Quantum Field Theory

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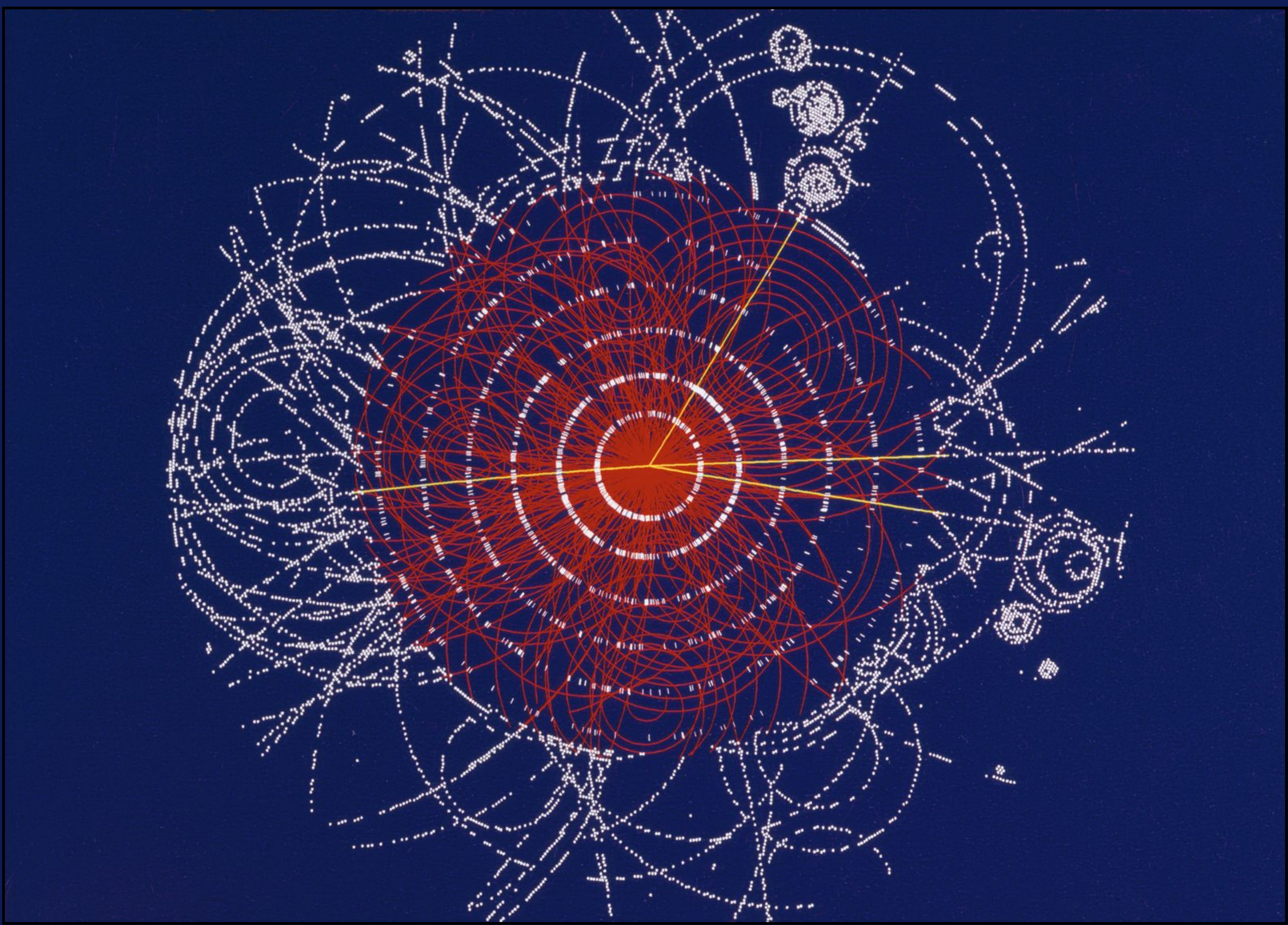

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Notation

Conventions

In these notes, the Lorentz–Minkowski metric $\eta_{\mu\nu}$ has signature $(+,-,-,-)$ and Greek indices generally run over spacetime coordinates, while Latin indices are general $\mathbb{N}_0$-indices defined in each context. Repeated indices are generally summed over, unless otherwise specified. The n-dimensional Levi–Civita symbol $\epsilon^{i_1 \cdots i_n}$ is defined with the convention $\epsilon^{01\cdots n} = +1$.

All quantities are expressed in natural units, defined by $\hbar \equiv c \equiv 1$: in particular, the "dimensions" of a quantity generally refers to its mass dimension.

The normal ordering and the time ordering of an expression are respectively denoted by the normal-ordering operator $\mathfrak{N}$ and by the time-ordering operator $\mathfrak{T}$.

Given $\alpha \in \mathbb{C}^{n\times n}$, with $n \in \mathbb{N}_0$, its complex conjugate is denoted as α^*, its transpose as $\alpha^\intercal$ and its Hermitian conjugate as $\alpha^\dagger \coloneqq (\alpha^*)^\intercal$. Given a Dirac spinor $\Psi \in \mathbb{C}^4$, its Dirac dual (or Dirac adjoint) is defined as $\bar{\Psi} \coloneqq \Psi^\dagger \gamma^0$.

The Landau symbol for a function $f : D \subseteq \mathbb{R} \to \mathbb{C}$ is defined by the condition $\exists M \in \mathbb{R} : |o(f(x))| \leq M\,|f(x)| \;\; \forall x \in D$.

Mathematical notation

The empty set is denoted by $\varnothing$ and the power set of a set A by $\wp(A) \coloneqq \{B : B \subseteq A\}$. The counting numbers are $\mathbb{N} \equiv \{1, 2, 3, \dots\}$, and the natural numbers are defined by $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$.

The imaginary unit is denoted by i and the unit quaternions by $\mathrm{i}, \mathrm{j}, \mathrm{k}$, so that $\mathbb{C}(\mathbb{R}) = \mathrm{span}(1, \mathrm{i})$ and $\mathbb{H}(\mathbb{R}) = \mathrm{span}(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$.

The permutation group of n objects, i.e. the n^{th} symmetric group, is denoted by S_n.

Given two $\mathbb{K}$-vector spaces V and W, with $\mathbb{K}$ a generic field, the space of all $\mathbb{K}$-linear applications $f : V \to W$ is denoted by $\mathrm{Hom}_{\mathbb{K}}(V, W)$: in particular, $\mathrm{Hom}_{\mathbb{K}}(V) \equiv \mathrm{End}(V)$. The subset of $\mathrm{End}(V)$ of all automorphisms of V is the automorphism group $\mathrm{Aut}(V)$, which is a group under composition of morphisms.

Given a manifold $\mathcal{M}$, the space of all smooth scalar functions on $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$, the space of all vector fields on $\mathcal{M}$ by $\mathfrak{X}(\mathcal{M})$, the space of all p-forms on $\mathcal{M}$ by $\bigwedge^p(\mathcal{M})$ and the Grassmann algebra of $\mathcal{M}$ by $\bigwedge(\mathcal{M}) \coloneqq \bigoplus_{k=0}^n \bigwedge^k(\mathcal{M})$.

The exterior derivative is denoted by d, the partial derivative by ∂, the nabla operator by $\boldsymbol{\nabla} \equiv (\partial_1, \partial_2, \partial_3)$, the Laplacian operator by $\triangle \equiv \boldsymbol{\nabla}^2$ and the D'Alembert operator by $\square \equiv \partial_0^2 - \triangle$.

A list of "important" Lie groups:

$\mathrm{GL}(n, \mathbb{K}) \coloneqq \{\mathrm{A} \in \mathbb{K}^{n\times n} : \det \mathrm{A} \neq 0\}$ general linear group (Lie group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$)

$\mathrm{SL}(n, \mathbb{K}) \coloneqq \{\mathrm{A} \in \mathrm{GL}(n, \mathbb{K}) : \det \mathrm{A} = 1\}$ special linear group (Lie group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$)

$\mathrm{O}(n) \coloneqq \{\mathrm{A} \in \mathbb{R}^{n\times n} : \mathrm{A}\mathrm{A}^\intercal = \mathrm{A}^\intercal\mathrm{A} = \mathrm{I}_n\}$ orthogonal group

$\mathrm{SO}(n) \coloneqq \{\mathrm{A} \in \mathrm{O}(n) : \det \mathrm{A} = 1$ special orthogonal group

$\mathrm{U}(n) \coloneqq \{\mathrm{A} \in \mathbb{C}^{n\times n} : \mathrm{A}\mathrm{A}^\dagger = \mathrm{A}^\dagger\mathrm{A} = \mathrm{I}_n\}$ unitary group

$\mathrm{SU}(n) \coloneqq \{\mathrm{A} \in \mathrm{U}(n) : \det \mathrm{A} = 1$ special unitary group

Given a Lie group G, its associated Lie algebra is denoted by $\mathfrak{g}$.

Physical notation

The Lorentz group is denoted by $\mathrm{SO}^+(1,3)$, the Lorentz algebra by $\mathfrak{so}^+(1,3)$, the Poincaré group by $\mathrm{ISO}^+(1,3)$, the Poincaré algebra by $\mathfrak{iso}^+(1,3)$ and the Dirac algebra by $\mathfrak{cl}_{1,3}(\mathbb{C})$.
Hilbert and Fock spaces are generally denoted respectively by $\mathscr{H}$ and $\mathscr{F}$, the Hamiltonian and the Lagrangian of a system by H and L, and the Hamiltonian density and the Lagrangian density of a filed theory by $\mathcal{H} : H = \int \mathrm{d}^3x\, H$ and $\mathcal{L} : L = \int \mathrm{d}^3x\, L$. The action functional is defined as $\mathcal{S} \coloneqq \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}$.
The matrix element of a scattering process is generally denoted by $\mathcal{M}$, and its amplitude by $\mathcal{A} \equiv |\mathcal{M}|^2$.
The parity operator is denoted by $\mathcal{P}$, the charge-conjugation operator by $\mathcal{C}$ and the time-reversal operator by $\mathcal{T}$.
For a general gauge theory, the covariant derivative is denoted by $\mathcal{D}_\mu$, the connection (or gauge field) by $\mathcal{A}_\mu$ and the field-strength tensor by $\mathcal{F}_{\mu\nu}$. For the particular case of QED, these quantities are denoted respectively by D_μ, A_μ and $F_{\mu\nu}$.
The Feynman propagator in position and momentum space is denoted by $\Delta(x,y) \equiv \Delta(x-y)$ and $\tilde{\Delta}(p)$, the Dirac propagator by $\Sigma(x,y) \equiv \Sigma(x-y)$ and $\tilde{\Sigma}(p)$, and the photon propagator by $\Delta(x,y) \equiv \Delta(x-y)$ and $\tilde{\Delta}(p)$.

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Part I

Field Theory

Chapter 1

Classical Field Theory

§1.1 Continuous limit

§1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x-axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a. Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n, one can write the Lagrangian for a **harmonic crystal** as:

$$L = \sum_n \left[\frac{1}{2}\dot{\eta}_n^2 - \frac{\lambda}{2}\left(\eta_n - \eta_{n-1}\right)^2 \right] \tag{1.1}$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda\left(\eta_{n+1} - 2\eta_n + \eta_{n-1}\right) \tag{1.2}$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{\mathrm{i}(kn - \omega t)} \tag{1.3}$$

where the dispersion relation is:

$$\omega^2 = 2\lambda\left(1 - \cos k\right) \approx \lambda k^2 \tag{1.4}$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \qquad \Longrightarrow \qquad k_m = \frac{2\pi m}{N}, \; m = 0, 1, \ldots, N-1 \tag{1.5}$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} \left[\mathcal{A}_m e^{\mathrm{i}(k_m n - \omega_m t)} + \mathcal{A}^* e^{-\mathrm{i}(k_m n - \omega_m t)} \right] \tag{1.6}$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjucated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \tag{1.7}$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = \mathrm{i}\hbar\delta_{jk}$. Implementing time evolution with the Heisenberg picture[1]:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = \mathrm{i}\hbar\delta_{jk} \tag{1.8}$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce **annihilation** and **creation operators**[2] $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} \left[e^{\mathrm{i}(k_m n - \omega_m t)}\hat{a}_m + e^{-\mathrm{i}(k_m n - \omega_m t)}\hat{a}_m^\dagger\right] \tag{1.9}$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{\mathrm{i}k_m(n-n')} = N\delta_{nn'} \tag{1.10}$$

The Hamiltonian of the system can be written as:

$$\hat{H} = \sum_n \left[\frac{1}{2}\hat{\pi}_n^2 + \frac{\lambda}{2}\left(\hat{\eta}_n - \hat{\eta}_{n-1}\right)^2\right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger\hat{a}_m + \frac{1}{2}\right) \tag{1.11}$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m \left|0\right\rangle \quad \forall m = 0, 1, \ldots, N-1 \tag{1.12}$$

$$\left|n_0, n_1, \ldots, n_{N-1}\right\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} \left|0\right\rangle \tag{1.13}$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2}\sum_{m=0}^{N-1} \hbar\omega_m \tag{1.14}$$

$$E_{n_0, n_1, \ldots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \tag{1.15}$$

This Hilbert space is called **Fock space** and the excited states **phonons**: these can be thought as "particles" and n_m is the number of phonons in the m^{th} normal mode.

[1]Recall that $\hat{\mathcal{O}}(t) = e^{\frac{\mathrm{i}}{\hbar}\hat{H}t}\hat{\mathcal{O}}(0)e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}$ and $\frac{\mathrm{d}\hat{\mathcal{O}}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{\mathcal{O}}]$.

[2]For a harmonic oscillator $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{p}(t)$ and $\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) + \hat{a}^\dagger(t)\right] \qquad \hat{p}(t) = -\mathrm{i}\omega\sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) - \hat{a}^\dagger(t)\right]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. The time evolution $\hat{a}(t) = e^{-\mathrm{i}\omega t}\hat{a}(0)$ ensures that $\hat{H}$ is times-independent.

§1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \to 0$ and $N \to \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x,t)$ dependent on the continuous real coordinate $x \in [0,R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x}\right)^2 \qquad \sum_n \longrightarrow \frac{1}{a}\int_0^R \mathrm{d}x$$

Proposition 1.1.1 (Dirac delta)

In the continuous limit:

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x-x') = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{\mathrm{i}k(x-x')}$$

Proof. By direct calculation:

$$a \sum_n f(an)\frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R \mathrm{d}x\, f(x)\delta(x-y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \to kx$, symmetrizing $k_m \in [-\pi,\pi]$ (instead of $[0,2\pi]$) one finds:

$$\delta(x-x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na}\sum_m e^{\mathrm{i}k_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{\mathrm{i}k(x-x')}$$

where integration limits are $\pm\frac{\pi}{a} \to \pm\infty$. □

Proposition 1.1.2

The inverse Fourier transform of the Dirac delta reads:

$$\int_0^R \mathrm{d}x\, e^{\mathrm{i}(k-k')x} = 2\pi\delta(k-k') \tag{1.16}$$

Proof. Denoting the Fourier transform by the operator $\mathfrak{F}$, the thesis is a consequence of $\mathfrak{F}^{-1}\mathfrak{F} = \mathbb{1}$. □

By these relations, it can be seen that $\frac{\mathrm{d}k}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k+\mathrm{d}k$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.

In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}\rho_0(\partial_t\eta)^2 - \frac{\kappa}{2}(\partial_x\eta)^2\right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x,t) \equiv \sqrt{\rho_0}\eta(x,t)$ and $\kappa = c^2\rho_0$ and adding a pinning term $\propto \phi^2$, the

Lagrangian can be written as:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \frac{m^2c^4}{2}\phi^2\right] \tag{1.17}$$

The classical equation of motion of this field yields:

$$\partial_t^2\phi = c^2\partial_x^2\phi - m^2c^4\phi \tag{1.18}$$

The solutions of this wave equation can be written as:

$$\phi(x,t) = e^{\mathrm{i}(kx-\omega_k t)} \tag{1.19}$$

with dispersion relation:

$$\omega_k^2 = c^2k^2 + m^2c^4 \tag{1.20}$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x,t) \coloneqq \frac{\partial L}{\partial(\partial_t\phi)} = \partial_t\phi(x,t) \tag{1.21}$$

The classical Hamiltonian can then be found as:

$$H = \int_0^R \mathrm{d}x \left[\frac{1}{2}\Pi^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2c^4}{2}\phi^2\right] \tag{1.22}$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\sqrt{\frac{\hbar}{2\omega_k}}\left[e^{\mathrm{i}(kx-\omega_k t)}\hat{a}_k + e^{-\mathrm{i}(kx-\omega_k t)}\hat{a}_k^\dagger\right] \tag{1.23}$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi\delta(k-k') \tag{1.24}$$

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = \mathrm{i}\hbar\delta(x-x') \tag{1.25}$$

The quantum Hamiltonian can be written as:

$$\hat{H} = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\frac{1}{2}\hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger\right) = E_0 + \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\hbar\omega_k\hat{a}_k^\dagger\hat{a}_k \tag{1.26}$$

The ground-state energy can be computed from Eq. 1.14, defining $\mathrm{Vol} \coloneqq 2\pi\delta(k=0)$:

$$E_0 = \mathrm{Vol}\int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\frac{1}{2}\hbar\omega_k \tag{1.27}$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as **Casimir energy**.

§1.2 Spacetime symmetries

§1.2.1 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu}x^\mu x^\nu$, i.e. the orthogonal group $\mathrm{O}(1,3)$. The condition that $\Lambda^\mu{}_\nu$ must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \tag{1.28}$$

This implies that $\det\Lambda = \pm 1$: a transformation with $\det\Lambda = -1$ can always be written as the product of a transformation with $\det\Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates, hence one further defines the special orthogonal group $\mathrm{SO}(1,3) := \{\Lambda \in \mathrm{O}(1,3) : \det\Lambda = 1\}$.
Writing explicitly the temporal component $1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$, it is clear that $(\Lambda^0{}_0)^2 \geq 1$. Therefore, $\mathrm{O}(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0{}_0 \geq 1$ and the non-orthochronous component with $\Lambda^0{}_0 \leq 1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a dircrete transformation which reverses the sign of the temporal component.

Definition 1.2.1 (Lorentz group)
The **Lorentz group** $\mathrm{SO}^+(1,3)$ is the orthochronous component of $\mathrm{SO}(1,3)$.

The discrete transformations are factored out of the Lorentz group: these are **parity** and **time reversal**, represented as $\mathcal{P}^\mu{}_\nu = \mathrm{diag}\,(+1,-1,-1,-1)$ and $\mathcal{T}^\mu_\nu = \mathrm{diag}\,(-1,+1,+1,+1)$. Applying these discrete transformations in all combinations ($\mathbb{1}$, $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{PT}$) one gets the four disconnected components of $\mathrm{O}(1,3)$, which are not simply connected. This means that Lorentz invariance does not include parity and time reversal invariance.
Considering an infinitesimal Lorentz transformation and applying Eq. 1.28:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \qquad \Longrightarrow \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x,y), (y,z) and (z,x) planes and the 3 rapidities of hyperbolic rotations in the (t,x), (t,y) and (t,z) planes.

Proposition 1.2.1
The Lorentz group is a non-compact Lie group.

Proof. Spherical and hyperbolic rotations are continuous and differentiable functions respectively of angles and of rapidities, but while angles vary in $[0,2\pi)$, rapidities vary in $\mathbb{R}$, hence the differentiable manifold associated to $\mathrm{SO}^+(1,3)$ is not compact. □

§1.2.1.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \mathrm{SO}^+(1,3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.29}$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given an n-dimensional representation of $\mathrm{SO}^+(1,3)$, then $[J^{\mu\nu}]^i{}_j \in \mathbb{C}^{n\times n}$ and $[\Lambda]^i{}_j \in \mathbb{R}^{n\times n}$: for example, the $n=1$ representation acts on **scalars**, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \ \forall \mu,\nu = 0,\dots,3$ and $\Lambda \equiv 1$.

4-vectors The $n=4$ representation[3]acts on **contravariant 4-vectors** v^μ, which transform according to $v^\mu \mapsto \Lambda^\mu{}_\nu v^\nu$, and **covariant 4-vectors** v_μ, which transform according to $v_\mu \mapsto \Lambda_\mu{}^\nu v_\nu$. In this representation, the generators are represented as $\mathbb{C}^{4\times4}$ matrices:

$$[J^{\mu\nu}]^\rho{}_\sigma = \mathrm{i}\left(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma\right) \tag{1.30}$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}^+(1,3)$, called the **Lorentz algebra**, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = \mathrm{i}\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \tag{1.31}$$

It is convenient to rearrange the 6 independent components of $J^{\mu\nu}$ into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad\qquad K^i := J^{i0} \tag{1.32}$$

The $\mathfrak{so}^+(1,3)$ algebra can then be rewritten as:

$$[J^i, J^j] = \mathrm{i}\epsilon^{ijk}J^k \qquad\qquad [J^i, K^j] = \mathrm{i}\epsilon^{ijk}K^k \qquad\qquad [K^i, K^j] = -\mathrm{i}\epsilon^{ijk}J^k \tag{1.33}$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^i are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \qquad\qquad \eta^i := \omega^{i0} \tag{1.34}$$

so that:

$$\Lambda = \exp\left[-\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{J} + \mathrm{i}\boldsymbol{\eta}\cdot\mathbf{K}\right] \tag{1.35}$$

where $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}^3$ and $\mathbf{J}, \mathbf{K} \in (\mathbb{R}^{4\times4})^3$. This definition reflect the *alias* interpretation: angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts which increase velocities with respect to said frame.

§1.2.1.2 Tensor Representations

Given the action of $\mathrm{SO}^+(1,3)$ on covariant and contravariant vectors, it is clear to see that a generic (p,q)-tensor transforms as:

$$T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q} \mapsto \Lambda^{\mu_1}{}_{\alpha_1}\dots\Lambda^{\mu_p}{}_{\alpha_p}\Lambda_{\nu_1}{}^{\beta_1}\dots\Lambda_{\nu_q}{}^{\beta_q}T^{\alpha_1\dots\alpha_p}{}_{\beta_1\dots\beta_q} \tag{1.36}$$

This shows that the representation of the Lorentz group which acts on (p,q)-tensors, although being of degree $n = 4^{p+q}$, is reducible into the direct product of $p+q$ 4-dimensional representations.

[3]To be precise, contravariant 4-vectors are elements of $V \equiv \mathbb{R}^{1,3}$, so they transform according $v_\mu \mapsto \tilde{v}_\mu \equiv [\Lambda]^\mu{}_\nu v^\nu$, while covariant 4-vectors are elements of V^*, hence they require the transposed endomorphism (see Def. 1.4.2 in [5]) and inversely transform according to $\tilde{v}_\mu \mapsto v_\mu \equiv [\Lambda^\intercal]_\mu{}^\nu \tilde{v}_\nu := \tilde{v}_\nu[\Lambda]^\nu{}_\mu$. To invert this relation note that, from Eq. 1.28, $[\Lambda^{-1}]^\mu{}_\nu = [\Lambda]_\nu{}^\mu$, thus the transformation law is $v_\mu \mapsto \tilde{v}_\mu \equiv [\Lambda]_\mu{}^\nu v_\nu$. The identification $[\Lambda]^\mu{}_\nu \equiv \Lambda^\mu{}_\nu$ is an abuse of notation.

Moreover, consider the action of the Lorentz group on $(2,0)$-tensors: being $T^{\mu\nu} \mapsto \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu}T^{\mu\nu} \mapsto \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9} \tag{1.37}$$

These are irreducible representations which act on S, $A^{\mu\nu}$ and $\bar{S}^{\mu\nu} \equiv S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right)$ and $S^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right)$.

Decomposition under rotations Restricting the action of $\mathrm{SO}^+(1,3)$ to its subgroup $\mathrm{SO}(3)$, tensors can be decomposed according to irreducible representations of $\mathrm{SO}(3)$, which are labelled by the angular momentum $j \in \mathbb{N}_0$ and are of degree $n = 2j+1$. Also recall the Clebsch–Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \tag{1.38}$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$ (abuse of notation to denote that α transforms according to the $\mathbf{0}$ representation). A 4-vector v^μ is irreducible under the action of $\mathrm{SO}^+(1,3)$, but under $\mathrm{SO}(3)$ it is decomposed into v^0 and $\mathbf{v}$, so $v^\mu \in \mathbf{0} \oplus \mathbf{1}$. A $(2,0)$-tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.37: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2}\epsilon^{ijk}A^{jk}$, so $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ under spatial rotations.
Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$, $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$, $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$ and $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk}A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

Example 1.2.1 (Gravitational waves)

Gravitational waves in **de Donder gauge** are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two **invariant tensors** under $\mathrm{SO}^+(1,3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.28, and the Levi–Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\sigma}{}_{\gamma}\Lambda^{\rho}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} = (\det\Lambda)\,\epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

Note that, on the contrary, discrete transformations invert the handedness of spacetime, hence the Levi–Civita symbol changes sign: this hints to the facts that $\epsilon^{\mu\nu\rho\sigma}$ is a *tensor density*, since its transformation law is proportional to a power of the determinant of the coordinate transformation (this is true for general differentiable manifolds).

§1.2.1.3 Spinorial representations

The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ both have three generators defined by:

$$[\mathrm{X}_i, \mathrm{X}_j] = \mathrm{i}\epsilon^{ijk}\mathrm{X}_k \tag{1.39}$$

which means that SU(2) and SO(3) are indistinguishable by infinitesimal transformations; however, they are globally different, as SO(3) rotations are periodic by 2π, while SU(2) rotations are periodic by 4π: in particular, it can be shown that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$, i.e. SU(2) is the universal cover of SO(3). This means that SU(2) representations can be labelled by $j \in \frac{1}{2}\mathbb{N}_0$, where half-integer spin representations are known as **spinorial representations**: they act on spinors, i.e. objects which change sign under rotations of 2π (thus not suitable to represent SO(3)).

Example 1.2.2 (Composition of two spin-$\frac{1}{2}$)

The $\frac{\mathbf{1}}{\mathbf{2}}$ representation of SU(2) is a 2-dimensional representation where $J^i = \frac{\sigma^i}{2}$: the Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\epsilon^{ijk}\sigma^k$, thus the $\mathfrak{su}(2)$ algebra is satisfied. Denoting the $m = \pm\frac{1}{2}$ states in the $\frac{\mathbf{1}}{\mathbf{2}}$ representation as $|\uparrow\rangle$ and $|\downarrow\rangle$, the Clebsch–Gordan decomposition $\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0} \oplus \mathbf{1}$ yields the triplet ($j = 1$) $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$ and the singlet ($j = 0$) $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Proposition 1.2.2 (Lorentz algebra)

$$\mathfrak{so}^+(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{1.40}$$

Proof. Given the $\mathfrak{so}^+(1,3)$ algebra in Eq. 1.32, it is possible to define:

$$\mathbf{J}_\pm \coloneqq \frac{1}{2}(\mathbf{J} \pm \mathrm{i}\mathbf{K}) \quad \Longrightarrow \quad [\mathbf{J}^i_\pm, \mathbf{J}^j_\pm] = \mathrm{i}\epsilon^{ijk}\mathbf{J}^k_\pm \qquad [\mathbf{J}^i_\pm, \mathbf{J}^j_\mp] = \mathbf{0} \tag{1.41}$$

These are two commuting $\mathfrak{su}(2)$ algebras, thus proving the thesis[a]. □

[a] To be precise, $\mathfrak{so}^+(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$, while $\mathfrak{so}^+(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ since complexification yields the following isomorphisms (with Eq. 1.41):

$$\mathfrak{so}^+(1,3) \hookrightarrow \mathfrak{so}^+(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \hookleftarrow \mathfrak{sl}(2,\mathbb{C})$$

Indeed, note that $\mathrm{SO}^+(1,3)$ is not globally equivalent to $\mathrm{SU}(2) \times \mathrm{SU}(2)$: instead, $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(4)$, while the universal cover of $\mathrm{SO}^+(1,3)$ is $\mathrm{SL}(2,\mathbb{C})$, as $\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

By Prop. 1.2.2, representations of $\mathrm{SO}^+(1,3)$ can be labelled by $(\mathbf{j}_-, \mathbf{j}_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$, with each index labelling a representation of SU(2): as $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, the $(\mathbf{j}_-, \mathbf{j}_+)$ representation contains states with all possible spins $|j_+ - j_-| \leq j \leq j_+ + j_-$, and it is a representation of degree $n = (2j_- + 1)(2j_+ + 1)$.
More formally, the $(\mathbf{j}_-, \mathbf{j}_+)$ representation is $\rho_{(j_-,j_+)} : \mathfrak{so}^+(1,3) \to \mathfrak{gl}(V_{(j_-,j_+)})$, where $V_{(j_-,j_+)}$ is a $(2j_- + 1)(2j_+ + 1)$-dimensional $\mathbb{C}$-vector space, defined up to similarity as (inverting Eq. 1.41):

$$\begin{aligned} \rho_{(j_-,j_+)}(J^i) &= J^i_{(j_+)} \otimes \mathrm{I}_{(2j_-+1)} + \mathrm{I}_{(2j_++1)} \otimes J^i_{(j_-)} \\ \rho_{(j_-,j_+)}(K^i) &= -\mathrm{i}\left[J^i_{(j_+)} \otimes \mathrm{I}_{(2j_-+1)} - \mathrm{I}_{(2j_++1)} \otimes J^i_{(j_-)}\right] \end{aligned} \tag{1.42}$$

where $\mathbf{J}_{(j)}$ is the $(2j+1)$-dimensional irreducible representation of $\mathfrak{su}(2)$ and $\otimes$ is the tensor product (see Def. A.1.2). For example, the trivial representation is $(\mathbf{0}, \mathbf{0})$, as both $\mathbf{J}_\pm = \mathbf{0}$ and $\mathbf{J} = \mathbf{K} = \mathbf{0}$: this is the representation which acts on scalars.

Weyl representations $\left(\frac{1}{2},\mathbf{0}\right)$ and $\left(\mathbf{0},\frac{1}{2}\right)$ are 2-dimensional spinorial representations. These representations act on different spinors $(\psi_{\mathrm{L}})_\alpha, (\psi_{\mathrm{R}})_\beta \in \mathbb{C}^2$, with $\alpha, \beta = 1, 2$, which are called **left-** and **right-handed Weyl spinors**.
In $\left(\frac{1}{2},\mathbf{0}\right)$ the generators are $\mathbf{J}_- = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{J}_+ = \mathbf{0}$, while in $\left(\mathbf{0},\frac{1}{2}\right)$ they are $\mathbf{J}_- = \mathbf{0}$ and $\mathbf{J}_+ = \frac{\boldsymbol{\sigma}}{2}$, thus, by Eq. 1.42, $\mathbf{J}_{\mathrm{L}} = \mathbf{J}_{\mathrm{R}} = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{K}_{\mathrm{L}} = -\mathbf{K}_{\mathrm{R}} = \mathrm{i}\frac{\boldsymbol{\sigma}}{2}$, so that from Eq. 1.35:

$$\psi_{\mathrm{L}} \mapsto \Lambda_{\mathrm{L}}\psi_{\mathrm{L}} = \exp\left[(-\mathrm{i}\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_{\mathrm{L}} \tag{1.43}$$

$$\psi_{\mathrm{R}} \mapsto \Lambda_{\mathrm{R}}\psi_{\mathrm{R}} = \exp\left[(-\mathrm{i}\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_{\mathrm{R}} \tag{1.44}$$

Note that the generators K^i are not hermitian, as expected from Prop. A.2.3. Furthermore, as $\psi_{\mathrm{L/R}} \in \mathbb{C}^2$, then $\Lambda_{\mathrm{L/R}} \in \mathbb{C}^{2\times 2}$ (in general the spinorial representations are complex representations).

Proposition 1.2.3

Given $\psi_{\mathrm{L}} \in \left(\frac{1}{2},\mathbf{0}\right)$ and $\psi_{\mathrm{R}} \in \left(\mathbf{0},\frac{1}{2}\right)$, then $\sigma^2\psi_{\mathrm{L}}^* \in \left(\mathbf{0},\frac{1}{2}\right)$ and $\sigma^2\psi_{\mathrm{R}}^* \in \left(\frac{1}{2},\mathbf{0}\right)$.

Proof. Recall that for Pauli matrices $\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^*$, so $\sigma^2\Lambda_{\mathrm{L}}^*\sigma^2 = \Lambda_{\mathrm{R}}$ and:

$$\sigma^2\psi_{\mathrm{L}}^* \mapsto \sigma^2\left(\Lambda_{\mathrm{L}}\psi_{\mathrm{L}}\right)^* = \left(\sigma^2\Lambda_{\mathrm{L}}^*\sigma^2\right)\sigma^2\psi_{\mathrm{L}}^* = \Lambda_{\mathrm{R}}\sigma^2\psi_{\mathrm{L}}^* \quad\Longrightarrow\quad \sigma^2\psi_{\mathrm{L}}^* \in \left(\mathbf{0},\frac{1}{2}\right)$$

where $\sigma^2\sigma^2 = \mathrm{I}_2$ was used. The proof for $\sigma^2\psi_{\mathrm{R}}^*$ is analogous. □

This allows to define **charge conjugation** on Weyl spinors:

$$\psi_{\mathrm{L}}^c := \mathrm{i}\sigma^2\psi_{\mathrm{L}}^* \qquad\qquad \psi_{\mathrm{R}}^c := -\mathrm{i}\sigma^2\psi_{\mathrm{R}}^* \tag{1.45}$$

As of Prop. 1.2.3, charge conjugation transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the i factor ensures that applying this operator twice yields the identity operator.

Dirac representation $\left(\frac{1}{2},\mathbf{0}\right) \oplus \left(\mathbf{0},\frac{1}{2}\right)$ is a 4-dimensional complex representation: this representation acts on bispinors of the form $\Psi = ((\psi_{\mathrm{L}})_\alpha, (\xi_{\mathrm{R}})_\beta)^\intercal \in \mathbb{C}^4$, called **Dirac spinors**, with transformation $\Lambda_{\mathrm{D}} = \mathrm{diag}\left(\Lambda_{\mathrm{L}}, \Lambda_{\mathrm{R}}\right) \in \mathbb{C}^{4\times 4}$.

Definition 1.2.2 (Charge conjugation on Dirac spinors)

On Dirac spinors, the **charge conjugation operator** is defined as:

$$\Psi^c := \begin{pmatrix} -\mathrm{i}\sigma^2\xi_{\mathrm{R}}^* \\ \mathrm{i}\sigma^2\psi_{\mathrm{L}}^* \end{pmatrix} = -\mathrm{i}\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\Psi^* \tag{1.46}$$

This definition reflect the fact that charge conjugation changes the representation of the Weyl spinors (Prop. 1.2.3), i.e. the chirality of the components of the Dirac spinor.

Observation 1.2.1 (4-vector representation)

The other 4-dimensional representation of the Lorentz group is $\left(\frac{1}{2},\frac{1}{2}\right)$. To see that this is, in fact, the 4-vector representation, first consider the double-covering $\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

Lemma 1.2.1 (Double-covering of Lorentz group)

$$\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \tag{1.47}$$

Proof. Define the set $\mathcal{H}(2) \coloneqq \{\mathrm{H} \in \mathbb{C}^{2\times 2} : \mathrm{H}^\dagger = \mathrm{H}\}$ of all $\mathbb{C}^{2\times 2}$ Hermitian matrices. Trivially $\mathcal{H}(2) \cong \mathbb{R}^{1,3}$, since:

$$\mathbb{R}^{1,3} \ni v^\mu = (v^0, v^1, v^2, v^3) \longleftrightarrow \begin{pmatrix} v^0 - v^3 & -v^1 + \mathrm{i}v^2 \\ -v^1 - \mathrm{i}v^2 & v^0 + v^3 \end{pmatrix} = V \in \mathcal{H}(2)$$

where the isomorphisms are $V = v_\mu \sigma^\mu$, $v^\mu = \frac{1}{2}\operatorname{tr}(\bar{\sigma}^\mu V)$, with $\sigma^\mu \equiv (\mathrm{I}_2, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu \equiv (\mathrm{I}_2, -\boldsymbol{\sigma})$, since $\{\mathrm{I}_2, \sigma^1, \sigma^2, \sigma^3\}$ is a basis of $\mathcal{H}(2)$.
Now, consider $\mathrm{M} \in \mathrm{SL}(2,\mathbb{C})$ and define its linear action on $\mathcal{H}(2)$ as $V \mapsto \mathrm{M}V\mathrm{M}^\dagger$: then, $\mathrm{SL}(2,\mathbb{C})$ is an endomorphism group of $\mathcal{H}(2)$, since its action preserves Hermiticity, and $\det V \mapsto \det V$ as $\det \mathrm{M} = 1$ by definition. But $\det V = v_\mu v^\mu$, hence the action of $\mathrm{SL}(2,\mathbb{C})$ leaves the Minkowski norm invariant: it is thus possible to define a group homomorphism $\Phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3)$. WTS $\Phi(\mathrm{SL}(2,\mathbb{C})) \subseteq \mathrm{SO}^+(1,3)$: since $\mathrm{SL}(2,\mathbb{C})$ is simply connected (it is homeomorphic to $\mathbb{S}^3 \times \mathbb{R}^3$) and $\mathrm{I}_2 \in \mathrm{SL}(2,\mathbb{C})$, then $\Phi(\mathrm{SL}(2,\mathbb{C}))$ must be connected, as Φ can be shown to be continuous, and it must contain the identity, but the only connected compontent of $\mathrm{O}(1,3)$ which satisfies these conditions is $\mathrm{SO}^+(1,3)$, hence $\Phi(\mathrm{SL}(2,\mathbb{C})) \subseteq \mathrm{SO}^+(1,3)$. Then:

- surjectivity: to show that Φ surjective, recall that $\mathrm{SO}^+(1,3)$ is generated by spatial (trigonometric) rotations and boosts (i.e. hyperbolic rotations). The former are clearly realized by Φ, since $\mathrm{SU}(2) < \mathrm{SL}(2,\mathbb{C})$ and $\Phi|_{\mathrm{SU}(2)}$ is the standard double-covering map for $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$. Then, by polar decomposition, any $\mathrm{M} \in \mathrm{SL}(2,\mathbb{C})$ can be written as $\mathrm{M} = \mathrm{UP}$, with $\mathrm{U} \in \mathrm{SU}(2)$ and $\mathrm{P} = \exp(\mathbf{a} \cdot \boldsymbol{\sigma})$, $\mathbf{a} \in \mathbb{R}^3$: since P represents[a] a boost along the direction $\hat{\mathbf{a}}$ of rapidity $2a$, all generators of $\mathrm{SO}^+(1,3)$ are represented by Φ, i.e. $\Im\{\Phi\} = \mathrm{SO}^+(1,3)$.

- $\ker \Phi = \{\mathrm{I}_2, -\mathrm{I}_2\}$: solving $\Phi(\mathrm{M}) = \mathbb{1}_{\mathrm{SO}^+(1,3)}$ means that $\mathrm{M}V\mathrm{M}^\dagger = V \; \forall V \in \mathcal{H}(2)$. $V = \mathrm{I}_2$ results in $\mathrm{M} \in \mathrm{SU}(2) < \mathrm{SL}(2,\mathbb{C})$, hand $V = \sigma_i$ that M commutes with all the Pauli matrices: this means that M commutes with all elements of $\mathcal{H}(2)$, hence it is a multiple of I_2 by Schur's Lemma (applicable since the Pauli matrices form the basis of an irreducible representation of $\mathrm{SU}(2)$), and $\det \mathrm{M} = 1 \implies \mathrm{M} = \pm \mathrm{I}_2$.

Since $\{\mathrm{I}_2, -\mathrm{I}_2\} \cong \mathbb{Z}_2$, by the first isomorphism theorem $\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. □

[a] First, note that, given a versor $\hat{\mathbf{n}} \in \mathbb{R}^3$, it is always possible to find a rotation $\mathrm{U}_n \in \mathrm{SU}(2)$ such that $\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} = \mathrm{U}_n \sigma^3 \mathrm{U}_n^\dagger$, hence $\mathrm{P} = \mathrm{U}_a \exp(a\sigma^3)\mathrm{U}_a^\dagger$, thus WTS $\exp(a\sigma^3) = \operatorname{diag}(e^a, e^{-a})$ is a boost along the z-axis:

$$\begin{bmatrix} e^a & 0 \\ 0 & e^{-a} \end{bmatrix} \begin{pmatrix} v^0 - v^3 & -v^1 + \mathrm{i}v^2 \\ -v^1 - \mathrm{i}v^2 & v^0 + v^3 \end{pmatrix} \begin{bmatrix} e^a & 0 \\ 0 & e^{-a} \end{bmatrix} = \begin{pmatrix} e^{2a}(v^0 - v^3) & -v^1 + \mathrm{i}v^2 \\ -v^1 - \mathrm{i}v^2 & e^{-2a}(v^0 + v^3) \end{pmatrix}$$

i.e. $v^0 \pm v^3 \mapsto e^{\mp 2a}(v^0 \pm v^3)$ and $(v^1, v^2) \mapsto (v^1, v^2)$, which is equivalent to:

$$\begin{pmatrix} v^0 \\ v^3 \end{pmatrix} \mapsto \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{pmatrix} v^0 \\ v^3 \end{pmatrix}$$

which is exactly the Lorentz transformation realizing a boost along the z-axis.

As of Lemma 1.2.1, given $\Lambda \in \mathrm{SO}^+(1,3)$, it is possible to associate to it $\Lambda_s \in \mathrm{SL}(2,\mathbb{C})$ such that:

$$\Lambda^\mu{}_\nu \sigma^\nu = \Lambda_s \sigma^\mu \Lambda_s^\dagger \tag{1.48}$$

Recalling the proof of Prop. 1.2.2 $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$, hence:

$$\Lambda_s = \exp\left[(\mathbf{a} + \mathrm{i}\mathbf{b}) \cdot \frac{\boldsymbol{\sigma}}{2}\right] \quad \Longrightarrow \quad \Lambda_s^\dagger = \exp\left[(-\mathbf{a} + \mathrm{i}\mathbf{b}) \cdot \frac{\boldsymbol{\sigma}^\intercal}{2}\right]$$

where $\sigma^2 \boldsymbol{\sigma} \sigma^2 = -\boldsymbol{\sigma}^\intercal$ was used. Since $\exp(\mathrm{i}\alpha\sigma^k) = \mathrm{I}_2 \cos\alpha - \mathrm{i}\sigma^k \sin\alpha$, the exponential is periodic of 2π in $\mathbf{b}$, so identify $\mathbf{a} \equiv -\boldsymbol{\eta}$ and $\mathbf{b} \equiv -\boldsymbol{\theta}$, i.e. $\Lambda_s \equiv \Lambda_L$ and $\Lambda_s^\dagger = \sigma^2 \Lambda_R^\intercal \sigma^2$: it is then clear that the representation of $\mathrm{SO}^+(1,3)$ acting on 4-vectors is $\left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right)$, with[4]:

$$(\psi \otimes \xi) \equiv \psi \xi^\intercal \sigma^2 \quad \Longrightarrow \quad \Lambda(\psi \otimes \xi) = \Lambda_s (\psi \otimes \xi) \Lambda_s^\dagger = (\Lambda_L \psi) \otimes (\Lambda_R \xi)$$

Note that the restriction of $(\psi \otimes \xi)$ being Hermitian correctly reduces the real dimension of the basis space from 8 to 4 (i.e. the basis space from $\mathbb{C}^2 \otimes \mathbb{C}^2$ to $\mathcal{H}(2)$). Now, given the canonical isomorphism $V \otimes W \cong W \otimes V$ for generic $\mathbb{K}$-vector spaces, the following equivalence relation holds[5]:

$$\left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) \cong \left(\tfrac{\mathbf{1}}{\mathbf{2}} \otimes \mathbf{0}, \mathbf{0} \otimes \tfrac{\mathbf{1}}{\mathbf{2}}\right) \cong \left(\tfrac{\mathbf{1}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}\right)$$

where Clebsch–Gordan decomposition was used. This shows that $\left(\tfrac{\mathbf{1}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}\right)$ is the representation of $\mathrm{SO}^+(1,3)$ acting on 4-vectors.
Moreover, by Clebsch–Gordan decomposition, $\left(\tfrac{\mathbf{1}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) \cong \mathbf{0} \oplus \mathbf{1}$, which is exactly the representation acting on 4-vectors ($v^0 \in \mathbf{0}$ and $\mathbf{v} \in \mathbf{1}$).

Parity Note that $\mathcal{P}\mathbf{K} = -\mathbf{K}$, as the velocity of the boost gets reversed, while $\mathcal{P}\mathbf{J} = \mathbf{J}$: this means that $\mathcal{P}\mathbf{J}_\pm = \mathbf{J}_\mp$, i.e. parity exchanges a $(\mathbf{j}_-, \mathbf{j}_+)$ representation into a $(\mathbf{j}_+, \mathbf{j}_-)$ representation. Therefore, a $(\mathbf{j}_-, \mathbf{j}_+)$ representation of $\mathrm{SO}^+(1,3)$ is a basis for the representation of the parity transformation if and only if $j_- = j_+$.

Example 1.2.3 (Parity on spinors)

While Weyl spinors (separately) are not a basis for a representation of the parity operator, Dirac spinors are.

§1.2.1.4 Dirac algebra

The Dirac algebra is $\mathfrak{cl}_{1,3}(\mathbb{C}) \cong \mathfrak{cl}_{1,3}(\mathbb{R}) \otimes \mathbb{C}$ (complexification). This Clifford algebra admits a matrix representation via $\gamma^\mu \in \mathbb{C}^{4\times4}$ $\forall \mu = 0,1,2,3$ such that:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathrm{I}_4 \tag{1.49}$$

[5] To be precise, this is a representation equivalent to $\left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right)$, which would act on $[\psi \otimes \xi]_{(ij)} = \psi_i \xi_j$.

[5] In particular, a consequence is that, given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and given $\rho_1 : \mathfrak{g} \to V_1$, $\rho_2 : \mathfrak{g} \to V_2$, $\pi_1 : \mathfrak{h} \to W_1$ and $\pi_2 : \mathfrak{h} \to W_2$ their representations on $\mathbb{C}$-vector spaces, then $(\rho_1 \otimes \pi_1) \otimes (\rho_2 \otimes \pi_2) : (\mathfrak{g} \times \mathfrak{h}) \times (\mathfrak{g} \times \mathfrak{h}) \to \mathfrak{gl}((V_1 \otimes W_1) \otimes (V_2 \otimes W_2))$ and $(\rho_1 \otimes \rho_2) \otimes (\pi_1 \otimes \pi_2) : (\mathfrak{g} \times \mathfrak{g}) \times (\mathfrak{h} \times \mathfrak{h}) \to \mathfrak{gl}((V_1 \otimes V_2) \otimes (W_1 \otimes W_2))$ are equivalent.

The basis of this algebra is then given by:

I_4	1 matrix
γ^μ	4 matrices
$\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]}$	6 matrices
$\gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \mathrm{i}\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$	4 matrices
$\gamma^{\mu\nu\rho\sigma} \equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\gamma^5$	1 matrix

where $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ is an additional gamma matrix. Note that this matrix is not in the basis of $\mathfrak{cl}_{1,3}(\mathbb{C})$. It has the following properties:

$$(\gamma^5)^\dagger = \gamma^5 \qquad (\gamma^5)^2 = \mathrm{I}_4 \qquad \{\gamma^5, \gamma^\mu\} = 0$$

Therefore, the Dirac algebra is a 16-dimensional unital associative $\mathbb{C}$-algebra, and its generic component is:

$$\Gamma = a\mathrm{I}_4 + a_\mu\gamma^\mu + \frac{1}{2}a_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{3!}a_{\mu\nu\rho}\gamma^{\mu\nu\rho} + \frac{1}{4!}a_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma} \tag{1.50}$$

where each coefficient is totally anti-symmetric.
The quadratic subspace $\mathfrak{cl}^{(2)}_{1,3}(\mathbb{C})$ of the Dirac algebra is of particular interest. By convention, define its 6 generators as $\sigma^{\mu\nu} := \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$: it is straightforward to show that these generators satisfy Eq. 1.31, thus they give a 4-dimensional representation of $\mathfrak{so}^+(1,3)$.

Proposition 1.2.4 (Dirac algebra and Dirac representation)
The generators of $\mathfrak{cl}^{(2)}_{1,3}(\mathbb{C})$ also give the $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ representation of $\mathfrak{so}^+(1,3)$.

Moreover, $\mathfrak{cl}^{(2)}_{1,3}(\mathbb{C}) \cong \mathrm{Spin}(1,3)$ (see §A.3.1.1). Given a spinor $\Psi \in \mathbb{C}^4$, this spin group acts as:

$$\Psi \mapsto \Lambda_{\frac{1}{2}}\Psi = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right]\Psi$$

Using the property $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, its easy to see that $(\sigma^{\mu\nu})^\dagger = -\gamma^0\sigma^{\mu\nu}\gamma^0$, so, defining the **Dirac dual** $\bar{\Psi} := \Psi^\dagger\gamma^0$, this transforms via the inverse transformation:

$$\bar{\Psi} \mapsto \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}$$

Dirac bilinears The fact that Ψ and $\bar{\Psi}$ transform inversely allows to define objects with particular transformation relations.

Theorem 1.2.1 (Lorentz index of gamma matrices)
The gamma matrices transform under the $\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ representation of $\mathrm{SO}^+(1,3)$ as:

$$\Lambda^\mu{}_\nu\gamma^\nu = \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} \tag{1.51}$$

Proof. Recalling the explicit form of the generators in Eq. 1.30:

$$[J^{\rho\sigma}]^\mu{}_\nu\gamma^\nu = \mathrm{i}\left(\eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho\right) = \frac{\mathrm{i}}{2}\left(\gamma^\sigma\gamma^\rho\gamma^\mu + \gamma^\sigma\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\rho\gamma^\mu\gamma^\sigma\right)$$

As $[\gamma^\mu, \gamma^\nu] = \{\gamma^\mu, \gamma^\nu\} - 2\gamma^\nu\gamma^\mu = 2\left(\eta^{\mu\nu} - \gamma^\nu\gamma^\mu\right)$:

$$\begin{aligned}[\gamma^\mu, \sigma^{\rho\sigma}] &= \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] = \frac{\mathrm{i}}{4}\left(\gamma^\rho[\gamma^\mu, \gamma^\sigma] + [\gamma^\mu, \gamma^\rho]\gamma^\sigma - [\gamma^\mu, \gamma^\sigma]\gamma^\rho - \gamma^\sigma[\gamma^\mu, \gamma^\rho]\right) \\ &= \frac{\mathrm{i}}{2}\left(-\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\rho\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu\gamma^\rho + \gamma^\sigma\gamma^\rho\gamma^\mu\right) = [J^{\rho\sigma}]^\mu{}_\nu\gamma^\nu\end{aligned}$$

Expanding the thesis at first order:

$$\left[\mathrm{I}_4 - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right]^\mu_{\ \nu}\gamma^\nu = \left[\mathrm{I}_4 + \frac{\mathrm{i}}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right]\gamma^\mu\left[\mathrm{I}_4 - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right]$$

which becomes:

$$-[J^{\rho\sigma}]^\mu{}_\nu\gamma^\nu = \sigma^{\rho\sigma}\gamma^\mu - \gamma^\mu\sigma^{\rho\sigma}$$

This equation has been verified, so the thesis holds. □

This result means that, although they are matrices, the μ in γ^μ is effectively a Lorentz index, thus allowing to construct dot products with 4-vectors; in this way, gamma matrices transform 4-vectors into operators on spinors[6]: $A^\mu \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$, but $\not{A} \equiv \gamma^\mu A_\mu \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$.
A **Dirac bilinear** is an object $\bar{\Psi}\Gamma\Psi$, with $\Gamma \in \mathfrak{cl}_{1,3}(\mathbb{C})$. There are 5 basic bilinears:

$$\bar{\Psi}\Psi \qquad \bar{\Psi}\gamma^\mu\Psi \qquad \bar{\Psi}\gamma^5\Psi \qquad \bar{\Psi}\gamma^5\gamma^\mu\Psi \qquad \bar{\Psi}\sigma^{\mu\nu}\Psi$$

According to their transformation relations under the Lorentz group, these bilears are respectively called scalar, vector, pseudo-scalar, pseudo-vector and tensor bilinear.

§1.2.1.5 Dual and adjoint representations

Recall that the adjoint representation of a Lie group is a representation acting on the associated Lie algebra: therefore, the adjoint representation of $\mathrm{SO}^+(1,3)$ is 6-dimensional and, by Prop. 1.2.2, it can be decomposed as $\mathbf{6} \cong (\mathbf{1}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{1})$.

Dual representations Consider a generic anti-symmetric tensor $A^{\mu\nu}$: its **dual tensor** is defined as $A_\star^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}$. Every anti-symmetric tensor can then be decomposed as:

$$A_\pm^{\mu\nu} \equiv \frac{1}{2}\left(A^{\mu\nu} \pm \mathrm{i}A_\star^{\mu\nu}\right) \tag{1.52}$$

which are respectively called the **self-dual** and **anti-self-dual part** of $A^{\mu\nu}$.

Lemma 1.2.2 ((Anti)-self-dual tensor)
An (anti)-self-dual tensor satisfies:

$$A^{\mu\nu} = \pm\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} \tag{1.53}$$

[6] More precisely: spinors are defined as left-modules (even more precisely, as minimal left-ideals: see Def. A.3.3 and Def. A.3.4) over $\mathfrak{cl}_{1,3}(\mathbb{C})$, and $\not{A} = \gamma^\mu A_\mu \in \mathfrak{cl}_{1,3}(\mathbb{C})$ (specifically $\mathfrak{cl}_{1,3}^{(1)}(\mathbb{C})$, with notation from Eq. A.12).

Proof. Consider $A^{\mu\nu} = \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}$:

$$A_+^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\right) = \frac{1}{2}\left(\mathrm{i}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\right) = A^{\mu\nu} \qquad A_-^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} - \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\right) = 0$$

The other case is trivially verified. □

Anti-symmetric tensors belong to the **6** representation, but this is a reducible representation. Indeed, the subspaces of self-dual and anti-self-dual tensors are invariant subspaces:

$$A^{\mu\nu} = \pm\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} \quad \Longrightarrow \quad \pm\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\left(\Lambda_\rho{}^\alpha\Lambda_\sigma{}^\beta A_{\alpha\beta}\right) = -\frac{1}{4}\underbrace{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}}_{\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}}\Lambda_\rho{}^\alpha\Lambda_\sigma{}^\beta A^{\gamma\delta} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta A^{\alpha\beta}$$

Therefore, $\mathbf{6} = \mathbf{3}\oplus\bar{\mathbf{3}} \cong (\mathbf{1},\mathbf{0})\oplus(\mathbf{0},\mathbf{1})$.

Adjoint representation To see explicitly that the adjoint representation of $\mathrm{SO}^+(1,3)$ is exactly **6**, recall that the adjoint representation is $\rho : \mathrm{SO}^+(1,3) \to \mathrm{GL}(\mathfrak{so}^+(1,3))$ and it acts on $\mathfrak{so}^+(1,3)$ as $\rho(\Lambda)M = \Lambda M\Lambda^{-1}$, where $\Lambda \in \mathrm{SO}^+(1,3)$ and $M \in \mathfrak{so}^+(1,3)$. As $[\Lambda^{-1}]^\mu{}_\nu = \Lambda_\nu{}^\mu$ by Eq. 1.28:

$$M^{\mu\nu} \mapsto \Lambda^\mu{}_\alpha M^{\alpha\beta}[\Lambda^{-1}]_\beta{}^\nu = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta M^{\alpha\beta}$$

As $M \in \mathfrak{so}^+(1,3)$ is anti-symmetric, this is the transformation relation of an anti-symmetric tensor: the adjoint representation is the $\mathbf{6} \cong (\mathbf{1},\mathbf{0})\oplus(\mathbf{0},\mathbf{1})$.

