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Infrared-Safe NLO Calculations with Massive Quarks: An Extension of the NSC Subtraction Formalism

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Abstract

The treatment of infrared divergences in Next-to-Leading Order (NLO) QCD calculations becomes significantly more complex when accounting for massive quarks, particularly in processes where mass effects cannot be neglected. We present a generalization of the Nested Soft-Collinear (NSC) subtraction scheme to incorporate arbitrary massive quark flavours, preserving the original framework's efficiency while systematically addressing mass-dependent divergences.
By removing the need for massless approximations, this work enables precision calculations in particle-production processes where quark mass effects are theoretically or phenomenologically relevant.

Contents

Chapter 1

Introduction

The Standard Model of Particle Physics (SM) is, as of now, the most complete theoretical framework in subatomic physics, describing all known elementary particles and fundamental interactions [19–21], except for the very weak gravitational force. Over the last fifty years, the SM has been continuously tested via experiments, especially in the context of particle colliders, and its validity has been confirmed by the agreement of its predictions with experimental observations: this process culminated in 2012 with the discovery of the Higgs boson [1, 2] at the Large Hadron Collider (LHC) at CERN.

Despite its success, there is strong evidence for the existence of Physics Beyond the SM (BSM): the most prominent indications include the existence of dark matter and dark energy, the observed matter-antimatter asymmetry and the non-vanishing neutrino masses. Contrary to earlier expectations, though, since its first run in 2009 the LHC has not yet detected any new particle, nor any confirmation of BSM physics: on the contrary, the huge amount of data collected in its three runs (currently Run 3 is ongoing) puts increasingly stricter exclusion limits to BSM models [3–6]. As a consequence, the masses of hypothesized new particles become so large that, although still not excluded, their frequent production at the LHC is hardly possible.

The lack of any observation of BSM physics at the LHC has sparked a change in the research paradigm in High-Energy Particle Physics. Substantial further increase in the energy of colliding particles at the LHC (or anywhere else) is currently not feasible, hence it is clear that BSM physics searches based on the idea of detectable resonant-like structures on top of flat backgrounds has to be supplemented by new search strategies. Indeed, new particles can still be produced at the LHC, though in a way which does not allow for their direct detection: undetected light particles could be hidden in complex final states, while heavy particles could be virtually produced for extremely short periods of time, before disappearing back into the quantum vacuum. In the latter case, these virtual particles could affect measurable properties, prompting their indirect detection as deviations from SM predictions.

Given this shift of focus towards higher experimental precision in collider physics, it is clear that reliable theoretical predictions of hadron-collision processes is needed.

Chapter 2

Preliminaries

§2.1 Renormalization scheme

The computation of NLO corrections to scattering processes often involves diverging loop amplitudes. In order to obtain finite results from these divergences, a renormalization scheme must be implemented.
As the generalized Catani's formula for virtual corrections is provided in [7] in a charge-unrenormalized (but mass-renormalized) way, it is necessary to carry out the renormalization procedure explicitly. To this end, we formally state the renormalization scheme adopted in this work.

§2.1.1 Dimensional regularization

In the evaluation of loop amplitudes, both UV- and IR-singularities are encountered. The most efficient way to simultaneously regularize both types of divergences is dimensional regularization, a regularization scheme first introduced by 't Hooft and Veltman in [8].
In general, the dimensional regularization scheme consists in the analytic continuation of loop momenta to $d = 4 - 2\epsilon$ dimensions, with $\epsilon \in \mathbb{C} : \Re\epsilon < 0$. This procedure turns loop integrals into meromorphic[1]functions of $\epsilon \in \mathbb{C}$, allowing for the isolation of divergences as poles in ϵ.
The dimensional regularization prescription leaves freedom in choosing the dimensionality of external momenta, as well as the number of polarizations of both external and internal particles, thus allowing for the definition of different regularization schemes. We choose to work with **conventional dimensional regularization** (CDR), in which all momenta and polarization are analytically continued to d dimensions, as opposed to the 't Hooft–Veltman scheme (HV), in which only internal momenta and polarizations are.
When considering non-chiral gauge theories like QCD, CDR is the most natural choice, as the main difference between CDR and HV is the treatment of purely 4-dimensional objects, i.e. γ^5 and $\epsilon_{\mu\nu\sigma\rho}$. In particular, in CDR both the Dirac algebra and Lorentz indices are analytically continued to d dimensions, leading to a mathematical inconsistency when $d \notin \mathbb{N}$.
The choice of CDR over HV is then clear: in QCD, the only pathological objects are encountered when considering chiral vertices (e.g. for pseudoscalar mesons) and electroweak interactions, and both can be handled via known prescriptions, e.g. the Breitenlohner-Maison/'t Hooft-Veltman (BMHV) scheme outlined in [9].

[1]Given an open set $D \subset \mathbb{C}$, then $f : D \to \mathbb{C}$ is *meromorphic* if it is holomorphic on $D - P$, where $P \subset D$ is a set of isolated points called *poles*. Recall that a function $f : D \to \mathbb{C}$ is *holomorphic* on D if it is complex differentiable at every point in D.

