General Relativity and Cosmology

Leonardo Cerasi

General Relativity and Cosmology

This document has been typeset by LaTeX with the `book` class.
Source code available on GitHub at LeonardoCerasi/notes.

Author's email: leonardo@cerasi.net

Notation

Conventions

In these notes, the Lorentz–Minkowski metric $\eta_{\mu\nu}$ has signature $(+,-,-,-)$ and Greek indices generally run over spacetime coordinates, while Latin indices are general $\mathbb{N}_0$-indices defined in each context. Repeated indices are generally summed over, unless otherwise specified. The n-dimensional Levi–Civita symbol $\epsilon^{i_1 \cdots i_n}$ is defined with the convention $\epsilon^{01\dots n} = +1$.
Given $\alpha \in \mathbb{C}^{n\times n}$, with $n \in \mathbb{N}_0$, its complex conjugate is denoted as α^*, its transpose as $\alpha^\intercal$ and its Hermitian conjugate as $\alpha^\dagger \coloneqq (\alpha^*)^\intercal$. Given a Dirac spinor $\Psi \in \mathbb{C}^4$, its Dirac dual (or Dirac adjoint) is defined as $\bar{\Psi} \coloneqq \Psi^\dagger \gamma^0$.
The Landau symbol for a function $f : D \subseteq \mathbb{R} \to \mathbb{C}$ is defined by the condition $\exists M \in \mathbb{R} : |o(f(x))| \leq M\,|f(x)|\ \forall x \in D$.

Mathematical notation

The empty set is denoted by $\varnothing$ and the power set of a set A by $\wp(A) \coloneqq \{B : B \subseteq A\}$. The counting numbers are $\mathbb{N} \equiv \{1,2,3,\dots\}$, and the natural numbers are defined by $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$.
The imaginary unit is denoted by i and the unit quaternions by $\mathrm{i}, \mathrm{j}, \mathrm{k}$, so that $\mathbb{C}(\mathbb{R}) = \mathrm{span}(1, \mathrm{i})$ and $\mathbb{H}(\mathbb{R}) = \mathrm{span}(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$.
The n-dimensional sphere is denoted by $\mathbb{S}^n$, while the n-dimensional disk by $\mathbb{D}^n$, so that $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$, where in general $\partial\Omega$ denotes the boundary of Ω.
The permutation group of n objects, i.e. the n^{th} symmetric group, is denoted by S_n.
Given two $\mathbb{K}$-vector spaces V and W, with $\mathbb{K}$ a generic field, the space of all $\mathbb{K}$-linear applications $f : V \to W$ is denoted by $\mathrm{Hom}_{\mathbb{K}}(V,W)$: in particular, $\mathrm{Hom}_{\mathbb{K}}(V) \equiv \mathrm{End}(V)$. The subset of $\mathrm{End}(V)$ of all automorphisms of V is the automorphism group $\mathrm{Aut}(V)$, which is a group under composition of morphisms.
Given a manifold $\mathcal{M}$, the space of all smooth scalar functions on $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$, the space of all vector fields on $\mathcal{M}$ by $\mathfrak{X}(\mathcal{M})$, the space of all p-forms on $\mathcal{M}$ by $\bigwedge^p(\mathcal{M})$ and the Grassmann algebra of $\mathcal{M}$ by $\bigwedge(\mathcal{M}) \coloneqq \bigoplus_{k=0}^n \bigwedge^k(\mathcal{M})$.
The exterior derivative is denoted by d, the partial derivative by ∂, the nabla operator by $\boldsymbol{\nabla} \equiv (\partial_1, \partial_2, \partial_3)$, the Laplacian operator by $\triangle \equiv \boldsymbol{\nabla}^2$ and the D'Alembert operator by $\Box \equiv \partial_0^2 - \triangle$.
A list of "important" Lie groups:
$\mathrm{GL}(n,\mathbb{K}) \coloneqq \{\mathrm{A} \in \mathbb{K}^{n\times n} : \det \mathrm{A} \neq 0\}$ general linear group (Lie group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$)
$\mathrm{SL}(n,\mathbb{K}) \coloneqq \{\mathrm{A} \in \mathrm{GL}(n,\mathbb{K}) : \det \mathrm{A} = 1\}$ special linear group (Lie group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$)
$\mathrm{O}(n) \coloneqq \{\mathrm{A} \in \mathbb{R}^{n\times n} : \mathrm{A}\mathrm{A}^\intercal = \mathrm{A}^\intercal\mathrm{A} = \mathrm{I}_n\}$ orthogonal group
$\mathrm{SO}(n) \coloneqq \{\mathrm{A} \in \mathrm{O}(n) : \det \mathrm{A} = 1$ special orthogonal group
$\mathrm{U}(n) \coloneqq \{\mathrm{A} \in \mathbb{C}^{n\times n} : \mathrm{A}\mathrm{A}^\dagger = \mathrm{A}^\dagger\mathrm{A} = \mathrm{I}_n\}$ unitary group
$\mathrm{SU}(n) \coloneqq \{\mathrm{A} \in \mathrm{U}(n) : \det \mathrm{A} = 1$ special unitary group
Given a Lie group G, its associated Lie algebra is denoted by $\mathfrak{g}$.

Contents

Introduction

Prova

prova

prova

prova

Part I

Differential Geometry

Chapter 1

Manifolds

§1.1 Topological spaces

Definition 1.1.1 (Topological space)

The **topology** $\mathcal{T}$ of a set X is a family of subsets of X, i.e. $\mathcal{T} \subseteq \wp(X)$, defined as **open sets**, with the following properties:

1. $\varnothing, X \in \mathcal{T}$;
2. $O_\alpha, O_\beta \in \mathcal{T} \implies O_\alpha \cap O_\beta \in \mathcal{T}$;
3. $\{O_\alpha\}_{\alpha\in\mathcal{I}} \subset \mathcal{T} \implies \bigcup_{\alpha\in\mathcal{I}} O_\alpha \in \mathcal{T}$.

A **topological space** M is a set of points, endowed with a topology $\mathcal{T}$.

Given a topological space $(M, \mathcal{T})$, $O \in \mathcal{T}$ is a **neighbourhood** of a point $p \in M$ if $p \in O$: then, $(M, \mathcal{T})$ is **Hausdorff** if $\forall p, q \in M \ \exists O_1, O_2 \in \mathcal{T}$ neighbourhoods of p and q respectively such that $O_1 \cap O_2 = \varnothing$.
Topological spaces allow to introduce the concept of continuity: given two topological spaces $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$, a map $f : M_1 \to M_2$ is **continuous** if $O \in \mathcal{T}_2 \Rightarrow f^{-1}(O) \in \mathcal{T}_1$.