Example 1.2.4 (Gauge theories)

In gauge theories (ex.: QED, QCD), the gauge potential A_μ transforms according to the fundamental representation $\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)$, while the field strength $F_{\mu\nu}$ according to the adjoint representation $(\mathbf{1},\mathbf{0})\oplus(\mathbf{0},\mathbf{1})$.

§1.2.1.6 Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields A scalar field transforms as:

$$\phi'(x') = \phi(x) \tag{1.54}$$

Consider an infinitesimal transformation $x'^\rho = x^\rho + \delta x^\rho$, with $\delta x^\rho = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^\rho{}_\sigma x^\sigma$ as of Eq. 1.28. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar representation of $\mathrm{SO}^+(1,3)$ is the trivial one ($J^{\mu\nu} = 0$).
However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$: while $\delta\phi$ studies only a single degree of freedom, as the point $\mathrm{P} \in \mathbb{R}^{1,3}$ is kept constant and only $\phi(\mathrm{P})$ can vary (i.e. the base space is one-dimensional), $\delta_0\phi$ studies $\phi(\mathrm{P})$ with P varying over $\mathbb{R}^{1,3}$, thus the base space is now a space of functions, which is infinite-dimensional. Therefore, $\delta\phi$ provides a finite-dimensional representation of the generators, while $\delta_0\phi$ an infinite-dimensional one.
To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho\partial_\rho\phi = \frac{\mathrm{i}}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^\rho{}_\sigma x^\sigma\partial_\rho\phi \equiv -\frac{\mathrm{i}}{2}\omega_{\mu\nu}L^{\mu\nu}\phi$$

Recalling Eq. 1.30, the generators can be expressed as:

$$L^{\mu\nu} := \mathrm{i}\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \tag{1.55}$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As $p^\mu = \mathrm{i}\partial^\mu$ (with signature $(+,-,-,-)$), it is clear that $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ is the orbital angular momentum.

Weyl fields A left-handed Weyl field transforms as:

$$\psi'_\mathrm{L}(x') = \Lambda_\mathrm{L}\psi_\mathrm{L}(x) \tag{1.56}$$

with Λ_L defined in Eq. 1.43, and similarly for right-handed Weyl fields. The infinite-dimensional representation of the Lorentz generators determined by Weyl spinors can be found as:

$$\begin{aligned}\delta_0\psi_\mathrm{L} &\equiv \psi'_\mathrm{L}(x) - \psi_\mathrm{L}(x) = \psi'_\mathrm{L}(x' - \delta x) - \psi_\mathrm{L}(x) \\ &= \psi'_\mathrm{L}(x') - \delta x^\rho \partial_\rho \psi_\mathrm{L}(x) - \psi_\mathrm{L}(x) = \left(\Lambda_\mathrm{L} - \mathrm{I}_2\right)\psi_\mathrm{L}(x) - \delta x^\rho \partial_\rho \psi_\mathrm{L}(x)\end{aligned}$$

The second term yields $L^{\mu\nu}$, while the first can be further elaborated by writing:

$$\Lambda_\mathrm{L} = e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{1.57}$$

Thus:

$$\delta_0\psi_\mathrm{L} = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_\mathrm{L}$$

where the angular momentum separates into the orbital and the spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1.58}$$

This separation is general: $L^{\mu\nu}$ is always expressed as in Eq. 1.55, while $S^{\mu\nu}$ depends on the specific representation. In the scalar representation $S^{\mu\nu} = 0$, while in the left/right-handed Weyl representation $S^{i0} = \pm\mathrm{i}\frac{\sigma^i}{2}$.

Dirac fields A Dirac field transforms as:

$$\Psi'(x') = \Lambda_\mathrm{D}\Psi(x) \tag{1.59}$$

where $\Lambda_\mathrm{D} = \mathrm{diag}(\Lambda_\mathrm{L}, \Lambda_\mathrm{R})$. The infinite-dimensional representation of the Lorentz generators determined by Dirac spinors is:

$$\delta_0\Psi \equiv \Psi'(x) - \Psi(x) = \Psi'(x') - \delta x^\rho \partial_\rho \Psi(x) - \Psi(x) = \left(\Lambda_\mathrm{D} - \mathrm{I}_4\right)\Psi(x) - \frac{\mathrm{i}}{2}\omega_{\mu\nu}L^{\mu\nu}\Psi(x)$$

The first term can be rewritten defining:

$$\Lambda_\mathrm{D} = e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}} \qquad S^{0i} \equiv -\frac{i}{2}\begin{bmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{bmatrix} \qquad S^{ij} = -\frac{1}{2}\epsilon^{ijk}\begin{bmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{bmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k$$

Therefore, a Dirac field transforms as:

$$\delta_0\Psi = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}J^{\mu\nu}\Psi \tag{1.60}$$

with clear orbital and spin angular components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1.61}$$

Vector fields A (contravariant) vector field transforms as:

$$V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} V^{\nu}(x) \tag{1.62}$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation.

§1.2.2 Poincaré group

Definition 1.2.3 (Poincaré group)
The **Poincaré group** is defined as $\mathrm{ISO}^+(1,3) \coloneqq \mathrm{T}^{1,3} \rtimes \mathrm{SO}^+(1,3)$, where $\mathrm{T}^{1,3} \cong \mathbb{R}^{1,3}$ is the group of translations of $\mathbb{R}^{1,3}$, with multiplication $(a^{\mu}, \Lambda^{\mu}{}_{\nu}) \cdot (\bar{a}^{\mu}, \bar{\Lambda}^{\mu}{}_{\nu}) \equiv (a^{\mu} + \Lambda^{\mu}{}_{\nu}\bar{a}^{\nu}, \Lambda^{\mu}{}_{\rho}\bar{\Lambda}^{\rho}{}_{\nu})$.

Given a translation $x^{\mu} \mapsto x^{\mu} + a^{\mu}$, the associated group element can be written as:

$$T = e^{-\mathrm{i}a_{\mu}P^{\mu}} \tag{1.63}$$

where P^{μ} is the 4-momentum operator. Clearly translations commute, and so do their generators; on the other hand, as $\mathbf{P}$ is a vector under rotations, while P^0 (energy) a scalar, one has:

$$[J^i, P^j] = \mathrm{i}e^{ijk}P^k \qquad\qquad [J^i, P^0] = 0$$

These equations uniquely determine the **Poincaré algebra** $\mathfrak{iso}^+(1,3)$:

$$\begin{aligned} [P^{\mu}, P^{\nu}] &= 0 \\ [J^{\mu\nu}, J^{\sigma\rho}] &= \mathrm{i}\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \\ [P^{\mu}, J^{\rho\sigma}] &= \mathrm{i}\left(\eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho}\right) \end{aligned} \tag{1.64}$$

It's easy to check that $[K^i, P^0] = \mathrm{i}P^i$, while $[J^i, P^0] = [P^i, P^0] = 0$: given that P^0 generates time translations, linear and angular momentum are conserved quantities, while $\mathbf{K}$ is not.

§1.2.2.1 Field representations

Fields provide an infinite-dimensional representation of the Lorentz group as $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $S^{\mu\nu}$ does not depend on x^{μ}, but only on the spin of the field.
To represent P^{μ} on fields, their transformation law must be specified: all fields are required to be scalars under translations, independently of their spin. This means that, given a generic field $\phi(x)$, under a translation $x' = x + a$ it transforms as $\phi'(x') = \phi(x)$, so, under an infinitesimal translation $x' = x + \varepsilon$:

$$\begin{aligned} \delta_0\phi \equiv \phi'(x) - \phi(x) = \phi'(x' - \varepsilon) - \phi(x) &= -\varepsilon^{\mu}\partial_{\mu}\phi(x) \\ = e^{-\mathrm{i}(-\varepsilon_{\mu})P^{\mu}}\phi'(x') - \phi(x) &= \left(e^{\mathrm{i}\varepsilon_{\mu}P^{\mu}} - \mathrm{I}\right)\phi(x) = \mathrm{i}\varepsilon_{\mu}P^{\mu}\phi(x) \end{aligned}$$

It is clear then that:

$$P^{\mu} = +\mathrm{i}\partial^{\mu} \tag{1.65}$$

Explicitly, $P^0 = \mathrm{i}\partial_t$ and $\mathbf{P} = -\mathrm{i}\boldsymbol{\nabla}$. It is trivial to check that these generators obey the Poincaré algebra.

§1.2.2.2 Particle representations

The Poincaré group can also be represented using the Hilbert space $\mathscr{H}$ of a free particle as a basis. Denoting a generic state as $|\mathbf{p}, s\rangle \in \mathscr{H}$, where $\mathbf{p}$ is the particle's momentum and s collectively labels all other quantum numbers, it is clear that $\mathscr{H}$ is infinite-dimensional, as $\mathbf{p}$ is a continuous unbounded variable.

Theorem 1.2.2 (Wigner's theorem)

On the Hilbert space of physical states, any symmetry transformation can be represented by a linear and unitary or anti-linear and anti-unitary operator.

By this theorem, Poincaré transformations can be represented by unitary matrices, i.e. $\mathbf{J}$, $\mathbf{K}$, $\mathbf{P}$ and P^0 can be represented by hermitian operators. These representations can be labeled by Casimir operators, which for $\mathrm{ISO}^+(1,3)$ are easily found as $P_\mu P^\mu$ and $W_\mu W^\mu$, where W^μ is the **Pauli-Lubański operator**:

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}J_{\nu\sigma}P_\rho \tag{1.66}$$

On single-particle states $P_\mu P^\mu = m^2$, while $W_\mu W^\mu$ can be conveniently computed in a particular frame (due to Lorentz invariance). If $m \neq 0$, this frame is the rest-frame of the particle:

$$W^\mu = -\frac{m}{2}\epsilon^{\mu\nu\sigma 0}J_{\nu\sigma} = \frac{m}{2}\delta^{\mu i}\epsilon^{ijk}J^{jk} = \delta^{\mu i} m J^i$$

Therefore, on single-particle states of mass m and spin j, the Casimir operator takes the form:

$$W_\mu W^\mu = -m^2 j\,(j+1) \tag{1.67}$$

If $m = 0$, the rest-frame does not exist, but it is possible to choose a frame where $P^\mu = (\omega, 0, 0, \omega)$, where $W^0 = W^3 = \omega J^3$, $W^1 = \omega\,(J^1 - K^2)$ and $W^2 = \omega\,(J^2 + K^1)$, so that:

$$W_\mu W^\mu = -\omega^2\left[\left(K^2 - J^1\right)^2 + \left(K^1 + J^2\right)^2\right] \tag{1.68}$$

It is clear that the $m \to 0$ limit is not trivial, and massive and massless representation need to be studied separately.

Massive representations Restricting to $m \in \mathbb{R}^+$ ($m^2 < 0$ states, called tachyons, are excluded), the massive representations are labeled by mass m and spin j: in fact, after a Lorentz transformation such that $P^\mu = (m, \mathbf{0})$, spatial rotations can still be performed, i.e. the subspace of single-particle states with momentum $P^\mu = (m, \mathbf{0})$ is still a basis for the representation of $\mathrm{SU}(2)$ (as spinors must be included too). The group of transformations which leaves invariant a certain choice of P^μ is called the **little group**, so $\mathrm{SU}(2)$ is the little group of massive single-particle states: massive representations are labelled by m and j, which means that massive particles of spin j have $2j+1$ degrees of freedom.

Massless representations The little group for $P^\mu = (\omega, 0, 0, \omega)$ clearly is $\mathrm{SO}(2)$, the group of rotations in the (x, y) plane generated by J^3: as for any abelian group, its irreducible representations are one-dimensional, and they are labeled by the eigenvalue h of J^3, which represents the angular momentum in the direction of propagation of the particle and is called **helicity**. Helicity can be shown to be quantized as $h \in \frac{1}{2}\mathbb{Z}_0$ (by topologic considerations on $\mathrm{ISO}^+(1,3) \equiv \mathbb{R}^4 \times \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$).

As a consequence, massless particles only have one degree of freedom and are characterized by their helicity h. As $\mathrm{SO}(2) \cong \mathrm{U}(1)$, on a state of helicity h the little group is represented as:

$$U(\theta) = e^{-\mathrm{i}h\theta} \tag{1.69}$$

Although massless particles with opposite helicities are logically two different species of particles, it can be written as $h = \hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$ (unit vectors), so h is a pseudoscalar such that $h \mapsto -h$ under parity: this means that, if the interaction conserves parity, h and $-h$ must be symmetric.

Example 1.2.5 (Gauge bosons)

The electromagnetic and gravitational interactions conserve parity, thus photons and gravitons must be a basis for the representation of both $\mathrm{ISO}^{+}(1,3)$ and parity: photons can have $h = \pm 1$ (left- and right-handed), while gravitons have $h = \pm 2$.

Example 1.2.6 (Neutrinos)

Neutrinos only interact via the weak interaction, which does not conserve parity, and in fact the two states $h = \pm\frac{1}{2}$ are different particles: neutrinos have $h = -\frac{1}{2}$, while antineutrinos have $h = +\frac{1}{2}$.

§1.3 Classical equations of motion

Consider a **local field theory** of fields $\{\phi_i(x)\}_{i\in\mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkowski spacetime. Its Lagrangian takes the form:

$$L = \int \mathrm{d}^3x \, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.70}$$

where $\mathcal{L}$ is the **Lagrangian density** of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int \mathrm{d}t \, L = \int \mathrm{d}^4x \, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.71}$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows dropping all boundary terms.

Theorem 1.3.1 (Euler-Lagrange equations)

The **stationary action principle** $\delta\mathcal{S} = 0$ determines the classical equations of motion:

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0 \tag{1.72}$$

Proof. Varying Eq. 1.71:

$$\delta\mathcal{S} = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi_i)\right] = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi_i = 0$$

□

Corollary 1.3.1.1

Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.

Proof. This is a consequence of Stokes theorem:

$$\int_\Sigma \mathrm{d}^4x \, \partial_\mu K^\mu = \int_{\partial\Sigma} \mathrm{d}A \, n_\mu K^\mu$$

with n_μ normal vector to the $\partial\Sigma$ hypersurface. □

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)} \tag{1.73}$$

$$\mathcal{H} = \sum_{i\in\mathcal{I}} \Pi_i(x)\partial_0\phi(x) - \mathcal{L} \tag{1.74}$$

Unlike the Hamiltonian formalism, the Lagrangian formalism keeps Lorentz covariance explicit.

§1.3.1 Noether's theorem

Definition 1.3.1 (Infinitesimal transformation)

Given a field theory with fields $\{\phi_i\}_{i=1,\dots,k}$ and action $\mathcal{S}[\phi]$, an **infinitesimal transformation** parametrized by $\{\varepsilon^a\}_{a=1,\dots,N} : |\varepsilon^a| \ll 1$ is defined by two sets of functions $\{A^\mu_a(x)\}_{a=1,\dots,N}$ and $\{F_{i,a}(\phi, \partial\phi)\}_{i=1,\dots,k;\, a=1,\dots,N}$ such that:

$$\begin{aligned} x^\mu \mapsto x'^\mu &= x^\mu + \varepsilon^a A^\mu_a(x) \\ \phi_i(x) \mapsto \phi'_i(x) &= \phi_i(x) + \varepsilon^a F_{i,a}(\phi, \partial\phi) \end{aligned} \tag{1.75}$$

Definition 1.3.2 (Symmetry transformation)

An infinitesimal transformation is a **symmetry transformation** if it leaves $\mathcal{S}[\phi]$ invariant, regardless of ϕ being a solution of the equations of motion. It can further be classified as:

- **global symmetry**, if $\varepsilon^a \equiv \text{const.}$;
- **local symmetry**, if $\varepsilon^a = \varepsilon^a(x)$.

Symmetry transformations which leave spacetime unchanged, i.e. with $A^\mu_a(x) = 0$, are called **internal symmetries**.

Theorem 1.3.2 (Noether's theorem)

Given a global (but not local) symmetry parametrized by N generators, then there are N conserved currents $\{j^\mu_a(\phi)\}_{a=1,\dots,N}$ such that:

$$\partial_\mu j^\mu_a(\phi^{\text{cl}}) = 0 \tag{1.76}$$

where ϕ^{cl} is a classical solution of the equations of motion.

Proof. First, consider an infinitesimal transformation with slowly-varying parameters, that is $l\,|\partial_\mu \varepsilon^a| \ll |\varepsilon^a|$ (l characteristic scale of the field theory): being it not a local symmetry, $\delta\mathcal{S} \neq 0$ at $o(\varepsilon)$ and:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x\ [\varepsilon^a(x) K_a(\phi) - (\partial_\mu \varepsilon^a(x))\, j^\mu_a(\phi) + o(\partial\partial\varepsilon)] + o(\varepsilon^2)$$

If $\varepsilon^a \equiv \text{const.}$ (global symmetry) then $\delta\mathcal{S}[\phi] = 0\,\forall\phi$, therefore $K_a(\phi) = 0\,\forall\phi$ (independent of ε). Assuming $\varepsilon^a(x) \to 0$ sufficiently fast as $x \to \infty$, then integration by parts yields:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x\, \varepsilon^a(x) \partial_\mu j^\mu_a(\phi) + o(\partial\partial\varepsilon) + o(\varepsilon^2)$$

This expression is independent of the choice of ϕ. Moreover, note that Eq. 1.75 can be rewritten as an internal transformation by setting:

$$\phi_i(x) \mapsto \phi'_i(x) = \phi_i(x - \varepsilon^a A_a) + \varepsilon^a F_{i,a} = \phi_i(x) + \varepsilon^a F_{i,a} - \varepsilon^a A^\mu_a \partial_\mu \phi_i \equiv \phi_i(x) + \delta\phi_i(x)$$

$\delta\phi_i(x)$ vanishes at infinity, therefore it is the kind of variation used to derive the equations of motion: choosing $\phi \equiv \phi^{\text{cl}}$ classical solution then implies $\delta\mathcal{S} = 0$ independently of ε, i.e. the thesis. □

These are often called **Noether currents**, and the associated **Noether charges** are defined as:

$$Q_a := \int \mathrm{d}^3x\, j_a^0(t, \mathbf{x}) \tag{1.77}$$

These are time-independent, as $\partial_0 Q_a = \int \mathrm{d}^3x\, \partial_0 j_a^0 = -\int \mathrm{d}^3x\, \partial_i j_a^i$: on all spacetime it vanishes by divergence theorem (fields vanish at infinity), but on a finite volume it yields a boundary term interpreted as the incoming and outgoing flux.

Proposition 1.3.1 (Noether currents)

The explicit expression of Noether currents is:

$$j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left[A_a^\nu(x)\partial_\nu \phi_i - F_{i,a}(\phi, \partial\phi)\right] - A_a^\mu(x)\mathcal{L} \tag{1.78}$$

Proof. Varying the action at $o(\partial\varepsilon)$:

$$\delta_\varepsilon \mathcal{S} = \delta_\varepsilon \int \mathrm{d}^4x\, \mathcal{L} = \int \left[\delta_\varepsilon(\mathrm{d}^4x)\mathcal{L} + \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta_\varepsilon \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta_\varepsilon(\partial_\mu \phi_i)\right)\right]$$

The Jacobian of Eq. 1.75 gives $\mathrm{d}^4x \mapsto \mathrm{d}^4x\,(1 + A_a^\mu \partial_\mu \varepsilon^a) + o(\varepsilon)$, while $\delta_\varepsilon \phi_i$ is not $o(\partial\varepsilon)$ and:

$$\delta_\varepsilon(\partial_\mu \phi_i) = \frac{\partial \phi'_i}{\partial x'^\mu} - \frac{\partial \phi_i}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\left(\phi_i + \varepsilon^a F_{i,a}\right) - \frac{\partial \phi_i}{\partial x^\mu} = -(\partial_\mu \varepsilon^a)\left(A_a^\nu \partial_\nu \phi_i - F_{i,a}\right) + o(\varepsilon)$$

The thesis follows from $\delta_\varepsilon \mathcal{S} = -\int \mathrm{d}^4x\, (\partial_\mu \varepsilon^a) j_a^\mu + o(\partial\partial\varepsilon) + o(\varepsilon^2)$. □

If the considered infinitesimal transformation is not a global symmetry, then $\delta_\varepsilon \mathcal{S}$ has a non-vanishing $o(\varepsilon)$ term which gives rise to a quasi-conserved current:

$$\partial_\mu j_\mu^a = -(\delta_a \mathcal{L})_{\text{global}} \tag{1.79}$$

§1.3.1.1 Energy-momentum tensor

Consider spacetime translations: as all fields must be scalars under these transformations, they define a Noether current. In particular, translations have $A_\nu^\mu = \delta_\nu^\mu$ and $F_{i,\mu} = 0$ (the parameter index is a Lorentz index), so the conserved current is the **energy-momentum tensor** $j_\nu^\mu \equiv \theta^\mu{}_\nu$:

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i - \eta^{\mu\nu}\mathcal{L} \tag{1.80}$$

which is covariantly conserved on classical solutions of the equations of motion. The conserved Noether charge associated to the energy-momentum tensor is the **four-momentum**:

$$P^\mu := \int \mathrm{d}^3x\, \theta^{0\mu} \tag{1.81}$$

The energy-momentum tensor so defined is not symmetric, however it can be made to via a tensor $A^{\rho\mu\nu}$ which is anti-symmetric w.r.t. (ρ, μ): $T^{\mu\nu} \equiv \theta^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$ is physically equivalent from $\theta^{\mu\nu}$, as the second term is a vanishing spatial divergence in the definition of P^μ and is contracted to 0 in the conservation law.

§1.3.2 Scalar fields

§1.3.2.1 Real scalar fields

Consider a real scalar field ϕ: a non-trivial Poincaré-invariant action must contain $\partial_\mu\phi$ and must saturate each Lorentz index. For example:

$$\mathcal{S}[\phi] = \frac{1}{2}\int \mathrm{d}^4x \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \tag{1.82}$$

The resulting equation of motion is the **Klein-Gordon equation**:

$$\left(\Box + m^2\right)\phi = 0 \tag{1.83}$$

where $\Box \equiv \partial_\mu\partial^\mu$. A plane wave $e^{\pm \mathrm{i}p_\mu x^\mu}$ is a solution if $p^2 = m^2$, so the KG equation imposes the relativistic dispersion relation and m can be interpreted as the mass. As ϕ must be real, the general solution is a superposition of waves:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p}e^{-\mathrm{i}p_\mu x^\mu} + a^*_\mathbf{p}e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{1.84}$$

The positive energy solution has $E_\mathbf{p} = +\sqrt{\mathbf{p}^2+m^2}$, but it contains both **positive** and **negative frequency modes** $e^{\mp\mathrm{i}p_\mu x^\mu}$, while the $(2E_\mathbf{p})^{-1/2}$ factor is a convenient normalization of the $a_\mathbf{p}$ coefficients. The overall normalization of $\mathcal{S}[\phi]$ does not influence the equations of motion, however it is important for obtaining a positive-definite Hamiltonian. As the momentum conjugate to ϕ is $\Pi_\phi = \partial_0\phi$, the Hamiltonian density is found as:

$$\mathcal{H} = \frac{1}{2}\left[\Pi_\phi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right] \tag{1.85}$$

The energy-momentum tensor is computed to be:

$$\theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{1.86}$$

It is trivial to see that $\theta^{00} = \mathcal{H}$: the Hamiltonian is the conserved charge related to the invariance under time translations.
To compute the conserved currents associated to Lorentz invariance, it is convenient to label the transformation parameters $\omega^{\mu\nu}$ by an anti-symmetric pair of Lorentz indices, so that Eq. 1.75 become:

$$x^\mu \mapsto x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}\left(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho\right) \equiv x^\mu + \frac{1}{2}A^\mu_{(\rho\sigma)}\omega^{\rho\sigma}$$

As $F_{i,a} = 0$, from Eq. 1.86 the conserved currents are:

$$j^{(\rho\sigma)\mu} = x^\rho\theta^{\mu\sigma} - x^\sigma\theta^{\mu\rho} \tag{1.87}$$

For spatial rotations, the conserved charge is:

$$M^{ij} = \int \mathrm{d}^3x\left(x^i\theta^{0j} - x^j\theta^{0i}\right) = \int \mathrm{d}^3x\,\partial_0\phi\left(x^i\partial^j - x^j\partial^i\right)\phi = \frac{\mathrm{i}}{2}\int \mathrm{d}^3x\left[\phi L^{ij}(\partial_0\phi) - (\partial_0\phi)L^{ij}\phi\right]$$

where integration by parts was carried and L^{ij} is defined by Eq. 1.55. This can be generalized.

Definition 1.3.3 (Scalar product)

Given two real scalar fields ϕ_1 and ϕ_2, their **scalar product** is defined as:

$$\langle\phi_1|\phi_2\rangle := \frac{\mathrm{i}}{2}\int \mathrm{d}^3x\, \phi_1 \overleftrightarrow{\partial_0} \phi_2 \tag{1.88}$$

where $f\overleftrightarrow{\partial_\mu}g := f\partial_\mu g - (\partial_\mu f)g$.

Proposition 1.3.2

If ϕ_1 and ϕ_2 are KG solutions, then $\langle\phi_1|\phi_2\rangle$ is time-independent.

Proof. By the KG equation:

$$\partial_0\left[\phi_1\partial_0\phi_2 - (\partial_0\phi_1)\phi_2\right] = \phi_1\partial_0^2\phi_2 - (\partial_0^2\phi_1)\phi_2 = \phi_1\triangle\phi_2 - (\triangle\phi_1)\phi_2$$

which vanishes after integration by parts. □

Note that this scalar product is not positive-definite.

Theorem 1.3.3 (Conserved charges)

Given a symmetry represented by a Lie group and a representation $L^{\mu\nu}$ of its generators as operators acting on fields, the value of the associated conserved charges on a solution ϕ of the equations of motion is:

$$M^{\mu\nu} = \langle\phi|L^{\mu\nu}|\phi\rangle \tag{1.89}$$

Example 1.3.1 (Four-momentum)

Applying Th. 1.3.3 to four-momentum $P^\mu = \langle\phi|\mathrm{i}\partial^\mu|\phi\rangle$; for example, the $\mu = 0$ component is:

$$\begin{aligned} P^0 &= \langle\phi|\mathrm{i}\partial^0|\phi\rangle = \langle\phi|\mathrm{i}\partial_0|\phi\rangle = \frac{\mathrm{i}}{2}\int \mathrm{d}^3x\left[\phi(\mathrm{i}\partial_0)\partial_0\phi - (\partial_0\phi)\mathrm{i}\partial_0\phi\right] \\ &= \frac{1}{2}\int \mathrm{d}^3x\left[-\phi\partial_0^2\phi + (\partial_0\phi)^2\right] = \frac{1}{2}\int \mathrm{d}^3x\left[-\phi\left(\triangle - m^2\right)\phi + (\partial_0\phi)^2\right] \\ \text{(int. by parts)} &= \frac{1}{2}\int \mathrm{d}^3x\left[(\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 + (\partial_0\phi)^2\right] = \frac{1}{2}\int \mathrm{d}^3x\,\theta^{00} \end{aligned}$$

The free KG action can be generalized to a self-interacting real scalar field introducing a general potential:

$$\mathcal{S}[\phi] = \int \mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \tag{1.90}$$

The quadratic term in $V(\phi)$ is the mass term, while higher-order terms describe the self-interaction.

§1.3.2.2 Complex scalar fields

Consider now two real scalar fields ϕ_1, ϕ_2 with the same mass m and combine them into a single complex scalar field $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2)$. The KG action is the sum of the single actions and may be written as:

$$\mathcal{S}[\phi] = \int \mathrm{d}^4x \left(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\right) \tag{1.91}$$

Considering ϕ and ϕ^* as independent variables one obtains the KG equation, which yields the same mode expansion as Eq. 1.84:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \left[a_\mathbf{p}e^{-\mathrm{i}p_\mu x^\mu} + b^*_\mathbf{p}e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{1.92}$$

Now $a_\mathbf{p}$ and $b_\mathbf{p}$ are independent, since there's no reality condition on ϕ.

Electric charge An interesting property of the complex KG field is the existence of a global U(1) symmetry of the action: $\mathcal{S}[\phi] \mapsto \mathcal{S}[\phi]$ under $\phi(x) \mapsto e^{\mathrm{i}\theta}\phi(x)$. The associated Noether current can be computed from Eq. 1.78, using $\phi_i = (\phi, \phi^*)$, $A^\nu_a = 0$ and $F_{i,a} = (\mathrm{i}, -\mathrm{i})$:

$$j_\mu = -\mathrm{i}\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) = \mathrm{i}\phi^*\overleftrightarrow{\partial_\mu}\phi \tag{1.93}$$

The conserved U(1) charge is then $Q_{\mathrm{U}(1)} = \mathrm{i}\int \mathrm{d}^3x\,\phi^*\overleftrightarrow{\partial_0}\phi = \langle\phi|\phi\rangle$, which is consistent with Th. 1.3.3 as the generator of U(1) is the identity operator.

§1.3.3 Spinor fields

§1.3.3.1 Weyl fields

Consider the theory of a left/right-handed Weyl field: recalling that $\psi^\dagger_\mathrm{L}\sigma^\mu\psi_\mathrm{L}$ and $\psi_\mathrm{R}\bar\sigma^\mu\psi_\mathrm{R}$ are 4-vectors, with $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ and $\bar\sigma^\mu \equiv (1, -\boldsymbol{\sigma})$, the kinetic term of the Lorentz-invariant Lagrangian can be written as:

$$\mathcal{L}_\mathrm{L} = \mathrm{i}\psi^\dagger_\mathrm{L}\bar\sigma^\mu\partial_\mu\psi_\mathrm{L} \qquad \mathcal{L}_\mathrm{R} = \mathrm{i}\psi^\dagger_\mathrm{R}\sigma^\mu\partial_\mu\psi_\mathrm{R} \tag{1.94}$$

The i factor ensures the reality of the Lagrangian, as σ matrices are hermitian. The Lagrangian does not depend on $\partial_\mu\psi^*$, thus the Euler-Lagrange equations are:

$$(\partial_0 - \sigma^i\partial_i)\psi_\mathrm{L} = 0 \qquad (\partial_0 + \sigma^i\partial_i)\psi_\mathrm{R} = 0 \tag{1.95}$$

These are known as **Weyl equations**: by $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\epsilon^{ijk}\sigma^k$, these equations imply two massless KG equations (assuming regular functions, so that $\partial_i\partial_j = \partial_j\partial_i$). Considering positive-energy plane-wave solutions of the form $\psi_\mathrm{L}(x) = u_\mathrm{L}\exp(-\mathrm{i}p_\mu x^\mu) = u_\mathrm{L}\exp(-\mathrm{i}Et + \mathrm{i}\mathbf{p}\cdot\mathbf{x})$, where u_L is a constant spinor, Weyl equations become:

$$\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E}u_\mathrm{L} = -u_\mathrm{L} \qquad \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E}u_\mathrm{R} = u_\mathrm{R}$$

As these are massless fields $E = |\mathbf{p}|$, and since for $s = \frac{1}{2}$ fields $\mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$ these equations become:

$$(\hat{\mathbf{p}}\cdot\mathbf{J})u_\mathrm{L} = -\frac{1}{2}u_\mathrm{L} \qquad (\hat{\mathbf{p}}\cdot\mathbf{J})u_\mathrm{R} = \frac{1}{2}u_\mathrm{R}$$

These show that left/right-handed Weyl massless Weyl spinors have helicity $h = \mp\frac{1}{2}$, consistent with the fact that massless particles are helicity eigenstates.

The energy-momentum tensor can be computed from Eq. 1.80, noting that on classical solutions (by Weyl equations) the Lagrangian of the theory vanishes:

$$\theta^{\mu\nu} = \mathrm{i}\psi_\mathrm{L}^\dagger \bar{\sigma}^\mu \partial^\nu \psi_\mathrm{L} \qquad \theta^{\mu\nu} = \mathrm{i}\psi_\mathrm{R}^\dagger \sigma^\mu \partial^\nu \psi_\mathrm{R} \tag{1.96}$$

Moreover, note that the Lagrangian is invariant under a global U(1) internal transformation with Noether currents and conserved charges:

$$j^\mu = \psi_\mathrm{L}^\dagger \bar{\sigma}^\mu \psi_\mathrm{L} \qquad j^\mu = \psi_\mathrm{R}^\dagger \sigma^\mu \psi_\mathrm{R}$$
$$Q_{\mathrm{U}(1)} = \int \mathrm{d}^3x\, \psi_\mathrm{L}^\dagger \psi_\mathrm{L} \qquad Q_{\mathrm{U}(1)} = \int \mathrm{d}^3x\, \psi_\mathrm{R}^\dagger \psi_\mathrm{R}$$

Weyl Lagrangians are not invariant under parity, as $\psi_\mathrm{L} \leftrightarrow \psi_\mathrm{R}$.

Example 1.3.2 (Neutrinos)

Neutrinos are divided into three leptonic families: ν_e, ν_μ and ν_τ. Although they are massive $s = \frac{1}{2}$, in most contexts their mass can be neglected, thus they can be described by Weyl spinors: in particular, neutrinos by left-handed Weyl spinors and antineutrinos by right-handed Weyl spinors.

§1.3.3.2 Dirac fields

Consider a theory with both a left-handed and a right-handed Weyl spinor: one can construct two new Lorentz scalars, $\psi_\mathrm{L}^\dagger \psi_\mathrm{R}$ and $\psi_\mathrm{R}^\dagger \psi_\mathrm{L}$, as from Eq. 1.43-1.44 it is easy to check that $\Lambda_\mathrm{L}^\dagger \Lambda_\mathrm{R} = \Lambda_\mathrm{R}^\dagger \Lambda_\mathrm{L} = \mathbb{1}$. Two real combinations are $\psi_\mathrm{L}^\dagger \psi_\mathrm{R} + \psi_\mathrm{R}^\dagger \psi_\mathrm{L}$ and $\mathrm{i}(\psi_\mathrm{L}^\dagger \psi_\mathrm{R} - \psi_\mathrm{R}^\dagger \psi_\mathrm{L})$: under parity $\psi_\mathrm{L} \leftrightarrow \psi_\mathrm{R}$, so the former is a scalar and the latter a pseudoscalar. The Dirac Lagrangian then is:

$$\mathcal{L}_\mathrm{D} = \mathrm{i}\psi_\mathrm{L}^\dagger \bar{\sigma}^\mu \partial_\mu \psi_\mathrm{L} + \mathrm{i}\psi_\mathrm{R}^\dagger \sigma^\mu \partial_\mu \psi_\mathrm{R} - m(\psi_\mathrm{L}^\dagger \psi_\mathrm{R} + \psi_\mathrm{R}^\dagger \psi_\mathrm{L}) \tag{1.97}$$

This Lagrangian is invariant under parity, as $\bar{\sigma}^\mu \partial_\mu \leftrightarrow \sigma^\mu \partial_\mu$ ($\partial_i \mapsto -\partial_i$). Considering ψ_i and ψ_i^* independent, the variation w.r.t. the latter yields:

$$\mathrm{i}\bar{\sigma}^\mu \partial_\mu \psi_\mathrm{L} = m\psi_\mathrm{R} \qquad \mathrm{i}\sigma^\mu \partial_\mu \psi_\mathrm{R} = m\psi_\mathrm{L} \tag{1.98}$$

which is the **Dirac equation** in terms of Weyl spinors.

Proposition 1.3.3

The Dirac equation implies two massive Klein-Gordon equations.

Proof. Applying $\mathrm{i}\sigma^\mu \partial_\mu$ to the first equation:

$$-\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu \psi_\mathrm{L} = m\mathrm{i}\sigma^\mu \partial_\mu \psi_\mathrm{R} = m^2 \psi_\mathrm{L}$$

Assuming that Ψ_L satisfies the hypotheses of Schwarz's Lemma, then $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, and $\sigma^\mu \bar{\sigma}^\nu$ can be replaced by $\frac{1}{2}(\sigma^\nu \bar{\sigma}^\mu + \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2}(2\eta^{\mu\nu}) = \eta^{\mu\nu}$, yielding the thesis. □

The m parameter can thus be interpreted as a mass, and now the two spinors are no longer helicity eigenstates. It is convenient to rewrite this theory in terms of a Dirac spinor, which defines the chiral representation:

$$\Psi = \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix}$$

The chiral representation of the Dirac algebra $\mathfrak{cl}_{1,3}(\mathbb{C})$ is defined as[7]:

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$$

This allows to rewrite Dirac equation as:

$$(\mathrm{i}\not{\partial} - m)\Psi = 0 \tag{1.100}$$

In the chiral representation, the Dirac adjoint simply is $\bar{\Psi} = (\psi_\mathrm{R}^\dagger, \psi_\mathrm{L}^\dagger)$, so the Dirac Lagrangian can be rewritten as:

$$\mathcal{L}_\mathrm{D} = \bar{\Psi}(\mathrm{i}\not{\partial} - m)\Psi \tag{1.101}$$

The γ^5 matrix in the chiral representation allows to define the projectors on left- and right-handed Weyl spinors:

$$\gamma^5 \equiv \begin{bmatrix} -\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{bmatrix} \qquad \Longrightarrow \qquad \frac{1-\gamma^5}{2}\Psi = \begin{pmatrix} \psi_\mathrm{L} \\ 0 \end{pmatrix} \qquad \frac{1+\gamma^5}{2}\Psi = \begin{pmatrix} 0 \\ \psi_\mathrm{R} \end{pmatrix} \tag{1.102}$$

Proposition 1.3.4 (Unitary equivalence)

Given a constant $\mathrm{U} \in \mathbb{C}^{2\times 2} : \mathrm{U}^\dagger = \mathrm{U}^{-1}$, then under $\Psi \mapsto \mathrm{U}\Psi$ the Lagrangian is invariant for $\gamma^\mu \mapsto \mathrm{U}\gamma^\mu\mathrm{U}^\dagger$.

Proof. Inserting $\Psi = \mathrm{U}^\dagger\Psi'$ (and $\bar{\Psi} = \Psi'^\dagger\mathrm{U}\gamma^0$) in the Dirac Lagrangian:

$$\mathcal{L}_\mathrm{D} = \Psi'^\dagger\mathrm{U}\gamma^0(\mathrm{i}\gamma^\mu\partial_\mu - m)\mathrm{U}^\dagger\Psi' = \Psi'^\dagger\mathrm{U}\gamma^0\mathrm{U}^\dagger(\mathrm{i}\mathrm{U}\gamma^\mu\mathrm{U}^\dagger\partial_\mu - m)\Psi' = \bar{\Psi}'(\mathrm{i}\gamma'^\mu\partial_\mu - m)\Psi'$$

where $\bar{\Psi}' \equiv \Psi'^\dagger\gamma'^0$. The Lagrangian is thus unchanged. □

It can be shown that the Dirac algebra is invariant under the transformation in Prop. 1.3.4, thus it defines an equivalent representation of the algebra.

Proposition 1.3.5 (Lorentz invariance)

The Dirac equation is Lorentz invariant.

Proof. Recalling Eq. 1.51:

$$\begin{aligned}
(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\Psi(x) &\mapsto \left(\mathrm{i}\gamma^\mu[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\left(\mathrm{i}\gamma^\mu[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left(\mathrm{i}\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left(\mathrm{i}\Lambda^\mu{}_\sigma\gamma^\sigma[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left(\mathrm{i}\gamma^\nu\partial_\nu - m\right)\Psi(\Lambda^{-1}x)
\end{aligned}$$

[7]In terms of tensor products, the chiral representation of the Dirac algebra can be written as:

$$\gamma^0 = \sigma^1 \otimes \mathrm{I}_2 \qquad \boldsymbol{\gamma} = \mathrm{i}\sigma^2 \otimes \boldsymbol{\sigma} \qquad \gamma^5 = -\sigma^3 \otimes \mathrm{I}_2 \tag{1.99}$$

Since 0 is a Lorentz scalar, this transformation law implies that $\left(\mathrm{i}\not{\partial} - m\right)\Psi(y) = 0$, where $y = \Lambda^{-1}x$ are the coordinates in the new reference frame. This shows that the Dirac equation is Lorentz invariant. □

Solutions The general solution to the Dirac equation is a superposition of plane waves, both with positive and negative-frequency modes respectively written as $\Psi(x) = u(p)e^{-\mathrm{i}p_\mu x^\mu}$ and $\Psi(x) = v(p)e^{\mathrm{i}p_\mu x^\mu}$, where $u(p)$ and $v(p)$ are Dirac spinors which, in the chiral representation, have both a left-handed and a right-handed Weyl spinor component. The Dirac equation than becomes:

$$\left(\not{p} - m\right)u(p) = 0 \qquad \left(\not{p} + m\right)v(p) = 0 \tag{1.103}$$

Considering the massive positive-frequency solution in the rest frame, i.e. $p^\mu = (m, \mathbf{0})$, one finds $(\gamma^0 - 1)u(p) = 0$, which yields $u_\mathrm{L} = u_\mathrm{R}$: the KG equation imposes the mass-shell condition $p^2 = m^2$, but the Dirac equation, being a first-order equation, halves the number of independent degrees of freedom. A convenient normalization is:

$$u(p_0) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

where $\xi : \xi^\dagger\xi = 1$ is a two-component spinor which gives the spin orientation of the solution: $\xi = \binom{1}{0}$ for the spin-up solution and $\xi = \binom{0}{1}$ for the spin-down one.
The solution in a generic frame is obtained via a boost of the rest-frame solution, using transformation properties of left/right-handed spinors.

Proposition 1.3.6

The solution to the Dirac equation in a generic frame where $\mathbf{p} \parallel \mathbf{e}_z$ is given by:

$$u(p) = \begin{pmatrix}\begin{bmatrix}\sqrt{E-p^3} & 0\\ 0 & \sqrt{E+p^3}\end{bmatrix}\xi\\ \begin{bmatrix}\sqrt{E+p^3} & 0\\ 0 & \sqrt{E-p^3}\end{bmatrix}\xi\end{pmatrix} \tag{1.104}$$

Proof. Consider a boost along $\mathbf{e}_z$, parametrized by rapidity η, by Eq. 1.35:

$$\begin{pmatrix}E\\ p^3\end{pmatrix} = \exp\left(\eta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\begin{pmatrix}m\\ 0\end{pmatrix} = \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cosh\eta + \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\sinh\eta\right)\begin{pmatrix}m\\ 0\end{pmatrix} = \begin{pmatrix}m\cosh\eta\\ m\sinh\eta\end{pmatrix}$$

Thus, $E + p^3 = e^\eta m$ and $E - p^3 = e^{-\eta}m$. By Eqq. 1.43-1.44, then:

$$\begin{aligned}
u(p) &= \exp\left(-\frac{\eta}{2}\begin{bmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{bmatrix}\right)\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} = \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cosh\frac{\eta}{2} - \begin{bmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{bmatrix}\sinh\frac{\eta}{2}\right)\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{bmatrix}e^{\eta/2}\,\frac{1-\sigma^3}{2} + e^{-\eta/2}\,\frac{1+\sigma^3}{2} & 0\\ 0 & e^{\eta/2}\,\frac{1+\sigma^3}{2} + e^{-\eta/2}\,\frac{1-\sigma^3}{2}\end{bmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{bmatrix}\sqrt{E+p^3}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + \sqrt{E-p^3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & 0\\ 0 & \sqrt{E+p^3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \sqrt{E-p^3}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\end{bmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}
\end{aligned}$$

which is the thesis. □

This expression can be generalized for an arbitrary direction of $\mathbf{p}$:

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi^s \end{pmatrix} \tag{1.105}$$

where the square root of a matrix is understood as taking the positive square root of its eigenvalues, and the polarization of ξ^s is made explicit.
It is convenient to work with specific spinors ξ: a useful choice are eigenvectors of σ^3, in particular $\xi^1 = \binom{1}{0}$ (spin-up) and $\xi^2 = \binom{0}{1}$ (spin-down). In the ultra-relativistic (or massless) limit $p^\mu \to (E, 0, 0, E)$, so u^1 has only the right-handed component and u^2 only the left-handed one.
The negative-frequency solution is equivalent:

$$v^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \eta^s \end{pmatrix} \tag{1.106}$$

where the spin states are the same: $\eta^1 = \binom{1}{0}$ and $\eta^2 = \binom{0}{1}$.
Recalling the Dirac adjoint $\bar{u}^s(p) = u^{s\dagger}(p)\gamma^0, \bar{v}^s(p) = v^{s\dagger}(p)\gamma^0$ and the normalization choice $\xi^{r\dagger}\xi^s = \delta^{rs}, \eta^{r\dagger}\eta^s = \delta^{rs}$, several properties follow:

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \qquad \bar{v}^r(p)v^s(p) = -2m\delta^{rs} \tag{1.107}$$

$$u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs} \qquad v^{r\dagger}(p)v^s(p) = 2E_{\mathbf{p}}\delta^{rs} \tag{1.108}$$

$$\bar{u}^r(p)v^s(p) = 0 \qquad \bar{v}^r(p)u^s(p) = 0 \tag{1.109}$$

$$\bar{u}^r(p)\gamma^\mu u^s(p) = 2p^\mu\delta^{rs} \qquad \bar{v}^r(p)\gamma^\mu v^s(p) = 2p^\mu\delta^{rs} \tag{1.110}$$

Note that $\bar{u}u \in \mathbb{C}$, while $u\bar{u} \in \mathbb{C}^{4\times4}$.

Proposition 1.3.7 (Spinor sums)
The sum over possible polarizations of a fermion yields:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \tag{1.111}$$

Proof. Using $\sum_{s=1,2} \xi^s \xi^{s\dagger} = \mathrm{I}_2$:

$$\begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \sum_s \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p_\mu \bar{\sigma}^\mu} & \xi^{s\dagger}\sqrt{p_\mu \sigma^\mu} \end{pmatrix} \\ &= \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\sqrt{p_\mu \bar{\sigma}^\mu} & \sqrt{p_\mu \sigma^\mu}\sqrt{p_\mu \sigma^\mu} \\ \sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\mu \bar{\sigma}^\mu} & \sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\mu \sigma^\mu} \end{bmatrix} = \begin{bmatrix} m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & m \end{bmatrix} \end{aligned}$$

where the identity $(p_\mu \sigma^\mu)(p_\mu \bar{\sigma}^\mu) = p^2 = m^2$ was used. □

Chiral symmetry Consider the massless Dirac Lagrangian. In this case, the action has two global internal symmetries:

$$\psi_{\mathrm{L}} \mapsto e^{\mathrm{i}\theta_{\mathrm{L}}}\psi_{\mathrm{L}} \qquad \psi_{\mathrm{R}} \mapsto e^{\mathrm{i}\theta_{\mathrm{R}}}\psi_{\mathrm{R}}$$

Therefore, this theory is symmetric under $\mathrm{U}(1) \times \mathrm{U}(1)$. These can be rewritten as two distinct transformations of the Dirac spinor:

$$\Psi \mapsto e^{\mathrm{i}\alpha}\Psi \tag{1.112}$$

$$\Psi \mapsto e^{\mathrm{i}\beta\gamma^5}\Psi \tag{1.113}$$

where $\alpha \equiv \theta_{\mathrm{L}} = \theta_{\mathrm{R}}$ and $\beta \equiv -\theta_{\mathrm{L}} = \theta_{\mathrm{R}}$ respectively (to prove that the latter is a symmetry, from $\{\gamma^5, \gamma^\mu\} = 0$ it follows that $\gamma^\mu e^{\mathrm{i}\beta\gamma^5} = e^{-\mathrm{i}\beta\gamma^5}\gamma^\mu$). The former symmetry is called the vector $\mathrm{U}(1)_{\mathrm{V}}$ symmetry, with conserved Noether current the **vector current** $j^\mu_{\mathrm{V}} := \bar{\Psi}\gamma^\mu\Psi$, while the latter is called the axial $\mathrm{U}(1)_{\mathrm{A}}$ symmetry, with conserved **axial current** $j^\mu_{\mathrm{A}} := \bar{\Psi}\gamma^\mu\gamma^5\Psi$.
While the vector symmetry is always a symmetry of the Dirac field, the axial symmetry is only for the massless Dirac field, as:

$$\partial_\mu j^\mu_{\mathrm{A}} = 2\mathrm{i}m\bar{\Psi}\gamma^5\Psi \tag{1.114}$$

Chapter 2

Canonical Quantization

§2.1 Scalar fields

As for the quantization of a classical system in Quantum Mechanics, the quantization of a scalar field theory is performed promoting $\phi(t, \mathbf{x})$ and $\Pi(t, \mathbf{x})$ to hermitian operators in the Heisenberg picture and imposing the canonical equal-time commutation relation:

$$[\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = \mathrm{i}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{2.1}$$

while of course $[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = 0$.

§2.1.1 Real scalar fields

A real scalar field is promoted to a real hermitian operator. In particular, by Eq. 1.84:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-\mathrm{i}p_\mu x^\mu} + a_{\mathbf{p}}^\dagger e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0 = E_{\mathbf{p}}} \tag{2.2}$$

In terms of creation and annihilation operators, the commutator Eq. 2.1 reads:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{2.3}$$

while $[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0$. These can be regarded as the creation and annihilation operators of a collection of harmonic oscillators, one for each value of the momentum $\mathbf{p}$: the **Fock space** of the real scalar field can thus be constructed analogously to the Hilbert space of the harmonic oscillator.
Defining the **vacuum state** $|0\rangle : a_{\mathbf{p}} |0\rangle = 0 \ \forall \mathbf{p}$, suitably normalized as $\langle 0|0\rangle = 1$, the generic state of the Fock space is:

$$|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = \sqrt{2E_{\mathbf{p}_1}} \ldots \sqrt{2E_{\mathbf{p}_n}}\, a_{\mathbf{p}_1}^\dagger \ldots a_{\mathbf{p}_n}^\dagger |0\rangle \tag{2.4}$$

Proposition 2.1.1 (Normalization)

For one-particle states:

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = 2E_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \tag{2.5}$$

Proof. By Eq. 2.3, $\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = \sqrt{2E_{\mathbf{p}_1}}\sqrt{2E_{\mathbf{p}_2}} \langle 0| a_{\mathbf{p}_1} a_{\mathbf{p}_2}^\dagger |0\rangle$, but $\langle 0| a_{\mathbf{p}_1} a_{\mathbf{p}_2}^\dagger |0\rangle = \langle 0| [a_{\mathbf{p}_1}, a_{\mathbf{p}_2}^\dagger] |0\rangle$, hence $\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = 2E_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)$. □

Lemma 2.1.1

The combination $E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ is Lorentz invariant.

Proof. Recall that:

$$\delta(f(x)-f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0) \tag{2.6}$$

A Lorentz boost along $\mathbf{e}_i$ yields $p_i' = \gamma(p_i + \beta E)$ and $E' = \gamma(E+\beta p_i)$, so:

$$\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\frac{\mathrm{d}p_i'}{\mathrm{d}p_i} = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\gamma\left(1+\beta\frac{\mathrm{d}E}{\mathrm{d}p_i}\right)$$

Note that $p^2 = E^2 - \mathbf{p}^2$ is a Lorentz invariant, thus $E\mathrm{d}E = p_i\mathrm{d}p_i$, so:

$$\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\frac{\gamma(E+\beta p_i)}{E} = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\frac{E'}{E}$$

Clearly, then, $E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ is Lorentz invariant. □

This explains the choice of normalization.

Proposition 2.1.2 (KG Hamiltonian)

The Hamiltonian of a real scalar field can be written as:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]\right) \tag{2.7}$$

Proof. First of all, from Eq. 2.2:

$$\begin{aligned}\Pi(t,\mathbf{x}) = \partial_0\phi(t,\mathbf{x}) &= -\mathrm{i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu} - a_{\mathbf{p}}^\dagger e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}} \\ &= -\mathrm{i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} - a_{-\mathbf{p}}^\dagger e^{\mathrm{i}E_{\mathbf{p}}t}\right]e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\end{aligned}$$

$$\boldsymbol{\nabla}\phi(t,\mathbf{x}) = \mathrm{i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{\mathrm{i}E_{\mathbf{p}}t}\right]e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$$

Inserting these expressions in Eq. 1.85:

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\Bigg\{&-\frac{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}{2}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} - a_{-\mathbf{p}}^\dagger e^{\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-\mathrm{i}E_{\mathbf{q}}t} - a_{-\mathbf{q}}^\dagger e^{\mathrm{i}E_{\mathbf{q}}t}\right] + \\ &+\frac{-\mathbf{p}\cdot\mathbf{q}+m^2}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-\mathrm{i}E_{\mathbf{q}}t} + a_{-\mathbf{q}}^\dagger e^{\mathrm{i}E_{\mathbf{q}}t}\right]\Bigg\}e^{\mathrm{i}(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\end{aligned}$$

Recall the identity:

$$\int\frac{\mathrm{d}^3x}{(2\pi)^3}e^{\mathrm{i}(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.8}$$

Then (using $E_{-\mathbf{p}} = E_{\mathbf{p}}$):

$$\begin{aligned}
H &= \int \mathrm{d}^3x\, \mathcal{H} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \mathrm{d}^3q \Bigg\{ -\frac{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}{2}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} - a^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-\mathrm{i}E_{\mathbf{q}}t} - a^\dagger_{-\mathbf{q}}e^{\mathrm{i}E_{\mathbf{q}}t}\right] + \\
&\quad + \frac{-\mathbf{p}\cdot\mathbf{q}+m^2}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-\mathrm{i}E_{\mathbf{q}}t} + a^\dagger_{-\mathbf{q}}e^{\mathrm{i}E_{\mathbf{q}}t}\right]\Bigg\}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Bigg\{ -\frac{E_{\mathbf{p}}}{2}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} - a^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{-\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} - a^\dagger_{\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right] + \\
&\quad + \frac{\mathbf{p}^2+m^2}{2E_{\mathbf{p}}}\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{-\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t} + a^\dagger_{\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\Bigg\} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\Bigg\{ -\left[a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2\mathrm{i}E_{\mathbf{p}}t} - a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} - a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}}e^{2\mathrm{i}E_{\mathbf{p}}t}\right] \\
&\quad + \left[a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2\mathrm{i}E_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}}e^{2\mathrm{i}E_{\mathbf{p}}t}\right]\Bigg\} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left[a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right] = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right)
\end{aligned}$$

□

The second term in the Hamiltonian Eq. 2.7 is the sum of the zero-point energy of all oscillators and is proportional to $(2\pi)^3\delta^{(3)}(0) \to V$, thus:

$$E_{\text{vac}} = \frac{V}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}$$

This energy shows two divergences: the one coming from the infinite-volume limit (i.e. small momentum), regularized introducing an **infrared cutoff** in the form of a finite volume, and the one from the ultra-relativistic limit (i.e. large momentum), regularized introducing an **ultraviolet cutoff** in the form of a maximum momentum Λ. These divergences are retained in the expression for E_{vac}, as $E_{\text{vac}} \sim V$ and $E_{\text{vac}} \sim \Lambda^4$, but can be ignored (when ignoring gravity) since experiments are only sensitive to energy differences.
Discarding the zero-point energy, the Hamiltonian becomes:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}a^\dagger_{\mathbf{p}}a_{\mathbf{p}} \equiv \mathfrak{N}\frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right) \tag{2.9}$$

where the **normal ordering operator** $\mathfrak{N}$ was introduced, which acts by moving all creation operators to the left and all annihilation operators to the right (ex.: $\mathfrak{N}\{a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\} = a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$). It is now straightforward to compute the energy of a generic state in the Fock space, as $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ is just a number operator:

$$H\left|\mathbf{p}_1,\dots,\mathbf{p}_n\right\rangle = (E_{\mathbf{p}_1} + \dots + E_{\mathbf{p}_n})\left|\mathbf{p}_1,\dots,\mathbf{p}_n\right\rangle \tag{2.10}$$

Computing the spatial momentum from Eq. 1.81 as $P^i = \mathfrak{N}\int \mathrm{d}^3x\,\theta^{0i} = \int \mathrm{d}^3x\,\mathfrak{N}\{\partial_0\phi\partial^i\phi\}$:

$$P^i = \int \frac{\mathrm{d}^3p}{(2\pi)^3}p^i a^\dagger_{\mathbf{p}}a_{\mathbf{p}} \tag{2.11}$$

Therefore, the state $a^\dagger_{\mathbf{p}}|0\rangle$ can correctly be interpreted as a one-particle state with momentum $\mathbf{p}$, mass m and energy $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. The generic state in the Fock space is a multiparticle state, with total energy and momentum the sum of the individual energies and momenta.
Finally, note that creation operators commute between themselves, hence multiparticle states are symmetric under exchange of pairs of particles, i.e. they obey the Bose-Einstein statistics: this agrees with the fact that quanta of a scalar field have no intrinsic spin. i.e. are spin-0 particles.