§2.1.2 Minimal subtraction

Once regularized, UV-divergences have to be removed via renormalization of fields and coupling constants. As a result of the renormalization procedure, a running coupling $\alpha_{\mathrm{s}}(\mu^2)$ is introduced, and its definition in terms of the bare coupling $\alpha_{\mathrm{s,b}}$ depends both on the regularization and the renormalization schemes.
In this work, we renormalize the coupling in a standard way (as in [10]) using the **modified minimal-subtraction scheme** ($\overline{\mathrm{MS}}$), which directly subtracts UV-divergences from the coupling:

$$\alpha_{\mathrm{s,b}} S_\epsilon = \alpha_{\mathrm{s}}(\mu^2)\mu^{2\epsilon}\left[1 - \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\frac{\beta_0}{\epsilon} + o(\alpha_{\mathrm{s}}^2)\right] \tag{2.1}$$

where μ is an arbitrary renormalization scale, S_ϵ is the typical phase-space volume factor in dimensional regularization:

$$S_\epsilon \equiv (4\pi)^\epsilon\, e^{-\gamma_{\mathrm{E}}\epsilon} \tag{2.2}$$

with $\gamma_{\mathrm{E}} = 0.5772\ldots$ the Euler-Mascheroni constant, and β_0 is the leading-order coefficient of the QCD β-function:

$$\beta_0 := \frac{11}{6}C_{\mathtt{A}} - \frac{2}{3}T_{\mathrm{R}} n_q \tag{2.3}$$

where $C_{\mathtt{A}}$ and T_{R} are linked to the gauge group $\mathrm{SU}(n_c)$ (see §2.2) and n_q is the number of active quark flavours at the considered energy scale[2].
An important clarification about the dimensionality of $\alpha_{\mathrm{s,b}}$ and α_{s} is needed, due to the presence of $\mu^{2\epsilon}$ in Eq. 2.1. Consider the QCD Lagrangian, i.e. a Yang-Mills Lagrangian with gauge group $\mathrm{SU}(n_c)$ and n quark species (see e.g. Chapter 15 of [11], or §2.2.1):

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \bar{\Psi}\left(i\not{D} - m\right)\Psi \tag{2.4}$$

with covariant derivative and field-strength tensor defined as:

$$D_\mu := \partial_\mu - igA^a_\mu T_a \qquad\qquad F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{2.5}$$

where $\{T_a\}_{a=1,\ldots,n_c^2-1} \subset \mathbb{C}^{n\times n}$ is a Hermitian representation of $\mathfrak{su}(n_c)$ and Ψ is a n-spinor. Recall that the structure constants $f^{abc} \in \mathbb{R}$ of $\mathfrak{su}(n_c)$ are $f^{abc} = \epsilon^{abc}$.
In dimensional regularization, the action remains a dimensionless quantity, hence, given $\mathcal{S} = \int \mathrm{d}^d x\,\mathcal{L}$ and that in natural units ($c = \hbar = 1$) all dimensions can be expressed as mass dimensions (since $[T] = [L] = [M]^{-1}$), the Lagrangian must have dimension $[\mathcal{L}] = d$, as $[\mathrm{d}^d x] = -d$. It is now trivial to verify the following dimensions:

$$[\Psi] = \frac{d-1}{2} \qquad\qquad [A^a_\mu] = \frac{d-2}{2} \qquad\qquad [g] = \frac{4-d}{2} = \epsilon$$

This shows that, in dimensional regularization, $[\alpha_{\mathrm{s,b}}] = 2\epsilon$. In order to work with dimensionless quantities, then, in Eq. 2.1 we chose to extract the mass dimension from α_{s}.
In general, we consider amplitudes $\mathcal{M}_m$ involving m external QCD partons (gluons and quarks), with momenta $\{p\} \equiv \{p_1, \ldots, p_m\}$, and an arbitrary number of colorless particles (photons, leptons, ...). Dependence on the momenta and quantum numbers of colorless particles is always

[2] Out of $n = n_f + n_F$ total quark flavours (n_f massless and n_F massive quark flavours), we generally consider an energy scale such that $n_q = n_f$, unless otherwise specified.

understood and not explicitly shown. The $\overline{\text{MS}}$-renormalized amplitude has the following perturbative expansion in α_{s}:

$$\mathcal{M}_m(\alpha_{\text{s}}(\mu^2), \mu^2; \{p\}) = \left(\frac{\alpha_{\text{s}}(\mu^2)}{2\pi}\right)^q \left[\mathcal{M}_m^{(0)}(\mu^2; \{p\}) + \frac{\alpha_{\text{s}}(\mu^2)}{2\pi}\mathcal{M}_m^{(1)}(\mu^2; \{p\}) + o(\alpha_{\text{s}}^2)\right] \qquad (2.6)$$

where the overall power is, in general, $q \in \frac{1}{2}\mathbb{N}_0$. Note that, although spoiled of UV-divergences, these amplitudes are still IR-singular as $\epsilon \to 0$.

§2.2 Colour-space formalism

We consider a generalized QCD with gauge group SU(n_c), with n_c colours and $n = n_f + n_F$ quark flavours (see footnote 2). To handle the colour structure of QCD amplitudes, we adopt the colour-space formalism as in [12].

§2.2.1 Gauge theories

In order to better understand the colour-space formalism, it is useful to state how a general gauge theory is defined, and then analyze the specific case of a SU(n_c) gauge theory.

