

# Mathematical Reference

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Part I

# Multilinear Algebra



# Vector Spaces and Applications

## §1.1 Matrices

### Definition 1.1.1 (Matrix)

Given a field  $\mathbb{K}$  and  $n, m \in \mathbb{N}$ , an  $n \times m$  **matrix** on  $\mathbb{K}$  is the object:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \equiv [a_{ij}]_{j=1, \dots, m}^{i=1, \dots, n} \quad : \quad a_{ij} \in \mathbb{K} \quad \forall i = 1, \dots, n, j = 1, \dots, m$$

The set of all  $n \times m$  matrices on  $\mathbb{K}$  is denoted by  $\mathbb{K}^{n \times m}$ .

When the dimensions of the matrix  $A$  are unambiguous, we simply write  $A = [a_{ij}]$ . We say that an  $n \times n$  matrix is a **square matrix**, an  $n \times 1$  matrix is a **column vector** and a  $1 \times n$  matrix is a **row vector**.

It is possible to define three operations between matrices:

- sum  $+$  :  $\mathbb{K}^{n \times m} \times \mathbb{K}^{n \times m} \rightarrow \mathbb{K}^{n \times m} : [a_{ij}]_{j=1, \dots, m}^{i=1, \dots, n} + [b_{ij}]_{j=1, \dots, m}^{i=1, \dots, n} \mapsto [a_{ij} + b_{ij}]_{j=1, \dots, m}^{i=1, \dots, n}$
- product by a scalar  $\cdot$  :  $\mathbb{K} \times \mathbb{K}^{n \times m} \rightarrow \mathbb{K}^{n \times m} : \alpha \cdot [a_{ij}]_{j=1, \dots, m}^{i=1, \dots, n} = [\alpha a_{ij}]_{j=1, \dots, m}^{i=1, \dots, n}$
- product  $\cdot$  :  $\mathbb{K}^{n \times p} \times \mathbb{K}^{p \times m} \rightarrow \mathbb{K}^{n \times m} : [a_{ij}]_{j=1, \dots, p}^{i=1, \dots, n} \cdot [b_{ij}]_{j=1, \dots, m}^{i=1, \dots, p} = [c_{ij}]_{j=1, \dots, m}^{i=1, \dots, n}, c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$

Note that  $\alpha a_{ij}$  is the  $\mathbb{K}$ -product.

### Proposition 1.1.1

$(\mathbb{K}^{n \times m}, +)$  is an abelian group.

*Proof.* The matrix sum is equivalent to the  $\mathbb{K}$ -sum of corresponding elements, which is associative and commutative. The neutral element is the zero matrix  $0_{n \times m} = [0]_{j=1, \dots, m}^{i=1, \dots, n}$ , while the inverse element is  $-A = [-a_{ij}]_{j=1, \dots, m}^{i=1, \dots, n}$ .  $\square$

**Theorem 1.1.1**

$(\mathbb{K}^{n \times n}, +, \cdot)$  is a non-commutative ring.

*Proof.* By Prop. 1.1.1,  $(\mathbb{K}^{n \times n}, +)$  is an abelian group. It is trivial to show the associativity and distributivity of the matrix product, i.e.:

1.  $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ ,  $\lambda(A \cdot B) = (\lambda A) \cdot B = A \cdot (\lambda B) \quad \forall A, B, C \in \mathbb{K}^{n \times n}, \lambda \in \mathbb{K}$
2.  $A \cdot (B + C) = A \cdot B + A \cdot C$ ,  $(A + B) \cdot C = A \cdot C + B \cdot C \quad \forall A, B, C \in \mathbb{K}^{n \times n}$

Finally, the neutral element of the matrix product is the identity matrix  $I_n = [\delta_{ij}]_{i,j=1,\dots,n}$ .  $\square$

**Definition 1.1.2 (Transposed matrix)**

Given a matrix  $A \in \mathbb{K}^{n \times m}$ , its **transpose** is defined as  $A^\top \in \mathbb{K}^{m \times n} : [a_{ij}^\top]_{j=1,\dots,m}^{i=1,\dots,n} = [a_{ji}]_{i=1,\dots,m}^{j=1,\dots,n}$ .