§2.1.1.1 Ladder operators

Recalling Def. 1.3.3, it is possible to relate the Fourier modes through the KG inner product, defined as:

$$\langle f, g\rangle := \mathrm{i}\int \mathrm{d}^3x\, f^*(x)\overleftrightarrow{\partial_0}g(x) = \mathrm{i}\int \mathrm{d}^3x\, [f^*(x)\partial_0 g(x) - \partial_0 f^*(x) g(x)] \equiv \mathrm{i}\int \mathrm{d}^3x\, W_0[f^*, g](x) \tag{2.12}$$

where $W_0[f^*, g](x)$ is the t-**Wrońskian**[1] of $f^*(x), g(x)$. Although $W_0[f^*, g](x)$ is time-dependent, by Prop. 1.3.2, if $f^*(x), g(x)$ are solutions of the KG equation, then $\langle f^*, g\rangle$ is time-independent.

Proposition 2.1.3 (Orthonormality of Fourier modes)
Given positive- and negative-frequency Fourier modes:

$$f_{\mathbf{p}}(x) \equiv \frac{e^{-\mathrm{i}p_\mu x^\mu}}{\sqrt{(2\pi)^3 2E_{\mathbf{p}}}} \qquad f^*_{\mathbf{p}}(x) \equiv \frac{e^{\mathrm{i}p_\mu x^\mu}}{\sqrt{(2\pi)^3 2E_{\mathbf{p}}}} \tag{2.14}$$

then:

$$\langle f_{\mathbf{p}}, f_{\mathbf{q}}\rangle = -\langle f^*_{\mathbf{p}}, f^*_{\mathbf{q}}\rangle = \delta^{(3)}(\mathbf{p}-\mathbf{q}) \qquad \langle f_{\mathbf{p}}, f^*_{\mathbf{q}}\rangle = 0 \tag{2.15}$$

Proof. By direct calculation (using Eq. 2.8):

$$\begin{aligned}\langle f_{\mathbf{p}}, f_{\mathbf{q}}\rangle &= \mathrm{i}\int \mathrm{d}^3x\, f^*_{\mathbf{p}}(x)\overleftrightarrow{\partial_0} f_{\mathbf{q}}(x) = \mathrm{i}\int \frac{\mathrm{d}^3x}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} e^{\mathrm{i}(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}\left(e^{\mathrm{i}E_{\mathbf{p}}t}\overleftrightarrow{\partial}_0 e^{-\mathrm{i}E_{\mathbf{q}}t}\right)\\ &= \mathrm{i}\frac{\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}(-\mathrm{i}E_{\mathbf{q}} - \mathrm{i}E_{\mathbf{p}})e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t} = \delta^{(3)}(\mathbf{p}-\mathbf{q})\end{aligned}$$

as $\delta(x-x_0)f(x) = \delta(x-x_0)f(x_0)$. Other products are equivalent. □

Using the KG inner product it is possible to explicit the ladder operators.

[1] Given $\{f_k(x)\}_{k=1,\dots,N} \subset \mathcal{C}^N(I,\mathbb{C})$, with $I \subset \mathbb{R}^n$ and $N \in \mathbb{N}$, their (i^{th}-) *Wrońskian* is $W_i \in \mathcal{C}(I,\mathbb{C})$ defined as:

$$W[f_1,\dots,f_N](x) := \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_N(x)\\ \partial_i f_1(x) & \partial_i f_2(x) & \dots & \partial_i f_N(x)\\ \vdots & \vdots & \ddots & \vdots\\ \partial_i^{N-1} f_1(x) & \partial_i^{N-1} f_2(x) & \dots & \partial_i^{N-1} f_N(x)\end{vmatrix} \tag{2.13}$$

If $\{f_k(x)\}_{k=1,\dots,N}$ are linearly dependent, so are the columns of the Wrońskian (as ∂_i is a linear operator), hence it vanishes: to show that a set of differentiable functions is linearly independent on a given set, it suffices to show that their Wrońskian does not vanish identically on said set (although it may vanish on a zero-measure subset).

Theorem 2.1.1 (Ladder operators)
The ladder operators for a real scalar field are:

$$\sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}} = \mathrm{i}\int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}\phi(x) \qquad \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}} = -\mathrm{i}\int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}\phi(x) \tag{2.16}$$

Proof. Eq. 2.2 can be cast in the form:

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3p\, [f_{\mathbf{p}}(x)a_{\mathbf{p}} + f^*_{\mathbf{p}}(x)a^\dagger_{\mathbf{p}}] \quad\Longrightarrow\quad \langle f_{\mathbf{p}}, \phi\rangle = \frac{a_{\mathbf{p}}}{\sqrt{(2\pi)^3}}$$

by Prop. 2.1.3. Then, inserting Eq. 2.14 yields the thesis ($a^\dagger_{\mathbf{p}}$ is analogous). □

§2.1.2 Complex scalar fields

When considering a complex scalar field, Eq. 1.92 becomes:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu} + b^\dagger_{\mathbf{p}}e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}} \tag{2.17}$$

$$\phi^\dagger(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left[a^\dagger_{\mathbf{p}}e^{\mathrm{i}p_\mu x^\mu} + b_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}} \tag{2.18}$$

Now there are two independent sets of creation/annihilation operators, which obey the canonical commutation relation:

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.19}$$

with all other commutators vanishing. The Fock space is constructed by defining a vacuum state $|0\rangle : a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}}|0\rangle = 0$ and then acting repeatedly with both creation operators. With normal ordering, one finds:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right) \tag{2.20}$$

$$P^i = \int \frac{\mathrm{d}^3p}{(2\pi)^3}p^i\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right) \tag{2.21}$$

The quanta of a complex scalar field are given by two different species of particles with the same mass.

Proposition 2.1.4 ($\mathrm{U}(1)$ charge)
The $\mathrm{U}(1)$ charge of the quantized complex scalar field is:

$$Q_{\mathrm{U}(1)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right) \tag{2.22}$$

Proof. By Eq. 1.93:

$$
\begin{aligned}
Q_{\mathrm{U}(1)} = \mathrm{i}\int \mathrm{d}^3x\,\phi^\dagger\overleftrightarrow{\partial_0}\phi = \mathrm{i}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\times \\
\times\Big\{\left[a^\dagger_{\mathbf{q}}e^{\mathrm{i}q_\mu x^\mu}+b_{\mathbf{q}}e^{-\mathrm{i}q_\mu x^\mu}\right]\partial_0\left(a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu}+b^\dagger_{\mathbf{p}}e^{\mathrm{i}p_\mu x^\mu}\right)+ \\
-\partial_0\left(a^\dagger_{\mathbf{q}}e^{\mathrm{i}q_\mu x^\mu}+b_{\mathbf{q}}e^{-\mathrm{i}q_\mu x^\mu}\right)\left[a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu}+b^\dagger_{\mathbf{p}}e^{\mathrm{i}p_\mu x^\mu}\right]\Big\} \\
=\int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\times \\
\times\Big\{E_{\mathbf{p}}\left[a^\dagger_{\mathbf{q}}e^{\mathrm{i}q_\mu x^\mu}+b_{\mathbf{q}}e^{-\mathrm{i}q_\mu x^\mu}\right]\left[a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu}-b^\dagger_{\mathbf{p}}e^{\mathrm{i}p_\mu x^\mu}\right]+ \\
+E_{\mathbf{q}}\left[a^\dagger_{\mathbf{q}}e^{\mathrm{i}q_\mu x^\mu}-b_{\mathbf{q}}e^{-\mathrm{i}q_\mu x^\mu}\right]\left[a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu}+b^\dagger_{\mathbf{p}}e^{\mathrm{i}p_\mu x^\mu}\right]\Big\} \\
=\int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\times \\
\times\Big\{E_{\mathbf{p}}\left[a^\dagger_{\mathbf{q}}e^{\mathrm{i}E_{\mathbf{q}}t}+b_{-\mathbf{q}}e^{-\mathrm{i}E_{\mathbf{q}}t}\right]\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t}-b^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]+ \\
+E_{\mathbf{q}}\left[a^\dagger_{\mathbf{q}}e^{\mathrm{i}E_{\mathbf{q}}t}-b_{-\mathbf{q}}e^{-\mathrm{i}E_{\mathbf{q}}t}\right]\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t}+b^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\Big\}e^{\mathrm{i}(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
=\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\Big\{\left[a^\dagger_{\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}+b_{-\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t}-b^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]+ \\
+\left[a^\dagger_{\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}-b_{-\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t}\right]\left[a_{\mathbf{p}}e^{-\mathrm{i}E_{\mathbf{p}}t}+b^\dagger_{-\mathbf{p}}e^{\mathrm{i}E_{\mathbf{p}}t}\right]\Big\} \\
=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[a^\dagger_{\mathbf{p}}a_{\mathbf{p}}-b_{-\mathbf{p}}b^\dagger_{-\mathbf{p}}\right]=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}}-b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right)
\end{aligned}
$$

Applying normal ordering yields the thesis. □

While normal ordering was justified when considering the Hamiltonian on the grounds that the vacuum energy is unobservable, a charged vacuum would have observable effects; however when promoting ϕ to a quantum operator, the expression $\phi^\dagger\overleftrightarrow{\partial_0}\phi$ presents an ordering ambiguity (ex.: $\phi^\dagger(\partial_0\phi)$ or $(\partial_0\phi)\phi^\dagger$), which is removed requiring the charge of the vacuum to vanish.

Being $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}b_{\mathbf{p}}$ number operators, the U(1) charge is equal to the number of quanta created by $a^\dagger_{\mathbf{p}}$ minus the number of quanta created by $b^\dagger_{\mathbf{p}}$, integrated over all momenta: in particular, $a^\dagger_{\mathbf{p}}|0\rangle$ and $b^\dagger_{\mathbf{p}}|0\rangle$ are both spin-zero particles of mass m and momentum $\mathbf{p}$, but they respectively have charges $Q_{\mathrm{U}(1)}=+1$ and $Q_{\mathrm{U}(1)}=-1$. This allows to properly interpret the negative-energy solutions of the KG equations: they are positive-energy particles with opposite U(1) charge and are called **antiparticles**.

For a real scalar field, the reality condition reads $a_{\mathbf{p}}=b_{\mathbf{p}}$, thus it describes a field whose particle is its own antiparticle, and it is symmetric under any U(1) symmetry.

§2.2 Spinor fields

A principle of QFT is the **spin-statistic theorem**: integer-spin fields are to be quantized imposing equal-time commutation relations, while half-integer-spin with equal-time anticommutation relations.

§2.2.1 Dirac fields

From the Dirac Lagrangian Eq. 1.101, the conjugate momentum to the Dirac field Ψ is computed as:

$$\Pi_\Psi = \mathrm{i}\bar{\Psi}\gamma^0 = \mathrm{i}\Psi^\dagger \tag{2.23}$$

Imposing the canonical anticommutation relation, according to the spin-statistic theorem:

$$\{\Psi_a(t,\mathbf{x}), \Psi_b^\dagger(t,\mathbf{y})\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab} \tag{2.24}$$

where $a, b = 1, 2, 3, 4$ are Dirac indices. Expanding the free Dirac field in plane waves:

$$\Psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{\mathbf{p},s} u^s(p) e^{-\mathrm{i}p_\mu x^\mu} + b^\dagger_{\mathbf{p},s} v^s(p) e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{2.25}$$

$$\bar{\Psi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a^\dagger_{\mathbf{p},s} \bar{u}^s(p) e^{\mathrm{i}p_\mu x^\mu} + b_{\mathbf{p},s} \bar{v}^s(p) e^{-\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{2.26}$$

where the spinor wave functions $u^s(p), v^s(p)$ are given by Eq. 1.105-1.106. Translating Eq. 2.24 in terms of creation/annihilation operators:

$$\{a_{\mathbf{p},s}, a^\dagger_{\mathbf{q},r}\} = \{b_{\mathbf{p},s}, b^\dagger_{\mathbf{q},r}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{sr} \tag{2.27}$$

The Fock space is again constructed defining a vacuum state $|0\rangle : a_{\mathbf{p},s}|0\rangle = b_{\mathbf{p},s}|0\rangle = 0$ and then acting repeatedly on it with $a^\dagger_{\mathbf{p}.s}, b^\dagger_{\mathbf{p},s}$:

$$\begin{aligned}
&|(\mathbf{p}_1, s_1), \dots, (\mathbf{p}_n, s_n); (\mathbf{q}_1, r_1), \dots, (\mathbf{q}_m, r_m)\rangle \\
&\qquad = \sqrt{2E_{\mathbf{p}_1}} \dots \sqrt{2E_{\mathbf{p}_n}} \sqrt{2E_{\mathbf{q}_1}} \dots \sqrt{2E_{\mathbf{q}_m}} a^\dagger_{\mathbf{p}_1,s_1} \dots a^\dagger_{\mathbf{p}_n,s_n} b^\dagger_{\mathbf{q}_1,r_1} \dots b^\dagger_{\mathbf{q}_m,r_m} |0\rangle
\end{aligned}$$

As these operators anticommute, states in this Fock space are antisymmetric under the exchange of particles, therefore spin-$\frac{1}{2}$ obey the Fermi-Dirac statistics (as of the spin-statistic theorem).

Proposition 2.2.1 (Dirac Hamiltonian)
The Hamiltoniana for a Dirac field Ψ is:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{s=1,2} E_\mathbf{p} \left(a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right) \tag{2.28}$$

Proof. By Eq. 2.23, the Hamiltonian density is:

$$\mathcal{H} = \Pi_\Psi \partial_0 \Psi - \mathcal{L}_\mathrm{D} = \mathrm{i}\Psi^\dagger \partial_0 \Psi - \bar{\Psi}\left(\mathrm{i}\gamma^0\partial_0 + \mathrm{i}\gamma^i\partial_i - m\right)\Psi = \bar{\Psi}\left(-\mathrm{i}\gamma^i\partial_i + m\right)\Psi$$

Therefore, using Eq. 2.25-2.26 and Eq. 2.8:

$$H = \int \mathrm{d}^3x\,\bar{\Psi}\left(-\mathrm{i}\gamma^i\partial_i + m\right)\Psi = \int \mathrm{d}^3x\,\bar{\Psi}\left(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right)\Psi$$

$$= \mathfrak{N}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s}\bar{u}^s(p)e^{\mathrm{i}p_\mu x^\mu} + b_{\mathbf{p},s}\bar{v}^s(p)e^{-\mathrm{i}p_\mu x^\mu}\right] \times$$
$$\times\left(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right)\left[a_{\mathbf{q},r}u^r(q)e^{-\mathrm{i}q_\mu x^\mu} + b^\dagger_{\mathbf{q},r}v^r(q)e^{\mathrm{i}q_\mu x^\mu}\right]$$

$$= \mathfrak{N}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s}\bar{u}^s(p)e^{\mathrm{i}p_\mu x^\mu} + b_{\mathbf{p},s}\bar{v}^s(p)e^{-\mathrm{i}p_\mu x^\mu}\right] \times$$
$$\times\left[\left(\boldsymbol{\gamma}\cdot\mathbf{q} + m\right)a_{\mathbf{q},r}u^r(q)e^{-\mathrm{i}q_\mu x^\mu} + \left(-\boldsymbol{\gamma}\cdot\mathbf{q} + m\right)b^\dagger_{\mathbf{q},r}v^r(q)e^{\mathrm{i}q_\mu x^\mu}\right]$$

Using the Dirac equation in the form $(\not{p} - m)u(p) = (\not{p} + m)v(p) = 0$:

$$\left(\boldsymbol{\gamma}\cdot\mathbf{q} + m\right)u(q) = \gamma^0 E_{\mathbf{q}}u(q) \qquad \left(-\boldsymbol{\gamma}\cdot\mathbf{q} + m\right)v(q) = -\gamma^0 E_{\mathbf{q}}v(q)$$

Therefore, omitting the constraint $p^0 = E_{\mathbf{p}}$ in the spinors' arguments and using Eq. 2.8:

$$H = \mathfrak{N}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s}\bar{u}^s(\mathbf{p})e^{\mathrm{i}E_{\mathbf{p}}t} + b_{-\mathbf{p},s}\bar{v}^s(-\mathbf{p})e^{-\mathrm{i}E_{\mathbf{p}}t}\right] \times$$
$$\times\,\gamma^0 E_{\mathbf{q}}\left[a_{\mathbf{q},r}u^r(\mathbf{q})e^{-iE_{\mathbf{q}}t} - b^\dagger_{-\mathbf{q},r}v^r(-\mathbf{q})e^{\mathrm{i}E_{\mathbf{q}}t}\right]e^{\mathrm{i}(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}$$

$$= \mathfrak{N}\int \frac{\mathrm{d}^3p}{2(2\pi)^3} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s}\bar{u}^s(\mathbf{p})e^{\mathrm{i}E_{\mathbf{p}}t} + b_{-\mathbf{p},s}\bar{v}^s(-\mathbf{p})e^{-\mathrm{i}E_{\mathbf{p}}t}\right] \times$$
$$\times\,\gamma^0\left[a_{\mathbf{p},r}u^r(\mathbf{p})e^{-\mathrm{i}E_{\mathbf{p}}t} - b^\dagger_{-\mathbf{p},s}v^r(-\mathbf{p})e^{\mathrm{i}E_{\mathbf{p}}t}\right]$$

$$= \mathfrak{N}\int \frac{\mathrm{d}^3p}{2(2\pi)^3} \sum_{s=1,2}\sum_{r=1,2} \Big[a^\dagger_{\mathbf{p},s}a_{\mathbf{p},r}u^{s\dagger}(\mathbf{p})u^r(\mathbf{p}) + b_{-\mathbf{p},s}a_{\mathbf{p},r}v^{s\dagger}(-\mathbf{p})u^r(\mathbf{p})e^{-2\mathrm{i}E_{\mathbf{p}}t} +$$
$$- a^\dagger_{\mathbf{p},s}b^\dagger_{-\mathbf{p},r}u^{s\dagger}(\mathbf{p})v^r(-\mathbf{p})e^{2\mathrm{i}E_{\mathbf{p}}t} - b_{-\mathbf{p},s}b^\dagger_{-\mathbf{p},r}v^{s\dagger}(-\mathbf{p})v^r(-\mathbf{p})\Big]$$

Lemma 2.2.1

$$u^{s\dagger}(\mathbf{p})v^r(-\mathbf{p}) = v^{s\dagger}(-\mathbf{p})u^r(\mathbf{p}) = 0$$

Using Lemma 2.2.1, Eq. 1.108 and the antisymmetry $\mathfrak{N}\{b_{\mathbf{p},s}b^\dagger_{\mathbf{p},s}\} = -b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}$:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{s=1,2} E_{\mathbf{p}}\mathfrak{N}\{a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} - b_{\mathbf{p},s}b^\dagger_{\mathbf{p},s}\} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{s=1,2} E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}\right)$$

which completes the proof. □

It can be seen that, using anticommutators, the Hamiltonian and its interpretation are analogous to that of the complex scalar field: if commutators were used, instead, one would get a final $-b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}$ term, which is problematic as it yields an energy unbounded from below[2].
The momentum operator too is analogous to that of the complex scalar field, with the additional

[2]The spin-statistics theorem is implied by Lorentz invariance, positive energies and causality (see [1]).

spin degree of freedom.

Proposition 2.2.2 (Momentum operator)
The momentum operator of a Dirac field Ψ is:

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \mathbf{p} \left(a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s} \right) \tag{2.29}$$

Proof. By Eq. 1.80, the $0i$-component of energy-momentum tensor of the Dirac Lagrangian is:

$$\theta^{0i} = \frac{\partial \mathcal{L}_D}{\partial(\partial_0 \Psi)} \partial^i \Psi = \bar{\Psi} i\gamma^0 \partial^i \Psi = -\Psi^\dagger i \partial_i \Psi$$

Thus, according to Eq. 1.81:

$$\begin{aligned}
\mathbf{P} = \mathfrak{N} \int d^3x \int \frac{d^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s} u^{s\dagger}(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} v^{s\dagger}(p) e^{-ip_\mu x^\mu} \right] \times \\
\times (-i\boldsymbol{\nabla}) \left[a_{\mathbf{q},r} u^r(q) e^{-iq_\mu x^\mu} + b^\dagger_{\mathbf{q},r} v^r(q) e^{iq_\mu x^\mu} \right] \\
= \mathfrak{N} \int d^3x \int \frac{d^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s} u^{s\dagger}(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} v^{s\dagger}(p) e^{-ip_\mu x^\mu} \right] \times \\
\times \mathbf{q} \left[a_{\mathbf{q},r} u^r(q) e^{-iq_\mu x^\mu} - b^\dagger_{\mathbf{q},r} v^r(q) e^{iq_\mu x^\mu} \right] \\
= \mathfrak{N} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2E_\mathbf{p}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s} u^{s\dagger}(\mathbf{p}) e^{iE_\mathbf{p}t} + b_{-\mathbf{p},s} v^{s\dagger}(-\mathbf{p}) e^{-iE_\mathbf{p}t} \right] \times \\
\times \left[a_{\mathbf{p},r} u^r(\mathbf{p}) e^{-iE_\mathbf{p}t} - b^\dagger_{-\mathbf{p},r} v^r(-\mathbf{p}) e^{iE_\mathbf{p}t} \right]
\end{aligned}$$

Using Lemma 2.2.1, Eq. 1.108 and the antisymmetry $\mathfrak{N} b_{\mathbf{p},s} b^\dagger_{\mathbf{p},s} = -b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}$:

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \mathbf{p} \mathfrak{N}\{ a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b_{\mathbf{p},s} b^\dagger_{\mathbf{p},s} \} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \mathbf{p} \left(a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s} \right)$$

which is the thesis. □

Thus, both $a^\dagger_{\mathbf{p},s}$ and $b^\dagger_{\mathbf{p},s}$ create particles with energy $+E_\mathbf{p}$ and momentum $\mathbf{p}$: these are respectively called **fermions** and **antifermions**.

§2.2.1.1 Quantum numbers

Under a generic Lorentz transformation, the Dirac field Ψ transforms according to Eq. 1.60, i.e. $\Psi'(x) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \Psi(x)$, where the Lorentz generator $J^{\mu\nu}$ is defined in Eq. 1.61.

Proposition 2.2.3 (Angular momentum)
The conserved charge associated to the infinitesimal rotation Eq. 1.60 is:

$$\mathbf{J} = \int d^3x \, \Psi^\dagger \left(\mathbf{x} \times (-i\boldsymbol{\nabla}) + \tfrac{1}{2}\boldsymbol{\Sigma} \right) \Psi \tag{2.30}$$

Proof. Consider a rotation of θ around the z-axis: it is described by $\omega_{12} = -\omega_{21} = \theta$, so, by Eq. 1.60:

$$\delta_0\Psi = \theta\left(x^1\partial^2 - x^2\partial^1 - \tfrac{\mathrm{i}}{2}\Sigma^3\right)\Psi = -\theta\left(x^1\partial_2 - x^2\partial_1 + \tfrac{\mathrm{i}}{2}\Sigma^3\right)\Psi \equiv \theta\Delta\Psi$$

Using the same notation of Def. 1.3.1, $\epsilon \equiv \theta$ and $F = \Delta\Psi$, therefore, by Eq. 1.78, the temporal component of the conserved Noether current is (without negative sign, as it is mathematically equivalent):

$$j^0 = \frac{\partial\mathcal{L}_\mathrm{D}}{\partial(\partial_0\Psi)}\Delta\Psi = -\mathrm{i}\Psi^\dagger\left((\mathbf{x}\times\boldsymbol{\nabla})^3 + \tfrac{\mathrm{i}}{2}\Sigma^3\right)\Psi$$

As the associated Noether charge is $J^3 = \int \mathrm{d}^3x\, j^0$, this can be generalized to rotations around the x-axis and the y-axis, yielding:

$$\mathbf{J} = \int \mathrm{d}^3x\,\Psi^\dagger\left(\mathbf{x}\times(-\mathrm{i}\boldsymbol{\nabla}) + \tfrac{1}{2}\boldsymbol{\Sigma}\right)\Psi$$

which is the thesis. □

For non-relativistic fermions, the first term gives the orbital angular momentum, while the second term gives the spin angular momentum. For relativistic fermions, this division is not straightforward.
To determine the spin of fermions, it is sufficient to consider them at rest. The spin along the z-axis is given by the S_z operator (at $t = 0$):

$$\begin{aligned}
S_z &= \int \mathrm{d}^3x \int\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}} e^{\mathrm{i}(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \times \\
&\quad \times \sum_{t=1,2}\sum_{r=1,2}\left[a^\dagger_{\mathbf{p},t}u^{t\dagger}(\mathbf{p}) + b_{-\mathbf{p},t}v^{t\dagger}(-\mathbf{p})\right]\frac{\Sigma^3}{2}\left[a_{\mathbf{q},r}u^r(\mathbf{q}) + b^\dagger_{-\mathbf{q},r}v^r(-\mathbf{q})\right] \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_\mathbf{p}}\sum_{t=1,2}\sum_{r=1,2}\left[a^\dagger_{\mathbf{p},t}u^{t\dagger}(\mathbf{p}) + b_{-\mathbf{p},t}v^{t\dagger}(-\mathbf{p})\right]\frac{\Sigma^3}{2}\left[a_{\mathbf{p},r}u^r(\mathbf{p}) + b^\dagger_{-\mathbf{p},r}v^r(-\mathbf{p})\right]
\end{aligned}$$

Noting that $S_z|0\rangle = 0$ (by definition of vacuum), then $S_z a^\dagger_{\mathbf{0},s}|0\rangle = \{S_z, a^\dagger_{\mathbf{0},s}\}|0\rangle$; the only non-zero terms are those proportional to $u^\dagger\Sigma^3 u$ and $v^\dagger\Sigma^3 v$, but the latter vanishes as $\{a, b\} = 0$, so the only term remaining is:

$$\{a^\dagger_{\mathbf{p},t}a_{\mathbf{p},r}, a^\dagger_{\mathbf{0},s}\} = a^\dagger_{\mathbf{p},t}\{a_{\mathbf{p},r}, a^\dagger_{\mathbf{0},s}\} = (2\pi)^3\delta^{(3)}(\mathbf{p})\delta_{rs}a^\dagger_{\mathbf{p},t}$$

The S_z operator thus acts as (recall Eq. 1.105 with $p^\mu = (m, 0, 0, 0)$):

$$S_z a^\dagger_{\mathbf{0},s}|0\rangle = \frac{1}{2E_\mathbf{p}}\sum_{t=1,2}u^{t\dagger}(\mathbf{0})\frac{\Sigma^3}{2}u^s(\mathbf{0})a^\dagger_{\mathbf{0},t}|0\rangle = \frac{1}{4}\left(2\xi^{1\dagger}\sigma^3\xi^s a^\dagger_{\mathbf{0},1} + 2\xi^{2\dagger}\sigma^3\xi^s a^\dagger_{\mathbf{0},2}\right)|0\rangle$$

Using $\sigma^3\xi^1 = +\xi^1$ and $\sigma^3\xi^2 = -\xi^2$, by $\xi^{r\dagger}\xi^s = \delta^{rs}$ one gets:

$$S_z a^\dagger_{\mathbf{0},1}|0\rangle = +\frac{1}{2}a^\dagger_{\mathbf{0},1}|0\rangle \qquad S_z a^\dagger_{\mathbf{0},2}|0\rangle = -\frac{1}{2}a^\dagger_{\mathbf{0},2}|0\rangle$$

which means that fermions are spin-$\frac{1}{2}$ particles, with $a^\dagger_{\mathbf{0},1}$ creating $s = +\frac{1}{2}$ fermions and $a^\dagger_{\mathbf{0},2}$ creating $s = -\frac{1}{2}$ fermions. Conversely, it is equivalent to show that $b^\dagger_{\mathbf{0},1}$ creates $s = -\frac{1}{2}$

antifermions and $b^\dagger_{\mathbf{0},2}$ creates $s=+\frac{1}{2}$ antifermions (as b and $b^\dagger$ are not in normal order in S_z, so there's an extra negative sign due to antisymmetry).
Another important conserved Noether charge of Dirac theory is that associated to the vector current $j^\mu_{\mathrm{V}} = \bar\Psi\gamma^\mu\Psi$ (recall Eq. 1.112), which is (using $j^0_{\mathrm{V}} = \Psi^\dagger\Psi$):

$$Q_{\mathrm{U}(1)_{\mathrm{V}}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} - b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}\right) \tag{2.31}$$

This means that fermions have $\mathrm{U}(1)_{\mathrm{V}}$ charge of $+1$, while antifermions of -1.

state	S_z	$Q_{\mathrm{U}(1)_{\mathrm{V}}}$
$a^\dagger_{\mathbf{p},1}\lvert 0\rangle$	$+\frac{1}{2}$	$+1$
$a^\dagger_{\mathbf{p},2}\lvert 0\rangle$	$-\frac{1}{2}$	$+1$
$b^\dagger_{\mathbf{p},1}\lvert 0\rangle$	$-\frac{1}{2}$	-1
$b^\dagger_{\mathbf{p},2}\lvert 0\rangle$	$+\frac{1}{2}$	-1

Table 2.1: Quantum numbers for fermions in the Dirac theory.

§2.2.1.2 Ladder operators

As for the KG field, the ladder operators of the Dirac field can be made explicit too.

Proposition 2.2.4 (Ladder operators)
The ladder operators for a Dirac field are:

$$\sqrt{2E_{\mathbf{p}}}a_{\mathbf{p},s} = \int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu}\bar u^s(p)\gamma^0\Psi(x) \qquad \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p},s} = \int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu}\bar\Psi(x)\gamma^0 u^s(p) \tag{2.32}$$

$$\sqrt{2E_{\mathbf{p}}}b^\dagger_{\mathbf{p},s} = \int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu}\bar v^s(p)\gamma^0\Psi(x) \qquad \sqrt{2E_{\mathbf{p}}}b_{\mathbf{p},s} = \int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu}\bar\Psi(x)\gamma^0 v^s(p) \tag{2.33}$$

Proof. First, cast Eq. 2.25 in the form:

$$\Psi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}} \sum_{s=1,2}\left[a_{\mathbf{k},s}u^s(\mathbf{k})e^{-\mathrm{i}E_{\mathbf{k}}t} + b^\dagger_{-\mathbf{k},s}v^s(-\mathbf{k})e^{\mathrm{i}E_{\mathbf{k}}t}\right]e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

Then:

$$\begin{aligned}
\int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu}\Psi(x) &= \int \frac{\mathrm{d}^3x\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}} \sum_{s=1,2}\left[a_{\mathbf{k},s}u^s(\mathbf{k})e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{k}})t} + b^\dagger_{-\mathbf{k},s}v^s(-\mathbf{k})e^{\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right]e^{\mathrm{i}(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} \\
&= \int \frac{\mathrm{d}^3k}{\sqrt{2E_{\mathbf{k}}}}\delta^{(3)}(\mathbf{k}-\mathbf{p}) \sum_{s=1,2}\left[a_{\mathbf{k},s}u^s(\mathbf{k})e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{k}})t} + b^\dagger_{-\mathbf{k},s}v^s(-\mathbf{k})e^{\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{r=1,2}\left[a_{\mathbf{p},r}u^r(\mathbf{p}) + b^\dagger_{-\mathbf{p},r}v^r(-\mathbf{p})e^{2\mathrm{i}E_{\mathbf{p}}t}\right]
\end{aligned}$$

Lemma 2.2.2

$$\bar{u}^s(\mathbf{p})\gamma^0 v^r(-\mathbf{p}) = 0 \qquad \bar{v}^s(\mathbf{p})\gamma^0 u^r(-\mathbf{p}) = 0 \tag{2.34}$$

Using Lemma 2.2.2 and Eq. 1.110:

$$\int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu} \bar{u}^s(p)\gamma^0\Psi(x) = \sqrt{2E_\mathbf{p}} a_{\mathbf{p},s}$$

Analogously:

$$\int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu} \bar{v}^s(p)\gamma^0\Psi(x) = \sqrt{2E_\mathbf{p}} b^\dagger_{\mathbf{p},s}$$

To find the remaining ladder operators, either repeat the previous steps with $\bar{\Psi}(x)$ or use (as $(\gamma^\mu)^2 = \mathrm{I}_4$):

$$(\bar{u}^s\gamma^0\Psi)^\dagger = (u^{s\dagger}\Psi)^\dagger = \Psi^\dagger u^s = \bar{\Psi}\gamma^0 u^s \tag{2.35}$$

This yields the thesis. □

§2.2.2 Massless Weyl fields

The quantization of massless Weyl fields follows immediately from that of Dirac fields, and its useful to use Dirac notation:

$$\Psi_\mathrm{L} \equiv \begin{pmatrix} \psi_\mathrm{L} \\ 0 \end{pmatrix} \qquad \Psi_\mathrm{R} \equiv \begin{pmatrix} 0 \\ \psi_\mathrm{R} \end{pmatrix}$$

Consider Ψ_L. As for Eq. 2.25:

$$\Psi_\mathrm{L}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{\mathbf{p},s} u^s_\mathrm{L}(p) e^{-\mathrm{i}p_\mu x^\mu} + b^\dagger_{\mathbf{p},s} v^s_\mathrm{L}(p) e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0 = E_\mathbf{p}}$$

where the Dirac spinors now have the right-hand component vanishing, in the chiral representation. By Eq. 1.104-1.106, in the massless (ultra-relativistic) limit $p^\mu = (E, 0, 0, E)$, so spinors with $s = 1$ have only the right-handed component, while those with $s = 2$ only the left-handed one: therefore, only $s = 2$ spinors contribute to Ψ_L (and only $s = 1$ to Ψ_R).
By Tab. 2.1, it is clear that in this context $a^\dagger_{\mathbf{p},2}$ creates a particle with helicity $h = -\frac{1}{2}$, while $b^\dagger_{\mathbf{p},2}$ creates an antiparticle with $h = +\frac{1}{2}$: in general, a left-handed massless Weyl field describes particles with $h = -\frac{1}{2}$ and antiparticles with $h = +\frac{1}{2}$, while a right-handed one describes particles with $h = +\frac{1}{2}$ and antiparticles with $h = -\frac{1}{2}$.

§2.2.3 Discrete symmetries of fermionic fields

§2.2.3.1 Parity

Under parity $\mathcal{P}\mathbf{p} = -\mathbf{p}$ and $\mathcal{P}s = s$, so, for a generic particle of type a:

$$\mathcal{P}\,|a; \mathbf{p}, s\rangle = \eta_a\,|a; -\mathbf{p}, s\rangle \tag{2.36}$$

where η_a is a generic constant phase factor, since states in the Fock space which differ by a phase still represent the same physical state. As $\mathcal{P}^2 = \mathbb{1}$, this means that $\eta_a^2 = \pm 1$, as observables are built from an even number of fermionic ladder operators.

Non-Majorana fermions It is possible to prove that, for non-Majorana spin-$\frac{1}{2}$ fermions, it is possible to redefine $\mathcal{P}$ so that $\eta_a = +1$, i.e. $\eta_a = \pm 1$.

Proposition 2.2.5

$$\mathcal{P}a^\dagger_{\mathbf{p},s}\mathcal{P} = \eta_a a^\dagger_{-\mathbf{p},s} \qquad \mathcal{P}b^\dagger_{\mathbf{p},s}\mathcal{P} = \eta_b b^\dagger_{-\mathbf{p},s} \tag{2.37}$$

Proof. For a multiparticle state one must have:

$$\mathcal{P}a^\dagger_{\mathbf{p},s}b^\dagger_{\mathbf{q},r}|0\rangle = \eta_a\eta_b a^\dagger_{-\mathbf{p},s}b^\dagger_{-\mathbf{q},r}|0\rangle$$

Therefore:

$$\mathcal{P}a^\dagger_{\mathbf{p},s} = \eta_a a^\dagger_{-\mathbf{p},s}\mathcal{P} \qquad \mathcal{P}b^\dagger_{\mathbf{p},s} = \eta_b b^\dagger_{-\mathbf{p},s}\mathcal{P}$$

Using $\mathcal{P}^2 = \mathbb{1}$ yields the thesis. □

According to Wigner's theorem (Th. 1.2.2), this symmetry can be represented by a unitary operator, i.e. $\mathcal{P}^\dagger = \mathcal{P}^{-1}$, but $\mathcal{P}^2 = \mathbb{1}$, therefore $\mathcal{P}^\dagger = \mathcal{P}$. Then, Eq. 2.37 is valid for $a_{\mathbf{p},s}, b_{\mathbf{p},s}$ too, thus parity acts as:

$$\Psi(x) \mapsto \Psi'(x') = \mathcal{P}\Psi(x)\mathcal{P} \tag{2.38}$$

Explicitly:

$$\begin{aligned}\mathcal{P}\Psi(x)\mathcal{P} &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left[\eta_a a_{-\mathbf{p},s}u^s(p)e^{-\mathrm{i}p_\mu x^\mu} + \eta_b b^\dagger_{-\mathbf{p},s}v^s(p)e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}} \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left[\eta_a a_{-\mathbf{p},s}u^s(p)e^{-\mathrm{i}E_{\mathbf{p}}t+\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \eta_b b^\dagger_{-\mathbf{p},s}v^s(p)e^{\mathrm{i}E_{\mathbf{p}}t-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\right]_{p^0=E_{\mathbf{p}}}\end{aligned}$$

Setting $\mathbf{p}' \equiv -\mathbf{p}, \mathbf{x}' \equiv -\mathbf{x}$ and noting that, by Eq. 1.104-1.106 $\mathbf{p} \mapsto -\mathbf{p}$ exchanges the left-handed and the right-handed compontents of the spinors, i.e. $u^s(p) \mapsto \gamma^0 u^s(p)$ and $v^s(p) \mapsto -\gamma^0 v^s(p)$:

$$\begin{aligned}\mathcal{P}\Psi(x)\mathcal{P} &= \int \frac{\mathrm{d}^3p'}{(2\pi)^3\sqrt{E_{\mathbf{p}'}}} \sum_{s=1,2}\left[\eta_a a_{\mathbf{p}',s}\gamma^0 u^s(p')e^{-\mathrm{i}E_{\mathbf{p}'}t+\mathrm{i}\mathbf{p}'\cdot\mathbf{x}'} - \eta_b b^\dagger_{\mathbf{p}',s}\gamma^0 v^s(p')e^{\mathrm{i}E_{\mathbf{p}'}t-\mathrm{i}\mathbf{p}'\cdot\mathbf{x}'}\right]_{p'^0=E_{\mathbf{p}'}} \\ &= \gamma^0\int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left[\eta_a a_{\mathbf{p},s}u^s(p)e^{-\mathrm{i}p_\mu x'^\mu} - \eta_b b^\dagger_{\mathbf{p},s}v^s(p)e^{\mathrm{i}p_\mu x'^\mu}\right]_{p^0=E_{\mathbf{p}}}\end{aligned}$$

Requiring that Ψ is a representation of parity, up to a phase, means $\eta_a = -\eta_b$, so that:

$$\mathcal{P}\Psi(t,\mathbf{x})\mathcal{P} = \eta_a\gamma^0\Psi(t,-\mathbf{x}) \tag{2.39}$$

This, in the chiral representation, agrees with the fact that parity exchanges left-handed and right-handed Weyl spinors. The η_a factor cancels in any fermion bilinear involving only one type of particles; however, the relative phase factors of different particles can be observed: in particular, the opposite intrinsic parity of fermions and antifermions.

Spin-0 bosons As already noted, the scalar complex field is similar to the Dirac field, apart for the absence of spinors in the expansions of $\phi(x)$: in particular, this means that scalar fields have no relative negative signe between η_a and η_b, so a quantized complex scalar field gives a representation of parity if $\eta_a = \eta_b$, and the intrinsic parity of spin-0 particle and antiparticles is equal.

§2.2.3.2 Charge conjugation

Recall Eq. 1.46: charge conjugation acts on the classical Dirac field as $\Psi \mapsto -\mathrm{i}\gamma^2\Psi^*$.

Definition 2.2.1 (Quantized charge conjugation)
The **charge conjugation operator** is defined as:

$$\mathcal{C}a_{\mathbf{p},s}\mathcal{C} = \eta_c b_{\mathbf{p},s} \qquad \mathcal{C}b_{\mathbf{p},s}\mathcal{C} = \eta_c a_{\mathbf{p},s} \tag{2.40}$$

with $\eta_c = \pm 1$ for simplicity.

Thus, $\mathcal{C}^2 = \mathbb{1}$ too, and its physical interpretation is the exchange of particles and antiparticles, while leaving $\mathbf{p}$ and s unchanged: as $a_{\mathbf{p},s}$ and $b_{\mathbf{p},s}$ create particles with opposite spin, this means that charge conjugation reverses the helicity of particles.

Lemma 2.2.3 (Charge conjugation on spinors)

$$\mathcal{C}u^s(p)\mathcal{C} = -\mathrm{i}\gamma^2\left[v^s(p)\right]^* \qquad \mathcal{C}v^s(p)\mathcal{C} = -\mathrm{i}\gamma^2\left[u^s(p)\right]^* \tag{2.41}$$

Proposition 2.2.6 (Charge conjugation on Dirac fields)
Given a Dirac field $\Psi(x)$:

$$\mathcal{C}\Psi(x)\mathcal{C} = -\mathrm{i}\eta_c\gamma^2[\Psi(x)]^* \tag{2.42}$$

Proof. Using Eq. 2.40-2.41:

$$\begin{aligned}
\mathcal{C}\Psi(x)\mathcal{C} &= \eta_c \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} \left[b_{\mathbf{p},s}\mathcal{C}u^s(p)\mathcal{C}e^{-\mathrm{i}p_\mu x^\mu} + a^\dagger_{\mathbf{p},s}\mathcal{C}v^s(p)\mathcal{C}e^{\mathrm{i}p_\mu x^\mu}\right] \\
&= -\mathrm{i}\eta_c\gamma^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} \left[b_{\mathbf{p},s}[v^s(p)]^*e^{-\mathrm{i}p_\mu x^\mu} + a^\dagger_{\mathbf{p},s}[u^s(p)]^*e^{\mathrm{i}p_\mu x^\mu}\right] \\
&= -\mathrm{i}\eta_c\gamma^2[\Psi(x)]^*
\end{aligned}$$

which is the thesis. □

As for parity, the transformation of quantized fields is analogous to that of classical fields, with an additional quantum phase factor which depends on the particle type.

Proposition 2.2.7 (Charge conjugation of the vector current)

$$\mathcal{C}\left(\bar{\Psi}\gamma^\mu\Psi\right)\mathcal{C} = -\bar{\Psi}\gamma^\mu\Psi \tag{2.43}$$

§2.2.3.3 Time reversal

Theorem 2.2.1 (Anti-unitary time reversal)
Time reversal cannot be implemented as a linear unitary operator.

Proof. Assume that the time reversal operator $\mathcal{T}$ is linear and unitary; then, as it must be a symmetry of the free Dirac Lagrangian, $[\mathcal{T}, H] = 0$, so:

$$\begin{aligned}\mathcal{T}\Psi(t,\mathbf{x})\mathcal{T}\left|0\right\rangle &= \mathcal{T}e^{\mathrm{i}Ht}\Psi(\mathbf{x})e^{-\mathrm{i}Ht}\mathcal{T}\left|0\right\rangle = e^{\mathrm{i}Ht}\mathcal{T}\Psi(\mathbf{x})\mathcal{T}e^{-\mathrm{i}Ht}\left|0\right\rangle = e^{\mathrm{i}Ht}\mathcal{T}\Psi(\mathbf{x})\mathcal{T}\left|0\right\rangle \\ &= \Psi(-t,\mathbf{x})\left|0\right\rangle = e^{-\mathrm{i}Ht}\Psi(\mathbf{x})e^{\mathrm{i}Ht}\left|0\right\rangle = e^{-\mathrm{i}Ht}\Psi(\mathbf{x})\left|0\right\rangle\end{aligned}$$

assuming $H\left|0\right\rangle = 0$. But the first line is a sum of negative frequencies only, while the second line is a sum of positive frequencies only, which is absurd. □

Time reversal is an example of operator which, according to Wigner's theorem (Th. 1.2.2), is represented as an anti-linear anti-unitary operator, i.e. an operator such that:

$$\langle a|\mathcal{T}^\dagger\mathcal{T}|b\rangle = \langle a|b\rangle^* \qquad \mathcal{T}\lambda\left|a\right\rangle = \lambda^*\mathcal{T}\left|a\right\rangle \qquad \forall\left|a\right\rangle,\left|b\right\rangle\in\mathscr{H}\,,\ \forall\lambda\in\mathbb{C}$$

In particular, anti-linearity (second property) solves the absurdum in the proof of Th. 2.2.1. Time reversal is expected to reverse the spin of particles, and this can be used to construct $\mathcal{T}$. First, define:

$$\xi^{-s} \equiv -\mathrm{i}\sigma^2[\xi^s]^* \tag{2.44}$$

that is, $\xi^{-1} = \binom{0}{1}$ and $\xi^{-2} = \binom{-1}{0}$. This allows redefining $\eta^s \equiv \xi^{-s}$, and also $\xi^{-(-s)} = -\xi^s$.

Lemma 2.2.4 (Reversed spinors)

$$u^{-s}(-\mathbf{p}) = -\gamma^1\gamma^3[u^s(\mathbf{p})]^* \qquad v^{-s}(-\mathbf{p}) = -\gamma^1\gamma^3[v^s(\mathbf{p})]^* \tag{2.45}$$

Proof. Defining $\tilde{p}^\mu \equiv (p^0, -\mathbf{p})$ and recalling that $\boldsymbol{\sigma}\sigma^2 = -\sigma^2\boldsymbol{\sigma}^*$:

$$u^{-s}(-\mathbf{p}) = \begin{pmatrix}\sqrt{\tilde{p}^\mu\sigma_\mu}(-\mathrm{i}\sigma^2[\xi^s]^*) \\ \sqrt{\tilde{p}^\mu\bar{\sigma}_\mu}(-\mathrm{i}\sigma^2[\xi^s]^*)\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\sigma^2\sqrt{p^\mu\sigma^*_\mu}[\xi^s]^* \\ -\mathrm{i}\sigma^2\sqrt{p^\mu\bar{\sigma}^*_\mu}[\xi^s]^*\end{pmatrix} = -\mathrm{i}\begin{pmatrix}\sigma^2 & 0 \\ 0 & \sigma^2\end{pmatrix}[u^s(\mathbf{p})]^*$$

Noting that $\mathrm{i}\,\mathrm{diag}(\sigma^2,\sigma^2) = \gamma^1\gamma^3$ completes the proof. On the other hand:

$$\begin{aligned}v^{-s}(-\mathbf{p}) &= \begin{pmatrix}\sqrt{\tilde{p}^\mu\sigma_\mu}(-\xi^s) \\ -\sqrt{\tilde{p}^\mu\bar{\sigma}_\mu}(-\xi^s)\end{pmatrix} = \begin{pmatrix}\sigma^2\sqrt{p^\mu\sigma^*_\mu}\sigma^2(-\xi^s) \\ -\sigma^2\sqrt{p^\mu\bar{\sigma}^*_\mu}\sigma^2(-\xi^s)\end{pmatrix} \\ &= -\mathrm{i}\begin{pmatrix}\sigma^2 & 0 \\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\sqrt{p^\mu\sigma^*_\mu}(-\mathrm{i}\sigma^2[\xi^s]^*) \\ -\sqrt{p^\mu\bar{\sigma}^*_\mu}(-\mathrm{i}\sigma^2[\xi^s]^*)\end{pmatrix} = -\gamma^1\gamma^3[v^s(-\mathbf{p})]^*\end{aligned}$$

□

It is useful to define the reversed ladder operators:

$$a_{\mathbf{p},-s} \equiv (a_{\mathbf{p},2}, -a_{\mathbf{p},1}) \qquad b_{\mathbf{p},-s} \equiv (b_{\mathbf{p},2}, -b_{\mathbf{p},1}) \tag{2.46}$$

Definition 2.2.2 (Time reversal)

The **time reversal operator** is defined as:

$$\mathcal{T}a_{\mathbf{p},s}\mathcal{T} = a_{-\mathbf{p},-s} \qquad \mathcal{T}b_{\mathbf{p},s}\mathcal{T} = b_{-\mathbf{p},-s} \tag{2.47}$$

An additional overall phase is irrelevant. Moreover, as other discrete symmetries, $\mathcal{T}^2 = \mathbb{1}$.

	$\bar{\Psi}\Psi$	$\bar{\Psi}\gamma^5\Psi$	$\bar{\Psi}\gamma^\mu\Psi$	$\bar{\Psi}\gamma^\mu\gamma^5\Psi$	$\bar{\Psi}\sigma^{\mu\nu}\Psi$	∂_μ
$\mathcal{C}$	$+1$	$+1$	-1	$+1$	-1	$+1$
$\mathcal{P}$	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
$\mathcal{T}$	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
$\mathcal{CPT}$	$+1$	$+1$	-1	-1	$+1$	-1

Table 2.2: Eigenvalues of fermion bilinears and derivative operator, with the shorthand $(-1)^\mu \equiv (+1,-1,-1,-1)$.

Proposition 2.2.8 (Time reversal on Dirac fields)
Given a Dirac field $\Psi(x)$:

$$\mathcal{T}\Psi(t,\mathbf{x})\mathcal{T} = -\gamma^1\gamma^3\Psi(-t,\mathbf{x}) \tag{2.48}$$

Proof. Using Lemma 2.2.4 and defining $\tilde{p}^\mu \equiv (p^0, -\mathbf{p})$, $\tilde{x}^\mu \equiv (-t, \mathbf{x})$, so that $p^\mu x_\mu = -\tilde{p}^\mu \tilde{x}_\mu$:

$$\begin{aligned}
\mathcal{T}\Psi(x)\mathcal{T} &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \mathcal{T}\left[a_{\mathbf{p},s}u^s(p)e^{-\mathrm{i}p_\mu x^\mu} + b^\dagger_{\mathbf{p},s}v^s(p)e^{\mathrm{i}p_\mu x^\mu}\right]\mathcal{T} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{-\mathbf{p},-s}[u^s(p)]^* e^{\mathrm{i}p_\mu x^\mu} + b^\dagger_{-\mathbf{p},-s}[v^s(p)]^* e^{-\mathrm{i}p_\mu x^\mu}\right] \\
&= -\gamma^3\gamma^1 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{\tilde{\mathbf{p}},-s}u^{-s}(\tilde{p})e^{-\mathrm{i}\tilde{p}_\mu \tilde{x}^\mu} + b^\dagger_{\tilde{\mathbf{p}},-s}v^{-s}(\tilde{p})e^{\mathrm{i}\tilde{p}_\mu \tilde{x}^\mu}\right] \\
&= \gamma^1\gamma^3 \int \frac{-\mathrm{d}^3\tilde{p}}{(2\pi)^3\sqrt{2E_{\tilde{\mathbf{p}}}}} \sum_{s=1,2} \left[a_{\tilde{\mathbf{p}},s}u^s(\tilde{p})e^{-\mathrm{i}\tilde{p}_\mu \tilde{x}^\mu} + b^\dagger_{\tilde{\mathbf{p}},s}v^s(\tilde{p})e^{\mathrm{i}\tilde{p}_\mu \tilde{x}^\mu}\right] = -\gamma^1\gamma^3\Psi(\tilde{x})
\end{aligned}$$

□

§2.2.3.4 CPT symmetry

In order to study the invariance properties of fermionic Lagrangians, it is necessary to state the transformation relations of fermion bilinears (see Sec. 3.6 of [4] for details).
This shows that it is not possible to construct a Lorentz-invariant Lagrangian which violates CPT symmetry: this is an example of the **CPT theorem**, which states that, independently of the spin of the particle, a (locale) Lorentz-invariant field theory with a hermitian Hamiltonian cannot violate CPT symmetry. A consequence of this theorem is that particles and antiparticles have exactly the same mass, which has been so far empirically confirmed.

§2.3 Electromagnetic field

§2.3.1 Maxwell theory

The electromagnetic field is described by a 4-vector A_μ, the **gauge potential**. From this, the field strength tensor is defined as:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.49}$$

which is related to the electric and magnetic fields as $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B^k$. The Lagrangian of the free electromagnetic field is:

$$\mathcal{L}_{\mathrm{M}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.50}$$

The associated equations of motion are:

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.51}$$

Moreover, defining $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ (the Hodge dual), it is trivial to check that, by Schwarz lemma:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.52}$$

Eq. 2.51-2.52 are exactly Maxwell equations in the absence of sources: when written in terms of $\mathbf{E}$ and $\mathbf{B}$, Eq. 2.51 gives the equations for $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{B}$, while Eq. 2.52 those for $\boldsymbol{\nabla} \times \mathbf{E}$ and $\boldsymbol{\nabla} \cdot \mathbf{B}$.

§2.3.1.1 Gauge invariance

A crucial local symmetry of the Maxwell Lagrangian is the symmetry under local gauge transformations like:

$$A_\mu(x) \mapsto A_\mu(x) - \partial_\mu \alpha(x) \tag{2.53}$$

with arbitrary $\alpha \in \mathcal{C}^\infty(\mathbb{R}^{1,3})$. Considering the free electromagnetic field, the global version of this transformation (that is, α independent of x) yields no conserved charge, as the associated Noether current vanishes identically.