§2.2.1.1 Yang-Mills Lagrangian

A quantum field theory can be built starting from its symmetry properties: in particular, specifying a group of local transformations, the **gauge group**, under which the theory must be invariant. Historically, the idea of gauge theories was first explored by Yang and Mills in [13], with the aim of studying isotopic gauge invariance for the nucleon, and then generalized by Utiyama in [14]. A modern treatment of gauge theories can be found in Chapter 15 of [15], which we follow for our discussion.
Consider n fermionic fields $\{\psi_k(x)\}_{k=1,\dots,n}$ and an n-spinor $\Psi(x)$ defined as:

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} \qquad (2.7)$$

As a gauge group, consider a d-dimensional Lie group G: in particular, take G to be a simply-connected Lie group, so that each element can be expressed via the exponential map, and compact too, so that its representations are unitary. Then, consider $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ a representation of the associated Lie algebra $\mathfrak{g}$, so that the action of G on Ψ can be expressed as:

$$\Psi(x) \mapsto V(x)\Psi(x) \qquad V(x) := \exp\left[i\theta_a(x)T^a\right] \qquad (2.8)$$

where the Lie parameters $\{\theta_a(x)\}_{a=1,\dots,d} \subset \mathcal{C}^\infty(\mathbb{R}^{1,3})$ so define a local gauge transformation. The aim is to define a Lagrangian which is invariant under this transformation, i.e. the Lagrangian of a (local) gauge theory.
Simple terms invariant under global phase rotations, like the fermion mass term $m\bar{\Psi}\Psi$, are of course invariant under Eq. 2.8 too, but derivatives need a careful treatment: indeed, the limit-definition of a derivative involves fields at different spacetime points, which have different

transformations according to Eq. 2.8. In order to define a derivative of Ψ, it is necessary to introduce a factor to subtract values of $\Psi(x)$ in a meaningful way, so consider $\mathrm{U}(y,x) \in \mathrm{U}(n)$: $\mathrm{U}(x,x) = 1$ and which transforms under the action of G as:

$$\mathrm{U}(y,x) \mapsto V(y)\mathrm{U}(y,x)V^{\dagger}(x) \tag{2.9}$$

By the unitarity of the representations of G, it is clear that $\mathrm{U}(y,x)\Psi(x)$ and $\Psi(y)$ have the same transformation law, so they can be meaningfully subtracted. Then, given $n^\mu \in \mathbb{R}^{1,3}$, the covariant derivative of a fermionic field $\Psi(x)$ along n^μ is defined as:

$$n^\mu D_\mu \Psi(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\Psi(x+\varepsilon n) - \mathrm{U}(x+\varepsilon n, x)\Psi(x)\right] \tag{2.10}$$

where $\mathrm{U}(y,x)$ is defined in Eq. 2.9. To make this definition explicit, it is necessary to get an expression of $\mathrm{U}(y,x)$ at infinitesimally-separted points. Given the unitarity of $\mathrm{U}(y,x)$, it can be expressed through the generators $\{T^a\}_{a=1,\dots,d}$ as:

$$\mathrm{U}(x+\varepsilon n, x) = \mathrm{I}_n + ig\varepsilon n^\mu A^a_\mu(x) T_a + o(\varepsilon^2) \tag{2.11}$$

where $g \in \mathbb{R}$ is a constant. The new vector field $A^a_\mu(x)$ (actually, d different vector fields) is a **connection**, and it allows to express the covariant derivative as (directly from Eq. 2.10):

$$D_\mu = \partial_\mu - igA^a_\mu T_a \tag{2.12}$$

It is straightforward to show that $D_\mu \Psi$ transforms in the same way as Ψ.
The gauge-invariant Lagrangian can thus be built using covariant derivatives (minimal coupling prescription), but there needs to be included a kinetic term for the connection, i.e. a gauge-invariant term depending on $A^a_\mu(x)$ only. This term can be found considering the commutator of covariant derivatives:

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}T_a \tag{2.13}$$

with the **field-strength tensor** defined as:

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{2.14}$$

Note that the field-strength tensor is not itself a gauge-invariant quantity, as really there are d different field-strength tensors; however, it is straightforward to construct gauge-invariant combinations of $F^a_{\mu\nu}$. In fact, in general the following theorem holds: any globally-symmetric function of Ψ, $F^a_{\mu\nu}$ and their covariant derivatives is also locally-symmetric, i.e. gauge-invariant (see Chapter 15 of [15]).
Usually, the following gauge-invariant term is taken as kinetic term for the gauge field (i.e. the connection $A^a_\mu(x)$):

$$\mathrm{tr}\{(F^a_{\mu\nu}T_a)^2\} = 2F^a_{\mu\nu}F^{\mu\nu}_a \tag{2.15}$$

This allows defining the simplest non-Abelian gauge theory, **Yang-Mills theory** without fermionic species:

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a \tag{2.16}$$

To account for fermions interacting with the gauge field (i.e. the connection $A^a_\mu(x)$), the Dirac Lagrangian with minimal coupling is added:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \bar{\Psi}\left(i\not{D} - m\right)\Psi \tag{2.17}$$

§2.2.1.2 Gauge group $\mathrm{SU}(n)$

The $\mathrm{SU}(n)$ group is the group of unitary transformations of n-dimensional complex vectors. Its (faithful) fundamental representation thus is:

$$\mathrm{SU}(n) = \{\mathrm{U} \in \mathbb{C}^{n\times n} : \mathrm{U}\mathrm{U}^\dagger = \mathrm{U}^\dagger\mathrm{U} = \mathrm{I}_n \wedge \det \mathrm{U} = +1\}$$

