2 + r^2 d\Omega_2^2$$

Solving the Einstein field equations requires that $f(\tilde{t},r) = h(\tilde{t})f(r)$ and $g(\tilde{t},r) = g(r)$, thus redefining $h(\tilde{t})d\tilde{t}^2 = dt^2$ yields:

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_2^2$$

Note that, even though time-independence was not assumed, it was derived from the field equations: this metric has the timelike Killing vector $K = \partial_t$, besides those from the SO(3) isometry. At this point, the field equations require that $f(r) = g(r)^{-1}$, thus this metric reduces to that considered in Sec. 4.1: the Schwarzschild metric is the most general solution with $\Lambda = 0$. □

Therefore, the Schwarzschild metric does not only describe a black hole, but it describes the spacetime outside any non-rotating spherically-symmetric object, even in the presence of time-dependence (ex.: collapsing star).

Definition 6.1.1. A spacetime is said to be *stationary* if it admits a globally-timelike Killing vector field K. In addition, being t the coordinate along the integral curves of K, if it is invariant under $t \to -t$, then it is said to be *static*.

The definition of static spacetime rules out $dtdx$ cross-terms in the metric.

Proposition 6.1.1. *A spherically-symmetric metric must be static.*

Proof. By Birkhoff theorem. □

6.1.2 Horizon

The Schwarzschild metric diverges at $r=0$ and $r = 2GM \equiv R_s$. To distinguish between true singularities and coordinate singularities, the usual way is defining a coordinate-independent scalar quantity and studying it at divergence points.
Due to the vaccum field equations, the simplest scalars R and $R_{\mu\nu}R^{\mu\nu}$ both vanish: the nect simplest curvature scalar is the *Krentschmann scalar* $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, which for the Schwarzschild metric is:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{6.2}$$

At $r = 2GM$, the Krentschmann scalar is $\sim 1/(GM)^4$, suggesting that this is only a coordinate singularity: it turns out that's the case, and indeed the surface $r = R_s$ is known as the *event horizon* of the black hole. Note that, interestingly, heavier black holes have smaller curvature at the horizon. On the contrary, $r=0$ is a true singularity, simply known as the *singularity* of the (classical) black hole.

6.1.2.1 Near horizon limit

To study the spacetime in the vicinity of the horizon, set $r = 2GM + \eta$, with $\eta \ll 2GM$. Moreover, consider $\eta \in \mathbb{R}^+$, thus restricting to the spacetime just outside the horizon. The metric then becomes at first order in η:

$$ds^2 = -\frac{\eta}{2GM}dt^2 + \frac{2GM}{\eta}d\eta^2 + (2GM)^2 d\Omega_2^2$$

Spacetime is thus decomposed in a direct product of $\mathbb{S}^2_{2GM}$ and a 2d Lorentzian manifold. Focusing on the latter, set $\rho^2 = 8GM\eta$, so that:

$$ds^2 = -\left(\frac{\rho}{4GM}\right)^2 dt^2 + d\rho^2 \tag{6.3}$$

This is called *Rindler metric* and it is, in fact, just Minkowski spacetime:

$$T = \rho \sinh\left(\frac{t}{4GM}\right) \qquad X = \rho\cosh\left(\frac{t}{4GM}\right) \qquad \Rightarrow \qquad ds^2 = -dT^2 + dX^2$$

These are precisely the coordinates Eqq. 1.33-1.34 experienced by an observer undergoing constant acceleration $a = 1/(4GM)$: this makes sense, as an observer sitting at constant ρ, i.e. constant r, must accelerate in order not to fall into the black hole.
The outside of the black hole is $(\rho, t) \in \mathbb{R}^+ \times \mathbb{R}$, corresponding to $X > |T|$. The horizon $\rho = 0$, instead, is mapped to the origin $T = X = 0$ of Minkowski space. However, note that the time coordinate is undefined at the origin since $g_{00} = 0$: scaling $t \to \infty$ and $\rho \to 0$ keeping $\rho e^{\pm t/4GM}$ fixed makes it clear that the horizon actually corresponds to the lines:

$$r = 2GM \quad \Rightarrow \quad X = \pm T$$

Therefore, the event horizon is not a timelike surface (like the surface of a star), but a null surface. Although the outside of the event horizon is only described by $X > |T|$, it is nonetheless possible to extend the metric to the whole Minkowski space $X, T \in \mathbb{R}$: this makes it clear that the event horizon is not a real singularity, as nothing particular happens at $X = \pm T$ (see Fig. 6.1). Zooming out, however, the peculiar properties of the horizon start to emerge from a global perspective.