Theorem 2.3.1 (Radiation gauge)

In the absence of sources, it is always possible to choose the **radiation gauge**:

$$A_0 = 0 \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{2.54}$$

Proof. Starting from a general gauge potential A_μ, the condition $A_0 = 0$ is achieved through:

$$A_\mu \mapsto A_\mu - \partial_\mu \int_{t_0}^{t} \mathrm{d}\tau \, A_0(\tau, \mathbf{x})$$

Then, $A_0 = 0$ will remain unchanged if another gauge transformation with $\alpha(x) = \alpha(\mathbf{x})$ is performed. Consider:

$$\alpha(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 y}{4\pi\, |\mathbf{x} - \mathbf{y}|} \partial_i A^i(t, \mathbf{y})$$

which is independent of t since $E^i = -\partial_0 A^i$, as $A_0 = 0$, so $\partial_i E^i = 0$ implies $\partial_0 \partial_i A^i = 0$. Recall the identity:

$$\triangle_{\mathbf{x}} \frac{1}{|\mathbf{x}-\mathbf{y}|} = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{2.55}$$

Thus:

$$\boldsymbol{\nabla}\cdot\mathbf{A} \mapsto \boldsymbol{\nabla}\cdot\mathbf{A} - \triangle_{\mathbf{x}}\alpha = \partial_i A^i(t,\mathbf{x}) - \partial_i A^i(t,\mathbf{x}) = 0$$

□

The radiation gauge clearly implies the **Lorentz gauge**:

$$\partial_\mu A^\mu = 0 \tag{2.56}$$

In this gauge, the equations of motions Eq. 2.51 become:

$$\Box A^\mu = 0 \tag{2.57}$$

which are massless KG equations for each component of the gauge potential. Plane-wave solutions take the form:

$$A_\mu(x) = 2\Re\{\epsilon_\mu(k)e^{-\mathrm{i}k_\mu x^\mu}\} \tag{2.58}$$

where $\epsilon_\mu(x)$ is the **polarization vector**. Then, Eq. 2.57 gives $k^2 = 0$, while the chosen radiation gauge implies $\epsilon_0 = 0$ and $\boldsymbol{\epsilon}\cdot\mathbf{k} = 0$: therefore, an electromagnetic wave has only two degrees of freedom, represented by a polarization vector $\boldsymbol{\epsilon}$ perpendicular to the direction of propagation.

Example 2.3.1 (Linear and circular polarization)

Given an electromagnetic wave with $\hat{\mathbf{k}} = \hat{\mathbf{e}}_z$, then there are two possible polarization directions perpendicular to $\hat{\mathbf{e}}_z$. A possible choice are **linear polarization vectors**:

$$\boldsymbol{\epsilon}_1 = (1,0,0) \qquad \boldsymbol{\epsilon}_2 = (0,1,0) \tag{2.59}$$

Another choice is a linear combination of these vectors, the **circular polarization vectors**:

$$\boldsymbol{\epsilon}_\mathrm{R} = \frac{\boldsymbol{\epsilon}_1 + \mathrm{i}\boldsymbol{\epsilon}_2}{\sqrt{2}} \qquad \boldsymbol{\epsilon}_\mathrm{L} = \frac{\boldsymbol{\epsilon}_1 - \mathrm{i}\boldsymbol{\epsilon}_2}{\sqrt{2}} \tag{2.60}$$

which are helicity eigenstates respectively with $h = +1$ and $h = -1$.

The advantage of the radiation gauge is that it exposes clearly the physical degrees of freedom of the electromagnetic field, while sacrificing explicit Lorentz covariance; on the other hand, the Lorentz gauge retains the explicit Lorentz covariance, at the cost of redundant degrees of freedom.

§2.3.1.2 Energy-momentum tensor

By Eq. 1.80, writing Eq. 2.50 explicitly in terms of A_μ, the energy-momentum tensor of the electromagnetic field is:

$$\theta^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}F^2 \tag{2.61}$$

with $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$. To show the gauge-invariance of this tensor, recall Eq. 2.51:

$$\theta^{\mu\nu} \mapsto \theta^{\mu\nu} + F^{\mu\rho}\partial^\nu\partial_\rho\alpha \qquad \Longrightarrow \qquad P^\mu \mapsto P^\mu + \int \mathrm{d}^3x\,\partial_\rho(F^{0\rho}\partial^\mu\alpha) = P^\mu + \int \mathrm{d}^3x\,\partial_i(F^{0i}\partial^\mu\alpha) = P^\mu$$

where the last term is a total spatial derivative, hence vanishing by divergence theorem provided that the field decreases sufficiently fast at infinity. To improve the energy-momentum tensor, add $\partial_\rho(F^{\mu\rho}A^\nu)$, which is covariantly conserved by itself and whose $\mu = 0$ component is a total spatial derivative, so to obtain:

$$T^{\mu\nu} = F^{\mu\rho}F_\rho{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^2 \tag{2.62}$$

which is explicitly gauge-invariant and yields the usual expressions for the energy density $T^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$ and the momentum density $T^{0i} = (\mathbf{E}\times\mathbf{B})^i$.
In a general field theory, the observable quantities are the charges, not the currents: two Lagrangian densities which differ by a total 4-divergence are physically equivalent and give the same equations of motion, but the conserved currents obtained through Noether theorem are different, while the associated Noether charges are the same.

§2.3.1.3 Matter coupling

In the presence of an external current j^μ, Eq. 2.52 is not modified, as it is a consequence of the definition of $F^{\mu\nu}$ (assuming regular gauge fields), while Eq. 2.51 becomes:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{2.63}$$

By Schwarz lemma, this equation is consistent only if $\partial_\mu j^\mu = 0$. This can be understood in light of gauge invariance, considering the action:

$$\mathcal{S}_\mathrm{M} = -\int \mathrm{d}^4x \left[\frac{1}{4}F^2 + j^\mu A_\mu\right] \tag{2.64}$$

A gauge transformation $A_\mu \mapsto A_\mu - \partial_\mu\alpha$ implies $\mathcal{S}_\mathrm{M} \mapsto \mathcal{S}_\mathrm{M} + \int \mathrm{d}^4x\, j^\mu\partial_\mu\alpha$: integrating by parts, it is clear that $\mathcal{S}_\mathrm{M}$ is gauge invariant only if $\partial_\mu j^\mu = 0$.

Dirac field The coupling of the electromagnetic field to the Dirac field is an example of the general procedure of writing a gauge-invariant action for a gauge theory. In particular, consider a theory with a global U(1) invariance, which is a symmetry of the free Dirac action by Eq. 1.112. Now generalize to a local U(1) symmetry:

$$\Psi(x) \mapsto e^{\mathrm{i}q\alpha(x)}\Psi(x) \tag{2.65}$$

with $q \in \mathbb{R}$. This no longer is a symmetry of the Dirac action, however it can be combined with Eq. 2.53 defining the **covariant derivative**:

$$D_\mu\Psi := (\partial_\mu + \mathrm{i}qA_\mu)\Psi \tag{2.66}$$

Proposition 2.3.1

$$D_\mu\Psi(x) \mapsto e^{\mathrm{i}q\alpha(x)}D_\mu\Psi(x) \tag{2.67}$$

Proof. $D_\mu\Psi \mapsto [\partial_\mu + \mathrm{i}q(A_\mu - \partial_\mu\alpha)]\, e^{\mathrm{i}q\alpha}\Psi = e^{\mathrm{i}q\alpha}\left[\mathrm{i}q\partial_\mu\alpha + \partial_\mu + \mathrm{i}qA_\mu - \mathrm{i}q\partial_\mu\alpha\right]\Psi = e^{\mathrm{i}q\alpha}D_\mu\Psi$ □

The Lagrangian with a local U(1) symmetry is found replacing $\partial_\mu \mapsto D_\mu$ (**minimal coupling**): the global symmetry is gauged to a local symmetry, resulting in a gauge theory with gauge field A_μ. Applying this to Eq. 1.101:

$$\mathcal{L}_\mathrm{D} = \bar{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi - qA_\mu\bar{\Psi}\gamma^\mu\Psi \tag{2.68}$$

where $j^\mu_\mathrm{V} := \bar{\Psi}\gamma^\mu\Psi$ is the Noether current associated to the global U(1) symmetry. The associated conserved charge then is:

$$Q = \int \mathrm{d}^3x\, \bar{\Psi}\gamma^0\Psi = \int \mathrm{d}^3x\, \Psi^\dagger\Psi \tag{2.69}$$

Complex scalar field A complex scalar field has a global U(1) symmetry $\phi \mapsto e^{\mathrm{i}q\alpha}\phi$, thus the covariant derivative is identical to Eq. 2.67 and the gauged Lagrangian reads (recall Eq. 1.91):

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi + \mathrm{i}qA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + q^2\,|\phi|^2\,A_\mu A^\mu - m^2\phi^*\phi \tag{2.70}$$

where $j_\mu := \mathrm{i}\phi^*\overleftrightarrow{\partial_\mu}\phi$ is the Noether current associated to the global U(1) symmetry.

Higher order interaction terms Although a real scalar field cannot be coupled to the electromagnetic field through the minimal coupling (as the real condition imposes $q = 0$, i.e. a neutral field), interaction terms are possible via higher order terms, as $\mathcal{L}_\mathrm{int} \sim \phi F_{\mu\nu}F^{\mu\nu}$ or $\mathcal{L}_\mathrm{int} \sim \phi\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$.
The same is possible for the Dirac field too, for example with $\mathcal{L}_\mathrm{int} \sim \bar{\Psi}\sigma^{\mu\nu}\Psi F^{\mu\nu}$: note, however, that these non-minimal couplings have dimensional coupling constants (with dimension of the inverse of a mass), which have a less fundamental significance than dimensionless coupling constant.

Example 2.3.2 (Neutral pions)

The neutral pion π^0 is described by a pseudoscalar field, thus its interaction with the electromagnetic field needs to be a pseudoscalar term, like $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ (this gives a good phenomenological description), as opposed to parity-invariant terms like $F_{\mu\nu}F^{\mu\nu}$.

§2.3.2 Quantization

Due to gauge symmetry, the gauge field gives a redundant physical description, therefore the quantization procedure can be carried in two different ways: fixing the gauge, thus working with only physical degrees of freedom but at the cost of loosing explicit Lorentz invariance, or considering the whole A_μ, hence carrying spurious degrees of freedom.

§2.3.2.1 Quantization in the radiation gauge

As of Eq. 2.54, Maxwell equations Eq. 2.57 read $\Box\mathbf{A} = \mathbf{0}$, with general classical solution:

$$\mathbf{A}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}} \sum_{\lambda=1,2} \left[\boldsymbol{\epsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-\mathrm{i}p_\mu x^\mu} + \boldsymbol{\epsilon}^*(\mathbf{p},\lambda)a^*_{\mathbf{p},\lambda}e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=\omega_\mathbf{p}}$$

Inserting this expression into $\Box\mathbf{A} = \mathbf{0}$ results in the mass-shell condition $p^2 = 0$, i.e. $\omega_\mathbf{p} = |\mathbf{p}|$. On the other hand, the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ requires $\boldsymbol{\epsilon}\cdot\mathbf{p} = 0$: for each fixed $\mathbf{p}$, this solution has two orthonormal solutions labelled by $\lambda = 1, 2$, which describe the two physical degrees of freedom of the electromagnetic field. The classical solution can be promoted to a hermitian operator as:

$$\mathbf{A}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \sum_{\lambda=1,2} \left[\boldsymbol{\epsilon}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-\mathrm{i} p_\mu x^\mu} + \boldsymbol{\epsilon}^*(\mathbf{p}, \lambda) a^\dagger_{\mathbf{p},\lambda} e^{\mathrm{i} p_\mu x^\mu} \right]_{p^0 = \omega_{\mathbf{p}}} \tag{2.71}$$

and imposing the canonical commutation relations:

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta_{\lambda\lambda'} \tag{2.72}$$

In terms of commutators of A^i and conjugate momenta, consider that $\Pi_0 = 0$ (as $A_0 = 0$) and[3]:

$$\Pi^i = \frac{\delta}{\delta(\partial_0 A_i)} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \frac{\delta}{\delta(\partial_0 A_i)} \left(-\frac{1}{2} F_{0i} F^{0i} \right) = -F^{0i} = E^i$$

Lemma 2.3.1

$$\frac{1}{2} \sum_{\lambda=1,2} \left[\epsilon^i(\mathbf{k}, \lambda) \epsilon^{*j}(\mathbf{k}, \lambda) + \epsilon^{*i}(-\mathbf{k}, \lambda) \epsilon^j(-\mathbf{k}, \lambda) \right] = \delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \tag{2.73}$$

Proof. Trivially verified in a frame where $\mathbf{k} = (0, 0, k)$ choosing $\boldsymbol{\epsilon}(\mathbf{k}, 1) = (1, 0, 0)$ and $\boldsymbol{\epsilon}(\mathbf{k}, 2) = (0, 1, 0)$ and valid in any frame[a] is ensured as both sides transform as tensor under rotations. □

[a] In particular, the linear polarizations satisfy:

$$\sum_{\lambda=1,2} \epsilon^i(\mathbf{k}, \lambda) \epsilon^{*j}(\mathbf{k}, \lambda) = \delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \tag{2.74}$$

Lemma 2.3.2

$$[A^i(t, \mathbf{x}), E^j(t, \mathbf{y})] = -\mathrm{i} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) \tag{2.75}$$

The r.h.s. of this equation is similar to a Dirac delta and it is called **transverse Dirac delta**, defined in order to get $[\boldsymbol{\nabla} \cdot \mathbf{A}(t, \mathbf{x}), \mathbf{E}(t, \mathbf{y})] = \mathbf{0}$ (as $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), being the integrand proportional to:

$$k^i \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) = k^j - k^j = 0$$

Fock space The standard construction of the Fock space proceeds defining the vacuum state $a_{\mathbf{p},\lambda} |0\rangle = 0$. The Hamiltonian and the linear momentum are then found as:

$$H = \frac{1}{2} \mathfrak{N} \int \mathrm{d}^3 x \, \left[\mathbf{E}^2 + \mathbf{B}^2 \right] = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{\lambda=1,2} \omega_{\mathbf{k}} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} \tag{2.76}$$

[3] Note that Π^i is the momentum conjugate to A_i, while $\Pi_i = -\Pi^i$ is the one conjugate to A^i.

$$\mathbf{P} = \mathfrak{N} \int \mathrm{d}^3x\, \mathbf{E} \times \mathbf{B} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_{\lambda=1,2} \mathbf{k} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} \tag{2.77}$$

Therefore, $a^\dagger_{\mathbf{k},\lambda} |0\rangle$ describes a massless particle with energy $\omega_\mathbf{k}$ and momentum $\mathbf{k}$. To study spin, it is necessary to switch to circular polarizations:

$$a^\dagger_{\mathbf{k},\pm} := \frac{1}{\sqrt{2}} \left(a^\dagger_{\mathbf{k},1} \pm \mathrm{i} a^\dagger_{\mathbf{k},2} \right) \tag{2.78}$$

where $a_{\mathbf{k},1}, a_{\mathbf{k},2}$ are the linear polarizations $\boldsymbol{\epsilon}(\mathbf{k},1) = (1,0,0), \boldsymbol{\epsilon}(\mathbf{k},2) = (0,1,0)$. Linear polarizations are not helicity eigenstates, while circular polarizations are:

$$S^3 a^\dagger_{\mathbf{k},1} |0\rangle = +\mathrm{i} a^\dagger_{\mathbf{k},2} |0\rangle \qquad S^3 a^\dagger_{\mathbf{k},+} |0\rangle = +a^\dagger_{\mathbf{k},+} |0\rangle$$
$$S^3 a^\dagger_{\mathbf{k},2} |0\rangle = -\mathrm{i} a^\dagger_{\mathbf{k},1} |0\rangle \qquad S^3 a^\dagger_{\mathbf{k},-} |0\rangle = -a^\dagger_{\mathbf{k},-} |0\rangle$$

In conclusion, $a^\dagger_{\mathbf{k},\pm} |0\rangle$ describe massless particles with energy $\omega_\mathbf{k}$, momentum $\mathbf{k}$, spin 1 and helicity ± 1: these are photons.
Defining angular momentum and boost generators too in terms of ladder operators, it is possible to show that Lorentz invariance is preserved by the quantization procedure, although not explicitly.

Discrete transformations It is possible to define parity and charge conjugation on photon states. The electric field is a true vector, while the magnetic field is a pseudovector, thus the gauge potential is a true vector: $\mathcal{P}\mathbf{A}(t,\mathbf{x}) = -\mathbf{A}(t,-\mathbf{x})$. In terms of photon states:

$$\mathcal{P} |\gamma; \mathbf{k}, \mathbf{s}\rangle = - |\gamma; -\mathbf{k}, \mathbf{s}\rangle \tag{2.79}$$

so the intrinsic parity of physical photon states is -1.
As the fermionic current changes sign under charge conjugation (Eq. 2.43), it is a symmetry of the QED Lagrangian if $\mathcal{C} A^\mu \mathcal{C} = -A^\mu$, i.e. $\mathcal{C} a^\dagger_{\mathbf{k},\lambda} \mathcal{C} = -a^\dagger_{\mathbf{k},\lambda}$. As $\mathcal{C}|0\rangle = +|0\rangle$ and $\mathcal{C}^2 = \mathbb{1}$ by definition, then $\mathcal{C} a^\dagger_{\mathbf{k},\lambda} |0\rangle = \mathcal{C} a_{\mathbf{k},\lambda} \mathcal{C}\mathcal{C} |0\rangle = -a^\dagger_{\mathbf{k},\lambda} |0\rangle$, or:

$$\mathcal{C} |\gamma; \mathbf{k}, \mathbf{s}\rangle = - |\gamma; \mathbf{k}, \mathbf{s}\rangle \tag{2.80}$$

§2.3.2.2 Covariant quantization

The Maxwell Lagrangian Eq. 2.50 cannot be straightforwardly quantize, as Π^0 cannot be defined due to the absence of $\partial_0 A_0$ terms. The basic idea of the covariant quantization of the electromagnetic field, or Gupta-Bleuler quantization, is to start from a modified Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \tag{2.81}$$

Conjugate momenta are then found to be:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} \qquad \Longrightarrow \qquad \Pi^i = -F^{0i} = E^i \qquad \Pi^0 = -\partial_\mu A^\mu$$

Canonical commutation relations now take the form:

$$[A^\mu(t,\mathbf{x}), \Pi^\nu(t,\mathbf{y})] = \mathrm{i}\eta^{\mu\nu} \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{2.82}$$

The metric ensures Lorentz covariance. The equations of motion are $\Box A^\mu = 0$, thus the gauge field operators are:

$$A_\mu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \sum_{\lambda=0}^{3} \left[\epsilon_\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-\mathrm{i}p_\mu x^\mu} + \epsilon^*_\mu(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{\mathrm{i}p_\mu x^\mu} \right]_{p^0=\omega_{\mathbf{p}}} \tag{2.83}$$

$\Box A_\mu = 0$ imposes $p^2 = 0$. Note that the modified Lagrangian is not gauge invariant, so there is no constraint on ϵ^μ; in the frame $p^\mu = (p,0,0,p)$ a convenient choice of basis is $\epsilon^\mu(\mathbf{p},\lambda) = \delta^\mu_\lambda$, hence only $\lambda = 1,2$ satisfy $\epsilon_\mu p^\mu = 0$. Eq. 2.82 becomes:

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{p},\lambda'}] = -(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \eta_{\lambda\lambda'} \tag{2.84}$$

Note that the commutator for $\lambda = \lambda' = 0$ has a negative sign, which means that the norm is not positive defined on this Fock space (rendering impossible its interpretation as a probability):

$$\langle \mathbf{p},\lambda | \mathbf{p},\lambda \rangle = (2\omega_{\mathbf{p}}) \langle 0 | a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{p},\lambda} | 0 \rangle = (2\omega_{\mathbf{p}}) \langle 0 | [a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{p},\lambda}] | 0 \rangle = -2\omega_{\mathbf{p}} V \eta_{\lambda\lambda} \tag{2.85}$$

(where $V \equiv (2\pi)^3 \delta^{(3)}(\mathbf{0})$) which is negative for $\lambda = 0$. However, the only physical states are those associated with transverse polarization vectors, thus these problematic states can be shown to be unphysical.

To recover the correct description of QED from the modified Lagrangian, it is necessary to restrict the Fock space; in particular, any two physical states must satisfy:

$$\langle \text{phys}' | \partial_\mu A^\mu | \text{phys} \rangle = 0 \tag{2.86}$$

Note that $\partial_\mu A^\mu$ can be decomposed into its positive- and negative-frequency parts $\partial_\mu A^\mu = (\partial_\mu A^\mu)^+ + (\partial_\mu A^\mu)^-$ as:

$$(\partial_\mu A^\mu)^+ \equiv -\mathrm{i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \sum_{\lambda=0}^{3} p_\mu \epsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-\mathrm{i}p_\mu x^\mu} = (\partial_\mu A^\mu)^\dagger$$

Therefore, Eq. 2.86 is equivalent to:

$$(\partial_\mu A^\mu)^+ |\text{phys}\rangle = 0 \tag{2.87}$$

This is the definition of the physical subspace of the Fock space, as it preserves the linear structure of the physical Hilbert space. Now consider the most general superposition of polarization states with momentum $\mathbf{k}$, i.e. $|\mathbf{k}\rangle = \sum_{\lambda=0}^{3} c_\lambda a^\dagger_{\mathbf{k},\lambda} |0\rangle$, and choose the frame $k^\mu = (k,0,0,k)$: in this frame the physical-state condition reads $c_0 + c_3 = 0$. Thus, all transverse states $\lambda = 1,2$ are physical, while there is only one non-transverse state that remains:

$$|\phi\rangle \equiv (a^\dagger_{\mathbf{k},0} - a^\dagger_{\mathbf{k},3}) |0\rangle$$

The most general one-particle state of the physical subspace can then be written as $|\mathbf{k}_\mathrm{T}\rangle + c\,|\phi\rangle$. However:

$$\langle \phi | \phi \rangle = \langle 0 | (a_{\mathbf{k},0} - a_{\mathbf{k},3})(a^\dagger_{\mathbf{k},0} - a^\dagger_{\mathbf{k},3}) | 0 \rangle = \langle 0 | [a_{\mathbf{k},0}, a^\dagger_{\mathbf{k},0}] + [a_{\mathbf{k},3}, a^\dagger_{\mathbf{k},3}] | 0 \rangle = 0$$

This means that $|\phi\rangle$ is orthogonal to all physical states $|\varphi\rangle$. Moreover, only transverse states contribute to the energy and to the momentum, as $H, \mathbf{P} \sim -\eta^{\lambda\lambda'} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda'}$ and (as $c_0 + c_3 = 0$):

$$(a_{\mathbf{k},0} - a_{\mathbf{k},3}) |\psi\rangle = 0 \quad \Longrightarrow \quad \langle \varphi' | - a^\dagger_{\mathbf{k},0} a_{\mathbf{k},0} + a^\dagger_{\mathbf{k},3} a_{\mathbf{k},3} | \varphi \rangle = \langle \varphi' | (-a^\dagger_{\mathbf{k},0} + a^\dagger_{\mathbf{k},3}) a_{\mathbf{k},3} | \varphi \rangle = 0$$

These facts mean that $|\mathbf{k}_\mathrm{T}\rangle$ and $|\mathbf{k}_\mathrm{T}\rangle + c\,|\phi\rangle$ are physically indistinguishable: photons can then be identified as the equivalence classes with respect to the equivalence relation $|\psi\rangle \sim |\psi\rangle + c\,|\phi\rangle$. This procedure has eliminated the spurious degrees of freedom, thus showing the equivalence between covariant quantization and gauge quantization.

§2.3.2.3 Ladder operators

As for the KG field and the Dirac field, the ladder operators of the electromagnetic field can be made explicit too.

Proposition 2.3.2 (Ladder operators)
The ladder operators for the electromagnetic field are (in radiation gauge):

$$\sqrt{2\omega_{\mathbf{k}}}a_{\mathbf{k},\lambda} = \mathrm{i}\boldsymbol{\epsilon}(\mathbf{k},\lambda)\cdot\int \mathrm{d}^3x\, e^{\mathrm{i}k_\mu x^\mu}\overleftrightarrow{\partial_0}\mathbf{A}(x) \tag{2.88}$$

$$\sqrt{2\omega_{\mathbf{k}}}a^\dagger_{\mathbf{k},\lambda} = -\mathrm{i}\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)\cdot\int \mathrm{d}^3x\, e^{-\mathrm{i}k_\mu x^\mu}\overleftrightarrow{\partial_0}\mathbf{A}(x) \tag{2.89}$$

Proof. From Eq. 2.71:

$$\partial_0\mathbf{A}(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2\omega_{\mathbf{p}}}}\sum_{\lambda=1,2}\left[-\mathrm{i}\omega_{\mathbf{p}}\boldsymbol{\epsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-\mathrm{i}p_\mu x^\mu} + \mathrm{i}\omega_{\mathbf{p}}\boldsymbol{\epsilon}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=\omega_{\mathbf{p}}}$$

Therefore:

$$\begin{aligned}\int\mathrm{d}^3x\, e^{\mathrm{i}k_\mu x^\mu}\mathbf{A}(x) &= \int\mathrm{d}^3x\int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2\omega_{\mathbf{p}}}}\sum_{\lambda=1,2}\left[\boldsymbol{\epsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{\mathrm{i}(k-p)_\mu x^\mu} + \boldsymbol{\epsilon}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{\mathrm{i}(k+p)_\mu x^\mu}\right]\\ &= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\sum_{\lambda=1,2}\left[\boldsymbol{\epsilon}(\mathbf{k},\lambda)a_{\mathbf{k},\lambda} + \boldsymbol{\epsilon}^*(-\mathbf{k},\lambda)a^\dagger_{-\mathbf{k},\lambda}e^{2\mathrm{i}\omega_{\mathbf{k}}}\right]\end{aligned}$$

$$\int\mathrm{d}^3x\, e^{\mathrm{i}k_\mu x^\mu}\partial_0\mathbf{A}(x) = \frac{i\omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\sum_{\lambda=1,2}\left[-\boldsymbol{\epsilon}(\mathbf{k},\lambda)a_{\mathbf{k},\lambda} + \boldsymbol{\epsilon}^*(-\mathbf{k},\lambda)a^\dagger_{-\mathbf{k},\lambda}e^{2\mathrm{i}\omega_{\mathbf{k}}}\right]$$

These can be combined as:

$$\int\mathrm{d}^3x\, e^{\mathrm{i}k_\mu x^\mu}\overleftrightarrow{\partial_0}\mathbf{A}(x) = \frac{\mathrm{i}\omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\sum_{\lambda=1,2}\left[-2\boldsymbol{\epsilon}(\mathbf{k},\lambda)a_{\mathbf{k},\lambda}\right] = -\mathrm{i}\sqrt{2\omega_{\mathbf{k}}}\sum_{\lambda=1,2}\boldsymbol{\epsilon}(\mathbf{k},\lambda)a_{\mathbf{k},\lambda}$$

In the radiation gauge $\boldsymbol{\epsilon}(\mathbf{k},\lambda)\cdot\boldsymbol{\epsilon}(\mathbf{k},\sigma) = \delta_{\lambda\sigma}$, hence Eq. 2.88. Analogously:

$$\int\mathrm{d}^3x\, e^{-\mathrm{i}k_\mu x^\mu}\overleftrightarrow{\partial_0}\mathbf{A}(x) = \frac{\mathrm{i}\omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\sum_{\lambda=1,2}\left[2\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)a^\dagger_{\mathbf{k},\lambda}\right] = \mathrm{i}\sqrt{2\omega_{\mathbf{k}}}\sum_{\lambda=1,2}\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)a^\dagger_{\mathbf{k},\lambda}$$

This completes the proof. □

Writing them in Lorentz-covariant form (using $\epsilon^0 = \epsilon^3 = 0$):

$$\sqrt{2\omega_{\mathbf{p}}}a_{\mathbf{p},\lambda} = \mathrm{i}\epsilon^\mu(\mathbf{p},\lambda)\int\mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}A_\mu(x) \tag{2.90}$$

$$\sqrt{2\omega_{\mathbf{p}}}a^\dagger_{\mathbf{p},\lambda} = -\mathrm{i}\epsilon^{\mu*}(\mathbf{p},\lambda)\int\mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}A_\mu(x) \tag{2.91}$$

Part II

Interacting Fields

Chapter 3

Perturbative Approach

In the case of free Hamiltonians/Lagrangians, free-particle states are eigenstates of the Hamiltonian and they cannot interact. To describe interactions between particles, non-linear terms must be included in $\mathcal{H}/\mathcal{L}$, which will couple Fourier modes (and particles that occupy them) to one another.
To preserve causality, these interaction terms must include only products of fields at the same spacetime point. Moreover, the discussion is restricted to terms which do not include derivatives of fields, so that $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$.

Example 3.0.1 (ϕ^4 theory)
The so-called ϕ^4 theory is a scalar field theory with an interaction term of the kind:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{3.1}$$

for $\lambda \in \mathbb{R}$ a dimensionless coupling constant. This interaction describes the Higgs self-interaction in the Standard Model.

§3.1 Time evolution

Consider a system described by a Hamiltonian H. Two different ways of describing its time evolution are possible.

§3.1.1 Schrödinger picture

In the Schrödinger picture, states are treated as time-dependent, while operators are treated as time-independent. Consider an initial state $|\psi(0)\rangle \equiv |\psi\rangle$: its time-evolution is:

$$|\psi(t)\rangle = e^{-\mathrm{i}Ht}\,|\psi\rangle \tag{3.2}$$

In this picture, the amplitude for a process $|a(t_0)\rangle \to |b(t)\rangle$ is:

$$\mathcal{A} = \langle b|e^{-\mathrm{i}H(t-t_0)}|a\rangle =: \langle b|S(t,t_0)|a\rangle \tag{3.3}$$

In the $t - t_0 \to \infty$ limit, S is known as **S-matrix**, and scattering amplitudes simply are its elements. This can be seen as an operator which realizes time evolution, as:

$$|\psi(t)\rangle = S(t)\,|\psi\rangle \tag{3.4}$$

where $S(t) \equiv S(t,0)$.

Proposition 3.1.1 (Unitarity of time evolution)
$S(t, t_0)$ is a unitary operator on $\mathscr{H}$.

Proof. Consider a normalized initial state $|\psi\rangle : \langle\psi|\psi\rangle = 1$. Given a complete orthonormal eigenbasis $\{|n\rangle\}_{n\in\mathbb{N}}$ of $\mathscr{H}$, then it must be:

$$\sum_{n\in\mathbb{N}} |\langle n|S|\psi\rangle|^2 = 1 \tag{3.5}$$

This can also be written as:

$$\sum_{n\in\mathbb{N}} |\langle n|S|\psi\rangle|^2 = \sum_{n\in\mathbb{N}} \langle\psi|S^\dagger|n\rangle \langle n|S|\psi\rangle = \langle\psi|S^\dagger S|\psi\rangle \tag{3.6}$$

Being $|\psi\rangle$ arbitrary, it must be $S^\dagger S = SS^\dagger = \mathbb{1}_{\mathscr{H}}$. □

The unitarity of the S-matrix expresses the conservation of probabilities. Moreover, it can be decomposed as:

$$S = \mathrm{I} + \mathrm{i}T \tag{3.7}$$

where the **T-matrix** contains the information on the interactions, while the identity I only represents a part of the incident wave-packet unaffected by interactions.

§3.1.2 Heisenberg picture

In the Heisenberg picture, states are treated as time-independent, while operators are treated as time-dependent. This picture is better suited for QFT, as fields (which are operators) depend both on $\mathbf{x}$ and t, therefore the Heisenberg picture is natural in light of Lorentz covariance.
Given a state $|\psi(t)\rangle$ in the Schrödinger picture, in the Heisenberg picture it is defined as:

$$|\psi, t\rangle := e^{\mathrm{i}Ht} |\psi(t)\rangle \tag{3.8}$$

which is manifestly time-independent. An operator $\mathcal{O}$ in the Schrödinger picture instead becomes:

$$\mathcal{O}(t) := e^{\mathrm{i}Ht}\mathcal{O}e^{-\mathrm{i}Ht} \tag{3.9}$$

The label t in the definition of state is necessary to distinguish whose operator the state is eigenstate of: in fact, in general $\mathcal{O}(t) \neq \mathcal{O}(t')$ for $t \neq t'$, so they will have different eigenstates too.
In the Heisenberg picture, the S-matrix becomes:

$$\langle b|S(t, t_0)|a\rangle = \langle b, t|a, t_0\rangle \tag{3.10}$$

§3.2 Asymptotic theory

On a macroscopic scale, interaction times are extremely small. Therefore, it is convenient to make an assumption, the **adiabatic hypothesis**: while the interaction is described by $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$, where $\mathcal{H}_0$ is the free Hamiltonian, in the far past and the far future $\mathcal{H}_{\text{int}}$ is adiabatically turned off, i.e. $\mathcal{H} \to \mathcal{H}_0$ as $t \to \pm\infty$. Moreover, as the out-going states can represent in-coming states for a successive process, the two Fock spaces[1] must be isomorphic, i.e. $\mathscr{F}_{\text{in}} \cong \mathscr{F}_{\text{out}}$: in particular, this implies the (physical) uniqueness of the vacuum, as $|0_{\text{in}}\rangle = |0_{\text{out}}\rangle \equiv |0\rangle$. This isomorphism is realized by S-matrix $S \equiv S(+\infty, -\infty)$ (recall Eq. 3.3):

$$\langle \beta_{\text{out}} | \alpha_{\text{in}} \rangle = \langle \beta_{\text{in}} | S^\dagger | \alpha_{\text{in}} \rangle \quad \Longrightarrow \quad |\beta_{\text{out}}\rangle = S\, |\beta_{\text{in}}\rangle \quad \Longrightarrow \quad \phi_{\text{out}}(x) = S \phi_{\text{in}}(x) S^\dagger$$

where $\phi_{\text{in}}(x), \phi_{\text{out}}(x)$ are the free fields which generate $\mathscr{F}_{\text{in}}, \mathscr{F}_{\text{out}}$. Note that, to preserve covariance, the S-matrix must commute with Poincaré transformations. The adiabatic hypothesis asserts then:

$$\phi(x) \xrightarrow[t \to -\infty]{} \sqrt{Z} \phi_{\text{in}}(x) \qquad \qquad \phi(x) \xrightarrow[t \to +\infty]{} \sqrt{Z} \phi_{\text{out}}(x) \tag{3.12}$$

where $Z \in \mathbb{R}$ is a renormalization factor. These limits must be understood in the weak sense, as they are not operator equations, but they are only valid for each matrix element separately[2].

§3.2.1 LSZ reduction formula

§3.2.1.1 Scalar fields

Consider a scattering process of a single species of neutral scalar particles. The ladder operators of a free real scalar field can be expressed via Eq. 2.16, which is time-independent, and they can be generalized to an interacting theory, in which case they are time-dependent (as Prop. 1.3.2 is only valid if $\phi(x)$ satisfies the free KG equation).

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[1] Formally, given a single-particle Hilbert space $\mathscr{H}$, the Fock space is defined as the following completion:

$$\mathscr{F}_\nu(\mathscr{H}) := \overline{\bigoplus_{n=0}^{\infty} S_\nu \mathscr{H}^{\otimes n}} \tag{3.11}$$

where S_ν is the operator which symmetrizes ($\nu = +$, for bosons) or antisymmetrizes ($\nu = -$, for fermions) the tensors it acts upon. In general:

$$\mathscr{F}_\nu(\mathscr{H}) = \mathbb{C} \oplus \mathscr{H} \oplus S_\nu(\mathscr{H} \otimes \mathscr{H}) \oplus S_\nu(\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}) \oplus \dots$$

where $S_\nu \mathscr{H}^{\otimes n}$ consists of n-particle states ($\mathbb{C}$ is the vacuum). A general state then is:

$$|\Psi\rangle_\nu = \sum_{n=0}^{\infty} |\Psi_n\rangle_\nu = a\,|0\rangle \oplus \sum_i a_i\,|\psi_i\rangle \oplus \sum_{i,j} a_{ij}\,|\psi_i, \psi_j\rangle_\nu \oplus \dots$$

The need for this infinite sum to converge in $\mathscr{F}_\nu(\mathscr{H})$ is solved by the completion, as it restricts the Fock space only to states with a finite inner-product-induced norm:

$$\| |\Psi\rangle_\nu \|^2 = \sum_{n=0}^{\infty} \langle \Psi_n | \Psi_n \rangle_\nu < \infty$$

[2] If that wasn't the case, then canonical quantization would imply $Z = 1$, i.e. that $\phi(x)$ is a free field. For details, see Sec. 5.1.2 of [3].

Proposition 3.2.1 (Time-dependent ladder operators)
Given an interacting scalar field theory $H = H_0 + H_{\text{int}}$, the time-evolution of generalized ladder operators is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t} a_{\mathbf{p}}(t) = \mathrm{i}\,[H_{\text{int}}, a_{\mathbf{p}}(t)] \tag{3.13}$$

Proof. Recall the Heisenberg equation for the time-evolution of an operator $\mathcal{O}(t)$ (in the Heisenberg representation):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}(t) = \mathrm{i}[H, \mathcal{O}] + \frac{\partial}{\partial t}\mathcal{O}(t) \tag{3.14}$$

Using Eq. 2.16, the explicit time-dependece of $a_{\mathbf{p}}(t)$ is solved as:

$$\frac{\partial}{\partial t} a_{\mathbf{p}}(t) = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \frac{\partial}{\partial t} \int \mathrm{d}^3x \, [\partial_0\phi(x) - \mathrm{i}E_{\mathbf{p}}\phi(x)] \, e^{\mathrm{i}p_\mu x^\mu}$$

Recall that for a general function $f = f(x_1, \dots, x_n, t)$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial f}{\partial t} \tag{3.15}$$

Therefore, treating $\phi(x)$ and $\partial_0\phi(x)$ as time-dependent variables:

$$\begin{aligned}
\frac{\partial}{\partial t} a_{\mathbf{p}}(t) &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \left[\frac{\mathrm{d}}{\mathrm{d}t} - \partial_0\phi(x)\frac{\partial}{\partial\phi} - \partial_0^2\phi(x)\frac{\partial}{\partial(\partial_0\phi)}\right] \int \mathrm{d}^3x \, [\partial_0\phi(x) - \mathrm{i}E_{\mathbf{p}}\phi(x)] \, e^{\mathrm{i}p_\mu x^\mu} \\
&= \frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}}}} \int \mathrm{d}^3x \left[\partial_0^2\phi(x) - \mathrm{i}E_{\mathbf{p}}\partial_0\phi(x) + \mathrm{i}E_{\mathbf{p}}\partial_0\phi(x) - (\mathrm{i}E_{\mathbf{p}})^2\phi(x)\right] e^{\mathrm{i}p_\mu x^\mu} + \\
&\qquad\qquad - \frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}}}} \int \mathrm{d}^3x \left[-\mathrm{i}E_{\mathbf{p}}\partial_0\phi(x) + \partial_0^2\phi(x)\right] e^{\mathrm{i}p_\mu x^\mu} \\
&= \frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}}}} \int \mathrm{d}^3x \left[\mathrm{i}E_{\mathbf{p}}\partial_0\phi(x) - (\mathrm{i}E_{\mathbf{p}})^2\phi(x)\right] e^{ip_\mu x^\mu} = \mathrm{i}E_{\mathbf{p}} a_{\mathbf{p}}(t)
\end{aligned}$$

Next, by Eq. 2.7:

$$\begin{aligned}
[H, a_{\mathbf{p}}(t)] &= [H_0, a_{\mathbf{p}}(t)] + [H_{\text{int}}, a_{\mathbf{p}}(t)] \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3} E_{\mathbf{k}} [a^\dagger_{\mathbf{k}}(t) a_{\mathbf{k}}(t), a_{\mathbf{p}}(t)] + [H_{\text{int}}, a_{\mathbf{p}}(t)] \\
&= -\int \mathrm{d}^3k \, E_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{p}) a_{\mathbf{k}}(t) + [H_{\text{int}}, a_{\mathbf{p}}(t)] = -E_{\mathbf{p}} a_{\mathbf{p}}(t) + [H_{\text{int}}, a_{\mathbf{p}}(t)]
\end{aligned}$$

The first term cancels the partial derivative, yielding the thesis. □

Moreover, in the adiabatic hypothesis it is possible to define **in-** and **out-ladder operators**:

$$a_{\mathbf{p}}^{\text{in,out}} \equiv \frac{1}{\sqrt{Z}} \lim_{t \to \mp\infty} a_{\mathbf{p}}(t) \tag{3.16}$$

In terms of the free in- and out-fields, they can be expressed as:

$$\sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{\text{in,out}\dagger} = -\frac{\mathrm{i}}{\sqrt{Z}}\lim_{t\to\mp\infty}\int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}\phi(x) = -\mathrm{i}\int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial}_0\phi_{\text{in,out}}(x) \tag{3.17}$$

These operators respectively act on $\mathscr{F}_{\text{in,out}}$. Note that now these operators, although not explicitly time-dependent, depend implicitly on it as they are defined through the interacting field, i.e. they depend on the field at all times: the relation between in- and out-ladder operators is thus non-trivial.
Conversely, he in- and out-fields can be expressed in terms of in- and out-ladder operators:

$$\phi_{\text{in,out}}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}^{\text{in,out}}e^{-\mathrm{i}p_\mu x^\mu} + a_{\mathbf{p}}^{\text{in,out}\dagger}e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}} \tag{3.18}$$

Proposition 3.2.2 (Behavior of ladder operators)
The in- and out-ladder operators are well-defined ladder operators on $\mathscr{F}_{\text{in,out}}$, i.e.:

$$\hat{P}_\mu a_{\mathbf{k}}^{\text{in,out}\dagger}\left|0_{\text{in}}\right\rangle = k_\mu a_{\mathbf{k}}^{\text{in,out}\dagger}\left|0_{\text{out}}\right\rangle \tag{3.19}$$

Proof. The thesis is equivalent to[a]:

$$[\hat{P}_\mu, \phi_{\text{in,out}}(x)] = -\mathrm{i}\partial_\mu\phi_{\text{in,out}}(x) \tag{3.20}$$

To show this, first note that Eq. 3.13 is equivalent to:

$$a_{\mathbf{k}}(t) = a_{\mathbf{k}}^{\text{in}} + \mathrm{i}\int_{-\infty}^{t}\mathrm{d}\tau\,[H_{\text{int}}, a_{\mathbf{k}}(\tau)] \equiv a_{\mathbf{k}}^{\text{int}} + \mathrm{i}\int_{-\infty}^{t}\mathrm{d}\tau\int\frac{\mathrm{d}^3x}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}e^{\mathrm{i}k_\mu x^\mu}\chi[\phi(\tau,\mathbf{x})]$$

where $\chi[\phi]$ is a certain functional of ϕ. Then:

$$\begin{aligned}
\phi(x) &= \int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}e^{-\mathrm{i}k_\mu x^\mu} + a_{\mathbf{k}}^\dagger e^{\mathrm{i}k_\mu x^\mu}\right]_{k^0=E_{\mathbf{k}}} \\
&= \int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}^{\text{in}}e^{-\mathrm{i}k_\mu x^\mu} + a_{\mathbf{k}}^{\text{in}\dagger}e^{\mathrm{i}k_\mu x^\mu}\right]_{k^0=E_{\mathbf{k}}} + \\
&\quad + \mathrm{i}\int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\int_{-\infty}^{t}\mathrm{d}\tau\int\frac{\mathrm{d}^3y}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\left[\chi[\phi(\tau,\mathbf{y})]e^{-\mathrm{i}k_\mu(x^\mu-y^\mu)} - \text{h.c.}\right]_{k^0=E_{\mathbf{k}}} \\
&\equiv \phi_{\text{in}}(x) + \int_{y^0\in(-\infty,x^0]}\mathrm{d}^4y\, G_{\mathrm{S}}(x-y)\chi[\phi(y)]
\end{aligned}$$

Therefore, at every time the interacting field can be seen as the in-field plus a source term. Now, consider that:

$$[\hat{P}_\mu, \phi(x)] = -i\partial_\mu\phi(x)$$

As the source terms vanishes as $t\to-\infty$ (with $x^0\equiv t$), this equation holds for ϕ_{in} too, hence the thesis. The proof for ϕ_{out} is analogous. □

[a] Requires proof.

It is possible to make the relationship between in- and out-ladder operators explicit.

Lemma 3.2.1

$$\sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^{\text{in}\dagger} - a_{\mathbf{p}}^{\text{out}\dagger}) = \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{-\mathrm{i}p_\mu x^\mu}(\Box + m^2)\phi(x) \tag{3.21}$$

$$\sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^{\text{out}} - a_{\mathbf{p}}^{\text{in}}) = \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{\mathrm{i}p_\mu x^\mu}(\Box + m^2)\phi(x) \tag{3.22}$$

Proof. For any integrable $f(t, \mathbf{x})$ the following identity holds:

$$\left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right) \int \mathrm{d}^3x\, f(t, \mathbf{x}) = \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{\partial}{\partial t} \int \mathrm{d}^3x\, f(t, \mathbf{x}) \tag{3.23}$$

Applying this to $f(t, \mathbf{x}) = -\mathrm{i}Z^{-1/2}e^{-\mathrm{i}k_\mu x^\mu}\overleftrightarrow{\partial_0}\phi(x)$ and using Eq. 3.17:

$$\begin{aligned}
\sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^{\text{out}\dagger} - a_{\mathbf{p}}^{\text{in}\dagger}) &= -\frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, \partial_0 \left[e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}\phi(x)\right] \\
&= -\frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, \partial_0 \left[e^{-\mathrm{i}p_\mu x^\mu}\partial_0\phi(x) - \phi(x)\partial_0 e^{-\mathrm{i}p_\mu x^\mu}\right] \\
&= -\frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x \left[e^{-\mathrm{i}p_\mu x^\mu}\partial_0^2\phi(x) - \phi(x)\partial_0^2 e^{-\mathrm{i}p_\mu x^\mu}\right] \\
&= -\frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x \left[e^{-\mathrm{i}p_\mu x^\mu}\partial_0^2\phi(x) - \phi(x)(\boldsymbol{\nabla}^2 - m^2)e^{-\mathrm{i}p_\mu x^\mu}\right]
\end{aligned}$$

where the last line follows from $\partial_0^2 e^{-\mathrm{i}p_\mu x^\mu} = -p_0^2 e^{-\mathrm{i}p_\mu x^\mu} = (\mathbf{p}^2 - p^2)e^{-\mathrm{i}p_\mu x^\mu} = (\boldsymbol{\nabla}^2 - m^2)e^{-\mathrm{i}p_\mu x^\mu}$. In order to perform integration by parts, note that initial and final particle states are understood to be convoluted to form wave-packets, which are localized in space, while $\phi(x)$ is not localized in time; hence, the second term can be integrated by parts twice, resulting in:

$$\sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^{\text{in}\dagger} - a_{\mathbf{p}}^{\text{out}\dagger}) = \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{-\mathrm{i}p_\mu x^\mu}(\partial_0^2 - \boldsymbol{\nabla}^2 + m^2)\phi(x)$$

The other operator is found analogously. □

Note that if $\phi(x)$ were a free field, this operator would identically vanish: this is because the KG scalar product is time-independent in a free field theory, and so are ladder operators too. It is now possible to write the amplitude for a generic scattering process (i.e. S-matrix element). In particular, consider a process $|\mathbf{k}_1, \dots, \mathbf{k}_n; -\infty\rangle \to |\mathbf{p}_1, \dots, \mathbf{p}_m; +\infty\rangle$ with $\mathbf{k}_i \neq \mathbf{p}_j\ \forall i = 1, \dots, n,\ j = 1, \dots, m$: in this case, as there are no particles which remain unchanged by the interaction, the S-matrix reduces to the T-matrix.

Definition 3.2.1 (Bosonic chronological product)

Given two bosonic fields $f(x)$ and $g(x)$, their **chronological product** is defined as:

$$\begin{aligned}
\mathfrak{T}\{f(x)g(y)\} &:= \begin{cases} g(y)f(x) & y^0 > x^0 \\ f(x)g(y) & y^0 < x^0 \end{cases} \\
&= \theta(y^0 - x^0)g(y)f(x) + \theta(x^0 - y^0)f(x)g(y)
\end{aligned} \tag{3.24}$$

where $\theta(x)$ is the Heaviside distribution.

Theorem 3.2.1 (LSZ reduction formula for scalar fields)
The amplitude for the process $|\mathbf{k}_1, \dots, \mathbf{k}_n; -\infty\rangle \to |\mathbf{p}_1, \dots, \mathbf{p}_m; +\infty\rangle$, with $\mathbf{k}_i \neq \mathbf{p}_j \ \forall i = 1, \dots, n, j = 1, \dots, m$, is given by the **Lehmann–Symanzik–Zimmermann formula**:

$$\begin{aligned}\langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | \mathbf{k}_1, \dots, \mathbf{k}_n; -\infty\rangle &= \\ &= \prod_{i=1}^{n} \frac{k_i^2 - m^2}{\mathrm{i}\sqrt{Z}} \int \mathrm{d}^4 x_i \, e^{-\mathrm{i}k_{i\mu}x_i{}^\mu} \prod_{j=1}^{m} \frac{p_j^2 - m^2}{\mathrm{i}\sqrt{Z}} \int \mathrm{d}^4 y_j \, e^{+\mathrm{i}p_{j\mu}y_j{}^\mu} \times \\ &\quad \times \langle 0 | \mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\phi(y_1)\dots\phi(y_m)\}|0\rangle \end{aligned} \tag{3.25}$$

Proof. Using Eq. 3.17 it is possible to extract a particle from the initial state:

$$\begin{aligned}\langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | \mathbf{k}_1, \dots, \mathbf{k}_n; -\infty\rangle &= \sqrt{2E_{\mathbf{k}_1}} \, \langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | a^{\mathrm{in}}_{\mathbf{k}_1}{}^\dagger | \mathbf{k}_2, \dots, \mathbf{k}_n; -\infty\rangle \\ &= \sqrt{2E_{\mathbf{k}_1}} \, \langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | a^{\mathrm{in}}_{\mathbf{k}_1}{}^\dagger - a^{\mathrm{out}\dagger}_{\mathbf{k}_1} | \mathbf{k}_2, \dots, \mathbf{k}_n; -\infty\rangle\end{aligned}$$

as $a^{\mathrm{out}}_{\mathbf{k}_1} |\mathbf{p}_1, \dots, \mathbf{p}_n; +\infty\rangle = 0$ since $\mathbf{k}_i \neq \mathbf{p}_j$. Then, by Lemma 3.2.1:

$$\begin{aligned}\langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | \mathbf{k}_1, \dots, \mathbf{k}_n; -\infty\rangle &= \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4 x_1 \, e^{-\mathrm{i}k_{1\mu}x_1{}^\mu} (\Box_{x_1} + m^2) \times \\ &\quad \times \langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | \phi(x_1) | \mathbf{k}_2, \dots, \mathbf{k}_n; -\infty\rangle\end{aligned}$$

Using the same argument (noting that $\mathfrak{T}\{a^{\mathrm{in}}_{\mathbf{p}_1}\phi(x_1)\} = \phi(x_1)a^{\mathrm{in}}_{\mathbf{p}_1}$, $\mathfrak{T}\{a^{\mathrm{out}}_{\mathbf{p}_1}\phi(x_1)\} = a^{\mathrm{out}}_{\mathbf{p}_1}\phi(x_1)$):

$$\begin{aligned}&\langle \mathbf{p}_1, \dots, \mathbf{p}_m; +\infty | \phi(x_1) | \mathbf{k}_2, \dots, \mathbf{k}_n; -\infty\rangle = \\ &\quad = \sqrt{2E_{\mathbf{p}_1}} \, \langle \mathbf{p}_2, \dots, \mathbf{p}_m; +\infty | \mathfrak{T}\{(a^{\mathrm{out}}_{\mathbf{p}_1} - a^{\mathrm{in}}_{\mathbf{p}_1})\phi(x_1)\} | \mathbf{k}_2, \dots, \mathbf{k}_n; -\infty\rangle \\ &\quad = \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4 y_1 \, e^{+\mathrm{i}p_{1\mu}y_1{}^\mu} (\Box_{y_1} + m^2) \, \langle \mathbf{p}_2, \dots, \mathbf{p}_m; +\infty | \mathfrak{T}\{\phi(y_1)\phi(x_1)\} | \mathbf{k}_2, \dots, \mathbf{k}_n; -\infty\rangle\end{aligned}$$

This procedure can be iterated[a], obtaining:

$$\begin{aligned}\langle \mathbf{p}_1, \dots \mathbf{p}_m; +\infty | \mathbf{k}_1, \dots, \mathbf{k}_n; -\infty\rangle &= \prod_{i=1}^{n} \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4 x_i \, e^{-\mathrm{i}k_{i\mu}x_i{}^\mu} \prod_{j=1}^{m} \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4 y_j \, e^{+\mathrm{i}p_{j\mu}x_j{}^\mu} \times \\ &\quad \times (\Box_{x_i} + m^2)(\Box_{y_j} + m^2) \, \langle 0 | \mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\phi(y_1)\dots\phi(y_m)\}|0\rangle\end{aligned}$$

It is possible to define the **N-point Green function** as:

$$G(x_1, \dots, x_N) \equiv \langle 0 | \mathfrak{T}\{\phi(x_1)\dots\phi(x_N)\}|0\rangle \tag{3.26}$$

In terms of its (4-dimensional) Fourier transform it reads:

$$G(x_1, \dots, x_N) = \prod_{i=1}^{N} \int \frac{\mathrm{d}^4 \xi_i}{(2\pi)^4} e^{\mathrm{i}\xi_{i\mu}x_i{}^\mu} \tilde{G}(\xi_1, \dots, \xi_N)$$

Then:

$$(\Box_{x_j} + m^2) G(x_1, \dots, x_N) = -\prod_{i=1}^{N} \int \frac{\mathrm{d}^4 \xi_i}{(2\pi)^4} e^{\mathrm{i}\xi_{i\mu}x_i{}^\mu} (\xi_j^2 - m^2) \tilde{G}(\xi_1, \dots, \xi_N)$$

Substituting into the above expression:

$$\prod_{i=1}^{N}\frac{\mathrm{i}}{\sqrt{Z}}\int \mathrm{d}^4x_i e^{-\mathrm{i}k_{i\mu}x_i^{\mu}}(\Box_{x_i}+m^2)G(x_1,\ldots,x_N)=$$
$$=-\prod_{i=1}^{N}\frac{\mathrm{i}}{\sqrt{Z}}\int\frac{\mathrm{d}^4x_i\mathrm{d}^4\xi_i}{(2\pi)^4}e^{\mathrm{i}(\xi_i-k_i)_\mu x^\mu}(\xi_i^2-m^2)\tilde{G}(\xi_1,\ldots,\xi_N)$$
$$=\prod_{i=1}^{N}\frac{1}{\mathrm{i}\sqrt{Z}}\int \mathrm{d}^4\xi_i\,\delta^{(4)}(\xi_i-k_i)(\xi_i^2-m^2)\tilde{G}(\xi_1,\ldots,\xi_N)$$
$$=\prod_{i=1}^{N}\frac{k_i^2-m^2}{\mathrm{i}\sqrt{Z}}\tilde{G}(k_1,\ldots,k_N)=\prod_{i=1}^{N}\frac{k_i^2-m^2}{\mathrm{i}\sqrt{Z}}\int \mathrm{d}^4x_i\,e^{-\mathrm{i}k_{i\mu}x_i^{\mu}}G(x_1,\ldots,x_N)$$

Using $G(x_1,\ldots,x_n,y_1,\ldots,y_m)=\langle 0|\mathfrak{T}\{\phi(x_1)\ldots\phi(x_n)\phi(y_1)\ldots\phi(y_m)\}|0\rangle$ yields the thesis. □

[a] Technically, $\Box_{y_1}$ cannot be extracted from $\mathfrak{T}$, as it does not commute with the Heaviside distribution in its definition. However, the extraction of $\Box_{y_1}$ can be performed accounting for an additional term proportional to $\partial_0\theta({y_1}^0-{x_1}^0)[\partial_0\phi(y_1),\phi(x_1)]\sim\delta^{(4)}(y_1-x_1)$, which does not alter the singular structure of the LSZ formula, thus leaving the residue and the resulting amplitude unchanged.