The generators of $\mathrm{SU}(n)$ can be found setting $\mathrm{U} = \exp(i\theta_a T^a) = \mathrm{I}_n + i\theta_a T^a + o(\theta^2)$ and using $\mathrm{U}^\dagger\mathrm{U} = \mathrm{I}_n$:

$$T^a = T^{a\dagger} \tag{2.18}$$

Moreover, by the Jacobi formula $(\det A(t))\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \mathrm{tr}(A(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A(t))$ evaluated at $t = 0$:

$$\mathrm{tr}\, T^a = 0 \tag{2.19}$$

The traceless condition can be generalized to all semi-simple Lie algebras. Therefore, the generators of $\mathrm{SU}(n)$ are $\mathbb{C}^{n\times n}$ Hermitian traceless matrices: the dimension of $\mathfrak{su}(n)$ then is $n^2 - 1$.
In general, the adjoint representation of a Lie group is given by representing its generators (i.e. the basis of the Lie algebra) with the structure constants of the Lie algebra:

$$(T^b_{\mathrm{ad}})_{ac} \equiv \bar{T}^b_{ac} = i f^{abc} \tag{2.20}$$

which, in the case of $\mathrm{SU}(n)$, are $f^{abc} = \epsilon^{abc}$. Indeed, it can be shown by the Jacobi identity that the structure constants satisfy the Lie algebra. Moreover, since the structure constant are real, the adjoint representation is always a real representation: the adjoint representation of $\mathrm{SU}(n)$ has degree $n^2 - 1$.
Representation are labelled by their Casimir operators. For any simple Lie algebra, given a representation $\mathtt{r}$, a Casimir operator is defined as:

$$T^a_{\mathtt{r}} T^a_{\mathtt{r}} = C_2(\mathtt{r})\mathrm{I}_{n_{\mathtt{r}}} \tag{2.21}$$

This is called the **quadratic Casimir operator**, as it is associated to $T^2 \equiv T^a T^a$ (a Casimir operator since $[T^b, T^2] = i f^{bac}\{T^c, T^a\} = 0$ by antisymmetry). For the fundamental and the adjoint representations $\mathtt{n}$ and $\mathtt{g}$ of $\mathrm{SU}(n)$, the quadratic Casimir operators are:

$$C_{\mathtt{F}} \equiv C_2(\mathtt{n}) = T_{\mathrm{R}}\frac{n^2-1}{n} \qquad\qquad C_{\mathtt{A}} \equiv C_2(\mathtt{g}) = 2T_{\mathrm{R}} n \tag{2.22}$$

where T_{R} (usually taken to be $T_{\mathrm{R}} = \frac{1}{2}$) is the trace normalization of the generators in the fundamental representation:

$$\mathrm{tr}(T^a_{\mathtt{n}} T^b_{\mathtt{n}}) = T_{\mathrm{R}}\delta^{ab} \tag{2.23}$$

§2.2.2 QCD amplitudes

The m external partons in the amplitude $\mathcal{M}_m$ each carry two indices: a colour index and a spin index. Colour indices are denoted by $c_1, \ldots, c_m$: for gluons $c_i \equiv a_i \in \{1, \ldots, n_c^2 - 1\}$, as the field-strength tensor Eq. 2.5 transforms according to the adjoint representation of the gauge group, while for quarks $c_i \equiv \alpha_i \in \{1, \ldots, n_c\}$, as their Dirac fields transform according to the fundamental representation of the gauge group. Spin indices, on the other hand, are denoted

by $s_1, \ldots, s_m$, and they need to take into account how helicities change in CDR: for gluons $s_i \equiv \mu_i \in \{1, \ldots, d\}$, while for quarks $s_i \in \{1, 2\}$.
Consider the m-parton colour-space $\mathscr{H}_c$ and helicity-space $\mathscr{H}_s$, and introduce an orthonormal basis in each:

$$\{|c_1, \ldots, c_m\rangle\} \in \mathscr{H}_c \qquad \{|s_1, \ldots, s_m\rangle\} \in \mathscr{H}_s$$

Note that, being these finite-dimensional Hilbert spaces, the non-canonical (basis-dependent) isomorphisms $\mathscr{H}_c \leftrightarrow \mathscr{H}_c^*$ and $\mathscr{H}_s \leftrightarrow \mathscr{H}_s^*$ are well-defined[3].
Then, to explicit the colour-helicity structure of the m-parton amplitude, we define it as an abstract vector in $\mathscr{H}_c \otimes \mathscr{H}_s$, so that:

$$\mathcal{M}_m^{\{c_1,\ldots,c_m\},\{s_1,\ldots,s_m\}}(\{p_1, \ldots, p_m\}) \equiv \langle\{c_1, \ldots, c_m\}, \{s_1, \ldots, s_m\}|\mathcal{M}_m(\{p_1, \ldots, p_m\})\rangle \tag{2.24}$$

with:

$$|\{c_1, \ldots, c_m\}, \{s_1, \ldots, s_m\}\rangle \equiv |c_1, \ldots, c_m\rangle \otimes |s_1, \ldots, s_m\rangle$$

Hence, it is clear that the squared amplitude summed over colours and helicities is:

$$|\mathcal{M}_m|^2 = \langle\mathcal{M}_m|\mathcal{M}_m\rangle \tag{2.25}$$

To represent colour interactions at QCD vertices, we associate to each parton i a colour charge $\mathbf{T}_i = \{T_i^a\}_{a=1,\ldots,n_c^2-1}$ related to the emission of a gluon. The action of $\mathbf{T}_i$ onto $\mathscr{H}_c$ is defined by:

$$\langle c_1, \ldots, c_i, \ldots, c_m|T_i^a|b_1, \ldots, b_i, \ldots, b_m\rangle = \delta_{c_1,b_1} \ldots T^a_{c_i b_i} \ldots \delta_{c_m,b_m} \tag{2.26}$$

