Quantum Field Theory 1

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Contents

Part I

Field Theory

Chapter 1

Classical Field Theory

1.1 Continuous limit

1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x-axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a.
Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n, one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2}\dot{\eta}_n^2 - \frac{\lambda}{2}\left(\eta_n - \eta_{n-1}\right)^2 \right] \tag{1.1}$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda\left(\eta_{n+1} - 2\eta_n + \eta_{n-1}\right) \tag{1.2}$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \tag{1.3}$$

where the dispersion relation is:

$$\omega^2 = 2\lambda\left(1 - \cos k\right) \approx \lambda k^2 \tag{1.4}$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \qquad \Rightarrow \qquad k_m = \frac{2\pi m}{N}\,,\ m = 0, 1, \ldots, N-1 \tag{1.5}$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} \left[\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)} \right] \tag{1.6}$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjucated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \tag{1.7}$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*[1]:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \tag{1.8}$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators* [2]$\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} \left[e^{i(k_m n - \omega_m t)}\hat{a}_m + e^{-i(k_m n - \omega_m t)}\hat{a}_m^\dagger\right] \tag{1.9}$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \tag{1.10}$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2}\hat{\pi}_n^2 + \frac{\lambda}{2}\left(\hat{\eta}_n - \hat{\eta}_{n-1}\right)^2\right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}\right) \tag{1.11}$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m |0\rangle \quad \forall m = 0, 1, \ldots, N-1 \tag{1.12}$$

$$|n_0, n_1, \ldots, n_{N-1}\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} |0\rangle \tag{1.13}$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2}\sum_{m=0}^{N-1} \hbar\omega_m \tag{1.14}$$

$$E_{n_0, n_1, \ldots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \tag{1.15}$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as "particles" and n_m is the number of phonons in the m^{th} normal mode.

[1] Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t}\hat{\mathcal{O}}(0)e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{\mathrm{d}\hat{\mathcal{O}}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

[2] For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{p}(t)$ and $\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) + \hat{a}^\dagger(t)\right] \qquad \hat{p}(t) = -i\omega\sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) - \hat{a}^\dagger(t)\right]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger(t)\hat{a} + \frac{1}{2}\right)$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.

1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \to 0$ and $N \to \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x,t)$ dependent on the continuous real coordinate $x \in [0,R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x}\right)^2 \qquad\qquad \sum_n \longrightarrow \frac{1}{a}\int_0^R \mathrm{d}x$$

Proposition 1.1.1

In the continuous limit:

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x-x') = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

Proof. By direct calculation:

$$a\sum_n f(an)\frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R \mathrm{d}x\, f(x)\delta(x-y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \to kx$, symmetrizing $k_m \in [-\pi,\pi]$ (instead of $[0,2\pi]$) one finds:

$$\delta(x-x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na}\sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

where integration limits are $\pm\frac{\pi}{a} \to \pm\infty$. □

Proposition 1.1.2

The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R \mathrm{d}x\, e^{i(k-k')x} = 2\pi\delta(k-k')$$

By these relations, it can be seen that $\frac{\mathrm{d}k}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k+\mathrm{d}k$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.
In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}\rho_0(\partial_t\eta)^2 - \frac{\kappa}{2}(\partial_x\eta)^2\right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x,t) \equiv \sqrt{\rho_0}\eta(x,t)$ and $\kappa = c^2\rho_0$ and adding a pinning term $\propto \varphi^2$, the Lagrangian can be written as:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \frac{m^2c^4}{2}\phi^2\right] \tag{1.16}$$

The classical equation of motion of this field yields:

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi - m^2 c^4 \phi \tag{1.17}$$

The solutions of this wave equation can be written as:

$$\phi(x,t) = e^{i(kx - \omega_k t)} \tag{1.18}$$

with dispersion relation:

$$\omega_k^2 = c^2 k^2 + m^2 c^4 \tag{1.19}$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x,t) := \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi(x,t) \tag{1.20}$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R \mathrm{d}x \left[\frac{1}{2}\Pi^2 + \frac{c^2}{2}(\partial_x \phi)^2 + \frac{m^2 c^4}{2}\phi^2 \right] \tag{1.21}$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \sqrt{\frac{\hbar}{2\omega_k}} \left[e^{i(kx-\omega_k t)} \hat{a}_k + e^{-i(kx-\omega_k t)} \hat{a}_k^\dagger \right] \tag{1.22}$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi\delta(k - k') \tag{1.23}$$

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = i\hbar\delta(x - x') \tag{1.24}$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \frac{1}{2} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) = E_0 + \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k \tag{1.25}$$

The ground-state energy can be computed from Eq. 1.14, defining $\mathrm{Vol} := 2\pi\delta(k = 0)$:

$$E_0 = \mathrm{Vol} \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \frac{1}{2} \hbar\omega_k \tag{1.26}$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

1.2 Spacetime symmetries

1.2.1 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu}x^\mu x^\nu$, i.e. the orthogonal group $\mathrm{O}(1,3)$ (with signature $(+,-,-,-)$). The condition that $\Lambda^\mu{}_\nu$ must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \tag{1.27}$$

This implies that $\det\Lambda = \pm 1$: a transformation with $\det\Lambda = -1$ can always be written as the product of a transformation with $\det\Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines $\mathrm{SO}(1,3) \coloneqq \{\Lambda \in \mathrm{O}(1,3) : \det\Lambda = 1\}$.
Writing explicitly the temporal component $1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$, it is clear that $(\Lambda^0{}_0)^2 \geq 1$. Therefore, $\mathrm{O}(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0{}_0 \geq 1$ and the non-orthochronous component with $\Lambda^0{}_0 \leq 1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a dircrete transformation which reverses the sign of the temporal component.

Definition 1.2.1: Lorentz group

The *Lorentz group* $\mathrm{SO}^+(1,3)$ is the orthochronous component of $\mathrm{SO}(1,3)$.

The discrete transformations are factored out of the Lorentz group: these are *parity* and *time reversal*, which can be represented as $\mathcal{P}^\mu{}_\nu = \mathrm{diag}\,(+1,-1,-1,-1)$ and $\mathcal{T}^\mu_\nu = \mathrm{diag}\,(-1,+1,+1,+1)$. Applying these discrete transformations in all combinations (id, $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{PT}$) one gets the four disconnected components of $\mathrm{SO}(1,3)$, which are not simply connected. This means that Lorentz invariance does not include parity and time reversal invariance.
Considering the infinitesimal transformation and applying Eq. 1.27:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \qquad \Rightarrow \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x,y), (y,z) and (z,x) planes and the 3 rapidities of hyperbolic rotations in the (t,x), (t,y) and (t,z) planes.

Theorem 1.2.1

The Lorentz group is a non-compact Lie group.

Proof. Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in $[0,2\pi)$, rapidities vary in $\mathbb{R}$, so the differentiable manifold associated to $\mathrm{SO}^+(1,3)$ is not compact. □

1.2.1.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \mathrm{SO}^+(1,3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.28}$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a n-dimensional representation of $SO^+(1,3)$, both $[J^{\mu\nu}]^i{}_j$ and $[\Lambda]^i{}_j$ are $\mathbb{C}^{n\times n}$ matrices (Λ is real): for example, the $n=1$ representation acts on *scalars*, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \, \forall \mu, \nu = 0, \ldots, 3$.

4-vectors The $n=4$ representation acts on *contravariant 4-vectors* v^μ, which transform according to $v^\mu \mapsto \Lambda^\mu{}_\nu v^\nu$, and *covariant 4-vectors* v_μ, which transform according to $v_\mu \mapsto \Lambda_\mu{}^\nu v_\nu$. In this representation, the generators are represented as $\mathbb{C}^{4\times 4}$ matrices:

$$[J^{\mu\nu}]^\rho{}_\sigma = i\left(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma\right) \tag{1.29}$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}^+(1,3)$, called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \tag{1.30}$$

It is convenient to rearrange the 6 independent components of $J^{\mu\nu}$ into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad\qquad K^i := J^{i0} \tag{1.31}$$

The $\mathfrak{so}^+(1,3)$ can then be rewritten as:

$$[J^i, J^j] = i\epsilon^{ijk}J^k \qquad\qquad [J^i, K^j] = i\epsilon^{ijk}K^k \qquad\qquad [K^i, K^j] = -i\epsilon^{ijk}J^k \tag{1.32}$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^i are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \qquad\qquad \eta^i := \omega^{i0} \tag{1.33}$$

so that:

$$\Lambda = \exp\left[-i\boldsymbol{\theta}\cdot\mathbf{J} + i\boldsymbol{\eta}\cdot\mathbf{K}\right] \tag{1.34}$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts wich increase velocities with respect to said frame.

1.2.1.2 Tensor Representations

A generic (p,q)-tensor transforms as:

$$T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} \mapsto \Lambda^{\mu_1}{}_{\alpha_1}\ldots\Lambda^{\mu_p}{}_{\alpha_p}\Lambda_{\nu_1}{}^{\beta_1}\ldots\Lambda_{\nu_q}{}^{\beta_q}T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} \tag{1.35}$$

This shows that the representation of the Lorentz group which acts on (p,q)-tensors, although being of degree $n = 4^{p+q}$, is reducible into the direct product of $p+q$ 4-dimensional representations.
Moreover, consider the action of the Lorentz group on $(2,0)$-tensors: being $T^{\mu\nu} \mapsto \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu}T^{\mu\nu} \mapsto \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$\mathbf{4}\otimes\mathbf{4} = \mathbf{1}\oplus\mathbf{6}\oplus\mathbf{9} \tag{1.36}$$

These are irreducible representations which act on S, $A^{\mu\nu}$ and $\bar{S}^{\mu\nu} \equiv S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right)$ and $S^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right)$.

Decomposition under rotations Restricting the action to the SO(3) sub-group of $\mathrm{SO}^+(1,3)$, tensors can be decomposed according to irreducible representations of SO(3), which are labelled by the angular momentum $j \in \mathbb{N}_0$ and are of degree $n = 2j+1$. Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \tag{1.37}$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$. A 4-vector v^μ is irreducible under the action of $\mathrm{SO}^+(1,3)$, but under SO(3) it is decomposed into v^0 and $\mathbf{v}$, so $v^\mu \in \mathbf{0} \oplus \mathbf{1}$. A $(2,0)$-tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.36: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2}\epsilon^{ijk}A^{jk}$, so $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ under spatial rotations.
Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$, $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$, $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$ and $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk}A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

Example 1.2.1

Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two *invariant tensors* under $\mathrm{SO}^+(1,3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.27, and the Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\sigma{}_\gamma \Lambda^\rho{}_\delta \epsilon^{\alpha\beta\gamma\delta} = (\det\Lambda)\, \epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

1.2.1.3 Spinorial representations

The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same, which means that SU(2) and SO(3) are indistinguishable by infinitesimal transformations; however, they are globally different, as SO(3) rotations are periodic by 2π, while SU(2) rotations are periodic by 4π: in particular, it can be shown that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$, i.e. SU(2) is the universal cover of SO(3). This means that SU(2) representations can be labelled by $j \in \frac{1}{2}\mathbb{N}_0$, where half-integer spin representations are known as *spinorial representations*: they act on spinors, i.e. objects which change sign under rotations of 2π (thus not suitable to represent SO(3)).

Example 1.2.2

The $\mathbf{\frac{1}{2}}$ representation of SU(2) is a 2-dimensional representation where $J^i = \frac{\sigma^i}{2}$: Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, thus the $\mathfrak{su}(2)$ algebra is satisfied. Denoting the $m = \pm\frac{1}{2}$ states in the $\mathbf{\frac{1}{2}}$ representation as $|\uparrow\rangle$ and $|\downarrow\rangle$, the Clebsch-Gordan decomposition $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$ yields the triplet $(j=1)$ $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$ and the singlet $(j=0)$ $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Proposition 1.2.1

$$\mathfrak{so}^+(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{1.38}$$

Proof. Given the $\mathfrak{so}^+(1,3)$ algebra in Eq. 1.31, it is possible to define:

$$\mathbf{J}_\pm := \frac{1}{2}(\mathbf{J} \pm i\mathbf{K}) \qquad \Rightarrow \qquad [\mathbf{J}^i_\pm, \mathbf{J}^j_\pm] = i\epsilon^{ijk}\mathbf{J}^k_\pm \qquad [\mathbf{J}^i_\pm, \mathbf{J}^j_\mp] = \mathbf{0} \tag{1.39}$$

These are two commuting $\mathfrak{su}(2)$ algebras, thus proving the thesis[a]. □

[a] To be precise, $\mathfrak{so}^+(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$, but complexification yields the following isomorphisms (with Eq. 1.39):

$$\mathfrak{so}^+(1,3) \hookrightarrow \mathfrak{so}^+(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \hookleftarrow \mathfrak{sl}(2,\mathbb{C})$$

Indeed, note that $\mathrm{SO}^+(1,3)$ is not globally equivalent to $\mathrm{SU}(2) \times \mathrm{SU}(2)$: instead, $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(4)$, while the universal cover of $\mathrm{SO}^+(1,3)$ is $\mathrm{SL}(2,\mathbb{C})$, as $\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

By Prop. 1.2.1, representations of $\mathrm{SO}^+(1,3)$ can be labelled by $(\mathbf{j}_-, \mathbf{j}_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$, with each index labelling a representation of $\mathrm{SU}(2)$: as $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, the $(\mathbf{j}_-, \mathbf{j}_+)$ representation contains states with all possible spins $|j_+ - j_-| \leq j \leq j_+ + j_-$, and it is a representation of degree $n = (2j_- + 1)(2j_+ + 1)$. More formally, the $(\mathbf{j}_-, \mathbf{j}_+)$ representation is $\rho_{(j_-,j_+)} : \mathfrak{so}^+(1,3) \to \mathfrak{gl}(V_{(j_-,j_+)})$, where $V_{(j_-,j_+)}$ is a $(2j_- + 1)(2j_+ + 1)$-dimensional $\mathbb{C}$-vector space, defined as (recall Eq. 1.39):

$$\begin{aligned} \rho_{(j_-,j_+)}(J^i) &= J^i_{(j_+)} \otimes \mathrm{I}_{(2j_-+1)} + \mathrm{I}_{(2j_++1)} \otimes J^i_{(j_-)} \\ \rho_{(j_-,j_+)}(K^i) &= -i\left[J^i_{(j_+)} \otimes \mathrm{I}_{(2j_+1)} - \mathrm{I}_{(2j_++1)} \otimes J^i_{(j_-)}\right] \end{aligned} \tag{1.40}$$

where $\mathbf{J}_{(j)}$ is the $(2j+1)$-dimensional irreducible representation of $\mathfrak{su}(2)$ and $\otimes$ is the tensor product[3].