Going on mass-shell $p^2\to m^2$, the Green function (or **correlation function**) can be shown to have singularities $\sim(p^2-m^2)^{-1}$ for each particle involved: these poles are precisely cancelled by the factors in the LSZ reduction formula, thus leaving a finite result. In this sense, the amplitude of the process can be seen as the multi-pole residue of the Green function.

§3.2.1.2 Electromagnetic field

Consider now a scattering process of a species of massless spin-1 particles. In the adiabatic hypothesis (recall Eq. 2.90-2.91):

$$\sqrt{2\omega_{\mathbf{p}}}a_{\mathbf{p},\lambda}^{\text{in,out}\,\dagger}=-\mathrm{i}\epsilon^{\mu *}(\mathbf{p},\lambda)\int \mathrm{d}^3x\,e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}A_\mu^{\text{in,out}}(x)\tag{3.27}$$

Lemma 3.2.2

$$\sqrt{2\omega_{\mathbf{p}}}(a_{\mathbf{p},\lambda}^{\text{in}\,\dagger}-a_{\mathbf{p},\lambda}^{\text{out}\,\dagger})=\frac{\mathrm{i}}{\sqrt{Z}}\epsilon^{\mu *}(\mathbf{p},\lambda)\int \mathrm{d}^4x\,e^{-\mathrm{i}p_\mu x^\mu}\Box A_\mu(x)\tag{3.28}$$

$$\sqrt{2\omega_{\mathbf{p}}}(a_{\mathbf{p},\lambda}^{\text{out}}-a_{\mathbf{p},\lambda}^{\text{in}})=\frac{\mathrm{i}}{\sqrt{Z}}\epsilon^{\mu}(\mathbf{p},\lambda)\int \mathrm{d}^4x\,e^{-\mathrm{i}p_\mu x^\mu}\Box A_\mu(x)\tag{3.29}$$

Proof. Using Eq. 3.23:

$$\sqrt{2\omega_{\mathbf{p}}}(a_{\mathbf{p},\lambda}^{\text{out}\,\dagger}-a_{\mathbf{p},\lambda}^{\text{in}\,\dagger})=-\frac{\mathrm{i}}{\sqrt{Z}}\epsilon^{\mu *}(\mathbf{p},\lambda)\int \mathrm{d}^4x\,\partial_0\left[e^{-\mathrm{i}p_\mu x^\mu}\overleftrightarrow{\partial_0}A_\mu(x)\right]$$
$$=-\frac{\mathrm{i}}{\sqrt{Z}}\epsilon^{\mu *}(\mathbf{p},\lambda)\int \mathrm{d}^4x\left[e^{-\mathrm{i}p_\mu x^\mu}\partial_0^2A_\mu(x)-A_\mu(x)\partial_0^2e^{-\mathrm{i}p_\mu x^\mu}\right]$$
$$=-\frac{\mathrm{i}}{\sqrt{Z}}\epsilon^{\mu *}(\mathbf{p},\lambda)\int \mathrm{d}^4x\left[e^{-\mathrm{i}p_\mu x^\mu}\partial_0^2A_\mu(x)-A_\mu(x)\boldsymbol{\nabla}^2e^{-\mathrm{i}p_\mu x^\mu}\right]$$

where the last line follows from $\partial_0^2 e^{-\mathrm{i}p_\mu x^\mu} = -p_0^2 e^{-\mathrm{i}p_\mu x^\mu} = (\mathbf{p} - p^2)e^{-\mathrm{i}p_\mu x^\mu} = \boldsymbol{\nabla}^2 e^{-\mathrm{i}p_\mu x^\mu}$. By the same observation made in the proof of Lemma 3.2.1, the second term is integrated by parts twice, resulting in:

$$\sqrt{2\omega_{\mathbf{p}}}(a^{\text{in}\,\dagger}_{\mathbf{p},\lambda} - a^{\text{out}\,\dagger}_{\mathbf{p},\lambda}) = \frac{\mathrm{i}}{\sqrt{Z}} \epsilon^{\mu *}(\mathbf{p},\lambda) \int \mathrm{d}^4x\, e^{-\mathrm{i}p_\mu x^\mu} (\partial_0^2 - \boldsymbol{\nabla}^2) A_\mu(x)$$

The other operator follows analogously. □

Note that, like for scalar fields, these operators vanish identically when dealing with free fields. Consider now a scattering process $|\mathbf{k}_1, \ldots, \mathbf{k}_n; -\infty\rangle \to |\mathbf{p}_1, \ldots, \mathbf{p}_m; +\infty\rangle$ in which no particles remain unchanged by the interaction, i.e. $\mathbf{k}_i \neq \mathbf{p}_j\ \forall i = 1, \ldots, n, \forall j = 1, \ldots, m$, so that the S-matrix reduces to the T-matrix. Then, as for Th. 3.2.1, particles can be extracted from the generic amplitude as:

$$\begin{aligned} &\langle \mathbf{k}_1, \ldots, \mathbf{k}_n | \mathrm{i}T | \mathbf{p}_1, \ldots, \mathbf{p}_m \rangle \\ &\quad = \prod_{i=1}^n \frac{k_i^2}{\mathrm{i}\sqrt{Z}} \int \mathrm{d}^4x_i\, e^{-\mathrm{i}k_{i\mu} x_i^\mu} \sum_{\lambda_i=1,2} \epsilon^{\mu_i *}(\mathbf{k}_i, \lambda_i) \prod_{j=1}^m \int \mathrm{d}^4y_j\, e^{+\mathrm{i}p_{j\mu} x_j^\mu} \sum_{\sigma_j=1,2} \epsilon^{\nu_j}(\mathbf{p}_j, \sigma_j) \times \\ &\quad\quad \times \langle 0 | \mathfrak{T}\{A_{\mu_1}(x_1) \ldots A_{\mu_n}(x_n) A_{\nu_1}(y_1) \ldots A_{\nu_m}(y_m)\} | 0 \rangle \end{aligned} \tag{3.30}$$

§3.2.1.3 Spinor fields

Consider now a scattering process of a single species of spin-$\frac{1}{2}$ particles (and anti-particles). The adiabatic hypothesis now yields (recall Eq. 2.32-2.33):

$$\sqrt{2E_{\mathbf{p}}} a^{\text{in,out}\dagger}_{\mathbf{p},s} = \frac{1}{\sqrt{Z}} \lim_{t\to\mp\infty} \int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu} \bar{\Psi}(x)\gamma^0 u^s(p) = \int \mathrm{d}^3x\, e^{-\mathrm{i}p_\mu x^\mu} \bar{\Psi}_{\text{in,out}}(x)\gamma^0 u^s(p) \tag{3.31}$$

$$\sqrt{2E_{\mathbf{p}}} b^{\text{in,out}\dagger}_{\mathbf{p},s} = \frac{1}{\sqrt{Z}} \lim_{t\to\mp\infty} \int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu} \bar{v}^s(p)\gamma^0 \Psi(x) = \int \mathrm{d}^3x\, e^{\mathrm{i}p_\mu x^\mu} \bar{v}^s(p)\gamma^0 \Psi_{\text{in,out}}(x) \tag{3.32}$$

Lemma 3.2.3

$$\sqrt{2E_{\mathbf{p}}}(a^{\text{in}\,\dagger}_{\mathbf{p},s} - a^{\text{out}\dagger}_{\mathbf{p},s}) = \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, \bar{\Psi}(x)(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m) u^s(p) e^{-\mathrm{i}p_\mu x^\mu} \tag{3.33}$$

$$\sqrt{2E_{\mathbf{p}}}(a^{\text{out}}_{\mathbf{p},s} - a^{\text{in}}_{\mathbf{p},s}) = -\frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{\mathrm{i}p_\mu x^\mu} \bar{u}^s(p)(\mathrm{i}\overrightarrow{\partial\!\!\!/} - m)\Psi(x) \tag{3.34}$$

$$\sqrt{2E_{\mathbf{p}}}(b^{\text{in}\,\dagger}_{\mathbf{p},s} - b^{\text{out}\dagger}_{\mathbf{p},s}) = \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{-\mathrm{i}p_\mu x^\mu} \bar{v}^s(p)(\mathrm{i}\overrightarrow{\partial\!\!\!/} - m)\Psi(x) \tag{3.35}$$

$$\sqrt{2E_{\mathbf{p}}}(b^{\text{out}}_{\mathbf{p},s} - b^{\text{in}}_{\mathbf{p},s}) = -\frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x\, \bar{\Psi}(x)(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m) v^s(p) e^{\mathrm{i}p_\mu x^\mu} \tag{3.36}$$

Proof. Using Eq. 3.23:

$$\sqrt{2E_{\mathbf{p}}}({a^{\text{in}}_{\mathbf{p},s}}^\dagger - {a^{\text{out}}_{\mathbf{p},s}}^\dagger) = -\frac{1}{\sqrt{Z}}\int \mathrm{d}^4x\, \partial_0 \left[e^{-\mathrm{i}p_\mu x^\mu}\bar{\Psi}(x)\gamma^0 u^s(p)\right]$$
$$= -\frac{1}{\sqrt{Z}}\int \mathrm{d}^4x\, \bar{\Psi}(x)\left[-\mathrm{i}\gamma^0 p_0 + \gamma^0\overleftarrow{\partial}_0\right] u^s(p)e^{-\mathrm{i}p_\mu x^\mu}$$

By Eq. 1.103 $(\gamma^0 p_0 + \gamma^k p_k - m)u^s(p) = 0$, so:

$$\sqrt{2E_{\mathbf{p}}}({a^{\text{in}}_{\mathbf{p},s}}^\dagger - {a^{\text{out}}_{\mathbf{p},s}}^\dagger) = -\frac{1}{\sqrt{Z}}\int \mathrm{d}^4x\, \bar{\Psi}(x)\left[\mathrm{i}\gamma^k p_k - \mathrm{i}m + \gamma^0\overleftarrow{\partial}_0\right] u^s(p)e^{-\mathrm{i}p_\mu x^\mu}$$
$$= -\frac{1}{\sqrt{Z}}\int \mathrm{d}^4x\, \bar{\Psi}(x)\left[\gamma^0\overleftarrow{\partial}_0 - \gamma^k\overrightarrow{\partial}_k - \mathrm{i}m\right] u^s(p)e^{-\mathrm{i}p_\mu x^\mu}$$

Assuming that initial and final particle states are convoluted to form wave-packets, integartion by parts in spatial dimensions can be carried out, yielding:

$$\sqrt{2E_{\mathbf{p}}}({a^{\text{in}}_{\mathbf{p},s}}^\dagger - {a^{\text{out}}_{\mathbf{p},s}}^\dagger) = -\frac{1}{\sqrt{Z}}\int \mathrm{d}^4x\, \bar{\Psi}(x)\left[\gamma^0\overleftarrow{\partial}_0 + \gamma^k\overleftarrow{\partial}_k - \mathrm{i}m\right] u^s(p)e^{-\mathrm{i}p_\mu x^\mu}$$
$$= \frac{i}{\sqrt{Z}}\int \mathrm{d}^4x\, \bar{\Psi}(x)\left[\mathrm{i}\overleftarrow{\partial\!\!\!/} + m\right] u^s(p)e^{-\mathrm{i}p_\mu x^\mu}$$

Other operators are found analogously. □

Note that, like Eq. 3.21, these operators vanish identically when dealing with a free field. Consider now a general scattering process involving both fermions and anti-fermions (quantities denoted by a tilde) $|\mathbf{k}_1, \dots, \mathbf{k}_{n_1}, \tilde{\mathbf{k}}_1, \dots, \tilde{\mathbf{k}}_{n_2}; -\infty\rangle \to |\mathbf{p}_1, \dots, \mathbf{p}_{m_1}, \tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{m_2}; +\infty\rangle$ in which no particles remain unchanged by the interaction, i.e. $\mathbf{k}_i \neq \mathbf{p}_j, \tilde{\mathbf{k}}_k \neq \tilde{\mathbf{p}}_l\ \forall i = 1, \dots, n_1,\ j = 1, \dots, m_1,\ k = 1, \dots, n_2,\ l = 1, \dots, m_2$, so that the S-matrix reduces to the T-matrix. By the same reasoning of Th. 3.2.1, it is clear how to extract particles from this generic amplitude:

$$\langle\beta_{\text{fin}}; +\infty|\mathbf{k}, s; -\infty\rangle \mapsto \frac{\mathrm{i}}{\sqrt{Z}}\int \mathrm{d}^4x \sum_{a=1}^{4} \langle\beta_{\text{fin}}|[\bar{\Psi}(x)]_a|0\rangle\, [(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m)u^s(k)]_a e^{-\mathrm{i}k_\mu x^\mu} \tag{3.37}$$

$$\langle\beta_{\text{fin}}; +\infty|\tilde{\mathbf{k}}, \tilde{s}; -\infty\rangle \mapsto \frac{i}{\sqrt{Z}}\int \mathrm{d}^4\tilde{x} \sum_{a=1}^{4} e^{-i\tilde{k}_\mu\tilde{x}^\mu}[\bar{v}^{\tilde{s}}(\tilde{k})(\mathrm{i}\overrightarrow{\partial\!\!\!/} - m)]_a \,\langle\beta_{\text{fin}}|[\Psi(\tilde{x})]_a|0\rangle \tag{3.38}$$

$$\langle\mathbf{p}, s; +\infty|\alpha_{\text{in}}; -\infty\rangle \mapsto -\frac{\mathrm{i}}{\sqrt{Z}}\int \mathrm{d}^4y \sum_{a=1}^{4} e^{\mathrm{i}p_\mu y^\mu}[\bar{u}^s(p)(\mathrm{i}\overrightarrow{\partial\!\!\!/} - m)]_a \,\langle 0|[\Psi(y)]_a|\alpha_{\text{in}}\rangle \tag{3.39}$$

$$\langle\tilde{\mathbf{p}}, \tilde{s}; +\infty|\alpha_{\text{in}}; -\infty\rangle \mapsto -\frac{\mathrm{i}}{\sqrt{Z}}\int \mathrm{d}^4\tilde{y} \sum_{a=1}^{4} \langle 0|[\bar{\Psi}(\tilde{y})]_a|\alpha_{\text{in}}\rangle\, [(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m)v^{\tilde{s}}(\tilde{p})]_a e^{\mathrm{i}\tilde{p}_\mu\tilde{y}^\mu} \tag{3.40}$$

In general, the final "reduced" amplitude formula will contain a $(n_1 + n_2 + m_1 + m_2)$-point Green function like (omitting spinor indices):

$$\langle 0|\mathfrak{T}\{\bar{\Psi}(x_1)\dots\bar{\Psi}(x_{n_1})\Psi(\tilde{x}_1)\dots\Psi(\tilde{x}_{n_2})\Psi(y_1)\dots\Psi(y_{m_1})\bar{\Psi}(\tilde{y}_1)\dots\bar{\Psi}(\tilde{y}_{m_2})\}|0\rangle$$

§3.2.2 Correlation functions

With the LSZ reduction formulae, the problem of computing scattering amplitudes is reduced to that of computing correlation functions.
Consider a generic quantum field $\phi(x)$ described by a Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$: the exact form of $\phi(x)$ is in general too difficult to obtain, as it satisfies a complicated non-linear equation of motion, so it cannot be written through an expansion in plane waves. To compute correlation functions, then, it is convenient to define a field related to $\phi(x)$:

$$\phi_I(t, \mathbf{x}) := e^{\mathrm{i}H_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-\mathrm{i}H_0(t-t_0)} \tag{3.41}$$

This field evolves with H_0, i.e. is a free field, and it is called the **interaction picture** field. Being this a free field, it can be expanded as:

$$\phi_I(t, \mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}}e^{-\mathrm{i}p_\mu x^\mu} + a_{\mathbf{p}}^\dagger e^{\mathrm{i}p_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}},\, x^0=t-t_0} \tag{3.42}$$

Proposition 3.2.3 (Interaction picture)
The transformation from the interaction picture to the Heisenberg picture is given by a unitary operator:

$$\phi(t, \mathbf{x}) = \mathrm{U}^\dagger(t, t_0)\phi_I\mathrm{U}(t, t_0) \tag{3.43}$$

with:

$$\mathrm{U}(t, t_0) \equiv e^{\mathrm{i}H_0(t-t_0)}e^{-\mathrm{i}H(t-t_0)} \tag{3.44}$$

Proof. By direct calculation:

$$\begin{aligned}
\phi(t, \mathbf{x}) &= e^{\mathrm{i}H(t-t_0)}\phi(t_0, \mathbf{x})e^{-\mathrm{i}H(t-t_0)} \\
&= e^{\mathrm{i}H(t-t_0)}e^{-\mathrm{i}H_0(t-t_0)}\left[e^{\mathrm{i}H_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-\mathrm{i}H_0(t-t_0)}\right]e^{\mathrm{i}H_0(t-t_0)}e^{-\mathrm{i}H(t-t_0)} \\
&= e^{\mathrm{i}H(t-t_0)}e^{-\mathrm{i}H_0(t-t_0)}\phi_I(t, \mathbf{x})e^{\mathrm{i}H_0(t-t_0)}e^{-\mathrm{i}H(t-t_0)} \equiv \mathrm{U}^\dagger(t, t_0)\phi_I(t, \mathbf{x})\mathrm{U}(t, t_0)
\end{aligned}$$

Unitarity is obvious, as H is a Hermitian operator. □

Note that, since $[H_0, H_{\text{int}}] \neq 0$ in general, the two exponentials cannot be combined trivially (the Baker-Campbell-Hausdorff formula should be used instead).

Proposition 3.2.4 (Time evolution operator)
he time evolution operator is:

$$\mathrm{U}(t, t_0) = \mathfrak{T}\exp\left[-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\, H_I(\tau)\right] \tag{3.45}$$

where:

$$H_I(t) \equiv e^{\mathrm{i}H_0(t-t_0)}H_{\text{int}}e^{-\mathrm{i}H_0(t-t_0)} \tag{3.46}$$

Proof. First of all, the time evolution operator solves the Schrödinger equation:

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}\mathrm{U}(t,t_0) &= e^{\mathrm{i}H_0(t-t_0)}(H-H_0)e^{-\mathrm{i}H(t-t_0)} = e^{\mathrm{i}H_0(t-t_0)}H_{\text{int}}e^{-\mathrm{i}H(t-t_0)} \\ &= e^{\mathrm{i}H_0(t-t_0)}H_{\text{int}}e^{-\mathrm{i}H_0(t-t_0)}e^{\mathrm{i}H_0(t-t_0)}e^{-\mathrm{i}H(t-t_0)} \equiv H_I(t)\mathrm{U}(t,t_0)\end{aligned}$$

The solution to this equation which satisfies the initial condition $\mathrm{U}(t_0,t_0)=1$ is unique:

$$\begin{aligned}\mathrm{U}(t,t_0) = \mathfrak{T}\exp\left[-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\, H_I(\tau)\right] &= \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int_{[t_0,t]^n}\mathrm{d}t_1\dots\mathrm{d}t_n\,\mathfrak{T}\{H_I(t_1)\dots H_I(t_n)\} \\ &= \sum_{n=0}^{\infty}(-\mathrm{i})^n\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\dots\int_{t_0}^{t_{n-1}}\mathrm{d}t_n\, H_I(t_1)\dots H_I(t_n)\end{aligned}$$

It is in fact clear that:

$$\frac{\partial}{\partial t}\mathrm{U}(t,t_0) = \sum_{n=0}^{\infty}(-\mathrm{i})^n\int_{t_0}^{t}\mathrm{d}t_2\dots\int_{t_0}^{t_{n-1}}\mathrm{d}t_n\, H_I(t)H_I(t_2)\dots H_I(t_n) = -\mathrm{i}H_I(t)\mathrm{U}(t,t_0)$$

Hence, this is a solution to the above equation. □

Lemma 3.2.4

$$\mathrm{U}(t_1,t_0)\mathrm{U}(t_0,t_2) = \mathrm{U}(t_1,t_2) \tag{3.47}$$

Proof. Observe that:

$$\mathrm{i}\frac{\partial}{\partial t}[\mathrm{U}(t,t_0)\mathrm{U}(t_0,t_2)] = \left[\mathrm{i}\frac{\partial}{\partial t}\mathrm{U}(t,t_0)\right]\mathrm{U}(t_0,t_2) = H_I(t)\mathrm{U}(t,t_0)\mathrm{U}(t_0,t_2)$$

The boundary condition now reads $[\mathrm{U}(t,t_0)\mathrm{U}(t_0,t_2)]_{t=t_0} = \mathrm{U}(t_0,t_2)$, and the unique solution for this boundary condition is:

$$\mathrm{U}(t,t_0)\mathrm{U}(t_0,t_2) = \mathfrak{T}\exp\left[-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\, H_I(\tau)\right]$$

which is precisely $\mathrm{U}(t,t_2)$. □

It is now possible to compute correlation functions.

Theorem 3.2.2 (Correlation functions)

Given a quantum field $\phi(x)$ described by a Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$, then:

$$\langle 0|\mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \frac{\langle 0|\mathfrak{T}\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathcal{H}_I\right]\}|0\rangle}{\langle 0|\mathfrak{T}\{\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathcal{H}_I\right]\}|0\rangle} \tag{3.48}$$

Proof. Consider WLOG $\{t_k\}_{k=1,\dots,n} : t_i < t_j$. Using Eq. 3.43 and Lemma 3.2.4:

$$\begin{aligned}\langle 0|\phi(x_1)\dots\phi(x_n)|0\rangle &= \langle 0|\mathrm{U}^\dagger(t_1,t_0)\phi_I(x_1)\mathrm{U}(t_1,t_0)\mathrm{U}^\dagger(t_2,t_0)\dots\mathrm{U}^\dagger(t_n,t_0)\phi_I(x_n)\mathrm{U}(t_n,t_0)|0\rangle \\ &= \langle 0|\mathrm{U}^\dagger(t_1,t_0)\phi_I(x_1)\mathrm{U}(t_1,t_2)\dots\mathrm{U}(t_{n-1},t_n)\phi_I(x_n)\mathrm{U}(t_n,t_0)|0\rangle\end{aligned}$$

Consider $t \in \mathbb{R} : t \gg t_1 > \dots > t_n \gg -t$, so that $\mathrm{U}(t_n,t_0) = \mathrm{U}(t_n,-t)\mathrm{U}(-t,t_0)$ and $\mathrm{U}^\dagger(t_1,t_0) = \mathrm{U}^\dagger(t,t_0)\mathrm{U}(t,t_1)$. Therefore:

$$\begin{aligned}\langle 0|\phi(x_1)\dots\phi(x_n)|0\rangle = \langle 0|\,\mathrm{U}^\dagger(t,t_0)[\mathrm{U}(t,t_1)\phi_I(x_1)\mathrm{U}(t_1,t_2)\phi_I(x_2)\mathrm{U}(t_2,t_3)\dots \\ \dots\mathrm{U}(t_{n-1},t_n)\phi_I(x_n)\mathrm{U}(t_n,-t)]\mathrm{U}(-t,t_0)\,|0\rangle\end{aligned}$$

Note that this expression inside the square brackets is automatically time-ordered, so it may be rewritten as:

$$\begin{aligned}[\dots] &= \mathfrak{T}\{\phi_I(x_1)\dots\phi_I(x_n)\mathrm{U}(t,t_1)\mathrm{U}(t_1,t_2)\dots\mathrm{U}(t_n,-t)\} \\ &= \mathfrak{T}\left\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-\mathrm{i}\int_{-t}^{t}\mathrm{d}\tau\, H_I(\tau)\right]\right\}\end{aligned}$$

These considerations hold for arbitrary t_0, so it can be set $t_0 = -t \to -\infty$. Then:

$$\langle 0|\mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \langle 0|\mathrm{U}^\dagger(+\infty,-\infty)\mathfrak{T}\left\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathcal{H}_I\right]\right\}|0\rangle$$

Note that $\langle 0|\,\mathrm{U}^\dagger(\infty,-\infty)$ is the Hermitian conjugate of $\mathrm{U}(\infty,-\infty)\,|0\rangle$, i.e. the state obtained evolving in time the vacuum state. As it was already assumed, the initial-state vacuum coincides physically to the final-state vacuum, i.e.:

$$\mathrm{U}(\infty,-\infty)\,|0\rangle = e^{\mathrm{i}\alpha}\,|0\rangle \qquad \Longrightarrow \qquad e^{-\mathrm{i}\alpha} = \left(\langle 0|\mathfrak{T}\left\{\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathcal{H}_I\right]\right\}|0\rangle\right)^{-1}$$

Finally:

$$\langle 0|\mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \frac{\langle 0|\mathfrak{T}\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathcal{H}_I\right]\}|0\rangle}{\langle 0|\mathfrak{T}\{\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathcal{H}_I\right]\}|0\rangle}$$

which is the thesis. □

This allows to compute correlation functions from free fields, rather than interaction fields. Moreover, note that the functional dependence of $\mathcal{H}_I$ in terms of $\phi_I(x)$ is the same as that of $\mathcal{H}_{\text{int}}$ in terms of $\phi(x)$.

Example 3.2.1 (ϕ^4 potential)

Consider the ϕ^4-interaction Hamiltonian $\mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\phi^4$. Then:

$$\mathcal{H}_I = \frac{\lambda}{4!}e^{\mathrm{i}H_0(t-t_0)}\phi^4 e^{-\mathrm{i}H_0(t-t_0)} = \frac{\lambda}{4!}\left[e^{\mathrm{i}H_0(t-t_0)}\phi e^{-\mathrm{i}H_0(t-t_0)}\right]^4 = \frac{\lambda}{4!}\phi_I^4$$

Note that Eq. 3.48 is naturally suited for perturbative evaluation with respect to coupling constants which appear in $\mathcal{H}_{\text{int}}$, thanks to the exponential function.

§3.3 Feynman diagrams

The problem of computing scattering amplitudes has been further reduced to that of computing n-point Green functions of free (interaction picture) fields.

§3.3.1 Propagators

§3.3.1.1 Feynman propagator

Theorem 3.3.1 (Feynman propagator)
Given a real scalar field $\phi(x)$, the **Feynman propagator** is computed as:

$$\Delta(x-y) := \langle 0|\mathfrak{T}\{\phi_I(x)\phi_I(y)\}|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}e^{-\mathrm{i}p_\mu(x-y)^\mu} \tag{3.49}$$

with $\epsilon \to 0^+$.

Proof. First, decompose $\phi_I(x)$ into its positive- and negative-frequency parts (i.e. annihilation and creation parts):

$$\phi_I^+(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}a_\mathbf{p}e^{-\mathrm{i}p_\mu x^\mu} \qquad \phi_I^-(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}a_\mathbf{p}^\dagger e^{\mathrm{i}p_\mu x^\mu}$$

Clearly $\phi_I^+(x)\,|0\rangle = 0$ and $\langle 0|\,\phi_I^-(x) = 0$. Consider first the case $x^0 > y^0$:

$$\begin{aligned}
\mathfrak{T}\{\phi_I(x)\phi_I(y)\} &= \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) \\
&= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x),\phi_I^-(y)] \\
&= \mathfrak{N}\{\phi_I(x)\phi_I(y)\} + [\phi^+(x),\phi^-(y)]
\end{aligned}$$

Similarly, for $y^0 > x^0$ one has $\mathfrak{T}\{\phi_I(x)\phi_I(y)\} = \mathfrak{N}\{\phi_I(x)\phi_I(y)\} + [\phi^+(y),\phi^-(x)]$. Therefore, in general:

$$\begin{aligned}
\mathfrak{T}\{\phi_I(x)\phi_I(y)\} &= \mathfrak{N}\{\phi_I(x)\phi_I(y)\} + \theta(x^0-y^0)[\phi_I^+(x),\phi_I^-(y)] + \theta(y^0-x^0)[\phi_I^+(y),\phi_I^-(x)] \\
&\equiv \mathfrak{N}\{\phi_I(x)\phi_I(y)\} + \Delta(x-y)
\end{aligned}$$

Now, observe that $\langle 0|\mathfrak{N}\{\phi_I(x)\phi_I(y)\}|0\rangle = 0$, as there is always either an annihilation operator acting on $|0\rangle$ or a creation operator acting on $\langle 0|$. On the other hand, $\Delta(x-y) \in \mathbb{C}$, as $[a_\mathbf{p}, a_\mathbf{q}^\dagger] \in \mathbb{C}$, therefore $\langle 0|D_\mathrm{F}(x-y)|0\rangle = \Delta(x-y)\,\langle 0|0\rangle = \Delta(x-y)$. There only remains to compute the commutators:

$$\begin{aligned}
\Delta(x-y) &= \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\big[\theta(x^0-y^0)[a_\mathbf{p},a_\mathbf{q}^\dagger]e^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}q_\mu y^\mu} + \\
&\qquad\qquad + \theta(y^0-x^0)[a_\mathbf{p},a_\mathbf{q}^\dagger]e^{-\mathrm{i}p_\mu y^\mu+\mathrm{i}q_\mu x^\mu}\big] \\
&= \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^3\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[\theta(x^0-y^0)e^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}q_\mu y^\mu} + \theta(y^0-x^0)e^{-\mathrm{i}p_\mu y^\mu+\mathrm{i}q_\mu x^\mu}\right] \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_\mathbf{p}}\left[\theta(x^0-y^0)e^{-\mathrm{i}p_\mu(x-y)^\mu} + \theta(y^0-x^0)e^{\mathrm{i}p_\mu(x-y)^\mu}\right]
\end{aligned}$$

To show that this is equivalent to the thesis, note that Eq. 3.49 can be rewritten as:

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int_{\mathbb{R}} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon} e^{-ip^0(x^0-y^0)}$$

as $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$. The integral in dp^0 can be computed in the complex p^0-plane[a]: the iϵ-prescription slightly displaces the poles from the real axis[b], and in particular that in $p^0 = +E_{\mathbf{p}}$ is slightly below it while that in $p^0 = -E_{\mathbf{p}}$ is slightly above it. If $x^0 - y^0 > 0$, choose a semicircular (clockwise) contour in the lower-half plane, so that only the $p^0 = E_{\mathbf{p}}$ singularity is enclosed:

$$\Delta(x-y)|_{x^0>y^0} = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{i}{2\pi} \left[(-2\pi i)\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}\right] = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-ip_\mu(x-y)^\mu}$$

If $x^0 - y^0 < 0$, choose a semicircular (counter-clockwise) contour in the upper-half plane:

$$\Delta(x-y)|_{y^0>x^0} = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{i}{2\pi} \left[(2\pi i)\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{-2E_{\mathbf{p}}}\right] = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{ip_\mu(x-y)^\mu}$$

where in the last integral $\mathbf{p} \mapsto -\mathbf{p}$ was renamed. The proof is complete. □

[a] As the integrand is $\sim (p^0)^{-2}$, the contribution of the curved part of a semicircular contour vanishes, only leaving the integral over the real axis.

[b] The poles are $p^0 = \pm\sqrt{E_{\mathbf{p}}^2 - i\epsilon} = \pm\left(E_{\mathbf{p}} - \frac{i\epsilon}{2E_{\mathbf{p}}}\right) + o(\epsilon^2)$.

The expression for the Feynman propagator in momentum space is trivially found:

$$\tilde{\Delta}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{3.50}$$

Moreover, the Feynman propagator is just a Green function for the KG operator:

$$(\Box_x + m^2)\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}(-p^2 + m^2)e^{-ip_\mu(x-y)^\mu} = -i\delta^{(4)}(x-y)$$

§3.3.1.2 Dirac propagator

To explicitly compute the propagator for the Dirac field, it is necessary to include fermionic anti-symmetry when resolving time-ordering and normal-ordering, i.e. a factor of (-1) for each exchange of fermionic operators, e.g. $\mathfrak{T}\{\Psi(x_1)\Psi(x_2)\Psi(x_3)\Psi(x_4)\} = (-1)^3\Psi(x_3)\Psi(x_1)\Psi(x_4)\Psi(x_2)$ if $x_3^0 > x_1^0 > x_4^0 > x_2^0$, or $\mathfrak{N}\{a_{\mathbf{p}}a_{\mathbf{q}}a_{\mathbf{r}}^\dagger\} = (-1)^2 a_{\mathbf{r}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}} = (-1)^3 a_{\mathbf{r}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}$ (equal expressions).

Definition 3.3.1 (Fermionic time-ordering)
Given a Dirac field $\Psi(x)$, the **time-ordering operator** acts as:

$$\mathfrak{T}\{\Psi(x)\bar{\Psi}(y)\} := \begin{cases} \Psi(x)\bar{\Psi}(y) & x^0 > y^0 \\ -\bar{\Psi}(y)\Psi(x) & y^0 > x^0 \end{cases} \tag{3.51}$$

Theorem 3.3.2 (Dirac propagator)
Given a Dirac field $\Psi(x)$, the **Dirac propagator** is computed as:

$$\Sigma(x-y) := \langle 0|\mathfrak{T}\{\Psi_I(x)\bar{\Psi}_I(y)\}|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon}e^{-\mathrm{i}p_\mu(x-y)^\mu} \tag{3.52}$$

Proof. First, decompose $\Psi_I(x)$ and $\Psi_I(y)$ into positive- and negative-frequency parts:

$$\Psi_I^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2} a_{\mathbf{p},s}u^s(p)e^{-\mathrm{i}p_\mu x^\mu} \quad \Psi_I^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2} b^\dagger_{\mathbf{p},s}v^s(p)e^{\mathrm{i}p_\mu x^\mu}$$
$$\bar{\Psi}_I^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2} b_{\mathbf{p},s}\bar{v}^s(p)e^{-\mathrm{i}p_\mu x^\mu} \quad \bar{\Psi}_I^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2} a^\dagger_{\mathbf{p},s}\bar{u}^s(p)e^{\mathrm{i}p_\mu x^\mu}$$

Clearly $\Psi_I^+(x)\left|0\right\rangle = 0 = \bar{\Psi}_I^+(x)\left|0\right\rangle$ and $\left\langle 0\right|\Psi_I^-(x) = 0 = \left\langle 0\right|\bar{\Psi}_I^-(x)$. Consider first the case $x^0 > y^0$:

$$\begin{aligned}
\mathfrak{T}\{\Psi_I(x)\bar{\Psi}_I(y)\} &= \Psi_I^+(x)\bar{\Psi}_I^+(y) + \Psi_I^+(x)\bar{\Psi}_I^-(y) + \Psi_I^-(x)\bar{\Psi}_I^+(y) + \Psi_I^-(x)\bar{\Psi}_I^-(y)\\
&= \Psi_I^+(x)\bar{\Psi}_I^+(y) - \bar{\Psi}_I^-(y)\Psi_I^+(x) + \Psi_I^-(x)\bar{\Psi}_I^+(y) + \Psi_I^-(x)\bar{\Psi}_I^-(y) + \{\Psi_I^+(x),\bar{\Psi}_I^-(y)\}\\
&\equiv \mathfrak{N}\{\Psi_I(x)\bar{\Psi}_I(y)\} + \{\Psi_I^+(x),\bar{\Psi}_I^-(y)\}
\end{aligned}$$

Similarly, for $y^0 > x^0$:

$$\begin{aligned}
\mathfrak{T}\{\Psi_I(x)\bar{\Psi}_I(y)\} &= -\bar{\Psi}_I^+(y)\Psi_I^+(x) - \bar{\Psi}_I^-(y)\Psi_I^+(x) - \bar{\Psi}_I^+(y)\Psi_I^-(x) - \bar{\Psi}_I^-(y)\Psi_I^-(x)\\
&= \Psi_I^+(x)\bar{\Psi}_I^+(y) - \bar{\Psi}_I^-(y)\Psi_I^+(x) + \Psi_I^-(x)\bar{\Psi}_I^+(y) + \Psi_I^-(x)\bar{\Psi}_I^-(y) - \{\bar{\Psi}_I^+(y),\Psi_I^-(x)\}\\
&\equiv \mathfrak{N}\{\Psi_I(x)\bar{\Psi}_I(y)\} - \{\bar{\Psi}_I^+(y),\Psi_I^-(x)\}
\end{aligned}$$

Therefore, in general[a]:

$$\begin{aligned}
\mathfrak{T}\{\Psi_I(x)\bar{\Psi}_I(y)\} &= \mathfrak{N}\{\Psi_I(x)\bar{\Psi}_I(y)\} + \theta(x^0-y^0)\{\Psi_I^+(x),\bar{\Psi}_I^-(y)\} - \theta(y^0-x^0)\{\bar{\Psi}_I^+(y),\Psi_I^-(x)\}\\
&= \mathfrak{N}\{\Psi_I(x)\bar{\Psi}_I(y)\} + \Sigma(x-y)
\end{aligned}$$

By the same observations as in the previous proof $\langle 0|\mathfrak{T}\{\Psi_I(x)\bar{\Psi}_I(y)\}|0\rangle = \Sigma(x-y)$, so explicitly computing the anti-commutators:

$$\begin{aligned}
\Sigma(x-y) &= \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{s=1,2}\sum_{r=1,2}\Big[\theta(x^0-y^0)\{a_{\mathbf{p},s},a^\dagger_{\mathbf{q},r}\}u^s(p)\bar{u}^r(q)e^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}q_\mu y^\mu}+\\
&\qquad - \theta(y^0-x^0)\{b_{\mathbf{p},s},b^\dagger_{\mathbf{q},r}\}v^r(q)\bar{v}^s(p)e^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}q_\mu y^\mu}\Big]\\
&= \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{s=1,2}\sum_{r=1,2}\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{sr}\Big[\theta(x^0-y^0)u^s(p)\bar{u}^r(q)e^{-\mathrm{i}p_\mu y^\mu+\mathrm{i}q_\mu x^\mu}+\\
&\qquad - \theta(y^0-x^0)v^r(q)\bar{v}^s(p)e^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}q_\mu y^\mu}\Big]\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\sum_{s=1,2}\Big[\theta(x^0-y^0)u^s(p)\bar{u}^s(p)e^{-\mathrm{i}p_\mu(x-y)^\mu} - \theta(y^0-x^0)v^s(p)\bar{v}^s(p)e^{\mathrm{i}p_\mu(x-y)^\mu}\Big]
\end{aligned}$$

Recalling Eq. 1.111:

$$\Sigma(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[\theta(x^0-y^0)(\not{p}+m)e^{-\mathrm{i}p_\mu(x-y)^\mu} - \theta(y^0-x^0)(\not{p}-m)e^{\mathrm{i}p_\mu(x-y)^\mu}\Big]$$

By the same technique in the previous proof, to show that this expression is equal to the thesis, rewrite Eq. 3.52 as:

$$\Sigma(x-y)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int\frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{i}(p^0\gamma^0-\mathbf{p}\cdot\boldsymbol{\gamma}+m)}{(p^0)^2-E_{\mathbf{p}}^2+\mathrm{i}\epsilon}e^{-\mathrm{i}p^0(x^0-y^0)}$$

The $i\epsilon$-prescription shifts the pole in $p^0>0$ below the real axis and that in $p^0<0$ above it. If $x^0>y^0$, chose a semicircular (clockwise) contour in the lower-hald plane, so that only the $p^0=E_{\mathbf{p}}$ singularity is enclosed:

$$\begin{aligned}\Sigma(x-y)|_{x^0>y^0}&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{\mathrm{i}}{2\pi}\left[(-2\pi\mathrm{i})\frac{E_{\mathbf{p}}\gamma^0-\mathbf{p}\cdot\boldsymbol{\gamma}+m}{2E_{\mathbf{p}}}e^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}\right]\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^32E_{\mathbf{p}}}e^{\mathrm{i}p_\mu\cdot(x-y)^\mu}(E_{\mathbf{p}}\gamma^0-\mathbf{p}\cdot\boldsymbol{\gamma}+m)=\int\frac{\mathrm{d}^3p}{(2\pi)^32E_{\mathbf{p}}}e^{\mathrm{i}p_\mu\cdot(x-y)^\mu}(\not{p}+m)\end{aligned}$$

If instead $y^0>x^0$, chose a semicircular (counter-clockwise) contour in the upper-half plane:

$$\begin{aligned}\Sigma(x-y)|_{x^0<y^0}&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{\mathrm{i}}{2\pi}\left[(2\pi\mathrm{i})\frac{-E_{\mathbf{p}}\gamma^0-\mathbf{p}\cdot\boldsymbol{\gamma}+m}{-2E_{\mathbf{p}}}e^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}\right]\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^32E_{\mathbf{p}}}e^{\mathrm{i}p_\mu\cdot(x-y)^\mu}(-E_{\mathbf{p}}\gamma^0+\mathbf{p}\cdot\boldsymbol{\gamma}+m)\\&=-\int\frac{\mathrm{d}^3p}{(2\pi)^32E_{\mathbf{p}}}e^{\mathrm{i}p_\mu\cdot(x-y)^\mu}(\not{p}-m)\end{aligned}$$

where in the second equality $\mathbf{p}\mapsto-\mathbf{p}$ was renamed. The proof is complete. □

[a]To be precise, all these quantities should have spinor indices: indeed, $\Psi\bar{\Psi}\in\mathbb{C}^{4\times4}$ while $\bar{\Psi}\Psi\in\mathbb{C}$, so the anti-commutator $\{\Psi,\bar{\Psi}\}$ is only defined with spinor indices, i.e. $\{\Psi_\alpha,\bar{\Psi}_\beta\}$. To shorten the notation, spinor indices are suppressed, and $\{\Psi,\bar{\Psi}\},\{\bar{\Psi},\Psi\}$ are treated as $\mathbb{C}^{4\times4}$ matrices, always writing Dirac adjoints on the right in the explicit expressions.

Note that $\Sigma(x-y)\in\mathbb{C}^{4\times4}$, with two Dirac indices: $\Sigma_{\alpha\beta}(x-y)\equiv\langle0|\mathfrak{T}\{\Psi_{I,\alpha}(x)\bar{\Psi}_{I,\beta}(y)\}|0\rangle$. It is straightforward to see that the Feynman propagator in momentum space is:

$$\tilde{\Sigma}(x-y)=\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon}\tag{3.53}$$

Lemma 3.3.1 (Dirac propagator in momentum space)

$$\tilde{\Sigma}(x-y)=\frac{\mathrm{i}}{\not{p}-m}\tag{3.54}$$

Proof. $(\not{p}+m)(\not{p}-m)=\gamma^\mu\gamma^\nu p_\mu p_\nu-m^2=\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}p_\mu p_\nu-m^2=\eta^{\mu\nu}p_\mu p_\nu-m^2=p^2-m^2$. □

This is interpreted as the inverse matrix $\mathrm{i}(\not{p}-m)^{-1}$. As for the Feynman propagator, the Dirac propagator too is a Green function for the Dirac operator:

$$(\mathrm{i}\not{\partial}_x-m)\Sigma(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{\not{p}-m}(\gamma^\mu p_\mu-m)e^{-\mathrm{i}p_\mu(x-y)^\mu}=\mathrm{i}\delta^{(4)}(x-y)$$

§3.3.2 Wick's theorem

It is possible to simplify the computation of a general n-point Green function in terms of 2-point Green functions. In the following, the subscript I for the interaction-picture field is omitted.

§3.3.2.1 Bosonic fields

Definition 3.3.2 (Contraction)

Given a bosonic field $\phi(x)$, its **contraction** is defined as:

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) := \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & y^0 > x^0 \end{cases} \tag{3.55}$$

Recalling the proof of Th. 3.3.1, it is clear that:

$$\mathfrak{T}\{\phi(x)\phi(y)\} = \mathfrak{N}\{\phi(x)\phi(y)\} + \contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) \tag{3.56}$$

This relation can be generalized to n-point Green functions.

Theorem 3.3.3 (Wick's theorem)

Given a bosonic field $\phi(x)$:

$$\mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\} = \mathfrak{N}\{\phi(x_1)\dots\phi(x_n)\ + \text{all possible contractions}\} \tag{3.57}$$

where both partially-contracted and fully-contracted terms are considered.

Proof. Use induction on n. Eq. 3.56 proves the $n = 2$ case, so assume the theorem true for generic $n-1$ and consider the n case. WLOG $x_1^0 \geq \dots \geq x_n^0$ (if that's not the case, just relabel the points, as it does not affect Eq. 3.57), so, using the inductive step:

$$\begin{aligned}\mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\} &= \phi(x_1)\mathfrak{T}\{\phi(x_2)\dots\phi(x_n)\} \\ &= (\phi^+(x_1) + \phi^-(x_1))\mathfrak{N}\{\phi(x_2)\dots\phi(x_n) + \text{all contr. without } \phi(x_1)\}\end{aligned}$$

$\phi^-(x_1)$ is already in normal order (on the left), while $\phi^+(x_1)$ must be commuted with all other $\phi(x_i)$. For the term without contractions:

$$\begin{aligned}\phi^+(x_1)\mathfrak{N}\{\phi(x_2)\dots\phi(x_n)\} &= \mathfrak{N}\{\phi(x_2)\dots\phi(x_n)\}\phi^+(x_1) + [\phi^+(x_1), \mathfrak{N}\{\phi(x_2)\dots\phi(x_n)\}] \\ &= \mathfrak{N}\{\phi^+(x_1)\phi(x_2)\dots\phi(x_n)\} \\ &\qquad + \mathfrak{N}\{\dots + \phi(x_2)\dots[\phi^+(x_2), \phi^-(x_i)]\dots\phi(x_n) + \dots\} \\ &= \mathfrak{N}\{\phi^+(x_1)\phi(x_2)\dots\phi(x_n) + \dots \contraction{}{\phi}{^+(x_1)\phi(x_2)\dots}{\phi}\phi^+(x_1)\phi(x_2)\dots\phi(x_i)\dots\phi(x_n) + \dots\}\end{aligned}$$

Analogously, all possible contractions are obtained, proving the theorem. □

Note that only fully-contracted terms contribute to the n-point Green function: as every term with at least an uncontracted field vanishes when put between $\langle 0| \cdot |0\rangle$, as there will be an annihilation operator acting on $|0\rangle$ or a creation operator acting on $\langle 0|$.

Example 3.3.1 (4-point Green function)

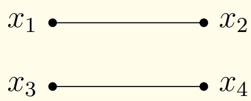
Applying Wick's theorem to a 4-point Green function yields:

$$
\begin{aligned}
\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = \mathfrak{N}\{&\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)+\\
&+ \wick{\c1\phi(x_1)\c1\phi(x_2)\phi(x_3)\phi(x_4)} + \wick{\c1\phi(x_1)\phi(x_2)\c1\phi(x_3)\phi(x_4)} + \wick{\c1\phi(x_1)\phi(x_2)\phi(x_3)\c1\phi(x_4)}\\
&+ \wick{\phi(x_1)\c1\phi(x_2)\c1\phi(x_3)\phi(x_4)} + \wick{\phi(x_1)\c1\phi(x_2)\phi(x_3)\c1\phi(x_4)} + \wick{\phi(x_1)\phi(x_2)\c1\phi(x_3)\c1\phi(x_4)}\\
&+ \wick{\c1\phi(x_1)\c1\phi(x_2)\c2\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c2\phi(x_3)\c1\phi(x_4)}
\end{aligned}
$$

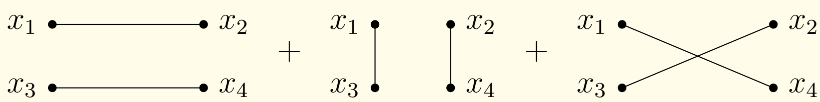
Only the last three terms contribute to the 4-point Green function. If $\phi(x)$ is a scalar field, then:

$$
\begin{aligned}
\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = \Delta(x_1-x_2)\Delta(x_3-x_4) + \Delta(x_1-x_3)\Delta(x_2-x_4)+\\
+\Delta(x_1-x_4)\Delta(x_2-x_3)
\end{aligned}
$$

This expression has immediate physical interpretation: indeed, $\Delta(x_1-x_2)$ can be interpreted as the amplitude for the propagation of a particle from spacetime point x_1 to x_2, while $\Delta(x_1-x_2)\Delta(x_3-x_4)$ is the amplitude for the propagation of a particle from x_1 to x_2 and one from x_3 to x_3, without interacting with each other.
It is possible to associate intuitive graphs, called **Feynman diagrams**, to these kind of expressions; for example:

$\Delta(x_1-x_2)\Delta(x_3-x_4) =$ 

Then, the 4-point Green function can be represented as:

$\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} =$ 

Using this formalism to compute Eq. 3.48, when expanding the exponential in powers of $\mathcal{H}_I$, each term contains fields at the same spacetime point: this gives rise to less trivial Feynman diagrams.

§3.3.2.2 Fermionic fields

Definition 3.3.3 (Contraction)

Given a fermionic $\Psi(x)$, its **contraction** is defined as:

$$
\wick{\c1\Psi(x)\c1{\bar\Psi}(y)} := \begin{cases} \{\Psi^+(x), \bar\Psi^-(y)\} & x^0 > y^0 \\ -\{\bar\Psi^+(y), \Psi^-(x)\} & y^0 > x^0 \end{cases} \tag{3.58}
$$

and $\wick{\c1\Psi(x)\c1\Psi(y)} = \wick{\c1{\bar\Psi}(x)\c1{\bar\Psi}(y)} = 0$.

By the proof of Th. 3.3.2, it is clear that:

$$\mathfrak{T}\{\Psi(x)\bar{\Psi}(y)\} = \mathfrak{N}\{\Psi(x)\bar{\Psi}(y)\} + \overline{\Psi(x)\bar{\Psi}}(y) \tag{3.59}$$

Wick's theorem remains basically unchanged for fermions.

Theorem 3.3.4 (Wick's theorem)
Given a fermionic field $\Psi(x)$:

$$\mathfrak{T}\{\Psi(x_1)\bar{\Psi}(x_2)\dots\} = \mathfrak{N}\{\Psi(x_1)\bar{\Psi}(x_2)\dots + \text{all possible contractions}\} \tag{3.60}$$

where both partially-contracted and fully-contracted terms are considered.

Proof. Inserting everything inside $\mathfrak{N}\{\cdot\}$ ensures that the correct signs are considered due to fermionic anti-commutation, therefore the proof is analogous to that of Th. 3.3.3. □

As for the bosonic counterparts, only fully-contracted terms contribute to the n-point Green function, as each partially-contracted term will have either an annihilation operator acting on $|0\rangle$ or a creation operator acting on $\langle 0|$ (or both).