So, $\{T^a_{c_i b_i}\}_{a=1,\ldots,n_c^2-1}$ form a vector with respect to the colour index a of the emitted gluon, and they are matrices in different representations of $\mathrm{SU}(n_c)$, depending on the parton i:

- if i is a gluon, then $T^a_{cb} \equiv i f_{cab}$ (adjoint representation);
- if i is a final-state quark, then $T^a_{\alpha\beta} \equiv t^a_{\alpha\beta}$ (fundamental representation), while if it is a final-state antiquark $T^a_{\alpha\beta} \equiv -t^a_{\alpha\beta}$;
- if i is an initial-state quark, by crossing-symmetry $T^a_{\alpha\beta} \equiv -t^a_{\alpha\beta}$, while if it is an initial-state antiquark $T^a_{\alpha\beta} \equiv t^a_{\alpha\beta}$.

The algebra of these colour-charge operators is easily determined. First of all, we set:

$$\mathbf{T}_i \cdot \mathbf{T}_j \equiv \sum_{a=1}^{n_c^2-1} T_i^a T_i^b \tag{2.27}$$

Then, by the action Eq. 2.26, it is clear that charges associated to different partons commute, i.e.:

$$\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i \qquad \forall i \neq j \in \{1, \ldots, m\} \tag{2.28}$$

[3] Given two $\mathbb{K}$-vector spaces V, W, the set of all $\mathbb{K}$-linear functions $V \to W$ is denoted by $\mathrm{Hom}(V, W)$; for finite-dimensional spaces $\dim_{\mathbb{K}} \mathrm{Hom}(V, W) = \dim_{\mathbb{K}} V \cdot \dim_{\mathbb{K}} W$. The *dual space* is defined as $V^* := \mathrm{Hom}(V, \mathbb{K})$. If V is finite-dimensional, given a basis $\{v_i\}_{i=1,\ldots,n} \subset V$, with $n = \dim_{\mathbb{K}} V$, then a basis $\{\omega^1, \ldots, \omega^n\} \subset V^*$ is defined by $\omega^i(v_j) = \delta^i_j$, and the function $\varphi : V \to V^* : v_i \mapsto \omega^i$ is a *non-canonical isomorphism* $V \leftrightarrow V^*$.
If V is infinite-dimensional, instead, given a basis $\{v_i\}_{i\in\mathcal{I}} \subset V$, the above construction only allows to define linearly-independent subsets of V^*, which are not granted to be bases.

Moreover, by Eq. 2.21,2.27:

$$\mathbf{T}_i^2 = C_i \, \mathrm{id}_{\mathscr{H}_c} \tag{2.29}$$

with $C_i \equiv C_{\mathtt{F}}$ if i is a quark/antiquark and $C_i \equiv C_{\mathtt{A}}$ if it is a gluon. Finally, as each vector $|\mathcal{M}_m\rangle$ is a colour-singlet, colour conservation implies:

$$\sum_{i=1}^{m} \mathbf{T}_i \, |\mathcal{M}_m\rangle = 0 \tag{2.30}$$

This allows to partially (or fully, if $m = 2$ or $m = 3$, as in Appendix A of [12]) factorize the colour-charge algebra in terms of quadratic Casimir operators.

Chapter 3

NSC Subtraction Scheme

The aim of the NSC subtraction scheme (SS) is to compute integrated subtraction terms which account for QCD corrections to the inclusive[1]production of jets in a hadron collider, i.e. to the process:

$$p + p \to X + N\,\text{jets} \tag{3.1}$$

Here, X is a colour-neutral system. The hadron-scale physics is known to be separated from the parton-scale physics (see Section 1.1 of [16]): this makes it possible for us to only manipulate partonic cross-sections, as the hadronic cross-section for the considered process can be factorized in terms of partonic cross-sections via the parton distribution functions (PDFs):

$$\mathrm{d}\sigma(P_1, P_2) = \sum_{a,b} \int_{[0,1]^2} \mathrm{d}\xi_1 \mathrm{d}\xi_2\, f_a(\xi_1, \mu_\mathrm{F}^2) f_b(\xi_2, \mu_\mathrm{F}^2)\, \mathrm{d}\hat{\sigma}_{a,b}(\xi_1 P_1, \xi_2 P_2, \alpha_\mathrm{s}(\mu^2), \mu^2, \mu_\mathrm{F}^2) \tag{3.2}$$

where μ is the renormalization scale, μ_F is the factorization scale (which is set to $\mu_\mathrm{F} = \mu$ from now on) and the sum runs over all initial-state massless partons a and b which contribute to the production of the considered final state.
Denoting the partons' momenta as $p_i \equiv \xi_i P_i$, $i = 1, 2$, and suppressing the explicit dependence on the running coupling and the renormalization scale, it is possible to expand the partonic cross-section as a power series in $\alpha_\mathrm{s}(\mu^2)$:

$$\mathrm{d}\hat{\sigma}_{a,b}(p_1, p_2) = \sum_{n \in \mathbb{N}_0} \mathrm{d}\hat{\sigma}_{a,b}^{(n)}(p_1, p_2) \tag{3.3}$$

where each term is $\mathrm{d}\hat{\sigma}^{(n)} \sim \alpha_\mathrm{s}^{n_0+n}$, with $n_0 \in \mathbb{N}$ giving the LO-dependence on $\alpha_\mathrm{s}(\mu^2)$. Each term of this expansion will have a different multiplicity of the final state, due to the increasing number of corrections and their different nature. At leading-order (see §A.1.1):