A square matrix  $A \in \mathbb{K}^{n \times n}$  is said **symmetric** if  $A^\top = A$  or **antisymmetric** if  $A^\top = -A$ , and it is **diagonal** if  $a_{ij} = 0 \quad \forall i \neq j \in \{1, \dots, n\}$ .

**Definition 1.1.3 (Inverse matrix)**

A square matrix  $A \in \mathbb{K}^{n \times n}$  is **invertible** if  $\exists A^{-1} \in \mathbb{K}^{n \times n} : A^{-1} \cdot A = A \cdot A^{-1} = I_n$ .

**Example 1.1.1 (Non-invertible matrix)**

The matrix  $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$  is non-invertible, as  $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 2\alpha & 2\beta \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \forall \alpha, \beta, \gamma, \delta \in \mathbb{R}$ .

**Definition 1.1.4 (General linear group)**

The **general linear group**  $GL(n, \mathbb{K})$  is defined as the subset of  $\mathbb{K}^{n \times n}$  of all invertible matrices.

Note that  $GL(1, \mathbb{K}) = \mathbb{K} - \{0\}$ .

**Theorem 1.1.2**

$(GL(n, \mathbb{K}), \cdot)$  is a non-abelian group.

*Proof.* The neutral element is  $I_n$ , as  $I_n^{-1} = I_n \implies I_n \in GL(n, \mathbb{K})$ , while the existence of the inverse is granted by definition. We only have to show closure under matrix multiplication:

$$(AB)^{-1} = B^{-1}A^{-1} \iff I_n = A \cdot A^{-1} = AI_nA^{-1} = ABB^{-1}A^{-1} = (AB)(AB)^{-1}$$

Hence,  $A, B \in GL(n, \mathbb{K}) \implies AB \in GL(n, \mathbb{K})$ .  $\square$

### §1.1.1 Linear systems of equations

A **linear equation** with  $n \in \mathbb{N}$  variables and  $\mathbb{K}$ -coefficients is an expression of the form:

$$a_1x_1 + \cdots + a_nx_n = b \quad a_i, b \in \mathbb{K} \quad \forall i = 1, \dots, n$$

A **solution** of the equation is an  $n$ -tuple  $(\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{K}^n$  which satisfies this expression.

#### Definition 1.1.5 (Linear system of equations)

A linear system of equations (or simply **linear system**) is a collection of  $m$  linear equations with  $n$  variables:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \iff \mathbf{Ax} = \mathbf{b}$$

where we defined:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{K}^{m \times n} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{K}^{m \times 1} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{K}^{n \times 1}$$

Two linear systems with the same set of solutions are called **equivalent systems**: note that two equivalent systems must have the same number of variables, but not necessarily the same number of equations.

Based on the cardinality of its solution set, a linear system is said to be **impossible** if it has no solutions, **determined** if it has one solution and **undetermined** if it has infinitely-many solutions. Moreover, if the solution set can be parametrized by  $k \in \mathbb{N}_0$  variables, the system is of kind  $\infty^k$ : a determined system is of kind  $\infty^0$ .

Linear systems can be systematically solved applying a reduction algorithm to their corresponding matrices: **Gauss algorithm**. Starting with a general composed matrix  $[\mathbf{A}|\mathbf{b}] \in \mathbb{K}^{m \times (n+1)}$ , first we multiply the first row by  $a_{11}^{-1}$ , so that:

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right] \longrightarrow \left[ \begin{array}{cccc|c} 1 & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

Then, at each row  $R_2, \dots, R_m$  we apply the transformation  $R_k \mapsto R_k - a_{k1}R_1$ , so that:

$$\left[ \begin{array}{cccc|c} 1 & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right] \longrightarrow \left[ \begin{array}{cccc|c} 1 & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a'_{m2} & \cdots & a'_{mn} & b'_m \end{array} \right]$$

Reiterating this process to progressively smaller submatrices, the algorithm yields the general transformation:

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right] \longrightarrow \left[ \begin{array}{cccc|c} 1 & a'_{12} & \dots & a'_{1n} & b'_1 \\ 0 & 1 & \dots & a'_{2n} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b'_m \end{array} \right]$$

As these are linear transformations, the two matrices represent equivalent linear systems: the transformed linear system is substantially easier to solve, and its solution set is a solution set of the starting linear system too.