[3] Given two $\mathbb{K}$-vector spaces V, W with bases $\{\mathbf{e}_i\}_{i=1,\dots,n}, \{\mathbf{f}_j\}_{i=1,\dots,m}$, the tensor product $V \otimes W$ is defined as:

$$\mathbf{v} \otimes \mathbf{w} = \sum_{i=1}^{n}\sum_{j=1}^{m} v^i w^j \mathbf{e}_i \otimes \mathbf{f}_j \qquad \Longleftrightarrow \qquad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ \vdots \\ v_1 w_m \\ \vdots \\ v_n w_1 \\ \vdots \\ v_n w_m \end{pmatrix}$$

Moreovere, given $\mathbb{K}$-linear maps $f : V \to V', g : W \to W'$ represented by $F \in \mathbb{K}^{n' \times n}, G \in \mathbb{K}^{m' \times m}$, the tensor product $f \otimes g : V \otimes W \to V' \otimes W'$ is defined as $F \otimes G \in \mathbb{K}^{(n'm') \times (nm)}$:

$$\begin{bmatrix} f_{1,1} & \dots & f_{1,n} \\ \vdots & \ddots & \vdots \\ f_{n',1} & \dots & f_{n',n} \end{bmatrix} \otimes \begin{bmatrix} g_{1,1} & \dots & g_{1,m} \\ \vdots & \ddots & \vdots \\ g_{m',1} & \dots & g_{m',m} \end{bmatrix} = \begin{bmatrix} f_{1,1}g_{1,1} & \dots & f_{1,1}g_{1,m} & & f_{1,n}g_{1,1} & \dots & f_{1,n}g_{1,m} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ f_{1,1}g_{m',1} & \dots & f_{1,1}g_{m',m} & & f_{1,n}g_{m',1} & \dots & f_{1,n}g_{m',m} \\ & \vdots & & \ddots & & \vdots & \\ f_{n',1}g_{1,1} & \dots & f_{n',1}g_{1,m} & & f_{n',n}g_{1,1} & \dots & f_{n',n}g_{1,m} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ f_{n',1}g_{m',1} & \dots & f_{n',1}g_{m',m} & & f_{n',n}g_{m',1} & \dots & f_{n',n}g_{m',m} \end{bmatrix}$$

Trivial representation The trivial representation is $(\mathbf{0}, \mathbf{0})$, as both $\mathbf{J}_\pm = 0$ and $\mathbf{J} = \mathbf{K} = 0$: this is the representation which acts on scalars.

Weyl representations $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ and $\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ are 2-dimensional spinorial representations. These representations act on different spinors $(\psi_\mathrm{L})_\alpha, (\psi_\mathrm{R})_\beta \in \mathbb{C}^2$, with $\alpha, \beta = 1, 2$, which are called *left-* and *right-handed Weyl spinors*.
In $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ the generators are $\mathbf{J}_- = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{J}_+ = \mathbf{0}$, while in $\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ they are $\mathbf{J}_- = \mathbf{0}$ and $\mathbf{J}_+ = \frac{\boldsymbol{\sigma}}{2}$, thus, by Eq. 1.40, $\mathbf{J}_\mathrm{L} = \mathbf{J}_\mathrm{R} = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{K}_\mathrm{L} = -\mathbf{K}_\mathrm{R} = i\frac{\boldsymbol{\sigma}}{2}$, so that:

$$\psi_\mathrm{L} \mapsto \Lambda_\mathrm{L}\psi_\mathrm{L} = \exp\left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_\mathrm{L} \tag{1.41}$$

$$\psi_\mathrm{R} \mapsto \Lambda_\mathrm{R}\psi_\mathrm{R} = \exp\left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_\mathrm{R} \tag{1.42}$$

Note that the generators K^i are not hermitian, as expected from Prop. A.1.3. Furthermore, as $\psi_{\mathrm{L/R}} \in \mathbb{C}^2$, then $\Lambda_{\mathrm{L/R}} \in \mathbb{C}^{2\times 2}$ (in general the spinorial representations are complex representations).

Proposition 1.2.2

Given $\psi_\mathrm{L} \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ and $\psi_\mathrm{R} \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$, then $\sigma^2\psi_\mathrm{L}^* \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ and $\sigma^2\psi_\mathrm{R}^* \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$.

Proof. Recall that for Pauli matrices $\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^*$, so $\sigma^2\Lambda_\mathrm{L}^*\sigma^2 = \Lambda_\mathrm{R}$ and:

$$\sigma^2\psi_\mathrm{L}^* \mapsto \sigma^2\left(\Lambda_\mathrm{L}\psi_\mathrm{L}\right)^* = \left(\sigma^2\Lambda_\mathrm{L}^*\sigma^2\right)\sigma^2\psi_\mathrm{L}^* = \Lambda_\mathrm{R}\sigma^2\psi_\mathrm{L}^* \quad \Rightarrow \quad \sigma^2\psi_\mathrm{L}^* \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$$

where $\sigma^2\sigma^2 = \mathrm{I}_2$ was used. The proof for $\sigma^2\psi_\mathrm{R}^*$ is analogous. $\square$

This allows to define *charge conjugation* on Weyl spinors:

$$\psi_\mathrm{L}^c := i\sigma^2\psi_\mathrm{L}^* \qquad\qquad \psi_\mathrm{R}^c := -i\sigma^2\psi_\mathrm{R}^* \tag{1.43}$$

As of Prop. 1.2.2, charge conjugation transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the i factor ensures that applying this operator twice yields the identity operator.

Dirac representation $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ is a 4-dimensional complex representation: this representation acts on bispinors of the form $\Psi = ((\psi_\mathrm{L})_\alpha, (\xi_\mathrm{R})_\beta)^\intercal \in \mathbb{C}^4$, called *Dirac spinors*, with transformation $\Lambda_\mathrm{D} = \mathrm{diag}\left(\Lambda_\mathrm{L}, \Lambda_\mathrm{R}\right) \in \mathbb{C}^{4\times 4}$.

Definition 1.2.2: Charge conjugation on Dirac spinors

On Dirac spinors, the *charge conjugation operator* is defined as:

$$\Psi^c := \begin{pmatrix} -i\sigma^2\xi_\mathrm{R}^* \\ i\sigma^2\psi_\mathrm{L}^* \end{pmatrix} = -i\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\Psi^* \tag{1.44}$$

This definition reflect the fact that charge conjugation changes the representation of the Weyl spinors (Prop. 1.2.2).

4-vector representation To see how $\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ is the 4-vector representation, consider the space $\mathcal{H}(2) \coloneqq \{\mathrm{H} \in \mathbb{C}^{2\times 2} : \mathrm{H}^\dagger = \mathrm{H}\}$ of all $\mathbb{C}^{2\times 2}$ Hermitian matrices.

Lemma 1.2.1: Minkowski space and Hermitian matrices

$$\mathcal{H}(2) \cong \mathbb{R}^{1,3} \tag{1.45}$$

Proof.

$$\mathbb{R}^{1,3} \ni v^\mu = (v^0, v^1, v^2, v^3) \longleftrightarrow \begin{pmatrix} v^0 - v^3 & -v^1 + iv^2 \\ -v^1 - iv^2 & v^0 + v^3 \end{pmatrix} = V \in \mathcal{H}(2)$$

where the isomorphisms are $V = v_\mu \sigma^\mu$, $v^\mu = \frac{1}{2}\operatorname{tr}(\bar{\sigma}^\mu V)$, with $\sigma^\mu \equiv (\mathrm{I}_2, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (\mathrm{I}_2, -\boldsymbol{\sigma})$. □

As of Lemma 1.2.1, given $\Lambda \in \mathrm{SO}^+(1,3)$ acting on 4-vectors as $v^\mu \mapsto \Lambda^\mu{}_\nu v^\nu$, it is isomorphic to the action of $\Lambda_\mathrm{s} \in \mathrm{SL}(2,\mathbb{C})$ on Hermitian matrices as $V \mapsto \Lambda_\mathrm{s} V \Lambda_\mathrm{s}^\dagger$: indeed, this transformation preserves Hermiticity and the Minkowski norm $\det V = v_\mu v^\mu$ (as $\det \Lambda_\mathrm{s} = 1$). This transformation can be explicited as:

$$\Lambda^\mu{}_\nu \sigma^\nu = \Lambda_\mathrm{s} \sigma^\mu \Lambda_\mathrm{s}^\dagger \tag{1.46}$$

By Eq. A.4 and Prop. 1.2.1, $\mathfrak{sl}(2,\mathbb{C}) = \{\mathrm{X} \in \mathbb{C}^{2\times 2} : \operatorname{tr}\mathrm{X} = 0\}$ and an $\mathbb{R}$-basis is $\{\frac{\boldsymbol{\sigma}}{2}, i\frac{\boldsymbol{\sigma}}{2}\}$, then:

$$\Lambda_\mathrm{s} = \exp\left[-i\left(\boldsymbol{\theta} + i\boldsymbol{\eta}\right)\cdot\frac{\boldsymbol{\sigma}}{2}\right] = \Lambda_\mathrm{R} \qquad \Rightarrow \qquad \Lambda_\mathrm{s}^\dagger = \exp\left[\left(i\boldsymbol{\theta} + \boldsymbol{\eta}\right)\cdot\frac{\boldsymbol{\sigma}}{2}\right] = \sigma^2 \Lambda_\mathrm{L}^\intercal \sigma^2$$

where $\sigma^2 \boldsymbol{\sigma} \sigma^2 = -\boldsymbol{\sigma}^\intercal$ was used. This is exactly the $\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ representation, where the left component is considered in a unitarily-equivalent form $\psi_\mathrm{L}' \coloneqq \psi_\mathrm{L}^\intercal \sigma^2$ (so that $\psi_\mathrm{L}' \mapsto \psi_\mathrm{L}' \Lambda_\mathrm{s}^\dagger$ by transposing Eq. 1.41). Moreover, note that by Clebsch-Gordan decomposition $\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right) \cong \mathbf{0} \oplus \mathbf{1}$, which is exactly the representation which acts on 4-vectors ($v^0 \in \mathbf{0}$ and $\mathbf{v} \in \mathbf{1}$).

Parity Note that $\mathcal{P}\mathbf{K} = -\mathbf{K}$, as the velocity of the boost gets reversed, while $\mathcal{P}\mathbf{J} = \mathbf{J}$: this means that $\mathcal{P}\mathbf{J}_\pm = \mathbf{J}_\mp$, i.e. parity exchanges a $(\mathbf{j}_-, \mathbf{j}_+)$ representation into a $(\mathbf{j}_+, \mathbf{j}_-)$ representation. Therefore, a $(\mathbf{j}_-, \mathbf{j}_+)$ representation of $\mathrm{SO}^+(1,3)$ is a basis for the representation of the parity transformation iff $j_- = j_+$.

Example 1.2.3: Parity on spinors

While Weyl spinors (separately) are not a basis for a representation of the parity operator, Dirac spinors are.

1.2.1.4 Dirac algebra

The Dirac algebra is $\mathfrak{cl}_{1,3}(\mathbb{C}) \cong \mathfrak{cl}_{1,3}(\mathbb{R}) \otimes \mathbb{C}$ (complexification). This Clifford algebra admits a matrix representation via $\gamma^\mu \in \mathbb{C}^{4\times 4}$ $\forall \mu = 0,1,2,3$ such that:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathrm{I}_4 \tag{1.47}$$

The basis of this algebra is then given by:

I_4	1 matrix
γ^μ	4 matrices
$\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]}$	6 matrices
$\gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$	4 matrices
$\gamma^{\mu\nu\rho\sigma} \equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$	1 matrix

where $\gamma^5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3$ is an additional gamma matrix. Note that this matrix is not in the basis of $\mathfrak{cl}_{1,3}(\mathbb{C})$. It has the following properties:

$$(\gamma^5)^\dagger = \gamma^5 \qquad (\gamma^5)^2 = \mathrm{I}_4 \qquad \{\gamma^5, \gamma^\mu\} = 0$$

Therefore, the Dirac algebra is a 16-dimensional unital associative $\mathbb{C}$-algebra, and its generic component is:

$$\Gamma = a\mathrm{I}_4 + a_\mu\gamma^\mu + \frac{1}{2}a_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{3!}a_{\mu\nu\rho}\gamma^{\mu\nu\rho} + \frac{1}{4!}a_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma} \tag{1.48}$$

where each coefficient is totally anti-symmetric.
The quadratic subspace $\mathfrak{cl}^{(2)}_{1,3}(\mathbb{C})$ of the Dirac algebra is of particular interest. By convention, define its 6 generators as $\sigma^{\mu\nu} \coloneqq \frac{i}{4}[\gamma^\mu, \gamma^\nu]$: it is straightforward to show that these generators satisfy Eq. 1.30, thus they give a 4-dimensional representation of $\mathfrak{so}^+(1,3)$.

Proposition 1.2.3

The generators of $\mathfrak{cl}^{(2)}_{1,3}(\mathbb{C})$ also give the $\left(\frac{1}{2}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{1}{2}\right)$ representation of $\mathfrak{so}^+(1,3)$.

Moreover, $\mathfrak{cl}^{(2)}_{1,3}(\mathbb{C}) \cong \mathrm{Spin}(1,3)$ (Sec. A.1.3.1). Given a spinor $\Psi \in \mathbb{C}^4$, this spin group acts as:

$$\Psi \mapsto \Lambda_{\frac{1}{2}}\Psi = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right]\Psi$$

Using the property $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, its easy to see that $(\sigma^{\mu\nu})^\dagger = -\gamma^0\sigma^{\mu\nu}\gamma^0$, so, defining the *Dirac dual* $\bar{\Psi} \coloneqq \Psi^\dagger\gamma^0$, this transforms via the inverse transformation:

$$\bar{\Psi} \mapsto \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}$$

Dirac bilinears The fact that Ψ and $\bar{\Psi}$ transform inversely allows to define objects with particular transformation relations.

Theorem 1.2.2: Lorentz index of gamma matrices

The gamma matrices transform under the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of $\mathrm{SO}^+(1,3)$ as:

$$\Lambda^\mu{}_\nu\gamma^\nu = \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} \tag{1.49}$$

Proof. Recalling the explicit form of the generators in Eq. 1.29:

$$[J^{\rho\sigma}]^{\mu}{}_{\nu}\gamma^{\nu} = i\left(\eta^{\rho\mu}\gamma^{\sigma} - \eta^{\sigma\mu}\gamma^{\rho}\right) = \frac{i}{2}\left(\gamma^{\sigma}\gamma^{\rho}\gamma^{\mu} + \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu} - \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\right)$$

As $[\gamma^{\mu}, \gamma^{\nu}] = \{\gamma^{\mu}, \gamma^{\nu}\} - 2\gamma^{\nu}\gamma^{\mu} = 2\left(\eta^{\mu\nu} - \gamma^{\nu}\gamma^{\mu}\right)$:

$$\begin{aligned}[\gamma^{\mu}, \sigma^{\rho\sigma}] &= \frac{i}{4}[\gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\rho}] = \frac{i}{4}\left(\gamma^{\rho}[\gamma^{\mu}, \gamma^{\sigma}] + [\gamma^{\mu}, \gamma^{\rho}]\gamma^{\sigma} - [\gamma^{\mu}, \gamma^{\sigma}]\gamma^{\rho} - \gamma^{\sigma}[\gamma^{\mu}, \gamma^{\rho}]\right) \\ &= \frac{i}{2}\left(-\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu} - \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma} + \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho} + \gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}\right) = [J^{\rho\sigma}]^{\mu}{}_{\nu}\gamma^{\nu}\end{aligned}$$

Expanding the thesis at first order:

$$\left[\mathrm{I}_4 - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right]^{\mu}{}_{\nu}\gamma^{\nu} = \left[\mathrm{I}_4 + \frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right]\gamma^{\mu}\left[\mathrm{I}_4 - \frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right]$$

which becomes:

$$-[J^{\rho\sigma}]^{\mu}{}_{\nu}\gamma^{\nu} = \sigma^{\rho\sigma}\gamma^{\mu} - \gamma^{\mu}\sigma^{\rho\sigma}$$

This equation has been verified, so the thesis holds. □

This result means that, although they are matrices, the μ in γ^{μ} is effectively a Lorentz index, thus allowing to construct dot products with 4-vectors; in this way, gamma matrices transform 4-vectors into operators on spinors[4]: $A^{\mu} \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$, but $\not{A} \equiv \gamma^{\mu}A_{\mu} \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$.
A *Dirac bilinear* is an object $\bar{\Psi}\Gamma\Psi$, with $\Gamma \in \mathfrak{cl}_{1,3}(\mathbb{C})$. There are 5 basic bilinears:

$$\bar{\Psi}\Psi \qquad \bar{\Psi}\gamma^{\mu}\Psi \qquad \bar{\Psi}\gamma^{5}\Psi \qquad \bar{\Psi}\gamma^{5}\gamma^{\mu}\Psi \qquad \bar{\Psi}\sigma^{\mu\nu}\Psi$$

According to their transformation relations under the Lorentz group, these bilears are respectively called scalar, vector, pseudo-scalar, pseudo-vector and tensor bilinear.