$$\mathrm{d}\hat{\sigma}_{a,b}^{(0)}(p_1, p_2) := \frac{\mathcal{N}}{2\hat{s}} \int \mathrm{d}\mathbf{\Phi}_n \left|\mathcal{M}_m^{(0)}(p_1, p_2, p_X, p_\mathcal{H})\right|^2 \mathcal{O}_m(p_\mathcal{H}, p_X) \tag{3.4}$$

where $\hat{s} \equiv 2p_1 \cdot p_2$ is the partonic CM energy squared, $\mathcal{H}$ is the set of all final-state partons (with $p_\mathcal{H}$ its total momentum) and the normalization factor $\mathcal{N}$ includes all necessary symmetry factors (e.g. $(N_g!)^{-1}$, with N_g number of resolved gluons in the final state), as well as averaging factors for initial-state colours and helicities.

[1]Inclusive jet production denotes the theoretical prediction (or experimental measurement) of the cross-section for the production of jets of given kinematics, while summing/integrating over all other final-state radiation and particles.

Note that $\mathcal{O}_m$ is an IR-finite measurement function defining the observable, which ensures that the final state contains at least N resolved jets: in particular, if the energy of a final-state parton vanishes (soft limit), or if two partons become collinear to one another (collinear limit), then $\mathcal{O}_{m+n} \to \mathcal{O}_{m+n-1}$ for $n \in \mathbb{N}$, and $\mathcal{O}_m \to 0$.
Non-trivial combinations of different-multiplicity final states emerges already at next-to-leading order:

$$\mathrm{d}\hat{\sigma}^{(1)}_{a,b}(p_1,p_2) = \mathrm{d}\hat{\sigma}^{\mathrm{R}}_{a,b}(p_1,p_2) + \mathrm{d}\hat{\sigma}^{\mathrm{V}}_{a,b}(p_1,p_2) + \mathrm{d}\hat{\sigma}^{\mathrm{C}}_{a,b}(p_1,p_2) \tag{3.5}$$

where:

$$\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{a,b}(p_1,p_2) := \frac{\mathcal{N}}{2\hat{s}}\int \mathrm{d}\mathbf{\Phi}_{m+1}\left|\mathcal{M}^{(0)}_{m+1}(p_1,p_2,p_X,p_{\mathcal{H}})\right|^2 \mathcal{O}_{m+1}(p_{\mathcal{H}},p_X) \tag{3.6}$$

$$\mathrm{d}\hat{\sigma}^{\mathrm{V}}_{a,b}(p_1,p_2) := \frac{\mathcal{N}}{2\hat{s}}\int \mathrm{d}\mathbf{\Phi}_{m}\, 2\Re\,\langle \mathcal{M}^{(0)}_{m}|\mathcal{M}^{(1)}_{m}\rangle\, \mathcal{O}_{m}(p_{\mathcal{H}},p_X) \tag{3.7}$$

$$\mathrm{d}\hat{\sigma}^{\mathrm{C}}_{a,b}(p_1,p_2) := \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\frac{1}{\epsilon}\sum_c \int_0^1 \frac{\mathrm{d}z}{z}\left[\hat{P}^{(0)}_{c,a}(z)\mathrm{d}\hat{\sigma}^{(0)}_{c,b}(zp_1,p_2) + \hat{P}^{(0)}_{c,b}(z)\mathrm{d}\hat{\sigma}^{(0)}_{a,c}(p_1,zp_2)\right] \tag{3.8}$$

In Eq. 3.6 $\mathcal{H}$ contains $m+1$ partons, while in Eq. 3.7 it only contains m partons. The Altarelli-Parisi splitting kernels are listed in §B.2, and proof of Eq. 3.8 is provided in SECTION.
The rest of this chapter is devoted to the extrapolation of IR-singularities from Eq. 3.6-3.8, prooving their cancellation and providing the associated integrated counterterms.

Chapter 4

NSC SS with Massive Quarks

Appendices

Appendix A

Mathematical reference

§A.1 Phase-space parametrization

In dimensional regularization with $d = 4 - 2\epsilon$, we define the measure on the phase space of a parton i to be:

$$[\mathrm{d}p_i] \equiv \frac{\mathrm{d}^{d-1}p_i}{(2\pi)^{d-1}2E_i}\theta(E_{\max} - E_i) \tag{A.1}$$

Note that $E_{\max}$ is an upper bound on the energies of individual partons: it is an arbitrary parameter to be taken sufficiently large as to be greater or equal to the maximal energy that a final-state parton can reach.
This measure can be cast in a more useful form introducing a suitable parametrization of the phase space: in particular, given that $\mathbb{R}^n - \{\mathbf{0}\} \cong \mathbb{R}^+ \times \mathbb{S}^{n-1}$, it is convenient to introduce hyperspherical coordinates on the $\mathbb{S}^{d-2}$ component of the phase space.