### Definition 1.1.6 (Character)

Given a matrix  $M \in \mathbb{K}^{n \times m}$ , its **character**  $\text{car}(M)$  is the number of non-zero rows remaining after Gauss reduction.

It can be proven that the character is independent of the operations performed during the reduction algorithm.

### Theorem 1.1.3 (Rouché–Capelli theorem)

A linear system  $A\mathbf{x} = \mathbf{b}$  has solutions only if  $\text{car}(A) = \text{car}([A|\mathbf{b}])$ . Moreover, if the system has solutions, then it is of kind  $\infty^{n-r}$ , with  $n$  number of variables and  $r = \text{car}(A)$ .

## §1.2 Vector spaces

### Definition 1.2.1 (Vector space)

Given a set  $V \neq \emptyset$  and a field  $\mathbb{K}$ , then  $V$  is a  **$\mathbb{K}$ -vector space** if there exist two operations:

$$+ : V \times V \rightarrow V : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} + \mathbf{w} \quad \cdot : \mathbb{K} \times V \rightarrow V : (\lambda, \mathbf{v}) \mapsto \lambda \cdot \mathbf{v}$$

such that  $(V, +)$  is an abelian group and the following properties hold  $\forall \lambda, \mu \in \mathbb{K}, \mathbf{v}, \mathbf{w} \in V$ :

1.  $(\lambda + \mu) \cdot (\mathbf{v} + \mathbf{w}) = \lambda \cdot \mathbf{v} + \mu \cdot \mathbf{v} + \lambda \cdot \mathbf{w} + \mu \cdot \mathbf{w}$
2.  $(\lambda \cdot \mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v}) = \mu \cdot (\lambda \cdot \mathbf{v})$
3.  $1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v}$

Note that there are three unique neutral elements:  $0_{\mathbb{K}}$ ,  $1_{\mathbb{K}}$  and  $0_V \equiv \mathbf{0}$ . In the following, the multiplication symbol  $\cdot$  is suppressed, as the factors clarify which multiplication is occurring ( $\cdot : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K}$  or  $\cdot : \mathbb{K} \times V \rightarrow V$ , which have the same neutral element  $1_{\mathbb{K}}$ ).

### Example 1.2.1 (Complex numbers)

$V = \mathbb{C}$  is a vector space both for  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{K} = \mathbb{C}$ , although they are different objects.



**Example 1.2.2** (Field as vector space)

$V = \mathbb{K}$  is a  $\mathbb{K}$ -vector space. Note that, in this case,  $0_{\mathbb{K}} \equiv 0_V$ .

Note that, by the uniqueness of  $0_V$ , then  $\forall \mathbf{v} \in V \exists! -\mathbf{v} \in V : \mathbf{v} + (-\mathbf{v}) = 0_V$ , so the following cancellation rule holds  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ :

$$\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{v} \implies \mathbf{u} = \mathbf{w} \quad (1.1)$$

We can now state some basic properties of vector spaces.

**Lemma 1.2.1** (Basic properties)

Given a  $\mathbb{K}$ -vector space  $V$ , then  $\forall \lambda \in \mathbb{K}, \mathbf{v} \in V$ :

- |  |   |
|--|---|
| a. $0_{\mathbb{K}} \cdot \mathbf{v} = 0_V$                     | c. $\lambda \cdot 0_V = 0_V$  |
| b. $(-\lambda) \cdot \mathbf{v} = -(\lambda \cdot \mathbf{v})$ | d. $\lambda \cdot \mathbf{v} = 0_V \iff \lambda = 0_{\mathbb{K}} \vee \mathbf{v} = 0_V$ |

*Proof.* Respectively:

- Consider  $c \in \mathbb{K} - \{0_{\mathbb{K}}\}$ ; then  $c\mathbf{v} + 0_V = c\mathbf{v} = (c + 0_{\mathbb{K}})\mathbf{v} = c\mathbf{v} + 0_{\mathbb{K}} \cdot \mathbf{v}$ , which by Eq. 1.1 proves  $0_{\mathbb{K}} \cdot \mathbf{v} = 0_V$ .
- $\lambda\mathbf{v} + (-\lambda)\mathbf{v} = (\lambda - \lambda)\mathbf{v} = 0_{\mathbb{K}} \cdot \mathbf{v} = 0_V$ , which by the uniqueness of the negative element proves  $(-\lambda)\mathbf{v} = -(\lambda\mathbf{v})$ .
- $\lambda \cdot 0_V = \lambda(\mathbf{v} - \mathbf{v}) = \lambda\mathbf{v} + \lambda \cdot (-1_{\mathbb{K}}) \cdot \mathbf{v} = \lambda\mathbf{v} + (-\lambda)\mathbf{v} = \lambda\mathbf{v} - (\lambda\mathbf{v}) = 0_V$
- $\lambda = 0_{\mathbb{K}}$  is trivial, so consider  $\lambda \neq 0_{\mathbb{K}}$ ; then  $\exists! \lambda^{-1} \in \mathbb{K} : \lambda^{-1} \cdot \lambda = 1_{\mathbb{K}}$ , so  $0_V = \lambda^{-1} \cdot 0_V = \lambda^{-1} \cdot (\lambda\mathbf{v}) = (\lambda^{-1} \cdot \lambda)\mathbf{v} = 1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v}$ , i.e.  $\mathbf{v} = 0_V$ .

□

**§1.2.1 Subspaces****Definition 1.2.2** (Subspace)

Given a  $\mathbb{K}$ -vector space  $V$  and a subset  $U \subseteq V : U \neq \emptyset$ , then  $U$  is a **subspace** of  $V$  if it is closed under  $+: U \times U \rightarrow U$  and  $\cdot: \mathbb{K} \times U \rightarrow U$ .

**Lemma 1.2.2**

If  $U$  is a subspace of  $V(\mathbb{K})$ , then  $0_V \in U$ .

*Proof.* By definition  $U \neq \emptyset \implies \exists \mathbf{v} \in U$ . By the closure condition  $\lambda\mathbf{v} \in U \forall \lambda \in \mathbb{K}$ , hence taking  $\lambda = 0_{\mathbb{K}}$  proves the thesis. □

A typical strategy to prove that  $U$  is a subspace of  $V(\mathbb{K})$  is showing the closure properties, while to prove that it is *not* a subspace we usually show that  $0_V \notin U$ .

**Example 1.2.3** (Polynomial subspaces)

Given  $V = \mathbb{K}[x]$ , then  $U = \mathbb{K}_n[x]$  is a subspace  $\forall n \in \mathbb{N}_0$ .

An important concept to analyze vector spaces is that of linear combination. Given two sets  $\{\lambda_k\}_{k=1,\dots,n} \subset \mathbb{K}$  and  $\{\mathbf{v}_k\}_{k=1,\dots,n} \subset V$ , their **linear combination** is:

$$\sum_{k=1}^n \lambda_k \mathbf{v}_k = \lambda_1 \mathbf{v}_1 + \dots + \lambda_n \mathbf{v}_n \in V \quad (1.2)$$

**Proposition 1.2.1** (Subspaces and linear combinations)

Given a  $\mathbb{K}$ -vector space  $V$  and  $U \subset V : U \neq \emptyset$ , then  $U$  is a subspace of  $V$  if and only if it is closed under linear combinations, that is:

$$\{\lambda_k\}_{k=1,\dots,n} \subset \mathbb{K}, \{\mathbf{v}_k\}_{k=1,\dots,n} \subset U \implies \sum_{k=1}^n \lambda_k \mathbf{v}_k \in U$$

*Proof.* First, note that the general case of linear combinations of  $n$  vectors can be reduced to the case of 2 vectors.

( $\Rightarrow$ ) Being  $U$  a subspace, it is closed under  $+$  :  $U \times U \rightarrow U$  and  $\cdot$  :  $\mathbb{K} \times U \rightarrow U$ ; then, by definition  $\lambda, \mu \in \mathbb{K}, \mathbf{v}, \mathbf{w} \in U \implies \lambda \mathbf{v} + \mu \mathbf{w} \in U$ .

( $\Leftarrow$ ) Given  $\lambda \in \mathbb{K}$  and  $\mathbf{v}, \mathbf{w} \in U$ , then  $\mathbf{v} + \mathbf{w} = 1_{\mathbb{K}} \mathbf{v} + 1_{\mathbb{K}} \mathbf{w}$  and  $\lambda \mathbf{v} = \lambda \mathbf{v} + 0_{\mathbb{K}} \mathbf{w}$ , hence closure under linear combinations implies closure under  $+$  :  $U \times U \rightarrow U$  and  $\cdot$  :  $\mathbb{K} \times U \rightarrow U$ .  $\square$

Generally, it is easier to show closure under linear combinations rather than under addition and scalar multiplication.

