NLO QCD with Massive Quarks

An extension of the NSC subtraction scheme

Bachelor Degree in Physics

Leonardo Cerasi (11410A)

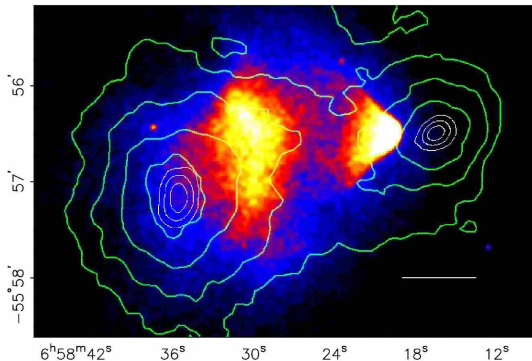UNIVERSITÀ DEGLI STUDI DI MILANO

Main BSM evidence

- dark matter and dark energy
- matter-antimatter asymmetry
- neutrino masses

Figure from Clowe et al. 2006.
Offset between the observed baryonic mass distribution and the gravitational potential in the Bullet Cluster (1E 0657-56).
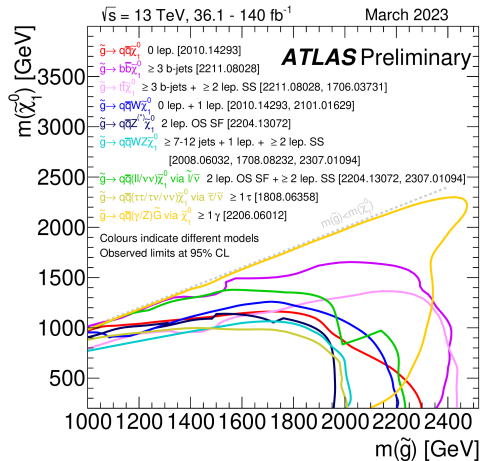
Precision estimates at the LHC
BSM constraints and shift in research paradigm

Main BSM proposals
- supersymmetric models (MSSM, ...)
- dark matter models (WIMPs, axions, ...)
- extended gauge sectors ($SO(10)$, ...)
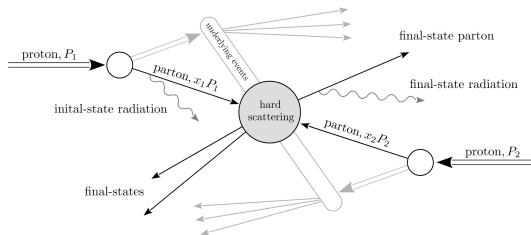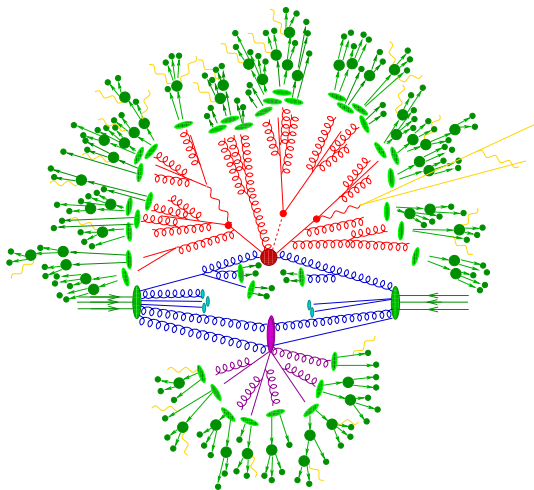- SM Effective Field Theory (SMEFT)

Figure from ATLAS PUB Note 2023-025.
Exclusion limits in the $\tilde{g} - \tilde{\chi}_1^0$ mass plane for various models for the decay of the gluino to the lightest supersymmetric particle.

Precision estimates at the LHC
Factorization theorem and perturbative QCD

Figures from Höche 2015 (left) and Asteriadis 2021 (right).
Hadronization of jets produced in hadronic scattering and detail of the underlying hard partonic scattering.