Definition 1.1.2 (Homeomorphism)

Given two topological spaces $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$, a map $f : M_1 \to M_2$ is a **homeomorphism** if it is bijective and bicontinuous, i.e. both f and f^{-1} are continuous.

§1.2 Differentiable Manifolds

Definition 1.2.1 (Differentiable manifold)

An n-dimensional **differentiable manifold** $\mathcal{M}$ is a Hausdorff topological space such that:

1. $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall p \in \mathcal{M} \ \exists O \in \mathcal{T}(\mathcal{M}) : p \in O \wedge \exists \varphi : O \to U \in \mathcal{T}(\mathbb{R}^n)$ homeomorphism;
2. given $O_\alpha, O_\beta \in \mathcal{T}(\mathcal{M}) : O_\alpha \cap O_\beta \neq \varnothing$, the corresponding maps $\varphi_\alpha : O_\alpha \to U_\alpha$ and $\varphi_\beta : O_\beta \to U_\beta$ must be *compatible*, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(O_\alpha \cap O_\beta) \to \varphi_\beta(O_\alpha \cap O_\beta)$ and its inverse must be smooth (of $\mathcal{C}^\infty$ class).

The maps φ_α are called **charts** and a collection of compatible charts is called an **atlas**: a maximal atlas $\mathcal{A}$ is an atlas such that $\bigcup_{\alpha\in\mathcal{I}} O_\alpha = \mathcal{M}$. Two atlases are compatible if each chart of one atlas is compatible with every chart of the other: they define the same differentiable structure on the manifold.
Each chart φ_α provides a coordinate system on O_α: $\varphi_\alpha(p) = (x^1(p), \dots, x^\mu(p), \dots, x^n(p))$. The transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are therefore coordinate transformations on overlapping regions.

Example 1.2.1 (Spheres)

$\mathbb{S}^n$ is a differentiable manifold for $n \in \mathbb{N}$. In particular, to define a differentiable structure on $\mathbb{S}^1$, an atlas of two charts is needed: the standard parametrization $\vartheta \in [0, 2\pi)$ is not a well-defined chart because $[0, 2\pi)$ is not an open set in the Euclidean topology of $\mathbb{R}$, therefore the elimination of a point is necessary; usually, the two charts of the atlas are defined by $\vartheta_1 \in (0, 2\pi)$, excluding $(1, 0)$ (in the embedding space $\mathbb{R}^2$), and $\vartheta_2 \in (-\pi, \pi)$, excluding $(-1, 0)$: they are evidently compatible, thus they form a maximal atlas.

In the remainder of these notes, $\mathcal{M}$ is always taken to be an n-dimensional differentiable manifold.

§1.2.1 Maps between manifolds

Locally mapping $\mathcal{M}$ to $\mathbb{R}^n$ allows for the extension of the concepts of Analysis from $\mathbb{R}^n$ to $\mathcal{M}$.

Definition 1.2.2 (Smooth maps)

function $f : \mathcal{M} \to \mathbb{R}$ on a differentiable manifold $(\mathcal{M}, \mathcal{A})$ is **smooth** if $f \circ \varphi_\alpha^{-1} : U_\alpha \to \mathbb{R}$ is smooth for all charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$.
A map $f : \mathcal{M} \to \mathcal{N}$ between two differentiable manifolds $(\mathcal{M}, \mathcal{A}_1), (\mathcal{N}, \mathcal{A}_2)$ is **smooth** if $\psi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1} : U_{\alpha_1} \to V_{\alpha_2}$ is smooth for all charts $(U_{\alpha_1}, \varphi_{\alpha_1}) \in \mathcal{A}_1, (V_{\alpha_2}, \varphi_{\alpha_2}) \in \mathcal{A}_2$.

A smooth homeomorphism $f : \mathcal{M} \to \mathcal{N}$ between two differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$ is called a **diffeomorphism**.

Proposition 1.2.1 (Diffeomorphic manifolds)

If $\mathcal{M}$ and $\mathcal{N}$ are **diffeomorphic**, then $\dim_{\mathbb{R}} \mathcal{M} = \dim_{\mathbb{R}} \mathcal{N}$.

Example 1.2.2 (Diffenetiable structures)

$\mathbb{S}^7$ can be covered by multiple incompatible atlases: the resulting manifolds are homeomorphic but not diffeomorphic.
$\mathbb{R}^n$ has a unique differentiable structure for all $n \in \mathbb{N}$, except for $n = 4$: $\mathbb{R}^4$ can be covered by infinitely-many incompatible atlases.

§1.3 Tangent spaces

The notions of calculus can be defined on a differential manifold $(\mathcal{M}, \mathcal{A})$ with the notion of tangent spaces. Indeed, the derivative of a function $f : \mathcal{M} \to \mathbb{R}$ at a point $p \in \mathcal{M}$, covered by the chart (φ, U), is defined as:

$$\left.\frac{\partial f}{\partial x^\mu}\right|_p := \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu}\right|_{\varphi(p)} \tag{1.1}$$

Evidently, this definition depends on the choise of coordinates x^μ, thus it depends on the chart.

§1.3.1 Tangent vectors

Definition 1.3.1 (Space of smooth functions)

The set of all smooth functions on $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$.

Definition 1.3.2 (Tangent vector)

A **tangent vector** to $\mathcal{M}$ in $p \in \mathcal{M}$ is an operator $X_p : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ such that:

1. $X_p(f+g) = X_p(f) + X_p(g)\ \forall f, g \in \mathcal{C}^\infty(\mathcal{M})$;
2. $X_p(f) = 0$ for all constant functions $f \in \mathcal{C}^\infty(\mathcal{M})$;
3. $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)\ \forall f, g \in \mathcal{C}^\infty(\mathcal{M})$.

Conditions 2. and 3. trivially imply that $X_p(\alpha f) = \alpha X_p(f)\ \forall \alpha \in \mathbb{R}$, which means that X_p is a linear operator, i.e. $X_p \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{C}^\infty(\mathcal{M}), \mathbb{R})$. Moreover, it is simple to check that $\partial_\mu|_p$ satisfies the conditions of Def. 1.3.2.