**Lemma 1.2.3** (Intersection of subspaces)

Given two subspaces of  $V$ ,  $V_1, V_2$  of  $V(\mathbb{K})$ , then  $V_1 \cap V_2$  is still a subset of  $V(\mathbb{K})$ .

*Proof.* Being  $V_1, V_2$  subspaces, both  $V_1$  and  $V_2$  are closed under linear combinations, so  $V_1 \cap V_2$  is too, as  $\mathbf{v} \in V_1 \cap V_2 \implies \mathbf{v} \in V_1 \wedge \mathbf{v} \in V_2$ .  $\square$

On the other hand, in general  $V_1 \cup V_2$  is not a subspace. As a counterexample, consider e.g.  $V = \text{Vect}_0(\mathbb{E}^3)$ , the plane  $\pi : z = 0$  and the line  $r : (x, y, z) = (0, 0, t), t \in \mathbb{R}$ ; then, consider the subspaces  $V_1 = \text{Vect}_0(\pi), V_2 = \text{Vect}_0(r)$ : their union is clearly not closed under addition, as:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in V_1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in V_2 \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin V_1 \cup V_2$$

**Definition 1.2.3** (Sum of subspaces)

Given a  $\mathbb{K}$ -vector space  $V$  and two subspaces  $V_1, V_2$ , their **sum** is defined as:

$$V_1 + V_2 := \{\mathbf{w} \in V : \mathbf{w} = \mathbf{u} + \mathbf{v}, \mathbf{u} \in V_1, \mathbf{v} \in V_2\}$$

This is a **direct sum**, denoted by  $V_1 \oplus V_2$ , if every  $\mathbf{w} \in V_1 + V_2$  has a unique expression as  $\mathbf{w} = \mathbf{u} + \mathbf{v}$ ,  $\mathbf{u} \in V_1$ ,  $\mathbf{v} \in V_2$ .

Trivially  $V_1, V_2 \subseteq V_1 + V_2$ .

**Lemma 1.2.4** (Direct sum as disjoint sum)

Given two subspaces  $V_1, V_2$  of  $V(\mathbb{K})$ , then  $V_1 + V_2 = V_1 \oplus V_2 \iff V_1 \cap V_2 = \{\mathbf{0}\}$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $\exists \mathbf{v} \in V_1 \cap V_2 : \mathbf{v} \neq \mathbf{0}$ ; then  $\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v}$ , i.e. the expression of  $\mathbf{v} \in V_1 + V_2$ , but the expression of  $\mathbf{v} \in V_1 \oplus V_2$  must be unique, hence  $\mathbf{v} = \mathbf{0} \dashv$ .

( $\Leftarrow$ ) Suppose  $\exists \mathbf{w} \in V_1 + V_2 : \mathbf{w} = \mathbf{u}_1 + \mathbf{v}_1 = \mathbf{u}_2 + \mathbf{v}_2$ ,  $\mathbf{u}_1 \neq \mathbf{u}_2 \in V_1$ ,  $\mathbf{v}_1 \neq \mathbf{v}_2 \in V_2$ ; then  $V_1 \ni \mathbf{u}_1 - \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{v}_1 \in V_2 \implies \mathbf{v}_2 - \mathbf{v}_1 \in V_1$ , so  $\mathbf{v}_2 - \mathbf{v}_1 \in V_1 \cap V_2$ , but  $V_1 \cap V_2 = \{\mathbf{0}\}$ , hence  $\mathbf{v}_2 = \mathbf{v}_1$  and idem for  $\mathbf{u}_1 = \mathbf{u}_2 \dashv$ .  $\square$

The sum of subspaces preserves the subspace structure, contrary to the simple union.

**Proposition 1.2.2** (Sum as subspace)

Given a  $\mathbb{K}$ -vector space and two subspaces  $V_1, V_2$ , their sum  $V_1 + V_2$  is still a subspace of  $V$ .