§3.3.2.3 Scattering in $\lambda\phi^4$-theory

Consider a real scalar field theory with $\mathcal{H}_I = \frac{\lambda}{4!}\phi^4$ and consider a scattering process with two initial particles with momenta $\mathbf{k}_1, \mathbf{k}_2$ and two final particles with momenta $\mathbf{p}_1, \mathbf{p}_2$. Eq. 3.25 and Eq. 3.48 give the amplitude:

$$\begin{aligned}\langle \mathbf{p}_1, \mathbf{p}_2|\mathrm{i}T|\mathbf{k}_1, \mathbf{k}_2\rangle = &\frac{k_1^2-m^2}{\mathrm{i}\sqrt{Z}}\frac{k_2^2-m^2}{\mathrm{i}\sqrt{Z}}\frac{p_1^2-m^2}{\mathrm{i}\sqrt{Z}}\frac{p_2^2-m^2}{\mathrm{i}\sqrt{Z}}\int\prod_{i=1}^{4}\mathrm{d}^4x_i\, e^{\mathrm{i}(p_{1\mu}x_1{}^\mu+p_{2\mu}x_2{}^\mu-p_{3\mu}x_3{}^\mu-p_{4\mu}x_4{}^\mu)}\times\\ &\times\frac{\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\exp\left[-\mathrm{i}\frac{\lambda}{4!}\int \mathrm{d}^4x\,\phi(x)\right]\}|0\rangle}{\langle 0|\mathfrak{T}\{\exp\left[-\mathrm{i}\frac{\lambda}{4!}\int \mathrm{d}^4x\,\phi(x)\right]\}|0\rangle}\end{aligned}$$

To compute this expression up to $o(\lambda)$, consider that for $\lambda\phi^4$-theory $Z = 1 + o(\lambda^2)$, thus it is safe to set $Z \equiv 1$.
The $o(\lambda^0)$ term is obtained setting $\lambda = 0$, i.e. no coupling, but in the absence of coupling there is no scattering either and the amplitude must be trivial. Recalling Ex. 3.3.1:

$$\begin{aligned}&\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\, e^{\mathrm{i}(p_{1\mu}x_1{}^\mu+p_{2\mu}x_2{}^\mu-p_{3\mu}x_3{}^\mu-p_{4\mu}x_4{}^\mu)}\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle\\ &\quad=\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\, e^{\mathrm{i}(p_{1\mu}x_1{}^\mu+p_{2\mu}x_2{}^\mu-p_{3\mu}x_3{}^\mu-p_{4\mu}x_4{}^\mu)}\left[\Delta(x_1-x_2)\Delta(x_3-x_4)+\dots\right]\\ &\quad=\left[\int \mathrm{d}^4x\mathrm{d}^4X\, e^{\mathrm{i}(p_1+p_2)_\mu X^\mu+\mathrm{i}(p_1-p_2)_\mu x^\mu/2}\Delta(x)\right]\left[\int \mathrm{d}^4y\mathrm{d}^4Y\, e^{-\mathrm{i}(k_1+k_2)_\mu Y^\mu-\mathrm{i}(k_1-k_2)_\mu y^\mu/2}\Delta(y)\right]+\dots\\ &\quad=(2\pi)^4\delta^{(4)}(p_1+p_2)\frac{\mathrm{i}}{\left(\frac{p_1-p_2}{2}\right)^2-m^2}(2\pi)^4\delta^{(4)}(k_1+k_2)\frac{\mathrm{i}}{\left(\frac{k_1-k_2}{2}\right)^2-m^2}+\dots\end{aligned}$$

with $x = x_1 - x_2, 2X = x_1 + x_2$ and $y = x_3 - x_4, 2Y = x_3 + x_4$. Then, using $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$:

$$\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\, e^{i(p_{1\mu}x_1{}^\mu + p_{2\mu}x_2{}^\mu - p_{3\mu}x_3{}^\mu - p_{4\mu}x_4{}^\mu)}\, \langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle$$
$$= (2\pi)^4\delta^{(4)}(p_1 + p_2)(2\pi)^4\delta^{(4)}(k_1 + k_2)\frac{\mathrm{i}}{p_1^2 - m^2}\frac{\mathrm{i}}{k_1^2 - m^2} + \dots$$

Note that the iϵ term must be retained while p is an integration variable, as it gives the prescription of going around the poles, but it can be dropped when p is the momentum on an external leg, i.e. fixed. This expression only has two poles, therefore:

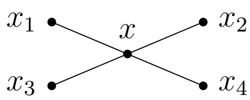
$$\langle \mathbf{p}_1, \mathbf{p}_2|\mathrm{i}T|\mathbf{k}_1, \mathbf{k}_2\rangle = -(2\pi)^8\delta^{(4)}(p_1 + p_2)\delta^{(4)}(k_1 + k_2)(p_2^2 - m^2)(k_2^2 - m^2) + \dots + o(\lambda)$$

These terms vanish when going on mass-shell, confirming that at $o(\lambda^0)$ there is no interaction, i.e. no contribution to the scattering amplitude. This is a general feature of $n \to m$ scattering amplitudes: *disconnected graphs do not contribute to the amplitude*, as they do not provide enough pole factors to cancel those in the LSZ formula.
The first non-trivial contribution is that at $o(\lambda)$. The numerator of Eq. 3.48 is:

$$\int \prod_{i=1}^{4} \mathrm{d}^4x_i\, e^{\mathrm{i}(p_{1\mu}x_1{}^\mu + p_{2\mu}x_2{}^\mu - p_{3\mu}x_3{}^\mu - p_{4\mu}x_4{}^\mu)} \left(-\mathrm{i}\frac{\lambda}{4!}\right) \int \mathrm{d}^4x\, \langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\}|0\rangle$$

The only non-vanishing contribution is the fully-contracted term:

$$\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle \quad =$$

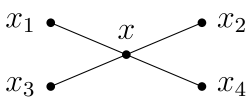


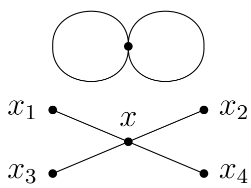
as it is the only connected diagram between the possible ones. Note that this diagram corresponds to 4! equal contractions (only one is shown), therefore the numerator becomes:

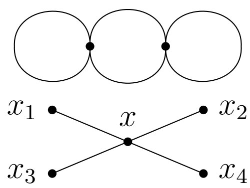
$$\begin{aligned}
&-\mathrm{i}\lambda \int \mathrm{d}^4x \prod_{i=1}^{4} \mathrm{d}^4x_i\, e^{\mathrm{i}(p_{1\mu}x_1{}^\mu + p_{2\mu}x_2{}^\mu - p_{3\mu}x_3{}^\mu - p_{4\mu}x_4{}^\mu)} \Delta(x_1 - x)\Delta(x_2 - x)\Delta(x_3 - x)D_{\mathrm{F}}(x_4 - x) \\
&= -\mathrm{i}\lambda\tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\tilde{\Delta}(k_1)\tilde{\Delta}(k_2) \int \mathrm{d}^4x\, e^{\mathrm{i}(p_1 + p_2 - k_1 - k_2)_\mu x^\mu} \\
&= -\mathrm{i}\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2)\frac{\mathrm{i}}{p_1^2 - m^2}\frac{\mathrm{i}}{p_2^2 - m^2}\frac{\mathrm{i}}{k_1^2 - m^2}\frac{\mathrm{i}}{k_2^2 - m^2}
\end{aligned}$$

There remains to evaluate the denominator: this term gives only **vacuum-to-vacuum transitions**, i.e. diagrams with no external legs. To account for these, note that each connected diagram which contributes to the numerator can be "dressed" with all possible vacuum-to-vacuum transitions; indeed, to be precise:

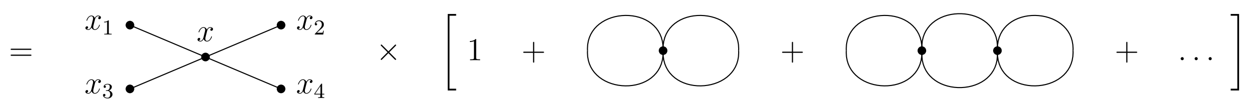
$$\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle =$$

$=$ 

$+$ 

$+$ 

$+ \quad \dots$



This is nothing but the perturbative expansion of the denominator of Eq. 3.48, therefore they cancel out leaving only the already computed diagram[3]. Thus, the transition amplitude at $o(\lambda)$ is:

$$\langle \mathbf{p}_1, \mathbf{p}_2 | \mathrm{i}T | \mathbf{k}_1, \mathbf{k}_2 \rangle = -\mathrm{i}\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \tag{3.61}$$

§3.3.2.4 Scattering in $\lambda\phi^3$-theory

To illustrate the case of internal lines, it is useful to consider a real scalar field theory with $\mathcal{H}_I = \frac{\lambda}{3!}\phi^3$: in this case, 3 lines meet at each vertex, instead of 4. The $2 \to 2$ scattering amplitude is:

$$\begin{aligned}\langle \mathbf{p}_1, \mathbf{p}_2 | \mathrm{i}T | \mathbf{k}_1, \mathbf{k}_2 \rangle = \prod_{i=1,2} \frac{p_i^2 - m^2}{\mathrm{i}\sqrt{Z}} \prod_{j=1,2} \frac{k_j^2 - m^2}{\mathrm{i}\sqrt{Z}} \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, e^{\mathrm{i}(p_{1\mu} x_1^\mu + p_{2\mu} x_2^\mu - k_{1\mu} x_3^\mu - k_{4\mu} x_4^\mu)} \times \\ \times \langle 0 | \mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \exp\left[-\mathrm{i}\frac{\lambda}{3!}\int \mathrm{d}^4 x \, \phi^3(x)\right]\} | 0 \rangle\end{aligned}$$

where the denominator was omitted by implicitly considering only connected graphs with 4 external legs. This amplitude does not have a $o(\lambda^0)$ contribution (as it only has disconnected diagrams) nor an $o(\lambda)$ contribution (as the product of an odd number of fields cannot be fully contracted). At $o(\lambda^2)$ the integral becomes:

$$\begin{aligned}\langle \mathbf{p}_1, \mathbf{p}_2 | \mathrm{i}T | \mathbf{k}_1, \mathbf{k}_2 \rangle = \frac{p_1^2 - m^2}{\mathrm{i}\sqrt{Z}} \frac{p_2^2 - m^2}{\mathrm{i}\sqrt{Z}} \frac{k_1^2 - m^2}{\mathrm{i}\sqrt{Z}} \frac{k_2^2 - m^2}{\mathrm{i}\sqrt{Z}} \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, e^{\mathrm{i}(p_{1\mu} x_1^\mu + p_{2\mu} x_2^\mu - k_{1\mu} x_3^\mu - k_{4\mu} x_4^\mu)} \times \\ \times \frac{1}{2!}\left(-\mathrm{i}\frac{\lambda}{3!}\right)^2 \int \mathrm{d}^4 x \mathrm{d}^4 y \, \langle 0 | \mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^3(x)\phi^3(y)\} | 0 \rangle\end{aligned}$$

In order to obtain a fully connected diagrams, there needs to be a contraction $\overbracket{\phi(x)\phi(y)}$ between the two vertices, creating an internal line. Note that the $1/2!$ factor accounts for the symmetry

[3]This too is a general statement. Consider a generic process with possible connected diagrams $\{C_i\}_{i \in \mathcal{I}}$ and possible vacuum-to-vacuum diagrams $\{V_j\}_{j \in \mathcal{J}}$. Then, as shown above, the contribution of a connected diagram C_i is really:

$$C_i \times \sum_{\{n_j\} \subset \mathbb{N}_0} \prod_{j \in \mathcal{J}} \frac{1}{n_j!} V_j^{n_j}$$

with $\{n_j\}_{j \in \mathcal{J}} \subset \mathbb{N}_0$ (the symmetry factor is due to the swap of identical diagrams). The sum of all non-vanishing diagrams thus is:

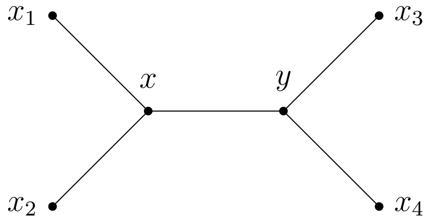
$$\sum(\text{diagrams}) = \sum_{i \in \mathcal{I}} C_i \times \sum_{\{n_j\} \subset \mathbb{N}_0} \prod_{j \in \mathcal{J}} \frac{1}{n_j!} V_j^{n_j} = \sum_{i \in \mathcal{I}} C_i \times \prod_{j \in \mathcal{J}} \sum_{n=1}^{\infty} \frac{1}{n!} V_j^n = \sum_{i \in \mathcal{I}} C_i \times \prod_{j \in \mathcal{J}} \exp V_j = \sum_{i \in \mathcal{I}} C_i \times \exp \sum_{j \in \mathcal{J}} V_j$$

which can be recast in the form:

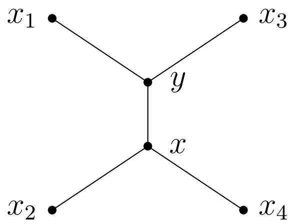
$$\sum(\text{diagrams}) = \sum(\text{connected}) \times \exp \sum(\text{vacuum-to-vacuum}) \tag{3.62}$$

The exponential exactly cancels $\langle 0 | \mathfrak{T}\{\exp\left[-\mathrm{i}\int \mathrm{d}^4 x \, \mathcal{H}_I\right]\} | 0 \rangle$, leaving only connected diagrams in the calculation.

$x \leftrightarrow y$, while $1/3!$ accounts for sets of equivalent contractions. Examples of possible diagrams are:



$$= \quad \Delta(x_1 - x)\Delta(x_2 - x)\Delta(x - y)\Delta(y - x_3)\Delta(y - x_4)$$

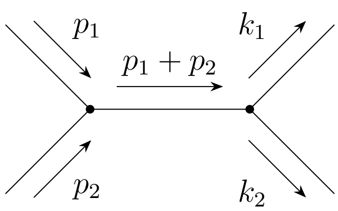


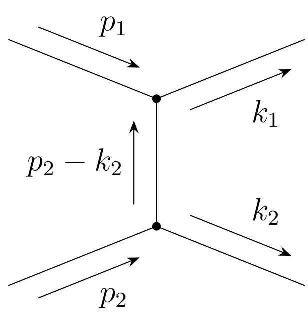
$$= \quad \Delta(x_1 - y)\Delta(x_2 - x)\Delta(x - y)\Delta(y - x_3)\Delta(x - x_4)$$

These diagrams are inequivalent, as they correspond to different sets of contractions. Computing for example the first one:

$$\begin{aligned}
&(-\mathrm{i}\lambda)^2 \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, e^{\mathrm{i}(p_{1\mu}x_1^\mu + p_{2\mu}x_2^\mu - k_{1\mu}x_3^\mu - k_{4\mu}x_4^\mu)} \times \\
&\qquad\qquad \times \int \mathrm{d}^4x \mathrm{d}^4 y \, \Delta(x_1 - x)\Delta(x_2 - x)\Delta(x - y)\Delta(y - x_3)\Delta(y - x_4) \\
&= (-\mathrm{i}\lambda)^2 \tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\tilde{\Delta}(k_1)\tilde{\Delta}(k_2) \int \mathrm{d}^4x \mathrm{d}^4 y \, e^{\mathrm{i}(p_1+p_2)_\mu x^\mu - i(k_1+k_2)_\mu y^\mu} \Delta(x - y) \\
&= (-\mathrm{i}\lambda)^2 \tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\tilde{\Delta}(k_1)\tilde{\Delta}(k_2) \int \mathrm{d}^4X \mathrm{d}^4 Y \, e^{\mathrm{i}(p_1+p_2-k_1-k_2)_\mu Y^\mu + \mathrm{i}(p_1+p_2+k_1+k_2)_\mu X^\mu/2} \Delta(X) \\
&= (-\mathrm{i}\lambda)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\Delta(p_1 + p_2)\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)
\end{aligned}$$

with $X = x - y, 2Y = x + y$. As expected, momentum-space propagators associated to external legs are canceled by pole terms in the LSZ formula: now there remains only the $\tilde{\Delta}(p_1 + p_2)$ factor associated to the internal line.
It is useful to work in momentum space, rather than coordinate space. For example, in momentum space the above diagrams become:





§3.3.2.5 General considerations

The example calculations carried for $\lambda\phi^3$- and $\lambda\phi^4$-theory hint to the general technique to compute general **tree diagrams**, i.e. connected diagrams without internal loops. For a scalar field theory, in momentum space:

1. draw all possible topologicaly-inequivalent connected graphs corresponding to the given initial and final states and multiply them by their symmetry factor;

2. each external leg is associated to a pole factor in the LSZ reduction formula, hence they can be omitted directly (“leg amputation”);

3. as there is always a momentum-conserving δ-function, it is convenient to define the matrix element $\mathcal{M}$ for the process $|\text{in}\rangle \to |\text{out}\rangle$ as:

$$\langle \mathbf{p}_1, \dots, \mathbf{p}_m | \mathrm{i}T | \mathbf{k}_1, \dots, \mathbf{k}_n \rangle \equiv (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{m} p_i - \sum_{j=1}^{n} k_j \right) \mathrm{i}\mathcal{M} \tag{3.63}$$

4. energy-momentum conservation must be imposed separately at each vertex;

5. each vertex is associated to a $-\mathrm{i}\lambda$ factor, where λ is the coupling constant of the interaction;

6. check if that symmetry factors cancel fully or partially with the $1/n!$ factor at $o(\lambda^n)$ and possible numerical factors in the definition of $\mathcal{H}_I$.

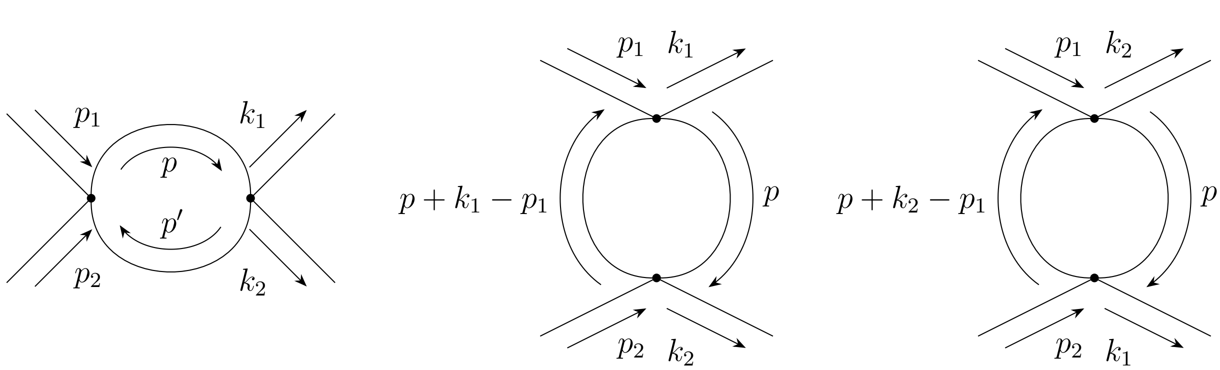
The fact that the considered diagrams are topologicaly inequivalent needs to be stressed out. Moreover, symmetry factors need to be computed carefully, as illustrated in §4.2.1.3.

§3.3.3 Loop diagrams

To illustrate how loop diagrams are treated, consider the $o(\lambda^2)$ corrections to the $2 \to 2$ scattering in $\lambda\phi^4$-theory:

$$\frac{1}{2!} \left(-\mathrm{i} \frac{\lambda}{4!} \right)^2 \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, e^{\mathrm{i}(p_{1\mu} x_1^\mu + p_{2\mu} x_2^\mu - k_{1\mu} x_3^\mu - k_{4\mu} x_4^\mu)} \times$$
$$\times \int \mathrm{d}^4 x \mathrm{d}^4 y \, \langle 0 | \mathfrak{T}\{ \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \phi^4(x) \phi^4(y) \} | 0 \rangle$$

There are three possible inequivalent full contractions, associated to the connected diagrams:



with $p' = p_1 + p_2 - p$. It is clear that energy conservation does not completely fix momenta of internal lines, and in particular of loops. Consider for example the first diagram: it is obtained contracting $\phi(x_1)$ and $\phi(x_2)$ with two $\phi(x)$ ($4{\cdot}3$ ways), $\phi(x_3)$ and $\phi(x_4)$ with two $\phi(y)$ ($4{\cdot}3$ ways) and two $\phi(x)$ with two $\phi(y)$ (2 ways), so, considering another 2 factor for $x \leftrightarrow y$ symmetry, this diagram has symmetry number $S = (4!)^2$. The resulting contribution is:

$$\begin{aligned}
&\frac{(-\mathrm{i}\lambda)^2}{2}\int\prod_{i=1}^{4}\mathrm{d}^4x_i\, e^{\mathrm{i}(p_{1\mu}x_1^\mu+p_{2\mu}x_2^\mu-k_{1\mu}x_3^\mu-k_{4\mu}x_4^\mu)}\int\mathrm{d}^4x\mathrm{d}^4y\,\Delta(x_1-x)\Delta(x_2-x)\Delta(x-y)\Delta(y-x)\\
&=\frac{(-\mathrm{i}\lambda)^2}{2}\tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)\int\mathrm{d}^4x\mathrm{d}^4y\,e^{\mathrm{i}(p_1+p_2)_\mu x^\mu-\mathrm{i}(k_1+k_2)_\mu y^\mu}\Delta^2(x-y)\\
&=\frac{(-\mathrm{i}\lambda)^2}{2}\tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\Delta}(k)\tilde{\Delta}(p_1+p_2-k)
\end{aligned}$$

Therefore, the additional feature of loop integrals is the integration over unfixed momenta, assigning the right propagators to each internal line. It is possible to define the **loop-amplitude** for $\lambda\phi^4$-theory:

$$\mathcal{A}(p) := \frac{(-\mathrm{i}\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\frac{\mathrm{i}}{(p-k)^2-m^2+\mathrm{i}\epsilon} \tag{3.64}$$

so that the $2\to2$ scattering amplitude at one-loop level (i.e. $o(\lambda^2)$) is:

$$\mathrm{i}\mathcal{M}_{2\to2} = -\mathrm{i}\lambda + \mathcal{A}(p_1+p_2) + \mathcal{A}(p_1-k_1) + \mathcal{A}(p_1-k_2) \tag{3.65}$$

§3.3.3.1 Divergences

The integral in Eq. 3.64 diverges at large k, which is an example of **UV divergence**. To study this divergence, first of all note that $(p-k)^2 \to k^2$ as $k\to\infty$, so that it is safe to set $p=0$. Due to the iϵ-presciption, in the complex k^0-plane the pole in $k^0>0$ is below the real axis, while that in $k^0<0$ is above it: the integration path can then be rotated counterclockwise from the real axis to the imaginary axis by $k^0 \mapsto \mathrm{i}k^0$, which is called a **Wick rotation**. The amplitude then becomes:

$$\mathcal{A}(0) = \frac{(-\mathrm{i}\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\right)^2 \mapsto \mathrm{i}\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2}$$

where $k^2 = k_0^2 + \mathbf{k}^2$ is now a Euclidean momentum. To regulate the UV divergence, a cutoff $\Lambda : k^2 < \Lambda^2$ is introduced, so that:

$$\begin{aligned}
\mathcal{A}(0) &= \mathrm{i}\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^4} + \text{finite terms} = \mathrm{i}\frac{\lambda^2}{2}\frac{1}{(2\pi)^4}2\pi^2\int^{\Lambda}\frac{\mathrm{d}k}{k} + \text{finite terms}\\
&= \frac{\mathrm{i}\lambda^2}{16\pi^2}\log\Lambda + \text{finite terms}
\end{aligned}$$

where $2\pi^2$ is the solid angle in $\mathbb{R}^4$. It can be shown that, for general p, the singular part of $\mathcal{A}(p)$ does not depend on p; hence, it can be written:

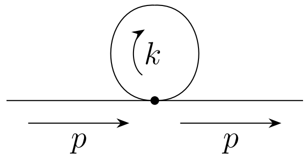
$$\mathrm{i}\mathcal{M}_{2\to2} = -\mathrm{i}\lambda + \mathrm{i}\lambda^2\left(\beta_0\log\Lambda + \text{finite terms}\right)$$

where:

$$\beta_0 \equiv \frac{3}{16\pi^2} \tag{3.66}$$

Another example of loop divergence in $\lambda\phi^4$-theory is that associated to the 2-point Green function at $o(\lambda^1)$:

$$\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi^4(x)\}|0\rangle = 4{\cdot}3{\cdot}\phi(x_1)\phi(x_2)\phi(x)\phi(x)\phi(x)\phi(x) = 4{\cdot}3\times$$

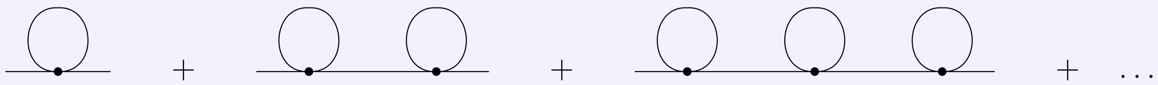


This diagram is typically called a **tadpole**. Its contribution can be computed in momentum performing a Wick rotation:

$$-\mathrm{i}B \equiv \frac{-\mathrm{i}\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+m^2} = -\mathrm{i}\frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\frac{\Lambda^2+m^2}{m^2}\right) \tag{3.67}$$

Note that the tadpole is independent of p, and it has both a quadratic and logarithmic UV divergence.

Lemma 3.3.2 (Tadpole resummation)
In $\lambda\phi^4$-theory, for a $|\mathbf{p}\rangle \to |\mathbf{p}\rangle$ scattering, tadpole diagrams collectively contribute as:



$$= \frac{\mathrm{i}}{p^2 - m^2 - B}$$

Proof. The represented diagram series reduces to a geometric series in momentum space:

$$\begin{aligned}&\tilde{\Delta}(p) + \tilde{\Delta}(p)(-\mathrm{i}B)\tilde{\Delta}(p) + \tilde{\Delta}(p)(-\mathrm{i}B)\tilde{\Delta}(p)(-\mathrm{i}B)\tilde{\Delta}(p) + \dots \\ &\quad = \tilde{\Delta}(p)\sum_{n=0}^{\infty}\left[-\mathrm{i}B\tilde{\Delta}(p)\right]^n = \tilde{\Delta}\frac{1}{1+\mathrm{i}B\tilde{\Delta}(p)} = \frac{\mathrm{i}}{p^2-m^2}\left[1 - \frac{B}{p^2-m^2}\right]^{-1} = \frac{\mathrm{i}}{p^2-m^2-B}\end{aligned}$$

which is the thesis. □

This shows that accounting for tadpole diagrams is equivalent to shifting the mass $m^2 \mapsto m^2 + B$.

§3.3.4 Feynman rules

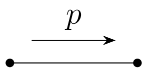
To compute amplitudes up to a desired order, it is not necessary to explicitly expanding the exponential each time and then performing Wick contractions, as these steps are summarized by a simple set of rules.

§3.3.4.1 $\lambda\phi^n$-theory

Consider a scalar field theory with Hamiltonian:

$$\mathcal{H} = \mathcal{H}_{\mathrm{KG}} + \frac{\lambda}{n!}\phi^n \tag{3.68}$$

The Feynman rules in momentum space for this theory are:

1. for each internal line: 

$= \tilde{\Delta}(p)$;
2. for each external leg: $= 1$;
3. for each vertex: $\bullet = -\mathrm{i}\lambda$;
4. impose momentum conservation at each vertex;
5. integrate over each undetermined loop momentum p with measure $\frac{\mathrm{d}^4 p}{(2\pi)^4}$;
6. divide by the symmetry factor of the diagram.

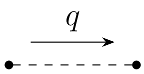
The problem of computing scattering amplitude is thus reduced to that of drawing every inequivalent fully-connected Feynman diagram (modulo void-to-void diagrams). These rules allow to compute the matrix element $\mathcal{M}$, which is related to the T-matrix element by Eq. 3.63 and to the S-matrix element by $S = \mathrm{I} + \mathrm{i}T$.

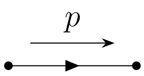
§3.3.4.2 Yukawa theory

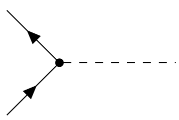
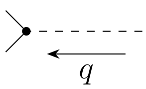
Consider a field theory with a real scalar field $\phi(x)$ and a Dirac field $\Psi(x)$. The **Yukawa Hamiltonian** is:

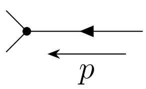
$$\mathcal{H} = \mathcal{H}_{\mathrm{KG}} + \mathcal{H}_{\mathrm{D}} + g\bar{\Psi}\Psi\phi \tag{3.69}$$

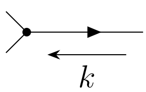
where g is a dimensionless coupling constant. This is a simplified model of QED. Note that the form of $\mathcal{H}_{\mathrm{int}}$ allows interaction vertices with two fermion lines and one scalar line. The Feynman rules in momentum space for this theory are:

1. propagators:
   (a) scalar: 

$= \dfrac{\mathrm{i}}{q^2 - m_\phi^2 + \mathrm{i}\epsilon} = \tilde{\Delta}(q)$;
   (b) fermion: 

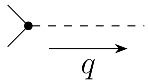
$= \dfrac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\epsilon} = \tilde{\Sigma}(p)$;
2. vertex:  $= -\mathrm{i}g$;
3. external legs:
   (a) initial state:
      i. scalar: 

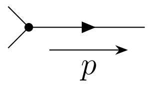
$= 1$;
      ii. fermion: 

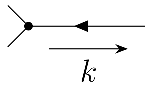
$= u^s(p)$;
      iii. anti-fermion: 

$= \bar{v}^s(k)$;

(b) final state:

i. scalar:  $= 1;$

ii. fermion:  $= \bar{u}^s(p);$

iii. anti-fermion:  $= v^s(k);$

4. impose momentum conservation at each vertex;

5. integrate over each undetermined loop momentum p with measure $\frac{\mathrm{d}^4 p}{(2\pi)^4}$ (each closed fermionic loop determines an additional (-1) factor due to anti-commutation);

6. divide by the symmetry factor of the diagram.

Note that for the Yukawa interaction $g\bar{\Psi}\Psi\phi$ the symmetry factor of each diagram is $S = 1$, as the three field in $\mathcal{H}_{\text{int}}$ cannot substitute for one another in contractions. Moreover, Dirac indices are always contracted together along fermionic lines.
It can be shown that an interaction mediated by a scalar field is always attractive, both for ff, $f\bar{f}$ and $\bar{f}\bar{f}$ scattering.

§3.3.4.3 Quantum Electrodynamics

Consider a field theory with a Dirac field $\Psi(x)$ and a vector (gauge) field $A_\mu(x)$. The **QED interaction Hamiltonian** is:

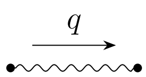
$$\mathcal{H}_{\text{int}} = e\bar{\Psi}\gamma^\mu\Psi A_\mu \tag{3.70}$$

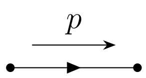
This interaction term determines interaction vertices with two fermionic lines and one vector line. The photon propagator is derived in §4.2.2:

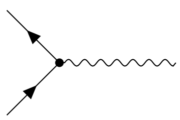
$$\Delta_{\mu\nu}(x-y) \coloneqq \langle 0|\mathfrak{T}\{A_\mu(x)A_\nu(y)\}|0\rangle \quad \Longrightarrow \quad \tilde{\Delta}(k) = \frac{-\mathrm{i}}{k^2 + \mathrm{i}\epsilon}\eta_{\mu\nu} \tag{3.71}$$

Note that the spatial components $A_i(x)$ have the same propagator as a massless real scalar field, while $A_0(x)$ has an additional negative sign. The Feynman rules in momentum space for QED are:

1. propagators:

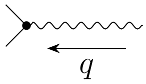
(a) vector: μ  $\nu \; = \tilde{\Delta}_{\mu\nu}(q);$

(b) fermion:  $= \tilde{\Sigma}(p);$

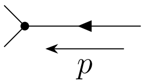
2. vertex:  $\mu \; = -\mathrm{i}e\gamma^\mu;$

3. external legs:

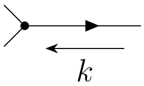
   (a) initial state:

      i. vector: 

$\mu \quad = \epsilon_\mu(q);$

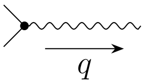
      ii. fermion: 

$= u^s(p);$

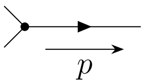
      iii. anti-fermion: 

$= \bar{v}^s(k);$

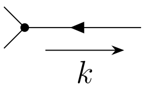
   (b) final state:

      i. scalar: 

$\mu \quad = \epsilon^*_\mu(q);$

      ii. fermion: 

$= \bar{u}^s(p);$

      iii. anti-fermion: 

$= v^s(k);$

4. impose momentum conservation at each vertex;

5. integrate over each undetermined loop momentum p with measure $\frac{\mathrm{d}^4p}{(2\pi)^4}$ (each closed fermionic loop determines an additional (-1) factor due to anti-commutation);

6. divide by the symmetry factor of the diagram.

Of course, when considering fermions with charge Q (in units of $|e|$), substitute $e \mapsto Q\,|e|$ in the above formulae.
It can be shown that the interaction mediated by a vector field is attractive for $f\bar{f}$ scattering and repulsive for ff and $\bar{f}\bar{f}$ scattering.

Chapter 4

Path Integral Quantization

§4.1 Path integrals

In contrast to canonical quantization, in which fields are promoted to operators, in path-integral quantization they remain ordinary functions. While the former allows for a more direct understanding of the notion of particle (thanks to ladder operators), the latter has the advantage of not being intrinsically perturbative, thus better describing theories with non-perturbative effects.

§4.1.1 Path integral in Quantum Mechanics

Consider a general (bosonic) quantum system, with Hermitian operators $\{\hat{q}_a, \hat{p}_a\}_{a=1,\dots,n}$ which satisfy the canonical commutation relations:

$$[\hat{q}_a, \hat{p}_b] = i\delta_{ab} \tag{4.1}$$

Eigenstates of these operators form two improperly-normalized complete orthonormal sets of eigenstates, with scalar product:

$$\langle q|p\rangle = \prod_{a=1}^{n} \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}q_a p_a} \equiv (2\pi)^{-n/2} e^{\mathrm{i}q\cdot p} \tag{4.2}$$

with $q \equiv \{q_1, \dots, q_n\}, p \equiv \{p_1, \dots, p_n\}$. The switch from the Schrödinger picture to the Heisenberg one is carried by the Hamiltonian $\hat{H}(\hat{q}, \hat{p})$ of the system, which is assumed to be normal-ordered with all q_a on the left of all p_a: in this picture, all of the above relations remain valid for equal-time states.

Theorem 4.1.1 (Hamiltonian path integral)
The amplitude of $|q_\mathrm{i}, t_\mathrm{i}\rangle \to |q_\mathrm{f}, t_\mathrm{f}\rangle$ is a functional integral:

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle = \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} \prod_{a=1}^{n} \mathrm{d}q_a \frac{\mathrm{d}p_a}{2\pi} \exp \mathrm{i} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \left[\sum_{b=1}^{n} \dot{q}_b(t) p_b(t) - H(q(t), p(t)) \right] \tag{4.3}$$

Proof. Consider a partition $\{t_m\}_{m=0,\dots,N} : t_m < t_j \; \forall m < j$ of the interval $[t_\mathrm{i}, t_\mathrm{f}]$, so that $t_0 \equiv t_\mathrm{i}$ and $t_N \equiv t_\mathrm{f}$: for simplicity, take them equally spaced, i.e. $t_m = t_0 + m\epsilon$, with

$N\epsilon = t_\mathrm{f} - t_\mathrm{i}$. The amplitude for $|q_\mathrm{i}, t_\mathrm{i}\rangle \to |q_\mathrm{f}, t_\mathrm{f}\rangle$ can thus be computed as:

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle = \int_{\mathbb{R}^{n(N-1)}} \prod_{m=0}^{N-1} \mathrm{d}q_m \, \langle q_{m+1}, t_{m+1}|q_m, t_m\rangle$$

where the completeness relation was repeatedly used. Switching to the Schrödinger picture:

$$\begin{aligned}
\langle q_{m+1}, t_{m+1}|q_m, t_m\rangle &= \langle q_{m+1}|e^{-\mathrm{i}\hat{H}(t_{m+1}-t_m)}|q_m\rangle \\
&= \langle q_{m+1}|e^{-\mathrm{i}\hat{H}\epsilon}|q_m\rangle = \int_{\mathbb{R}^n} \mathrm{d}p_m \, \langle q_{m+1}|p_m\rangle \, \langle p_m|e^{-\mathrm{i}\hat{H}\epsilon}|q_m\rangle \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{d}p_m \, e^{\mathrm{i}q_{m+1}\cdot p_m} \, \langle p_m|e^{-\mathrm{i}\hat{H}(\hat{q},\hat{p})\epsilon}|q_m\rangle \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{d}p_m \, e^{\mathrm{i}(q_{m+1}-q_m)\cdot p_m} e^{-\mathrm{i}H(q_m,p_m)\epsilon}
\end{aligned}$$

The amplitude is then rewritten as:

$$\begin{aligned}
\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n(N+1)}} \prod_{m=0}^{N-1} \mathrm{d}q_m \mathrm{d}p_m \, e^{\mathrm{i}[(q_{m+1}-q_m)\cdot p_m - H(q_m,p_m)\epsilon]} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n(N+1)}} \prod_{m=0}^{N-1} \mathrm{d}q_m \mathrm{d}p_m \exp \sum_{k=1}^{N-1} \mathrm{i}\epsilon \left[\frac{q_{k+1}-q_k}{\epsilon} \cdot p_k - H(q_k, p_k) \right]
\end{aligned}$$

Consider now the $\epsilon \to 0$ limit, i.e. $N \to \infty$: the integration is then performed on an infinite number of variables, hence becoming a functional integral. The integration is carried on the set of functions $\{q(t) : \mathbb{R}^n \to \mathbb{R}^n : q(t_\mathrm{i}) = q_\mathrm{i} \wedge q(t_\mathrm{f}) = q_\mathrm{f}\}$; moreover, note that, in the $\epsilon \to 0$ limit, the sum becomes a Riemann integral:

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle = \frac{1}{(2\pi)^n} \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} \mathrm{d}q \mathrm{d}p \exp \mathrm{i} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, [\dot{q}(t) \cdot p(t) - H(q(t), p(t))]$$

which is precisely the thesis. □

Proposition 4.1.1 (Lagrangian path integral)

For a p-quadratic Hamiltonian, the path integral can be written as:

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle = \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} [\mathrm{d}q] \, e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}[q,\dot{q}]} \tag{4.4}$$

where $\mathcal{S}[q, \dot{q}]$ is the action functional of the system and $[\mathrm{d}q]$ is a suitably-normalized integration measure.

Proof. Consider a system with a p-quadratic Hamiltonian $H(q,p) = \frac{p^2}{2} + V(q)$ (however, the thesis holds for a general quadratic form of p), so that the previous proof can be rewritten with a Gaussian integral (Eq. A.38):

$$
\begin{aligned}
\langle q_{m+1}, t_{m+1} | q_m, t_m \rangle &= \frac{1}{(2\pi)^n} e^{-\mathrm{i}\epsilon V(q_m)} \int_{\mathbb{R}^n} \mathrm{d}p_m \, e^{\mathrm{i}(q_{m+1}-q_m)\cdot p_m - \frac{\mathrm{i}\epsilon}{2} p_m^2} \\
&= \frac{1}{(2\pi)^n} e^{-\mathrm{i}\epsilon V(q_m)} \prod_{a=1}^{n} \int_{\mathbb{R}} \mathrm{d}p_{m,a} e^{\mathrm{i}(q_{m+1,a}-q_{m,a})p_{m,a} - \frac{\mathrm{i}\epsilon}{2} p_{m,a}^2} \\
&= \frac{1}{(2\pi)^n} e^{-\mathrm{i}\epsilon V(q_m)} \prod_{a=1}^{n} \sqrt{\frac{2\pi}{\mathrm{i}\epsilon}} e^{-\frac{1}{2\mathrm{i}\epsilon}(q_{m+1,a}-q_{m,a})^2} \\
&= \left(\frac{1}{2\pi \mathrm{i}\epsilon}\right)^{n/2} e^{\mathrm{i}\epsilon\left[\frac{1}{2}\left(\frac{q_{m+1}-q_m}{\epsilon}\right)^2 - V(q_m)\right]}
\end{aligned}
$$

Therefore:

$$
\langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i} \rangle = \left(\frac{1}{2\pi \mathrm{i}\epsilon}\right)^{nN/2} \int_{\mathbb{R}^{n(N-1)}} \prod_{m=0}^{N-1} \mathrm{d}q_m \exp \sum_{k=0}^{N-1} \mathrm{i}\epsilon \left[\frac{1}{2}\left(\frac{q_{k+1}-q_k}{\epsilon}\right)^2 - V(q_k)\right]
$$

In the $\epsilon \to 0$ limit, with the same considerations as above and absorbing the normalization in the integration measure, this becomes:

$$
\langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i} \rangle = \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} [dq] \exp\left[\mathrm{i} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, L(q(t), \dot{q}(t))\right] =: \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} [\mathrm{d}q] \, e^{\mathrm{i}\mathcal{S}[q,\dot{q}]}
$$

Reinstating explicitly $\hbar$ yields the thesis. □

This formulation elucidates the classical limit $\hbar \to 0$: while quantistically a particle explores all possible trajectories, which are then weighted by a phase $e^{\mathrm{i}\mathcal{S}}$, classically only trajectories which extremize the action are relevant, as those which are far from the extrema get extremely-oscillating phases under small deformations. Thus, the least-action principle has been recovered. The path integral is not only useful for computing amplitudes, but it is also used to express matrix elements on the coordinate-eigenbasis.

Lemma 4.1.1 (Time-ordered products)

Given $\{t_k\}_{k=1,\dots,N} \subset [t_\mathrm{i}, t_\mathrm{f}]$ and operators $\{\hat{\mathcal{O}}_k(\hat{q}(t))\}_{k=1,\dots,N} \equiv \{\hat{\mathcal{O}}_k(t)\}_{k=1,\dots,N}$, $N \in \mathbb{N}$, then:

$$
\int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} [\mathrm{d}q] \, \mathcal{O}_1(t_1) \dots \mathcal{O}_N(t_N) e^{\mathrm{i}\mathcal{S}} = \langle q_\mathrm{f}, t_\mathrm{f} | \mathfrak{T}\{\hat{\mathcal{O}}_1(t_1) \dots \hat{\mathcal{O}}_N(t_N)\} | q_\mathrm{i}, t_\mathrm{i} \rangle \tag{4.5}
$$

where $\mathfrak{T}$ is the time-ordered product, which is defined as:

$$
\mathfrak{T}\{f(x)g(y)\} := \begin{cases} f(x)g(y) & x^0 > y^0 \\ g(y)f(x) & y^0 > x^0 \end{cases} \tag{4.6}
$$

Proof. First, note that:

$$
\int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} [\mathrm{d}q] = \int_{\mathbb{R}^n} \mathrm{d}\bar{q} \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t)=\bar{q}} [\mathrm{d}q] \int_{q(t)=\bar{q}}^{q(t_\mathrm{f})=q_\mathrm{f}} [\mathrm{d}q]
$$

so that (due to the additivity of the action integral):

$$\int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}}[\mathrm{d}q]\,\mathcal{O}_k(t)e^{\mathrm{i}\mathcal{S}} = \int_{\mathbb{R}^n}\mathrm{d}\bar{q}\,\underbrace{\int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t)=\bar{q}}[\mathrm{d}q]\exp\left[\mathrm{i}\int_{t_\mathrm{i}}^{t}\mathrm{d}t\,L\right]}_{\langle\bar{q},t|q_\mathrm{i},t_\mathrm{i}\rangle}\mathcal{O}_k(\bar{q})\underbrace{\int_{q(t)=\bar{q}}^{q(t_\mathrm{f})=q_\mathrm{f}}[\mathrm{d}q]\exp\left[\mathrm{i}\int_{t}^{t_\mathrm{f}}\mathrm{d}t\,L\right]}_{\langle q_\mathrm{f},t_\mathrm{f}|\bar{q},t\rangle}$$

$$= \int_{\mathbb{R}^n}\mathrm{d}\bar{q}\,\langle q_\mathrm{f},t_\mathrm{f}|\bar{q},t\rangle\,\mathcal{O}_k(\bar{q})\,\langle\bar{q},t|q_\mathrm{i},t_\mathrm{i}\rangle = \int_{\mathbb{R}}\mathrm{d}\bar{q}\,\langle q_\mathrm{f},t_\mathrm{f}|\hat{\mathcal{O}}_k(t)|\bar{q},t\rangle\,\langle\bar{q},t|q_\mathrm{i},t_\mathrm{i}\rangle$$

$$= \langle q_\mathrm{f},t_\mathrm{f}|\hat{\mathcal{O}}_k(t)|q_\mathrm{i},t_\mathrm{i}\rangle$$

This links the matrix element of the operator $\hat{\mathcal{O}}_k(t)$ to the path integral of the function $\mathcal{O}_k(t)$. Consider now that, being $\mathcal{O}_k(t)\in\mathbb{C}$, i.e. commuting:

$$\int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}}[\mathrm{d}q]\,\mathcal{O}_1(t_1)\ldots\mathcal{O}_N(t_N)e^{\mathrm{i}\mathcal{S}} = \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}}[\mathrm{d}q]\,\mathcal{O}_{\pi(1)}(t_{\pi(1)})\ldots\mathcal{O}_{\pi(N)}(t_{\pi(N)})e^{i\mathcal{S}}$$

where $\pi\in S_N : t_{\pi(1)}\geq\cdots\geq t_{\pi(N)}$ is a time-ordering permutation. Applying the above result:

$$\int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}}[\mathrm{d}q]\,\mathcal{O}_1(t_1)\ldots\mathcal{O}_N(t_N)e^{\mathrm{i}\mathcal{S}} = \langle q_\mathrm{f},t_\mathrm{f}|\hat{\mathcal{O}}_{\pi(1)}(t_{\pi(1)})\ldots\hat{\mathcal{O}}_{\pi(N)}(t_{\pi(N)})|q_\mathrm{i},t_\mathrm{i}\rangle$$

But $\hat{\mathcal{O}}_{\pi(1)}(t_{\pi(1)})\ldots\hat{\mathcal{O}}_{\pi(N)}(t_{\pi(N)}) =: \mathfrak{T}\{\hat{\mathcal{O}}_1(t_1)\ldots\hat{\mathcal{O}}_N(t_N)\}$, thus completing the proof. $\square$

§4.1.2 Path integral in Quantum Field Theory

To generalize the path-integral formalism to QFT, consider the index a as running over points $\mathbf{x}$ in space and over a spin/helicity (i.e. species/helicity) index m:

$$\hat{q}_a(t)\longrightarrow\hat{q}_m(t,\mathbf{x}) \qquad\qquad |q,t\rangle\longrightarrow|q(\mathbf{x}),t\rangle$$

The generalizations of Eq. 4.3-4.4 is then straightforward:

$$\begin{aligned}\langle q_\mathrm{f}(\mathbf{x}),t_\mathrm{f}|q_\mathrm{i}(\mathbf{x}),t_\mathrm{i}\rangle &= \int_{q(t_\mathrm{i},\mathbf{x})=q_\mathrm{i}(\mathbf{x})}^{q(t_\mathrm{f},\mathbf{x})=q_\mathrm{f}(\mathbf{x})}\prod_m \mathrm{d}q(\mathbf{x})\frac{\mathrm{d}p(\mathbf{x})}{2\pi}\times \\ &\times\exp\mathrm{i}\int_{t_\mathrm{i}}^{t_\mathrm{f}}\mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}^3x\left[\sum_m\dot{q}_m(t,\mathbf{x})p_m(t,\mathbf{x})-\mathcal{H}[q(t,\mathbf{x}),p(t,\mathbf{x})]\right]\end{aligned} \tag{4.7}$$

$$\langle q_\mathrm{f}(\mathbf{x}),t_\mathrm{f}|q_\mathrm{i}(\mathbf{x}),t_\mathrm{i}\rangle = \int_{q(t_\mathrm{i},\mathbf{x})=q_\mathrm{i}(\mathbf{x})}^{q(t_\mathrm{f},\mathbf{x})=q_\mathrm{f}(\mathbf{x})}\mathcal{D}q\exp\mathrm{i}\int_{t_\mathrm{i}}^{t_\mathrm{f}}\mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}^3x\,\mathcal{L}[q(t,\mathbf{x}),\dot{q}(t,\mathbf{x})] \tag{4.8}$$

where $\mathcal{D}q$ generalizes the measure $[\mathrm{d}q]$. It is best to express the path integral not for coordinate eigenstates, but for experimentally-observed states: these are states at $t_\mathrm{i}\to-\infty, t_\mathrm{f}\to+\infty$ with a definite number of particles (of various types).

Theorem 4.1.2 ((Bosonic) Path integral)

The amplitude for the (bosonic) process $|\alpha_{\text{in}}\rangle \to |\beta_{\text{out}}\rangle$ reads:

$$\langle\beta_{\text{out}}|\alpha_{\text{in}}\rangle = \int \mathcal{D}q \exp\left[\mathrm{i}\int_{\mathbb{R}^4} \mathrm{d}t\mathrm{d}^3x\, \mathcal{L}[q(t,\mathbf{x}),\dot{q}(t,\mathbf{x})]\right] \langle\beta_{\text{out}}|q(\mathbf{x}),+\infty\rangle \langle q(\mathbf{x}),-\infty|\alpha_{\text{in}}\rangle \quad (4.9)$$

where the functional integral is now unconstrained.

Proof. Using Eq. 4.8:

$$\begin{aligned}\langle\beta_{\text{out}}|\alpha_{\text{in}}\rangle &= \int \mathrm{d}q_{\mathrm{i}}(\mathbf{x})\mathrm{d}q_{\mathrm{f}}(\mathbf{x}) \langle\beta_{\text{out}}|q_{\mathrm{f}}(\mathbf{x}),+\infty\rangle \langle q_{\mathrm{f}}(\mathbf{x}),+\infty|q_{\mathrm{i}}(\mathbf{x}),-\infty\rangle \langle q_{\mathrm{i}}(\mathbf{x}),-\infty\rangle \\ &= \int \mathrm{d}q_{\mathrm{i}}(\mathbf{x})\mathrm{d}q_{\mathrm{f}}(\mathbf{x}) \int_{q(-\infty,\mathbf{x})=q_{\mathrm{i}}(\mathbf{x})}^{q(+\infty,\mathbf{x})=q_{\mathrm{f}}(\mathbf{x})} \mathcal{D}q \exp\left[\mathrm{i}\int_{\mathbb{R}^4} \mathrm{d}t\mathrm{d}^3x\, \mathcal{L}[q(t,\mathbf{x}),\dot{q}(t,\mathbf{x})]\right] \times \\ &\qquad \times \langle\beta_{\text{out}}|q_{\mathrm{f}}(\mathbf{x}),+\infty\rangle \langle\alpha_{\text{in}}|q_{\mathrm{i}}(\mathbf{x}),-\infty\rangle\end{aligned}$$

This is equivalent to having an unconstrained functional integral, hence the thesis. □

This result is easily translated into Hamiltonian form. The expression for the wavefunctions which appear in Eq. 4.9 depends on the nature of the quantum field $q_m(t,\mathbf{x})$ considered (which, in this case, is constrained to be bosonic as of Eq. 4.1).
Note that Lemma 4.1.1 still holds: it allows to compute n-point Green functions via path integrals.

Proposition 4.1.2 (Correlation functions)
Given a (bosonic) quantum field $q(x)$, described by a Lagrangian $\mathcal{L}$, the n-point Green function can be computed as:

$$\langle 0|\mathfrak{T}\{\hat{q}(x_1)\ldots\hat{q}(x_n)\}|0\rangle = \frac{\int \mathcal{D}q\, q(x_1)\ldots q(x_n) e^{\mathrm{i}\mathcal{S}}}{\int \mathcal{D}q\, e^{\mathrm{i}\mathcal{S}}} \quad (4.10)$$

where $\hat{q}$ denotes the field in the Heisenberg picture and the integrals are performed over field configurations $q(x) : q(\pm\infty,\mathbf{x}) = 0$.

Proof. By Lemma 4.1.1:

$$\int_{q(-\infty,\mathbf{x})=0}^{q(+\infty,\mathbf{x})=0} \mathcal{D}q\, q(x_1)\ldots q(x_n) e^{\mathrm{i}\mathcal{S}} = \langle 0,+\infty|\mathfrak{T}\{\hat{q}(x_1)\ldots\hat{q}(x_n)\}|0,-\infty\rangle$$

As the final-state vacuum is assumed to physically coincide with the initial-state vacuum, then:

$$|0,+\infty\rangle = e^{\mathrm{i}\alpha}|0,-\infty\rangle \quad\Longrightarrow\quad e^{-\mathrm{i}\alpha} = \langle 0,+\infty|0,-\infty\rangle$$

Inserting $\langle 0,+\infty| = \langle 0,-\infty|\, e^{-\mathrm{i}\alpha}$ in the above equation and noting that:

$$\langle 0,+\infty|0,-\infty\rangle = \int_{q(-\infty,\mathbf{x})=0}^{q(+\infty,\mathbf{x})=0} \mathcal{D}q\, e^{\mathrm{i}\mathcal{S}} \quad (4.11)$$

yields the thesis ($|0,-\infty\rangle \equiv |0\rangle$). □

This is the equivalent of Eq. 3.48 evaluated with path integrals: in particular, it is clear that the denominator Eq. 4.11 gives the vacuum-to-vacuum transition amplitudes, "undressing" the numerator of vacuum-to-vacuum disconnected diagrams. Moreover, note how the normalization of $\mathcal{D}q$ carries no physical significance, as it simplifies in the fraction.

§4.2 Functional quantization of fields

§4.2.1 Scalar fields

First of all, consider a free scalar field theory:

$$G(x_1, \dots, x_n) = \frac{\int \mathcal{D}\phi\, \phi(x_1) \dots \phi(x_n) e^{\mathrm{i}\mathcal{S}}}{\int \mathcal{D}\phi\, e^{\mathrm{i}\mathcal{S}}} \qquad\qquad \mathcal{S} = \frac{1}{2} \int \mathrm{d}^4x \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$$

An integral so defined is not assured to converge, as the oscillating factor $e^{i\mathcal{S}}$ is not necessarily sufficient to provide convergence over large fluctuations, i.e. over field configurations with large action value. To ensure convergence, it is conventional to add a small convergence factor to the free Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{\mathrm{i}\epsilon}{2}\phi^2 \tag{4.12}$$

with $\epsilon \to 0^+$. This renders $G(x_1, \dots, x_n)$ a Gaussian integral, ensuring its convergence. The convergence factor can be viewed as an imaginary displacement of the mass $m^2 \mapsto m^2 - \mathrm{i}\epsilon$: this is precisely the $\mathrm{i}\epsilon$-prescription for the Feynman propagator.

Definition 4.2.1 (Scalar generating functional)

Given a scalar field theory with Lagrangian $\mathcal{L}$, the **generating functional** is defined as:

$$Z[J] := \int \mathcal{D}\phi \,\exp\left[\mathrm{i} \int \mathrm{d}^4x \left(\mathcal{L} + J(x)\phi(x)\right)\right] \tag{4.13}$$

where $J(x)$ is a scalar field.

The generating functional is a functional of the **source field** $J(x)$, and its functional derivatives[1] are the correlation functions of the theory.

Lemma 4.2.1 (Correlation functions from generating functionals)

$$G(x_1, \dots, x_n) = \frac{1}{Z[0]} \left[\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right) \dots \left(-\mathrm{i}\frac{\delta}{\delta J(x_n)}\right) Z[J]\right]_{J=0} \tag{4.16}$$

[1] The *functional derivative* is a formal manipulation which, given $J : \mathbb{R}^n \to \mathbb{R}$, is defined by:

$$\frac{\delta}{\delta J(x)} J(y) \equiv \delta^{(n)}(x - y) \tag{4.14}$$

It is clear, then, that given $f : \mathbb{R}^n \to \mathbb{C}^m$:

$$\frac{\delta}{\delta J(x)} \int_{\mathbb{R}^n} \mathrm{d}^n y\, J(y) f(y) = f(x) \tag{4.15}$$

It also follows the usual derivative rules.

Proof. Using Eq. 4.15:

$$\frac{1}{Z[0]}\left[\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\dots\left(-\mathrm{i}\frac{\delta}{\delta J(x_n)}\right)Z[J]\right]_{J=0}$$
$$=\frac{1}{Z[0]}\int \mathrm{d}^4x\left[\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\dots\left(-\mathrm{i}\frac{\delta}{\delta J(x_n)}\right)e^{\mathrm{i}\int \mathrm{d}^4y\, J(y)\phi(y)}\right]_{J=0}e^{\mathrm{i}\mathcal{S}}$$
$$=\frac{1}{Z[0]}\int \mathrm{d}^4x\,\phi(x_1)\dots\phi(x_n)e^{\mathrm{i}\mathcal{S}}$$

which is precisely Eq. 4.10. □

§4.2.1.1 Free theory

The generating functional for a free scalar theory can be written explicitly.

Proposition 4.2.1 (Free generating functional)
The generating functional for a free KG field ϕ is:

$$Z_0[J]=Z_0[0]\exp\left[-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right] \tag{4.17}$$

where $\Delta(x-y)$ is the Feynman propagator.

Proof. Integrating by parts the kinetic term of the action:

$$\int \mathrm{d}^4x\,[\mathcal{L}_0+J\phi]=\int \mathrm{d}^4x\left[-\frac{1}{2}\phi(x)(\Box+m^2-\mathrm{i}\epsilon)\phi(x)+J(x)\phi(x)\right]\equiv-\frac{1}{2}\langle\phi|\hat{A}|\phi\rangle+\langle J|\phi\rangle$$

(with abuse of notation) where the shifted KG operator $\hat{A}\equiv\Box+m^2-i\epsilon$ was introduced (Hermitian as $\epsilon\to 0^+$). Introducing a momentum-space basis:

$$|k\rangle:\langle x|k\rangle=e^{-\mathrm{i}k_\mu x^\mu}\qquad\qquad \int_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}|k\rangle\langle k|=\mathbb{1} \tag{4.18}$$

it is possible to switch to momentum space through a 4D Fourier transform[a]:

$$\phi(x)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\langle x|k\rangle\langle k|\phi\rangle=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\phi(k)e^{-\mathrm{i}k_\mu x^\mu}\qquad\qquad \hat{A}|k\rangle=(-k^2+m^2-\mathrm{i}\epsilon)|k\rangle$$

It is then possible to rewrite the integral above in momentum space:

$$\int \mathrm{d}^4x\,[\mathcal{L}_0+J\phi]=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{1}{2}\phi^*(k)(k^2-m^2+\mathrm{i}\epsilon)\phi(k)+J^*(k)\phi(k)\right]$$

Note that $\phi^*(k)=\phi(-k)$, $J^*(k)=J(-k)$ and $\langle J|x\rangle=\langle x|J\rangle$. To further simplify the integral, define a shifted field:

$$|\chi\rangle=|\phi\rangle-\hat{A}^{-1}|J\rangle$$

The Jacobian of this transformation is trivially 1, thus:

$$\int \mathrm{d}^4x\,[\mathcal{L}_0+J\phi]=-\frac{1}{2}(\langle\chi|+\langle J|\hat{A}^{-1})\hat{A}(|\chi\rangle+\hat{A}^{-1}|J\rangle)+\langle J|\chi\rangle+\langle J|\hat{A}^{-1}|J\rangle$$
$$=-\frac{1}{2}\langle\chi|\hat{A}|\chi\rangle+\frac{1}{2}\langle J|\hat{A}^{-1}|J\rangle$$

As the second term is independent of the free field, it is then clear that:

$$Z_0[J] = \int \mathcal{D}\chi \, \exp\left[-\frac{\mathrm{i}}{2}\langle\chi|\hat{A}|\chi\rangle + \frac{\mathrm{i}}{2}\langle J|\hat{A}^{-1}|J\rangle\right] = Z[0]\exp\left[\frac{i}{2}\langle J|\hat{A}^{-1}|J\rangle\right]$$

Recall that the Green's function of the KG operator is the Feynman propagator Eq. 3.49, thus:

$$\hat{A}^{-1}J(x) = \mathrm{i}\int \mathrm{d}^4y\, \Delta(x-y)J(y)$$

as $\hat{A}\Delta(x-y) = -\mathrm{i}\delta^{(4)}(x-y)$. Then:

$$\exp\left[\frac{\mathrm{i}}{2}\langle J|\hat{A}^{-1}|J\rangle\right] = \exp\left[\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, J(x)\hat{A}^{-1}J(x)\right] = \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J(x)\Delta(x-y)J(y)\right]$$

This completes the proof. □

[a] The Fourier transform is on all spacetime dimensions, as all possible paths contribute to the path integral, not just the solutions to the equations of motion.