Theorem 1.3.1 (Tangent space)

The set $T_p\mathcal{M}$ of all tangent vectors at a point $p \in \mathcal{M}$ forms an n-dimensional space, called **tangent space**, and $\{\partial_\mu|_p\}_{\mu=1,\dots,n}$ is a basis of such space.

Proof. Defining $f \circ \varphi^{-1} \equiv F : U \subset \mathcal{M} \to \mathbb{R}$, with $f : \mathcal{M} \to \mathcal{M}$ and $(\varphi, U) \in \mathcal{A}$, it can be shown that, in some neighbourhood of p (not necessarily U), F cal always be written as:

$$F(x) = F(x^\mu(p)) + (x^\mu - x^\mu(p))\, F_\mu(x)$$

for some functions $\{F_\mu\}_{\mu=1,\dots,n}$ (e.g. $F(x) = F(0) + x\int_0^1 \mathrm{d}t\, F(xt)$). Applying $\partial_\mu|_{x(p)}$:

$$\left.\frac{\partial F}{\partial x^\mu}\right|_{x(p)} = F_\mu(x(p))$$

Defining $f_\mu \equiv F_\mu \circ \varphi$, for any $q \in \mathcal{M}$ in an appropriate neighbourhood of p:

$$f(q) = f(p) + (x^\mu(q) - x^\mu(p))\, f_\mu(q)$$

Moreover, remembering Eq. Eq. 1.1:

$$f_\mu(p) = F_\mu \circ \varphi(p) = F_\mu(x(p)) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x(p)} = \left.\frac{\partial f}{\partial x^\mu}\right|_p$$

Using these facts, the action of a tangent vector can be written explicitly:

$$\begin{aligned} X_p(f) &= X_p\left(f(p) + (x^\mu - x^\mu(p))\, f_\mu\right) \\ &= X_p\left(f(p)\right) + X_p\left((x^\mu - x^\mu(p))\right) f_\mu(p) + (x^\mu - x^\mu(p))\,(p) X_p\left(f_\mu\right) \\ &= X_p\left(x^\mu\right) f_\mu(p) \end{aligned}$$

because $f(p)$ is a constant and $\left(x^\mu - x^\mu(p)\right)(p) = x^\mu(p) - x^\mu(p) = 0$. Therefore, remembering the expression for $f_\mu(p)$:

$$X_p = X_p(x^\mu)\left.\frac{\partial}{\partial x^\mu}\right|_p \equiv X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p$$

Thus, $T_p\mathcal{M} = \langle \{\partial_\mu|_p\}_{\mu=1,\dots,n}\rangle$. To check for linear independence, suppose $\alpha = \alpha^\mu \partial_\mu|_p \equiv 0$: acting on $f = x^\nu$, it gives $\alpha(f) = \alpha_\mu \partial_\mu(x^\nu)|_p = \alpha_\nu = 0$. This concludes the proof. □

§1.3.1.1 Changing coordinates

Although $\partial_\mu|_p$ depends on the choice of coordinates (it is a **coordinate basis**), the existence of X_p is independent of that choice.
If two different charts (φ, U) and $(\tilde{\varphi}, V)$ intersect in a neighbourhood of $p \in U \cap V$, the transition from x^μ to y^μ can be expressed as:

$$X_p(f) = X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p = X^\mu \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\varphi(p)} \left.\frac{\partial f}{\partial y^\nu}\right|_p \tag{1.2}$$

This equation can have two interpretations, namely the alibi interpretation:

$$\left.\frac{\partial}{\partial x^\mu}\right|_p = \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\varphi(p)} \left.\frac{\partial}{\partial y^\nu}\right|_p \tag{1.3}$$

and the alias interpretation:

$$\tilde{X}^\nu = X^\mu \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\varphi(p)} \tag{1.4}$$

Components of vectors which transform this way are called **contravariant**.

§1.3.1.2 Curves

Consider a smooth curve on $\mathcal{M}$, i.e. a smooth map $\sigma : I \in \mathcal{T}(\mathbb{R}) \to \mathcal{M}$, WLOG parametrized as $\sigma(t) : \sigma(0) = p \in \mathcal{M}$; with a given chart (φ, U), this curve becomes $\varphi \circ \sigma : I \to \mathbb{R}^n$, parametrized by $x^\mu(t)$. The tangent vector to the curve in p is:

$$X_p = \left.\frac{dx^\mu(t)}{dt}\right|_{t=0} \left.\frac{\partial}{\partial x^\mu}\right|_p \tag{1.5}$$

This operator, applied to a function $f \in \mathcal{C}^\infty(\mathcal{M})$, computes the directional derivative of f along the curve. It can be showed that every tangent vector can be written as in Eq. Eq. 1.5, therefore the tangent space can be seen as the space of all possible tangent vectors to curves passing through p.
It must be noted that tangent spaces at different points are entirely different spaces: there is no way to directly compare vectors between them.

§1.3.2 Vector fields

Definition 1.3.3 (Vector fields)

A **vector field** X is a smooth map $X : p \in \mathcal{M} \mapsto X_p \in T_p\mathcal{M}$. The space of all vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$.

Note that a vector field can also be viewed as a smooth map $X : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, since $(X(f))(p) = X_p(f) \in \mathbb{R}$. Given a chart (φ, U), a vector field X can be expressed as:

$$X = X^\mu \frac{\partial}{\partial x^\mu} \tag{1.6}$$

with $X^\mu \in \mathcal{C}^\infty(\mathcal{M})$. This expression is only defined on U.

§1.3.2.1 Lie brakets

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, their product is clearly not a vector field, as it does not satisfy Leibniz' rule:

$$XY(fg) = XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) \neq XY(f)g + fXY(g)$$

where $XY(f) \equiv X(Y(f))$.

Definition 1.3.4 (Lie brackets)
Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, their commutator (or **Lie bracket**) is defined as:

$$[X, Y](f) = XY(f) - YX(f) \tag{1.7}$$

With a given chart:

$$\begin{aligned} [X, Y](f) &= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial f}{\partial x^\nu}\right) - Y^\mu \frac{\partial}{\partial x^\mu}\left(X^\nu \frac{\partial f}{\partial x^\nu}\right) \\ &= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial f}{\partial x^\nu} \end{aligned}$$

therefore:

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial x^\nu} \tag{1.8}$$

Theorem 1.3.2 (Jacobi identity)
Given $X, Y, Z \in \mathfrak{X}(\mathcal{M})$, then:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.9}$$

With Lie brackets, $\mathfrak{X}(\mathcal{M})$ can be given the structure of a Lie algebra.