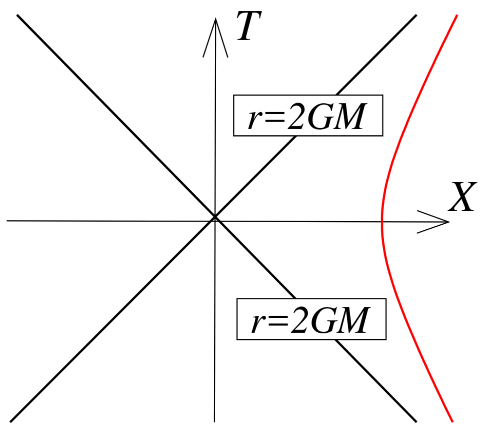


Figure 6.1: Rindler spacetime, with null lines in black and a line at constant $r > 2GM$ in red.

6.1.3 Eddington-Finkelstein coordinates

Both the Schwarzschild coordinates and the Rindler coordinates do not cover the whole manifold. To find a set of coordinates that does, first introduce a new radial coordinate:

$$dr_*^2 = \left(1 - \frac{2GM}{r}\right)^{-2} dr^2 \quad \Rightarrow \quad r_* = r + 2GM \log\left(\frac{r - 2GM}{2GM}\right) \tag{6.4}$$

This is the *Regge-Wheeler radial coordinate* and it maps the region outside the horizon $r \in (2GM, \infty)$ to $r_* \in \mathbb{R}$: as an observer approaches the horizon, r changes increasingly slow varying r_*, since $\frac{dr}{dr_*} \to 0$ as $r \to 2GM$. This coordinate is suited to describe the path of lightrays travelling in the radial direction:

$$ds^2 = 0 \quad \Rightarrow \quad \frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right) \quad \Rightarrow \quad \frac{dr_*}{dt} = \pm 1 \quad \Rightarrow \quad t \pm r_* = \text{const.}$$

These null radial geodesics correspond to an ingoing lightray for the positive sign and an outgoing lightray for the negative sign (r_* must decrease/increase as t increases). Next, introduce a pair of null coordiantes:

$$v = t + r_* \qquad u = t - r_* \tag{6.5}$$

These allow to extend the Schwarzschild solution beyond the horizon.

6.1.3.1 Ingoing coordinates

Replace t with $t = v - r_*(r)$:

$$dt = dv - \left(1 - \frac{2GM}{r}\right) dr$$

The Schwarzschild metric then becomes, in coordinates (v, r):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dv\, dr + r^2 d\Omega_2^2 \tag{6.6}$$

This is the Schwarzschild black hole in *ingoing Eddington-Finkelstein coordinates.* Note that there's no more singularity at $r = 2GM$; however, the dv^2 term vanishes at the horizon and becomes negative for $r < 2GM$. Nonetheless, the metric is still non-degenerate thanks to the cross-term:

$$\det g = \det \begin{bmatrix} -(1 - \frac{2GM}{r}) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\vartheta \end{bmatrix} = -r^4 \sin^2\vartheta$$

Thus, the metric is non-degenerate and with Lorentzian signature for all values of $r \in \mathbb{R}_+$ (except for the known poles of $\mathbb{S}^2$): this means that the radial coordinate can be extended past the horizon. The globally-timelike Killing vector field $K = \partial_t$ of the Schwarzschild metric is retained in the ingoing Eddington-Finkelstein coordinates as $K = \partial_v$, however now it is no longer globally-timelike: it remains so outside the horizon, where $g_{vv} < 0$, but become spacelike inside it, where $g_{vv} > 0$. This means that the full black hole geometry is not time-independent.

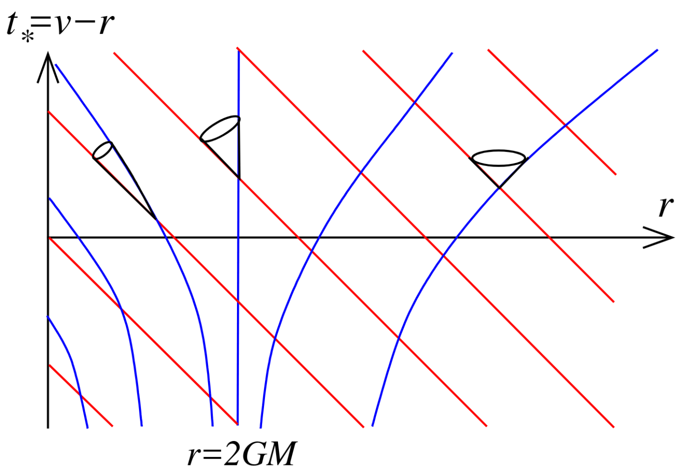


Figure 6.2: Finkelstein diagram in ingoing coordinates: ingoing null geodesics are shown in red, outgoing ones in blue.

