Quantum Field Theory 1
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Part I

Field Theory

Chapter 1

Classical Field Theory

1.1 Continuous limit

1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x-axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a.

Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n, one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2}\dot{\eta}_n^2 - \frac{\lambda}{2}\left(\eta_n - \eta_{n-1}\right)^2 \right] \tag{1.1}$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda\left(\eta_{n+1} - 2\eta_n + \eta_{n-1}\right) \tag{1.2}$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \tag{1.3}$$

where the dispersion relation is:

$$\omega^2 = 2\lambda\left(1 - \cos k\right) \approx \lambda k^2 \tag{1.4}$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \qquad \Rightarrow \qquad k_m = \frac{2\pi m}{N} \, , \, m = 0, 1, \ldots, N-1 \tag{1.5}$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} \left[\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)} \right] \tag{1.6}$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjugated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \tag{1.7}$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*[1]:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \tag{1.8}$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators* [2] $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} \left[e^{i(k_m n - \omega_m t)} \hat{a}_m + e^{-i(k_m n - \omega_m t)} \hat{a}_m^\dagger \right] \tag{1.9}$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \tag{1.10}$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2}\hat{\pi}_n^2 + \frac{\lambda}{2}(\hat{\eta}_n - \hat{\eta}_{n-1})^2 \right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) \tag{1.11}$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m |0\rangle \quad \forall m = 0, 1, \ldots, N-1 \tag{1.12}$$

$$|n_0, n_1, \ldots, n_{N-1}\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} |0\rangle \tag{1.13}$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2} \sum_{m=0}^{N-1} \hbar\omega_m \tag{1.14}$$

$$E_{n_0, n_1, \ldots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \tag{1.15}$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as "particles" and n_m is the number of phonons in the mth normal mode.

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[1] Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t} \hat{\mathcal{O}}(0) e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{\mathrm{d}\hat{\mathcal{O}}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

[2] For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{p}(t)$ and $\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}} \left[\hat{a}(t) + \hat{a}^\dagger(t) \right] \qquad \hat{p}(t) = -i\omega\sqrt{\frac{\hbar}{2\omega}} \left[\hat{a}(t) - \hat{a}^\dagger(t) \right]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega \left(\hat{a}^\dagger(t)\hat{a} + \frac{1}{2} \right)$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.

1.1.2   One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \to 0$ and $N \to \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x, t)$ dependent on the continuous real coordinate $x \in [0, R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x}\right)^2 \qquad\qquad \sum_n \longrightarrow \frac{1}{a} \int_0^R \mathrm{d}x$$

Proposition 1.1.1

In the continuous limit:
$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x - x') = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

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Proof. By direct calculation:

$$a \sum_n f(an)\frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R \mathrm{d}x \, f(x)\delta(x - y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \to kx$, symmetrizing $k_m \in [-\pi, \pi]$ (instead of $[0, 2\pi]$) one finds:

$$\delta(x - x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na} \sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

where integration limits are $\pm\frac{\pi}{a} \to \pm\infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 1.1.2

The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R \mathrm{d}x \, e^{i(k-k')x} = 2\pi\delta(k - k')$$

By these relations, it can be seen that $\frac{\mathrm{d}k}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k + \mathrm{d}k$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.

In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}\rho_0(\partial_t \eta)^2 - \frac{\kappa}{2}(\partial_x \eta)^2\right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x, t) \equiv \sqrt{\rho_0}\eta(x, t)$ and $\kappa = c^2\rho_0$ and adding a pinning term $\propto \varphi^2$, the Lagrangian can be written as:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{c^2}{2}(\partial_x \phi)^2 - \frac{m^2 c^4}{2}\phi^2\right] \tag{1.16}$$

The classical equation of motion of this field yields:

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi - m^2 c^4 \phi \tag{1.17}$$

The solutions of this wave equation can be written as:

$$\phi(x,t) = e^{i(kx - \omega_k t)} \tag{1.18}$$

with dispersion relation:

$$\omega_k^2 = c^2 k^2 + m^2 c^4 \tag{1.19}$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x,t) := \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi(x,t) \tag{1.20}$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R \mathrm{d}x \left[\frac{1}{2}\Pi^2 + \frac{c^2}{2}(\partial_x \phi)^2 + \frac{m^2 c^4}{2}\phi^2 \right] \tag{1.21}$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \sqrt{\frac{\hbar}{2\omega_k}} \left[e^{i(kx - \omega_k t)} \hat{a}_k + e^{-i(kx - \omega_k t)} \hat{a}_k^\dagger \right] \tag{1.22}$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi \delta(k - k') \tag{1.23}$$

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = i\hbar \delta(x - x') \tag{1.24}$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \frac{1}{2} \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) = E_0 + \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \tag{1.25}$$

The ground-state energy can be computed from Eq. 1.14, defining $\mathrm{Vol} := 2\pi \delta(k = 0)$:

$$E_0 = \mathrm{Vol} \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \frac{1}{2} \hbar \omega_k \tag{1.26}$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

1.2 Spacetime symmetries

1.2.1 Lie groups

Definition 1.2.1: Lie group

A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\ldots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d-dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0,\ldots,0) = e$.

Definition 1.2.2: Representation

Given a group G and a vector space $V(\mathbb{K})$, a *representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Given the isomorphism $\mathrm{GL}(V) \to \mathbb{K}^{n \times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to *de facto* represent G as matrices acting on elements of V, i.e. $\rho : G \to \mathbb{K}^{n \times n}$.

Theorem 1.2.1

Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ as:

$$\rho(g(\theta)) = e^{i\theta_a T^a} \tag{1.27}$$

where $\{T^a\}_{a=1,\ldots,d} \subset \mathbb{C}^{n \times n}$ are the *generators* of G on V.

Definition 1.2.3: Lie algebra

Given a Lie group G with generators $\{T^a\}_{a=1,\ldots,d} \subset \mathbb{C}^{n \times n}$ on $V(\mathbb{C})$, its *Lie algebra* is:

$$[T^a, T^b] = i f^{ab}_{\ \ c} T^c \tag{1.28}$$

where $f^{ab}_{\ \ c}$ are called the *structure constants*.

Proposition 1.2.1

The Lie algebra of a Lie group is independent of the representation.

Proposition 1.2.2

Any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

> **Definition 1.2.4: Casimir operator**
>
> Given a Lie group with generators $\{T^a\}_{a=1,\ldots,d} \subset \mathbb{C}^{n \times n}$ on $V(\mathbb{C})$, a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation, Casimir operators are operators proportional to id_V, and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

> **Proposition 1.2.3**
>
> A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

1.2.2 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu} x^\mu x^\nu$, i.e. the orthogonal group $O(1,3)$ (with signature $(+,-,-,-)$). The condition that $\Lambda^\mu{}_\nu$ must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \tag{1.29}$$

This implies that $\det \Lambda = \pm 1$: a transformation with $\det \Lambda = -1$ can always be written as the product of a transformation with $\det \Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines $SO(1,3) \coloneqq \{\Lambda \in O(1,3) : \det \Lambda = 1\}$.

Writing explicitly the temporal component $1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$, it is clear that $(\Lambda^0{}_0)^2 \geq 1$. Therefore, $O(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0{}_0 \geq 1$ and the non-orthochronous component with $\Lambda^0{}_0 \leq 1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a dircrete transformation which reverses the sign of the temporal component.

> **Definition 1.2.5: Lorentz group**
>
> The *Lorentz group* $SO^+(1,3)$ is the orthochronous component of $SO(1,3)$.

The discrete transformations are factored out of the Lorentz group: these are *parity* and *time reversal*, which can be represented as $\mathcal{P}^\mu{}_\nu = \mathrm{diag}\,(+1,-1,-1,-1)$ and $\mathcal{T}^\mu{}_\nu = \mathrm{diag}\,(-1,+1,+1,+1)$. Applying these discrete transformations in all combinations (id, $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{PT}$) one gets the four disconnected components of $SO(1,3)$, which are not simply connected. This means that Lorentz invariance does not include parity and time reversal invariance.

Considering the infinitesimal transformation and applying Eq. 1.29:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \qquad \Rightarrow \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x,y), (y,z) and (z,x) planes and the 3 rapidities of hyperbolic rotations in the (t,x), (t,y) and (t,z) planes.

> ### Theorem 1.2.2
>
> The Lorentz group is a non-compact Lie group.

Proof. Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in $[0, 2\pi)$, rapidities vary in $\mathbb{R}$, so the differentiable manifold associated to $\mathrm{SO}^+(1,3)$ is not compact. $\qquad\square$

1.2.2.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \mathrm{SO}^+(1,3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.30}$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a n-dimensional representation of $\mathrm{SO}^+(1,3)$, both $[J^{\mu\nu}]^i_{\ j}$ and $[\Lambda]^i_{\ j}$ are $\mathbb{C}^{n\times n}$ matrices (Λ is real): for example, the $n = 1$ representation acts on *scalars*, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \,\forall \mu, \nu = 0, \ldots, 3$.

4-vectors The $n = 4$ representation acts on *contravariant 4-vectors* v^μ, which transform according to $v^\mu \mapsto \Lambda^\mu_{\ \nu}v^\nu$, and *covariant 4-vectors* v_μ, which transform according to $v_\mu \mapsto \Lambda_\mu^{\ \nu}v_\nu$. In this representation, the generators are represented as $\mathbb{C}^{4\times 4}$ matrices:

$$[J^{\mu\nu}]^\rho_{\ \sigma} = i\left(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma\right) \tag{1.31}$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}^+(1,3)$, called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \tag{1.32}$$

It is convenient to rearrange the 6 components of $J^{\mu\nu}$ into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad\qquad K^i := J^{i0} \tag{1.33}$$

The $\mathfrak{so}^+(1,3)$ can then be rewritten as:

$$[J^i, J^j] = i\epsilon^{ijk}J^k \qquad [J^i, K^j] = i\epsilon^{ijk}K^k \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k \tag{1.34}$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^i are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \qquad\qquad \eta^i := \omega^{i0} \tag{1.35}$$

so that:

$$\Lambda = \exp\left[-i\boldsymbol{\theta}\cdot\mathbf{J} + i\boldsymbol{\eta}\cdot\mathbf{K}\right] \tag{1.36}$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts wich increase velocities with respect to said frame.

1.2.2.2 Tensor Representations

A generic (p, q)-tensor transforms as:

$$T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \mapsto \Lambda^{\mu_1}{}_{\alpha_1} \ldots \Lambda^{\mu_p}{}_{\alpha_p} \Lambda_{\nu_1}{}^{\beta_1} \ldots \Lambda_{\nu_q}{}^{\beta_q} T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} \tag{1.37}$$

The representation of the Lorentz group which acts on (p, q)-tensors is of degree $n = 4^{p+q}$, however it is reducible into the direct product of $p + q$ 4-dimensional representations as of Eq. 1.38.