1.2.1.5 Dual and adjoint representations

Recall that the adjoint representation of a Lie group is a representation acting on the associated Lie algebra: therefore, the adjoint representation of $\mathrm{SO}^{+}(1,3)$ is 6-dimensional and, by Prop. 1.2.1, it can be decomposed as $\mathbf{6} \cong (\mathbf{1}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{1})$.

Dual representations Consider a generic anti-symmetric tensor $A^{\mu\nu}$: its *dual tensor* is defined as $A^{\mu\nu}_{\star} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}$. Every anti-symmetric tensor can then be decomposed as:

$$A^{\mu\nu}_{\pm} \equiv \frac{1}{2}\left(A^{\mu\nu} \pm iA^{\mu\nu}_{\star}\right) \tag{1.50}$$

[4]More precisely: spinors are defined as left-modules (even more precisely, as minimal left-ideals) over $\mathfrak{cl}_{1,3}(\mathbb{C})$, and $\not{A} = \gamma^{\mu}A_{\mu} \in \mathfrak{cl}_{1,3}(\mathbb{C})$ (specifically $\mathfrak{cl}^{(1)}_{1,3}(\mathbb{C})$, with notation from Eq. A.10).
Given a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a *left-module* M is a $\mathbb{K}$-vector space equipped with a $\mathbb{K}$-bilinear multiplication map $\mathcal{A} \times M \ni (a, m) \mapsto am \in M$ such that $(ab)m = a(bm) \wedge 1_{\mathcal{A}}m = m \;\forall a, b \in \mathcal{A}, \forall m \in M$.
Given a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a *left-ideal* is a subalgebra $\mathcal{I} \subset \mathcal{A}$ such that $aj \in \mathcal{I} \;\forall a \in \mathcal{A}, \forall j \in \mathcal{I}$. A left-ideal is said minimal if it is non-trivial and does not contain any non-trivial sub-left-ideal. A left-ideal of $\mathcal{A}$ can be viewed as a left-module on $\mathcal{A}$.

which are respectively called the *self-dual* and *anti-self-dual part* of $A^{\mu\nu}$.

Lemma 1.2.2: (Anti)-self-dual tensor

An (anti)-self-dual tensor satisfies:

$$A^{\mu\nu} = \pm\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} \tag{1.51}$$

Proof. Consider $A^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}$:

$$A_+^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\right) = \frac{1}{2}\left(i\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\right) = A^{\mu\nu} \qquad A_-^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\right) = 0$$

The other case is trivially verified. □

Anti-symmetric tensors belong to the **6** representation, but this is a reducible representation. Indeed, the subspaces of self-dual and anti-self-dual tensors are invariant subspaces:

$$A^{\mu\nu} = \pm\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} \qquad \Rightarrow \qquad \pm\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\left(\Lambda_\rho{}^\alpha\Lambda_\sigma{}^\beta A_{\alpha\beta}\right) = -\frac{1}{4}\underbrace{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}}_{\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}}\Lambda_\rho{}^\alpha\Lambda_\sigma{}^\beta A^{\gamma\delta} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta A^{\alpha\beta}$$

Therefore, $\mathbf{6} = \mathbf{3} \oplus \bar{\mathbf{3}} \cong (\mathbf{1},\mathbf{0}) \oplus (\mathbf{0},\mathbf{1})$.

Dual representation To see explicitely that the adjoint representation of $\mathrm{SO}^+(1,3)$ is exactly **6**, recall that the adjoint representation is $\rho : \mathrm{SO}^+(1,3) \to \mathrm{GL}(\mathfrak{so}^+(1,3))$ and it acts on $\mathfrak{so}^+(1,3)$ as $\rho(\Lambda)M = \Lambda M \Lambda^{-1}$, where $\Lambda \in \mathrm{SO}^+(1,3)$ and $M \in \mathfrak{so}^+(1,3)$. As $[\Lambda^{-1}]^\mu{}_\nu = \Lambda_\nu{}^\mu$ by Eq. 1.27:

$$M^{\mu\nu} \mapsto \Lambda^\mu{}_\alpha M^{\alpha\beta}[\Lambda^{-1}]_\beta{}^\nu = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta M^{\alpha\beta}$$

As $M \in \mathfrak{so}^+(1,3)$ is anti-symmetric, this is the transformation relation of an anti-symmetric tensor: the adjoint representation is the $\mathbf{6} \cong (\mathbf{1},\mathbf{0}) \oplus (\mathbf{0},\mathbf{1})$.

Example 1.2.4: Gauge theories

In gauge theories (ex.: QED, QCD), the gauge potential A_μ transforms according to the fundamental representation $\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)$, while the field strength $F_{\mu\nu}$ according to the adjoint representation $(\mathbf{1},\mathbf{0}) \oplus (\mathbf{0},\mathbf{1})$.

1.2.1.6 Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields A scalar field transforms as:

$$\phi'(x') = \phi(x) \tag{1.52}$$

Consider an infinitesimal transformation $x'^\rho = x^\rho + \delta x^\rho$, with $\delta x^\rho = -\frac{i}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^\rho{}_\sigma x^\sigma$ as of Eq. 1.27. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar

representation of $SO^+(1,3)$ is the trivial one ($J^{\mu\nu} = 0$).
However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$: while $\delta\phi$ studies only a single degree of freedom, as the point $P \in \mathbb{R}^{1,3}$ is kept constant and only $\phi(P)$ can vary (i.e. the base space is one-dimensional), $\delta_0\phi$ studies $\phi(P)$ with P varying over $\mathbb{R}^{1,3}$, thus the base space is now a space of functions, which is infinite-dimensional. Therefore, $\delta\phi$ provides a finite-dimensional representation of the generators, while $\delta_0\phi$ an infinite-dimensional one.
To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho \partial_\rho \phi = \frac{i}{2}\omega_{\mu\nu} \left[J^{\mu\nu}\right]^\rho{}_\sigma x^\sigma \partial_\rho \phi \equiv -\frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}\phi$$

Recalling Eq. 1.29, the generators can be expressed as:

$$L^{\mu\nu} := i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \tag{1.53}$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As $p^\mu = i\partial^\mu$ (with signature $(+,-,-,-)$), it is clear that $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ is the orbital angular momentum.

Weyl fields A left-handed Weyl field transforms as:

$$\psi'_{\mathrm{L}}(x') = \Lambda_{\mathrm{L}}\psi_{\mathrm{L}}(x) \tag{1.54}$$

with Λ_{L} defined in Eq. 1.41, and similarly for right-handed Weyl fields. The infinite-dimensional representation of the Lorentz generators determined by Weyl spinors can be found as:

$$\begin{aligned}\delta_0\psi_{\mathrm{L}} &\equiv \psi'_{\mathrm{L}}(x) - \psi_{\mathrm{L}}(x) = \psi'_{\mathrm{L}}(x' - \delta x) - \psi_{\mathrm{L}}(x) \\ &= \psi'_{\mathrm{L}}(x') - \delta x^\rho \partial_\rho \psi_{\mathrm{L}}(x) - \psi_{\mathrm{L}}(x) = \left(\Lambda_{\mathrm{L}} - \mathrm{I}_2\right)\psi_{\mathrm{L}}(x) - \delta x^\rho \partial_\rho \psi_{\mathrm{L}}(x)\end{aligned}$$

The second term yields $L^{\mu\nu}$, while the first can be further elaborated by writing:

$$\Lambda_{\mathrm{L}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{1.55}$$

Thus:

$$\delta_0\psi_{\mathrm{L}} = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_{\mathrm{L}}$$

where the angular momentum separates into the orbital and the spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1.56}$$

This separation is general: $L^{\mu\nu}$ is always expressed as in Eq. 1.53, while $S^{\mu\nu}$ depends on the specific representation. In the scalar representation $S^{\mu\nu} = 0$, while in the left/right-handed Weyl representation $S^{i0} = \pm i\frac{\sigma^i}{2}$.

Dirac fields A Dirac field transforms as:

$$\Psi'(x') = \Lambda_{\mathrm{D}}\Psi(x) \tag{1.57}$$

where $\Lambda_{\mathrm{D}} = \mathrm{diag}(\Lambda_{\mathrm{L}}, \Lambda_{\mathrm{R}})$. The infinite-dimensional representation of the Lorentz generators determined by Dirac spinors is:

$$\delta_0\Psi \equiv \Psi'(x) - \Psi(x) = \Psi'(x') - \delta x^\rho \partial_\rho \Psi(x) - \Psi(x) = \left(\Lambda_{\mathrm{D}} - \mathrm{I}_4\right)\Psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\Psi(x)$$

The first term can be rewritten defining:

$$\Lambda_{\mathrm{D}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \qquad S^{0i} \equiv -\frac{i}{2}\begin{bmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{bmatrix} \qquad S^{ij} = -\frac{1}{2}\epsilon^{ijk}\begin{bmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{bmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k$$

Therefore, a Dirac field transforms as:

$$\delta_0\Psi = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\Psi \tag{1.58}$$

with clear orbital and spin angular components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1.59}$$

Vector fields A (contravariant) vector field transforms as:

$$V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}V^{\nu}(x) \tag{1.60}$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the $\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)$ representation.

1.2.2 Poincaré group

Definition 1.2.3: Poincaré group

The *Poincaré group* is defined as $\mathrm{ISO}^+(1,3) := \mathrm{T}^{1,3} \rtimes \mathrm{SO}^+(1,3)$, where $\mathrm{T}^{1,3} \cong \mathbb{R}^{1,3}$ is the group of translations of $\mathbb{R}^{1,3}$, with multiplication $(a^\mu, \Lambda^\mu{}_\nu) \cdot (\bar{a}^\mu, \bar{\Lambda}^\mu{}_\nu) \equiv (a^\mu + \Lambda^\mu{}_\nu \bar{a}^\nu, \Lambda^\mu{}_\rho \bar{\Lambda}^\rho{}_\nu)$.

Given a translation $x^\mu \mapsto x^\mu + a^\mu$, the associated group element can be written as:

$$T = e^{-ia_\mu P^\mu} \tag{1.61}$$

where P^μ is the 4-momentum operator. Clearly translations commute, and so do their generators; on the other hand, as $\mathbf{P}$ is a vector under rotations, while P^0 (energy) a scalar, one has:

$$[J^i, P^j] = ie^{ijk}P^k \qquad\qquad [J^i, P^0] = 0$$

These equations uniquely determine the *Poincaré algebra* $\mathfrak{iso}^+(1,3)$:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [J^{\mu\nu}, J^{\sigma\rho}] &= i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \\ [P^\mu, J^{\rho\sigma}] &= i\left(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right) \end{aligned} \tag{1.62}$$

It's easy to check that $[K^i, P^0] = iP^i$, while $[J^i, P^0] = [P^i, P^0] = 0$: given that P^0 generates time translations, linear and angular momentum are conserved quantities, while $\mathbf{K}$ is not.

1.2.2.1 Field representations

Fields provide an infinite-dimensional representation of the Lorentz group as $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $S^{\mu\nu}$ does not depend on x^μ, but only on the spin of the field.
To represent P^μ on fields, their transformation law must be specified: all fields are required to be scalars under translations, independently of their spin. This means that, given a generic field $\phi(x)$, under a translation $x' = x + a$ it transforms as $\phi'(x') = \phi(x)$, so, under an infinitesimal translation $x' = x + \varepsilon$:

$$\begin{aligned} \delta_0\phi \equiv \phi'(x) - \phi(x) = \phi'(x' - \varepsilon) - \phi(x) &= -\varepsilon^\mu \partial_\mu \phi(x) \\ &= e^{-i(-\varepsilon_\mu)P^\mu}\phi'(x') - \phi(x) = \left(e^{i\varepsilon_\mu P^\mu} - \mathrm{I}\right)\phi(x) = i\varepsilon_\mu P^\mu \phi(x) \end{aligned}$$

It is clear then that:

$$P^\mu = +i\partial^\mu \tag{1.63}$$

Explicitly, $P^0 = i\partial_t$ and $\mathbf{P} = -i\boldsymbol{\nabla}$. It is trivial to check that these generators obey the Poincaré algebra.

1.2.2.2 Particle representations

The Poincaré group can also be represented using the Hilbert space $\mathscr{H}$ of a free particle as a basis. Denoting a generic state as $|\mathbf{p}, s\rangle \in \mathscr{H}$, where $\mathbf{p}$ is the particle's momentum and s collectively labels all other quantum numbers, it is clear that $\mathscr{H}$ is infinite-dimensional, as $\mathbf{p}$ is a continuous unbounded variable.

Theorem 1.2.3: Wigner's theorem

On the Hilbert space of physical states, any symmetry transformation can be represented by a linear and unitary or anti-linear and anti-unitary operator.

By this theorem, Poincaré transformations can be represented by unitary matrices, i.e. $\mathbf{J}$, $\mathbf{K}$, $\mathbf{P}$ and P^0 can be represented by hermitian operators. These representations can be labeled by Casimir operators, which for $\mathrm{ISO}^+(1,3)$ are easily found as $P_\mu P^\mu$ and $W_\mu W^\mu$, where W^μ is the *Pauli-Lubanski operator*:

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}J_{\nu\sigma}P_\rho \tag{1.64}$$

On single-particle states $P_\mu P^\mu = m^2$, while $W_\mu W^\mu$ can be conveniently computed in a particular frame (due to Lorentz invariance). If $m \neq 0$, this frame is the rest-frame of the particle:

$$W^\mu = -\frac{m}{2}\epsilon^{\mu\nu\sigma 0}J_{\nu\sigma} = \frac{m}{2}\delta^{\mu i}\epsilon^{ijk}J^{jk} = \delta^{\mu i}mJ^i$$

Therefore, on single-particle states of mass m and spin j, the Casimir operator takes the form:

$$W_\mu W^\mu = -m^2 j\,(j+1) \tag{1.65}$$

If $m = 0$, the rest-frame does not exist, but it is possible to choose a frame where $P^\mu = (\omega, 0, 0, \omega)$, where $W^0 = W^3 = \omega J^3$, $W^1 = \omega\,(J^1 - K^2)$ and $W^2 = \omega\,(J^2 + K^1)$, so that:

$$W_\mu W^\mu = -\omega^2\left[\left(K^2 - J^1\right)^2 + \left(K^1 + J^2\right)^2\right] \tag{1.66}$$

It is clear that the $m \to 0$ limit is not trivial, and massive and massless representation need to be studied separately.