Recall that ingoing lightrays follow geodesics given by $v = \text{const.}$, while outgoing lightrays follow $u = t - r_* = \text{const.}$. As a function of (v, r_*), the latter reads $v = 2r_* + \text{const.}$, so in (v, r) coordinates:

$$v = 2r + 4GM \log\left(\frac{r - 2GM}{2GM}\right) + \text{const.}$$

This clearly has domain $r \in (2GM, \infty)$, therefore the Regge-Wheeler radial coordinate needs a redefinition:

$$r_* = r + 2GM \log\left|\frac{r - 2GM}{2GM}\right|$$

This means that now r_* is a multi-valued function: $r \in (2GM, \infty)$ is mapped to $r_* \in \mathbb{R}$, while $r \in (0, 2GM)$ is mapped to $r_* \in \mathbb{R}_-$, with the singularity at $r_* = 0$. Outgoing lightrays inside the horizon will then follow geodesics given by:

$$v = 2r + 4GM \log\left(\frac{2GM - r}{2GM}\right) + \text{const.}$$

Outgoing null geodesics at the horizon are easily found: the dv^2 term in the metric Eq. 6.6 vanishes and the surface $r = 2GM$ is itself a null geodesic (as event horizons are always null surfaces).
All of this information about lightrays can be pictured in a *Finkelstein diagram*, which shows null geodesics in a radial-temporal diagram. Since r_* is multi-valued, r is chosen as radial coordinate: as a consequence, one needs to define a new temporal coordinate $t_* : v = t + r_* = r_* + r$, i.e. $t_* = v - r$. The Finkelstein diagram in ingoing coordinates is shown in Fig. 6.2: ingoing null geodesics are $v = \text{const.}$, i.e. $t_* = \text{const.} - r$, hence travelling at 45°, while outgoing ones are given by the previously found expressions. Note that only outgoing null geodesics outside of the horizon effectively "go out", since $r \to \infty$ as $t \to \infty$ (so $t_* \to \infty$): those inside the horizon move inexorably towards the curvature singularity at $r = 0$ as time passes, and each one of them reaches it in a finite time t_*. The boundary between these two regions are the null geodesics which run along the horizon at $r = 2GM$.
This analysis can be extended to massice particles: they must move along timelike geodesics inside their future lightcone, which is determined by a pair of ingoing and outgoing future-pointing null

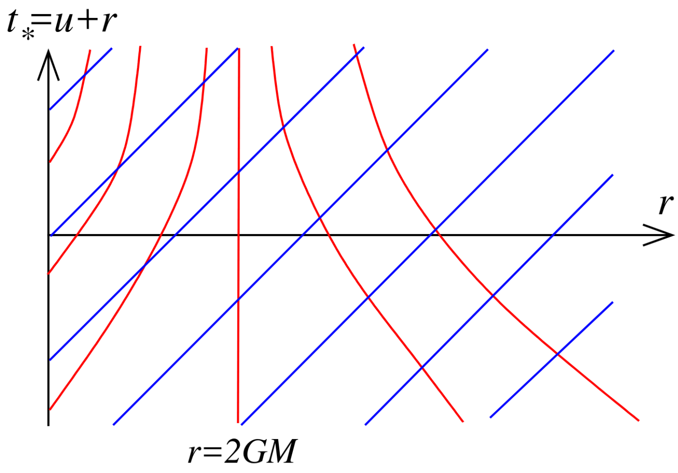


Figure 6.3: Finkelstein diagram in outgoing coordinates: ingoing null geodesics are shown in red, outgoing ones in blue.

geodesics. These future lightcones are also shown in Fig. 6.2: the name "black hole" captures the causal structure of the spacetime, as no massive particles nor lightrays can escape outside of the event horizon, thus an outside observer can know nothing of the inside of the black hole.
Consider two observers, one at constant $r > 2GM$ and the other moving towards the singularity: as the second approaches the event horizon, light signals emitted by it will take longer and longer to reach the first, and will appear progressively more redshifted due to the increasingly strong gravitational well. Thus, the outside observer will see the ingoing observer forever, but progressively more slowed down and redshifted, and will never see it passing the event horizon.

6.1.3.2 Outgoing coordinates

A different extension of the exterior of the Schwarzschild black hole is obtained by replacing t with $u = t - r_*$, i.e. $t = u + r_*(r)$:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - 2du\, dr + r^2 d\Omega_2^2 \tag{6.7}$$

This is the Schwarzschild black hole in *outgoing Eddington-Finkelstein coordinates.* Once again, the metric is smooth and non-degenerate at the horizon, thus it's possible to extend $r \in \mathbb{R}_+$. However, now $r \in (0, 2GM)$ describe a different part of spacetime from the analogous region in ingoing coordinates. To see this, set coordinates $(r, t_*) : t_* = u + r$ and draw the Finkelstein diagram (Fig. 6.3): outgoing null geodesics hence travel at 45°, effectively "going out" regardless of their position with respect to the horizon. On the contrary, ingoing null geodesics now don't "go in": those starting outside of the horizon pile up at the horizon itself, unable to cross it, while those starting inside the horizon move outwards and pile up at it, still unable to cross it.
Studying lightcones, it's easy to see that massive particles inside the horizon are inexorably expelled outside of it in a finite amount of time t_*. This is clearly a very different physics from a black hole: the metric Eq. 6.7 is that of a *white hole*, an object which expells any matter inside. A white hole can be viewed as the time reversal of a black hole: this is to be traced to the negative sign in the cross-term and can be seen by flipping Fig. 6.3 upside-down, getting Fig. 6.2.
White holes are perfectly acceptable solutions to the Einstein field equations, and their existence is to

be expected from the time-reversal invariance of the field equations. However, they're not physically relevant, as they cannot be formed from collapsing matter.

