UNIVERSITÀ DEGLI STUDI DI MILANO

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Infrared-Safe NLO Calculations with Massive Quarks: An Extension of the NSC Subtraction Formalism

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Abstract

The treatment of infrared divergences in Next-to-Leading Order (NLO) QCD calculations becomes significantly more complex when accounting for massive quarks, particularly in processes where mass effects cannot be neglected. We present a generalization of the Nested Soft-Collinear (NSC) subtraction scheme to incorporate arbitrary massive quark flavors, preserving the original framework's efficiency while systematically addressing mass-dependent divergences.
By removing the need for massless approximations, this work enables precision calculations in particle production processes where quark mass effects are theoretically or phenomenologically relevant.

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§2.1 Renormalization scheme

The computation of NLO corrections to scattering processes often involves diverging loop amplitudes. In order to obtain finite results from these divergences, a renormalization scheme must be implemented.
As the generalized Catani's formula for virtual corrections is provided in [1] in a charge-unrenormalized (but mass-renormalized) way, it is necessary to carry out the renormalization procedure explicitly. To this end, we formally state the renormalization scheme adopted in this work.

§2.1.1 Dimensional regularization

In the evaluation of loop amplitudes, both UV- and IR-singularities are encountered. The most efficient way to simultaneously regularize both types of divergences is dimensional regularization, a regularization scheme first introduced by 't Hooft and Veltman in [2].
In general, the dimensional regularization scheme consists in the analytic continuation of loop momenta to $d = 4 - 2\epsilon$ dimensions, with $\epsilon \in \mathbb{C} : \Re\epsilon < 0$. This procedure turns loop integrals into meromorphic[1]functions of $\epsilon \in \mathbb{C}$, allowing for the isolation of divergences as poles in ϵ.
The dimensional regularization prescription leaves freedom in choosing the dimensionality of external momenta, as well as the number of polarizations of both external and internal particles, thus allowing for the definition of different regularization schemes. We choose to work with **conventional dimensional regularization** (CDR), in which all momenta and polarization are analytically continued to d dimensions, as opposed to the 't Hooft–Veltman scheme (HV), in which only internal momenta and polarizations are.
When considering non-chiral gauge theories like QCD, CDR is the most natural choice, as the main difference between CDR and HV is the treatment of purely 4-dimensional objects, i.e. γ^5 and $\epsilon_{\mu\nu\sigma\rho}$. In particular, in CDR both the Dirac algebra and Lorentz indices are analytically continued to d dimensions, leading to a mathematical inconsistency when $d \notin \mathbb{N}$.

Observation 2.1.1 (Inconsistency in CDR)
In 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ with metric signature $\eta = (+, -, -, -)$, the Dirac

[1]Given an open set $D \subset \mathbb{C}$, then $f : D \to \mathbb{C}$ is *meromorphic* if it is holomorphic on $D - P$, where $P \subset D$ is a set of isolated points called *poles*. Recall that a function $f : D \to \mathbb{C}$ is *holomorphic* on D if it is complex differentiable at every point in D.

algebra is defined as $\mathfrak{cl}_{1,3}(\mathbb{C}) \cong \mathfrak{cl}_{1,3}(\mathbb{R}) \otimes \mathbb{C}$ (complexification[2]). This Clifford algebra admits a matrix representation with generators $\{\gamma^\mu\}_{\mu=0,1,2,3} \subset \mathbb{C}^{4\times 4}$ such that:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathrm{I}_4 \tag{2.1}$$

As in CDR Lorentz indices too are d-dimensional, the Dirac algebra becomes $\mathfrak{cl}_{1,d-1}(\mathbb{C}) \simeq \mathfrak{cl}_{1,d-1}(\mathbb{R}) \otimes \mathbb{C}$, where usual Minkowski space $\mathbb{R}^{1,3}$ is replaced by the pseudo-Euclidean space $\mathbb{R}^{1,d-1}$ with metric signature $\eta = (+,-,\dots,-)$. This structure, however, is ill-defined, as for $d \in \mathbb{N}$ then $\dim_{\mathbb{C}} \mathfrak{cl}_{1,d-1}(\mathbb{C}) = 2^d$, but a finite-dimensional algebra of dimension 2^d with $d \notin \mathbb{N}$ is meaningless.
To solve this issue, in CDR spinor indices remain 4-dimensional, i.e. we consider a matrix representation generated by $\{\gamma^\mu\}_{\mu=0,\dots,d-1} \subset \mathbb{C}^{4\times 4}$ and impose the formal relations Eq. 2.1. Consistency is achieved by analytical continuation of trace identities: indeed, traces obtained by recursively applying Eq. 2.1 (like $\mathrm{tr}\{\gamma^\mu\gamma^\nu\} = 4\eta^{\mu\nu}$) are still valid in d-dimensions, as the only dependence on dimension comes from contractions such as $\eta^\mu{}_\mu = d$.
A fatal inconsistency of CDR arises when considering γ^5. In $\mathfrak{cl}_{1,3}(\mathbb{C})$, this matrix is defined as $\gamma^5 := \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3$ and has the property $\{\gamma^5, \gamma^\mu\} = 0 \ \forall\mu = 0,1,2,3$, which allows to prove the following identity:

$$\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = -4i\epsilon^{\mu\nu\rho\sigma} \tag{2.2}$$