Massive representations Restricting to $m \in \mathbb{R}^+$ ($m^2 < 0$ states, called tachyons, are excluded), the massive representations are labeled by mass m and spin j: in fact, after a Lorentz transformation such that $P^\mu = (m, \mathbf{0})$, spatial rotations can still be performed, i.e. the subspace of single-particle states with momentum $P^\mu = (m, \mathbf{0})$ is still a basis for the representation of $\mathrm{SU}(2)$ (as spinors must be included too). The group of transformations which leaves invariant a certain choice of P^μ is called the *little group*, so $\mathrm{SU}(2)$ is the little group of massive single-particle states: massive representations are labelled by m and j, which means that massive particles of spin j have $2j+1$ degrees of freedom.

Massless representations The little group for $P^\mu = (\omega, 0, 0, \omega)$ clearly is $\mathrm{SO}(2)$, the group of rotations in the (x, y) plane generated by J^3: as for any abelian group, its irreducible representations are one-dimensional, and they are labeled by the eigenvalue h of J^3, which represents the angular momentum in the direction of propagation of the particle and is called *helicity*. Helicity can be shown to be quantized as $h \in \frac{1}{2}\mathbb{Z}_0$ (by topologic considerations on $\mathrm{ISO}^+(1,3) \equiv \mathbb{R}^4 \times \mathrm{SL}(2)\mathbb{C}/\mathbb{Z}_2$).
As a consequence, massless particles only have one degree of freedom and are characterized by their helicity h. As $\mathrm{SO}(2) \cong \mathrm{U}(1)$, on a state of helicity h the little group is represented as:

$$U(\theta) = e^{-ih\theta} \tag{1.67}$$

Although massless particles with opposite helicities are logically two different species of particles, it can be written as $h = \hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$ (unit vectors), so h is a pseudoscalar such that $h \mapsto -h$ under parity: this means that, if the interaction conserves parity, h and $-h$ must be symmetric.

Example 1.2.5

The electromagnetic and gravitational interactions conserve parity, thus photons and gravitons must be a basis for the representation of both $\mathrm{ISO}^+(1,3)$ and parity: photons can have $h = \pm 1$ (left- and right-handed), while gravitons have $h = \pm 2$.

Example 1.2.6

Neutrinos only interact via the weak interaction, which does not conserve parity, and in fact the two states $h = \pm\frac{1}{2}$ are different particles: neutrinos have $h = -\frac{1}{2}$, while antineutrinos have $h = +\frac{1}{2}$.

1.3 Classical equations of motion

Consider a *local field theory* of fields $\{\phi_i(x)\}_{i\in\mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkoski spacetime. Its Lagrangian takes the form:

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.68}$$

where $\mathcal{L}$ is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.69}$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

Theorem 1.3.1

The *stationary action principle* $\delta\mathcal{S} = 0$ determines the classical equations of motion:

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0 \tag{1.70}$$

Proof. Varying Eq. 1.69:

$$\delta\mathcal{S} = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi_i)\right] = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi_i = 0$$

□

Corollary 1.3.1.1

Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.

Proof. This is a consequence of Stokes theorem:

$$\int_\Sigma \mathrm{d}^4x\, \partial_\mu K^\mu = \int_{\partial\Sigma} \mathrm{d}A\, n_\mu K^\mu$$

□

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \tag{1.71}$$

$$\mathcal{H} = \sum_{i\in\mathcal{I}} \Pi_i(x)\partial_0\phi(x) - \mathcal{L} \tag{1.72}$$

Unlike the Hamiltonian formalism, the Lagrangian formalism keeps Lorentz covariance explicit.

1.3.1 Noether's theorem

Definition 1.3.1: Infinitesimal transformation

Given a field theory with fields $\{\phi_i\}_{i=1,\dots,k}$ and action $\mathcal{S}[\phi]$, an *infinitesimal transformation* parametrized by $\{\varepsilon^a\}_{a=1,\dots,N} : |\varepsilon^a| \ll 1$ is defined by two sets of functions $\{A^\mu_a(x)\}_{a=1,\dots,N}$ and $\{F_{i,a}(\phi, \partial\phi)\}_{i=1,\dots,k;\, a=1,\dots,N}$ such that:

$$\begin{aligned} x^\mu \mapsto x'^\mu &= x^\mu + \varepsilon^a A^\mu_a(x) \\ \phi_i(x) \mapsto \phi'_i(x) &= \phi_i(x) + \varepsilon^a F_{i,a}(\phi, \partial\phi) \end{aligned} \tag{1.73}$$

Definition 1.3.2: Symmetry transformation

An infinitesimal transformation is a *symmetry transformation* if it leaves $\mathcal{S}[\phi]$ invariant, regardless of ϕ being a solution of the equations of motion. It can further be classified as:

- *global symmetry*, if $\varepsilon^a \equiv$ const.;
- *local symmetry*, if $\varepsilon^a = \varepsilon^a(x)$.

Symmetry transformations which leave spacetime unchanged, i.e. with $A^\mu_a(x) = 0$, are called *internal symmetries*.

Theorem 1.3.2: Noether's theorem

Given a global (but not local) symmetry parametrized by N generators, then there are N conserved currents $\{j^\mu_a(\phi)\}_{a=1,\dots,N}$ such that:

$$\partial_\mu j^\mu_a(\phi^{\mathrm{cl}}) = 0 \tag{1.74}$$

where ϕ^{cl} is a classical solution of the equations of motion.

Proof. First, consider an infinitesimal transformation with slowly-varying parameters, i.e. $l\,|\partial_\mu \varepsilon^a| \ll |\varepsilon^a|$ (l characteristic scale of the field theory): being it not a local symmetry, $\delta\mathcal{S} \neq 0$ at $o(\varepsilon)$ and:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x\, \left[\varepsilon^a(x) K_a(\phi) - (\partial_\mu \varepsilon^a(x))\, j^\mu_a(\phi) + o(\partial\partial\varepsilon)\right] + o(\varepsilon^2)$$

If $\varepsilon^a \equiv$ const. (global symmetry) then $\delta\mathcal{S}[\phi] = 0\,\forall\phi$, therefore $K_a(\phi) = 0\,\forall\phi$ (independent of ε). Assuming $\varepsilon^a(x) \to 0$ sufficiently fast as $x \to \infty$, then integration by parts yields:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x\, \varepsilon^a(x) \partial_\mu j^\mu_a(\phi) + o(\partial\partial\varepsilon) + o(\varepsilon^2)$$

This expression is independent of the choice of ϕ. Moreover, note that Eq. 1.73 can be rewritten as an internal transformation by setting:

$$\phi_i(x) \mapsto \phi'_i(x) = \phi_i(x - \varepsilon^a A_a) + \varepsilon^a F_{i,a} = \phi_i(x) + \varepsilon^a F_{i,a} - \varepsilon^a A^\mu_a \partial_\mu \phi_i \equiv \phi_i(x) + \delta\phi_i(x)$$

$\delta\phi_i(x)$ vanishes at infinity, therefore it is the kind of variation used to derive the equations of motion: choosing $\phi \equiv \phi^{\text{cl}}$ classical solution then implies $\delta\mathcal{S} = 0$ independently of ε, i.e. the thesis. □

These are often called *Noether currents*, and the associated *Noether charges* are defined as:

$$Q_a := \int \mathrm{d}^3x\, j^0_a(t, \mathbf{x}) \tag{1.75}$$

These are time-independent, as $\partial_0 Q_a = \int \mathrm{d}^3x\, \partial_0 j^0_a = -\int \mathrm{d}^3x\, \partial_i j^i_a$: on all spacetime it vanishes by divergence theorem (fields vanish at infinity), but on a finite volume it yields a boundary term interpreted as the incoming and outgoing flux.

Proposition 1.3.1: Noether currents

The explicit expression of Noether currents is:

$$j^\mu_a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \left[A^\nu_a(x)\partial_\nu\phi_i - F_{i,a}(\phi, \partial\phi)\right] - A^\mu_a(x)\mathcal{L} \tag{1.76}$$

Proof. Varying the action at $o(\partial\varepsilon)$:

$$\delta_\varepsilon\mathcal{S} = \delta_\varepsilon \int \mathrm{d}^4x\, \mathcal{L} = \int \left[\delta_\varepsilon(\mathrm{d}^4x)\mathcal{L} + \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta_\varepsilon\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta_\varepsilon(\partial_\mu\phi_i)\right)\right]$$

The Jacobian of Eq. 1.73 gives $\mathrm{d}^4x \mapsto \mathrm{d}^4x\,(1 + A^\mu_a\partial_\mu\varepsilon^a) + o(\varepsilon)$, while $\delta_\varepsilon\phi_i$ is not $o(\partial\varepsilon)$ and:

$$\delta_\varepsilon(\partial_\mu\phi_i) = \frac{\partial\phi'_i}{\partial x'^\mu} - \frac{\partial\phi_i}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}(\phi_i + \varepsilon^a F_{i,a}) - \frac{\partial\phi_i}{\partial x^\mu} = -(\partial_\mu\varepsilon^a)\,(A^\nu_a\partial_\nu\phi_i - F_{i,a}) + o(\varepsilon)$$

The thesis follows from $\delta_\varepsilon\mathcal{S} = -\int \mathrm{d}^4x\,(\partial_\mu\varepsilon^a) j^\mu_a + o(\partial\partial\varepsilon) + o(\varepsilon^2)$. □

If the considered infinitesimal transformation is not a global symmetry, then $\delta_\varepsilon \mathcal{S}$ has a non-vanishing $o(\varepsilon)$ term which gives rise to a quasi-conserved current:

$$\partial_\mu j^a_\mu = -(\delta_a \mathcal{L})_{\text{global}} \tag{1.77}$$

1.3.1.1 Energy-momentum tensor

Consider spacetime translations: as all fields must be scalars under these transformations, they define a Noether current. In particular, translations have $A^\mu_\nu = \delta^\mu_\nu$ and $F_{i,\mu} = 0$ (the parameter index is a Lorentz index), so the conserved current is the *energy-momentum tensor* $j^\mu_\nu \equiv \theta^\mu{}_\nu$:

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i - \eta^{\mu\nu} \mathcal{L} \tag{1.78}$$

which is covariantly conserved on classical solutions of the equations of motion. The conserved Noether charge associated to the energy-momentum tensor is the *four-momentum*:

$$P^\mu := \int \mathrm{d}^3 x \, \theta^{0\mu} \tag{1.79}$$

The energy-momentum tensor so defined is not symmetric, however it can be made to via a tensor $A^{\rho\mu\nu}$ which is anti-symmetric w.r.t. (ρ, μ): $T^{\mu\nu} \equiv \theta^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$ is physically equivalent from $\theta^{\mu\nu}$, as the second term is a vanishing spatial divergence in the definition of P^μ and is contracted to 0 in the conservation law.

1.3.2 Scalar fields

1.3.2.1 Real scalar fields

Consider a real scalar field ϕ: a non-trivial Poincaré-invariant action must contain $\partial_\mu \phi$ and must saturate each Lorentz index. For example:

$$\mathcal{S}[\phi] = \frac{1}{2} \int \mathrm{d}^4 x \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \tag{1.80}$$

The resulting equation of motion is the *Klein-Gordon equation*:

$$\left(\Box + m^2 \right) \phi = 0 \tag{1.81}$$

where $\Box \equiv \partial_\mu \partial^\mu$. A plane wave $e^{\pm i p_\mu x^\mu}$ is a solution if $p^2 = m^2$, so the KG equation imposes the relativistic dispersion relation and m can be interpreted as the mass. As ϕ must be real, the general solution is a superposition of waves:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2 E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i p_\mu x^\mu} + a^*_{\mathbf{p}} e^{i p_\mu x^\mu} \right]_{p^0 = E_{\mathbf{p}}} \tag{1.82}$$

The positive energy solution has $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$, but it contains both *positive* and *negative frequency modes* $e^{\mp i p_\mu x^\mu}$, while the $(2 E_{\mathbf{p}})^{-1/2}$ factor is a convenient normalization of the $a_{\mathbf{p}}$ coefficients. The overall normalization of $\mathcal{S}[\phi]$ does not influence the equations of motion, however it is important for obtaining a positive-definite Hamiltonian. As the momentum conjugate to ϕ is $\Pi_\phi = \partial_0\phi$, the Hamiltonian density is found as:

$$\mathcal{H} = \frac{1}{2}\left[\Pi_\phi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right] \tag{1.83}$$

The energy-momentum tensor is computed to be:

$$\theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{1.84}$$

It is trivial to see that $\theta^{00} = \mathcal{H}$: the Hamiltonian is the conserved charge related to the invariance under time translations.
To compute the conserved currents associated to Lorentz invariance, it is convenient to label the transformation parameters $\omega^{\mu\nu}$ by an anti-symmetric pair of Lorentz indices, so that Eq. 1.73 become:

$$x^\mu \mapsto x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}\left(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho\right) \equiv x^\mu + \frac{1}{2}A^\mu_{(\rho\sigma)}\omega^{\rho\sigma}$$

As $F_{i,a} = 0$, from Eq. 1.84 the conserved currents are:

$$j^{(\rho\sigma)\mu} = x^\rho\theta^{\mu\sigma} - x^\sigma\theta^{\mu\rho} \tag{1.85}$$

For spatial rotations, the conserved charge is:

$$M^{ij} = \int \mathrm{d}^3x\,\left(x^i\theta^{0j} - x^j\theta^{0i}\right) = \int \mathrm{d}^3x\,\partial_0\phi\left(x^i\partial^j - x^j\partial^i\right)\phi = \frac{i}{2}\int \mathrm{d}^3x\,\left[\phi L^{ij}(\partial_0\phi) - (\partial_0\phi)L^{ij}\phi\right]$$

where integration by parts was carried and L^{ij} is defined by Eq. 1.53. This can be generalized.

Definition 1.3.3: Scalar product

Given two real scalar fields ϕ_1 and ϕ_2, their *scalar product* is defined as:

$$\langle\phi_1|\phi_2\rangle := \frac{i}{2}\int \mathrm{d}^3x\,\phi_1\overleftrightarrow{\partial_0}\phi_2 \tag{1.86}$$

where $f\overleftrightarrow{\partial_\mu}g := f\partial_\mu g - (\partial_\mu f)g$.

Proposition 1.3.2

If ϕ_1 and ϕ_2 are KG solutions, then $\langle\phi_1|\phi_2\rangle$ is time-independent.

Proof. By the KG equation:

$$\partial_0\left[\phi_1\partial_0\phi_2 - (\partial_0\phi_1)\phi_2\right] = \phi_1\partial_0^2\phi_2 - (\partial_0^2\phi_1)\phi_2 = \phi_1\nabla^2\phi_2 - (\nabla^2\phi_1)\phi_2$$

which vanishes after integration by parts. □

Note that this scalar product is not positive-definite.