6.1.4 Kruskal spacetime

To better understand how the region parametrized by $r \in (0, 2GM]$ corresponds to two different parts of spacetime, one needs to introduce coordinates which cover the entire spacetime, including both black and white holes.
First, rewrite the Schwarzschild metric in (u, v) coordinates:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv + r^2 d\Omega_2^2$$

where $r = r(u - v)$. There's a degeneracy at $r = 2GM$ which can be solved introducing the *Kruskal-Szekeres coordinates*:

$$U = -\exp\left(-\frac{u}{4GM}\right) \qquad V = \exp\left(\frac{v}{4GM}\right) \tag{6.8}$$

Both U and V are null coordinates, and the exterior of the Schwarzschild balck hole is parametrized by $U < 0, V > 0$. The metric becomes:

$$ds^2 = -\frac{32(GM)^3}{r} e^{-\frac{r}{2GM}} dU\, dV + r^2 d\Omega_2^2 \tag{6.9}$$

This is known as the metric of *Kruskal spacetime*. The function $r = r(U, V)$ can be found inverting:

$$UV = -\exp\left(\frac{r_*}{2GM}\right) = \frac{2GM - r}{2GM} \exp\left(\frac{r}{2GM}\right) \tag{6.10}$$

On the other hand, the function $t = t(U, V)$ can be found inverting:

$$\frac{U}{V} = -\exp\left(-\frac{t}{2GM}\right) \tag{6.11}$$

The original Schwarzschild metric, which covers $U < 0, V > 0$, can now be extended to $(U, V) \in \mathbb{R}^2$, without any divergence at $r = 2GM$.

Proposition 6.1.2. *Kruskal spacetime is the maximal extension of the Schwarzschild metric.*

Proof. To check whether a given spacetime can be further extended, one needs to look at all possible geodesics: if they are defined for infinite affine parameter, then tey can escape to infinity; however, if they alt at some finite affine parameter, that is either a coordinate singularity, hinting that the spacetime can in fact be extended, or true singularity. A spacetime is maximally extended if any geodesics that alt at a finite affine parameter do so at a true singularity, and that's the case for Kruskal spacetime. □

The extension process allows to solve the field equations in some particular region (open subset) of spacetime: being the metric components real analytic functions, this is sufficient to determine the metric in all of spacetime.

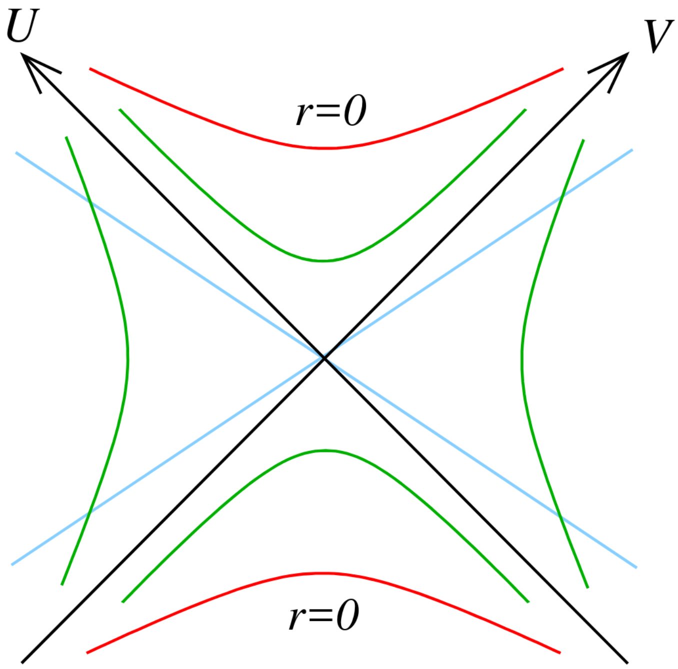


Figure 6.4: Kruskal diagram for the Schwarzschild balck hole: green lines are $r =$ const. and blue lines $t =$ const., while singularities are in red and horizons in black.