§1.3.2.2 Integral curves

Definition 1.3.5 (Flow)
A **flow** on $\mathcal{M}$ is a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$, labelled by $t \in \mathbb{R}$, with group structure: $\sigma_0 = \mathbb{1}_\mathcal{M}$ and $\sigma_s \circ \sigma_t = \sigma_{s+t}$, thus $\sigma_{-t} = \sigma_t^{-1}$.

Such flows give rise to streamlines on the manifold: these streamlines are required to be smooth. Defining $x^\mu(\sigma_t) \equiv x^\mu(t)$, a vector field can be defined by the tangent to the streamlines at each point on the manifold:

$$X^\mu(x^\mu(t)) = \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} \tag{1.10}$$

The inverse reasoning is also possible: given $X \in \mathfrak{X}(\mathcal{M})$, the streamlines defined by Eq. 1.10 are the **integral curves** of X.

Proposition 1.3.1 (Infinitesimal flow)

The **infinitesimal flow** generated by $X \in \mathfrak{X}(\mathcal{M})$ is:

$$x^\mu(t) = x^\mu(0) + tX^\mu(x(t)) + o(t) \tag{1.11}$$

A vector field which generates a flow defined for all $t \in \mathbb{R}$ is called **complete**.

Theorem 1.3.3

f $\mathcal{M}$ is compact, then all $X \in \mathfrak{X}(\mathcal{M})$ are complete.

Example 1.3.1 (Integral curves on the 2-sphere)

On $\mathbb{S}^2$, the flow generated by $X = \partial_\phi$ is described by $\dot\phi = 1, \dot\theta = 0$, thus $\theta(t) = \theta_0$ and $\phi(t) = \phi_0 + t$: the flow lines are lines of constant latitude.

§1.3.3 Lie derivative

Defining calculus for vector fields requires a way to compare vectors of different tangent spaces.

Definition 1.3.6 (Pull-back)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a function $f : \mathcal{N} \to \mathbb{R}$, the **pull-back** of f is the function $\varphi^* f : \mathcal{M} \to \mathbb{R}$ such that $\varphi^* f(p) = f(\varphi(p))$.

Definition 1.3.7 (Push-forward)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$, the **push-forward** of X is the vector field $\varphi_* X \in \mathfrak{X}(\mathcal{N})$ such that $\varphi_* X(f) = X(\varphi^* f)$.

This last equality must be evaluated at the appropriate points: $[\varphi_* X(f)](\varphi(p)) = [X(\varphi^* f)](p)$. With the appropriate charts on $\mathcal{M}$ and $\mathcal{N}$, the definitions above can be rewritten with coordinates:

$$\varphi^* f(x) = f(y(x)) \tag{1.12}$$

$$\varphi_* X(f) = X^\mu \frac{\partial f(y(x))}{\partial x^\mu} = X^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial f(y)}{\partial y^\alpha} \tag{1.13}$$

The notions of pull-back and push-forward allow to compare tangent vectors at neighbouring points and, in particular, to define the derivative along a vector field.

Definition 1.3.8 (Lie derivative of functions)

Given a function $f : \mathcal{M} \to \mathbb{R}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$, the derivative of f along X, called **Lie derivative**, is defined as:

$$\mathcal{L}_X f(x) := \lim_{t\to 0} \frac{f(\sigma_t(x)) - f(x)}{t} = \left.\frac{df(\sigma_t(x))}{dt}\right|_{t=0} \tag{1.14}$$

where σ_t is the flow generated by X.

Lemma 1.3.1

$$\mathcal{L}_X f = X(f) \tag{1.15}$$

Proof. $\mathcal{L}_X f = \frac{df(\sigma_t)}{dt} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu(t)}{dt} = X^\mu \frac{\partial f}{\partial x^\mu} = X(f)$. □

The Lie derivative can be extended to vector fields.

Definition 1.3.9 (Lie derivative of vector fields)

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, the **Lie derivative** of Y along X is defined as:

$$\mathcal{L}_X Y_p := \lim_{t\to 0} \frac{((\sigma_{-t})_* Y)_p - Y_p}{t} \tag{1.16}$$

where σ_t is the flow generated by X.

The use of the inverse flow σ_{-t} is necessary because to evaluate the vector field $\mathcal{L}_X Y$ at the point $p \in \mathcal{M}$, the tangent vector $Y_{\sigma_t(p)} \in T_{\sigma_t(p)}\mathcal{M}$ must be "pushed-back" to $T_p\mathcal{M} = T_{\sigma_0(p)}\mathcal{M}$. With $t \to 0$, the infinitesimal flow σ_{-t} is, according to Eq. 1.11, $x^\mu(t) = x^\mu(0) - tX^\mu + o(t)$, therefore the Lie derivative of basis tangent vectors can be expressed as:

$$(\sigma_{-t})_* \partial_\mu = \frac{\partial x^\nu(t)}{\partial x^\mu}\frac{\partial}{\partial x^\nu(t)} = \left(\delta^\nu_\mu - t\frac{\partial X^\nu}{\partial x^\mu} + o(t)\right)\partial_\nu(t) \quad \Longrightarrow \quad \mathcal{L}_X \partial_\mu = -\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu \tag{1.17}$$

Moreover, by the Jacobi identity it follows that:

$$\mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z = \mathcal{L}_{[X,Y]} Z \tag{1.18}$$

Lemma 1.3.2

$$\mathcal{L}_X Y = [X, Y] \tag{1.19}$$

Proof. $\mathcal{L}_X Y = \mathcal{L}_X(Y^\mu \partial_\mu) = (\mathcal{L}_X Y^\mu)\,\partial_\mu + Y^\mu\,(\mathcal{L}_X \partial_\mu) = X^\nu \frac{\partial Y^\mu}{\partial x^\nu}\partial_\mu - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\partial_\nu = [X, Y]$. □

§1.4 Tensors

§1.4.1 Dual Spaces

Definition 1.4.1 (Dual space)

Given a vector space V, its **dual space** V^* is the space of all linear maps $f : V \to \mathbb{R}$.