Theorem 1.3.3: Conserved charges

Given a symmetry represented by a Lie group and a representation $L^{\mu\nu}$ of its generators as operators acting on fields, the value of the associated conserved charges on a solution ϕ of the equations of motion is:

$$M^{\mu\nu} = \langle \phi | L^{\mu\nu} | \phi \rangle \tag{1.87}$$

Example 1.3.1: Four-momentum

Applying Th. 1.3.3 to four-momentum $P^\mu = \langle \phi | i\partial^\mu | \phi \rangle$; for example, the $\mu = 0$ component is:

$$\begin{aligned}
P^0 = \langle \phi | i\partial^0 | \phi \rangle = \langle \phi | i\partial_0 | \phi \rangle &= \frac{i}{2} \int \mathrm{d}^3x \, [\phi(i\partial_0)\partial_0\phi - (\partial_0\phi) i\partial_0\phi] \\
&= \frac{1}{2} \int \mathrm{d}^3x \left[-\phi\partial_0^2\phi + (\partial_0\phi)^2 \right] = \frac{1}{2} \int \mathrm{d}^3x \left[-\phi \left(\nabla^2 - m^2 \right) \phi + (\partial_0\phi)^2 \right] \\
\text{(int. by parts)} &= \frac{1}{2} \int \mathrm{d}^3x \left[(\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 + (\partial_0\phi)^2 \right] = \frac{1}{2} \int \mathrm{d}^3x \, \theta^{00}
\end{aligned}$$

The free KG action can be generalized to a self-interacting real scalar field introducing a general potential:

$$\mathcal{S}[\phi] = \int \mathrm{d}^4x \left[\frac{1}{2} \partial_\mu\phi\partial^\mu\phi - V(\phi) \right] \tag{1.88}$$

The quadratic term in $V(\phi)$ is the mass term, while higher-order terms describe the self-interaction.

1.3.2.2 Complex scalar fields

Consider now two real scalar fields ϕ_1, ϕ_2 with the same mass m and combine them into a single complex scalar field $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. The KG action is the sum of the single actions and may be written as:

$$\mathcal{S}[\phi] = \int \mathrm{d}^4x \left(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \right) \tag{1.89}$$

Considering ϕ and ϕ^* as independent variables one obtains the KG equation, which yields the same mode expansion as Eq. 1.82:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \left[a_\mathbf{p} e^{-ip_\mu x^\mu} + b_\mathbf{p}^* e^{ip_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \tag{1.90}$$

Now $a_\mathbf{p}$ and $b_\mathbf{p}$ are independent, since there's no reality condition on ϕ.

Electric charge An interesting property of the complex KG field is the existence of a global U(1) symmetry of the action: $\mathcal{S}[\phi] \mapsto \mathcal{S}[\phi]$ under $\phi(x) \mapsto e^{i\theta}\phi(x)$. The associated Noether current can be computed from Eq. 1.76, using $\phi_i = (\phi, \phi^*)$, $A_a^\nu = 0$ and $F_{i,a} = (i, -i)$:

$$j_\mu = -i \left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi \right) = i\phi^* \overleftrightarrow{\partial_\mu} \phi \tag{1.91}$$

The conserved U(1) charge is then $Q_{\mathrm{U}(1)} = i \int \mathrm{d}^3x \, \phi^* \overleftrightarrow{\partial_0} \phi = \langle \phi | \phi \rangle$, which is consistent with Th. 1.3.3 as the generator of U(1) is the identity operator.

1.3.3 Spinor fields

1.3.3.1 Weyl fields

Consider the theory of a left/right-handed Weyl field: recalling that $\psi_{\mathrm{L}}^\dagger \sigma^\mu \psi_{\mathrm{L}}$ and $\psi_{\mathrm{R}} \bar{\sigma}^\mu \psi_{\mathrm{R}}$ are 4-vectors, with $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$, the kinetic term of the Lorentz-invariant lagrangian can be written as:

$$\mathcal{L}_{\mathrm{L}} = i\psi_{\mathrm{L}}^\dagger \bar{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}} \qquad \mathcal{L}_{\mathrm{R}} = i\psi_{\mathrm{R}}^\dagger \sigma^\mu \partial_\mu \psi_{\mathrm{R}} \tag{1.92}$$

The i factor ensures the reality of the Lagrangian, as σ matrices are hermitian. The Lagrangian does not depend on $\partial_\mu \psi^*$, thus the Euler-Lagrange equations are:

$$(\partial_0 - \sigma^i \partial_i)\psi_{\mathrm{L}} = 0 \qquad (\partial_0 + \sigma^i \partial_i)\psi_{\mathrm{R}} = 0 \tag{1.93}$$

These are known as *Weyl equations*: by $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, these equations imply two massless KG equations (assuming regular functions, so that $\partial_i \partial_j = \partial_j \partial_i$). Considering positive-energy plane-wave solutions of the form $\psi_{\mathrm{L}}(x) = u_{\mathrm{L}} \exp(-ip_\mu x^\mu) = u_{\mathrm{L}} \exp(-iEt + i\mathbf{p}\cdot\mathbf{x})$, where u_{L} is a constant spinor, Weyl equations become:

$$\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E} u_{\mathrm{L}} = -u_{\mathrm{L}} \qquad \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E} u_{\mathrm{R}} = u_{\mathrm{R}}$$

As these are massless fields $E = |\mathbf{p}|$, and since for $s = \frac{1}{2}$ fields $\mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$ these equations become:

$$(\hat{\mathbf{p}} \cdot \mathbf{J}) u_{\mathrm{L}} = -\frac{1}{2} u_{\mathrm{L}} \qquad (\hat{\mathbf{p}} \cdot \mathbf{J}) u_{\mathrm{R}} = \frac{1}{2} u_{\mathrm{R}}$$

These show that left/right-handed Weyl massless Weyl spinors have helicity $h = \mp\frac{1}{2}$, consistent with the fact that massless particles are helicity eigenstates.
The energy-momentum tensor can be computed from Eq. 1.78, noting that on classical solutions (by Weyl equations) the Lagrangian of the theory vanishes:

$$\theta^{\mu\nu} = i\psi_{\mathrm{L}}^\dagger \bar{\sigma}^\mu \partial^\nu \psi_{\mathrm{L}} \qquad \theta^{\mu\nu} = i\psi_{\mathrm{R}}^\dagger \sigma^\mu \partial^\nu \psi_{\mathrm{R}} \tag{1.94}$$

Moreover, note that the Lagrangian is invariant under a global U(1) internal transformation with Noether currents and conserved charges:

$$j^\mu = \psi_{\mathrm{L}}^\dagger \bar{\sigma}^\mu \psi_{\mathrm{L}} \qquad j^\mu = \psi_{\mathrm{R}}^\dagger \sigma^\mu \psi_{\mathrm{R}}$$

$$Q_{\mathrm{U}(1)} = \int \mathrm{d}^3x\, \psi_{\mathrm{L}}^\dagger \psi_{\mathrm{L}} \qquad Q_{\mathrm{U}(1)} = \int \mathrm{d}^3x\, \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{R}}$$

Weyl Lagrangians are not invariant under parity, as $\psi_{\mathrm{L}} \leftrightarrow \psi_{\mathrm{R}}$.

Example 1.3.2: Neutrinos

Neutrinos are divided into three leptonic families: ν_e, ν_μ and ν_τ. Although they are massive $s = \frac{1}{2}$, in most contexts their mass can be neglected, thus they can be described by Weyl spinors: in particular, neutrinos by left-handed Weyl spinors and antineutrinos by right-handed Weyl spinors.

1.3.3.2 Dirac fields

Consider a theory with both a left-handed and a right-handed Weyl spinor: one can construct two new Lorentz scalars, $\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{R}}$ and $\psi_{\mathrm{R}}^\dagger\psi_{\mathrm{L}}$, as from Eqq. 1.41-1.42 it is easy to check that $\Lambda_{\mathrm{L}}^\dagger\Lambda_{\mathrm{R}} = \Lambda_{\mathrm{R}}^\dagger\Lambda_{\mathrm{L}} = \mathrm{id}$. Two real combinations are $\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^\dagger\psi_{\mathrm{L}}$ and $i(\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{R}} - \psi_{\mathrm{R}}^\dagger\psi_{\mathrm{L}})$: under parity $\psi_{\mathrm{L}} \leftrightarrow \psi_{\mathrm{R}}$, so the former is a scalar and the latter a pseudoscalar. The Dirac Lagrangian then is:

$$\mathcal{L}_{\mathrm{D}} = i\psi_{\mathrm{L}}^\dagger\bar{\sigma}^\mu\partial_\mu\psi_{\mathrm{L}} + i\psi_{\mathrm{R}}^\dagger\sigma^\mu\partial_\mu\psi_{\mathrm{R}} - m(\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^\dagger\psi_{\mathrm{L}}) \tag{1.95}$$

This Lagrangian is invariant under parity, as $\bar{\sigma}^\mu\partial_\mu \leftrightarrow \sigma^\mu\partial_\mu$ ($\partial_i \mapsto -\partial_i$). Considering ψ_i and ψ_i^* independent, the variation w.r.t. the latter yields:

$$i\bar{\sigma}^\mu\partial_\mu\psi_{\mathrm{L}} = m\psi_{\mathrm{R}} \qquad\qquad i\sigma^\mu\partial_\mu\psi_{\mathrm{R}} = m\psi_{\mathrm{L}} \tag{1.96}$$

which is the *Dirac equation* in terms of Weyl spinors.

Proposition 1.3.3

Dirac equation implies two massive Klein-Gordon equations.

Proof. Applying $i\sigma^\mu\partial_\mu$ to the first equation:

$$-\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\psi_{\mathrm{L}} = mi\sigma^\mu\partial_\mu\psi_{\mathrm{R}} = m^2\psi_{\mathrm{L}}$$

Assuming $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, $\sigma^\mu\bar{\sigma}^\nu$ can be replaced by $\frac{1}{2}(\sigma^\nu\bar{\sigma}^\mu + \sigma^\nu\bar{\sigma}^\mu) = \frac{1}{2}(2\eta^{\mu\nu}) = \eta^{\mu\nu}$, yielding the thesis. $\square$

The m parameter can thus be interpreted as a mass, and now the two spinors are no longer helicity eigenstates. It is convenient to rewrite this theory in terms of a Dirac spinor, which defines the chiral representation:

$$\Psi = \begin{pmatrix}\psi_{\mathrm{L}} \\ \psi_{\mathrm{R}}\end{pmatrix}$$

The chiral representation of the Dirac algebra $\mathfrak{cl}_{1,3}(\mathbb{C})$ is then found to be:

$$\gamma^\mu := \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$$

This allows to rewrite Dirac equation as:

$$(i\not{\partial} - m)\Psi = 0 \tag{1.97}$$

In the chiral representation, the Dirac adjoint simply is $\bar{\Psi} = (\psi_{\mathrm{R}}^\dagger, \psi_{\mathrm{L}}^\dagger)$, so the Dirac Lagrangian can be rewritten as:

$$\mathcal{L}_{\mathrm{D}} = \bar{\Psi}(i\not{\partial} - m)\Psi \tag{1.98}$$

The γ^5 matrix in the chiral representation is:

$$\gamma^5 = \begin{bmatrix} -\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{bmatrix}$$

This allows to define projectors on Weyl spinors:

$$\frac{1-\gamma^5}{2}\Psi = \begin{pmatrix}\psi_{\mathrm{L}} \\ 0\end{pmatrix} \qquad\qquad \frac{1+\gamma^5}{2}\Psi = \begin{pmatrix}0 \\ \psi_{\mathrm{R}}\end{pmatrix} \tag{1.99}$$

Proposition 1.3.4: Unitary equivalence

Given a constant $\mathrm{U} \in \mathbb{C}^{2\times 2} : \mathrm{U}^\dagger = \mathrm{U}^{-1}$, then under $\Psi \mapsto \mathrm{U}\Psi$ the Lagrangian is invariant for $\gamma^\mu \mapsto \mathrm{U}\gamma^\mu\mathrm{U}^\dagger$.

Proof. Inserting $\Psi = \mathrm{U}^\dagger\Psi'$ (and $\bar{\Psi} = \Psi'^\dagger\mathrm{U}\gamma^0$) in the Dirac Lagrangian:

$$\mathcal{L}_\mathrm{D} = \Psi'^\dagger\mathrm{U}\gamma^0(i\gamma^\mu\partial_\mu - m)\mathrm{U}^\dagger\Psi' = \Psi'^\dagger\mathrm{U}\gamma^0\mathrm{U}^\dagger(i\mathrm{U}\gamma^\mu\mathrm{U}^\dagger\partial_\mu - m)\Psi' = \bar{\Psi}'(i\gamma'^\mu\partial_\mu - m)\Psi'$$

where $\bar{\Psi}' \equiv \Psi'^\dagger\gamma'^0$. The Lagrangian is thus unchanged. □

It can be shown that the Dirac algebra is invariant under the transformation in Prop. 1.3.4, thus it defines an equivalent representation of the algebra.

Proposition 1.3.5: Lorentz invariance

Dirac equation is Lorentz invariant.

Proof. Recalling Eq. 1.49:

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\,\Psi(x) \mapsto& \left(i\gamma^\mu[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\left(i\gamma^\mu[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left(i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left(i\Lambda^\mu{}_\sigma\gamma^\sigma[\Lambda^{-1}]^\nu{}_\mu\partial_\nu - m\right)\Psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left(i\gamma^\nu\partial_\nu - m\right)\Psi(\Lambda^{-1}x) = 0
\end{aligned}$$

□

Solutions The general solution to the Dirac equation is a superposition of plane waves, both with positive and negative-frequency modes:

$$\Psi(x) = u(p)e^{-ip_\mu x^\mu} \qquad\qquad \Psi(x) = v(p)e^{ip_\mu x^\mu}$$

where $u(p)$ and $v(p)$ are Dirac spinors, which in the chiral representation both have a left-handed and a right-handed Weyl spinor component. The Dirac equation than becomes:

$$\left(\not{p} - m\right)u(p) = 0 \qquad\qquad \left(\not{p} + m\right)v(p) = 0$$

Considering the massive positive-frequency solution in the rest frame, i.e. $p^\mu = (m, \mathbf{0})$, one finds $(\gamma^0 - 1)u(p) = 0$, which yields $u_\mathrm{L} = u_\mathrm{R}$: the KG equation imposes the mass-shell condition $p^2 = m^2$, but the Dirac equation, being a first-order equation, halves the number of independent degrees of freedom. A convenient normalization is:

$$u(p_0) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}$$

where $\xi : \xi^\dagger\xi = 1$ is a two-component spinor which gives the spin orientation of the solution: $\xi = \binom{1}{0}$ for the spin-up solution and $\xi = \binom{0}{1}$ for the spin-down one.

The solution in a generic frame is obtained via a boost of the rest-frame solution, using transformation properties of left/right-handed spinors.