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Hadronic and partonic physics decouple in hard scattering processes, and we can formulate a factorization theorem:

$$
\mathrm{d}\sigma_{h_1,h_2}(P_1, P_2) = \sum_{a,b} \int_{[0,1]^2} \frac{\mathrm{d}\xi_1}{\xi_1} \frac{\mathrm{d}\xi_2}{\xi_2} f_a^{(h_1)}(\xi_1, \mu_{\mathsf{F}}^2) f_b^{(h_2)}(\xi_2, \mu_{\mathsf{F}}^2) \times
$$

$$
\times \, \mathrm{d}\hat{\sigma}_{a,b}(\xi_1 P_1, \xi_2 P_2, \alpha_{\mathsf{s}}, \mu_{\mathsf{R}}^2, \mu_{\mathsf{F}}^2) \left[1 + o\left(\frac{\Lambda_{\mathsf{QCD}}^n}{Q^n} \right) \right]
$$

Asymptotic freedom allows for a perturbative analysis of the hard partonic scattering:

$$
\mathrm{d}\hat{\sigma}_{a,b}(p_1, p_2) = \sum_{n \in \mathbb{N}_0} \mathrm{d}\hat{\sigma}_{a,b}^{(n)}(p_1, p_2)
$$

$n \geq 1$ are denoted by $\mathsf{N}^n\mathsf{LO}$ QCD corrections.
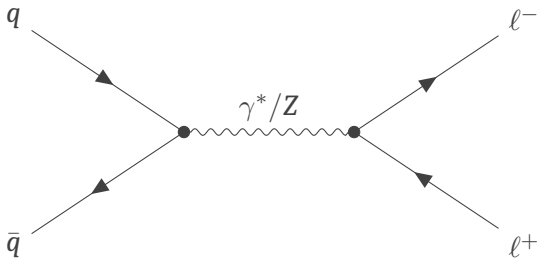
Our focus is on NLO QCD corrections. Consider e.g. the Drell-Yan process:

LO process
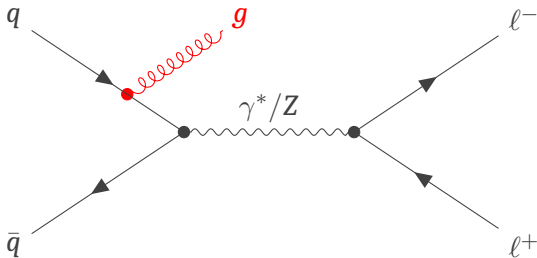
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Virtual correction

Main difficulty: infrared singularities in particular kinematic regimes.

Example in real corrections:

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Soft singularity: $E_k \to 0$

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Collinear singularity: $\theta \to 0$

The key idea to regularize IR divergences is dimensional regularization:

$$d = 4 - 2\epsilon \quad , \quad \epsilon \in \mathbb{C} : \Re\epsilon < 0$$

Then, soft and collinear singularities are expressed as poles in ϵ:

$$\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k} \left|\mathcal{A}(p_k)\right|^2 \sim \int_0^{\mathcal{E}} \frac{\mathrm{d}E_k}{E_k^{5-d}} \int_0^{\pi} \mathrm{d}\theta \frac{\sin^{d-3}\theta}{1-\cos\theta} \left|\mathcal{A}_0\right|^2$$

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soft singularity $= -\dfrac{\mathcal{E}^{-2\epsilon}}{2\epsilon}$

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$$\begin{array}{c} \text{soft singularity} \\ \text{collinear singularity} \end{array} = -\frac{\mathcal{E}^{-2\epsilon}}{2\epsilon} \quad = -\frac{2^{-2+\epsilon}}{\epsilon}$$

ϵ-poles can be extracted from partonic cross-sections via subtraction methods.
General idea using a regular function $f(x)$:

$$I = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} f(x) = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} \left[f(x) - f(0)\right] + f(0) \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}$$

• $\dfrac{f(x) - f(0)}{x^{1+\epsilon}}$ regular at $x = 0$, so it can be numerically integrated with $\epsilon \to 0$;

• $f(0) \displaystyle\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} = -\dfrac{f(0)}{\epsilon}$ contains the explicit ϵ-pole.