Given a basis $\{\mathbf{e}_\mu\}_{\mu=1,\dots,n}$ of V, its **dual basis** $\{\mathbf{f}^\mu\}_{\mu=1,\dots,n}$ of V^* can be defined by:

$$\mathbf{f}^\nu(\mathbf{e}_\mu) = \delta^\nu_\mu \tag{1.20}$$

A general vector in V can be written as $X = X^\mu \mathbf{e}_\mu$, thus, according to Eq. 1.20, $X^\mu = \mathbf{f}^\mu(X)$. Clearly, the map $f : \mathbf{e}_\mu \mapsto \mathbf{f}^\mu$ is a non-canonical isomorphism between V and V^*, hence $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*$.

Proposition 1.4.1 (Dual of the dual)

$$(V^*)^* \cong V \tag{1.21}$$

Proof. The natural isomorphism between $(V^*)^*$ and V is basis-independent: suppose $X \in V$ and $\omega \in V^*$, so that $\omega(X) \in \mathbb{R}$; X can be viewed as $X \in (V^*)^*$ by setting $X(\omega) \equiv \omega(X)$. □

§1.4.2 Cotangent vectors

Definition 1.4.2 (Cotangent space)

Given a differentiable manifold $(\mathcal{M}, \mathcal{A})$ and a point $p \in \mathcal{M}$, the **cotangent space** to $\mathcal{M}$ at p is defined as $T^*_p\mathcal{M} := (T_p\mathcal{M})^*$.

Elements of $T^*_p\mathcal{M}$ are called cotangent vectors (or **covectors**).

Definition 1.4.3 (Covector field)

A covector field (or **1-form**) is a smooth map $\omega : p \in \mathcal{M} \mapsto \omega_p \in T^*_p\mathcal{M}$. The space of all 1-forms on $\mathcal{M}$ is denoted by $\bigwedge^1(\mathcal{M})$.

Note that a 1-form can also be viewed as a smooth map $\omega : \mathfrak{X}(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, since $(\omega(X))(p) = \omega_p(X_p) \in \mathbb{R}$.

Proposition 1.4.2

$\{\mathrm{d}x^\mu\}_{\mu=1,\dots,n}$ is a basis of $\bigwedge^1(\mathcal{M})$, dual to the basis $\{\partial_\mu\}_{\mu=1,\dots,n}$ of $\mathfrak{X}(\mathcal{M})$.

Proof. Consider $f \in \mathcal{C}^\infty(\mathcal{M})$ and define $\mathrm{d}f \in \bigwedge^1(\mathcal{M})$ by $\mathrm{d}f(X) = X(f)$: taking $f = x^\mu$ and $X = \partial_\mu$, $\mathrm{d}f(X) = \partial_\nu(x^\mu) = \delta^\mu_\nu$, therefore $\{\mathrm{d}x^\mu\}_{\mu=1,\dots,n}$ is the dual basis of $\bigwedge^1(\mathcal{M})$. □

This is also confirmed by $\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu$. These are coordinate bases: in fact, given two different charts $(\varphi, U), (\tilde{\varphi}, V)$:

$$\mathrm{d}y^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}x^\nu}\mathrm{d}x^\nu \tag{1.22}$$

which is the inverse of Eq. 1.3 (not evaluated at a specific point). This ensures that:

$$\mathrm{d}y^\mu\left(\frac{\partial}{\partial y^\nu}\right) = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\nu}\mathrm{d}x^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^\nu} = \delta^\mu_\nu$$

A 1-form $\omega \in \bigwedge^1(\mathcal{M})$ can thus be expressed both as $\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\mu \mathrm{d}y^\mu$, with:

$$\tilde{\omega}_\omega = \frac{\partial x^\nu}{\partial y^\mu}\omega_\nu \tag{1.23}$$

Components of 1-forms which transform this way are called **covariant**.

Definition 1.4.4 (Pull-back of 1-forms)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a 1-form $\omega \in \Lambda^1(\mathcal{N})$, the **pull-back** of ω is the 1-form $\varphi^*\omega \in \bigwedge^1(\mathcal{M})$ such that $\varphi^*\omega(X) = \omega(\varphi_* X)$.

With the appropriate charts on $\mathcal{M}$ and $\mathcal{N}$, the definition above can be rewritten with coordinates:

$$\varphi^*\omega = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu} \mathrm{d}x^\mu \tag{1.24}$$

Definition 1.4.5 (Lie derivative of 1-forms)

Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a 1-form $\omega \in \bigwedge^1(\mathcal{M})$, the **Lie derivative** of ω along X is defined as:

$$\mathcal{L}_X \omega_p := \lim_{t \to 0} \frac{(\sigma_t^* \omega)_p - \omega_p}{t} \tag{1.25}$$

where σ_t is the flow generated by X.

In contrast with the Lie derivative of a vector field, which pushes forward with σ_{-t} (i.e. pushes back), the Lie derivative of a 1-form pulls back with σ_t: this results in the difference of a minus sign with respect to Eq. 1.17, giving:

$$\mathcal{L}_X \mathrm{d}x^\mu = \frac{\partial X^\mu}{\partial x^\nu} \mathrm{d}x^\nu \tag{1.26}$$

Therefore, on a general 1-form $\omega = \omega_\mu \mathrm{d}x^\mu$:

$$\mathcal{L}_X \omega = \left(X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu \right) \mathrm{d}x^\mu \tag{1.27}$$

§1.4.3 Tensor fields

Definition 1.4.6 (Tensor)

A **tensor** of rank (r, s) at a $p \in \mathcal{M}$ of a differentiable manifold $(\mathcal{M}, \mathcal{A})$ is a multilinear map defined as:

$$T_p : \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r} \times \overbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}^{s} \to \mathbb{R} \tag{1.28}$$

For example, a cotangent vector $\omega_p \in T_p^*\mathcal{M}$ is a tensor of rank $(1, 0)$, while a tangent vector $X_p \in T_p\mathcal{M}$ is a tensor of rank $(0, 1)$.

Definition 1.4.7 (Tensor field)

A **tensor field** of rank (r, s) is a smooth map $T : p \in \mathcal{M} \mapsto T_p$ tensor of rank (r, s) at p. It can also be viewed as a smooth map $T : [\bigwedge^1(\mathcal{M})]^r \times [\mathfrak{X}(\mathcal{M})]^s \to \mathcal{C}^\infty(\mathcal{M})$.