This construction cannot be generalized consistently to $d \notin \mathbb{N}$. To show this, assume a d-dimensional generalization $\gamma^5 \in \mathbb{C}^{4\times 4} : \{\gamma^5, \gamma^\mu\} = 0 \ \forall\mu = 0,\dots,d-1$, so that $\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n} = (-1)^n\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^5$. By the cyclicity of the trace, then:

$$(1-(-1)^n)\,\mathrm{tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n}\} = 0 \tag{2.3}$$

Consider $n = d$: clearly $1-(-1)^d = 1 - e^{i\pi d} \neq 0$ for $d \notin \mathbb{N}$, so $\mathrm{tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_d}\} = 0$. This is an open contradiction to Eq. 2.2, as analytic continuation should continuously preserve the top product of the algebra as $\epsilon \to 0$.
This contradiction is the explicit manifestation of a more profound topological issue of analytically continuing the number of dimensions: the Levi-Civita symbol in $d = 4$ is linked[3]to the Grassmann algebra $\bigwedge(\mathbb{R}^{1,3})$, and in particular to its top-form, but $\bigwedge^k(\mathbb{R}^d)$ is only defined for $d \in \mathbb{N}$, so the top exterior subspace $\bigwedge^d(\mathbb{R}^{1,d-1})$ is meaningless for $d \notin \mathbb{N}$ and the Levi-Civita symbol cannot be analytically continued to $d = 4-2\epsilon$ dimensions.

As of Obs. 2.1.1, it is now clear why we choose to adopt CDR: in QCD, the only pathological objects are encountered when considering chiral vertices (e.g. for pseudoscalar mesons) and electroweak interactions, and both can be handled via known prescriptions for the γ^5.

[3]Given an n-dimensional vector space $V(\mathbb{K})$ with a quadratic form q, associated linear form ω and orthogonal basis $\{e_i\}_{i=1,\dots,n}$, and a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a *Clifford mapping* is an injective $\mathbb{K}$-linear map $\rho : V \to \mathcal{A} : \mathit{1} \notin \rho(V) \wedge \rho(x)^2 = -q(x)\mathit{1} \ \forall x \in V$. If $\rho(V)$ generates $\mathcal{A}$, then $(\mathcal{A}, \rho)$ is a *Clifford algebra* for (V, q), and is denoted by $\mathfrak{cl}(V)$. It can be shown with simple algebraic manipulation that $\{\rho(x), \rho(y)\} = 2\omega(x,y)\mathit{1} \ \forall x, y \in V$. $\mathfrak{cl}_{m,n}(\mathbb{R})$ denotes the Clifford algebra associated to $\mathbb{R}^{m,n}$ with canonical pseudo-Euclidean quadratic form.

[3]Given an n-dimensional vector space $V(\mathbb{K})$, its *Grassmann algebra* (or exterior algebra) is the $\mathbb{N}_0$-graded algebra $\bigwedge(V) = \bigoplus_{k=0}^n \bigwedge^k(V)$ of k-forms. It can be shown that $\dim_{\mathbb{K}} \bigwedge^k(V) = \binom{n}{k}$, hence the top exterior subspace $\bigwedge^n(V)$ is 1-dimensional: indeed, given a basis $\{e_i\}_{i=1,\dots,n} \subset V$, it is $\bigwedge^n(V) = \langle e_1 \wedge \dots \wedge e_n\rangle$, and the Levi-Civita symbol is normalized so that $\epsilon^{1\cdots n}$ has the same sign of $e_1 \wedge \dots \wedge e_n$.

§2.1.2 Minimal subtraction

Once regularized, UV-divergences have to be removed via renormalization of fields and coupling constants. As a result of the renormalization procedure, a running coupling $\alpha_{\mathrm{s}}(\mu^2)$ is introduced, and its definition in terms of the bare coupling $\alpha_{\mathrm{s,b}}$ depends both on the regularization and the renormalization schemes.
In this work, we renormalize the coupling in a standard way (as in [3]) using the **modified minimal-subtraction scheme** ($\overline{\mathrm{MS}}$), which directly subtracts UV-divergences from the coupling:

$$\alpha_{\mathrm{s,b}} S_\epsilon = \alpha_{\mathrm{s}}(\mu^2)\mu^{2\epsilon}\left[1 - \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\frac{\beta_0}{\epsilon} + \mathcal{O}(\alpha_{\mathrm{s}}^2)\right] \tag{2.4}$$

where μ is an arbitrary renormalization scale, S_ϵ is the typical phase-space volume factor in dimensional regularization:

$$S_\epsilon \equiv (4\pi)^\epsilon \, e^{-\gamma_{\mathrm{E}}\epsilon} \tag{2.5}$$

with $\gamma_{\mathrm{E}} = 0.5772\ldots$ is the Euler-Mascheroni constant, and β_0 is the first coefficient of the QCD β-function:

$$\beta_0 := \frac{11}{6}C_{\mathtt{A}} - \frac{2}{3}T_{\mathrm{R}} n_q \tag{2.6}$$

where $C_{\mathtt{A}}$ and T_{R} are linked to the gauge group $\mathrm{SU}(n)$, with $n = n_f + n_F$ number of quark flavors (n_f massless and n_F massive quark flavors, see §2.2), and n_q is the number of active quark flavors at the considered energy scale.