Moreover, consider the action of the Lorentz group on $(2, 0)$-tensors: being $T^{\mu\nu} \mapsto \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu} T^{\mu\nu} \mapsto \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$4 \otimes 4 = 1 \oplus 6 \oplus 9 \tag{1.38}$$

These are irreducible representations which, given a generic tensor $T^{\mu\nu}$, act on S, $A^{\mu\nu}$ and $S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$ and $S^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$.

Decomposition under rotations Restricting the action to the SO(3) sub-group of $\mathrm{SO}^+(1, 3)$, tensors can be decomposed according to irreducible representations of SO(3), which are labelled by the angular momentum $j \in \mathbb{N}_0$ and are of degree $n = 2j + 1$. Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \tag{1.39}$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$. A 4-vector v^μ is irreducible under the action of $\mathrm{SO}^+(1, 3)$, but under SO(3) it is decomposed into v^0 and $\mathbf{v}$, so $v^\mu \in \mathbf{0} \oplus \mathbf{1}$. A $(2, 0)$-tensor then is:

$$T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) = (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1})$$
$$= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2})$$

This is equivalent to Eq. 1.38: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2}\epsilon^{ijk}A^{jk}$, so $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ under spatial rotations.

Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$, $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$, $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$ and $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk}A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

> **Example 1.2.1**
>
> Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two *invariant tensors* under $\mathrm{SO}^+(1, 3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.29, and the Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \Lambda^{\sigma}{}_{\gamma} \Lambda^{\rho}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = (\det \Lambda) \epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

1.2.2.3   Spinorial representations

The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same, which means that SU(2) and SO(3) are indistinguishable by infinitesimal transformations; however, they are globally different, as SO(3) rotations are periodic by 2π, while SU(2) rotations are periodic by 4π: in particular, it can be shown that $SO(3) \cong SU(2)/\mathbb{Z}_2$, i.e. SU(2) is the universal cover of SO(3). This means that SU(2) representations can be labelled by $j \in \frac{1}{2}\mathbb{N}_0$, where half-integer spin representations are known as *spinorial representations*: they act on spinors, i.e. objects which change sign under rotations of 2π (thus not suitable to represent SO(3)).

Example 1.2.2

The $\frac{1}{2}$ representation of SU(2) is a 2-dimensional representation where $J^i = \frac{\sigma^i}{2}$: Pauli matrices satisfy $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, thus the $\mathfrak{su}(2)$ algebra is satisfied. Denoting the $m = \pm\frac{1}{2}$ states in the $\frac{1}{2}$ representation as $|\uparrow\rangle$ and $|\downarrow\rangle$, the Clebsch-Gordan decomposition $\frac{1}{2} \otimes \frac{1}{2} = \mathbf{0} \oplus \mathbf{1}$ yields the triplet $(j = 1)$ $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$ and the singlet $(j = 0)$ $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Proposition 1.2.4

The Lorentz algebra $\mathfrak{so}^+(1,3)$ can be decomposed as $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

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Proof. Given the $\mathfrak{so}^+(1,3)$ algebra in Eq. 1.33, it is possible to define:

$$\mathbf{J}_\pm := \frac{1}{2}\left(\mathbf{J} \pm i\mathbf{K}\right)$$

The Lie algebra then becomes:

$$[\mathbf{J}_\pm^i, \mathbf{J}_\pm^j] = i\epsilon^{ijk}\mathbf{J}_\pm^k \qquad\qquad [\mathbf{J}_\pm^i, \mathbf{J}_\mp^j] = 0$$

These are two commuting $\mathfrak{so}(2)$ algebras, thus proving the thesis. $\square$

As observed before, this does not imply that $SO^+(1,3)$ is globally equivalent to $SU(2) \times SU(2)$: in fact, $SU(2) \times SU(2)/\mathbb{Z}_2 \cong SO(4)$, while the universal cover of $SO^+(1,3)$ is $SL(2,\mathbb{C})$, as it can be shown that $SO^+(1,3) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$.

By Prop. 1.2.4, representations of $SO^+(1,3)$ can be labelled by $(j_-, j_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$, with each index labelling a representation of SU(2): as $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, the (j_-, j_+) representation contains states with all possible spins $|j_+ - j_-| \le j \le j_+ + j_-$, and it is a representation of degree $n = (2j_- + 1)(2j_+ + 1)$. $(\mathbf{0}, \mathbf{0})$ is the trivial (scalar) representation, as both $\mathbf{J}_\pm = 0$ and $\mathbf{J} = \mathbf{K} = 0$.

$\left(\frac{1}{2}, \mathbf{0}\right)$ and $\left(\mathbf{0}, \frac{1}{2}\right)$ are 2-dimensional spinorial representations. These representations act on different spinors $(\psi_\mathrm{L})_\alpha \in \left(\frac{1}{2}, \mathbf{0}\right)$ and $(\psi_\mathrm{R})_\alpha \in \left(\mathbf{0}, \frac{1}{2}\right)$, with $\alpha = 1, 2$, which are called *left-* and *right-handed Weyl spinors*. In $\left(\frac{1}{2}, \mathbf{0}\right)$ the generators are $\mathbf{J}_- = \frac{\sigma}{2}$ and $\mathbf{J}_+ = \mathbf{0}$, while in $\left(\mathbf{0}, \frac{1}{2}\right)$ they are $\mathbf{J}_- = \mathbf{0}$ and $\mathbf{J}_+ = \frac{\sigma}{2}$, thus one finds $\mathbf{J}_\mathrm{L} = \mathbf{J}_\mathrm{R} = \frac{\sigma}{2}$ and $\mathbf{K}_\mathrm{L} = -\mathbf{K}_\mathrm{R} = i\frac{\sigma}{2}$, so that:

$$\psi_\mathrm{L} \mapsto \Lambda_\mathrm{L}\psi_\mathrm{L} = \exp\left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_\mathrm{L} \tag{1.40}$$

$$\psi_\mathrm{R} \mapsto \Lambda_\mathrm{R}\psi_\mathrm{R} = \exp\left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_\mathrm{R} \tag{1.41}$$

Note that the generators K^i are not hermitian, as expected from Prop. 1.2.3. Furthermore, note that $\Lambda_{L,L} \in \mathbb{C}^{2\times 2}$, therefore $\psi_{L,R} \in \mathbb{C}^2$.

Proposition 1.2.5

Given $\psi_L \in \left(\frac{1}{2}, \mathbf{0}\right)$ and $\psi_R \in \left(\mathbf{0}, \frac{1}{2}\right)$, then $\sigma^2 \psi_L^* \in \left(\mathbf{0}, \frac{1}{2}\right)$ and $\sigma^2 \psi_R^* \in \left(\frac{1}{2}, \mathbf{0}\right)$.

- -

Proof. Recall that for Pauli matrices $\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$, so $\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$ and:

$$\sigma^2 \psi_L^* \mapsto \sigma^2 \left(\Lambda_L \psi_L\right)^* = \left(\sigma^2 \Lambda_L^* \sigma^2\right) \sigma^2 \psi_L^* = \Lambda_R \sigma^2 \psi_L^* \quad\Rightarrow\quad \sigma^2 \psi_L^* \in \left(\mathbf{0}, \frac{1}{2}\right)$$

where $\sigma^2 \sigma^2 = I_2$ was used. The proof for $\sigma^2 \psi_R^*$ is analogous. $\square$

Definition 1.2.6: Charge conjugation

On Weyl spinors, the *charge conjugation operator* is defined as:

$$\psi_L^c := i\sigma^2 \psi_L^* \qquad\qquad \psi_R^c := -i\sigma^2 \psi_R^* \tag{1.42}$$

By Prop. 1.2.5, charge conjugation changes transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the i factor ensures that applying this operator twice yields the identity operator.

$\left(\frac{1}{2}, \frac{1}{2}\right)$ is a 4-dimensional complex representation: as $j = 0, 1$, this representation acts on complex 4-vectors of the form $\Psi = ((\psi_L)_\alpha, (\xi_R)_\beta) \in \mathbb{C}^4$, called *Dirac spinors*, and $\Lambda_D = \text{diag}\left(\Lambda_L, \Lambda_R\right) \in \mathbb{C}^{4\times 4}$. To explicit this relation, set $\psi_R \equiv i\sigma^2 \psi_L^*$, $\xi_L \equiv -i\sigma^2 \xi_R^*$ and $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$: it can be shown, then, that $\xi_R^\dagger \sigma^\mu \psi_R$ and $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ are contravariant 4-vectors. Although these 4-vectors are complex by construction, being the matrix $\Lambda^\mu{}_\nu$ which represents the Lorentz transformation of a 4-vector real, a reality condition $v_\mu^* = v_\mu$ is Lorentz invariant.

Parity   Note that $\mathcal{P}\mathbf{K} = -\mathbf{K}$, as the velocity of the boost gets reversed, while $\mathcal{P}\mathbf{J} = \mathbf{J}$: this means that $\mathcal{P}\mathbf{J}_\pm = \mathbf{J}_\mp$, i.e. parity exchanges a $(\mathbf{j}_-, \mathbf{j}_+)$ representation into a $(\mathbf{j}_+, \mathbf{j}_-)$ representation. Therefore, a $(\mathbf{j}_-, \mathbf{j}_+)$ representation of $SO^+(1,3)$ is a basis for the representation of the parity transformation iff $j_- = j_+$.

Example 1.2.3

Weyl spinors (separately) are not a basis for a representation of the parity transformation, but Dirac spinors are.

1.2.2.4   Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields   A scalar field transforms as:

$$\phi'(x') = \phi(x) \tag{1.43}$$

Consider an infinitesimal transformation $x'^\rho = x^\rho + \delta x^\rho$, with $\delta x^\rho = -\frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}]^\rho{}_\sigma x^\sigma$ as of Eq. 1.30. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar representation of $SO^+(1,3)$ is the trivial one ($J^{\mu\nu} = 0$).

However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$: while $\delta\phi$ studies only a single degree of freedom, as the point $P \in \mathbb{R}^{1,3}$ is kept constant and only $\phi(P)$ can vary (i.e. the base space is one-dimensional), $\delta_0\phi$ studies $\phi(P)$ with P varying over $\mathbb{R}^{1,3}$, thus the base space is now a space of functions, which is infinite-dimensional. Therefore, $\delta\phi$ provides a finite-dimensional representation of the generators, while $\delta_0\phi$ an infinite-dimensional one.

To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho \partial_\rho \phi = \frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}]^\rho{}_\sigma x^\sigma \partial_\rho \phi \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi$$

Recalling Eq. 1.31, the generators can be expressed as:

$$L^{\mu\nu} := i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \tag{1.44}$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As $p^\mu = i\partial^\mu$ (with signature $(+,-,-,-)$), it is clear that $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ is the orbital angular momentum.