6.1.4.1 Kruskal diagram

To see where the event horizon is mapped in Kruskal spacetime:

$$r = 2GM \qquad \Rightarrow \qquad U = 0 \quad \vee \quad V = 0$$

This means that the event horizon no longer is a null surface, but two null surfaces intersecting at $U = V = 0$, in agreement with the near horizon limit (Rindler space): the null surface $U = 0$ is the horizon of the black hole, known as the *future horizon*; the null surface $V = 0$ is the horizon of the white hole, known as the *past horizon*. On the other hand, the singularity becomes:

$$r = 0 \qquad \Rightarrow \qquad UV = 1$$

This hyperbola has two disconnected components: one $(U, V > 0)$ corresponding to the singularity of the black hole and one $(U, V < 0)$ to the singularity of the white hole.
All of this information can be represented in a *Kruskal diagram*, as shown in Fig. 6.4: the U and V axes are at 45°, reflecting that they are null lines, and these are the two horizons. In this diagram, the vertical direction can be viewed as time $T = \frac{1}{2}(V + U)$ and the horizontal one as a spatial coordinate $X = \frac{1}{2}(V - U)$. This diagram makes it clear that the black hole and the white hole cohabit the same spacetime. Lines of constant r and t correspond respectively to lines of constant UV and U/V, as from Eqq. 6.10-6.11.
The Kruskal diagram makes it clear that, once an observer crosses the event horizon, the singularity is unavoidable: the $r = 0$ are not timelike worldlines, but *the singularity is spacelike*. Therefore, for the in-falling observer, after crossing the horizon the singularity of the black hole does not lie in a spatial direction, but it lies in the future. Similarly, the singularity of the white hole lies in the past. This can be formally seen using the Killing vector field $K = \partial_t$ of the Schwarzschild solution. Outside of

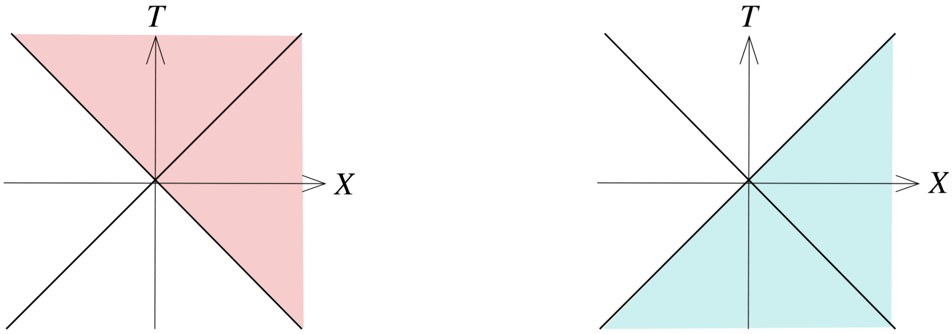


Figure 6.5: Regions of Kruskal spacetime covered by ingoing and outgoing Eddington-Finkelstein coordinates respectively.

the horizon K is timelike, therefore it defines the conserved energy of geodesics; in Kruskal-Szekeres coordinates:

$$K = \partial_t = \frac{\partial V}{\partial t}\partial_V + \frac{\partial U}{\partial t}\partial_U = \frac{1}{4GM}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right) \quad \Rightarrow \quad g_{\mu\nu}K^\mu K^\nu = -\left(1 - \frac{2GM}{r}\right)$$

Ouside of the horizon $K_\mu K^\mu < 0$ as expected. But inside, with $r < 2GM$, the Killing vector field is spacelike: therefore, by Def. 6.1.1, the full Schwarzschild spacetime is not time-independent, but it becomes clear only after crossing the event horizon.

Einstein-Rosen bridge With reference to the Kruskal diagram in Fig. 6.4, the right-hand quadrant is the exterior of the black hole, covered by the Schwarzschild coordinates, while the upper and lower quadrants are respectively the interior of the black hole and of the white hole, covered by ingoing and outgoing Eddington-Finkelstein coordinates (see Fig. 6.5). There remains the left-hand quadrant: this turns out to be just another copy of the exterior of the black hole, now covered by $U > 0, V < 0$.
The full spacetime hence contains two asymptotically flat regions, joined together by a black hole: note that it's not possible for an observer in one region to send a signal in the other, because the causal structure of the spacetime doesn't allow this.
To elucidate the spatial geometry connecting these two regions, one need to study the $t = 0$ hypersurface of Kruskal spacetime: this is a straight horizontal line passing through $U = V = 0$ in Fig. 6.4. In Schwarzschild coordinates, the geometry of the $t = 0$ hypersurface is given by:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \tag{6.12}$$

which is valid for $r > 2GM$. Kruskal spacetime presents two copies of this geometry, one in the right-hand quadrant and one in the left-hand one, which are glues together at $r = 2GM$ to give a wormhole-like geometry as in Fig. 6.6: this is called the *Einstein-Rosen bridge*. Note that it's not possible to travel through it, as the paths are spacelike, not timelike.
It's possible to write a metric which includes both sides. Define the radial coordinate ρ such that:

$$r = \rho\left(1 + \frac{GM}{2\rho}\right)^2 = \rho + GM + \frac{G^2M^2}{4\rho}$$