Given appropriate bases for vector fields $\{\mathbf{e}_\mu\}_{\mu=1,\ldots,n}$ and 1-forms $\{\mathbf{f}^\mu\}_{\mu=1,\ldots,n}$, the components of a tensor field are defined as:

$$T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} := T(\mathbf{f}^{\mu_1}, \ldots, \mathbf{f}^{\mu_r}, \mathbf{e}_{\nu_1}, \ldots, \mathbf{e}_{\nu_s}) \tag{1.29}$$

Proposition 1.4.3 (Components of tensor fields)

On an n-dimensional manifold, a (r, s) tensor field has n^{r+s} components, each being an element of $\mathcal{C}^\infty(\mathcal{M})$.

Consider two general basis transformations, for vector fields and 1-forms, described by invertible matrices A and B such that $\tilde{\mathbf{e}}_\mu = A^\nu{}_\mu \mathbf{e}_\nu$ and $\tilde{\mathbf{f}}^\mu = B^\mu{}_\nu \mathbf{f}^\nu$, with necessary condition $A^\mu{}_\nu B^\rho{}_\mu = \delta^\rho_\nu$ to ensure duality: this implies $\mathrm{B} = \mathrm{A}^{-1}$, i.e. covectors transform inversely with respect to vectors. Thus:

$$\tilde{T}^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = B^{\mu_1}{}_{\rho_1} \dots B^{\mu_r}{}_{\rho_r} A^{k_1}{}_{\nu_1} \dots A^{k_s}{}_{\nu_s} T^{\rho_1 \dots \rho_r}{}_{k_1 \dots k_s} \tag{1.30}$$

If the considered basis are coordinate basis, then $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}$ and $B^\mu{}_\nu = \frac{\partial y^\mu}{\partial x^\nu}$.

§1.4.4 Operations on tensors

Algebric addition and multiplication by functions are trivially defined on tensors of the same rank.

Proposition 1.4.4 (T)

e space of all (r, s) tensors at a point $p \in \mathcal{M}$ is a vector space.

Definition 1.4.8 (G)

ven two tensor fields S of rank (p, q) and T of rank (r, s), their *tensor product* is defined as:

$$\begin{aligned} S \otimes T(\omega_1, \dots, \omega_p, \eta_1, \dots, \eta_r, X_1, \dots, X_q, Y_1, \dots, Y_s) \\ = S(\omega_1, \dots, \omega_p, X_1, \dots, X_q) T(\eta_1, \dots, \eta_r, Y_1, \dots, Y_s) \end{aligned} \tag{1.31}$$

or, in components:

$$(S \otimes T)^{\mu_1 \dots \mu_p \nu_1 \dots \nu_r}{}_{\rho_1 \dots \rho_q \dots \sigma_1 \dots \sigma_s} = S^{\mu_1 \dots \mu_p}{}_{\rho_1 \dots \rho_q} T^{\nu_1 \dots \nu_r}{}_{\sigma_1 \dots \sigma_s} \tag{1.32}$$

It is also possible to contract tensor $((r, s) \mapsto (r-1, s-1))$: for example, given a rank $(2, 1)$ tensor, a rank $(1, 0)$ tensor can be defined as $S(\omega) = T(\omega, \mathbf{f}^\mu, \mathbf{e}_\mu)$, with components $S^\mu = T^{\nu\mu}{}_\mu$; it must be noted that, in general, $T^{\nu\mu}{}_\mu \neq T^{\mu\nu}{}_\mu$.

Definition 1.4.9 (G)

ven an object $T_{\mu_1 \dots \mu_n}$ dependent on some indices, its *symmetric* and *antisymmetric* parts are respectively defined as:

$$T_{(\mu_1 \dots \mu_n)} := \frac{1}{n!} \sum_{\sigma \in S^n} T_{\sigma(\mu_1) \dots \sigma(\mu_n)} \tag{1.33}$$

$$T_{[\mu_1 \dots \mu_n]} := \frac{1}{n!} \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) T_{\sigma(\mu_1) \dots \sigma(\mu_n)} \tag{1.34}$$

Conventionally, indices surrounded by || are not (anti-)symmetrized (ex: $T_{[\mu|\nu|\rho]} = \frac{1}{2}(T_{\mu\nu\rho} - T_{\rho\nu\mu})$). As previously seen, vector fields are pushed forward and 1-form are pulled back: tensors will thus behave in a mixed way.

Definition 1.4.10 (G)

ven a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a (r, s) tensor field T on $\mathcal{M}$, the *push-forward* of T is the (r, s) tensor field $\varphi_* T$ on $\mathcal{N}$ such that, for $\omega_j \in \Lambda^1(\mathcal{N})$ and $X_j \in \mathfrak{X}(\mathcal{N})$:

$$\varphi_* T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) = T(\varphi^* \omega_1, \ldots, \varphi^* \omega_r, \varphi_*^{-1} X_1, \ldots, \varphi_*^{-1} X_s) \tag{1.35}$$

Definition 1.4.11 (G)

ven a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a (r, s) tensor field T on $\mathcal{M}$, the *Lie derivative* of T along X is defined as:

$$\mathcal{L}_X T_p := \lim_{t \to 0} \frac{((\sigma_{-t})_* T)_p - T_p}{t} \tag{1.36}$$

where σ_t is the flow generated by X.

§1.5 Differential forms

Definition 1.5.1 (A)

otally anti-symmetric $(0, p)$ tensor is defined as a *p-form*. The set of all p-forms over a manifold $\mathcal{M}$ is denoted as $\bigwedge^p(\mathcal{M})$.

Proposition 1.5.1 (A)

p $-formhas$ $\binom{n}{p}$ independent components.

Proposition 1.5.2 (T)

e maximum degree of differential forms is $p = n \equiv \dim_{\mathbb{R}} \mathcal{M}$: forms in $\bigwedge^n(\mathcal{M})$ are called *top forms*.

Definition 1.5.2 (G)

ven $\omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, their *wedge product* is a $(p+q)$-form defined as:

$$(\omega \wedge \eta)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \frac{(p+q)!}{p! q!} \omega_{[\mu_1 \ldots \mu_p} \eta_{\nu_1 \ldots \nu_q]} \tag{1.37}$$

Example 1.5.1 (G)

ven $\omega, \eta \in \bigwedge^2(\mathcal{M})$, their wedge product is $(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu$.

Proposition 1.5.3 (G)

ven $\omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$:

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \tag{1.38}$$

Corollary 1.5.3.1 ()

$\wedge\omega = 0 \quad \forall\omega \in \bigwedge^p(\mathcal{M}) : p$ is odd.