Proposition 1.3.6

The solution to the Dirac equation in a generic frame where $\mathbf{p} \parallel \mathbf{e}_z$ is given by:

$$u(p) = \begin{pmatrix} \begin{bmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{bmatrix} \xi \\ \begin{bmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{bmatrix} \xi \end{pmatrix} \tag{1.100}$$

Proof. Consider a boost along $\mathbf{e}_z$, parametrized by rapidity η, by Eq. 1.34:

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \exp\left(\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{pmatrix} m \\ 0 \end{pmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cosh\eta + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sinh\eta\right) \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

Thus, $E+p^3 = e^{\eta}m$ and $E-p^3 = e^{-\eta}m$. By Eqq. 1.41-1.42, then:

$$\begin{aligned} u(p) &= \exp\left(-\frac{\eta}{2}\begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}\right)\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cosh\frac{\eta}{2} - \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}\sinh\frac{\eta}{2}\right)\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= \begin{bmatrix} e^{\eta/2}\frac{1-\sigma^3}{2} + e^{-\eta/2}\frac{1+\sigma^3}{2} & 0 \\ 0 & e^{\eta/2}\frac{1+\sigma^3}{2} + e^{-\eta/2}\frac{1-\sigma^3}{2} \end{bmatrix}\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= \begin{bmatrix} \sqrt{E+p^3}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{E-p^3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \sqrt{E+p^3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sqrt{E-p^3}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \end{aligned}$$

□

This expression can be generalized for an arbitrary direction of $\mathbf{p}$:

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\xi^s \end{pmatrix} \tag{1.101}$$

where the square root of a matrix is understood as taking the positive square root of its eigenvalues, and the polarization of ξ^s is made explicit.
It is convenient to work with specific spinors ξ: a useful choice are eigenvectors of σ^3, in particular $\xi^1 = \binom{1}{0}$ (spin-up) and $\xi^2 = \binom{0}{1}$ (spin-down). In the ultra-relativistic (or massless) limit $p^\mu \to (E, 0, 0, E)$, so u^1 has only the right-handed component and u^2 only the left-handed one.
The negative-frequency solution is equivalent:

$$v^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\eta^s \end{pmatrix} \tag{1.102}$$

where the spin states are the same: $\eta^1 = \binom{1}{0}$ and $\eta^2 = \binom{0}{1}$.
Recalling the Dirac adjoint $\bar{u}^s(p) = u^{s\dagger}(p)\gamma^0, \bar{v}^s(p) = v^{s\dagger}(p)\gamma^0$ and the normalization choice $\xi^{r\dagger}\xi^s = \delta^{rs}, \eta^{r\dagger}\eta^s = \delta^{rs}$, several properties follow:

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \qquad \bar{v}^r(p)v^s(p) = -2m\delta^{rs} \tag{1.103}$$

$$u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs} \qquad v^{r\dagger}(p)v^s(p) = 2E_{\mathbf{p}}\delta^{rs} \tag{1.104}$$

$$\bar{u}^r(p)v^s(p) = 0 \qquad \bar{v}^r(p)u^s(p) = 0 \tag{1.105}$$

$$\bar{u}^r(p)\gamma^\mu u^s(p) = 2p^\mu\delta^{rs} \qquad \bar{v}^r(p)\gamma^\mu v^s(p) = 2p^\mu\delta^{rs} \tag{1.106}$$

Note that $\bar{u}u \in \mathbb{C}$, while $u\bar{u} \in \mathbb{C}^{4\times4}$.

Proposition 1.3.7: Spinor sums

The sum over possible polarizations of a fermion yields:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \tag{1.107}$$

Proof. Using $\sum_{s=1,2}\xi^s\xi^{s\dagger} = \mathrm{I}_2$:

$$\sum_s u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix}\sqrt{p_\mu\sigma^\mu}\xi^s \\ \sqrt{p_\mu\bar{\sigma}^\mu}\xi^s\end{pmatrix}\begin{pmatrix}\xi^{s\dagger}\sqrt{p_\mu\bar{\sigma}^\mu} & \xi^{s\dagger}\sqrt{p_\mu\sigma^\mu}\end{pmatrix}$$

$$= \begin{bmatrix}\sqrt{p_\mu\sigma^\mu}\sqrt{p_\mu\bar{\sigma}^\mu} & \sqrt{p_\mu\sigma^\mu}\sqrt{p_\mu\sigma^\mu} \\ \sqrt{p_\mu\bar{\sigma}^\mu}\sqrt{p_\mu\bar{\sigma}^\mu} & \sqrt{p_\mu\bar{\sigma}^\mu}\sqrt{p_\mu\sigma^\mu}\end{bmatrix} = \begin{bmatrix} m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & m\end{bmatrix}$$

where the identity $(p_\mu\sigma^\mu)(p_\mu\bar{\sigma}^\mu) = p^2 = m^2$ was used. □

Chiral symmetry Consider the massless Dirac Lagrangian. In this case, the action has two global internal symmetries:

$$\psi_{\mathrm{L}} \mapsto e^{i\theta_{\mathrm{L}}}\psi_{\mathrm{L}} \qquad \psi_{\mathrm{R}} \mapsto e^{i\theta_{\mathrm{R}}}\psi_{\mathrm{R}}$$

Therefore, this theory is symmetric under $\mathrm{U}(1)\times\mathrm{U}(1)$. These can be rewritten as two distinct transformations of the Dirac spinor:

$$\Psi \mapsto e^{i\alpha}\Psi \tag{1.108}$$

$$\Psi \mapsto e^{i\beta\gamma^5}\Psi \tag{1.109}$$

where $\alpha \equiv \theta_{\mathrm{L}} = \theta_{\mathrm{R}}$ and $\beta \equiv -\theta_{\mathrm{L}} = \theta_{\mathrm{R}}$ respectively (to prove that the latter is a symmetry, from $\{\gamma^5, \gamma^\mu\} = 0$ it follows that $\gamma^\mu e^{i\beta\gamma^5} = e^{-i\beta\gamma^5}\gamma^\mu$). The former symmetry is called the vector $\mathrm{U}(1)_{\mathrm{V}}$ symmetry, with conserved Noether current the *vector current* $j^\mu_{\mathrm{V}} := \bar{\Psi}\gamma^\mu\Psi$, while the latter is called the axial $\mathrm{U}(1)_{\mathrm{A}}$ symmetry, with conserved *axial current* $j^\mu_{\mathrm{A}} := \bar{\Psi}\gamma^\mu\gamma^5\Psi$.
While the vector symmetry is always a symmetry of the Dirac field, the axial symmetry is only for the massless Dirac field, as:

$$\partial_\mu j^\mu_{\mathrm{A}} = 2im\bar{\Psi}\gamma^5\Psi \tag{1.110}$$

Chapter 2

Quantization of Free Fields

2.1 Scalar fields

As for the quantization of a classical system in Quantum Mechanics, the quantization of a scalar field theory is performed promoting $\phi(t,\mathbf{x})$ and $\Pi(t,\mathbf{x})$ to hermitian operators in the Heisenberg picture and imposing the canonical equal-time commutation relation:

$$[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{2.1}$$

while of course $[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})] = [\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})] = 0$.

2.1.1 Real scalar fields

A real scalar field is promoted to a real hermitian operator. In particular, by Eq. 1.82:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \left[a_\mathbf{p} e^{-ip_\mu x^\mu} + a_\mathbf{p}^\dagger e^{ip_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{2.2}$$

In terms of creation and annihilation operators, the commutator Eq. 2.1 reads:

$$[a_\mathbf{p}, a_\mathbf{p}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.3}$$

while $[a_\mathbf{p}, a_\mathbf{q}] = [a_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] = 0$. These can be regarded as the creation and annihilation operators of a collection of harmonic oscillators, one for each value of the momentum $\mathbf{p}$: the *Fock space* of the real scalar field can thus be constructed analogously to the Hilbert space of the harmonic oscillator.
Defining the *vacuum state* $|0\rangle : a_\mathbf{p}|0\rangle = 0 \ \forall \mathbf{p}$, suitably normalized as $\langle 0|0\rangle = 1$, the generic state of the Fock space is:

$$|\mathbf{p}_1,\dots,\mathbf{p}_n\rangle = \sqrt{2E_{\mathbf{p}_1}}\dots\sqrt{2E_{\mathbf{p}_n}}\, a_{\mathbf{p}_1}^\dagger \dots a_{\mathbf{p}_n}^\dagger |0\rangle \tag{2.4}$$

Proposition 2.1.1: Normalization

For one-particle states:

$$\langle \mathbf{p}_1|\mathbf{p}_2\rangle = 2E_{\mathbf{p}_1}(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2) \tag{2.5}$$

Proof. By Eq. 2.3:

$$\langle \mathbf{p}_1|\mathbf{p}_2\rangle = \sqrt{2E_{\mathbf{p}_1}}\sqrt{2E_{\mathbf{p}_2}}\,\langle 0|a_{\mathbf{p}_1}a_{\mathbf{p}_2}^\dagger|0\rangle = \sqrt{2E_{\mathbf{p}_1}}\sqrt{2E_{\mathbf{p}_2}}\,\langle 0|[a_{\mathbf{p}_1},a_{\mathbf{p}_2}^\dagger]|0\rangle = 2E_{\mathbf{p}_1}(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)$$

□

Lemma 2.1.1

The combination $E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ is Lorentz invariant.

Proof. Recall that:

$$\delta(f(x)-f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0) \tag{2.6}$$

A Lorentz boost along $\mathbf{e}_i$ yields $p_i' = \gamma(p_i+\beta E)$ and $E' = \gamma(E+\beta p_i)$, so:

$$\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\frac{\mathrm{d}p_i'}{\mathrm{d}p_i} = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\gamma\left(1+\beta\frac{\mathrm{d}E}{\mathrm{d}p_i}\right)$$

Note that $p^2 = E^2-\mathbf{p}^2$ is a Lorentz invariant, thus $E\mathrm{d}E = p_i\mathrm{d}p_i$, so:

$$\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\frac{\gamma(E+\beta p_i)}{E} = \delta^{(3)}(\mathbf{p}'-\mathbf{q}')\frac{E'}{E}$$

□

This explains the choice of normalization.

Proposition 2.1.2: KG Hamiltonian

The Hamiltonian of a real scalar field can be written as:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]\right) \tag{2.7}$$

Proof. First of all, from Eq. 2.2:

$$\begin{aligned}\Pi(t,\mathbf{x}) = \partial_0\phi(t,\mathbf{x}) &= -i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-ip_\mu x^\mu} - a_{\mathbf{p}}^\dagger e^{ip_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}}\\ &= -i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]e^{i\mathbf{p}\cdot\mathbf{x}}\end{aligned}$$

$$\boldsymbol{\nabla}\phi(t,\mathbf{x}) = i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]e^{i\mathbf{p}\cdot\mathbf{x}}$$

Inserting these expressions in Eq. 1.83:

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\Bigg\{&-\frac{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}{2}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-iE_{\mathbf{q}}t} - a_{-\mathbf{q}}^\dagger e^{iE_{\mathbf{q}}t}\right] + \\ &+\frac{-\mathbf{p}\cdot\mathbf{q}+m^2}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-iE_{\mathbf{q}}t} + a_{-\mathbf{q}}^\dagger e^{iE_{\mathbf{q}}t}\right]\Bigg\}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\end{aligned}$$

Recall the identity:

$$\int\frac{\mathrm{d}^3x}{(2\pi)^3}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.8}$$

Then (using $E_{-\mathbf{p}} = E_{\mathbf{p}}$):

$$\begin{aligned}
H &= \int \mathrm{d}^3x\, \mathcal{H} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \mathrm{d}^3q \Bigg\{ -\frac{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}{2}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a^\dagger_{-\mathbf{p}}e^{iE_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-iE_{\mathbf{q}}t} - a^\dagger_{-\mathbf{q}}e^{iE_{\mathbf{q}}t}\right] + \\
&\quad + \frac{-\mathbf{p}\cdot\mathbf{q} + m^2}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p}}e^{iE_{\mathbf{p}}t}\right]\left[a_{\mathbf{q}}e^{-iE_{\mathbf{q}}t} + a^\dagger_{-\mathbf{q}}e^{iE_{\mathbf{q}}t}\right]\Bigg\}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Bigg\{ -\frac{E_{\mathbf{p}}}{2}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a^\dagger_{-\mathbf{p}}e^{iE_{\mathbf{p}}t}\right]\left[a_{-\mathbf{p}}e^{-iE_{\mathbf{p}}t} - a^\dagger_{\mathbf{p}}e^{iE_{\mathbf{p}}t}\right] + \\
&\quad + \frac{\mathbf{p}^2 + m^2}{2E_{\mathbf{p}}}\left[a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p}}e^{iE_{\mathbf{p}}t}\right]\left[a_{-\mathbf{p}}e^{-iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p}}e^{iE_{\mathbf{p}}t}\right]\Bigg\} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\Bigg\{ -\left[a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} - a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} - a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}}e^{2iE_{\mathbf{p}}t}\right] \\
&\quad + \left[a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}}e^{2iE_{\mathbf{p}}t}\right]\Bigg\} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left[a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right] = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right)
\end{aligned}$$

□

The second term in the Hamiltonian Eq. 2.7 is the sum of the zero-point energy of all oscillators and is proportional to $(2\pi)^3\delta^{(3)}(0) \to V$, thus:

$$E_{\text{vac}} = \frac{V}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}$$

This energy shows two divergences: the one coming from the infinite-volume limit (i.e. small momentum), regularized introducing an *infrared cutoff* in the form of a finite volume, and the one from the ultra-relativistic limit (i.e. large momentum), regularized introducing an *ultraviolet cutoff* in the form of a maximum momentum Λ. These divergences are retained in the expression for E_{vac}, as $E_{\text{vac}} \sim V$ and $E_{\text{vac}} \sim \Lambda^4$, but can be ignored (when ignoring gravity) since experiments are only sensitive to energy differences.
Discarding the zero-point energy, the Hamiltonian becomes:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}a^\dagger_{\mathbf{p}}a_{\mathbf{p}} \equiv \mathfrak{N}\frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right) \tag{2.9}$$

where the *normal ordering operator* $\mathfrak{N}$ was introduced, which acts by moving all creation operators to the left and all annihilation operators to the right (ex.: $\mathfrak{N}a_{\mathbf{p}}a^\dagger_{\mathbf{p}} = a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$). It is now straightforward to compute the energy of a generic state in the Fock space, as $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ is just a number operator:

$$H\,|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle = (E_{\mathbf{p}_1} + \cdots + E_{\mathbf{p}_n})\,|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle \tag{2.10}$$

Computing the spatial momentum from Eq. 1.79 as $P^i = \mathfrak{N}\int \mathrm{d}^3x\,\theta^{0i} = \int \mathrm{d}^3x\,\mathfrak{N}\partial_0\phi\partial^i\phi$:

$$P^i = \int \frac{\mathrm{d}^3p}{(2\pi)^3}p^i a^\dagger_{\mathbf{p}}a_{\mathbf{p}} \tag{2.11}$$

Therefore, the state $a^\dagger_{\mathbf{p}} |0\rangle$ can correctly be interpreted as a one-particle state with momentum $\mathbf{p}$, mass m and energy $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. The generic state in the Fock space is a multiparticle state, with total energy and momentum the sum of the individual energies and momenta.
Finally, note that creation operators commute between themselves, hence multiparticle states are symmetric under exchange of pairs of particles, i.e. they obey the Bose-Einstein statistics: this agrees with the fact that quanta of a scalar field have no intrinsic spin. i.e. are spin-0 particles.