Weyl fields   A left-handed Weyl field transforms as:

$$\psi'_L(x') = \Lambda_L \psi_L(x) \tag{1.45}$$

with Λ_L defined in Eq. 1.40, and similarly for right-handed Weyl fields. The infinite-dimensional representation of the Lorentz generators determined by Weyl spinors can be found as:

$$\begin{aligned}
\delta_0\psi_L &\equiv \psi'_L(x) - \psi_L(x) = \psi'_L(x' - \delta x) - \psi_L(x) \\
&= \psi'_L(x') - \delta x^\rho \partial_\rho \psi_L(x) - \psi_L(x) = (\Lambda_L - I_2)\psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x)
\end{aligned}$$

The second term yields $L^{\mu\nu}$, while the first can be further elaborated by writing:

$$\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{1.46}$$

Thus:

$$\delta_0\psi_L = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_L$$

where the angular momentum separates into the orbital and the spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1.47}$$

This separation is general: $L^{\mu\nu}$ is always expressed as in Eq. 1.44, while $S^{\mu\nu}$ depends on the specific representation. In the scalar representation $S^{\mu\nu} = 0$, while in the left/right-handed Weyl representation $S^{i0} = \pm i\frac{\sigma^i}{2}$.

Vector fields   A (contravariant) vector field transforms as:

$$V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) \tag{1.48}$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation.

1.2.3 Poincaré group

> **Definition 1.2.7: Poincaré group**
>
> The *Poincaré group* is defined as $\mathrm{ISO}^+(1,3) := \mathrm{T}^{1,3} \rtimes \mathrm{SO}^+(1,3)$, where $\mathrm{T}^{1,3} \cong \mathbb{R}^{1,3}$ is the group of translations of $\mathbb{R}^{1,3}$, with multiplication $(a^\mu, \Lambda^\mu{}_\nu) \cdot (\bar{a}^\mu, \bar{\Lambda}^\mu{}_\nu) \equiv (a^\mu + \Lambda^\mu{}_\nu \bar{a}^\nu, \Lambda^\mu{}_\rho \bar{\Lambda}^\rho{}_\nu)$.

Given a translation $x^\mu \mapsto x^\mu + a^\mu$, the associated group element can be written as:

$$T = e^{-i a_\mu P^\mu} \tag{1.49}$$

where P^μ is the 4-momentum operator. Clearly translations commute, and so do their generators; on the other hand, as $\mathbf{P}$ is a vector under rotations, while P^0 (energy) a scalar, one has:

$$[J^i, P^j] = i e^{ijk} P^k \qquad\qquad [J^i, P^0] = 0$$

These equations uniquely determine the *Poincaré algebra* $\mathfrak{iso}^+(1,3)$:

$$[P^\mu, P^\nu] = 0$$
$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\rho} J^{\nu\sigma}\right) \tag{1.50}$$
$$[P^\mu, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho\right)$$

It's easy to check that $[K^i, P^0] = i P^i$, while $[J^i, P^0] = [P^i, P^0] = 0$: given that P^0 generates time translations, linear and angular momentum are conserved quantities, while $\mathbf{K}$ is not.

1.2.3.1 Field representations

Fields provide an infinite-dimensional representation of the Lorentz group as $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $S^{\mu\nu}$ does not depend on x^μ, but only on the spin of the field.

To represent P^μ on fields, their transformation law must be specified: all fields are required to be scalars under translations, independently of their spin. This means that, given a generic field $\phi(x)$, under a translation $x' = x + a$ it transforms as $\phi'(x') = \phi(x)$, so, under an infinitesimal translation $x' = x + \varepsilon$:

$$\delta_0 \phi \equiv \phi'(x) - \phi(x) = \phi'(x' - \varepsilon) - \phi(x) = -\varepsilon^\mu \partial_\mu \phi(x)$$
$$= e^{-i(-\varepsilon_\mu) P^\mu} \phi'(x') - \phi(x) = \left(e^{i\varepsilon_\mu P^\mu} - \mathrm{I}\right)\phi(x) = i\varepsilon_\mu P^\mu \phi(x)$$

It is clear then that:

$$P^\mu = +i\partial^\mu \tag{1.51}$$

Explicitly, $P^0 = i\partial_t$ and $\mathbf{P} = -i\boldsymbol{\nabla}$. It is trivial to check that these generators obey the Poincaré algebra.

1.2.3.2 Particle representations

The Poincaré group can also be represented using the Hilbert space $\mathscr{H}$ of a free particle as a basis. Denoting a generic state as $|\mathbf{p}, s\rangle \in \mathscr{H}$, where $\mathbf{p}$ is the particle's momentum and s collectively labels all other quantum numbers, it is clear that $\mathscr{H}$ is infinite-dimensional, as $\mathbf{p}$ is a continuous unbounded variable.

> ### Theorem 1.2.3: Wigner's theorem
>
> On the Hilbert space of physical states, any symmetry transformation can be represented by a linear and unitary or anti-linear and anti-unitary operator.

By this theorem, Poincaré transformations can be represented by unitary matrices, i.e. $\mathbf{J}$, $\mathbf{K}$, $\mathbf{P}$ and P^0 can be represented by hermitian operators. These representations can be labeled by Casimir operators, which for $\text{ISO}^+(1,3)$ are easily found as $P_\mu P^\mu$ and $W_\mu W^\mu$, where W^μ is the *Pauli-Lubanski operator*:

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}J_{\nu\sigma}P_\rho \tag{1.52}$$

On single-particle states $P_\mu P^\mu = m^2$, while $W_\mu W^\mu$ can be conveniently computed in a particular frame (due to Lorentz invariance). If $m \neq 0$, this frame is the rest-frame of the particle:

$$W^\mu = -\frac{m}{2}\epsilon^{\mu\nu\sigma 0}J_{\nu\sigma} = \frac{m}{2}\delta^{\mu i}\epsilon^{ijk}J^{jk} = \delta^{\mu i}mJ^i$$

Therefore, on single-particle states of mass m and spin j, the Casimir operator takes the form:

$$W_\mu W^\mu = -m^2 j(j+1) \tag{1.53}$$

If $m = 0$, the rest-frame does not exist, but it is possible to choose a frame where $P^\mu = (\omega, 0, 0, \omega)$, where $W^0 = W^3 = \omega J^3$, $W^1 = \omega(J^1 - K^2)$ and $W^2 = \omega(J^2 + K^1)$, so that:

$$W_\mu W^\mu = -\omega^2\left[\left(K^2 - J^1\right)^2 + \left(K^1 + J^2\right)^2\right] \tag{1.54}$$

It is clear that the $m \to 0$ limit is not trivial, and massive and massless representation need to be studied separately.

Massive representations Restricting to $m \in \mathbb{R}^+$ ($m^2 < 0$ states, called tachyons, are excluded), the massive representations are labeled by mass m and spin j: in fact, after a Lorentz transformation such that $P^\mu = (m, \mathbf{0})$, spatial rotations can still be performed, i.e. the subspace of single-particle states with momentum $P^\mu = (m, \mathbf{0})$ is still a basis for the representation of SU(2) (as spinors must be included too). The group of transformations which leaves invariant a certain choice of P^μ is called the *little group*, so SU(2) is the little group of massive single-particle states: massive representations are labelled by m and j, which means that massive particles of spin j have $2j + 1$ degrees of freedom.

Massless representations The little group for $P^\mu = (\omega, 0, 0, \omega)$ clearly is SO(2), the group of rotations in the (x, y) plane generated by J^3: as for any abelian group, its irreducible representations are one-dimensional, and they are labeled by the eigenvalue h of J^3, which represents the angular momentum in the direction of propagation of the particle and is called *helicity*. Helicity can be shown to be quantized as $h \in \frac{1}{2}\mathbb{Z}_0$ (by topologic considerations on $\text{ISO}^+(1,3) \equiv \mathbb{R}^4 \times \text{SL}(2,\mathbb{C})/\mathbb{Z}_2$).
As a consequence, massless particles only have one degree of freedom and are characterized by their helicity h. As $\text{SO}(2) \equiv \text{U}(1)$, on a state of helicity h the little group is represented as:

$$U(\theta) = e^{-ih\theta} \tag{1.55}$$

Although massless particles with opposite helicities are logically two different species of particles, it can be written as $h = \hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$ (unit vectors), so h is a pseudoscalar such that $h \mapsto -h$ under parity: this means that, if the interaction conserves parity, h and $-h$ must be symmetric.

> **Example 1.2.4**
>
> The electromagnetic and gravitational interactions conserve parity, thus photons and gravitons must be a basis for the representation of both $\text{ISO}^+(1,3)$ and parity: photons can have $h = \pm 1$ (left- and right-handed), while gravitons have $h = \pm 2$.

> **Example 1.2.5**
>
> Neutrinos only interact via the weak interaction, which does not conserve parity, and in fact the two states $h = \pm\frac{1}{2}$ are different particles: neutrinos have $h = -\frac{1}{2}$, while antineutrinos have $h = +\frac{1}{2}$.

1.3   Classical equations of motion

Consider a *local field theory* of fields $\{\phi_i(x)\}_{i\in\mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkoski spacetime. Its Lagrangian takes the form:

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.56}$$

where $\mathcal{L}$ is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.57}$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

> **Theorem 1.3.1**
>
> The *stationary action principle* $\delta\mathcal{S} = 0$ determines the classical equations of motion:
>
> $$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0 \tag{1.58}$$
>
> -
>
> *Proof.* Varying Eq. 1.57:
>
> $$\delta\mathcal{S} = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi_i)\right] = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi_i = 0$$
>
> $\square$

> **Corollary 1.3.1.1**
>
> Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.

Proof. This is a consequence of Stokes theorem:

$$\int_\Sigma \mathrm{d}^4x\, \partial_\mu K^\mu = \int_{\partial\Sigma} \mathrm{d}A\, n_\mu K^\mu$$

$\square$

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)} \tag{1.59}$$

$$\mathcal{H} = \sum_{i\in\mathcal{I}} \Pi_i(x)\partial_0\phi(x) - \mathcal{L} \tag{1.60}$$

Unlike the Hamiltonian formalism, the Lagrangian formalism keeps Lorentz covariance explicit.

1.3.1   Noether's theorem

Definition 1.3.1: Infinitesimal transformation

Given a field theory with fields $\{\phi_i\}_{i=1,\dots,k}$ and action $\mathcal{S}[\phi]$, an *infinitesimal transformation* parametrized by $\{\varepsilon^a\}_{a=1,\dots,N} : |\varepsilon^a| \ll 1$ is defined by two sets of functions $\{A_a^\mu(x)\}_{a=1,\dots,N}$ and $\{F_{i,a}(\phi,\partial\phi)\}_{i=1,\dots,k;\,a=1,\dots,N}$ such that:

$$x^\mu \mapsto x'^\mu = x^\mu + \varepsilon^a A_a^\mu(x)$$
$$\phi_i(x) \mapsto \phi_i'(x) = \phi_i(x) + \varepsilon^a F_{i,a}(\phi,\partial\phi) \tag{1.61}$$

Definition 1.3.2: Symmetry transformation

An infinitesimal transformation is a *symmetry transformation* if it leaves $\mathcal{S}[\phi]$ invariant, regardless of ϕ being a solution of the equations of motion. It can further be classified as:

- *global symmetry*, if $\varepsilon^a \equiv$ const.;

- *local symmetry*, if $\varepsilon^a = \varepsilon^a(x)$.