Proposition 1.5.4 (T)

e wedge product is associative.

Proposition 1.5.5 (I)

$\{\mathbf{f}^\mu\}_{\mu=1,\ldots,n}$ is a basis of $\bigwedge^1(\mathcal{M})$, then $\{\mathbf{f}^{\mu_1} \wedge \cdots \wedge \mathbf{f}^{\mu_p}\}_{\mu_1,\ldots,\mu_p=1,\ldots,n}$ is a basis of $\bigwedge^p(\mathcal{M})$.

Locally $\{dx^\mu\}_{\mu=1,\ldots,n}$ is a basis of $T_p^*\mathcal{M}$, thus a general p-form can be locally written as:

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{1.39}$$

Definition 1.5.3 (T)

e *exterior derivative* is a map $d : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p+1}(\mathcal{M})$ defined as:

$$(d\omega)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\omega_{\mu_2\ldots\mu_{p+1}]} \tag{1.40}$$

In local coordinates:

$$d\omega = \frac{1}{p!}\frac{\partial\omega_{\mu_1\ldots\mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{1.41}$$

Theorem 1.5.1 ()

$^2 = 0$.

Proof. Consequence of Schwarz lemma. □

Proposition 1.5.6 (G)

ven $\omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, then:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \tag{1.42}$$

Proposition 1.5.7 (G)

ven a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and $\omega \in \bigwedge^p(\mathcal{M})$, then $d(\varphi^*\omega) = \varphi^*(d\omega)$.

Corollary 1.5.7.1 (G)

ven $X \in \mathfrak{X}(\mathcal{M}), \omega \in \bigwedge^p(\mathcal{M})$, then $d(\mathcal{L}_X\omega) = \mathcal{L}_X(d\omega)$.

Definition 1.5.4 ()

$\in \bigwedge^p(\mathcal{M})$ is *closed* if $d\omega = 0$.

Definition 1.5.5 ()

$\in \bigwedge^p(\mathcal{M})$ is *exact* if $\exists \eta \in \bigwedge^{p-1}(\mathcal{M}) : \omega = d\eta$.

Theorem 1.5.2 ()

$\in \bigwedge^p(\mathcal{M})$ is exact $\Rightarrow$ it is closed.

Definition 1.5.6 (G)

ven a vector field $X \in \mathfrak{X}(\mathcal{M})$, the *interior product* determined by X is a map $\iota_X : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p-1}(\mathcal{M})$ defined as:

$$\iota_X \omega(Y_1, \dots, Y_{p-1}) := \omega(X, Y_1, \dots, Y_{p-1}) \tag{1.43}$$

On 0-forms (i.e. scalar functions), it is defined as $\iota_X f \equiv 0$.

Proposition 1.5.8 (G)

ven $X, Y \in \mathfrak{X}(\mathcal{M})$, then $\iota_X \iota_Y = -\iota_Y \iota_X$.

Proof. Consequence of the total anti-symmetry of p-forms. □

Proposition 1.5.9 (G)

ven $X \in \mathfrak{X}(\mathcal{M}), \omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, then:

$$\iota_X(\omega \wedge \eta) = \iota_X \omega \wedge \eta + (-1)^p \omega \wedge \iota_X \eta \tag{1.44}$$

Theorem 1.5.3 (G)

ven a vector field $X \in \mathfrak{X}(\mathcal{M})$, then:

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d \tag{1.45}$$

Proof. Consider $\omega \in \bigwedge^1(\mathcal{M})$:

$$\iota_X(d\omega) = \iota_X \frac{1}{2} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)\, dx^\mu \wedge dx^\nu = X^\mu \partial_\mu \omega_\nu dx^\nu - X^\nu \partial_\mu \omega_\nu dx^\mu$$

$$d(\iota_X \omega) = d(\omega_\mu X^\mu) = X^\mu \partial_\nu \omega_\mu dx^\nu + \omega_\mu \partial_\nu X^\mu dx^\nu$$

Thus, adding these expressions and recalling Eq. 1.27:

$$(d\iota_X + \iota_X d)\omega = (X^\mu \partial_\mu \omega_\nu + \omega_\mu \partial_\nu X^\mu)\, dx^\nu = \mathcal{L}_X \omega$$

□

§1.5.1 de Rham cohomology

While exact $\Rightarrow$ closed, the converse is not true, in general: it depends on the topological properties of the manifold.

Lemma 1.5.1 (I)

$\mathcal{M}$ is simply connected, then $\omega \in \bigwedge^p(\mathcal{M})$ closed $\Rightarrow \omega$ exact.

In general, it is always possible to choose a simply connected neighbourhood of a point $p \in \mathcal{M}$, in which every closed form is exact, but that may not always be possible globally.
It is convenient to set the notation $d_p \equiv d : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p+1}(\mathcal{M})$.

Definition 1.5.7 (T)

e set of all closed p-forms on $\mathcal{M}$ is denoted by $Z^p(\mathcal{M}) := \ker d_p$.

Definition 1.5.8 (T)

e set of all exact p-forms on $\mathcal{M}$ is denoted by $B^p(\mathcal{M}) := \operatorname{ran} d_{p-1}$.

Definition 1.5.9 (T)

o closed p-forms $\omega, \omega' \in Z^P(\mathcal{M})$ are said to be *equivalent* if $\omega = \omega' + \eta$ for some $\eta \in B^p(\mathcal{M})$.

Definition 1.5.10 (T)

e p^{th} *de Rham cohomolgy group* of a manifold $\mathcal{M}$ is defined to be:

$$H^p(\mathcal{M}) := Z^p(\mathcal{M})/B^p(\mathcal{M}) \tag{1.46}$$

Definition 1.5.11 (T)

e *Betti numbers* of a manifold $\mathcal{M}$ are defined as:

$$B_p := \dim_{\mathbb{R}} H^p(\mathcal{M}) \tag{1.47}$$

Theorem 1.5.4 (G)

ven a differentiable manifold, its Betti numbers are always finite.

$B_0 = 1$ for any connected manifold: there exist constant functions, which are manifestly closed and not exact, due to the non-existence of $-1 - forms$. Higher Betti numbers are non-zero only if the manifold has some non-trivial topology.