*Proof.* Consider  $\mathbf{a}, \mathbf{b} \in V_1 + V_2$  and define  $\mathbf{u}_{a,b} \in V_1, \mathbf{v}_{a,b} \in V_2 : \mathbf{a} = \mathbf{u}_a + \mathbf{v}_a \wedge \mathbf{b} = \mathbf{u}_b + \mathbf{v}_b$ : as  $V_1, V_2$  are subspaces, they are closed under linear combinations, so, given  $\lambda, \mu \in \mathbb{K}$ , then  $\lambda \mathbf{a} + \mu \mathbf{b} = (\lambda \mathbf{u}_a + \mu \mathbf{u}_b) + (\lambda \mathbf{v}_a + \mu \mathbf{v}_b) \equiv \mathbf{u} + \mathbf{v} \in V_1 + V_2$ , where  $\mathbf{u} \in V_1$  and  $\mathbf{v} \in V_2$ , which shows that  $V_1 + V_2$  too is closed under linear combinations and a subspace by Prop. 1.2.1.  $\square$

## §1.2.2 Bases

To give a more explicit description of vector spaces, we have to define the concept of basis and its properties.

### §1.2.2.1 Generators

**Definition 1.2.4** (Linear dependence)

Given a  $\mathbb{K}$ -vector space  $V$  and a subset  $\{\mathbf{v}_j\}_{j=1,\dots,k} \equiv S \subseteq V$ , then the vectors of  $S$  are:

- **linearly dependent** (LD) if  $\exists \{\lambda_j\}_{j=1,\dots,k} \subset \mathbb{K} - \{0\} : \lambda_1 \mathbf{v}_1 + \dots \lambda_k \mathbf{v}_k = \mathbf{0}$
- **linearly independent** (LI) if  $\lambda_1 \mathbf{v}_1 + \dots \lambda_k \mathbf{v}_k = \mathbf{0} \iff \lambda_j = 0 \forall j = 1, \dots, k$

## §1.3 Linear applications

## §1.4 Inner products

# **Appendices**



## Appendix A

# Logic

### §A.1 Binary relations

#### Definition A.1.1 (Binary relation)

Given two sets  $\mathcal{A}$ ,  $\mathcal{B}$  and their cartesian product  $\mathcal{A} \times \mathcal{B} := \{(a, b) : a \in \mathcal{A} \wedge b \in \mathcal{B}\}$ , a **binary relation**  $\mathfrak{R}$  is a subset of  $\mathcal{A} \times \mathcal{B}$ . Two elements  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$  are related, and we write  $a\mathfrak{R}b$ , if  $(a, b) \in \mathfrak{R} \subseteq \mathcal{A} \times \mathcal{B}$ .

If  $\mathcal{B} = \mathcal{A}$ , we say that  $\mathfrak{R}$  is a relation “on”  $\mathcal{A}$ .

#### Definition A.1.2 (Function)

A **function** between two sets  $\mathcal{A}$ ,  $\mathcal{B}$  is a relation  $\mathfrak{R}_f$  such that, given an element  $a \in \mathcal{A}$ , then there exists at most one element  $b \in \mathcal{B} : a\mathfrak{R}_f b$ .

We usually write  $b = f(a)$  in place of  $a\mathfrak{R}_f b$ .

#### Definition A.1.3 (Equivalence relation)

Given a set  $\mathcal{A}$ , a relation  $\mathfrak{R}$  on  $\mathcal{A}$  is an **equivalence relation** if it has the following properties:

1. reflexivity:  $a\mathfrak{R}a \ \forall a \in \mathcal{A}$ ;
2. symmetry:  $a\mathfrak{R}b \iff b\mathfrak{R}a \ \forall a, b \in \mathcal{A}$ ;
3. transitivity:  $a\mathfrak{R}b \wedge b\mathfrak{R}c \implies a\mathfrak{R}c \ \forall a, b, c \in \mathcal{A}$ .

#### Example A.1.1

Take  $\mathcal{A} = \mathbb{Z}$ . Then, the relation  $a\mathfrak{R}b \iff \exists k \in \mathbb{Z} : a - b = 2k$  is an equivalence relation:  $a - a = 2k$  with  $k = 0$  (reflexivity),  $a - b = 2k \iff b - a = 2h$  with  $h = -k$  (symmetry) and  $a - b = 2k, b - c = 2h \implies a - c = 2l$  with  $l = k + h$  (transitivity).