Our aim is finding the most general subtraction terms $f(0)$ for partonic scattering.

The Nested Soft-Collinear (NSC) Subtraction Scheme (SS) at NLO is modular and local:

- singular limits extracted by operators: S_i soft limit $E_i \to 0$, C_{ij} collinear limit $\theta_{ij} \to 0$;

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NSC subtraction scheme
Sequential extraction of singularities

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- IR-safe part of cross-section extracted by:

$$O_{\mathrm{NLO}}^{m} := \sum_{i \in \mathcal{H}_{m,0}^{n}} \overline{S}_m \overline{C}_{im} \omega^{mi}$$

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- counterterms contain the IR divergences.

The total ϵ-finite NLO cross-section can be written as:

$$\mathrm{d}\hat{\sigma}_{a,b}^{(1)} = \mathrm{d}\hat{\sigma}_{a,b}^{\mathsf{NLO,reg}} + \frac{1}{2\hat{s}} \sum_{n \in \mathcal{G}_m} \left[\frac{\alpha_{\mathsf{s}}(\mu_{\mathsf{R}}^2)}{2\pi} \left\langle I_{\mathsf{T}}^{(0)} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \right\rangle + \left\langle \mathcal{F}_{a,b}^{\mathsf{fin}}[\mathcal{X}_m^n] \right\rangle \right]$$

$$+ \frac{\alpha_{\mathsf{s}}(\mu_{\mathsf{R}}^2)}{2\pi} \sum_c \left[P_{f_c f_a}^{\mathsf{gen}} \otimes \mathrm{d}\hat{\sigma}_{c,b}^{(0)} + P_{f_c f_b}^{\mathsf{gen}} \otimes \mathrm{d}\hat{\sigma}_{a,c}^{(0)} \right]$$

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$$+ \frac{\alpha_{\mathsf{s}}(\mu_{\mathsf{R}}^2)}{2\pi} \sum_c \left[P^{\mathsf{gen}}_{f_c f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\mathsf{gen}}_{f_c f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]$$

- NLO final reminder, defined using $\mathsf{O}^{\mathsf{m}}_{\mathsf{NLO}}$;

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- NLO final reminder, defined using $\mathsf{O}_{\mathsf{NLO}}^{\mathsf{m}}$;
- finite counterterm from virtual corrections;
- finite counterterm from PDF renormalization;

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$$+ \frac{\alpha_{\mathsf{s}}(\mu_{\mathsf{R}}^2)}{2\pi} \sum_c \left[P_{f_c f_a}^{\mathsf{gen}} \otimes \mathrm{d}\hat{\sigma}_{c,b}^{(0)} + P_{f_c f_b}^{\mathsf{gen}} \otimes \mathrm{d}\hat{\sigma}_{a,c}^{(0)} \right]$$

- NLO final reminder, defined using $O_{\mathsf{NLO}}^{\mathsf{m}}$;
- finite counterterm from virtual corrections;
- finite counterterm from PDF renormalization;
- counterterm we are interested in.

Soft singularities

$$
\left\langle \, \mathsf{S_m}\Delta^{\mathfrak{m}} \left| \begin{array}{c} a \\ \\ b \end{array} \right. \right. \!\!\!\!\! \left. \vcenter{\hbox{\includegraphics{}}} \!\!\!\!\! \left. \begin{array}{c} \vdots \\ \mathfrak{m} \\ \vdots \end{array} \right|^2 \right\rangle = [\alpha_{\mathsf{s}}] \left\langle \, I_{\mathsf{S}}(\epsilon) \cdot \left| \begin{array}{c} a \\ \\ b \end{array} \right. \!\!\!\!\! \left. \begin{array}{c} \vdots \\ \vdots \end{array} \right|^2 \right\rangle
$$

$$
I_{\mathsf{S}}(\epsilon) := -\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \neq j \in \mathcal{X}_{m,0}^n} \eta_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j) \, {}_2F_1(1,1,1-\epsilon,1-\eta_{ij})
$$