Symmetry transformations which leave spacetime unchanged, i.e. with $A_a^\mu(x) = 0$, are called *internal symmetries*.

Theorem 1.3.2: Noether's theorem

Given a global (but not local) symmetry parametrized by N generators, then there are N conserved currents $\{j_a^\mu(\phi)\}_{a=1,\dots,N}$ such that:

$$\partial_\mu j_a^\mu(\phi^{\mathrm{cl}}) = 0 \tag{1.62}$$

where ϕ^{cl} is a classical solution of the equations of motion.

Proof. First, consider an infinitesimal transformation with slowly-varying parameters, i.e. $l\,|\partial_\mu\varepsilon^a| \ll |\varepsilon^a|$ (l characteristic scale of the field theory): being it not a local symmetry, $\delta\mathcal{S} \neq 0$ at $o(\varepsilon)$ and:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x \left[\varepsilon^a(x)K_a(\phi) - (\partial_\mu\varepsilon^a(x))\,j_a^\mu(\phi) + o(\partial\partial\varepsilon)\right] + o(\varepsilon^2)$$

If $\varepsilon^a \equiv$ const. (global symmetry) then $\delta\mathcal{S}[\phi] = 0 \,\forall\phi$, therefore $K_a(\phi) = 0 \,\forall\phi$ (independent of ε). Assuming $\varepsilon^a(x) \to 0$ sufficiently fast as $x \to \infty$, then integration by parts yields:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x \, \varepsilon^a(x)\partial_\mu j_a^\mu(\phi) + o(\partial\partial\varepsilon) + o(\varepsilon^2)$$

This expression is independent of the choice of ϕ. Moreover, note that Eq. 1.61 can be rewritten as an internal transformation by setting:

$$\phi_i(x) \mapsto \phi_i'(x) = \phi_i(x - \varepsilon^a A_a) + \varepsilon^a F_{i,a} = \phi_i(x) + \varepsilon^a F_{i,a} - \varepsilon^a A_a^\mu \partial_\mu \phi_i \equiv \phi_i(x) + \delta\phi_i(x)$$

$\delta\phi_i(x)$ vanishes at infinity, therefore it is the kind of variation used to derive the equations of motion: choosing $\phi \equiv \phi^{\mathrm{cl}}$ classical solution then implies $\delta\mathcal{S} = 0$ independently of ε, i.e. the thesis. $\qquad\square$

These are often called *Noether currents*, and the associated *Noether charges* are defined as:

$$Q_a := \int \mathrm{d}^3x \, j_a^0(t, \mathbf{x}) \tag{1.63}$$

These are time-independent, as $\partial_0 Q_a = \int \mathrm{d}^3x \, \partial_0 j_a^0 = -\int \mathrm{d}^3x \, \partial_i j_a^i$: on all spacetime it vanishes by divergence theorem (fields vanish at infinity), but on a finite volume it yields a boundary term interpreted as the incoming and outgoing flux.

Proposition 1.3.1: Noether currents

The explicit expression of Noether currents is:

$$j_a^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left[A_a^\nu(x)\partial_\nu\phi_i - F_{i,a}(\phi, \partial\phi)\right] - A_a^\mu(x)\mathcal{L} \tag{1.64}$$

Proof. Varying the action at $o(\partial\varepsilon)$:

$$\delta_\varepsilon\mathcal{S} = \delta_\varepsilon \int \mathrm{d}^4x\,\mathcal{L} = \int \left[\delta_\varepsilon(\mathrm{d}^4x)\mathcal{L} + \mathrm{d}^4x\left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta_\varepsilon\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta_\varepsilon(\partial_\mu\phi_i)\right)\right]$$

The Jacobian of Eq. 1.61 gives $\mathrm{d}^4x \mapsto \mathrm{d}^4x\,(1 + A_a^\mu\partial_\mu\varepsilon^a) + o(\varepsilon)$, while $\delta_\varepsilon\phi_i$ is not $o(\partial\varepsilon)$ and:

$$\delta_\varepsilon(\partial_\mu\phi_i) = \frac{\partial\phi_i'}{\partial x'^\mu} - \frac{\partial\phi_i}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}(\phi_i + \varepsilon^a F_{i,a}) - \frac{\partial\phi_i}{\partial x^\mu} = -(\partial_\mu\varepsilon^a)(A_a^\nu\partial_\nu\phi_i - F_{i,a}) + o(\varepsilon)$$

The thesis follows from $\delta_\varepsilon\mathcal{S} = -\int \mathrm{d}^4x\,(\partial_\mu\varepsilon^a)j_a^\mu + o(\partial\partial\varepsilon) + o(\varepsilon^2)$. $\qquad\square$

If the considered infinitesimal transformation is not a global symmetry, then $\delta_\varepsilon \mathcal{S}$ has a non-vanishing $o(\varepsilon)$ term which gives rise to a quasi-conserved current:

$$\partial_\mu j^a_\mu = -(\delta_a \mathcal{L})_{\text{global}} \tag{1.65}$$

1.3.1.1   Energy-momentum tensor

Consider spacetime translations: as all fields must be scalars under these transformations, they define a Noether current. In particular, translations have $A^\mu_\nu = \delta^\mu_\nu$ and $F_{i,\mu} = 0$ (the parameter index is a Lorentz index), so the conserved current is the *energy-momentum tensor* $j^\mu_\nu \equiv \theta^\mu{}_\nu$:

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \eta^{\mu\nu} \mathcal{L} \tag{1.66}$$

which is covariantly conserved on classical solutions of the equations of motion. The conserved Noether charge associated to the energy-momentum tensor is the *four-momentum*:

$$P^\mu := \int \mathrm{d}^3 x \, \theta^{0\mu} \tag{1.67}$$

The energy-momentum tensor so defined is not symmetric, however it can be made to via a tensor $A^{\rho\mu\nu}$ which is anti-symmetric w.r.t. (ρ, μ): $T^{\mu\nu} \equiv \theta^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$ is physically equivalent from $\theta^{\mu\nu}$, as the second term is a vanishing spatial divergence in the definition of P^μ and is contracted to 0 in the conservation law.

1.3.2   Scalar fields

1.3.2.1   Real scalar fields

Consider a real scalar field ϕ: a non-trivial Poincaré-invariant action must contain $\partial_\mu \phi$ and must saturate each Lorentz index. For example:

$$\mathcal{S}[\phi] = \frac{1}{2} \int \mathrm{d}^4 x \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \tag{1.68}$$

The resulting equation of motion is the *Klein-Gordon equation*:

$$\left(\Box + m^2 \right) \phi = 0 \tag{1.69}$$

where $\Box \equiv \partial_\mu \partial^\mu$. A plane wave $e^{\pm i p_\mu x^\mu}$ is a solution if $p^2 = m^2$, so the KG equation imposes the relativistic dispersion relation and m can be interpreted as the mass. As ϕ must be real, the general solution is a superposition of waves:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \left[a_\mathbf{p} e^{-i p_\mu x^\mu} + a^*_\mathbf{p} e^{i p_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \tag{1.70}$$

The positive energy solution has $E_\mathbf{p} = +\sqrt{\mathbf{p}^2 + m^2}$, but it contains both *positive* and *negative frequency modes* $e^{\mp i p_\mu x^\mu}$, while the $(2E_\mathbf{p})^{-1/2}$ factor is a convenient normalization of the $a_\mathbf{p}$ coefficients. The overall normalization of $\mathcal{S}[\phi]$ does not influence the equations of motion, however it is important

for obtaining a positive-definite Hamiltonian. As the momentum conjugate to ϕ is $\Pi_\phi = \partial_0 \phi$, the Hamiltonian density is found as:

$$\mathcal{H} = \frac{1}{2} \left[\Pi_\phi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 \right] \tag{1.71}$$

The energy-momentum tensor is computed to be:

$$\theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{1.72}$$

It is trivial to see that $\theta^{00} = \mathcal{H}$: the Hamiltonian is the conserved charge related to the invariance under time translations.

To compute the conserved currents associated to Lorentz invariance, it is convenient to label the transformation parameters $\omega^{\mu\nu}$ by an anti-symmetric pair of Lorentz indices, so that Eq. 1.61 become:

$$x^\mu \mapsto x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}\left(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho\right) \equiv x^\mu + \frac{1}{2}A^\mu_{(\rho\sigma)}\omega^{\rho\sigma}$$

As $F_{i,a} = 0$, from Eq. 1.64 the conserved currents are:

$$j^{(\rho\sigma)\mu} = x^\rho \theta^{\mu\sigma} - x^\sigma \theta^{\mu\rho} \tag{1.73}$$

For spatial rotations, the conserved charge is:

$$M^{ij} = \int \mathrm{d}^3x \left(x^i\theta^{0j} - x^j\theta^{0i}\right) = \int \mathrm{d}^3x\, \partial_0\phi\left(x^i\partial^j - x^j\partial^i\right)\phi = \frac{i}{2}\int \mathrm{d}^3x \left[\phi L^{ij}(\partial_0\phi) - (\partial_0\phi)L^{ij}\phi\right]$$

where integration by parts was carried and L^{ij} is defined by Eq. 1.44. This can be generalized.

Definition 1.3.3: Scalar product

Given two real scalar fields ϕ_1 and ϕ_2, their *scalar product* is defined as:

$$\langle\phi_1|\phi_2\rangle := \frac{i}{2}\int \mathrm{d}^3x\, \phi_1\overleftrightarrow{\partial_0}\phi_2 \tag{1.74}$$

where $f\overleftrightarrow{\partial_\mu}g := f\partial_\mu g - (\partial_\mu f)g$.

Proposition 1.3.2

If ϕ_1 and ϕ_2 are KG solutions, then $\langle\phi_1|\phi_2\rangle$ is time-independent.

- -

Proof. By the KG equation:

$$\partial_0\left[\phi_1\partial_0\phi_2 - (\partial_0\phi_1)\phi_2\right] = \phi_1\partial_0^2\phi_2 - (\partial_0^2\phi_1)\phi_2 = \phi_1\boldsymbol{\nabla}^2\phi_2 - (\boldsymbol{\nabla}^2\phi_1)\phi_2$$

which vanishes after integration by parts. $\qquad\square$

Note that this scalar product is not positive-definite.