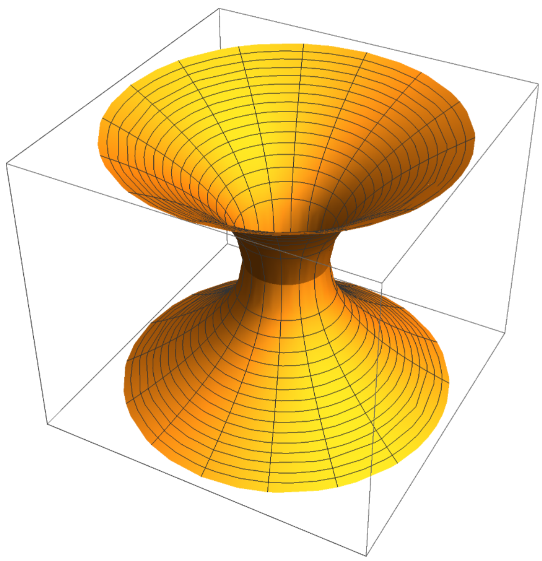
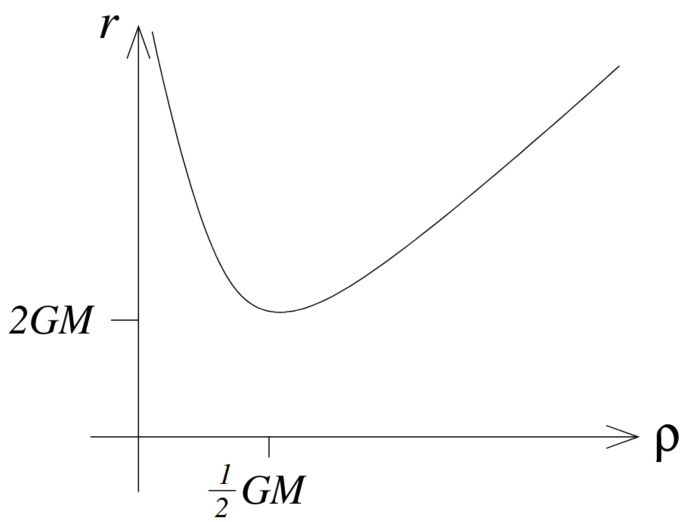



Figure 6.6: The Einstein-Rosen bridge and the ρ coordiante which parametrizes it.

Plotting $r = r(\rho)$, as in Fig. 6.6, it's clear that there are two values of ρ for each value of $r > 2GM$, while $r = 2GM$ only corresponds to $\rho = GM/2$: thus, is possible to parametrize one side of the wormhole with $\rho \in (0, GM/2)$ and the other with $\rho \in (GM/2, \infty)$. The metric in Eq. 6.12 becomes:

$$ds^2 = \left(1 + \frac{GM}{2\rho}\right)^4 \left[d\rho^2 + \rho^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right] \tag{6.13}$$

It is clear that, as $\rho \to \infty$, the (spatial) metric is that of $\mathbb{R}^3$. But the same is true for $\rho \to 0$: note that there's a symmetry $r \mapsto r$ for $\rho \mapsto G^2M^2/(4\rho)$, which swaps the two asymptotic spacetimes while leaving the midpoint at $\rho = GM/2$ invariant, i.e. it swaps the two sides of the wormhole.
Furthermore, the radius of the spatial $\mathcal{S}^2$ is minimum at $2GM$ in the midpoint $\rho = GM/2$ of the wormhole and increases in either direction: this middle point corresponds to the common point $U = V = 0$ of the two horizons and is called the *bifurcation sphere.*
Although there's no way that an observer in one quadrant can signal an observer in the other quadrant, there's one way for them to communicate: jump into the black hole and meet behind the respective horizons. However, as the white hole is thought to have no physical manifestation, the other universe in the left-hand quadrant of Kruskal space is thought to be a mathematical artifact too.