Hard-collinear singularities

$$\sum_{\rho=1}^{2n_f+1} \sum_{i \in \mathcal{H}^n_{m,0}(\mathfrak{m}_\rho)} \left\langle \bar{S}_\mathfrak{m} C_{i\mathfrak{m}_\rho} \Delta^{\mathfrak{m}_\rho} \left| \begin{array}{c} a \\ \\ b \end{array} \begin{array}{c} \vdots \\ \mathfrak{m}_\rho \\ i \\ \vdots \end{array} \right| \right\rangle^2 \sim [\alpha_s] \left\langle I_C(\epsilon) \cdot \left| \begin{array}{c} a \\ \\ b \end{array} \begin{array}{c} \vdots \\ [i\mathfrak{m}_\rho] \\ \vdots \end{array} \right| \right\rangle^2$$

$$I_C(\epsilon) := \sum_{i \in \mathcal{H}^n_{m,0}} \frac{\Gamma_{i,f_i}}{\epsilon} = \frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left[\gamma_i + 2\mathbf{T}_i^2 \log \frac{\mathcal{E}}{E_i} \right] + o(\epsilon^0)$$

Virtual singularities

$$\left\langle 2\Re \left\langle \substack{a \\ \text{1-L} \\ b} \right| \left| \substack{a \\ \\ b} \right\rangle \right\rangle = [\alpha_s] \left\langle I_V(\epsilon) \cdot \left| \substack{a \\ \\ b} \right|^2 \right\rangle$$

$$I_V(\epsilon) \equiv \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^{\epsilon} \cos\left(\lambda_{ij}\pi\epsilon\right) \left[\frac{1}{\epsilon^2} + \frac{1}{\mathbf{T}_i^2} \frac{\gamma_i}{\epsilon} \right]$$

$$I_{\text{S+V}} = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[-\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} K_{ij} + \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^{\epsilon} \cos(\lambda_{ij}\pi\epsilon) \left(\frac{1}{\epsilon^2} + \frac{1}{\mathbf{T}_i^2} \frac{\gamma_i}{\epsilon} \right) \right]$$

$$\equiv \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij} \, {}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) = 1 - \epsilon \log \eta_{ij} + \boldsymbol{o}(\epsilon^2)$$

$$= \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \left[-1 + \epsilon \log \eta_{ij} + \left(\frac{\mathcal{E}^2}{E_i E_j} \right)^{\epsilon} \eta_{ij}^{-\epsilon} \cos(\lambda_{ij}\pi\epsilon) \left(1 + \epsilon \frac{\gamma_i}{\mathbf{T}_i^2} \right) \right]$$

$$= \frac{1}{\epsilon} \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(L_i + L_j + \frac{\gamma_i}{\mathbf{T}_i^2} \right) + \boldsymbol{o}(\epsilon^0) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) + \boldsymbol{o}(\epsilon^0)$$

Compare:

$$I_{\mathsf{S+V}}(\epsilon) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}^2_i L_i \right) + I^{(0)}_{\mathsf{S+V}} + \boldsymbol{o}(\epsilon)$$

$$I_{\mathsf{C}}(\epsilon) = +\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}^2_i L_i \right) + I^{(0)}_{\mathsf{C}} + \boldsymbol{o}(\epsilon)$$

Hence:
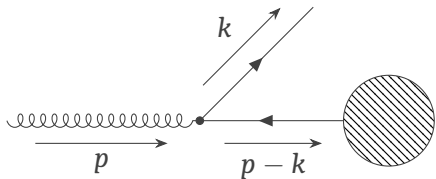
$$I_\mathsf{T}(\epsilon) \equiv I_\mathsf{S}(\epsilon) + I_\mathsf{C}(\epsilon) + I_\mathsf{V}(\epsilon) = I^{(0)}_\mathsf{T} + \boldsymbol{o}(\epsilon)$$

Massive partons do not determine soft or collinear singularities:

$$\sim \quad \frac{1}{(p-k)^2 - m^2} = -\frac{1}{2E_p \left(E_k - |\mathbf{k}| \cos\theta \right)}$$

Clearly, no more collinear singularities as $E_k \neq |\mathbf{k}|$ for $m \neq 0$.
The soft limit is non-singular even in the massless case.