Definition 1.5.12 (T)

e *Euler's character* of a manifold $\mathcal{M}$ is defined as:

$$\chi(\mathcal{M}) := \sum_{p \in \mathbb{N}_0} (-1)^p B_p \tag{1.48}$$

Example 1.5.2 (T)

e n-sphere $\mathbb{S}^n$ has only $B_0 = B_n = 1$, thus $\chi(\mathbb{S}^n) = 1 + (-1)^n$.

Example 1.5.3 (T)
e n-torus $\mathbb{T}^n$ has $B_p = \binom{n}{p}$, thus $\chi(\mathbb{T}^n) = 0$.

§1.5.2 Integration

Definition 1.5.13 (A)
volume form on an n-dimensional differentiable manifold $\mathcal{M}$ is a nowhere-venishing top form v, i.e. locally $v = v(x)dx^1 \wedge \cdots \wedge dx^n : v(x) \neq 0$. If such a form exists, the manifold is said to be *orientable*.

Definition 1.5.14 (G)
ven an orientable manifold $\mathcal{M}$ with volume form v, the orientation is:

- right-handed if $v(x) > 0$ locally on every neighbourhood of $\mathcal{M}$;
- left-handed if $v(x) < 0$ locally on every neighbourhood of $\mathcal{M}$;

To ensure that the handedness of the manifold doesn't change on overlapping charts:

$$v = v(x)\frac{\partial x^1}{\partial y^{\mu_1}}dy^{\mu_1} \wedge \cdots \wedge \frac{\partial x^n}{\partial x^{\mu_n}}dx^{\mu_n} = v(x)\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)dy^1 \wedge \cdots \wedge dy^n$$

It is therefore necessary that the two sets of coordinates on the overlapping region satisfy:

$$\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) > 0 \tag{1.49}$$

Non-orientable manifolds cannot be covered by overlapping charts satisfying this condition.

Example 1.5.4 (T)
e real projective space $\mathbb{RP}^n$ is orientable for odd n and non-orientable for even n.

Example 1.5.5 (T)
e complex projective space $\mathbb{CP}^n$ is orientable for all $n \in \mathbb{N}$.

Definition 1.5.15 (G)
ven a function $f : \mathcal{M} \to \mathbb{R}$ on an orientable manifold $\mathcal{M}$ with volume form v and a chart (φ, U) on $\mathcal{M}$ with coordinates $\{x^\mu\}_{\mu=1,\dots,n}$, the *integral* of f on $O = \varphi^{-1}(U) \subset \mathcal{M}$ is defined as:

$$\int_O fv := \int_U dx_1 \dots dx_n\, f(x)v(x) \tag{1.50}$$

It is clear that the volume form acts like a measure on the manifold. To integrate over the whole manifold, it must be divided up into different regions, each covered by a single chart.

Definition 1.5.16 (A)

k $-dimensionalmanifold\ \Sigma isasubmanifoldofan$ n $-dimensionalmanifold$ M $, with$ n $>$ k $, iftheree xistsaninjectivemap\ \varphi$: $\Sigma \to$M $suchthat\ \varphi_* : T_p(\Sigma) \to T_{\varphi(p)}(\mathcal{M})$ is injective.

Definition 1.5.17 (G)

ven a k-form $\omega \in \bigwedge^k(\mathcal{M})$, its integral over a k-dimensional submanifold Σ of $\mathcal{M}$ is defined as:

$$\int_{\varphi(\Sigma)} \omega := \int_\Sigma \varphi^* \omega \tag{1.51}$$

Example 1.5.6 (C)

nsider a 1-form $\omega \in \bigwedge^1(\mathcal{M})$ and a 1-dimensional submanifold γ of $\mathcal{M}$ described by a curve $\sigma : \gamma \to \mathcal{M} : x^\mu = \sigma^\mu(t)$: locally $\omega = \omega_\mu(x)dx^\mu$, thus the integral of ω on γ can be calculated as $\int_{\sigma(\gamma)} \omega = \int_\gamma \sigma^* \omega = \int_\gamma d\tau\, \omega_\mu(x) \frac{dx^\mu}{d\tau}$.

§1.5.2.1 Stokes' theorem

Integration can be generalized beyond smooth (i.e. differentiable) manifolds.

Definition 1.5.18 (A)

n-dimensional *manifold with boundary* is a Hausdorff topological space, equipped with a compatible maximal atlas, which is locally homeomorphic to $\mathbb{R}^{n-1} \times [a, \infty) : a \in \mathbb{R}$. The *boundary* $\partial\mathcal{M}$ is the 1-dimensional submanifold determined by $x^n = a$.

Theorem 1.5.5 (G)

ven an n-dimensional manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, then for any $\omega \in \bigwedge^{n-1}(\mathcal{M})$:

$$\int_\mathcal{M} d\omega = \int_{\partial\mathcal{M}} \omega \tag{1.52}$$

This important theorem unifies many different results.
Given the 1-dimensional manifold $I = [a, b] \subset \mathbb{R}$, then for any 0-form (i.e. scalar function) $\omega = \omega(x)$:

$$\int_I d\omega = \int_a^b \frac{d\omega}{dx} dx = \int_{\partial I} \omega = \omega(b) - \omega(a)$$

which is the fundamental theorem of calculus.
Given a 2-dimensional manifold with boundary $S \subset \mathbb{R}^2$ and a 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$, then $d\omega = (\partial_1\omega_2 - \partial_2\omega_1)dx^1 \wedge dx^2$ and:

$$\int_S d\omega = \int_S \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right) dx^1 dx^2 = \int_{\partial S} \omega = \int_{\partial S} \omega_1 dx^1 + \omega_2 dx^2$$

which is Green's theorem.
Given a 3-dimensional manifold with boundary $V \subset \mathbb{R}^3$ and a 2-form $\omega = \omega_1 dx^2 \wedge dx^3 + \omega_2 dx^3 \wedge$

$dx^1 + \omega_3 dx^1 \wedge dx^2$, then $d\omega = (\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3) dx^1 \wedge dx^2 \wedge dx^3$ and:

$$\int_V d\omega = \int_V \left(\frac{\partial \omega_1}{\partial x^1} + \frac{\partial \omega_2}{\partial x^2} + \frac{\partial \omega_3}{\partial x^3} \right) = \int_{\partial V} \omega = \int_{\partial V} \omega_1 dx^2 dx^3 + \omega_2 dx^3 dx^1 + \omega_3 dx^1 dx^2$$

which is Gauss' theorem.