Theorem 1.3.3: Conserved charges

Given a symmetry represented by a Lie group and a representation $L^{\mu\nu}$ of its generators as operators acting on fields, the value of the associated conserved charges on a solution ϕ of the equations of motion is:
$$M^{\mu\nu} = \langle \phi | L^{\mu\nu} | \phi \rangle \tag{1.75}$$

Example 1.3.1: Four-momentum

Applying Th. 1.3.3 to four-momentum $P^\mu = \langle \phi | i \partial^\mu | \phi \rangle$; for example, the $\mu = 0$ component is:

$$P^0 = \langle \phi | i \partial^0 | \phi \rangle = \langle \phi | i \partial_0 | \phi \rangle = \frac{i}{2} \int \mathrm{d}^3 x \left[\phi (i \partial_0) \partial_0 \phi - (\partial_0 \phi) i \partial_0 \phi \right]$$

$$= \frac{1}{2} \int \mathrm{d}^3 x \left[-\phi \partial_0^2 \phi + (\partial_0 \phi)^2 \right] = \frac{1}{2} \int \mathrm{d}^3 x \left[-\phi \left(\nabla^2 - m^2 \right) \phi + (\partial_0 \phi)^2 \right]$$

$$\text{(int. by parts)} = \frac{1}{2} \int \mathrm{d}^3 x \left[(\boldsymbol{\nabla} \phi)^2 + m^2 \phi^2 + (\partial_0 \phi)^2 \right] = \frac{1}{2} \int \mathrm{d}^3 x \, \theta^{00}$$

The free KG action can be generalized to a self-interacting real scalar field introducing a general potential:
$$\mathcal{S}[\phi] = \int \mathrm{d}^4 x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \tag{1.76}$$

The quadratic term in $V(\phi)$ is the mass term, while higher-order terms describe the self-interaction.

1.3.2.2 Complex scalar fields

Consider now two real scalar fields ϕ_1, ϕ_2 with the same mass m and combine them into a single complex scalar field $\phi \equiv \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$. The KG action is the sum of the single actions and may be written as:
$$\mathcal{S}[\phi] = \int \mathrm{d}^4 x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \tag{1.77}$$

Considering ϕ and ϕ^* as independent variables one obtains the KG equation, which yields the same mode expansion as Eq. 1.70:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2 E_\mathbf{p}}} \left[a_\mathbf{p} e^{-i p_\mu x^\mu} + b_\mathbf{p}^* e^{i p_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \tag{1.78}$$

Now $a_\mathbf{p}$ and $b_\mathbf{p}$ are independent, since there's no reality condition on ϕ.

Electric charge   An interesting property of the complex KG field is the existence of a global U(1) symmetry of the action: $\mathcal{S}[\phi] \mapsto \mathcal{S}[\phi]$ under $\phi(x) \mapsto e^{i\theta} \phi(x)$. The associated Noether current can be computed from Eq. 1.64, using $\phi_i = (\phi, \phi^*)$, $A_a^\nu = 0$ and $F_{i,a} = (i, -i)$:

$$j_\mu = -i \left(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi \right) = i \phi^* \overleftrightarrow{\partial_\mu} \phi \tag{1.79}$$

The conserved U(1) charge is then $Q_{\mathrm{U(1)}} = i \int \mathrm{d}^3 x \, \phi^* \overleftrightarrow{\partial_0} \phi = \langle \phi | \phi \rangle$, which is consistent with Th. 1.3.3 as the generator of U(1) is the identity operator.

1.3.3   Spinor fields

1.3.3.1   Weyl fields

Consider the theory of a left/right-handed Weyl field: recalling that $\psi_L^\dagger \sigma^\mu \psi_L$ and $\psi_R \bar\sigma^\mu \psi_R$ are 4-vectors, with $\sigma^\mu \equiv (1, \boldsymbol\sigma)$ and $\bar\sigma^\mu \equiv (1, -\boldsymbol\sigma)$, the kinetic term of the Lorentz-invariant lagrangian can be written as:

$$\mathcal{L}_L = i\psi_L^\dagger \bar\sigma^\mu \partial_\mu \psi_L \qquad\qquad \mathcal{L}_R = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \qquad\qquad (1.80)$$

The i factor ensures the reality of the Lagrangian, as σ matrices are hermitian. The Lagrangian does not depend on $\partial_\mu \psi^*$, thus the Euler-Lagrange equations are:

$$(\partial_0 - \sigma^i \partial_i)\psi_L = 0 \qquad\qquad (\partial_0 + \sigma^i \partial_i)\psi_R = 0 \qquad\qquad (1.81)$$

These are known as *Weyl equations*: by $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, these equations imply two massless KG equations (assuming regular functions, so that $\partial_i \partial_j = \partial_j \partial_i$). Considering positive-energy plane-wave solutions of the form $\psi_L(x) = u_L \exp(-ip_\mu x^\mu) = u_L \exp(-iEt + i\mathbf{p}\cdot\mathbf{x})$, where u_L is a constant spinor, Weyl equations become:

$$\frac{\mathbf{p}\cdot\boldsymbol\sigma}{E}u_L = -u_L \qquad\qquad \frac{\mathbf{p}\cdot\boldsymbol\sigma}{E}u_R = u_R$$

As these are massless fields $E = |\mathbf{p}|$, and since for $s = \frac{1}{2}$ fields $\mathbf{J} = \frac{\boldsymbol\sigma}{2}$ these equations become:

$$(\hat{\mathbf{p}}\cdot\mathbf{J})u_L = -\frac{1}{2}u_L \qquad\qquad (\hat{\mathbf{p}}\cdot\mathbf{J})u_R = \frac{1}{2}u_R$$

These show that left/right-handed Weyl massless Weyl spinors have helicity $h = \mp\frac{1}{2}$, consistent with the fact that massless particles are helicity eigenstates.

The energy-momentum tensor can be computed from Eq. 1.66, noting that on classical solutions (by Weyl equations) the Lagrangian of the theory vanishes:

$$\theta^{\mu\nu} = i\psi_L^\dagger \bar\sigma^\mu \partial^\nu \psi_L \qquad\qquad \theta^{\mu\nu} = i\psi_R^\dagger \sigma^\mu \partial^\nu \psi_R \qquad\qquad (1.82)$$

Moreover, note that the Lagrangian is invariant under a global U(1) internal transformation with Noether currents and conserved charges:

$$j^\mu = \psi_L^\dagger \bar\sigma^\mu \psi_L \qquad\qquad\qquad\qquad j^\mu = \psi_R^\dagger \sigma^\mu \psi_R$$

$$Q_{U(1)} = \int \mathrm{d}^3 x\, \psi_L^\dagger \psi_L \qquad\qquad\qquad\qquad Q_{U(1)} = \int \mathrm{d}^3 x\, \psi_R^\dagger \psi_R$$

Weyl Lagrangians are not invariant under parity, as $\psi_L \leftrightarrow \psi_R$.

> #### Example 1.3.2: Neutrinos
>
> Neutrinos are divided into three leptonic families: ν_e, ν_μ and ν_τ. Although they are massive $s = \frac{1}{2}$, in most contexts their mass can be neglected, thus they can be described by Weyl spinors: in particular, neutrinos by left-handed Weyl spinors and antineutrinos by right-handed Weyl spinors.

1.3.3.2   Dirac fields

Consider a theory with both a left-handed and a right-handed Weyl spinor: one can construct two new Lorentz scalars, $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$, as from Eqq. 1.40-1.41 it is easy to check that $\Lambda_L^\dagger \Lambda_R = \Lambda_R^\dagger \Lambda_L = \text{id}$. Two real combinations are $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ and $i(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L)$: under parity $\psi_L \leftrightarrow \psi_R$, so the former is a scalar and the latter a pseudoscalar. The Dirac Lagrangian then is:

$$\mathcal{L}_D = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \tag{1.83}$$

This Lagrangian is invariant under parity, as $\bar{\sigma}^\mu \partial_\mu \leftrightarrow \sigma^\mu \partial_\mu$ ($\partial_i \mapsto -\partial_i$). Considering ψ_i and ψ_i^* independent, the variation w.r.t. the latter yields:

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = m\psi_R \qquad\qquad i\sigma^\mu \partial_\mu \psi_R = m\psi_L \tag{1.84}$$

which is the *Dirac equation* in terms of Weyl spinors.

Proposition 1.3.3

Dirac equation implies two massive Klein-Gordon equations.

- -

Proof. Applying $i\sigma^\mu \partial_\mu$ to the first equation:

$$-\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu \psi_L = mi\sigma^\mu \partial_\mu \psi_R = m^2 \psi_L$$

Assuming $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, $\sigma^\mu \bar{\sigma}^\nu$ can be replaced by $\frac{1}{2}(\sigma^\nu \bar{\sigma}^\mu + \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2}(2\eta^{\mu\nu}) = \eta^{\mu\nu}$, yielding the thesis. $\qquad\square$

The m parameter can thus be interpreted as a mass, and now the two spinors are no longer helicity eigenstates. It is convenient to rewrite this theory in terms of a Dirac spinor, which defines the chiral representation:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

In this representation, the four γ matrices are introduced:

$$\gamma^\mu := \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$$

Proposition 1.3.4: Clifford algebra

The γ matrices satisfy the *Clifford algebra*:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1.85}$$

Introducing the Feynman slash notation, so that $\slashed{A} \equiv \gamma^\mu A_\mu$, the Dirac equation can be rewritten as:

$$(i\slashed{\partial} - m)\Psi = 0 \tag{1.86}$$

Defining the *Dirac adjoint* $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$, which in the chiral representation simply is $\bar{\Psi} = (\psi_R^\dagger, \psi_L^\dagger)$, the Dirac Lagrangian can be rewritten as:

$$\mathcal{L}_D = \bar{\Psi}(i\slashed{\partial} - m)\Psi \tag{1.87}$$

Another γ matrix is defined as $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, which in the chiral representation is:

$$\gamma^5 = \begin{bmatrix} -\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{bmatrix}$$

This hermitian matrix has properties $(\gamma^5)^2 = \mathrm{I}_4$ and $\{\gamma^5, \gamma^\mu\} = 0$, and allows to define projectors on Weyl spinors:

$$\frac{1-\gamma^5}{2}\Psi = \begin{pmatrix} \psi_\mathrm{L} \\ 0 \end{pmatrix} \qquad\qquad \frac{1+\gamma^5}{2}\Psi = \begin{pmatrix} 0 \\ \psi_\mathrm{R} \end{pmatrix} \qquad\qquad (1.88)$$

> ### Proposition 1.3.5: Unitary equivalence
>
> Given a constant $U \in \mathbb{C}^{2\times2} : U^\dagger = U^{-1}$, then under $\Psi \mapsto U\Psi$ the Lagrangian is invariant for $\gamma^\mu \mapsto U\gamma^\mu U^\dagger$.
>
> -
>
> *Proof.* Inserting $\Psi' = U^\dagger\Psi$ in the Dirac Lagrangian:
>
> $$\mathcal{L}'_\mathrm{D} = \Psi'^\dagger U\gamma^0(i\gamma^\mu\partial_\mu - m)U^\dagger\Psi' = \Psi'^\dagger U\gamma^0 U^\dagger(iU\gamma^\mu U^\dagger\partial_\mu - m)\Psi' = \bar{\Psi}'(i\gamma'^\mu\partial_\mu - m)\Psi'$$
>
> where $\bar{\Psi}' = \Psi'^\dagger\gamma'^0$. The Lagrangian is thus unchanged. $\qquad\qquad\square$

It can be shown that the Clifford algebra is invariant under the transformation in Prop. 1.3.5, thus it defines an equivalent representation of the algebra.