- In presence of final-state massive partons, $I_S(\epsilon)$ becomes:

$$I_S(\epsilon) = \frac{1}{2\epsilon} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i,j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) I_{i,j}(\epsilon)$$

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- ϵ^{-2}-poles of $I_S(\epsilon)$ are determined only by massless partons.

- In presence of final-state massive partons, $I_S(\epsilon)$ becomes:

$$I_V(\epsilon) := \Re I_1(\epsilon) = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{|s_{ij}|} \right)^{\epsilon} \left[\mathcal{V}_{i,j}(\epsilon) - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) \right] - \sum_{i \in \mathcal{X}_{m,0}^n} \Gamma_i(\epsilon)$$

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- $\mathcal{V}_{i,j}(\epsilon)$ depends on the nature (massive or massless) of partons i and j;
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- ϵ^{-2}-poles of $I_V(\epsilon)$ are determined only by massless partons.

It is convenient to group summations as follows:

$$I_{\text{S+V}}(\epsilon) = \sum_{i,j \in \mathcal{X}_{m,0}^n} I_{\text{S}}^{i,j}(\epsilon) + \sum_{i \neq j \in \mathcal{X}_{m,0}^n} \tilde{I}_{\text{V}}^{i,j}(\epsilon) - \sum_{i \in \mathcal{X}_{m,0}^n} \Gamma_i(\epsilon)$$

$$= \sum_{i \neq j \in \mathcal{X}_{m,0}^n} \left[I_{\text{S}}^{i,j}(\epsilon) + \tilde{I}_{\text{V}}^{i,j}(\epsilon) \right] + \sum_{i \in \mathcal{X}_{m,m}^n} \left[I_{\text{S}}^{i,i}(\epsilon) - \Gamma_i(\epsilon) \right] - \sum_{i \in \mathcal{H}_{m,0}^n} \Gamma_i(\epsilon)$$

Then, ϵ^{-2}-singularities are only present in the first sum:

$$I_{\text{S}}^{i,j}(\epsilon) + \tilde{I}_{\text{V}}^{i,j}(\epsilon) = \chi_{i,j} + \mathbb{K}_{i,j} + o(\epsilon)$$

It is convenient to group summations as follows:

$$I_{S+V}(\epsilon) = \sum_{i,j \in \mathcal{X}^n_{m,0}} I_S^{i,j}(\epsilon) + \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \tilde{I}_V^{i,j}(\epsilon) - \sum_{i \in \mathcal{X}^n_{m,0}} \Gamma_i(\epsilon)$$

$$= \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \left[I_S^{i,j}(\epsilon) + \tilde{I}_V^{i,j}(\epsilon) \right] + \sum_{i \in \mathcal{X}^n_{m,m}} \left[I_S^{i,i}(\epsilon) - \Gamma_i(\epsilon) \right] - \sum_{i \in \mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)$$

Then, ϵ^{-2}-singularities are only present in the first sum:

$$I_S^{i,j}(\epsilon) + \tilde{I}_V^{i,j}(\epsilon) = \chi_{i,j} + \mathbb{K}_{i,j} + o(\epsilon)$$

Laurent expansion: pole terms and ϵ-finite finite reminder.