2.1.2 Complex scalar fields

When considering a complex scalar field, Eq. 1.90 becomes:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b^\dagger_{\mathbf{p}} e^{ip_\mu x^\mu}\right]_{p^0 = E_{\mathbf{p}}} \tag{2.12}$$

$$\phi^\dagger(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left[a^\dagger_{\mathbf{p}} e^{ip_\mu x^\mu} + b_{\mathbf{p}} e^{-ip_\mu x^\mu}\right]_{p^0 = E_{\mathbf{p}}} \tag{2.13}$$

Now there are two independent sets of creation/annihilation operators, which obey the canonical commutation relation:

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{2.14}$$

with all other commutators vanishing. The Fock space is constructed by defining a vacuum state $|0\rangle : a_{\mathbf{p}} |0\rangle = b_{\mathbf{p}} |0\rangle = 0$ and then acting repeatedly with both creation operators. With normal ordering, one finds:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}\right) \tag{2.15}$$

$$P^i = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^i \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}\right) \tag{2.16}$$

The quanta of a complex scalar field are given by two different species of particles with the same mass.

Proposition 2.1.3: U(1) **charge**

The U(1) charge of the quantized complex scalar field is:

$$Q_{\mathrm{U}(1)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}}\right) \tag{2.17}$$

Proof. By Eq. 1.91:

$$Q_{\mathrm{U}(1)} = i \int \mathrm{d}^3 x\, \phi^\dagger \overleftrightarrow{\partial_0} \phi = i \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \times$$
$$\times \Big\{ \left[a^\dagger_{\mathbf{q}} e^{iq_\mu x^\mu} + b_{\mathbf{q}} e^{-iq_\mu x^\mu}\right] \partial_0 \left(a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b^\dagger_{\mathbf{p}} e^{ip_\mu x^\mu}\right) +$$
$$- \partial_0 \left(a^\dagger_{\mathbf{q}} e^{iq_\mu x^\mu} + b_{\mathbf{q}} e^{-iq_\mu x^\mu}\right) \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b^\dagger_{\mathbf{p}} e^{ip_\mu x^\mu}\right] \Big\}$$

$$
\begin{aligned}
Q_{\mathrm{U}(1)} &= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \times \\
&\quad \times \Big\{ E_{\mathbf{p}} \left[a^\dagger_{\mathbf{q}} e^{iq_\mu x^\mu} + b_{\mathbf{q}} e^{-iq_\mu x^\mu}\right] \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} - b^\dagger_{\mathbf{p}} e^{ip_\mu x^\mu}\right] + \\
&\quad + E_{\mathbf{q}} \left[a^\dagger_{\mathbf{q}} e^{iq_\mu x^\mu} - b_{\mathbf{q}} e^{-iq_\mu x^\mu}\right] \left[a_{\mathbf{p}} e^{-ip_\mu x^\mu} + b^\dagger_{\mathbf{p}} e^{ip_\mu x^\mu}\right] \Big\} \\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \times \\
&\quad \times \Big\{ E_{\mathbf{p}} \left[a^\dagger_{\mathbf{q}} e^{iE_{\mathbf{q}}t} + b_{-\mathbf{q}} e^{-iE_{\mathbf{q}}t}\right] \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} - b^\dagger_{-\mathbf{p}} e^{iE_{\mathbf{p}}t}\right] + \\
&\quad + E_{\mathbf{q}} \left[a^\dagger_{\mathbf{q}} e^{iE_{\mathbf{q}}t} - b_{-\mathbf{q}} e^{-iE_{\mathbf{q}}t}\right] \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} + b^\dagger_{-\mathbf{p}} e^{iE_{\mathbf{p}}t}\right] \Big\} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \Big\{ \left[a^\dagger_{\mathbf{p}} e^{iE_{\mathbf{p}}t} + b_{-\mathbf{p}} e^{-iE_{\mathbf{p}}t}\right] \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} - b^\dagger_{-\mathbf{p}} e^{iE_{\mathbf{p}}t}\right] + \\
&\quad + \left[a^\dagger_{\mathbf{p}} e^{iE_{\mathbf{p}}t} - b_{-\mathbf{p}} e^{-iE_{\mathbf{p}}t}\right] \left[a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} + b^\dagger_{-\mathbf{p}} e^{iE_{\mathbf{p}}t}\right] \Big\} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left[a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b_{-\mathbf{p}} b^\dagger_{-\mathbf{p}}\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b_{\mathbf{p}} b^\dagger_{\mathbf{p}}\right)
\end{aligned}
$$

Applying normal ordering yields the thesis. □

While normal ordering was justified when considering the Hamiltonian on the grounds that the vacuum energy is unobservable, a charged vacuum would have observable effects; however when promoting ϕ to a quantum operator, the expression $\phi^\dagger \overleftrightarrow{\partial_0} \phi$ presents an ordering ambiguity (ex.: $\phi^\dagger(\partial_0\phi)$ or $(\partial_0\phi)\phi^\dagger$), which is removed requiring the charge of the vacuum to vanish.
Being $a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}} b_{\mathbf{p}}$ number operators, the U(1) charge is equal to the number of quanta created by $a^\dagger_{\mathbf{p}}$ minus the number of quanta created by $b^\dagger_{\mathbf{p}}$, integrated over all momenta: in particular, $a^\dagger_{\mathbf{p}} |0\rangle$ and $b^\dagger_{\mathbf{p}} |0\rangle$ are both spin-zero particles of mass m and momentum $\mathbf{p}$, but they respectively have charges $Q_{\mathrm{U}(1)} = +1$ and $Q_{\mathrm{U}(1)} = -1$. This allows to properly interpret the negative-energy solutions of the KG equations: they are positive-energy particles with opposite U(1) charge and are called *antiparticles*.
For a real scalar field, the reality condition reads $a_{\mathbf{p}} = b_{\mathbf{p}}$, thus it describes a field whose particle is its own antiparticle, and it is symmetric under any U(1) symmetry.

2.2 Spinor fields

A principle of QFT is the *spin-statistic theorem*: integer-spin fields are to be quantized imposing equal-time commutation relations, while half-integer-spin with equal-time anticommutation relations.

2.2.1 Dirac fields

From the Dirac Lagrangian Eq. 1.98, the conjugate momentum to the Dirac field Ψ is computed as:

$$\Pi_\Psi = i\bar{\Psi}\gamma^0 = i\Psi^\dagger \tag{2.18}$$

Imposing the canonical anticommutation relation, according to the spin-statistic theorem:

$$\{\Psi_a(t,\mathbf{x}),\Psi_b^\dagger(t,\mathbf{y})\}=\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab} \tag{2.19}$$

where $a,b=1,2,3,4$ are Dirac indices. Expanding the free Dirac field in plane waves:

$$\Psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}\sum_{s=1,2}\left[a_{\mathbf{p},s}u^s(p)e^{-ip_\mu x^\mu}+b^\dagger_{\mathbf{p},s}v^s(p)e^{ip_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{2.20}$$

$$\bar\Psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}\sum_{s=1,2}\left[a^\dagger_{\mathbf{p},s}\bar u^s(p)e^{ip_\mu x^\mu}+b_{\mathbf{p},s}\bar v^s(p)e^{-ip_\mu x^\mu}\right]_{p^0=E_\mathbf{p}} \tag{2.21}$$

where the spinor wave functions $u^s(p),v^s(p)$ are given by Eqq. 1.101-1.102. Translating Eq. 2.19 in terms of creation/annihilation operators:

$$\{a_\mathbf{p},s,a^\dagger_{\mathbf{q},r}\}=\{b_{\mathbf{p},s},b^\dagger_{\mathbf{q},r}\}=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{sr} \tag{2.22}$$

The Fock space is again constructed defining a vacuum state $|0\rangle : a_{\mathbf{p},s}|0\rangle=b_{\mathbf{p},s}|0\rangle=0$ and then acting repeatedly on it with $a^\dagger_{\mathbf{p}.s},b^\dagger_{\mathbf{p},s}$:

$$\begin{aligned}&|(\mathbf{p}_1,s_1),\ldots,(\mathbf{p}_n,s_n);(\mathbf{q}_1,r_1),\ldots,(\mathbf{q}_m,r_m)\rangle\\&\qquad=\sqrt{2E_{\mathbf{p}_1}}\ldots\sqrt{2E_{\mathbf{p}_n}}\sqrt{2E_{\mathbf{q}_1}}\ldots\sqrt{2E_{\mathbf{q}_m}}a^\dagger_{\mathbf{p}_1,s_1}\ldots a^\dagger_{\mathbf{p}_n,s_n}b^\dagger_{\mathbf{q}_1,r_1}\ldots b^\dagger_{\mathbf{q}_m,r_m}|0\rangle\end{aligned}$$

As these operators anticommute, states in this Fock space are antisymmetric under the exchange of particles, therefore spin-$\frac{1}{2}$ obey the Fermi-Dirac statistics (as of the spin-statistic theorem).

Proposition 2.2.1: Dirac Hamiltonian

The Hamiltoniana for a Dirac field Ψ is:

$$H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{s=1,2}E_\mathbf{p}\left(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s}+b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}\right) \tag{2.23}$$

Proof. By Eqq. 2.18, the Hamiltonian density is:

$$\mathcal{H}=\Pi_\Psi\partial_0\Psi-\mathcal{L}_\mathrm{D}=i\Psi^\dagger\partial_0\Psi-\bar\Psi\left(i\gamma^0\partial_0+i\gamma^i\partial_i-m\right)\Psi=\bar\Psi\left(-i\gamma^i\partial_i+m\right)\Psi$$

Therefore, using Eqq. 2.20-2.21 and Eq. 2.8:

$$\begin{aligned}H&=\int\mathrm{d}^3x\,\bar\Psi\left(-i\gamma^i\partial_i+m\right)\Psi=\int\mathrm{d}^3x\,\bar\Psi\left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m\right)\Psi\\&=\mathfrak{N}\int\mathrm{d}^3x\int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}\int\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}}\sum_{s=1,2}\sum_{r=1,2}\left[a^\dagger_{\mathbf{p},s}\bar u^s(p)e^{ip_\mu x^\mu}+b_{\mathbf{p},s}\bar v^s(p)e^{-ip_\mu x^\mu}\right]\times\\&\qquad\times\left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m\right)\left[a_{\mathbf{q},r}u^r(q)e^{-iq_\mu x^\mu}+b^\dagger_{\mathbf{q},r}v^r(q)e^{iq_\mu x^\mu}\right]\\&=\mathfrak{N}\int\mathrm{d}^3x\int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}\int\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}}\sum_{s=1,2}\sum_{r=1,2}\left[a^\dagger_{\mathbf{p},s}\bar u^s(p)e^{ip_\mu x^\mu}+b_{\mathbf{p},s}\bar v^s(p)e^{-ip_\mu x^\mu}\right]\times\\&\qquad\times\left[\left(\boldsymbol{\gamma}\cdot\mathbf{q}+m\right)a_{\mathbf{q},r}u^r(q)e^{-iq_\mu x^\mu}+\left(-\boldsymbol{\gamma}\cdot\mathbf{q}+m\right)b^\dagger_{\mathbf{q},r}v^r(q)e^{iq_\mu x^\mu}\right]\end{aligned}$$

Using the Dirac equation in the form $(\not{p}-m)u(p) = (\not{p}+m)v(p) = 0$:

$$(\boldsymbol{\gamma}\cdot\mathbf{q}+m)\,u(q) = \gamma^0 E_{\mathbf{q}} u(q) \qquad (-\boldsymbol{\gamma}\cdot\mathbf{q}+m)\,v(q) = -\gamma^0 E_{\mathbf{q}} v(q)$$

Therefore, omitting the constraint $p^0 = E_{\mathbf{p}}$ in the spinors' arguments and using Eq. 2.8:

$$\begin{aligned}
H = \mathfrak{N}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s}\bar{u}^s(\mathbf{p})e^{iE_{\mathbf{p}}t} + b_{-\mathbf{p},s}\bar{v}^s(-\mathbf{p})e^{-iE_{\mathbf{p}}t}\right]\times \\
\times\,\gamma^0 E_{\mathbf{q}}\left[a_{\mathbf{q},r}u^r(\mathbf{q})e^{-iE_{\mathbf{q}}t} - b^\dagger_{-\mathbf{q},r}v^r(-\mathbf{q})e^{iE_{\mathbf{q}}t}\right] e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \\
= \mathfrak{N}\int \frac{\mathrm{d}^3p}{2(2\pi)^3}\sum_{s=1,2}\sum_{r=1,2}\left[a^\dagger_{\mathbf{p},s}\bar{u}^s(\mathbf{p})e^{iE_{\mathbf{p}}t} + b_{-\mathbf{p},s}\bar{v}^s(-\mathbf{p})e^{-iE_{\mathbf{p}}t}\right]\times \\
\times\,\gamma^0\left[a_{\mathbf{p},r}u^r(\mathbf{p})e^{-iE_{\mathbf{p}}t} - b^\dagger_{-\mathbf{p},s}v^r(-\mathbf{p})e^{iE_{\mathbf{p}}t}\right] \\
= \mathfrak{N}\int \frac{\mathrm{d}^3p}{2(2\pi)^3}\sum_{s=1,2}\sum_{r=1,2}\Big[a^\dagger_{\mathbf{p},s}a_{\mathbf{p},r}u^{s\dagger}(\mathbf{p})u^r(\mathbf{p}) + b_{-\mathbf{p},s}a_{\mathbf{p},r}v^{s\dagger}(-\mathbf{p})u^r(\mathbf{p})e^{-2iE_{\mathbf{p}}t} + \\
- a^\dagger_{\mathbf{p},s}b^\dagger_{-\mathbf{p},r}u^{s\dagger}(\mathbf{p})v^r(-\mathbf{p})e^{2iE_{\mathbf{p}}t} - b_{-\mathbf{p},s}b^\dagger_{-\mathbf{p},r}v^{s\dagger}(-\mathbf{p})v^r(-\mathbf{p})\Big]
\end{aligned}$$

Lemma 2.2.1

$$u^{s\dagger}(\mathbf{p})v^r(-\mathbf{p}) = v^{s\dagger}(-\mathbf{p})u^r(\mathbf{p}) = 0$$

Using Lemma 2.2.1, Eq. 1.104 and the antisymmetry $\mathfrak{N}b_{\mathbf{p},s}b^\dagger_{\mathbf{p},s} = -b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}$:

$$H = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{s=1,2}E_{\mathbf{p}}\mathfrak{N}\left[a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} - b_{\mathbf{p},s}b^\dagger_{\mathbf{p},s}\right] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{s=1,2}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}\right)$$

□

It can be seen that, using anticommutators, the Hamiltonian and its interpretation are analogous to that of the complex scalar field: if commutators were used, instead, one would get a final $-b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}$ term, which is problematic as it yields an energy unbounded from below[1].
The momentum operator too is analogous to that of the complex scalar field, with the additional spin degree of freedom.

Proposition 2.2.2: Momentum operator

The momentum operator of a Dirac field Ψ is:

$$\mathbf{P} = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{s=1,2}\mathbf{p}\left(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}\right) \tag{2.24}$$

[1]This hints to a more profound meaning of the spin-statistic theorem: as shown by Pauli in [1], this theorem is implied by the conditions of Lorentz invariance, positive energies and causality.