Solutions   The general solution to the Dirac equation is a superposition of plane waves, both with positive and negative-frequency modes:

$$\Psi(x) = u(p)e^{-ip_\mu x^\mu} \qquad\qquad \Psi(x) = v(p)e^{ip_\mu x^\mu}$$

where $u(p)$ and $v(p)$ are Dirac spinors, which in the chiral representation both have a left-handed and a right-handed Weyl spinor component. The Dirac equation than becomes:

$$\left(\not{p} - m\right)u(p) = 0 \qquad\qquad \left(\not{p} + m\right)v(p) = 0$$

Considering the massive positive-frequency solution in the rest frame, i.e. $p^\mu = (m, \mathbf{0})$, one finds $(\gamma^0 - 1)u(p) = 0$, which yields $u_\mathrm{L} = u_\mathrm{R}$: the KG equation imposes the mass-shell condition $p^2 = m^2$, but the Dirac equation, being a first-order equation, halves the number of independent degrees of freedom. A convenient normalization is:

$$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

where $\xi : \xi^\dagger\xi = 1$ is a two-component spinor which gives the spin orientation of the solution: $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the spin-up solution and $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the spin-down one.
The solution in a generic frame is obtained via a boost of the rest-frame solution, using transformation properties of left/right-handed spinors.

Proposition 1.3.6

The solution to the Dirac equation in a generic frame where $\mathbf{p} \parallel \mathbf{e}_z$ is given by:

$$u(p) = \begin{pmatrix} \begin{bmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{bmatrix} \xi \\ \begin{bmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{bmatrix} \xi \end{pmatrix} \tag{1.89}$$

Proof. Consider a boost along $\mathbf{e}_z$, parametrized by rapidity η, by Eq. 1.36:

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \exp\left(\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{pmatrix} m \\ 0 \end{pmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cosh \eta + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sinh \eta \right) \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ m \sinh \eta \end{pmatrix}$$

Thus, $E + p^3 = e^\eta m$ and $E - p^3 = e^{-\eta} m$. By Eqq. 1.40-1.41, then:

$$u(p) = \exp\left(-\frac{\eta}{2} \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix} \right) \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cosh \frac{\eta}{2} - \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix} \sinh \frac{\eta}{2} \right) \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{bmatrix} e^{\eta/2} \frac{1 - \sigma^3}{2} + e^{-\eta/2} \frac{1 + \sigma^3}{2} & 0 \\ 0 & e^{\eta/2} \frac{1 + \sigma^3}{2} + e^{-\eta/2} \frac{1 - \sigma^3}{2} \end{bmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{bmatrix} \sqrt{E + p^3} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{E - p^3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \sqrt{E + p^3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sqrt{E - p^3} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$\square$

This expression can be generalized for an arbitrary direction of $\mathbf{p}$:

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi^s \end{pmatrix} \tag{1.90}$$

where the square root of a matrix is understood as taking the positive square root of its eigenvalues, and the polarization of ξ^s is made explicit.

It is convenient to work with specific spinors ξ: a useful choice are eigenvectors of σ^3, in particular $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin-up) and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (spin-down). In the ultra-relativistic (or massless) limit $p^\mu \to (E, 0, 0, E)$, so u^1 has only the right-handed component and u^2 only the left-handed one.

The negative-frequency solution is equivalent:

$$v^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \eta^s \end{pmatrix} \tag{1.91}$$

where the spin states are the same: $\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Recalling the Dirac adjoint $\bar{u}^s(p) = u^{s\dagger}(p)\gamma^0$, $\bar{v}^s(p) = v^{s\dagger}(p)\gamma^0$ and the normalization choice $\xi^{r\dagger}\xi^s = \delta^{rs}$, $\eta^{r\dagger}\eta^s = \delta^{rs}$, several properties follow:

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \qquad \qquad \bar{v}^r(p)v^s(p) = -2m\delta^{rs} \tag{1.92}$$

$$u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs} \qquad \qquad v^{r\dagger}(p)v^s(p) = 2E_{\mathbf{p}}\delta^{rs} \tag{1.93}$$

$$\bar{u}^r(p)v^s(p) = 0 \qquad\qquad \bar{v}^r(p)u^s(p) = 0 \tag{1.94}$$

$$\bar{u}^r(p)\gamma^\mu u^s(p) = 2p^\mu\delta^{rs} \qquad\qquad \bar{v}^r(p)\gamma^\mu v^s(p) = 2p^\mu\delta^{rs} \tag{1.95}$$

Note that $\bar{u}u \in \mathbb{C}$, while $u\bar{u} \in \mathbb{C}^{4\times4}$.

Proposition 1.3.7: Spinor sums

The sum over possible polarizations of a fermion yields:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \qquad\qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \tag{1.96}$$

Proof. Using $\sum_{s=1,2}\xi^s\xi^{s\dagger} = \mathrm{I}_2$:

$$\sum_s u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\xi^s \\ \sqrt{p_\mu\bar{\sigma}^\mu}\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p_\mu\bar{\sigma}^\mu} & \xi^{s\dagger}\sqrt{p_\mu\sigma^\mu} \end{pmatrix}$$

$$= \begin{bmatrix} \sqrt{p_\mu\sigma^\mu}\sqrt{p_\mu\bar{\sigma}^\mu} & \sqrt{p_\mu\sigma^\mu}\sqrt{p_\mu\sigma^\mu} \\ \sqrt{p_\mu\bar{\sigma}^\mu}\sqrt{p_\mu\bar{\sigma}^\mu} & \sqrt{p_\mu\bar{\sigma}^\mu}\sqrt{p_\mu\sigma^\mu} \end{bmatrix} = \begin{bmatrix} m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & m \end{bmatrix}$$

where the identity $(p_\mu\sigma^\mu)(p_\mu\bar{\sigma}^\mu) = p^2 = m^2$ was used. □

Dirac bilinears   Expressions of the form $\bar{\Psi}\Gamma\Psi$, with constant $\Gamma \in \mathbb{C}^{4\times4}$, are known as *Dirac bilinears*, and their properties can be studied by expressing Γ on a particular basis of $\mathbb{C}^{4\times4}$, given by the 16 matrices:

I_4	1 matrix
γ^μ	4 matrices
$\gamma^{\mu\nu} \equiv \dfrac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} \equiv -i\sigma^{\mu\nu}$	6 matrices
$\gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$	4 matrices
$\gamma^{\mu\nu\rho\sigma} \equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$	1 matrix

The most general fermion bilinear is a linear combination of these basis bilinears:

$$\bar{\Psi}\Psi \qquad \bar{\Psi}\gamma^\mu\Psi \qquad \bar{\Psi}\gamma^5\Psi \qquad \bar{\Psi}\gamma^5\gamma^\mu\Psi \qquad \bar{\Psi}\sigma^{\mu\nu}\Psi$$

According to their transformation relations under the Lorentz group, these bilears are respectively called scalar, vector, pseudo-scalar, pseudo-vector and tensor bilinear.

Proposition 1.3.8: Lorentz transformations

In the chiral representation, the Lorentz transformation $\Psi \mapsto \Lambda_\mathrm{D}\Psi$ is represented as:

$$\Lambda_\mathrm{D} = \exp\left[-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right] \tag{1.97}$$

This representation of the Lorentz transformation obeys unitary equivalence (Prop. 1.3.5).

Chiral symmetry   Consider the massless Dirac Lagrangian. In this case, the action has two global internal symmetries:

$$\psi_{\mathrm{L}} \mapsto e^{i\theta_{\mathrm{L}}}\psi_{\mathrm{L}} \qquad\qquad \psi_{\mathrm{R}} \mapsto e^{i\theta_{\mathrm{R}}}\psi_{\mathrm{R}}$$

Therefore, this theory is symmetric under $\mathrm{U}(1) \times \mathrm{U}(1)$. These can be rewritten as two distinct transformations of the Dirac spinor:

$$\Psi \mapsto e^{i\alpha}\Psi \tag{1.98}$$

$$\Psi \mapsto e^{i\beta\gamma^5}\Psi \tag{1.99}$$

where $\alpha \equiv \theta_{\mathrm{L}} = \theta_{\mathrm{R}}$ and $\beta \equiv -\theta_{\mathrm{L}} = \theta_{\mathrm{R}}$ respectively (to prove that the latter is a symmetry, from $\{\gamma^5, \gamma^\mu\} = 0$ it follows that $\gamma^\mu e^{i\beta\gamma^5} = e^{-i\beta\gamma^5}\gamma^\mu$). The former symmetry is called the vector $\mathrm{U}(1)_{\mathrm{V}}$ symmetry, with conserved Noether current the *vector current* $j_{\mathrm{V}}^\mu := \bar{\Psi}\gamma^\mu\Psi$, while the latter is called the axial $\mathrm{U}(1)_{\mathrm{A}}$ symmetry, with conserved *axial current* $j_{\mathrm{A}}^\mu := \bar{\Psi}\gamma^\mu\gamma^5\Psi$.

While the vector symmetry is always a symmetry of the Dirac field, the axial symmetry is only for the massless Dirac field, as:

$$\partial_\mu j_{\mathrm{A}}^\mu = 2im\bar{\Psi}\gamma^5\Psi \tag{1.100}$$

Chapter 2

Quantization of Free Fields

2.1 Scalar fields

As for the quantization of a classical system in Quantum Mechanics, the quantization of a scalar field theory is performed promoting $\phi(t, \mathbf{x})$ and $\Pi(t, \mathbf{x})$ to hermitian operators in the Heisenberg picture and imposing the canonical equal-time commutation relation:

$$[\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{2.1}$$

while of course $[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = 0$.