Observation A.1.1 (Hyperspherical coordinates)

In general, $\mathbb{S}^n$ can be described as a surface embedded in $\mathbb{R}^{n+1}$, defined by:

$$\sum_{i=1}^{n+1} x_i^2 = 1 \tag{A.2}$$

Therefore, $\mathbb{S}^n$ can be parametrized by **hyperspherical coordinates** $\{\varphi_1, \dots, \varphi_{n-1}, \varphi_n\} \subset [0,\pi]^{n-1} \times [0, 2\pi)$ as:

$$\begin{aligned}
x_1 &= \cos\varphi_1 \\
x_2 &= \sin\varphi_1 \cos\varphi_2 \\
x_3 &= \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\vdots \\
x_{n-1} &= \sin\varphi_1 \dots \sin\varphi_{n-2} \cos\varphi_{n-1} \\
x_n &= \sin\varphi_1 \dots \sin\varphi_{n-2} \sin\varphi_{n-1} \cos\varphi_n \\
x_{n+1} &= \sin\varphi_1 \dots \sin\varphi_{n-2} \sin\varphi_{n-1} \sin\varphi_n
\end{aligned} \tag{A.3}$$

It is then possible to define the hyperspherical measure on $\mathbb{S}^n$:

$$\mathrm{d}\Omega_n \equiv \sin^{n-1}\varphi_1 \sin^{n-2}\varphi_2 \dots \sin^2\varphi_{n-2} \sin\varphi_{n-1}\, \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \dots \mathrm{d}\varphi_{n-2} \mathrm{d}\varphi_{n-1} \mathrm{d}\varphi_n \tag{A.4}$$

It is clear that this is a recursive relation, as:

$$\mathrm{d}\Omega_n = \sin^{n-1}\varphi\,\mathrm{d}\varphi\,\mathrm{d}\Omega_{n-1} \tag{A.5}$$

Using Eq. A.5 (with $\sin\varphi\,\mathrm{d}\varphi = \mathrm{d}\cos\varphi$), we can express the measure $\mathrm{d}^{d-1}p_i$ as:

$$\mathrm{d}^{d-1}p_i = |\mathbf{p}_i|^{d-2}\,\mathrm{d}\,|\mathbf{p}_i|\,\sin^{d-4}\varphi\,\mathrm{d}\cos\varphi\,\mathrm{d}\Omega_{d-2} \tag{A.6}$$

As we are only interested in integrations on phase spaces of real unresolved partons, which can only be massless, we can use the on-shell condition $p_i^2 = 0$ to express $|\mathbf{p}_i| = E_i$, so that the phase-space measure becomes:

$$[\mathrm{d}p_i] = \theta(E_{\max} - E_i)E_i^{d-3}\mathrm{d}E_i\,\sin^{d-4}\varphi\,\mathrm{d}\cos\varphi\,\frac{\mathrm{d}\Omega_{d-2}}{2(2\pi)^{d-1}} \tag{A.7}$$

with $E_i \in \mathbb{R}_0^+$ and $\varphi \in [0,\pi]$.

§A.1.1 Multi-particle phase space

When considering scattering processes, in general the final state is a multi-particle state, hence the measure on the final-state phase space must account for energy conservation too.

Theorem A.1.1 (Scattering cross-section)

Given a $2 \to m$ scattering process with well-defined initial momenta $p_\mathcal{A}$ and $p_\mathcal{B}$, then the **differential cross-section** is:

$$\mathrm{d}\sigma = \frac{1}{2E_\mathcal{A}2E_\mathcal{B}\,|\mathbf{v}_\mathcal{A} - \mathbf{v}_\mathcal{B}|}\prod_{k=1}^{m}\int\frac{\mathrm{d}^3p_k}{(2\pi)^3 2E_k}\,|\mathcal{M}(\mathcal{AB}\to\{f\})|^2\,(2\pi)^4\delta^{(4)}(p_\mathcal{A} + p_\mathcal{B} - \textstyle\sum_{i=1}^m p_i) \tag{A.8}$$

where $\mathcal{M}(\mathcal{AB}\to\{f\})$ is the matrix element of the scattering process and $\mathbf{v}_k \equiv \frac{\mathbf{p}_k}{E_k}$ is the velocity of the k^{th} particle.

Proof. See Chapter 4 of [15]. □

As we are only interested in massless initial-state partons, in the center-of-mass (CM) frame $p_{\mathcal{A},\mathcal{B}} = (E, \pm\mathbf{p})$, hence it is trivial to see that the flux factor in Eq. A.8 is just $2\hat{s} := 2(p_\mathcal{A} + p_\mathcal{B})^2$. The differential cross-section can then be rewritten as:

$$\mathrm{d}\sigma = \frac{1}{2\hat{s}}\int\mathrm{d}\mathbf{\Phi}_m\,|\mathcal{M}(\mathcal{AB}\to\{f\})|^2 \tag{A.9}$$

where the **invariant m-body phase space measure** is defined as:

$$\mathrm{d}\mathbf{\Phi}_m \equiv \prod_{k=1}^{m}[\mathrm{d}p_k](2\pi)^4\delta^{(4)}(p_\mathcal{A} + p_\mathcal{P} - \sum_{i=1}^{m}p_i) \tag{A.10}$$

§A.2 Angular integrals

Appendix B

Collection of relevant equations

In this Appendix, we provide definitions of relevant objects used in this work. To simply various formulas, we use a notation analogous to [17]:

$$\bar{z} \equiv 1 - z \qquad \mathcal{D}_n(z) \equiv \left[\frac{\log^n(1-z)}{1-z}\right]_+$$
$$L_i \equiv \log\frac{E_{\max}}{E_i} \qquad \mathscr{L}_i \equiv \log\frac{2E_i}{\mu} \qquad L_{\max} \equiv \log\frac{2E_{\max}}{\mu} \tag{B.1}$$