6.1.4.2 Penrose diagram

From the Kruskal diagram in Fig. 6.4, it's easy to draw the Penrose diagram for Kruskal spacetime. First, introduce new coordinates which cover the entire spacetime in a finite range:

$$U = \tan\tilde{U} \qquad V = \tan\tilde{V}$$

The new coordinates have range $(\tilde{U}, \tilde{V}) \in (-\frac{\pi}{2}, +\frac{\pi}{2})$ and the Kruskal metric becomes:

$$ds^2 = \frac{1}{\cos^2\tilde{U}\,\cos^2\tilde{V}}\left[-\frac{32(GM)^3}{r} e^{-\frac{r}{2GM}} d\tilde{U}\, d\tilde{V} + r^2\cos^2\tilde{U}\,\cos^2\tilde{V}\, d\Omega_2^2\right]$$

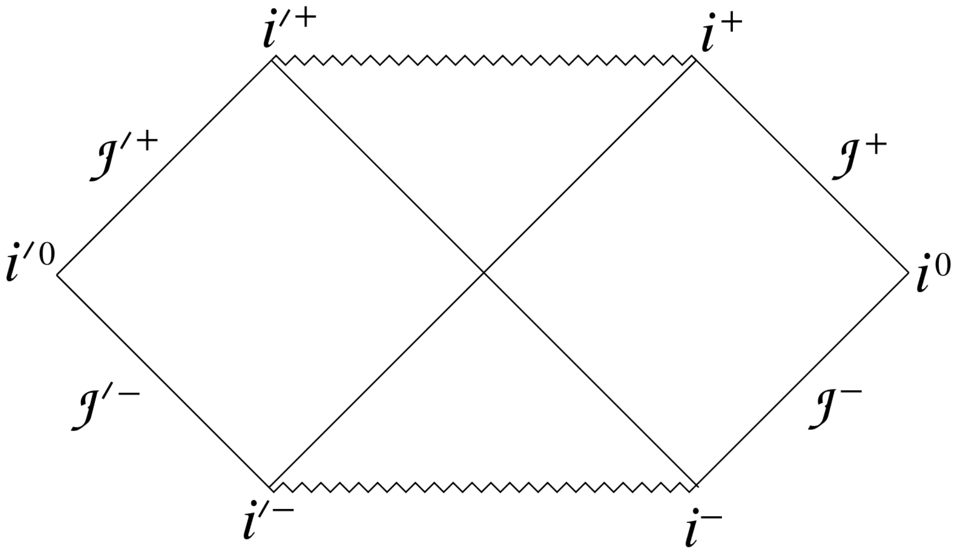


Figure 6.7: The Penrose diagram for Kruskal spacetime.

This metric is conformally equivalent to:

$$ds^2 = -\frac{32(GM)^3}{r} e^{-\frac{r}{2GM}} d\tilde{U}\, d\tilde{V} + r^2 \cos^2 \tilde{U}\, \cos^2 \tilde{V}\, d\Omega_2^2$$

However, the singularity at $r = 0$ must be retained: this is mapped to $UV = 1$, thus in the finite-range coordinates:

$$\tan \tilde{U}\, \tan \tilde{V} = 1 \quad \Leftrightarrow \quad \sin \tilde{U}\, \sin \tilde{V} - \cos \tilde{U}\, \cos \tilde{V} = 0 \quad \Leftrightarrow \quad \cos(\tilde{U} + \tilde{V}) = 0 \quad \Leftrightarrow \quad \tilde{U} + \tilde{V} = \pm\frac{\pi}{2}$$

These are straight horizontal lines in the Penrose diagram which chop the diamond-shaped diagram (as that in Fig. 4.5): the resulting diagram is shown in Fig. 6.7, with singularities represented by jagged lines.

The Penrose diagram contains the same information as the Kruskal diagram in Fig. 6.4, but it allows to state a more rigorous definition of black hole. Restricting to aymptotically flat spacetimes, the two asymptotic regions contain both null infinities $\mathcal{I}^+$ and $\mathcal{I}^-$: the black hole region is then defined as the set of points which cannot send signals to $\mathcal{I}^+$. The boundary of the black hole region is the *future event horizon* $\mathcal{H}^+$: equivalently, $\mathcal{H}^+$ is the boundary of the causal past of $\mathcal{I}^+$. With reference to FIg. 6.7, the black hole associated to $\mathcal{I}^+$ is the upper quadrant and the left quadrant, while the black hole associated to $\mathcal{I}^-$ is the upper quadrant and the right quadrant.

Importantly, note that to define a black hole one needs to know the whole spacetime, as all lightrays must be run backwards from $\mathcal{I}^+$ to determine $\mathcal{H}^+$ as their boundary: there's no reference to any spacelike hypersurface at constant coordinate time, thus an observer can't really know if it is inside a black hole unless by knowing the entire future evolution of the spacetime.

Equivalently, it's possible to define the white hole region and the *past event horizon* as the region that cannot receive signals from $\mathcal{I}^-$ and its boundary.