2.1.1 Real scalar fields

A real scalar field is promoted to a real hermitian operator. In particular, by Eq. 1.70:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} + a_{\mathbf{p}}^\dagger e^{ip_\mu x^\mu} \right]_{p^0 = E_{\mathbf{p}}} \tag{2.2}$$

In terms of creation and annihilation operators, the commutator Eq. 2.1 reads:

$$[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{2.3}$$

while $[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0$. These can be regarded as the creation and annihilation operators of a collection of harmonic oscillators, one for each value of the momentum $\mathbf{p}$: the *Fock space* of the real scalar field can thus be constructed analogously to the Hilbert space of the harmonic oscillator. Defining the *vacuum state* $|0\rangle : a_{\mathbf{p}} |0\rangle = 0 \ \forall \mathbf{p}$, suitably normalized as $\langle 0|0\rangle = 1$, the generic state of the Fock space is:

$$|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} \, a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \tag{2.4}$$

Proposition 2.1.1: Normalization

For one-particle states:

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = 2E_{\mathbf{p}_1}(2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \tag{2.5}$$

Proof. By Eq. 2.3:

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = \sqrt{2E_{\mathbf{p}_1}} \sqrt{2E_{\mathbf{p}_2}} \langle 0| a_{\mathbf{p}_1} a_{\mathbf{p}_2}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}_1}} \sqrt{2E_{\mathbf{p}_2}} \langle 0|[a_{\mathbf{p}_1}, a_{\mathbf{p}_2}^\dagger]|0\rangle = 2E_{\mathbf{p}_1}(2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)$$

$\square$

Lemma 2.1.1

The combination $E_{\mathbf{p}}\delta^{(3)}(\mathbf{p} - \mathbf{q})$ is Lorentz invariant.

Proof. Recall that:

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0) \tag{2.6}$$

A Lorentz boost along $\mathbf{e}_i$ yields $p'_i = \gamma(p_i + \beta E)$ and $E' = \gamma(E + \beta p_i)$, so:

$$\delta^{(3)}(\mathbf{p} - \mathbf{q}) = \delta^{(3)}(\mathbf{p}' - \mathbf{q}')\frac{\mathrm{d}p'_i}{\mathrm{d}p_i} = \delta^{(3)}(\mathbf{p}' - \mathbf{q}')\gamma\left(1 + \beta\frac{\mathrm{d}E}{\mathrm{d}p_i}\right)$$

Note that $p^2 = E^2 - \mathbf{p}^2$ is a Lorentz invariant, thus $E\mathrm{d}E = p_i\mathrm{d}p_i$, so:

$$\delta^{(3)}(\mathbf{p} - \mathbf{q}) = \delta^{(3)}(\mathbf{p}' - \mathbf{q}')\frac{\gamma(E + \beta p_i)}{E} = \delta^{(3)}(\mathbf{p}' - \mathbf{q}')\frac{E'}{E}$$

$\square$

This explains the choice of normalization.

Proposition 2.1.2: KG Hamiltonian

The Hamiltonian of a real scalar field can be written as:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\mathbf{p}}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]\right) \tag{2.7}$$

Proof. First of all, from Eq. 2.2:

$$\Pi(t, \mathbf{x}) = \partial_0\phi(t, \mathbf{x}) = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-ip_\mu x^\mu} - a_{\mathbf{p}}^\dagger e^{ip_\mu x^\mu}\right]_{p^0 = E_{\mathbf{p}}}$$

$$= -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\boldsymbol{\nabla}\phi(t, \mathbf{x}) = i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]e^{i\mathbf{p}\cdot\mathbf{x}}$$

Inserting these expressions in Eq. 1.71:

$$\mathcal{H} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\left\{-\frac{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}{2}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-iE_{\mathbf{q}}t} - a_{-\mathbf{q}}^\dagger e^{iE_{\mathbf{q}}t}\right] + \right.$$

$$\left. + \frac{-\mathbf{p}\cdot\mathbf{q} + m^2}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-iE_{\mathbf{q}}t} + a_{-\mathbf{q}}^\dagger e^{iE_{\mathbf{q}}t}\right]\right\}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}$$

Recall the identity:

$$\int \frac{\mathrm{d}^3x}{(2\pi)^3}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{2.8}$$

Then (using $E_{-\mathbf{p}} = E_{\mathbf{p}}$):

$$H = \int \mathrm{d}^3 x\, \mathcal{H}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \mathrm{d}^3 q \left\{ -\frac{\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}}{2} \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}} t} - a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t} \right] \left[a_{\mathbf{q}} e^{-iE_{\mathbf{q}} t} - a_{-\mathbf{q}}^\dagger e^{iE_{\mathbf{q}} t} \right] + \right.$$

$$\left. + \frac{-\mathbf{p}\cdot\mathbf{q} + m^2}{2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}} \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}} t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t} \right] \left[a_{\mathbf{q}} e^{-iE_{\mathbf{q}} t} + a_{-\mathbf{q}}^\dagger e^{iE_{\mathbf{q}} t} \right] \right\} \delta^{(3)}(\mathbf{p} + \mathbf{q})$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ -\frac{E_{\mathbf{p}}}{2} \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}} t} - a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t} \right] \left[a_{-\mathbf{p}} e^{-iE_{\mathbf{p}} t} - a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t} \right] + \right.$$

$$\left. + \frac{\mathbf{p}^2 + m^2}{2E_{\mathbf{p}}} \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}} t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t} \right] \left[a_{-\mathbf{p}} e^{-iE_{\mathbf{p}} t} + a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t} \right] \right\}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \left\{ -\left[a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}} t} - a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}} t} \right] \right.$$

$$\left. + \left[a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}} t} + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}} t} \right] \right\}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} \left[a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right] = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right)$$

$\square$

The second term in the Hamiltonian Eq. 2.7 is the sum of the zero-point energy of all oscillators and is proportional to $(2\pi)^3 \delta^{(3)}(0) \to V$, thus:

$$E_{\mathrm{vac}} = \frac{V}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}}$$

This energy shows two divergences: the one coming from the infinite-volume limit (i.e. small momentum), regularized introducing an *infrared cutoff* in the form of a finite volume, and the one from the ultra-relativistic limit (i.e. large momentum), regularized introducing an *ultraviolet cutoff* in the form of a maximum momentum Λ. These divergences are retained in the expression for E_{vac}, as $E_{\mathrm{vac}} \sim V$ and $E_{\mathrm{vac}} \sim \Lambda^4$, but can be ignored (when ignoring gravity) since experiments are only sensitive to energy differences.

Discarding the zero-point energy, the Hamiltonian becomes:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \equiv \mathcal{N} \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right) \tag{2.9}$$

where the *normal ordering operator* $\mathcal{N}$ was introduced, which acts by moving all creation operators to the left and all annihilation operators to the right (ex.: $\mathcal{N} a_{\mathbf{p}} a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$). It is now straightforward to compute the energy of a generic state in the Fock space, as $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ is just a number operator:

$$H \left| \mathbf{p}_1, \ldots, \mathbf{p}_n \right\rangle = \left(E_{\mathbf{p}_1} + \cdots + E_{\mathbf{p}_n} \right) \left| \mathbf{p}_1, \ldots, \mathbf{p}_n \right\rangle \tag{2.10}$$

Computing the spatial momentum from Eq. 1.67 as $P^i = \mathcal{N} \int \mathrm{d}^3 x\, \theta^{0i} = \int \mathrm{d}^3 x\, \mathcal{N} \partial_0 \phi \partial^i \phi$:

$$P^i = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^i a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \tag{2.11}$$

Therefore, the state $a_{\mathbf{p}}^{\dagger}|0\rangle$ can correctly be interpreted as a one-particle state with momentum $\mathbf{p}$, mass m and energy $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. The generic state in the Fock space is a multiparticle state, with total energy and momentum the sum of the individual energies and momenta.

Finally, note that creation operators commute between themselves, hence multiparticle states are symmetric under exchange of pairs of particles, i.e. they obey the Bose-Einstein statistics: this agrees with the fact that quanta of a scalar field have no intrinsic spin. i.e. are spin-0 particles.

2.1.2   Complex scalar fields

When considering a complex scalar field, Eq. 1.78 becomes:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b_{\mathbf{p}}^{\dagger} e^{ip_\mu x^\mu} \right]_{p^0 = E_{\mathbf{p}}} \tag{2.12}$$

$$\phi^{\dagger}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}}^{\dagger} e^{ip_\mu x^\mu} + b_{\mathbf{p}} e^{-ip_\mu x^\mu} \right]_{p^0 = E_{\mathbf{p}}} \tag{2.13}$$

Now there are two independent sets of creation/annihilation operators, which obey the canonical commutation relation:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] = [b_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{2.14}$$

with all other commutators vanishing. The Fock space is constructed by defining a vacuum state $|0\rangle : a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}}|0\rangle = 0$ and then acting repeatedly with both creation operators. With normal ordering, one finds:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} \right) \tag{2.15}$$

$$P^i = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^i \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} \right) \tag{2.16}$$

The quanta of a complex scalar field are given by two different species of particles with the same mass.

Proposition 2.1.3: U(1) charge

The U(1) charge of the quantized complex scalar field is:

$$Q_{\mathrm{U}(1)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} - b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} \right) \tag{2.17}$$

- -

Proof. By Eq. 1.79:

$$Q_{\mathrm{U}(1)} = i \int \mathrm{d}^3 x \, \phi^{\dagger} \overleftrightarrow{\partial_0} \phi = i \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \times$$

$$\times \left\{ \left[a_{\mathbf{q}}^{\dagger} e^{iq_\mu x^\mu} + b_{\mathbf{q}} e^{-iq_\mu x^\mu} \right] \partial_0 \left(a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b_{\mathbf{p}}^{\dagger} e^{ip_\mu x^\mu} \right) + \right.$$

$$\left. - \partial_0 \left(a_{\mathbf{q}}^{\dagger} e^{iq_\mu x^\mu} + b_{\mathbf{q}} e^{-iq_\mu x^\mu} \right) \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b_{\mathbf{p}}^{\dagger} e^{ip_\mu x^\mu} \right] \right\}$$

$$Q_{U(1)} = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \times$$

$$\times \left\{ E_\mathbf{p} \left[a_\mathbf{q}^\dagger e^{iq_\mu x^\mu} + b_\mathbf{q} e^{-iq_\mu x^\mu} \right] \left[a_\mathbf{p} e^{-ip_\mu x^\mu} - b_\mathbf{p}^\dagger e^{ip_\mu x^\mu} \right] + \right.$$

$$\left. + E_\mathbf{q} \left[a_\mathbf{q}^\dagger e^{iq_\mu x^\mu} - b_\mathbf{q} e^{-iq_\mu x^\mu} \right] \left[a_\mathbf{p} e^{-ip_\mu x^\mu} + b_\mathbf{p}^\dagger e^{ip_\mu x^\mu} \right] \right\}$$

$$= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \times$$

$$\times \left\{ E_\mathbf{p} \left[a_\mathbf{q}^\dagger e^{iE_\mathbf{q}t} + b_{-\mathbf{q}} e^{-iE_\mathbf{q}t} \right] \left[a_\mathbf{p} e^{-iE_\mathbf{p}t} - b_{-\mathbf{p}}^\dagger e^{iE_\mathbf{p}t} \right] + \right.$$

$$\left. + E_\mathbf{q} \left[a_\mathbf{q}^\dagger e^{iE_\mathbf{q}t} - b_{-\mathbf{q}} e^{-iE_\mathbf{q}t} \right] \left[a_\mathbf{p} e^{-iE_\mathbf{p}t} + b_{-\mathbf{p}}^\dagger e^{iE_\mathbf{p}t} \right] \right\} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left\{ \left[a_\mathbf{p}^\dagger e^{iE_\mathbf{p}t} + b_{-\mathbf{p}} e^{-iE_\mathbf{p}t} \right] \left[a_\mathbf{p} e^{-iE_\mathbf{p}t} - b_{-\mathbf{p}}^\dagger e^{iE_\mathbf{p}t} \right] + \right.$$

$$\left. + \left[a_\mathbf{p}^\dagger e^{iE_\mathbf{p}t} - b_{-\mathbf{p}} e^{-iE_\mathbf{p}t} \right] \left[a_\mathbf{p} e^{-iE_\mathbf{p}t} + b_{-\mathbf{p}}^\dagger e^{iE_\mathbf{p}t} \right] \right\}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left[a_\mathbf{p}^\dagger a_\mathbf{p} - b_{-\mathbf{p}} b_{-\mathbf{p}}^\dagger \right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left(a_\mathbf{p}^\dagger a_\mathbf{p} - b_\mathbf{p} b_\mathbf{p}^\dagger \right)$$