Observation 2.1.2 (Dimensionality of coupling)

An important clarification about the dimensionality of $\alpha_{\mathrm{s,b}}$ and α_{s} is needed, due to the presence of $\mu^{2\epsilon}$ in Eq. 2.4.
Consider the QCD Lagrangian, i.e. a Yang-Mills Lagrangian with gauge group $\mathrm{SU}(n)$ (see e.g.):

In general, we consider amplitudes $\mathcal{M}_m$ involving m external QCD partons (gluons and quarks), with momenta $\{p\} \equiv \{p_1, \ldots, p_m\}$, and an arbitrary number of colorless particles (photons, leptons, ...). Dependence on the momenta and quantum numbers of colorless particles is always understood and not explicitly shown. The $\overline{\mathrm{MS}}$-renormalized amplitude has the following perturbative expansion in α_{s}:

$$\mathcal{M}_m(\alpha_{\mathrm{s}}(\mu^2), \mu^2; \{p\}) = \left(\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\right)^q \left[\mathcal{M}_m^{(0)}(\mu^2; \{p\}) + \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\mathcal{M}_m^{(1)}(\mu^2; \{p\}) + \mathcal{O}(\alpha_{\mathrm{s}}^2)\right] \tag{2.7}$$

where the overall power is, in general, $q \in \frac{1}{2}\mathbb{N}_0$. Note that, although spoiled of UV-divergences, these amplitudes are still IR-singular as $\epsilon \to 0$.

§2.2 Color space

Chapter 3

NSC Subtraction Scheme

Factorization of hadronic cross-section:

$$\sigma_{h_1,h_2}(P_1,P_2) = \sum_{a,b} \int_{[0,1]^2} \mathrm{d}x_1 \mathrm{d}x_2\, f_a^{(h_1)}(x_1,\mu_\mathrm{F}^2) f_b^{(h_2)}(x_2,\mu_\mathrm{F}^2) \hat{\sigma}_{a,b}(x_1 P_1, x_2 P_2, \alpha_\mathrm{s}(\mu^2), \mu^2, \mu_\mathrm{F}^2)$$

where μ is the renormalization scale and μ_F is the factorization scale.

$$\hat{\sigma}_{a,b}(p_1,p_2) = \sum_{n\in\mathbb{N}_0} \hat{\sigma}_{a,b}^{(n)}(p_1,p_2) \tag{3.1}$$

with $p_i \equiv x_i P_i$, $i = 1,2$.

$$\hat{\sigma}_{a,b}^{(0)}(p_1,p_2) := \frac{\mathcal{N}_{a,b}}{2\hat{s}} \int \mathrm{d}\mathbf{\Phi}_n \, \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(0)} \rangle \, \mathcal{F}_n \tag{3.2}$$

where $\mathcal{F}_n$ is an n-particle, IR-finite measurement function defining the observable.

$$\hat{\sigma}_{a,b}^{(1)}(p_1,p_2) = \hat{\sigma}_{a,b}^{\mathrm{R}}(p_1,p_2) + \hat{\sigma}_{a,b}^{\mathrm{V}}(p_1,p_2) + \hat{\sigma}_{a,b}^{\mathrm{C}}(p_1,p_2) \tag{3.3}$$

where:

$$\hat{\sigma}_{a,b}^{\mathrm{R}}(p_1,p_2) := \frac{\mathcal{N}_{a,b}}{2\hat{s}} \int \mathrm{d}\mathbf{\Phi}_{n+1} \, \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle \, \mathcal{F}_{n+1} \tag{3.4}$$

$$\hat{\sigma}_{a,b}^{\mathrm{V}}(p_1,p_2) := \frac{\mathcal{N}_{a,b}}{2\hat{s}} \int \mathrm{d}\mathbf{\Phi}_n \, 2\Re \, \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(1)} \rangle \, \mathcal{F}_n \tag{3.5}$$

$$\hat{\sigma}_{a,b}^{\mathrm{C}}(p_1,p_2) := \frac{\alpha_\mathrm{s}(\mu^2)}{2\pi} \frac{1}{\epsilon} \left(\frac{\mu^2}{\mu_\mathrm{F}^2}\right)^{\epsilon} \sum_c \int_0^1 \mathrm{d}z \left[\hat{P}_{c,a}^{(0)} \hat{\sigma}_{c,b}^{(0)}(zp_1,p_2) + \hat{P}_{c,b}^{(0)} \hat{\sigma}_{a,c}^{(0)}(p_1,zp_2) \right] \tag{3.6}$$

Chapter 4

NSC SS with Massive Quarks

Appendices

Appendix A

Draft appendix

§A.1 Draft

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