Therefore, the propagator is a bi-local distribution. It is possible to express the integral in momentum space too, using $\hat{A}^{-1}|k\rangle = (-k^2+m^2-\mathrm{i}\epsilon)^{-1}|k\rangle = \mathrm{i}\tilde{\Delta}(k)|k\rangle$:

$$Z_0[J] = Z_0[0]\exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} J^*(k)\tilde{\Delta}(k)J(k)\right] \tag{4.19}$$

Using Lemma A.4.2, it is possible to make $Z[0]$ explicit:

$$Z_0[0] = \int \mathcal{D}\phi\, \exp\left[-\frac{\mathrm{i}}{2}\langle\phi|\hat{A}|\phi\rangle\right] = \mathcal{N}(\det\hat{A})^{-1/2} \tag{4.20}$$

where $\det\hat{A}$ is known as a **functional determinant**. Although it has a physical meaning, as $Z_0[0] = \langle 0|0\rangle$ represents the vacuum-to-vacuum amplitude, the functional determinant cancels in the expression for the correlation functions ("undressing" of vacuum-to-vacuum diagrams).

§4.2.1.2 Interacting theory

Consider now a scalar field theory with interaction Lagrangian $\mathcal{L}_{\text{int}} = V[\phi]$. Lemma 4.2.1 is independent on whether the theory is free or not, so the generating integral needs only to be generalized in a way which suits the formalism of correlation functions and Wick's theorem.

Proposition 4.2.2 (Interacting generating functional)

The generating functional for an interacting KG field ϕ with $\mathcal{L}_{\text{int}} = -V[\phi]$ can be written as:

$$Z[J] = \sum_{n\in\mathbb{N}_0}\frac{(-\mathrm{i})^n}{n!}\left(\int \mathrm{d}^4x\, V\left[-\mathrm{i}\frac{\delta}{\delta J(x)}\right]\right)^n Z_0[J] \tag{4.21}$$

where $Z_0[J]$ is the generating functional for the free theory.

Proof. Manipulating Def. 4.2.1:

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\phi\, e^{\mathrm{i}\mathcal{S}+\mathrm{i}\int \mathrm{d}^4y\, J(y)\phi(y)} \\
&= \int \mathcal{D}\phi\, e^{\mathrm{i}\int \mathrm{d}^4y(\mathcal{L}_0+V[\phi]+J(y)\phi(y))} \\
&= \int \mathcal{D}\phi\, e^{\mathrm{i}\int \mathrm{d}^4y(\mathcal{L}_0+J(y)\phi(y))} \sum_{n=0}^{\infty} c_n \phi^n \\
&= \int \mathcal{D}\phi \sum_{n=0}^{\infty} c_n \left(-\mathrm{i}\frac{\delta}{\delta J}\right)^n e^{\mathrm{i}\int \mathrm{d}^4x(\mathcal{L}_0+J\phi)} \\
&= \int \mathcal{D}\phi e^{-\mathrm{i}\int \mathrm{d}^4x V[-\mathrm{i}\frac{\delta}{\delta J(x)}]} e^{\mathrm{i}\int \mathrm{d}^4y(\mathcal{L}_0+J(y)\phi(y))} \\
&= e^{-\mathrm{i}\int \mathrm{d}^4x V[-\mathrm{i}\frac{\delta}{\delta J(x)}]} Z_0[J]
\end{aligned}$$

Expanding the first exponential yields the thesis. □

Hence, in order to compute the correlation functions for the interacting theory, one only needs the generating functional of the free theory:

$$\begin{aligned}
&\langle 0|\mathfrak{T}\{\phi(x_1)\dots\phi(x_n)\}|0\rangle \\
&= \frac{1}{Z[0]}\left[\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\dots\left(-\mathrm{i}\frac{\delta}{\delta J(x_n)}\right)\sum_{k\in\mathbb{N}}\frac{(-\mathrm{i})^k}{k!}\left(\int \mathrm{d}^4x\, V\left[-\mathrm{i}\frac{\delta}{\delta J(x)}\right]\right)^k Z_0[J]\right]_{J=0}
\end{aligned} \tag{4.22}$$

This expression is analogous to Wick's theorem: due to the final condition $J = 0$, only terms which do not contain any $J(x)$ contribute to the correlation function, and these are precisely the fully-contracted terms. Moreover, as of Eq. 4.17, only terms with an even number of functional derivatives are non-vanishing, as an odd number of them results in terms with at least one $J(x)$: this allows the interaction potential to shape the interaction vertex.

Example 4.2.1 (Free scalar field theory)

Computing the 2-point Green function in a free scalar field theory is trivial with Eq. 4.17:

$$\begin{aligned}
\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\}|0\rangle &= \frac{1}{Z_0[0]}\left[\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\left(-\mathrm{i}\frac{\delta}{\delta J(x_2)}\right)Z_0[J]\right]_{J=0} \\
&= \left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\left(-\mathrm{i}\frac{\delta}{\delta J(x_2)}\right)\exp\left[-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J(x)\Delta(x-y)J(y)\right]\Big|_{J=0} \\
&= \frac{1}{Z_0[0]}\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\left[\mathrm{i}\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)\right]Z_0[J]\Big|_{J=0} \\
&= \frac{1}{Z_0[0]}\Big[\Delta(x_2-x_1)+ \\
&\quad -\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)\int \mathrm{d}^4y\,\Delta(x_1-y)J(y)\Big]Z_0[J]\Big|_{J=0} = \Delta(x_2-x_1)
\end{aligned}$$

This expression can be used to compute the 3-point Green function:

$$
\begin{aligned}
&\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle \\
&= \frac{1}{Z_0[0]}\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\left[\Delta(x_3-x_2)-\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)\int \mathrm{d}^4y\,\Delta(x_3-y)J(y)\right]Z_0[J]\Big|_{J=0} \\
&= \frac{1}{Z_0[0]}\left[\mathrm{i}\Delta(x_3-x_2)\int \mathrm{d}^4y\,\Delta(x_1-y)J(y)+\mathrm{i}\Delta(x_2-x_1)\int \mathrm{d}^4y\,\Delta(x_3-y)J(y)\right. \\
&\quad +\mathrm{i}\Delta(x_3-x_1)\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)+ \\
&\quad \left.-\mathrm{i}\int \mathrm{d}^4y\,\Delta(x_1-y)J(y)\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)\int \mathrm{d}^4y\,\Delta(x_3-y)J(y)\right]Z_0[J]\Big|_{J=0} = 0
\end{aligned}
$$

The first non-trivial application of Wick's theorem is the 4-point Green function:

$$
\begin{aligned}
&\langle 0|\mathfrak{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle \\
&\qquad= \frac{1}{Z_0[0]}\left(-\mathrm{i}\frac{\delta}{\delta J(x_1)}\right)\left[\mathrm{i}\Delta(x_4-x_3)\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)+\right. \\
&\qquad+\mathrm{i}\Delta(x_3-x_2)\int \mathrm{d}^4y\,\Delta(x_4-y)J(y)+\mathrm{i}\Delta(x_4-x_2)\int \mathrm{d}^4y\,\Delta(x_3-y)J(y)+ \\
&\qquad\left.-\mathrm{i}\int \mathrm{d}^4y\,\Delta(x_2-y)J(y)\int \mathrm{d}^4y\,\Delta(x_3-y)J(y)\int \mathrm{d}^4y\,\Delta(x_4-y)J(y)\right]Z_0[J]\Big|_{J=0} \\
&\qquad= \frac{1}{Z_0[0]}\left[\Delta(x_4-x_3)\Delta(x_2-x_1)+\Delta(x_3-x_2)\Delta(x_4-x_1)+\right. \\
&\qquad\quad\left.+\Delta(x_4-x_2)\Delta(x_3-x_1)+\left(\propto J^2\right)+\left(\propto J^4\right)\right]Z_0[J]\Big|_{J=0} \\
&\qquad= \Delta(x_4-x_3)\Delta(x_2-x_1)+\Delta(x_3-x_2)\Delta(x_4-x_1)+\Delta(x_4-x_2)\Delta(x_3-x_1)
\end{aligned}
$$

§4.2.1.3 Symmetry factors

Consider Eq. 4.21 for $\lambda\phi^n$-theory:

$$
\begin{aligned}
Z[J] &= Z_0[0]\sum_{v,p\in\mathbb{N}_0}\frac{1}{v!}\left[\int \mathrm{d}^4x\,\frac{-\mathrm{i}\lambda}{n!}\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)^n\right]^v\frac{1}{p!}\left[-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right]^p \\
&= Z_0[0]\sum_{v,p\in\mathbb{N}_0}\frac{1}{v!p!(n!)^v2^p}\left[-\mathrm{i}\lambda\int \mathrm{d}^4x\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)^n\right]^v\left[-\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right]^p
\end{aligned}
$$

Each term in this expansion is represented by a Feynman diagram with v vertices and p propagators. This expression helps clarifying how the symmetry factor of a diagram is computed: first of all, note that there are $n!$ ways to rearrange functional derivatives in each v-term, amounting to a total of $(n!)^v$ equivalent ways for each term of the expansion, paired to $v!$ ways to rearrange the vertices; then, there are also $p!$ ways to rearrange propagators, with 2^p equivalent choices of sides (in total for p propagators).
Although it would seem that the combinatorial factor exactly cancels the prefactor of each term in the above expansion, it needs to be noted that not all these rearrangements are actual distinct (but equivalent) diagrams. The actual combinatorial factor of the diagram is the

symmetry factor, which can be computed as:

$$S = 2^\beta g \prod_k (k!)^{\alpha_k} \tag{4.23}$$

where β is the number of lines which connects a vertex to itself, g is the number of permutations of vertices which leave the diagram unchanged, and α_k is the number of couples of vertices connected by k identical lines.

§4.2.2 Electromagnetic field

With the functional formalism, it is possible to prove Eq. 3.71. The action of the electromagnetic field (massless spin-1 field) can be expressed as:

$$\begin{aligned}
\mathcal{S} &= \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] = -\frac{1}{4}\int \mathrm{d}^4x \,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{2}\int \mathrm{d}^4x \,(\partial_\mu A_\nu \partial^\mu A^\nu - \partial^\mu A_\nu \partial^\nu A_\mu) = \frac{1}{2}\int \mathrm{d}^4x \,(A_\nu \Box A^\nu - A_\nu \partial^\mu\partial^\nu A_\mu) \\
&= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4k\mathrm{d}^4p}{(2\pi)^8} A_\nu(k)\left(e^{-\mathrm{i}k_\alpha x^\alpha}\Box e^{-\mathrm{i}p_\alpha x^\alpha}\eta^{\mu\nu} - e^{-\mathrm{i}k_\alpha x^\alpha}\partial^\mu\partial^\nu e^{-\mathrm{i}p_\alpha x^\alpha}\right) A_\mu(p) \\
&= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4k\mathrm{d}^4p}{(2\pi)^8} e^{-\mathrm{i}(k+p)_\alpha x^\alpha} A_\nu(k)\left(-p^2\eta^{\mu\nu} + p^\mu p^\nu\right) A_\mu(p) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} A_\mu(-k)\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu\right) A_\nu(k)
\end{aligned}$$

This kinetic term is orthogonal to k_μ, i.e. the action vanishes for all gauge fields of the form $A_\mu(k) = k_\mu \alpha(k)$: this is equivalent to saying that the Green's equation for this kinetic operator has no solution, as the latter is singular and therefore non-invertible. This problem is due to gauge invariance: indeed, $F_{\mu\nu}$ is invariant under (Eq. 2.53):

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \alpha(x) \tag{4.24}$$

and the non-invertibility arises from modes gauge-equivalent to $A_\mu(x) \equiv 0$ (obtained with the gauge transformation $\partial_\mu \alpha(x) = -A_\mu(x)$), i.e. non-physical degrees of freedom.
It is possible to isolate and quantize physical degrees of freedom, thanks to a technique due to Faddeev and Popov (introduced in [2]), which yields (see Sec. 9.4 of [4] for a detailed derivation):

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left[\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \tag{4.25}$$

where $\xi \in \mathbb{R}$ is a gauge-fixing parameter. The most useful choices are $\xi = 0$ (Landau gauge), $\xi = 1$ (Feynman gauge) and $\xi = 3$ (Yennie gauge): the Feynman gauge $\xi = 1$ is adopted from now on. In position space:

$$\Delta_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-\mathrm{i}k_\alpha x^\alpha} \frac{-\mathrm{i}\eta^{\mu\nu}}{k^2 + \mathrm{i}\epsilon} \tag{4.26}$$

The formalism of the generating functional can be extended to massless spin-1 fields too.

Definition 4.2.2 (Generating functional for the electromagnetic field)
Given a gauge field A_μ described by a Lagrangian $\mathcal{L}$, the **generating functional** is defined as:

$$Z[J] := \int \mathcal{D}A \exp\left[\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L} + J^\mu(x)A_\mu(x)\right)\right] \tag{4.27}$$

where $J^\mu(x)$ is a 4-vector field.

Lemma 4.2.2 (Correlation functions from generating functionals)

$$\langle 0|\mathfrak{T}\{A_{\mu_1}(x_1)\dots A_{\mu_n}(x_n)\}|0\rangle = \frac{1}{Z[0]}\left[\left(-\mathrm{i}\frac{\delta}{\delta J^{\mu_1}(x_1)}\right)\dots\left(-\mathrm{i}\frac{\delta}{\delta J^{\mu_n}(x_n)}\right)Z[J]\right]_{J=0} \tag{4.28}$$

Proof. The proof is analogous to that of Lemma 4.2.1, using $\frac{\delta}{\delta J^\mu(x)}J^\nu(y) = \delta^\nu_\mu\delta^{(4)}(x-y)$. □

Proposition 4.2.3 (Generating functional and the propagator)

$$Z[J] = Z[0]\exp\left[\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right] \tag{4.29}$$

Proof. The proof is analogous to that of Prop. 4.2.1. The extra negative sign is due to $\Delta_{\mu\nu}(x-y)$ solving:

$$\left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right)\Delta_{\mu\nu}(x-y) = \mathrm{i}\delta^{(4)}(x-y) \tag{4.30}$$

Note that the source term is $\mathrm{i}\delta^{(4)}(x-y)$ and not $-\mathrm{i}\delta^{(4)}(x-y)$ (as for the KG operator). □

§4.2.3 Spinor fields

To describe fermions in the functional formalism, it is necessary to define them through Grassmann fields (see §A.3.2.1) of the form:

$$\Psi(x) = \sum_i \theta_i\Psi_i(x)$$

where θ_i are Grassmann numbers and $\Psi_i(x)$ are a basis of Dirac spinor fields. This allows to extend the path-integral formalism to fermoinic (anti-commuting) fields.

Example 4.2.2 (2-point Green function)
Using Eq. 4.10, it is possible to recover the Dirac propagator:

$$\langle 0|\mathfrak{T}\{\Psi(x_1)\bar{\Psi}(x_2)\}|0\rangle = \frac{\int \Psi\mathcal{D}\mathcal{D}\bar{\Psi}\,\Psi(x_1)\bar{\Psi}(x_2)\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\Psi}(\mathrm{i}\partial\!\!\!/ - m + i\epsilon)\Psi\right]}{\int \mathcal{D}\mathcal{D}\Psi\bar{\Psi}\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\Psi}(\mathrm{i}\partial\!\!\!/ - m + i\epsilon)\Psi\right]}$$

where convergence is assured by the iϵ-prescription. Using Eq. A.37 for the numerator and Eq. A.36 for the denominator (note that $\bar{\Psi}$ is used, instead of Ψ^*, as they are unitarily equivalent), the functional determinant of the $\mathrm{i}\partial\!\!\!/ - m + \mathrm{i}\epsilon$ operator cancels out, leaving the matrix element of the inverse operator $\left(\mathrm{i}\partial\!\!\!/ - m + \mathrm{i}\epsilon\right)$ (more precisely, $\left[-\mathrm{i}\left(\mathrm{i}\partial\!\!\!/ - m + \mathrm{i}\epsilon\right)\right]^{-1}$). To compute this matrix element in momentum space, consider the representation of the fermionic field in it:

$$\Psi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\Psi(k)e^{-\mathrm{i}k_\mu x^\mu} \qquad \bar{\Psi}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{\Psi}(k)e^{\mathrm{i}k_\mu x^\mu} \tag{4.31}$$

Therefore, the matrix element is:

$$\langle 0|\mathfrak{T}\{\Psi(x_1)\bar{\Psi}(x_2)\}|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{-\mathrm{i}\left(\not{k} - m + \mathrm{i}\epsilon\right)}e^{-\mathrm{i}k_\mu(x_1-x_2)^\mu} = \Sigma(x_1 - x_2)$$

which is precisely the Dirac propagator (Eq. 3.52).

To derive the Feynman rules for a fermionic theory, it is possible to define a generating functional.

Definition 4.2.3 (Fermionic generating functional)
Given a theory $\mathcal{L}$ with a fermionic field Ψ, the **generating functional** is defined as:

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left[\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L} + \bar{\eta}(x)\Psi(x) + \bar{\Psi}(x)\eta(x)\right)\right] \tag{4.32}$$

where $\eta(x)$ is a Grassmann field.

Proposition 4.2.4 (Free generating functional)
The generating functional for a free fermionic field Ψ is:

$$Z_0[\bar{\eta}, \eta] = Z_0[0, 0] \exp\left[-\int \mathrm{d}^4x\mathrm{d}^4y\, \bar{\eta}(x)\Sigma(x-y)\eta(y)\right] \tag{4.33}$$

Proof. Rewriting the exponent of Eq. 4.32 in momentum space:

$$\begin{aligned}\int \mathrm{d}^4x \left(\mathcal{L}_0 + \bar{\eta}\Psi + \bar{\Psi}\eta\right) &= \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k\mathrm{d}^4p}{(2\pi)^8}\bar{\Psi}(k)e^{\mathrm{i}k_\mu x^\mu}(\mathrm{i}\not{\partial} - m + \mathrm{i}\epsilon)\Psi(p)e^{-\mathrm{i}p_\mu x^\mu} + \\ &+ \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k\mathrm{d}^4p}{(2\pi)^8}\bar{\eta}(k)e^{\mathrm{i}k_\mu x^\mu}\Psi(p)e^{-\mathrm{i}p_\mu x^\mu} \\ &+ \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k\mathrm{d}^4p}{(2\pi)^8}\bar{\Psi}(k)e^{\mathrm{i}k_\mu x^\mu}\eta(p)e^{-\mathrm{i}p_\mu x^\mu} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\bar{\Psi}(k)(\not{p} - m + \mathrm{i}\epsilon)\Psi(k) + \bar{\eta}(k)\Psi(k) + \bar{\Psi}(k)\eta(k)\right]\end{aligned}$$

Now, shift the field as:

$$\Psi \mapsto \Xi = \Psi + (\mathrm{i}\not{\partial} - m + \mathrm{i}\epsilon)^{-1}\eta \equiv \Psi + \hat{A}^{-1}\eta$$

The integral can then be rewritten as (recall the Hermitianity of the Dirac operator):

$$\begin{aligned}\int \mathrm{d}^4x \left(\mathcal{L}_0 + \bar{\eta}\Psi + \bar{\Psi}\eta\right) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\left(\bar{\Xi} - \bar{\eta}\hat{A}^{-1}\right)\hat{A}\left(\Xi - \hat{A}^{-1}\eta\right) + \bar{\eta}\Xi + \bar{\Xi}\eta - 2\bar{\eta}\hat{A}^{-1}\eta\right] \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\bar{\Xi}\hat{A}\Xi - \bar{\eta}\hat{A}^{-1}\eta\right] \\ &= \int \mathrm{d}^4x\,\bar{\Xi}(\mathrm{i}\not{\partial} - m + \mathrm{i}\epsilon)\Xi - \int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{\eta}(k)\frac{1}{\not{k} - m + \mathrm{i}\epsilon}\eta(k)\end{aligned}$$

Therefore, the generating functional can be expressed as (using the shift-invariance of Grassmann integrals):

$$\begin{aligned}Z_0[\bar{\eta}, \eta] &= \int \mathcal{D}\bar{\Xi}\mathcal{D}\Xi\, \exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\Xi}(\mathrm{i}\not{\partial} - m + \mathrm{i}\epsilon)\Xi\right]\exp\left[-\int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{\eta}(k)\frac{\mathrm{i}}{\not{k} - m + \mathrm{i}\epsilon}\eta(k)\right] \\ &= Z_0[0,0]\exp\left[-\int \mathrm{d}^4x\mathrm{d}^4y\,\bar{\eta}(x)\Sigma(x-y)\eta(y)\right]\end{aligned}$$

which is the thesis. □

As for the free scalar field, using Eq. A.36 makes $Z_0[0]$ an explicit functional determinant of the kinetic operator:

$$Z_0[0,0] = \mathcal{N}\det\hat{A} \tag{4.34}$$

To compute correlation functions from the generating functional, note that for Grassmann numbers:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\theta\eta = -\frac{\mathrm{d}}{\mathrm{d}\eta}\eta\theta = -\theta \tag{4.35}$$

Lemma 4.2.3 (Correlation functions from generating functionals)

$$\langle 0|\mathfrak{T}\{\Psi(x_1)\bar{\Psi}(x_2)\dots\}|0\rangle = \frac{1}{Z[0,0]}\left[\left(-\mathrm{i}\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\left(\mathrm{i}\frac{\delta}{\delta\eta(x_2)}\right)\dots Z[\bar{\eta},\eta]\right]_{\bar{\eta},\eta=0} \tag{4.36}$$

Proof. Analogous to the proof of Lemma 4.2.1 with Eq. 4.35. □

Finally, note that the anti-commuting nature of Grassmann numbers makes Eq. 3.51 trivial.

Example 4.2.3 (QED Feynman rules)

Consider the QED Lagrangian:

$$\mathcal{L}_{\mathrm{QED}} = \bar{\Psi}(\mathrm{i}\not{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.37}$$

Expanding the covariant derivative $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$:

$$\mathcal{L}_{\mathrm{QED}} = \bar{\Psi}(\mathrm{i}\not{\partial} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\Psi}\gamma^\mu\Psi A_\mu$$

It is then clear that this theory describes spin-$\frac{1}{2}$ (anti-)fermions which propagate with $\tilde{\Sigma}(p)$ and massless spin-1 gauge bosons which propagate with $\tilde{\Delta}_{\mu\nu}(k)$. The interaction vertex involves a fermion, an anti-fermion and a gauge boson, and it acts as $-\mathrm{i}e\int \mathrm{d}^4x\,\gamma^\mu$.

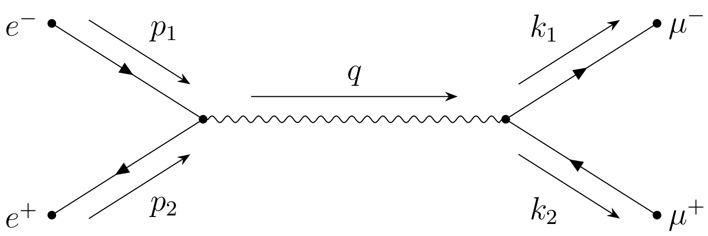
Chapter 5

Amplitudes in QED

§5.1 Tree-level amplitudes

§5.1.1 $e^+e^- \to \mu^+\mu^-$ scattering

The $e^+e^- \to \mu^+\mu^-$ scattering is the simplest process in QED and one of the most important in high-energy physics, as it is used to calibrate e^+e^- colliders.
Recalling the Feynman rules for QED from §3.3.4.3, the amplitude can be found from the only Feynman diagram of the process[1]:



with $q = p_1 + p_2 = k_1 + k_2$ by momentum conservation. By Eq. 3.63, then:

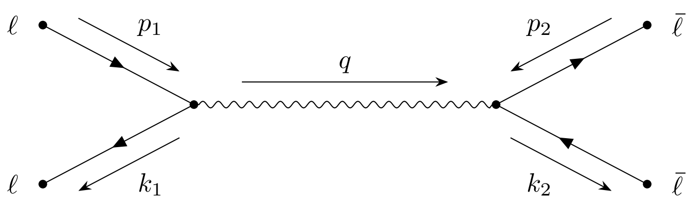
$$\mathrm{i}\mathcal{M} = \bar{u}^{s_1}(k_1)(-\mathrm{i}e\gamma^\mu)v^{s_2}(k_2)\frac{-\mathrm{i}\eta_{\mu\nu}}{q^2+\mathrm{i}\epsilon}\bar{v}^{r_2}(p_2)(-\mathrm{i}e\gamma^\nu)u^{r_1}(p_1) = \frac{\mathrm{i}e^2}{q^2}\bar{u}^{s_1}(k_1)\gamma^\mu v^{s_2}(k_2)\bar{v}^{r_2}(p_2)\gamma_\mu u^{r_1}(p_1)$$

Lemma 5.1.1 (Bi-spinor products)

$$(\bar{u}\gamma^\mu v)^* = \bar{v}\gamma^\mu u \tag{5.1}$$

Proof. $(\bar{u}\gamma^\mu v)^* = v^\dagger(\gamma^\mu)^\dagger\bar{u}^\dagger = v^\dagger(\gamma^\mu)^\dagger(u^\dagger\gamma^0)^\dagger = v^\dagger(\gamma^\mu)^\dagger\gamma^0 u = v^\dagger\gamma^0\gamma^\mu u = \bar{v}\gamma^\mu u.$ □

[1]Note that, if $\ell\bar{\ell} \to \ell\bar{\ell}$, with $\ell = e^-, \mu^-$, was considered instead, there would be a second Feynman diagram to computed:



with $q = p_1 - k_1 = k_2 - p_2$.

With this lemma, it is easy to see that (as $\bar{u}\gamma^\mu v \in \mathbb{C}$):

$$\begin{aligned}|\mathcal{M}_{r_1,r_2,s_1,s_2}|^2 &= \frac{e^4}{q^4}\bar{u}^{s_1}(k_1)\gamma^\mu v^{s_2}(k_2)\bar{v}^{r_2}(p_2)\gamma_\mu u^{r_1}(p_1)\bar{v}^{s_2}(k_2)\gamma^\nu u^{s_1}(k_1)\bar{u}^{r_1}(p_1)\gamma_\nu v^{r_2}(p_2)\\ &= \frac{e^4}{q^4}\bar{u}^{s_1}(k_1)\gamma^\mu v^{s_2}(k_2)\bar{v}^{s_2}(k_2)\gamma^\nu u^{s_1}(k_1)\bar{v}^{r_2}(p_2)\gamma_\mu u^{r_1}(p_1)\bar{u}^{r_1}(p_1)\gamma_\nu v^{r_2}(p_2)\end{aligned}$$

This matrix element has explicit dependence on spin states: although not impossible, it is difficult to experimentally retain control over them, both in the initial- and final-state. For this reason, it is convenient to consider a matrix element which is averaged over initial spin states and summed over final spin states (as usually the detector does not distinguish them):

$$|\mathcal{M}|^2 = \frac{1}{2}\sum_{r_1=1,2}\frac{1}{2}\sum_{r_2=1,2}\sum_{s_1}\sum_{s_2=1,2}|\mathcal{M}_{r_1,r_2,s_1,s_2}|^2 \tag{5.2}$$

Recalling Eq. 1.111 and writing spinor indices explicitly, the first half of the matrix element is:

$$\begin{aligned}&\frac{e^4}{4q^4}\sum_{s_1,s_2=1,2}\sum_{a,b,c,d=1}^{4}[\bar{u}^{s_1}(k_1)]_a\gamma^\mu_{ab}[v^{s_2}(k_2)]_b[\bar{v}^{s_2}(k_2)]_c\gamma^\nu_{cd}[u^{s_1}(k_1)]_d\\ &\qquad = \frac{e^4}{4q^4}\sum_{a,b,c,d=1}^{4}[\not{k}_1 + m_\mu]_{da}\gamma^\mu_{ab}[\not{k}_2 - m_\mu]_{bc}\gamma^\nu_{cd} = \frac{e^4}{4q^4}\operatorname{tr}\{(\not{k}_1 + m_\mu)\gamma^\mu(\not{k}_2 - m_\mu)\gamma^\nu\}\end{aligned}$$

The unpolarized matrix element then is:

$$|\mathcal{M}|^2 = \frac{e^4}{4q^4}\operatorname{tr}\{(\not{k}_1 + m_\mu)\gamma^\mu(\not{k}_2 - m_\mu)\gamma^\nu\}\operatorname{tr}\{(\not{p}_1 + m_e)\gamma_\mu(\not{p}_2 - m_e)\gamma_\nu\} \tag{5.3}$$

This approach is general: any QED amplitude involving external fermions can be simplified into traces of products of gamma matrices, when squared and summed/averaged over spin states, hence eliminating explicit dependence on spinors.

§5.1.1.1 Traces

Due to their importance in QED, traces of gamma matrices should be studied systematically.

Proposition 5.1.1 (Odd number of gamma matrices)
For any $n \in \mathbb{N}$ odd:

$$\operatorname{tr}\{\gamma^{\mu_1}\dots\gamma^{\mu_n}\} = 0 \tag{5.4}$$

Proof. As $\{\gamma^\mu, \gamma^5\} = 0$ and given the cyclic property of the trace:

$$\begin{aligned}\operatorname{tr}\{\gamma^{\mu_1}\dots\gamma^{\mu_n}\} &= \operatorname{tr}\{\gamma^5\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n}\} = -\operatorname{tr}\{\gamma^5\gamma^{\mu_1}\gamma^5\dots\gamma^{\mu_n}\}\\ &= (-1)^n\operatorname{tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^5\} = (-1)^n\operatorname{tr}\{\gamma^5\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n}\} = (-1)^n\operatorname{tr}\{\gamma^{\mu_1}\dots\gamma^{\mu_n}\}\end{aligned}$$

Therefore, the trace vanishes for any odd $n \in \mathbb{N}$. □

Proposition 5.1.2 (Even number of gamma matrices)

$$\operatorname{tr}\{\gamma^\mu\gamma^\nu\} = 4\eta^{\mu\nu} \tag{5.5}$$

$$\operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \tag{5.6}$$

Proof. Using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathrm{I}_4$:

$$\operatorname{tr}\{\gamma^\mu\gamma^\nu\} = \operatorname{tr}\{2\eta^{\mu\nu}\mathrm{I}_4 - \gamma^\nu\gamma^\mu\} = 8\eta^{\mu\nu} - \operatorname{tr}\{\gamma^\mu\gamma^\nu\}$$

$$\begin{aligned}\operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= \operatorname{tr}\{2\eta^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\} \\ &= \operatorname{tr}\{2\eta^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2\eta^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2\eta^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\} \\ &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\}\end{aligned}$$

which is the thesis. □

In general, the trace of the product of n gamma matrices can be reduced to traces of products of $n-2$ gamma matrices. Traces involving $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ are of interest too.

Proposition 5.1.3 (Odd number of gamma matrices)
For any $n \in \mathbb{N}$ odd:

$$\operatorname{tr}\{\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^5\} = 0 \tag{5.7}$$

Proof. As in the proof of Prop. 5.1.1:

$$\operatorname{tr}\{\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^5\} = (-1)^n \operatorname{tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^5\gamma^5\} = (-1)^n \operatorname{tr}\{\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^5\}$$

which vanishes for any odd $n \in \mathbb{N}$. □

Proposition 5.1.4 (Even number of gamma matrices)

$$\operatorname{tr}\gamma^5 = \operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^5\} = 0 \tag{5.8}$$

$$\operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\} = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma} \tag{5.9}$$

Proof. Recalling that $\{\gamma^5, \gamma^\mu\} = 0$:

$$\operatorname{tr}\gamma^5 = \operatorname{tr}\{\gamma^0\gamma^0\gamma^5\} = -\operatorname{tr}\{\gamma^0\gamma^5\gamma^0\} = -\operatorname{tr}\{\gamma^0\gamma^0\gamma^5\} = -\operatorname{tr}\gamma^5$$

Analogously, given $\alpha \neq \mu, \nu$ (so that $\{\gamma^\mu, \gamma^\alpha\} = \{\gamma^\nu, \gamma^\alpha\} = \{\gamma^5, \gamma^\alpha\} = 0$):

$$\operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^5\} = \operatorname{tr}\{\gamma^0\gamma^0\gamma^\mu\gamma^\nu\gamma^5\} = (-1)^3 \operatorname{tr}\{\gamma^0\gamma^\mu\gamma^\nu\gamma^5\gamma^0\} = -\operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^5\}$$

Consider now $\operatorname{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\}$: the above reasoning still applies if any couple of equal indices is present, therefore the trace must be proportional to $\epsilon^{\mu\nu\rho\sigma}$, which agrees with the anti-commutation relation of different gamma matrices. Computing $\operatorname{tr}\{\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\}$ gives the correct proportionality constant $-4\mathrm{i}$. □

It is useful to recall an identity for the Levi–Civita symbol.

Lemma 5.1.2 (Levi–Civita symbol)
Given $n \in \mathbb{N}$ and $\{i_k\}_{k=1,\dots,n}, \{j_k\}_{k=1,\dots,n} \subset \{1, \dots, n\}$, then:

$$\epsilon_{i_1 \dots i_k i_{k+1} \dots i_n} \epsilon^{i_1 \dots i_k j_{k+1} \dots j_n} = n! \delta^{j_{k+1} \dots j_n}_{i_{k+1} i_n} \tag{5.10}$$

where the **generalized Kronecker delta** is defined as:

$$\delta^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_p} := \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \dots & \delta^{\mu_p}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_p} & \dots & \delta^{\mu_p}_{\nu_p} \end{vmatrix} \tag{5.11}$$

Gamma matrices' contractions are important too.

Proposition 5.1.5 (Contractions)

$$\gamma^\mu \gamma_\mu = 4 \tag{5.12}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{5.13}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} \tag{5.14}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\nu \gamma^\rho \gamma^\sigma \tag{5.15}$$

§5.1.1.2 Unpolarized matrix element

It is now possible to evaluate the matrix element in Eq. 5.3. Explicitly, the traces are:

$$\begin{aligned} \mathrm{tr}\{(\not{k}_1 + m_\mu)\gamma^\mu(\not{k}_2 - m_\mu)\gamma^\nu\} &= \mathrm{tr}\{k_{1\alpha}k_{2\beta}\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu + m_\mu k_{2\beta}\gamma^\mu\gamma^\beta\gamma^\nu - k_{1\alpha}m_\mu\gamma^\alpha\gamma^\mu\gamma^\nu - m_\mu^2\gamma^\mu\gamma^\nu\} \\ &= k_{1\alpha}k_{2\beta}4\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\mu\beta}\right) - m_\mu^2 4\eta^{\mu\nu} \\ &= 4\left[{k_1}^\mu {k_2}^\nu + {k_1}^\nu {k_2}^\mu - \eta^{\mu\nu}\left(k_1 \cdot k_2 + m_\mu^2\right)\right] \end{aligned}$$

where $u \cdot v \equiv u^\mu v_\mu$. The other trace is analogous, so that:

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{4e^4}{q^4}\left[{k_1}^\mu {k_2}^\nu + {k_1}^\nu {k_2}^\mu - \eta^{\mu\nu}\left(k_1 \cdot k_2 + m_\mu^2\right)\right]\left[p_{1\mu}p_{2\mu} + p_{1\nu}p_{2\mu} - \eta_{\mu\nu}\left(p_1 \cdot p_2 + m_e^2\right)\right] \\ &= \frac{4e^4}{q^4}\left[(k_1 \cdot p_1)(k_2 \cdot p_2) + (k_1 \cdot p_2)(k_2 \cdot p_1) - (p_1 \cdot p_2)(k_1 \cdot k_2) - m_\mu^2(p_1 \cdot p_2) + \right. \\ &\quad (k_1 \cdot p_2)(k_2 \cdot p_1) + (k_1 \cdot p_1)(k_2 \cdot p_2) - (p_1 \cdot p_2)(k_1 \cdot k_2) - m_\mu^2(p_1 \cdot p_2) + \\ &\quad - 2(k_1 \cdot k_2)(p_1 \cdot p_2) - 2m_e^2(k_1 \cdot k_2) + \\ &\quad \left. + 4(k_1 \cdot k_2)(p_1 \cdot p_2) + 4m_e^2(k_1 \cdot k_2) + 4m_\mu^2(p_1 \cdot p_2) + 4m_\mu^2 m_e^2\right] \\ &= \frac{8e^4}{q^4}\left[(k_1 \cdot p_1)(k_2 \cdot p_2) + (k_1 \cdot p_2)(k_2 \cdot p_1) + m_\mu^2(p_1 \cdot p_2) + m_e^2(k_1 \cdot k_2) + 2m_\mu^2 m_e^2\right] \end{aligned}$$

As $m_e \sim 10^{-4} m_\mu$, it is possible to neglect terms which contain m_e, thus the unpolarized matrix element is:

$$|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(k_1 \cdot p_1)(k_2 \cdot p_2) + (k_1 \cdot p_2)(k_2 \cdot p_1) + m_\mu^2(p_1 \cdot p_2)\right] \tag{5.16}$$

Note that, for a generic $2 \to 2$ process, the unpolarized matrix element depends on 4 momenta, i.e. 16 variables. However, momentum conservation reduces this number by 4, as $k_1 + k_2 =$

$p_1 + p_2$ (4 equations), and mass-shell conditions on these 4 momenta reduces it further by 4. This leaves only 8 variables, but 6 of them are fixed by Lorentz invariance[2](from 6 independent generators of $SO^+(1,3)$): the unpolarized matrix element hence only depends on 2 independent variables.

Definition 5.1.1 (Mandelstam variables)

Given a $2 \to 2$ scattering such that $i_1(k_1) + i_2(k_2) \to f_1(p_1) + f_2(p_2)$, then the **Mandelstam variables** are defined as:

$$s := (p_1 + p_2)^2 = (k_1 + k_2)^2 \tag{5.17}$$

$$t := (k_1 - p_1)^2 = (k_1 - p_2)^2 \tag{5.18}$$

$$u := (k_1 - p_2)^2 = (k_2 - p_1)^2 \tag{5.19}$$

Proposition 5.1.6

Only two Mandelstam variables are independent, as:

$$s + t + u = m_{i_1}^2 + m_{i_2}^2 + m_{f_1}^2 + m_{f_2}^2 \tag{5.20}$$

Proof. By direct calculation, using momentum conservation:

$$\begin{aligned} s + t + u &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 + k_1^2 + p_1^2 - 2k_1 \cdot p_1 + k_2^2 + p_1^2 - 2k_2 \cdot p_1 \\ &= 3p_1^2 + p_2^2 + k_1^2 + k_2^2 - 2p_1 \cdot (k_1 + k_2 - p_2) = p_1^2 + p_2^2 + k_1^2 + k_2^2 \end{aligned}$$

Mass-shell conditions yield the thesis. □

It is possible to parametrize the 2 degrees of freedom of $e^+e^- \to \mu^+\mu^-$ scattering with two Mandelstam variables, which for this process can be expressed as:

$$s = 2(k_1 \cdot k_2 + m_\mu^2) = 2(p_1 \cdot p_2 + m_e^2)$$

$$t = -2k_1 \cdot p_1 + m_e^2 + m_\mu^2 = -2k_2 \cdot p_2 + m_e^2 + m_\mu^2$$

$$u = -2k_2 \cdot p_1 + m_e^2 + m_\mu^2 = -2k_1 \cdot p_2 + m_e^2 + m_\mu^2$$

Neglecting terms with m_e, it is possible to write:

$$|\mathcal{M}|^2 = \frac{8e^4}{s^2}\left[\frac{t - m_\mu^2}{2}\frac{t - m_\mu^2}{2} + \frac{u - m_\mu^2}{2}\frac{u - m_\mu^2}{2} + m_\mu^2 \frac{s}{2}\right]$$

which is:

$$|\mathcal{M}|^2 = \frac{2e^4}{s^2}\left[(t - m_\mu)^2 + (u - m_\mu)^2 + 2m_\mu^2 s\right] \tag{5.21}$$

To compare this theoretical prediction with experimental data, however, it is better to choose an energy and an angle as the 2 independent variables. To do so, it is necessary to choose a reference frame: a natural choise is the **center-of-mass frame**, defined as the frame in which $\mathbf{p}_1 = -\mathbf{p}_2$. In this RF, momenta are parametrized as (see Fig. 5.1):

$$p_1 = (E, E\hat{\mathbf{e}}_z) \qquad p_2 = (E, -E\hat{\mathbf{e}}_z) \qquad k_1 = (E, \mathbf{k}) \qquad k_2 = (E, -\mathbf{k})$$

[2] Note that momentum conservation is a consequence of traslational invariance, therefore it can be united to Lorentz invariance to get Poincaré invariance, and indeed $ISO^+(1,3)$ has $6 + 4 = 10$ independent generators.

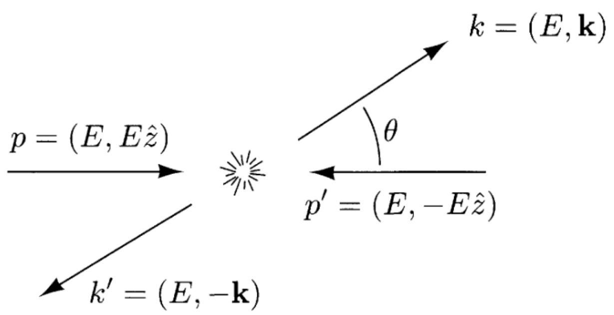


Figure 5.1: Center-of-mass RF for a $2 \to 2$ scattering process.

with:

$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2} \qquad \mathbf{k} \cdot \hat{\mathbf{e}}_z = |\mathbf{k}| \cos\theta$$

Note that this is the most genera expression for $2 \to 2$ scattering with $m_{i_1} = m_{i_2}$ and $m_{f_1} = m_{f_2}$. In this frame, the Mandelstam variables are:

$$s = 4E^2 \equiv E_{\text{cm}}^2$$

$$t = -2(E^2 - E\,|\mathbf{k}|\cos\theta) + m_e^2 + m_\mu^2 \approx -2E(E - |\mathbf{k}|\cos\theta) + m_\mu^2$$

$$u = -2(E^2 + E\,|\mathbf{k}|\cos\theta) + m_e^2 + m_\mu^2 \approx -2E(E - |\mathbf{k}|\cos\theta) + m_\mu^2$$

Inserting into Eq. 5.21:

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{2e^4}{16E^4}\left[4E^2(E - |\mathbf{k}|\cos\theta)^2 + 4E^2(E - |\mathbf{k}|\cos\theta)^2 + 8m_\mu^2 E^2\right] \\ &= \frac{e^4}{2E^2}\left[(E - |\mathbf{k}|\cos\theta)^2 + (E - |\mathbf{k}|\cos\theta)^2 + 2m_\mu^2\right] \\ &= e^4\left[1 + \frac{|\mathbf{k}|^2}{E^2}\cos^2\theta + \frac{m_\mu^2}{E^2}\right] = e^4\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \end{aligned}$$

§5.1.1.3 Unpolarized cross-section

The differential cross-section can be computed using Eq. B.2. In the case of $e^+e^- \to \mu^+\mu^-$ scattering, the flux factor is:

$$\Phi \equiv 4E_{\mathbf{p}_1}E_{\mathbf{p}_2}\left|\frac{\mathbf{p}_1}{E_{\mathbf{p}_1}} - \frac{\mathbf{p}_2}{E_{\mathbf{p}_2}}\right| = 4E^2\left|\frac{E\hat{\mathbf{e}}_z}{E} - \frac{-E\hat{\mathbf{e}}_z}{E}\right| = 8E^2$$

On the other hand, the invariant phase-space measure is:

$$\mathrm{d}\Pi_2 = \frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{\mathbf{k}_1}}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_{\mathbf{k}_2}}(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2)$$

Note that $p_1 + p_2 = (2E, \mathbf{0})$, therefore $\mathbf{k}_1 = -\mathbf{k}_2$, i.e. $E_{\mathbf{k}_1} = E_{\mathbf{k}_2}$, so:

$$\mathrm{d}\Pi_2 = \frac{\mathrm{d}^3k_1}{(2\pi)^2(2E_{\mathbf{k}_1})^2}\delta(2E - 2E_{\mathbf{k}_1}) = \frac{\mathbf{k}_1^2\mathrm{d}k_1\mathrm{d}\Omega_2}{16\pi^2E_{\mathbf{k}_1}^2}\delta(2E_{\mathbf{k}_1} - 2E)$$

where $\mathrm{d}\Omega_2$ is the measure on the 2-sphere[3].

Lemma 5.1.3 (Scaling property)
Given a differentiable $f : I \subset \mathbb{R} \to \mathbb{R}$ with roots $\{x_i\}_{i\in\mathcal{I}} \subset I : f'(x_i) \neq 0\ \forall i \in \mathcal{I}$, then:

$$\delta(f(x)) = \sum_{i\in\mathcal{I}} \frac{\delta(x_i)}{|f'(x_i)|} \tag{5.25}$$

Using Lemma 5.1.3, then:

$$\begin{aligned}
\mathrm{d}\Pi_2 &= \frac{\mathbf{k}_1^2 \mathrm{d}k_1 \mathrm{d}\Omega_2}{16\pi^2 E_{\mathbf{k}_1}^2}\delta(2\sqrt{m_\mu^2+\mathbf{k}_1^2}-2E) = \frac{\mathbf{k}_1^2 \mathrm{d}k_1 \mathrm{d}\Omega_2}{16\pi^2 E_{\mathbf{k}_1}^2}\frac{\delta(|\mathbf{k}_1|-\sqrt{E^2-m_\mu^2})}{2\frac{|\mathbf{k}_1|}{\sqrt{m_\mu^2+\mathbf{k}_1^2}}} \\
&= \frac{|\mathbf{k}_1|\,\mathrm{d}k_1 \mathrm{d}\Omega_2}{32\pi^2 E}\delta(|\mathbf{k}_1|-\sqrt{E^2-m_\mu^2})
\end{aligned}$$

Integrating over $|\mathbf{k}_1|$ and inserting into Eq. B.2 yields a general formula for $2 \to 2$ scattering:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_2} = \frac{1}{256\pi^2 E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\,|\mathcal{M}|^2 \tag{5.26}$$

Inserting the unpolarized matrix element:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_2} = \frac{e^4}{256\pi^2 E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \tag{5.27}$$

Proposition 5.1.7 (Total cross-section)

[3]In general, $\mathbb{S}^n$ can be described as a surface embedded in $\mathbb{R}^{n+1}$, defined by:

$$\sum_{i=1}^{n+1} x_i^2 = 1 \tag{5.22}$$

Therefore, $\mathbb{S}^n$ can be parametrized by *hyperspherical coordinates* $\{\varphi_1, \dots, \varphi_{n-1}, \varphi_n\} \subset [0,\pi]^{n-1} \times [0, 2\pi)$ as:

$$\begin{aligned}
x_1 &= \cos\varphi_1 \\
x_2 &= \sin\varphi_1 \cos\varphi_2 \\
x_3 &= \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\vdots \\
x_{n-1} &= \sin\varphi_1 \dots \sin\varphi_{n-2}\cos\varphi_{n-1} \\
x_n &= \sin\varphi_1 \dots \sin\varphi_{n-2}\sin\varphi_{n-1}\cos\varphi_n \\
x_{n+1} &= \sin\varphi_1 \dots \sin\varphi_{n-2}\sin\varphi_{n-1}\sin\varphi_n
\end{aligned} \tag{5.23}$$

It is then possible to define the *hyperspherical measure* on $\mathbb{S}^n$:

$$\mathrm{d}\Omega_n \equiv \sin^{n-1}\varphi_1 \sin^{n-2}\varphi_2 \dots \sin^2\varphi_{n-2}\sin\varphi_{n-1}\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2\dots\mathrm{d}\varphi_{n-2}\mathrm{d}\varphi_{n-1}\mathrm{d}\varphi_n \tag{5.24}$$

The **total cross-section** for $e^+e^- \to \mu^+\mu^-$ scattering is:

$$\sigma_{\text{tot}} = \frac{\pi\alpha^2}{3E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right) \tag{5.28}$$

Proof. Integrating Eq. 5.27:

$$\begin{aligned}\sigma_{\text{tot}} &:= \int d\Omega_2 \frac{d\sigma}{d\Omega_2} = \frac{\alpha^2}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \\ &= \frac{\alpha^2}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}2\pi\left[2\left(1+\frac{m_\mu^2}{E^2}\right)+\frac{2}{3}\left(1-\frac{m_\mu^2}{E^2}\right)\right] = \frac{\pi\alpha^2}{3E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)\end{aligned}$$

where $\alpha^2 \equiv \frac{e^2}{4\pi}$ was used. □

It is clear that $\sigma_{\text{tot}} \propto \frac{\alpha^2}{s}$: the $\alpha^2\,(1+o(\alpha^2))$ is understood from the structure of the interaction vertex, while the s^{-1} dependence can be derived from dimensional analysis and requiring σ_{tot} to be a scalar.

§5.1.1.4 Helicity structure

Proposition 5.1.8 (Helicity eigenstates)
In the high-energy limit, i.e. $m \approx 0$, chirality eigenstates are helicity eigenstates too:

$$\hat{h} = \frac{\gamma^5}{2} \tag{5.29}$$

Proof. By Eq. 1.103, in the high-energy limit $\not{p}u = \not{p}v = 0$, therefore (as $\gamma^0\gamma^0 = \mathrm{I}_4$):

$$\gamma^5\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}u = E\gamma^5 u \qquad \gamma^5\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}v = E\gamma^5 v$$

As $m=0$, then $\mathbf{p} = E\hat{\mathbf{p}}$. Moreover, in the chiral representation:

$$\gamma^5\gamma^0\gamma^i = \begin{bmatrix} -\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2\end{bmatrix}\begin{bmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0\end{bmatrix}\begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{bmatrix} = \begin{bmatrix} \sigma^i & 0 \\ 0 & \sigma^i\end{bmatrix} = \Sigma^i$$

Recalling that the spin operator is $\mathbf{S} = \frac{\boldsymbol{\Sigma}}{2}$ and that the helicity operator is $\hat{h} = \mathbf{S}\cdot\hat{\mathbf{p}}$, then by the above equations the helicity operator is proportional to the chirality operator, yielding the thesis. □

Recall Eq. 1.105-1.106. In the high-energy limit $m=0$, i.e. $\mathbf{p} = E\hat{\mathbf{p}}$, so that:

$$u(p) = \sqrt{E}\begin{pmatrix}\sqrt{\mathrm{I}_2 - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}\xi \\ \sqrt{\mathrm{I}_2 + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}\xi\end{pmatrix} \qquad v(p) = \sqrt{E}\begin{pmatrix}\sqrt{\mathrm{I}_2 - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}\eta \\ -\sqrt{\mathrm{I}_2 + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}\eta\end{pmatrix}$$

Consider chirality eigenstates $\xi_\pm : \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\xi_\pm = \pm\xi_\pm$ and $\eta_\pm : \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\eta_\pm = \mp\eta_\pm$, so that:

$$u_+(p) = \sqrt{2E}\begin{pmatrix}0\\ \xi_+\end{pmatrix} \quad u_-(p) = \sqrt{2E}\begin{pmatrix}\xi_-\\ 0\end{pmatrix} \quad v_+(p) = \sqrt{2E}\begin{pmatrix}\eta_+\\ 0\end{pmatrix} \quad v_+(p) = \sqrt{2E}\begin{pmatrix}0\\ -\eta_-\end{pmatrix}$$

As $\gamma^5 u_\pm = \pm u$ and $\gamma^5 v_\pm = \mp v_\pm$, these are clearly chirality eigenstates[4], hence they are helicity eigenstates too:

$$\hat{h} u_\pm = \pm\frac{1}{2} u_\pm \qquad \hat{h} v_\pm = \mp\frac{1}{2} v_\pm \tag{5.30}$$

Given the structure of the interaction vertex $e\bar{\Psi}\gamma^\mu\Psi$, the f and $\bar{f}$ which interact at a vertex have the same chirality:

$$\bar{\Psi}\gamma^\mu\Psi' = \begin{pmatrix}\psi_{\mathrm{L}}^\dagger & \psi_{\mathrm{R}}^\dagger\end{pmatrix}\begin{bmatrix}0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0\end{bmatrix}\begin{bmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{bmatrix}\begin{pmatrix}\psi'_{\mathrm{L}} \\ \psi'_{\mathrm{R}}\end{pmatrix} = \begin{pmatrix}\psi_{\mathrm{L}}^\dagger\bar{\sigma}^\mu\psi'_{\mathrm{L}} \\ \psi_{\mathrm{R}}^\dagger\sigma^\mu\psi'_{\mathrm{R}}\end{pmatrix}$$

which vanishes if f and $\bar{f}$ have opposite chirality. This means that f and $\bar{f}$ have opposite helicities, as per Eq. 5.30, i.e. parallel spins. In the case of $e^+e^- \to \mu^+\mu^-$ scattering, in the center-of-mass frame the e^+e^- pair must then annihilate in a photon with spin projection ± 1 along the $\hat{\mathbf{e}}_z$ axis, while the $\mu^+\mu^-$ pair must be created by a photon with spin projection ± 1 along the $\hat{\mathbf{k}} = (\sin\theta, 0, \cos\theta)$ axis ($\hat{\mathbf{e}}_y$ chosen such that $\hat{\mathbf{e}}_y \cdot \mathbf{k} = 0$). Recalling Eq. 2.60, there are two possible choices for the polarization vector of each photon:

$$\epsilon^\mu_{\text{in}} = \frac{1}{\sqrt{2}}(0, 1, \pm\mathrm{i}, 0) \qquad \epsilon^\mu_{\text{out}} = \frac{1}{\sqrt{2}}(0, \cos\theta, \pm\mathrm{i}, -\sin\theta)$$

where $\pm$ denotes a right/left-handed photon (i.e. $h = \pm 1$). The matrix element is proportional to the overlap between these polarization vectors, which is:

$$\epsilon^\mu_{\text{in}}\epsilon_{\text{out}\,\mu} = \frac{1}{2}\left(\pm_{\text{in}} \pm_{\text{out}} 1 - \cos\theta\right)$$

The amplitude is then proportional to the sum of the four possible square moduli:

$$|\mathcal{M}|^2 \propto \sum \left|\epsilon^\mu_{\text{in}}\epsilon_{\text{out}\,\mu}\right|^2 \propto (1+\cos\theta)^2 + (1-\cos\theta)^2 = \left(1+\cos^2\theta\right) \tag{5.31}$$

Therefore, the angular dependence of the cross-section too is fixed a priori by symmetry considerations.

[4] Recall Lemma 2.2.3: as $\{\gamma^5, \gamma^\mu\} = 0$, chirality anti-commutes with charge conjugation, therefore antifermions have flipped chirality.

Part III

Gauge Theories

Chapter 6

Non-Abelian Gauge Theories

In §2.3 it was shown how, from the gauge invariance Eq. 2.53 of Electrodynamics, it was necessary to introduce a covariant derivative Eq. 2.67, ultimately leading to the QED Lagrangian. The same process can be generalized to arbitrary groups of gauge transformations (**gauge groups**, for short), thus defining a field (gauge) theory starting from its symmetry properties.