Quadratic pole terms have a complex general form:

$$\chi_{i,j} \equiv \frac{1}{\epsilon^2}\left(\frac{1}{2}I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)}\right) + \frac{1}{\epsilon}\left(\frac{1}{2}I_{i,j}^{(0)} + \mathcal{V}_{i,j}^{(-1)} - \mathscr{L}_{\mathsf{m}}I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)}\log\frac{\mu^2}{|s_{ij}|}\right)$$

However, with further manipulation, the explicit structure is simple:

$$\chi_{i,j} = \begin{cases} 0 & i,j \text{ massive} \\ \frac{1}{\epsilon}L_j & i \text{ massive}, j \text{ massless} \\ \frac{1}{\epsilon}L_i & i \text{ massless}, j \text{ massive} \\ \frac{1}{\epsilon}\left(L_i + L_j\right) & i,j \text{ massless} \end{cases} \qquad L_k \equiv \log\frac{\mathcal{E}}{E_k}$$

The sum over $\mathcal{X}_{m,m}^n$ in $I_{\mathsf{S+V}}(\epsilon)$ is manifestly ϵ-finite:

$$\sum_{i \in \mathcal{X}_{m,m}^n} \left[I_{\mathsf{S}}^{i,i}(\epsilon) - \Gamma_i(\epsilon) \right] = \sum_{i \in \mathcal{X}_{m,m}^n} \left[C_{\mathrm{F}} \left(\frac{1}{\epsilon} - \frac{1}{\kappa_i} \log \frac{1 - \kappa_i}{1 + \kappa_i} - 2 \mathscr{L}_{\mathsf{m}} \right) \right.$$

$$\left. - C_{\mathrm{F}} \left(\frac{1}{\epsilon} + \frac{1}{2} \log \frac{m_{Q_i}^2}{\mu^2} - 2 \right) \right] \equiv \sum_{i \in \mathcal{X}_{m,m}^n} \text{Ц}_i$$

Finally, the poles in Catani's generalized anomalous dimensions are isolated by definition:

$$\sum_{i \in \mathcal{H}_{m,0}^n} \Gamma_i(\epsilon) = \sum_{i \in \mathcal{H}_{m,0}^n} \left(\frac{1}{\epsilon} \gamma_i + \text{ц}_i \right) \qquad \text{ц}_i \equiv -\delta_{f_i,g} \frac{2}{3} T_{\mathrm{R}} \sum_{\rho=1}^{n_F} \log \frac{m_{Q_\rho}^2}{\mu^2}$$

Puttin geverything together, using colour-conservation we find:

$$I_{\mathsf{S+V}}(\epsilon) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) + I_{\mathsf{S+V}}^{(0)} + o(\epsilon)$$

$$I_{\mathsf{C}}(\epsilon) = +\frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) + I_{\mathsf{C}}^{(0)} + o(\epsilon)$$

The total operator $I_{\mathsf{T}} \equiv I_{\mathsf{S}} + I_{\mathsf{C}} + I_{\mathsf{V}}$ is the ϵ-finite, and the integrated counterterms read:

$$I_{\mathsf{T}}^{(0)} = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \text{Ж}_{i,j} + \sum_{i \in \mathcal{X}_{m,m}^n} \text{Д}_i - \sum_{i \in \mathcal{H}_{m,0}^n} \text{Ц}_i - \sum_{i \in \mathcal{H}_{m,0}^n} \left[2\mathscr{L}_i \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) + 2\mathbf{T}_i^2 L_i^2 \right]$$

$$+ (N_q + N_{\bar{q}}) \frac{C_{\mathsf{F}}}{6} \left(39 - 4\pi^2 \right) + N_g \left[\frac{C_{\mathsf{A}}}{9} \left(67 - 6\pi^2 \right) - \frac{23}{9} T_{\mathsf{R}} n_f \right]$$

Conclusions
Results and future developments

Main results

- proof of IR-poles cancellation even with generic massive final-state partons
- computation of generalized integrated counterterms

Figure from CMS Report CERN–EP 2025–025. Example Feynman diagram of non-resonant mono-t production at tree-level mediated by a spin-1 boson M, which decays directly into a dark-matter pair $\chi\bar{\chi}$.

Future developments

- extension to NNLO in the NSC SS
- inclusion of initial-state massive partons
- resummation of massive logarithms
- application to heavy-quark processes