#### Definition A.1.4 (Equivalence class)

Given a set  $\mathcal{A}$  and an equivalence relation  $\mathfrak{R}$  on  $\mathcal{A}$ , then the **equivalence relation** of  $a \in \mathcal{A}$  is defined as  $[a]_{\mathfrak{R}} := \{b \in \mathcal{A} : a\mathfrak{R}b\}$ .

In absence of ambiguity, the subscript  $\mathfrak{R}$  is dropped, and the equivalence class  $a \in \mathcal{A}$  is simply denoted by  $[a]$ .

### Theorem A.1.1

Given a set  $\mathcal{A}$ , an **equivalence** relation  $\mathfrak{R}$  on  $\mathcal{A}$  and two elements  $a, b \in \mathcal{A}$ , then:

1.  $a \in [a]_{\mathfrak{R}}$ ;
2.  $a\mathfrak{R}b \implies [a]_{\mathfrak{R}} = [b]_{\mathfrak{R}}$ ;
3.  $a\not\mathfrak{R}b \implies [a]_{\mathfrak{R}} \cap [b]_{\mathfrak{R}} = \emptyset$ .

*Proof.* The first proposition is true by reflexivity. To prove the second proposition, let  $x \in [a]_{\mathfrak{R}}$ : then,  $x\mathfrak{R}a$ , but also  $x\mathfrak{R}b$  by transitivity, hence  $x \in [b]_{\mathfrak{R}}$ . This proves  $[b]_{\mathfrak{R}} \subseteq [a]_{\mathfrak{R}}$ , and the vice versa is equivalently proven, hence  $[a]_{\mathfrak{R}} = [b]_{\mathfrak{R}}$ . To prove the third proposition, suppose  $\exists x \in [b]_{\mathfrak{R}} \cap [a]_{\mathfrak{R}}$ : then,  $x\mathfrak{R}a \wedge x\mathfrak{R}b \implies a\mathfrak{R}b$  by transitivity, which is absurd.  $\square$

This theorem shows that an equivalence relation splits the set in separated equivalence classes.

### Definition A.1.5 (Partition)

Given a set  $\mathcal{X} \neq \emptyset$  and its power set  $\mathcal{P}(\mathcal{X}) := \{\mathcal{A} : \mathcal{A} \subseteq \mathcal{X}\}$ , a **partition** of  $\mathcal{X}$  is a collection of subsets  $\{\mathcal{A}_i\}_{i \in \mathcal{I}} \subseteq \mathcal{P}(\mathcal{X})$  which satisfies the following properties:

1.  $\mathcal{A}_i \neq \emptyset \ \forall i \in \mathcal{I}$ ;
2.  $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset \ \forall i \neq j \in \mathcal{I}$ ;
3.  $\mathcal{X} = \bigcup_{i \in \mathcal{I}} \mathcal{A}_i$ .

The equivalence classes determined by an equivalence relation form a partition of the set it is defined on.

### Definition A.1.6 (Quotient set)

Given a set  $\mathcal{A}$  and an equivalence relation  $\mathfrak{R}$  on  $\mathcal{A}$ , the **quotient set**  $\mathcal{A}/\mathfrak{R}$  is defined as the set of all equivalence classes of  $\mathcal{A}$  determined by  $\mathfrak{R}$ .

### Example A.1.2 ( $\mathbb{Z}$ as a quotient set)

The set  $\mathbb{Z}$  can be seen as a quotient set  $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\mathfrak{R}$  with  $(n, m)\mathfrak{R}(n', m') \iff n - m = n' - m'$ . Indeed, there are three kinds of equivalence classes:  $[(n, 0)] \equiv n$ ,  $[(0, n)] \equiv -n$  and  $[(0, 0)] \equiv 0$ .

### Example A.1.3 (Modular equivalence)

Given  $n \in \mathbb{N}$ , the **congruence modulo  $n$**  relation is an equivalence relation on  $\mathbb{Z}$  defined as  $a \equiv_n b \iff \exists k \in \mathbb{Z} : a - b = kn$ . This relation defines the quotient set  $\mathbb{Z}_n \equiv \mathbb{Z}/(\text{mod } n)$ , which in general is  $\mathbb{Z}_n = \{[0]_n, [1]_n, \dots, [n-1]_n\}$ .

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