§6.1 Yang-Mills Lagrangian

Consider $n \in \mathbb{N}$ fermionic fields $\{\psi_k(x)\}_{k=1,\dots,n}$ and an n-spinor $\Psi(x)$ defined as:

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} \tag{6.1}$$

As a gauge group, consider a d-dimensional Lie group G: WLOG, be G a simply-connected Lie group, so that each element can be expressed via the exponential map, and be it compact too, so that its representations are unitary. Then, consider $\{T^a\}_{a=1,\dots,d}$ an n-dimensional representation of the associated Lie algebra $\mathfrak{g}$, so that the action of G on Ψ can be expressed as:

$$\Psi(x) \mapsto V(x)\Psi(x) \qquad\qquad V(x) = \exp\left[\mathrm{i}\theta_a(x)T^a\right] \equiv e^{\mathrm{i}\alpha(x)} \tag{6.2}$$

where the Lie parameters $\{\theta_a(x)\}_{a=1,\dots,d} \subset \mathcal{C}^\infty(\mathbb{R}^{1,3})$ define a local gauge transformation. The aim is to define a Lagrangian which is invariant under this transformation, i.e. the Lagrangian of a (local) gauge theory.
Simple terms invariant under global phase rotations, like the fermion mass term $m\bar{\Psi}\Psi$, are of course invariant under Eq. 6.2 too, but derivatives need a careful treatment: indeed, the limit-definition of a derivative involves fields at different spacetime points, which have different transformations according to Eq. 6.2. In order to define a derivative of Ψ, it is necessary to introduce a factor to subtract values of $\Psi(x)$ in a meaningful way, so consider $\mathrm{U}(y,x) \in \mathrm{U}(n)$: $\mathrm{U}(x,x) = 1$ and which transforms under the action of G as:

$$\mathrm{U}(y,x) \mapsto e^{\mathrm{i}\alpha(y)}\mathrm{U}(y,x)e^{-\mathrm{i}\alpha(x)} \tag{6.3}$$

In general, $\mathrm{U}(y,x) \mapsto V(y)\mathrm{U}(y,x)V^\dagger(x)$. It is clear that $\mathrm{U}(y,x)\Psi(x)$ and $\Psi(y)$ have the same transformation law, so they can be meaningfully subtracted.

Definition 6.1.1 (Covariant derivative)

Given $n^\mu \in \mathbb{R}^{1,3}$, the **covariant derivative** of a fermionic field $\Psi(x)$ along n^μ is defined as:

$$n^\mu \mathcal{D}_\mu \Psi(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\Psi(x+\varepsilon n) - \mathrm{U}(x+\varepsilon n, x)\Psi(x)\right] \tag{6.4}$$

where $\mathrm{U}(y,x)$ is defined in Eq. 6.3.

To make this definition explicit, it is necessary to get an expression of $\mathrm{U}(y,x)$ for points at infinitesimal distance. Given the unitarity of $\mathrm{U}(y,x)$, it can be expressed through the generators $\{T^a\}_{a=1,\dots,d}$ as:

$$\mathrm{U}(x+\varepsilon n, x) = \mathrm{I}_n + \mathrm{i} g \varepsilon n^\mu \mathcal{A}^a_\mu(x) T_a + o(\varepsilon^2) \tag{6.5}$$

where $g \in \mathbb{R}$ is a constant. The new vector field $\mathcal{A}^a_\mu(x)$ (actually, d different vector fields) is a **connection**, and it allows to express the covariant derivative as (directly from Eq. 6.4):

$$\mathcal{D}_\mu = \partial_\mu - \mathrm{i} g \mathcal{A}^a_\mu T_a \tag{6.6}$$

Proposition 6.1.1

The covariant derivative $\mathcal{D}_\mu \Psi$ transforms as Ψ.

Proof. From Eq. 6.3-6.5 (recalling that ∂_μ is anti-Hermitian):

$$\begin{aligned}
\mathrm{I}_n + \mathrm{i} g \varepsilon n^\mu \mathcal{A}^a_\mu T_a &\mapsto V(x+\varepsilon n)\left(\mathrm{I}_n + \mathrm{i} g \varepsilon n^\mu \mathcal{A}^a_\mu T_a\right) V^\dagger(x) \\
&= \left[(1+\varepsilon n^\mu \partial_\mu) V(x)\right] V^\dagger(x) + V(x)\left(\mathrm{i} g \varepsilon n^\mu \mathcal{A}^a_\mu T_a\right) V^\dagger(x) + o(\varepsilon^2) \\
&= \mathrm{I}_n - \varepsilon n^\mu V(x) \partial_\mu V^\dagger(x) + V(x)\left(\mathrm{i} g \varepsilon n^\mu \mathcal{A}^a_\mu T_a\right) V^\dagger(x) + o(\varepsilon^2)
\end{aligned}$$

Hence, the connection transforms as:

$$\mathcal{A}^a_\mu(x) T_a \mapsto V(x)\left[\mathcal{A}^a_\mu(x) T_a + \frac{\mathrm{i}}{g}\partial_\mu\right] V^\dagger(x)$$

The derivative $\partial_\mu V^\dagger(x)$ is non-trivial to compute, as G is in general non-Abelian, hence the exponent does not necessarily commute with its derivative. At $o(\theta)$:

$$\begin{aligned}
\mathcal{A}^a_\mu(x) T_a \mapsto &\left(\mathrm{I}_n + \mathrm{i}\theta^b(x) T_b + o(\theta^2)\right)\left[\mathcal{A}^a_\mu(x) T_a + \frac{\mathrm{i}}{g}\partial_\mu\right]\left(\mathrm{I}_n - \mathrm{i}\theta^c(x) T_c + o(\alpha^2)\right) \\
&= \left(\mathrm{I}_n + \mathrm{i}\theta^b(x) T_b + o(\theta^2)\right)\left[\mathcal{A}^a_\mu(x) T_a - \mathrm{i}\mathcal{A}^a_\mu(x)\theta^c(x) T_a T_c + \frac{1}{g}\partial_\mu \theta^c(x) T_c + o(\theta^2)\right] \\
&= \mathcal{A}^a_\mu(x) T_a - \mathrm{i}\mathcal{A}^a_\mu(x)\theta^c(x) T_a T_c + \mathrm{i}\theta^b(x)\mathcal{A}^a_\mu(x) T_b T_a + \frac{1}{g}\partial_\mu \theta^c(x) T_c + o(\theta^2) \\
&= \mathcal{A}^a_\mu(x) T_a + f^{abc}\mathcal{A}^a_\mu(x)\theta^b(x) T_c + \frac{1}{g}\partial_\mu \theta^a(x) T_a + o(\theta^2)
\end{aligned}$$

Then:

$$\begin{aligned}
\mathcal{D}_\mu \Psi(x) \mapsto &\left[\partial_\mu - \mathrm{i} g \mathcal{A}^a_\mu(x) T_a - \mathrm{i} g f^{abc}\mathcal{A}^a_\mu(x)\theta^b(x) T_c - \mathrm{i}\partial_\mu \theta^a(x) T_a\right]\left(\mathrm{I}_n + \mathrm{i}\theta^a(x) T_a\right)\Psi(x) + o(\theta^2) \\
&= \left[\partial_\mu + \mathrm{i}\theta^a T_a \partial_\mu + \mathrm{i}\partial_\mu \theta^a T_a - \mathrm{i} g \mathcal{A}^a_\mu T_a + g\mathcal{A}^a_\mu \theta^b T_a T_b - \mathrm{i} g f^{abc}\mathcal{A}^a_\mu \theta^b T_c - \mathrm{i}\partial_\mu \theta^a T_a + o(\theta^2)\right]\Psi \\
&= \left[\partial_\mu + \mathrm{i}\theta^a T_a \partial_\mu - \mathrm{i} g \mathcal{A}^a_\mu T_a + g\mathcal{A}^a_\mu \theta^b T_a T_b - \mathrm{i} g f^{abc}\mathcal{A}^a_\mu \theta^b T_c + o(\theta^2)\right]\Psi
\end{aligned}$$

Recognizing $T_a T_b - \mathrm{i} f^{abc} T_c = T_b T_a$ allows to write:

$$\begin{aligned}\mathcal{D}_\mu \Psi(x) \mapsto & \left[\partial_\mu + \mathrm{i}\theta^a(x) T_a \partial_\mu - \mathrm{i}g\mathcal{A}^a_\mu(x) T_a + g\theta^b(x) T_b \mathcal{A}^a_\mu(x) T_a + o(\theta^2)\right] \Psi(x) \\ = & \left[\mathrm{I}_n + \mathrm{i}\theta^a(x) T_a + o(\theta^2)\right] \left(\partial_\mu - \mathrm{i}g\mathcal{A}^a_\mu(x) T_a\right) \Psi(x) = V(x)\mathcal{D}_\mu \Psi(x)\end{aligned}$$

which is the thesis. □

The gauge-invariant Lagrangian can thus be built using covariant derivatives (minimal coupling prescription), but there needs to be included a kinetic term for the connection, i.e. a gauge-invariant term depending on $\mathcal{A}^a_\mu(x)$ only.

Lemma 6.1.1 (Field-strength tensor)
The commutator of covariant derivatives reads:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathrm{i}g\mathcal{F}^a_{\mu\nu} T_a \tag{6.7}$$

with the **field-strength tensor** defined as:

$$\mathcal{F}^a_{\mu\nu} := \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu \tag{6.8}$$

Proof. By direct computation:

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu] &= [\partial_\mu, \partial_\nu] - \mathrm{i}g[\mathcal{A}^a_\mu, \partial_\nu] T_a - \mathrm{i}g[\partial_\mu, \mathcal{A}^a_\nu] T_a - g^2 \mathcal{A}^b_\mu \mathcal{A}^c_\nu [T_b, T_c] \\ &= -\mathrm{i}g\left(\mathcal{A}^a_\mu \partial_\nu - \partial_\nu \mathcal{A}^a_\mu - \mathcal{A}^a_\mu \partial_\nu\right) T_a - \mathrm{i}g\left(\mathcal{A}^a_\nu \partial_\mu - \partial_\mu \mathcal{A}^a_\nu - \mathcal{A}^a_\nu \partial_\mu\right) T_a - \mathrm{i}g^2 f^{bca} \mathcal{A}^b_\mu \mathcal{A}^b_\nu T_c \\ &= -\mathrm{i}g\left(\partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{bca} \mathcal{A}^b_\mu \mathcal{A}^c_\mu\right) T_a\end{aligned}$$

Using $f^{bca} = f^{abc}$, as it is always possible to choose generators such that the structure constants are completely antisymmetric[1], yields the thesis. □

Note that the field-strength tensor is not itself a gauge-invariant quantity, as really there are d different field-strength tensors. However, it is straightforward to construct gauge-invariant combinations of $\mathcal{F}^a_{\mu\nu}$.

Theorem 6.1.1 (Gauge invariance)
Any globally-symmetric function of Ψ, $\mathcal{F}^a_{\mu\nu}$ and their covariant derivatives is also locally-symmetric, i.e. gauge invariant.

Lemma 6.1.2 (Kinetic term)
The following term is gauge-invariant:

$$\mathrm{tr}\{(\mathcal{F}^a_{\mu\nu} T_a)^2\} = 2\mathcal{F}^a_{\mu\nu} \mathcal{F}^{\mu\nu}_a \tag{6.9}$$

This allows defining the simplest non-Abelian gauge theory, **Yang-Mills theory**:

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} \mathcal{F}^a_{\mu\nu} \mathcal{F}^{\mu\nu}_a \tag{6.10}$$

[1] Requires proof.

Appendices

Appendix A

Mathematical Reference

§A.1 Multilinear algebra

Definition A.1.1 (Tensor product of vectors)

Given two $\mathbb{K}$-vector spaces V, W with bases $\{\mathbf{e}_i\}_{i=1,\dots,n}$, $\{\mathbf{f}_j\}_{i=1,\dots,m}$, the **tensor product** $V \otimes W$ is defined as:

$$\mathbf{v} \otimes \mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{m} v^i w^j \mathbf{e}_i \otimes \mathbf{f}_j \quad \Longleftrightarrow \quad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ \vdots \\ v_1 w_m \\ \vdots \\ v_n w_1 \\ \vdots \\ v_n w_m \end{pmatrix} \tag{A.1}$$

Definition A.1.2 (Tensor product of applications)

Given $\mathbb{K}$-linear maps $f : V \to V'$ and $g : W \to W'$ represented by $F \equiv [f_{i,j}] \in \mathbb{K}^{n' \times n}$ and $G \equiv [g_{i,j}] \in \mathbb{K}^{m' \times m}$, the **tensor product** $f \otimes g : V \otimes W \to V' \otimes W'$ is defined as $F \otimes G \in \mathbb{K}^{(n'm') \times (nm)}$:

$$[F \otimes G] = \begin{bmatrix} f_{1,1}g_{1,1} & \cdots & f_{1,1}g_{1,m} & & f_{1,n}g_{1,1} & \cdots & f_{1,n}g_{1,m} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ f_{1,1}g_{m',1} & \cdots & f_{1,1}g_{m',m} & & f_{1,n}g_{m',1} & \cdots & f_{1,n}g_{m',m} \\ & \vdots & & \ddots & & \vdots & \\ f_{n',1}g_{1,1} & \cdots & f_{n',1}g_{1,m} & & f_{n',n}g_{1,1} & \cdots & f_{n',n}g_{1,m} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ f_{n',1}g_{m',1} & \cdots & f_{n',1}g_{m',m} & & f_{n',n}g_{m',1} & \cdots & f_{n',n}g_{m',m} \end{bmatrix}$$

It is clear that $(F \otimes G)(\mathbf{v} \otimes \mathbf{w}) = (F\mathbf{v}) \otimes (G\mathbf{w})$.

§A.2 Lie groups

Definition A.2.1 (Lie groups)

A **Lie group** is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d-dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0,\dots,0) = e$.

Definition A.2.2 (Representations)

Given a group G and a vector space $V(\mathbb{K})$, a **representation** of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

A representation ρ which is a isomorphism is called **faithful**. As $\mathrm{GL}(V) \cong \mathbb{K}^{n\times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to represent G as matrices acting on elements of V, i.e. $\rho : G \to \mathbb{K}^{n\times n}$.

Theorem A.2.1

Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ is:

$$\rho(g(\theta)) = e^{\mathrm{i}\theta_a T^a} \tag{A.2}$$

where $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ are the **generators** of G on V.

Definition A.2.3 (Lie algebras)

Given a Lie group G with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, its **Lie algebra** is:

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_c T^c \tag{A.3}$$

where $f^{ab}{}_c$ are called the **structure constants**.

The sum over repeated indices is understood.

Proposition A.2.1

The Lie algebra of a Lie group is independent of the representation.

Proposition A.2.2

Any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

Definition A.2.4 (Casimir operators)

Given a Lie group with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, a **Casimir operator** is an operator which commutes with each generator.

Given an irreducible representation on V, Casimir operators are operators proportional to $\mathbb{1}_V$, and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

Proposition A.2.3

A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

§A.2.1 Adjoint representation

To define the adjoint representation of a Lie group, it is necessary to give more formal definitions.

Definition A.2.5 (Lie algrebas)

An n-dimensional **$\mathbb{K}$-Lie algebra** $\mathfrak{g}$ is an n-dimensional $\mathbb{K}$-vector space equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

1. $[X, Y] = -[Y, X]\ \forall X, Y \in \mathfrak{g}$;
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ \forall X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

A Lie algebra $\mathfrak{g}$ is **commutative** if $[X, Y] = 0\ \forall X, Y \in \mathfrak{g}$. Note the link with Lie groups (to be formalized later): given a Lie group G , denoting as $\mathcal{G}$ the associated differentiable manifold, then the Lie algebra $\mathfrak{g}$ associated to G is $\mathfrak{g} := T_e\mathcal{G}$ (tangent space at the identity).

Example A.2.1 ($\mathbb{R}^3$ as a Lie algebra)

Let $\mathfrak{g} = \mathbb{R}^3$ and $[\cdot,\cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}, \mathbf{y}] \equiv \mathbf{x} \times \mathbf{y} \in \mathbb{R}^3$. Then $\mathfrak{g}$ is a Lie algebra.

Definition A.2.6 (Lie algebra morphisms)

Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a **Lie algebra homomorphism** if:

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]\ \forall X, Y \in \mathfrak{g}$$

If φ is bijective, then it is a **Lie algebra isomorphism**.

A Lie algebra isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a **Lie algebra automorphism** $\mathfrak{g} \in \operatorname{Aut}\mathfrak{g}$, where $\operatorname{Aut}\mathfrak{g}$ is the **automorphism group** of $\mathfrak{g}$ (a group under the composition of morphisms).

Definition A.2.7 (Adjoint map (pt. 1))

Let $\mathfrak{g}$ be a Lie group and, given $X \in \mathfrak{g}$, define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} : Y \mapsto \mathrm{ad}_X(Y) := [X, Y]$. Then the **adjoint map** on $\mathfrak{g}$ is the map $\mathrm{ad} : \mathfrak{g} \to \operatorname{End}\mathfrak{g} : X \mapsto \mathrm{ad}_X$.

By Jacobi identity, the adjoint map is a **derivation** of the Lie bracket, as:

$$\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$$

Proposition A.2.4

Given a Lie algebra $\mathfrak{g}$, the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ is a Lie algebra homomorphism.

Proof. Note that, by Jacobi identity:

$$\mathrm{ad}_{[X,Y]}(Z) = [[X,Y],Z] = [X,[Y,Z]] + [Y,[Z,X]]$$

Moreover:

$$[\mathrm{ad}_X, \mathrm{ad}_Y](Z) = [X,[Y,Z]] - [Y,[X,Z] = [X,[Y,Z]] + [Y,[Z,X]]$$

Therefore, $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$, which is the thesis. □

Let $\mathfrak{g}$ be an n-dimensional $\mathbb{K}$-Lie algebra and $\{X_i\}_{i=1,\dots,n} \subset \mathfrak{g}$ a basis of $\mathfrak{g}$. Then there are unique constants $c_{ijk} \in \mathbb{K}$:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k$$

known as **structure constants**.

Theorem A.2.2 (Lie algebras from Lie groups)

Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a Lie group. Then $\mathfrak{g} = \{X \in \mathbb{C}^{n\times n} : e^{tX} \in G \ \forall t \in \mathbb{R}\}$ is a Lie algebra.

Even if G is a complex Lie group, its Lie algebra can still be real.

Theorem A.2.3 (Induced Lie algebra homomorphism)

Let G, H be Lie groups, with Lie algebra $\mathfrak{g}, \mathfrak{h}$, and let $\varphi : G \to H$ be a Lie group homomorphism. Then there exists a unique $\mathbb{R}$-linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ such that:

$$\varphi(e^X) = e^{\Phi(X)} \ \forall X \in \mathfrak{g}$$

The map Φ as additional properties:

1. $\Phi(gXg^{-1}) = \varphi(g)\Phi(X)\varphi(g)^{-1} \ \forall X \in \mathfrak{g}, \forall g \in G$;
2. $\Phi([X,Y]) = [\Phi(X), \Phi(Y)] \ \forall X, Y \in \mathfrak{g}$ (Lie algebra homomorphism);
3. $\Phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi(e^{tX}) \ \forall X \in \mathfrak{g}$.

To phrase Th. A.2.3 in the language of manifolds, Φ is the **derivative** of φ at the identity: $\Phi = d\varphi_e$.

Definition A.2.8 (Adjoint map (pt. 2))

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. The **adjoint map** of $g \in G$ is the linear map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} : \mathrm{Ad}_g(X) := gXg^{-1}$.

As Ad_g is clearly invertible, with $\mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$, then $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g}) \ \forall g \in G$. Furthermore, it is clear that $\mathrm{Ad}_g([X,Y]) = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)] \ \forall X, Y \in \mathfrak{g}, \forall g \in G$, therefore each adjoint map is a Lie algebra homomorphism.

Proposition A.2.5 (Adjoint representation)

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{Ad}_g$ is a homomorphism.

Recalling Def. A.2.2, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$, called the **adjoint representation**.
As $\mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}(n, \mathbb{K})$ (with $n \equiv \dim_{\mathbb{K}} \mathfrak{g}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), it can be viewed as a Lie group itself, and its Lie algebra is $\mathfrak{gl}(\mathfrak{g})$. Thus, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a Lie group homomorphism (as it can be shown to be continuous).

Proposition A.2.6 (Adjoint maps)

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then, given the Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, the induced Lie algebra homomorphism is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $\mathrm{ad}_X(Y) = [X, Y]$.

Proof. By Th. A.2.3, the Lie algebra homomorphism induced by $\varphi \equiv \mathrm{Ad}$ is:

$$\Phi(X) \equiv \mathrm{ad}_X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{e^{tX}}$$

Hence:

$$\mathrm{ad}_X(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tX} Y e^{-tX} = [X, Y]$$

□

This result links the two adjoint maps in Def. A.2.7-A.2.8.

§A.2.2 $\mathrm{SU}(n)$ Lie group

The $\mathrm{SU}(n)$ group is the group of unitary transformations of n-dimensional complex vectors. Its (faithful) fundamental representation thus is:

$$\mathrm{SU}(n) = \{\mathrm{U} \in \mathbb{C}^{n\times n} : \mathrm{U}\mathrm{U}^\dagger = \mathrm{U}^\dagger\mathrm{U} = \mathrm{I}_n \wedge \det \mathrm{U} = +1\}$$

The generators of $\mathrm{SU}(n)$ can be found setting $\mathrm{U} = \exp(\mathrm{i}\theta_a T^a) = \mathrm{I}_n + \mathrm{i}\theta_a T^a$ and using $\mathrm{U}^\dagger\mathrm{U} = \mathrm{I}_n$:

$$T^a = T^{a\dagger} \tag{A.4}$$

Moreover, by the Jacobi formula $(\det A(t))\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \mathrm{tr}(A(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A(t))$ evaluated at $t = 0$:

$$\mathrm{tr}\, T^a = 0 \tag{A.5}$$

The traceless condition can be generalized to all semi-simple Lie algebras. Therefore, the generators of $\mathrm{SU}(n)$ are $\mathbb{C}^{n\times n}$ hermitian traceless matrices: the dimension of $\mathfrak{su}(n)$ then is $n^2 - 1$.
The adjoint representation can be is given by representing the generators of the Lie group (i.e. the basis of the Lie algebra) with the structure constants of the Lie algebra:

$$(T^b_{\mathrm{ad}})_{ac} = \mathrm{i} f^{ab}{}_c \tag{A.6}$$

Proposition A.2.7 (Structure constants)

The structure constants of a Lie algebra satisfy the Lie algebra.

Proof. As $[T^a, T^b] = if^{ab}{}_c T^c$, the Jacoby identity becomes (recalling that $f^{ab}{}_c$ is totally antisymmetric):

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$$
$$\iff f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0$$

The condition $([T^a_{\text{ad}}, T^c_{\text{ad}}])_{be} = if^{ac}{}_d (T^d_{\text{ad}})_{be}$ then gives:

$$f^{ba}{}_d f^{dc}{}_e - f^{bc}{}_d f^{da}{}_e = f^{ac}{}_d f^{bd}{}_e \qquad \iff \qquad f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0$$

These two expressions are equal, hence the thesis. □

Moreover, since the structure constant are real, the adjoint representation is always a real representation: the adjoint representation of $\mathrm{SU}(n)$ has degree $n^2 - 1$.
Representation are labelled by their Casimir operators. For any simple Lie algebra, given a representation $\mathtt{r}$, a Casimir operator is defined as:

$$T^a_{\mathtt{r}} T^a_{\mathtt{r}} = C_2(\mathtt{r}) \mathrm{I}_{n_{\mathtt{r}}} \tag{A.7}$$

This is called the **quadratic Casimir operator**, as it is associated to $T^2 \equiv T^a T^a$ (a Casimir operator since $[T^b, T^2] = \mathrm{i} f^{ba}{}_c \{T^c, T^a\} = 0$ by antisymmetry).

Proposition A.2.8 (Quadratic Casimir operator)

For the fundamental and the adjoint representations $\mathtt{n}$ and $\mathtt{g}$ of $\mathrm{SU}(n)$, the quadratic Casimir operator is:

$$C_2(\mathtt{n}) = T_{\mathrm{R}} \frac{n^2 - 1}{n} \qquad\qquad C_2(\mathtt{g}) = 2T_{\mathrm{R}} n \tag{A.8}$$

where T_{R} is the normalization factor defined as:

$$\mathrm{tr}(T^a_{\mathtt{n}} T^b_{\mathtt{n}}) = T_{\mathrm{R}} \delta^{ab} \tag{A.9}$$

Proof. It is always possible to chose generators such that Eq. A.9 holds[a], hence, contracting it with δ^{ab} (with $a, b = 1, \ldots, n^2 - 1$, as they label the basis of $\mathfrak{su}(n)$) and recalling Eq. A.7:

$$C_2(\mathtt{n}) n = T_{\mathrm{R}}(n^2 - 1)$$

To compute the Casimir operator for the adjoint representation, first consider the decomposition of the direct product of two representations:

$$\mathtt{r}_1 \otimes \mathtt{r}_2 = \bigoplus_i \mathtt{r}_i$$

In this representation $T^a_{\mathtt{r}_1 \otimes \mathtt{r}_2} = T^a_{\mathtt{r}_1} \otimes \mathbb{1}_{\mathtt{r}_2} + \mathbb{1}_{\mathtt{r}_1} \otimes T^a_{\mathtt{r}_2}$, and it acts on tensor objects Ξ_{pq} whose first index transforms according to $\mathtt{r}_1$ and the second index according to $\mathtt{r}_2$. Recalling that $\mathrm{tr}\, T^a = 0$:

$$\begin{aligned}\mathrm{tr}(T^a_{\mathtt{r}_1 \otimes \mathtt{r}_2})^2 &= \mathrm{tr}((T^a_{\mathtt{r}_1})^2 \otimes \mathbb{1}_{\mathtt{r}_2} + 2T^a_{\mathtt{r}_1} \otimes T^a_{\mathtt{r}_2} + \mathbb{1}_{\mathtt{r}_1} \otimes (T^a_{\mathtt{r}_2})^2) \\ &= \mathrm{tr}(C_2(\mathtt{r}_1)\mathbb{1}_{\mathtt{r}_1} \otimes \mathbb{1}_{\mathtt{r}_2}) + \mathrm{tr}(C_2(\mathtt{r}_2)\mathbb{1}_{\mathtt{r}_1} \otimes \mathbb{1}_{\mathtt{r}_2}) = (C_2(\mathtt{r}_1) + C_2(\mathtt{r}_2)) n_{\mathtt{r}_1} n_{\mathtt{r}_2}\end{aligned}$$

However, by the decomposition above:

$$\mathrm{tr}(T^a_{\mathtt{r}_1\otimes\mathtt{r}_2})^2 = \sum_i C_2(\mathtt{r}_i) n_{\mathtt{r}_i}$$

Consider $\mathtt{n}\otimes\mathtt{n}^*$, where $\mathtt{n}^*$ is the complex conjugate of the fundamental representation (for complex representations, $\mathtt{r}$ and $\mathtt{r}^*$ are generally inequivalent representations): then Ξ_{pq} contains a term proportional to the invariant δ_{pq}, while the other n^2-1 independent components transform as a general $n\times n$ traceless tensor, i.e. under the adjoint representation of $\mathrm{SU}(n)$ (as of Eq. A.4-A.5), thus $\mathtt{n}\otimes\mathtt{n}^* = \mathbf{1}\oplus\mathtt{g}$ and the above identity becomes:

$$(C_2(\mathbf{1}) + C_2(\mathtt{g}))(n^2-1) = (C_2(\mathtt{n}) + C_2(\mathtt{n}^*))n^2$$

Using $C_2(\mathbf{1}) = 0$ (as all generators are trivially zero) and $C_2(\mathtt{n}^*) = C_2(\mathtt{n})$:

$$C_2(\mathtt{g})(n^2-1) = 2T_{\mathrm{R}}\frac{n^2-1}{n}n^2$$

which completes the proof. □

[a]Requires proof.

§A.2.2.1 $\mathrm{SU}(2)$ Lie group

The fundamental representation of $\mathrm{SU}(2)$ is $T^a_{\mathbf{2}} = \frac{\sigma^a}{2}$, while $\mathfrak{su}(2)$ is defined by commutators $[T^a_{\mathbf{2}}, T^b_{\mathbf{2}}] = \mathrm{i}\epsilon^{abc}T^c_{\mathbf{2}}$ (as $\sigma^a\sigma^b = \delta^{ab}\mathrm{I}_2 + \mathrm{i}\epsilon^{abc}\sigma^c$). The adjoint representation then is:

$$(T^a_{\mathtt{g}})_{ij} = \mathrm{i}\epsilon^{iaj} \tag{A.10}$$

Explicitly:

$$T^1_{\mathtt{g}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix} \qquad T^2_{\mathtt{g}} = \begin{bmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{bmatrix} \qquad T^3_{\mathtt{g}} = \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As an aside, these are exactly the generators of the fundamental representation of $\mathrm{SO}(3)$: this is due to the adjoint map of $\mathrm{SU}(2)$ being also the double-covering map on $\mathrm{SO}(3)\cong\mathrm{SU}(2)/\mathbb{Z}_2$.

§A.3 Algebras

Definition A.3.1 (Associative algebras)

An n-dimensional **associative $\mathbb{K}$-algebra** $\mathcal{A}$ is an n-dimensional vector space $V(\mathbb{K})$ equipped with a bilinear map $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto ab \in \mathcal{A}$ such that:

1. $(ab)c = a(bc)\ \forall a, b, c \in \mathcal{A}$ (associativity);
2. $a(b + c) = ab + ac \wedge (a + b)c = ac + bc\ \forall a, b, c \in \mathcal{A}$;
3. $\lambda(ab) = (\lambda a)b = a(\lambda b)\ \forall \lambda \in \mathbb{K}, \forall a, b \in \mathcal{A}$.

An algebra is said to be **unital** if $\exists \mathit{1} \in \mathcal{A} : \mathit{1}a = a\mathit{1} = a\ \forall a \in \mathcal{A}$, called **identity element**.

Definition A.3.2 (Algebra morphisms)

Given two associative $\mathbb{K}$-algebras $\mathcal{A}, \mathcal{B}$, a $\mathbb{K}$-linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is an **algebra morphism** if:

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in \mathcal{A}$$

If $\varphi(\mathit{1}_{\mathcal{A}}) = \mathit{1}_{\mathcal{B}}$, then φ is a **unital morphism**. An algebra morphism $\varphi : \mathcal{A} \to \mathcal{A}$ is an **endomorphism**, and if $\varphi^2 = \mathbb{1}_{\mathcal{A}}$ it is an **involution**.

Proposition A.3.1 (Even subalgebras)

Given a unital associative $\mathbb{K}$-algebra $\mathcal{A}$ and an involution $\varphi \in \operatorname{End} \mathcal{A}$, then:

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$$

where, defining $\pi := \frac{1}{2}(\mathbb{1}_{\mathcal{A}} + \varphi)$:

$$\mathcal{A}^+ := \pi(\mathcal{A}) = \{a \in \mathcal{A} : \varphi(a) = a\}$$

$$\mathcal{A}^- := (\mathbb{1}_{\mathcal{A}} - \pi)(\mathcal{A}) = \{a \in \mathcal{A} : \varphi(a) = -a\}$$

As $\mathcal{A}^+\mathcal{A}^+, \mathcal{A}^-\mathcal{A}^- \subset \mathcal{A}^+$ and $\mathcal{A}^+\mathcal{A}^-, \mathcal{A}^-\mathcal{A}^+ \subset \mathcal{A}^-$, then $\mathcal{A}^+$ is a subalgebra of $\mathcal{A}$, the **even subalgebra**.

Definition A.3.3 (Modules)

Given a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a **right-module** M is a $\mathbb{K}$-vector space equipped with a $\mathbb{K}$-bilinear multiplication map $\mathcal{A} \times M \ni (a, m) \mapsto am \in M$ such that:

1. $(ab)m = a(bm)\ \forall a, b \in \mathcal{A}, \forall m \in M$;
2. $\mathit{1}_{\mathcal{A}}m = m\ \forall m \in M$.

Definition A.3.4 (Ideals)

Given a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a **right-ideal** is a subalgebra $\mathcal{I} \subset \mathcal{A}$ such that:

1. $aj \in \mathcal{I}\ \forall a \in \mathcal{A}, \forall j \in \mathcal{I}$.

A right-ideal is said minimal if it is non-trivial and does not contain any non-trivial sub-right-ideal.

The analogous left- definitions are clear. A left-ideal of $\mathcal{A}$ can be viewed as a right-module on $\mathcal{A}$.

§A.3.1 Clifford algebras

Definition A.3.5 (Clifford algebras)
Given an n-dimensional vector space $V(\mathbb{K})$ with a quadratic form q, associated linear form[1] ω and orthogonal basis $\{e_i\}_{i=1,\dots,n}$, and a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a **Clifford mapping** is an injective $\mathbb{K}$-linear map $\rho : V \to \mathcal{A} : \mathit{1} \notin \rho(V) \wedge \rho(x)^2 = -q(x)\mathit{1}\ \forall x \in V$.
If $\rho(V)$ generates $\mathcal{A}$, then $(\mathcal{A}, \rho)$ is a **Clifford algebra** for (V, q), and is denoted by $\mathfrak{cl}(V)$.

Lemma A.3.1

$$\{\rho(x), \rho(y)\} = -2\omega(x, y)\mathit{1} \quad \forall x, y \in V$$

Proof. By direct computation:

$$\rho(x)\rho(y)+\rho(y)\rho(x) = \rho(x+y)^2 - \rho(x)^2 - \rho(y)^2 = -\left(q(x+y) - q(x) - q(y)\right)\mathit{1} = -2\omega(x,y)\mathit{1}$$

which is the thesis. □

Setting for ease of reading $\rho(x) \equiv x$, it is clear that $x \perp y \implies xy = -yx$.
More intuitively, the Clifford algebra $\mathfrak{cl}(V)$ can be seen as the associative algebra generated by V setting $xy = -\omega(x, y)\mathit{1}\ \forall x, y \in V$, so that:

$$\{x, y\} = 2\omega(x, y)\mathit{1} \quad \forall x, y \in V \tag{A.11}$$

In general, $\mathfrak{cl}_{m,n}(\mathbb{R})$ denotes the Clifford algebra associated to $\mathbb{R}^{m,n}$ with quadratic form:

$$q(x_1, \dots, x_m, y_1, \dots, y_n) = \sum_{i=1}^{m} x_i^2 - \sum_{j=1}^{n} y_j^2$$

Example A.3.1 (Complex numbers)
Via Clifford algebras, $\mathbb{C} \cong \mathfrak{cl}_{0,1}(\mathbb{R})$. In fact, $\mathbb{R}^{0,1}$ has orthonormal basis $\{e_1\}$ such that $q(e_1) = -1$, i.e. $e_1^2 = -\mathit{1}$, so elements of the Clifford algebra are generated by $\{\mathit{1}, e_1\}$: identifying $e_1 \equiv \mathrm{i}$ gives the desired isomorphism.

Example A.3.2 (Quaternions)
$\mathbb{H} \cong \mathfrak{cl}_{0,2}(\mathbb{R})$. Indeed, $\mathbb{R}^{0,2}$ has orthonormal basis $\{e_1, e_2\} : e_1^2 = e_2^2 = -\mathit{1}$; moreover, as $e_1 \perp e_2$, then $e_1e_2 = -e_2e_1$ by Eq. A.11, so elements of $\mathfrak{cl}_{0,2}(\mathbb{R})$ are generated by $\{\mathit{1}, e_1, e_2, e_1e_2\}$:

[1] Given a vector space $V(\mathbb{K})$, a quadratic form is a map $q : V \to \mathbb{K}$ such that $q(\lambda x) = \lambda^2 q(x)\ \forall \lambda \in \mathbb{K}, \forall x \in V$, and the associated bilinear form is a $\mathbb{K}$-bilinear map $\omega : V \times V \to \mathbb{K}$ such that $\omega(x, y) = \frac{1}{2}\left(q(x + y) - q(x) - q(y)\right)$, which is manifestly symmetric and $q(x) = \omega(x, x)$.

setting $e_1 \equiv \mathrm{i}$, $e_2 \equiv \mathrm{j}$ and $e_1 e_2 = \mathrm{k}$ yields the result.

§A.3.1.1 Spin groups

Given an n-dimensional $\mathbb{K}$-vector space, the Clifford algebra $\mathfrak{cl}(V)$ is finite dimensional and is naturaly $\mathbb{N}_0$-graded[2]as:

$$\mathfrak{cl}(V) = \bigoplus_{i=0}^{n} \mathfrak{cl}^{(i)}(V) \tag{A.12}$$

where $\mathfrak{cl}^{(0)}(V) = \mathbb{K}$, $\mathfrak{cl}^{(1)}(V) = V$ and $\mathfrak{cl}^{(2)}(V) \equiv \mathrm{Spin}(V)$ is the **spin group** of V. The spin group is a Lie group and, via its natural action on V, can be shown to be $\mathrm{Spin}(V) \cong \mathfrak{so}(V)$.

§A.3.2 Grassmann algebras

For the definition of the tensor product, recall Eq. A.1.

Definition A.3.6 (Tensor algebras)

Given a vector space $V(\mathbb{K})$, for all $k \in \mathbb{N}$ define the k^{th} **tensor power**:

$$T^k V := V^{\otimes k} \equiv \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$$

By convention $T^0 V \equiv \mathbb{K}$. The **tensor algebra** (or free algebra) of V is then defined as:

$$T(V) := \bigoplus_{k=0}^{\infty} T^k V \tag{A.13}$$

Given $k \in \mathbb{N}$, $T^k V$ is a $\mathbb{K}$-vector consisting of all tensors on V of order $(k, 0)$ (hence, $T^0 V \equiv \mathbb{K}$ must be seen as a 1D $\mathbb{K}$-vector space). Expanding the direct sum, the tensor algebra is:

$$T(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots \tag{A.14}$$

and it is a unital associative algebra with multiplication determined by the canonical isomorphism $T^k V \otimes T^\ell V \to T^{k+\ell} V$: in particular, $T(V)$ is an $\mathbb{N}_0$-graded associative algebra.

Definition A.3.7 (Grassmann algebras)

Given a vector space $V(\mathbb{K})$ and the two-sided ideal $\mathcal{I}$ generated by[a] $\{x^{\otimes k} : x \in V, k \in \mathbb{N}_{\geq 2}\}$, the **Grassman algebra** (or exterior algebra) of V is defined as the quotient algebra:

$$\bigwedge(V) := T(V)/\mathcal{I} \tag{A.16}$$

The **exterior product** of two elements $\alpha, \beta \in \bigwedge(V)$ is defined as:

$$\alpha \wedge \beta \equiv \alpha \otimes \beta \mod \mathcal{I} \tag{A.17}$$

[2]Given an index set $\mathcal{I}$ and a $\mathbb{K}$-vector space, the latter is *$\mathcal{I}$-graded* if there exists a family of subspaces $\{V_i\}_{i\in\mathcal{I}}$ of V such that:

$$V = \bigoplus_{i\in\mathcal{I}} V_i$$

[a]Given a unital associative algebra $\mathcal{A}$ and a subset $X \subset \mathcal{A}$, the two-sided ideal $\mathcal{I}$ of $\mathcal{A}$ generated by X is:

$$\mathcal{I} := \left\{ \sum_{i=1}^{k} a_i x_i b_i : a_i, b_i \in \mathcal{A}, x \in X \right\} \tag{A.15}$$

The quotient algebra is found analogously to the quotient group. Note that the exterior product is an **alternating product**, as by definition $\omega \wedge \omega = 0 \; \forall \omega \in \Lambda(V)$. In general, by definition $x_1 \wedge \cdots \wedge x_k = 0$ if $x_i = x_j \in V$ for some $i \neq j \in [1, k]$.

Proposition A.3.2 (Anticommutativity)
Given a vector space $V(\mathbb{K})$, a k-ple $\{x_i\}_{i=1,\dots,k}$, $k \in \mathbb{N}$, and $\sigma \in S^k$, then:

$$x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)} = \operatorname{sgn} \sigma \, x_1 \wedge \cdots \wedge x_k \tag{A.18}$$

Proof. For $k = 2$:

$$0 = (x + y) \wedge (x + y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$$

from which $x \wedge y = -y \wedge x$. □

The Grassmann algebra is $\mathbb{N}_0$-graded too. Defining the k^{th} **exterior power** $\bigwedge^k(V)$ as the $\mathbb{K}$-vector subspace of $\bigwedge(V)$ spanned by elements of the form $x_1 \wedge \cdots \wedge x_k$, with $x_i \in V$, then:

$$\bigwedge(V) = \bigoplus_{k=0}^{\infty} \bigwedge^k(V) \tag{A.19}$$

If $\omega \in \bigwedge^k(V)$ can be expressed as the exterior product of k elements of V, then ω is said to be **decomposable**.

Theorem A.3.1 (Dimension of the Grassmann algebra)
Given an n-dimensional vector space $V(\mathbb{K})$, then $\bigwedge^k(V) = \{0\} \; \forall k > \dim_{\mathbb{K}} V$, i.e.:

$$\bigwedge(V) = \bigoplus_{k=0}^{n} \bigwedge^k(V) \tag{A.20}$$

Proof. Given a basis $\{e_i\}_{i=1,\dots,n}$ of V, WTS $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq n}$ is a basis of $\bigwedge^k(V)$. Consider $x_1 \wedge \cdots \wedge x_k \in \bigwedge^k(V)$, with $\{x_i\}_{i=1,\dots,k} \subset V$. Then, denoting teh component of x_i along e_j as x_i^j, it is clear that:

$$x_1 \wedge \cdots \wedge x_k = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} x_1^{i_1} \dots x_k^{i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

Note that every term with $e_{i_\ell} = e_{i_m}$ for some $\ell \neq m \in [1, k]$ vanishes, while the remaining terms can be reordered so that $1 \leq i_1 \leq \cdots \leq i_k \leq n$: this shows that $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq n}$ is a basis of $\bigwedge^k(V)$. It is then clear that if $k > n$, then there will be at least two equal basis vector in every basis element: therefore, by the alternating nature of the exterior product, $\bigwedge^k(V) = 0$. □

It can be shown that $\dim_{\mathbb{K}} \bigwedge^k(V) = \binom{n}{k}$, so $\dim_{\mathbb{K}} \bigwedge(V) = 2^n$. The graded structure of the Grassmann algebra is ensured by:

$$\bigwedge^k(V) \wedge \bigwedge^p(V) \subset \bigwedge^{k+p}(V) \tag{A.21}$$

This is an anticommutative grading, as:

$$\alpha \in \bigwedge^k(V), \beta \in \bigwedge^p(V) \quad \Longrightarrow \quad \alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha \tag{A.22}$$

§A.3.2.1 Grassmann numbers

Given an n-dimensional vector field $V(\mathbb{C})$ (possibly $n = \infty$), the elements of $\bigwedge(V)$ are called **Grassmann numbers** (or supernumbers). The general form of a Grassmann number is:

$$z = c_0 + \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} c_{i_1 \dots i_k} \theta_{i_1} \dots \theta_{i_k} \equiv z_{\mathrm{B}} + z_{\mathrm{S}} \tag{A.23}$$

where $c_{i_1 \dots i_k} \in \mathbb{C}$ are completely-antisymmetric tensors of rank $(k, 0)$ and $\theta_i \theta_j \equiv \theta_i \wedge \theta_j$ (basis of the Grassmann algebra). z_{B} is called the **body** and z_{S} the **soul** of the supernumber z.

Proposition A.3.3 (Soul)

If $n < \infty$, then the soul of a supernumber is nilpotent:

$$z_{\mathrm{S}}^{n+1} = 0 \tag{A.24}$$

Lemma A.3.2 (Equations)

If $n < \infty$, then:

$$\theta_i z = 0 \ \forall i = 1, \dots, n \quad \Longrightarrow \quad z = c\theta_1 \dots \theta_n \ , \ c \in \mathbb{C} \tag{A.25}$$

If $n = \infty$, instead:

$$\theta_a z = 0 \ \forall a \in \mathbb{N} \quad \Longrightarrow \quad z = 0 \tag{A.26}$$

In analogy to Hermitian conjugation, an **involution** (or conjugation) is defined for supernumbers:

$$(\theta_i \theta_j)^* = \theta_j \theta_i = -\theta_i \theta_j \tag{A.27}$$

known as the deWitt convention. Moreover, note that products of an odd number of Grassmann variables anticommute with each other: these are called **a-numbers**. On the other hand, products of an even number of Grassmann variables commute with each other (and with every Grassmann number): these are called **c-numbers**. This decomposition induces a Z_2-grading of the algebra, showing that Grassmann algebras are supercommutative algebras[3]. Note that c-numbers form a subalgebra of $\bigwedge(V)$, while a-numbers do not (they are only a subspace).

[3] A *superalgebra* is a Z_2-graded associative algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with bilinear multiplication $\mathcal{A} \times \mathcal{A} \rightarrowtail \mathcal{A}$ such that $\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$, where indices read modulo 2. Elements of each $\mathcal{A}_i$ are called homogeneous.
A *supercommutative algebra* is a superalgebra such that any two homogeneous elements x, y satisfy $yx = (-1)^{|x||y|} xy$, with $|x|$ denoting the grade of x, i.e. $x \in \mathcal{A}_i \implies |x| = i \mod 2$.

§A.3.2.2 Dual numbers

The case $n = 1$ is of particular interest: the elements of the exterior algebra of a 1-dimensional complex vector space are called **dual numbers**.
As for Grassmann variabales $\theta\eta = -\epsilon\theta$, so that θ^2, every analytic function reduces to a linear one on dual numbers (through its Taylor series): $f(\theta) = A + B\theta$, with $A, B \in \mathbb{C}$, is the most general function on dual numbers. Clearly, derivation is defined as:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(A + B\theta) \equiv B \tag{A.28}$$

On the other hand, integration (only considered on the whole domain of θ) is less trivial and can be defined imposing two conditions:

1. linearity: $\int \mathrm{d}\theta\,[af(\theta) + bg(\theta)] = a\int \mathrm{d}\theta\, f(\theta) + b\int \mathrm{d}\theta\, g(\theta)$ $\forall a, b \in \mathbb{C}$;
2. shift-invariance: $\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, f(\theta + \eta)$.

The only linear function satisfying the second constraint is the constant function, thus integration is conventionally defined as:

$$\int \mathrm{d}\theta\,(A + B\theta) \equiv B \tag{A.29}$$

Therefore, on Grassmann numbers, integration and derivation are the same thing: indeed, if it where $\mathrm{I} = \mathrm{D}^{-1}$, as $\mathrm{D}^2 = 0$, then $0 = \mathrm{I}\mathrm{D}^2 = \mathrm{D}$, which is absurd.
Integration can be extended to complex integration. Parting from deWitt's convention, define:

$$\theta = \frac{\theta_1 + \mathrm{i}\theta_2}{\sqrt{2}} \qquad\qquad \theta^* = \frac{\theta_1 - \mathrm{i}\theta_2}{\sqrt{2}} \tag{A.30}$$

These satisfy (simple check):

$$(\theta\eta)^* = \eta^*\theta^* \tag{A.31}$$

θ and θ^* are distinct Grassmann numbers, therefore $\mathrm{d}\theta_1\mathrm{d}\theta_2 = \frac{1}{\mathrm{i}}\mathrm{d}\theta^*\mathrm{d}\theta = -\frac{1}{\mathrm{i}}\mathrm{d}\theta\mathrm{d}\theta^*$. The order ambiguity of integrating over multiple Grassmann numbers is solved setting the convention of performing the innermost integral first:

$$\int \mathrm{d}\theta\mathrm{d}\eta\,\eta\theta = +1 \tag{A.32}$$

Lemma A.3.3 (Gaussian integral)
Given $b \in \mathbb{C}$:

$$\int \mathrm{d}\theta^*\mathrm{d}\theta\, e^{-\theta^* b\theta} = b \tag{A.33}$$

Proof. $\int \mathrm{d}\theta^*\mathrm{d}\theta\, e^{-\theta^* b\theta} = \int \mathrm{d}\theta^*\mathrm{d}\theta\,(1 - \theta^* b\theta) = \int \mathrm{d}\theta^*\mathrm{d}\theta\,(1 + b\theta\theta^*) = b.$ □

Introducing an extra factor of $\theta\theta^*$ in the integrand yields an extra b^{-1} factor (as in standard Gaussian integrals):

$$\int \mathrm{d}\theta^*\mathrm{d}\theta\,\theta\theta^* e^{-\theta^* b\theta} = 1 \tag{A.34}$$

Proposition A.3.4 (Unitary invariance)
Given $\mathrm{U} \in \mathrm{U}(n)$ and setting $\theta'_i = \sum_{j=1}^{n} \mathrm{U}_{ij}\theta_j$, then:

$$\int \prod_{i=1}^{n} \mathrm{d}\theta_i^* \theta_i \, f(\theta) = \int \prod_{i=1}^{n} \mathrm{d}\theta'^*_i \theta'_i \, f(\theta') \tag{A.35}$$

Proof. By the Prop. A.3.2:

$$\prod_{i=1}^{n} \theta'_i = \sum_{i_1,\ldots,i_n=1}^{n} \frac{1}{n!}\epsilon^{i_1\ldots i_n}\theta'_{i_1}\ldots\theta'_{i_n} = \sum_{i_1,\ldots,i_n=1}^{n} \sum_{j_1,\ldots,j_n=1}^{n} \frac{1}{n!}\epsilon^{i_1\ldots i_n}\mathrm{U}_{i_1 j_1}\ldots\mathrm{U}_{i_n j_n}\theta_{j_1}\ldots\theta_{j_n}$$
$$= \sum_{i_1,\ldots,i_n=1}^{n} \sum_{j_1,\ldots,j_n=1}^{n} \frac{1}{n!}\epsilon^{i_1\ldots i_n}\epsilon^{j_1\ldots j_n}\mathrm{U}_{i_1 j_1}\ldots\mathrm{U}_{i_n j_n}\prod_{k=1}^{n}\theta_k = \det \mathrm{U}\prod_{i=1}^{n}\theta_i$$

Equivalently:

$$\prod_{i=1}^{n} \theta'^*_i = (\det \mathrm{U})^* \prod_{i=1}^{n} \theta_i^*$$

As $\mathrm{U} \in \mathrm{U}(n)$, $(\det \mathrm{U})(\det \mathrm{U})^* = 1$, so:

$$\prod_{i=1}^{n} \theta'^*_i \theta'_i = \prod_{i=1}^{n} \theta_i^* \theta_i$$

The same holds for the measure. As the only term of $f(\theta)$ which survives the integration has exactly one factor of each θ_i^* and one of each θ_i, the integral remains unchanged. □

With unitary invariance, it is possible to extend complex integration to the multi-dimensional case.

Lemma A.3.4 (Gaussian integral)
Given $\mathrm{B} \in \mathbb{C}^{n\times n}$ Hermitian:

$$\int \prod_{i=1}^{n} \mathrm{d}\theta_i^* \mathrm{d}\theta_i \, e^{-\sum_{j,k=1}^{n}\theta_j^* \mathrm{B}_{jk}\theta_k} = \det \mathrm{B} \tag{A.36}$$

Proof. Being B Hermitian, it can be diagonalized by a unitary transformation, so, given its eigenvalues $\{b_i\}_{i=1,\ldots,n} \subset \mathbb{C}$:

$$\int \prod_{i=1}^{n} \mathrm{d}\theta_i^* \mathrm{d}\theta_i \, e^{-\sum_{j,k=1}^{n}\theta_j^* \mathrm{B}_{jk}\theta_k} = \int \prod_{i=1}^{n} \mathrm{d}\theta'^*_i \mathrm{d}\theta'_i \, e^{-\sum_{j=1}^{n}\theta'^*_j b_j \theta'_j} = \prod_{i=1}^{n} \int \mathrm{d}\theta_i^* \mathrm{d}\theta_i \, e^{-\theta_i^* b_i \theta_i} = \prod_{i=1}^{n} b_i$$

By Binet's theorem, this is precisely $\det \mathrm{B}$. □

With extra factors, the standard Gaussian behavior is recovered:

$$\int \prod_{i=1}^{n} \mathrm{d}\theta_i^* \mathrm{d}\theta_i \, \theta_p \theta_q e^{-\sum_{j,k=1}^{n}\theta_j^* B_{jk}\theta_k} = (\det \mathrm{B}) \left[\mathrm{B}^{-1}\right]_{pq} \tag{A.37}$$

§A.4 Gaussian integrals

Lemma A.4.1

Given $\alpha, \beta \in \mathbb{C} : \Re\alpha \geq 0$:

$$\int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{1}{2}\alpha x^2 + \beta x} = \sqrt{\frac{2\pi}{\alpha}} e^{\frac{\beta^2}{2\alpha}} \tag{A.38}$$

Proposition A.4.1 (Generalized Gaussian integral)

Given a non-singular matrix $\mathrm{A} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{J} \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, e^{-\frac{1}{2}\mathbf{x}^\intercal \mathrm{A} \mathbf{x} + \mathbf{J} \cdot \mathbf{x}} = \sqrt{\frac{(2\pi)^n}{\det \mathrm{A}}} e^{\frac{1}{2}\mathbf{J}^\intercal \mathrm{A}^{-1} \mathbf{J}} \tag{A.39}$$

Lemma A.4.2

Given a non-singular matrix $\mathrm{A} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{J} \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, e^{-\frac{\mathrm{i}}{2}\mathbf{x}^\intercal \mathrm{A} \mathbf{x} + \mathrm{i}\mathbf{J} \cdot \mathbf{x}} = \sqrt{\frac{(2\pi \mathrm{i})^n}{\det \mathrm{A}}} e^{-\frac{\mathrm{i}}{2}\mathbf{J}^\intercal \mathrm{A}^{-1} \mathbf{J}} \tag{A.40}$$

Appendix B

Scattering Processes

§B.1 Cross-sections and S-matrix

The cross-section generalizes the notion of geometric area of an object. Consider a flux of particles (particles per unit area per unit time):

$$j = \frac{\mathrm{d}n}{\mathrm{d}S\mathrm{d}t}$$

Then, the number of incident particles diffused by a disk of section S in a time Δt is:

$$N = jS\Delta t$$

The (classical) **cross-section** of the disk is then defined as:

$$\sigma := \frac{N}{j\Delta t} \tag{B.1}$$

From a quantum point of view, N is the probability of the transition $|\text{in}\rangle \to |\text{out}\rangle$ between the initial and final states, thus:

$$\mathrm{d}\sigma = \frac{1}{j\Delta t} \left|\langle \text{out}(f)|\text{in}\rangle\right|^2 \mathrm{d}f$$

where f is some quantity which the final state depends on.

§B.1.1 Classical and quantum definition

§B.1.2 Phase-space integration

Proposition B.1.1 (Scattering cross-section)

Given a $2 \to n$ scattering process with well-defined initial momenta $p_{\mathcal{A}}$ and $p_{\mathcal{B}}$, then the **differential cross-section** is:

$$\mathrm{d}\sigma = \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}\left|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}\right|} \prod_{k=1}^{n} \frac{\mathrm{d}^3 p_k}{(2\pi)^3 2E_k} \left|\mathcal{M}(\mathcal{A}\mathcal{B} \to \{f\})\right|^2 (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} + p_{\mathcal{B}} - \textstyle\sum_{i=1}^n p_i) \tag{B.2}$$

where $\mathcal{M}(\mathcal{A}\mathcal{B} \to \{f\})$ is the matrix element of the scattering process and $\mathbf{v}_k \equiv \frac{\mathbf{p}_k}{E_k}$ is the velocity of the k^{th} particle.

This cross-section is differential in $3n-4$ variables, as the momentum-conserving $\delta^{(4)}$ allows to perform 4 integrations. The integration measure is defined as the **invariant n-body phase space**:

$$\int \mathrm{d}\Pi_n \equiv \prod_{k=1}^{n} \int \frac{\mathrm{d}^3 p_k}{(2\pi)^3 2E_k} (2\pi)^4 \delta^{(4)}(P - \textstyle\sum_{i=1}^{n} p_i) \tag{B.3}$$

with P the total initial 4-momentum.

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