6.1.5 Weak cosmic censorship

Kruskal spacetime is unphysical in a number of ways. In fact, in reality, black holes do not emerge from white holes, but are formed by collapsing stars: the resulting Penrose diagram is rather different

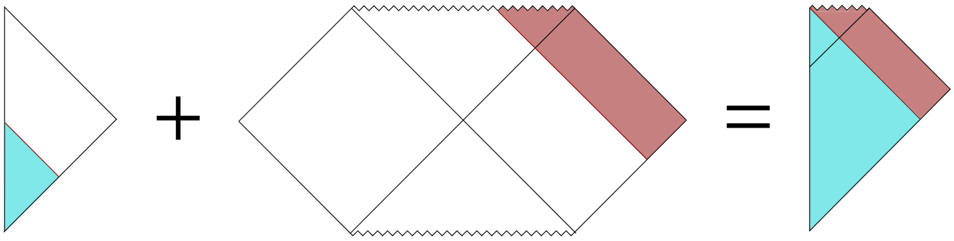


Figure 6.8: Joining two Penrose diagrams.

from that in Fig. 6.7. To understand the causal structure of such a black hole, consider the (unrealistic) situation of the spherically-symmetric collapse of a shell of matter: inside the shell, spacetime is flat, meanwhile on the outside it is described by Schwarzschild metric Eq. 6.1 (by Birkhoff theorem, time dependence doesn't change this fact). Furthermore, assume (unrealistically) that the shell is travelling at the speed of light, so that the Penrose diagrams for Minkowski spacetime and Kruskal spacetime can be glued as in Fig. 6.8: this is the Penrose diagram of a collapsing black hole.
Despite the number of unphysical assumptions, the Penrose diagram thus derived correctly describes the causal structure of a realistic spherically-symmetric collapsing star: as in Fig. 6.9, its surface follows a timelike trajectory.

6.1.5.1 Naked singularities

An important feature of the black hole is that the singularity remains shrouded behind the event horizon, so that an asymptotic observer cannot see it.
On the contrary, Einstein field equations also allow *naked singularities*, i.e. singularities without an horizon. For example, consider Schwarzschild metric with $M < 0$:

$$ds^2 = -\left(1 + \frac{2G\,|M|}{r}\right) dt^2 + \left(1 + \frac{2G\,|M|}{r}\right)^{-1} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

This metric presents no coordinate singularity at $r = 2G\,|M|$, and so no event horizon. The Penrose diagram for such a spacetime is found analogously to that of Minkwoski spacetime (Sec. 4.4.2.1), setting null coordinates $u = t - r$ and $v = t + r$. The resulting diagram, shown in Fig. 6.10, is

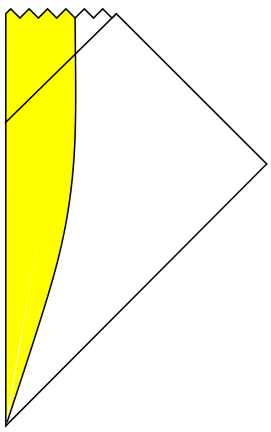


Figure 6.9: Penrose diagram for a collapsing star forming a black hole.

Figure 6.10: Penrose diagram for a negative mass black hole and unphysical collapse of a star.

identical to that of Minkowski spacetime (Fig. 4.7), with the difference that now $r = 0$ presents a curvature singularity not shielded by an horizon: it is therefore observed from $\mathcal{I}^+$.
An important conjecture in General Relativity, known as the *weak cosmic censorship conjecture*, states the following: given matter which obeys the dominant energy condition Eq. 4.55, generic smooth initial conditions on a spatial hypersurface for both the metric and matter fields in an asymptotically flat spacetime will not evolve to form naked singularities.
If this conjecture is true, then a dynamical evolution such as that in Fig. 6.10 must be ruled out: in this diagram, once the singularity forms, fields can no longer be evolved beyond the shown lightray; strictly speaking, this means that the dynamical evolution stops at the red line and cannot be extended further, thus abruptly ending $\mathcal{I}^+$.
There's no proof of the weak cosmic censorship conjecture, but only circumstantial evidence. However, there's one naked singularity that seems to be physical: the Big Bang singularity. In fact, since it is in the far past, it doesn't violate cosmic censorship.

6.1.6 de Sitter black holes

Schwarzschild metric Eq. 6.1 solves the Einstein field equations with $\Lambda = 0$, i.e. in asymptotically Minkowski spacetime. To generalize, consider $\Lambda \neq 0$, so that the field equations become $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and use the following (already seen) ansatz:

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right)$$

The field equations imply that:

$$f'' + \frac{2f'}{r} + \frac{f'^2}{f} = -\frac{\Lambda}{f} \qquad \wedge \qquad 1 - 2ff'r - f^2 = \Lambda r^2$$

The most general solution is:

$$f(r)^2 = 1 - \frac{2GM}{r} \mp \frac{r^2}{R^2} \qquad R^2 = \frac{3}{|\Lambda|} \tag{6.14}$$

where the negative sign is the solution with $\Lambda > 0$ and the positive sign with $\Lambda < 0$, hence describing black holes in de Sitter and anti-de Sidder spacetime respectively: in fact, if $2GM \ll R^2$, then for $r \ll R$ the metric is that of a Schwarzschild black hole in Minkowski spacetime.