§B.1 Useful constants

Denoting the colour-charge operators $\mathbf{T}_i$, with the conventional normalization $T_{\mathrm{R}} = \frac{1}{2}$ for $\mathrm{SU}(n_c)$, the squares of these operators are the quadratic Casimir operators of the corresponding representations:

$$\mathbf{T}_q^2 = \mathbf{T}_{\bar{q}}^2 = C_{\mathtt{F}} = \frac{n_c^2 - 1}{2n_c} \qquad \mathbf{T}_g^2 = C_{\mathtt{A}} = n_c \tag{B.2}$$

The quark and gluon anomalous dimensions are:

$$\gamma_q = \frac{3}{2}C_{\mathtt{F}} \qquad \gamma_q = \frac{11}{6}C_{\mathtt{A}} - \frac{2}{3}T_{\mathrm{R}}n_q \tag{B.3}$$

where n_q is the number of active flavours.
The strong coupling is renormalized in the $\overline{\mathrm{MS}}$ scheme, so that the bare and running couplings are related by:

$$\alpha_{\mathrm{s,b}}S_\epsilon = \alpha_{\mathrm{s}}(\mu^2)\mu^{2\epsilon}\left[1 - \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\frac{\beta_0}{\epsilon} + o(\alpha_{\mathrm{s}}^2)\right] \tag{B.4}$$

where $S_\epsilon \equiv (4\pi)^\epsilon e^{-\gamma_{\mathrm{E}}\epsilon}$ and:

$$\beta_0 = \frac{11}{6}C_{\mathtt{A}} - \frac{2}{3}T_{\mathrm{R}}n_q = \gamma_g \tag{B.5}$$

It is convenient to define a quantity related to the coupling constant:

$$[\alpha_{\mathrm{s}}] \equiv \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\frac{e^{\gamma_{\mathrm{E}}\epsilon}}{\Gamma(1-\epsilon)} \tag{B.6}$$

§B.2 Splitting functions

THIS TO BE PARTIALLY MOVED TO SECTION ON COLLINEAR SINGULARITIES
Consider the final-state splitting process $[i\mathfrak{m}]^* \to i(z)+\mathfrak{m}(1-z)$, where i and $\mathfrak{m}$ are two partons of flavours f_i and $f_\mathfrak{m}$ and $[i\mathfrak{m}]$ is the corresponding clustered parton of flavour $f_{[i\mathfrak{m}]}$. Recall that, given the interaction vertices determined by the QCD Lagrangian Eq. 2.17 (see FIGURE), a gluon clustered with any type of parton preserves the latter's flavours, while a quark clustered with an antiquark gives a gluon.
The energy fraction carried by the parton i is defined as $z \equiv 1 - E_\mathfrak{m}/E_{[i\mathfrak{m}]}$. As a consequence, the parton $\mathfrak{m}$ carries an energy fraction $1-z$. Denoting the spin-averaged fintal-state splitting functions as $P_{f_{[i\mathfrak{m}]}f_i}(z)$, they read:

$$P_{qq}(z) = C_\mathtt{F}\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right] \tag{B.7}$$

$$P_{qg}(z) = C_\mathtt{F}\left[\frac{1-(1-z)^2}{z} - \epsilon z\right] \equiv P_{qq}(1-z) \tag{B.8}$$

$$P_{gq}(z) = T_\mathrm{R}\left[1 - \frac{2z(1-z)}{1-\epsilon}\right] \tag{B.9}$$

$$P_{gg}(z) = 2C_\mathtt{A}\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right] \tag{B.10}$$

Now, consider instead the initial-state splitting process $i \to [i\mathfrak{m}]^* + \mathfrak{m}$, where i and $\mathfrak{m}$ are respectively an ingoing and outgoing parton, while the clustered parton $[i\mathfrak{m}]^*$ enters the hard scattering process. In this case, we define the z variable as $z \equiv 1 - E_\mathfrak{m}/E_i$. The spin- and color-averaged intial-state splitting functions, denoted as $P_{f_{[i\mathfrak{m}]}f_i,\mathrm{i}}(z)$, are:

$$P_{qq,\mathrm{i}} = -zP_{qq}(1/z) \equiv P_{qq}(z) \tag{B.11}$$

$$P_{qg,\mathrm{i}} = \left[\frac{2n_c}{2(1-\epsilon)(n_c^2-1)}\right] zP_{qg}(1/z) \equiv P_{gq}(z) \tag{B.12}$$

$$P_{gq,\mathrm{i}} = \left[\frac{2(1-\epsilon)(n_c^2-1)}{2n_c}\right] zP_{gq}(1/z) \equiv P_{qg}(z) \tag{B.13}$$

$$P_{gg,\mathrm{i}} = -zP_{gg}(1/z) \equiv P_{gg}(z) \tag{B.14}$$

Finally, the LO Altarelli-Parisi splitting kernels are:

$$\hat{P}^{(0)}_{qq}(z) = C_\mathtt{F}\left[2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z)\right] \tag{B.15}$$

$$\hat{P}^{(0)}_{qq}(z) = T_\mathrm{R}\left[(1-z)^2 + z^2\right] \tag{B.16}$$

$$\hat{P}^{(0)}_{qq}(z) = C_\mathtt{F}\left[\frac{1+(1-z)^2}{z}\right] \tag{B.17}$$

$$\hat{P}^{(0)}_{qq}(z) = 2C_\mathtt{A}\left[\mathcal{D}_0(z) + z(1-z) + \frac{1}{z} - 2\right] + \beta_0\delta(1-z) \tag{B.18}$$

All these splitting functions and kernels can be found in [18].

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