Applying normal ordering yields the thesis. $\square$

While normal ordering was justified when considering the Hamiltonian on the grounds that the vacuum energy is unobservable, a charged vacuum would have observable effects; however when promoting ϕ to a quantum operator, the expression $\phi^\dagger \overleftrightarrow{\partial_0} \phi$ presents an ordering ambiguity (ex.: $\phi^\dagger(\partial_0\phi)$ or $(\partial_0\phi)\phi^\dagger$), which is removed requiring the charge of the vacuum to vanish.

Being $a_\mathbf{p}^\dagger a_\mathbf{p}$ and $b_\mathbf{p}^\dagger b_\mathbf{p}$ number operators, the U(1) charge is equal to the number of quanta created by $a_\mathbf{p}^\dagger$ minus the number of quanta created by $b_\mathbf{p}^\dagger$, integrated over all momenta: in particular, $a_\mathbf{p}^\dagger |0\rangle$ and $b_\mathbf{p}^\dagger |0\rangle$ are both spin-zero particles of mass m and momentum $\mathbf{p}$, but they respectively have charges $Q_{U(1)} = +1$ and $Q_{U(1)} = -1$. This allows to properly interpret the negative-energy solutions of the KG equations: they are positive-energy particles with opposite U(1) charge and are called *antiparticles*.

For a real scalar field, the reality condition reads $a_\mathbf{p} = b_\mathbf{p}$, thus it describes a field whose particle is its own antiparticle, and it is symmetric under any U(1) symmetry.

2.2 Spinor fields

A principle of QFT is the *spin-statistic theorem*: integer-spin fields are to be quantized imposing equal-time commutation relations, while half-integer-spin with equal-time anticommutation relations.

2.2.1 Dirac fields

From the Dirac Lagrangian Eq. 1.87, the conjugate momentum to the Dirac field Ψ is computed as:

$$\Pi_\Psi = i\bar{\Psi}\gamma^0 = i\Psi^\dagger \tag{2.18}$$

Imposing the canonical anticommutation relation, according to the spin-statistic theorem:

$$\{\Psi_a(t, \mathbf{x}), \Psi_b^\dagger(t, \mathbf{y})\} = \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{ab} \tag{2.19}$$

where $a, b = 1, 2, 3, 4$ are Dirac indices. Expanding the free Dirac field in plane waves:

$$\Psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{\mathbf{p},s} u^s(p) e^{-ip_\mu x^\mu} + b_{\mathbf{p},s}^\dagger v^s(p) e^{ip_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \tag{2.20}$$

$$\bar{\Psi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{\mathbf{p},s}^\dagger \bar{u}^s(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} \bar{v}^s(p) e^{-ip_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \tag{2.21}$$

where the spinor wave functions $u^s(p), v^s(p)$ are given by Eqq. 1.90-1.91. Translating Eq. 2.19 in terms of creation/annihilation operators:

$$\{a_{\mathbf{p}}, s, a_{\mathbf{q},r}^\dagger\} = \{b_{\mathbf{p},s}, b_{\mathbf{q},r}^\dagger\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta_{sr} \tag{2.22}$$

The Fock space is again constructed defining a vacuum state $|0\rangle : a_{\mathbf{p},s}|0\rangle = b_{\mathbf{p},s}|0\rangle = 0$ and then acting repeatedly on it with $a_{\mathbf{p}.s}^\dagger, b_{\mathbf{p},s}^\dagger$. As these operators anticommute, states in this Fock space are antisymmetric under the exchange of particles, therefore spint-$\frac{1}{2}$ obey the Fermi-Dirac statistics (as of the spin-statistic theorem).

Proposition 2.2.1: Dirac Hamiltonian

The Hamiltoniana for a Dirac field Ψ is:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{s=1,2} E_\mathbf{p} \left[a_{\mathbf{p},s}^\dagger a_{\mathbf{p},s} + b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s} \right] \tag{2.23}$$

- -

Proof. By Eqq. 2.18, the Hamiltonian density is:

$$\mathcal{H} = \Pi_\Psi \partial_0 \Psi - \mathcal{L}_\mathrm{D} = i\Psi^\dagger \partial_0 \Psi - \bar{\Psi} \left(i\gamma^0 \partial_0 + i\gamma^i \partial_i - m \right) \Psi = \bar{\Psi} \left(-i\gamma^i \partial_i + m \right) \Psi$$

Therefore, using Eqq. 2.20-2.21 and Eq. 2.8:

$$H = \int \mathrm{d}^3 x \, \bar{\Psi} \left(-i\gamma^i \partial_i + m \right) \Psi = \int \mathrm{d}^3 x \, \bar{\Psi} \left(-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right) \Psi$$

$$= \mathcal{N} \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_\mathbf{q}}} \sum_{s=1,2} \sum_{r=1,2} \left[a_{\mathbf{p},s}^\dagger \bar{u}^s(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} \bar{v}^s(p) e^{-ip_\mu x^\mu} \right] \times$$

$$\times \left(-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m \right) \left[a_{\mathbf{q},r} u^r(q) e^{-iq_\mu x^\mu} + b_{\mathbf{q},r}^\dagger v^r(q) e^{iq_\mu x^\mu} \right]$$

$$= \mathcal{N} \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_\mathbf{q}}} \sum_{s=1,2} \sum_{r=1,2} \left[a_{\mathbf{p},s}^\dagger \bar{u}^s(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} \bar{v}^s(p) e^{-ip_\mu x^\mu} \right] \times$$

$$\times \left[(\boldsymbol{\gamma} \cdot \mathbf{q} + m) a_{\mathbf{q},r} u^r(q) e^{-iq_\mu x^\mu} + (-\boldsymbol{\gamma} \cdot \mathbf{q} + m) b_{\mathbf{q},r}^\dagger v^r(q) e^{iq_\mu x^\mu} \right]$$

Using the Dirac equation in the form $(\not{p} - m)u(p) = (\not{p} + m)v(p) = 0$:

$$(\boldsymbol{\gamma} \cdot \mathbf{q} + m) u(q) = \gamma^0 E_\mathbf{q} u(q) \qquad (-\boldsymbol{\gamma} \cdot \mathbf{q} + m) v(q) = -\gamma^0 E_\mathbf{q} v(q)$$

Therefore, omitting the constraint $p^0 = E_{\mathbf{p}}$ in the spinors' arguments and using Eq. 2.8:

$$H = \mathcal{N} \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \sum_{s=1,2} \sum_{r=1,2} \left[a_{\mathbf{p},s}^\dagger \bar{u}^s(\mathbf{p}) e^{iE_{\mathbf{p}}t} + b_{-\mathbf{p},s} \bar{v}^s(-\mathbf{p}) e^{-iE_{\mathbf{p}}t} \right] \times$$

$$\times \gamma^0 E_{\mathbf{q}} \left[a_{\mathbf{q},r} u^r(\mathbf{q}) e^{-iE_{\mathbf{q}}t} - b_{-\mathbf{q},r}^\dagger v^r(-\mathbf{q}) e^{iE_{\mathbf{q}}t} \right] e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}$$

$$= \mathcal{N} \int \frac{\mathrm{d}^3 p}{2(2\pi)^3} \sum_{s=1,2} \sum_{r=1,2} \left[a_{\mathbf{p},s}^\dagger \bar{u}^s(\mathbf{p}) e^{iE_{\mathbf{p}}t} + b_{-\mathbf{p},s} \bar{v}^s(-\mathbf{p}) e^{-iE_{\mathbf{p}}t} \right] \times$$

$$\times \gamma^0 \left[a_{\mathbf{p},r} u^r(\mathbf{p}) e^{-iE_{\mathbf{p}}t} - b_{-\mathbf{p},s}^\dagger v^r(-\mathbf{p}) e^{iE_{\mathbf{p}}t} \right]$$

$$= \mathcal{N} \int \frac{\mathrm{d}^3 p}{2(2\pi)^3} \sum_{s=1,2} \sum_{r=1,2} \left[a_{\mathbf{p},s}^\dagger a_{\mathbf{p},r} u^{s\dagger}(\mathbf{p}) u^r(\mathbf{p}) + b_{-\mathbf{p},s} a_{\mathbf{p},r} v^{s\dagger}(-\mathbf{p}) u^r(\mathbf{p}) e^{-2iE_{\mathbf{p}}t} + \right.$$

$$\left. - a_{\mathbf{p},s}^\dagger b_{-\mathbf{p},r}^\dagger u^{s\dagger}(\mathbf{p}) v^r(-\mathbf{p}) e^{2iE_{\mathbf{p}}t} - b_{-\mathbf{p},s} b_{-\mathbf{p},r}^\dagger v^{s\dagger}(-\mathbf{p}) v^r(-\mathbf{p}) \right]$$

> **Lemma 2.2.1**
>
> $$u^{s\dagger}(\mathbf{p}) v^r(-\mathbf{p}) = v^{s\dagger}(-\mathbf{p}) u^r(\mathbf{p}) = 0$$

Using Lemma 2.2.1, Eq. 1.93 and the antisymmetry $\mathcal{N} b_{\mathbf{p},s} b_{\mathbf{p},s}^\dagger = -b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s}$:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{s=1,2} E_{\mathbf{p}} \mathcal{N} \left[a_{\mathbf{p},s}^\dagger a_{\mathbf{p},s} - b_{\mathbf{p},s} b_{\mathbf{p},s}^\dagger \right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{s=1,2} E_{\mathbf{p}} \left[a_{\mathbf{p},s}^\dagger a_{\mathbf{p},s} + b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s} \right]$$

$\square$

It can be seen that, using anticommutators, the Hamiltonian and its interpretation are analogous to that of the complex scalar field: if commutators were used, instead, one would get a final $-b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s}$ term, which is problematic as it yields an energy unbounded from below.