Proof. By Eq. 1.78, the $0i$-component of energy-momentum tensor of the Dirac Lagrangian is:

$$\theta^{0i} = \frac{\partial \mathcal{L}_\mathrm{D}}{\partial(\partial_0 \Psi)}\partial^i \Psi = \bar{\Psi} i\gamma^0 \partial^i \Psi = -\Psi^\dagger i \partial_i \Psi$$

Thus, according to Eq. 1.79:

$$\begin{aligned}
\mathbf{P} &= \int \mathrm{d}^3x \, \Psi^\dagger (-i\boldsymbol{\nabla}) \Psi \\
&= \mathfrak{N} \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s} u^{s\dagger}(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} v^{s\dagger}(p) e^{-ip_\mu x^\mu}\right] \times \\
&\qquad\qquad \times (-i\boldsymbol{\nabla}) \left[a_{\mathbf{q},r} u^r(q) e^{-iq_\mu x^\mu} + b^\dagger_{\mathbf{q},r} v^r(q) e^{iq_\mu x^\mu}\right] \\
&= \mathfrak{N} \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s} u^{s\dagger}(p) e^{ip_\mu x^\mu} + b_{\mathbf{p},s} v^{s\dagger}(p) e^{-ip_\mu x^\mu}\right] \times \\
&\qquad\qquad \times \mathbf{q} \left[a_{\mathbf{q},r} u^r(q) e^{-iq_\mu x^\mu} - b^\dagger_{\mathbf{q},r} v^r(q) e^{iq_\mu x^\mu}\right] \\
&= \mathfrak{N} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathbf{p}}{2E_\mathbf{p}} \sum_{s=1,2}\sum_{r=1,2} \left[a^\dagger_{\mathbf{p},s} u^{s\dagger}(\mathbf{p}) e^{iE_\mathbf{p}t} + b_{-\mathbf{p},s} v^{s\dagger}(-\mathbf{p}) e^{-iE_\mathbf{p}t}\right] \times \\
&\qquad\qquad \times \left[a_{\mathbf{p},r} u^r(\mathbf{p}) e^{-iE_\mathbf{p}t} - b^\dagger_{-\mathbf{p},r} v^r(-\mathbf{p}) e^{iE_\mathbf{p}t}\right]
\end{aligned}$$

Using Lemma 2.2.1, Eq. 1.104 and the antisymmetry $\mathfrak{N} b_{\mathbf{p},s} b^\dagger_{\mathbf{p},s} = -b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}$:

$$\mathbf{P} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \mathbf{p} \mathfrak{N} \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b_{\mathbf{p},s} b^\dagger_{\mathbf{p},s}\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \mathbf{p} \left(a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right)$$

□

Thus, both $a^\dagger_{\mathbf{p},s}$ and $b^\dagger_{\mathbf{p},s}$ create particles with energy $+E_\mathbf{p}$ and momentum $\mathbf{p}$: these are respectively called *fermions* and *antifermions*.

2.2.1.1 Quantum numbers

Under a generic Lorentz transformation, the Dirac field Ψ transforms according to Eq. 1.58:

$$\Psi'(x) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \Psi(x) \simeq \left[\mathrm{I}_4 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right] \Psi(x) \tag{2.25}$$

where the Lorentz generator $J^{\mu\nu}$ is defined in Eq. 1.59.

Proposition 2.2.3: Angular momentum

The conserved charge associated to the infinitesimal rotation Eq. 2.25 is:

$$\mathbf{J} = \int \mathrm{d}^3x \, \Psi^\dagger \left(\mathbf{x} \times (-i\boldsymbol{\nabla}) + \tfrac{1}{2}\boldsymbol{\Sigma}\right) \Psi \tag{2.26}$$

Proof. Consider a rotation of θ around the z-axis: it is described by $\omega_{12} = -\omega_{21} = \theta$, so, by Eq. 1.58:

$$\delta_0 \Psi = \theta \left(x^1 \partial^2 - x^2 \partial^1 - \tfrac{i}{2}\Sigma^3\right) \Psi = -\theta \left(x^1 \partial_2 - x^2 \partial_1 + \tfrac{i}{2}\Sigma^3\right) \Psi \equiv \theta \Delta \Psi$$

Using the same notation of Def. 1.3.1, $\epsilon \equiv \theta$ and $F = \Delta\Psi$, therefore, by Eq. 1.76, the temporal component of the conserved Noether current is (without negative sign, as it is mathematically equivalent):

$$j^0 = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial(\partial_0 \Psi)} \Delta \Psi = -i \Psi^\dagger \left((\mathbf{x} \times \boldsymbol{\nabla})^3 + \tfrac{i}{2}\Sigma^3\right) \Psi$$

As the associated Noether charge is $J^3 = \int \mathrm{d}^3x \, j^0$, this can be generalized to rotations around the x-axis and the y-axis, yielding:

$$\mathbf{J} = \int \mathrm{d}^3 \, \Psi^\dagger \left(\mathbf{x} \times (-i\boldsymbol{\nabla}) + \tfrac{1}{2}\boldsymbol{\Sigma}\right) \Psi$$

□

For non-relativistic fermions, the first term gives the orbital angular momentum, while the second term gives the spin angular momentum. For relativistic fermions, this division is not straightforward. To determine the spin of fermions, it is sufficient to consider them at rest. The spin along the z-axis is given by the S_z operator (at $t = 0$):

$$\begin{aligned} S_z &= \int \mathrm{d}^3x \int\int \frac{\mathrm{d}^3p \, \mathrm{d}^3q}{(2\pi)^6 \sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \times \\ &\quad \times \sum_{t=1,2} \sum_{r=1,2} \left[a^\dagger_{\mathbf{p},t} u^{t\dagger}(\mathbf{p}) + b_{-\mathbf{p},t} v^{t\dagger}(-\mathbf{p})\right] \frac{\Sigma^3}{2} \left[a_{\mathbf{q},r} u^r(\mathbf{q}) + b^\dagger_{-\mathbf{q},r} v^r(-\mathbf{q})\right] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{t=1,2} \sum_{r=1,2} \left[a^\dagger_{\mathbf{p},t} u^{t\dagger}(\mathbf{p}) + b_{-\mathbf{p},t} v^{t\dagger}(-\mathbf{p})\right] \frac{\Sigma^3}{2} \left[a_{\mathbf{p},r} u^r(\mathbf{p}) + b^\dagger_{-\mathbf{p},r} v^r(-\mathbf{p})\right] \end{aligned}$$

Noting that $S_z \left|0\right\rangle = 0$ (by definition of vacuum), then $S_z a^\dagger_{\mathbf{0},s} \left|0\right\rangle = \{S_z, a^\dagger_{\mathbf{0},s}\} \left|0\right\rangle$; the only non-zero terms are those proportional to $u^\dagger \Sigma^3 u$ and $v^\dagger \Sigma^3 v$, but the latter vanishes as $\{a, b\} = 0$, so the only term remaining is:

$$\{a^\dagger_{\mathbf{p},t} a_{\mathbf{p},r}, a^\dagger_{\mathbf{0},s}\} = a^\dagger_{\mathbf{p},t} \{a_{\mathbf{p},r}, a^\dagger_{\mathbf{0},s}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p}) \delta_{rs} a^\dagger_{\mathbf{p},t}$$

The S_z operator thus acts as (recall Eq. 1.101 with $p^\mu = (m, 0, 0, 0)$):

$$S_z a^\dagger_{\mathbf{0},s} \left|0\right\rangle = \frac{1}{2E_{\mathbf{p}}} \sum_{t=1,2} u^{t\dagger}(\mathbf{0}) \frac{\Sigma^3}{2} u^s(\mathbf{0}) a^\dagger_{\mathbf{0},t} \left|0\right\rangle = \frac{1}{4} \left(2\xi^{1\dagger} \sigma^3 \xi^s a^\dagger_{\mathbf{0},1} + 2\xi^{2\dagger} \sigma^3 \xi^s a^\dagger_{\mathbf{0},2}\right) \left|0\right\rangle$$

Using $\sigma^3 \xi^1 = +\xi^1$ and $\sigma^3 \xi^2 = -\xi^2$, by $\xi^{r\dagger} \xi^s = \delta^{rs}$ one gets:

$$S_z a^\dagger_{\mathbf{0},1} \left|0\right\rangle = +\frac{1}{2} a^\dagger_{\mathbf{0},1} \left|0\right\rangle \qquad S_z a^\dagger_{\mathbf{0},2} \left|0\right\rangle = -\frac{1}{2} a^\dagger_{\mathbf{0},2} \left|0\right\rangle$$

which means that fermions are spin-$\frac{1}{2}$ particles, with $a^\dagger_{\mathbf{0},1}$ creating $s = +\frac{1}{2}$ fermions and $a^\dagger_{\mathbf{0},2}$ creating $s = -\frac{1}{2}$ fermions. Conversely, it is equivalent to show that $b^\dagger_{\mathbf{0},1}$ creates $s = -\frac{1}{2}$ antifermions and $b^\dagger_{\mathbf{0},2}$

creates $s = +\frac{1}{2}$ antifermions (as b and $b^\dagger$ are not in normal order in S_z, so there's an extra negative sign due to antisymmetry).
Another important conserved Noether charge of Dirac theory is that associated to the vector current $j^\mu_{\mathrm{V}} = \bar{\Psi}\gamma^\mu\Psi$ (recall Eq. 1.108), which is (using $j^0_{\mathrm{V}} = \Psi^\dagger\Psi$):

$$Q_{\mathrm{U(1)_V}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right) \tag{2.27}$$

This means that fermions have $\mathrm{U(1)_V}$ charge of $+1$, while antifermions of -1.

state	S_z	$Q_{\mathrm{U(1)_V}}$
$a^\dagger_{\mathbf{p},1}\|0\rangle$	$+\frac{1}{2}$	$+1$
$a^\dagger_{\mathbf{p},2}\|0\rangle$	$-\frac{1}{2}$	$+1$
$b^\dagger_{\mathbf{p},1}\|0\rangle$	$-\frac{1}{2}$	-1
$b^\dagger_{\mathbf{p},2}\|0\rangle$	$+\frac{1}{2}$	-1

Table 2.1: Quantum numbers for fermions in the Dirac theory.

2.2.2 Massless Weyl fields

The quantization of massless Weyl fields follows immediately from that of Dirac fields, and its useful to use Dirac notation:

$$\Psi_{\mathrm{L}} \equiv \begin{pmatrix} \psi_{\mathrm{L}} \\ 0 \end{pmatrix} \qquad\qquad \Psi_{\mathrm{R}} \equiv \begin{pmatrix} 0 \\ \psi_{\mathrm{R}} \end{pmatrix}$$

Consider Ψ_{L}. As for Eq. 2.20:

$$\Psi_{\mathrm{L}}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} \left[a_{\mathbf{p},s} u^s_{\mathrm{L}}(p) e^{-ip_\mu x^\mu} + b^\dagger_{\mathbf{p},s} v^s_{\mathrm{L}}(p) e^{ip_\mu x^\mu}\right]_{p^0=E_{\mathbf{p}}}$$

where the Dirac spinors now have the right-hand component vanishing, in the chiral representation. By Eqq. 1.100-1.101-1.102, in the massless (ultra-relativistic) limit $p^\mu = (E, 0, 0, E)$, so spinors with $s = 1$ have only the right-handed component, while those with $s = 2$ only the left-handed one: therefore, only $s = 2$ spinors contribute to Ψ_{L} (and only $s = 1$ to Ψ_{R}).
By Tab. 2.1, it is clear that in this context $a^\dagger_{\mathbf{p},2}$ creates a particle with helicity $h = -\frac{1}{2}$, while $b^\dagger_{\mathbf{p},2}$ creates an antiparticle with $h = +\frac{1}{2}$: in general, a left-handed massless Weyl field describes particles with $h = -\frac{1}{2}$ and antiparticles with $h = +\frac{1}{2}$, while a right-handed one describes particles with $h = +\frac{1}{2}$ and antiparticles with $h = -\frac{1}{2}$.

2.2.3 Discrete symmetries of fermionic fields

2.2.3.1 Parity

Under parity $\mathcal{P}\mathbf{p} = -\mathbf{p}$ and $\mathcal{P}s = s$, so, for a generic particle of type a:

$$\mathcal{P}\,|a; \mathbf{p}, s\rangle = \eta_a\,|a; -\mathbf{p}, s\rangle \tag{2.28}$$

where η_a is a generic constant phase factor, since states in the Fock space which differ by a phase still represent the same physical state. As $\mathcal{P}^2 = \mathrm{id}$, this means that $\eta_a^2 = \pm 1$, as observables are built from an even number of fermionic ladder operators.

Non-Majorana fermions It is possible to prove that, for non-Majorana spin-$\frac{1}{2}$ fermions, it is possible to redefine $\mathcal{P}$ so that $\eta_a = +1$, i.e. $\eta_a = \pm 1$.

Proposition 2.2.4

$$\mathcal{P} a^\dagger_{\mathbf{p},s} \mathcal{P} = \eta_a a^\dagger_{-\mathbf{p},s} \qquad \mathcal{P} b^\dagger_{\mathbf{p},s} \mathcal{P} = \eta_b b^\dagger_{-\mathbf{p},s} \tag{2.29}$$

Proof. For a multiparticle state one must have:

$$\mathcal{P} a^\dagger_{\mathbf{p},s} b^\dagger_{\mathbf{q},r} |0\rangle = \eta_a \eta_b a^\dagger_{-\mathbf{p},s} b^\dagger_{-\mathbf{q},r} |0\rangle$$

Therefore:

$$\mathcal{P} a^\dagger_{\mathbf{p},s} = \eta_a a^\dagger_{-\mathbf{p},s} \mathcal{P} \qquad \mathcal{P} b^\dagger_{\mathbf{p},s} = \eta_b b^\dagger_{-\mathbf{p},s} \mathcal{P}$$

Using $\mathcal{P}^2 = \mathrm{id}$ yields the thesis. □

According to Wigner's theorem (Th. 1.2.3), this symmetry can be represented by a unitary operator, i.e. $\mathcal{P}^\dagger = \mathcal{P}^{-1}$, but $\mathcal{P}^2 = \mathrm{id}$, therefore $\mathcal{P}^\dagger = \mathcal{P}$. Then, Eq. 2.29 is valid for $a_{\mathbf{p},s}, b_{\mathbf{p},s}$ too, thus parity acts as:

$$\Psi(x) \mapsto \Psi'(x') = \mathcal{P} \Psi(x) \mathcal{P} \tag{2.30}$$

Explicitly:

$$\begin{aligned}
\mathcal{P}\Psi(x)\mathcal{P} &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[\eta_a a_{-\mathbf{p},s} u^s(p) e^{-ip_\mu x^\mu} + \eta_b b^\dagger_{-\mathbf{p},s} v^s(p) e^{ip_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[\eta_a a_{-\mathbf{p},s} u^s(p) e^{-iE_\mathbf{p} t + i\mathbf{p}\cdot\mathbf{x}} + \eta_b b^\dagger_{-\mathbf{p},s} v^s(p) e^{iE_\mathbf{p} t - i\mathbf{p}\cdot\mathbf{x}} \right]_{p^0 = E_\mathbf{p}}
\end{aligned}$$

Setting $\mathbf{p}' \equiv -\mathbf{p}, \mathbf{x}' \equiv -\mathbf{x}$ and noting that, by Eqq. 1.100-1.102 $\mathbf{p} \mapsto -\mathbf{p}$ exchanges the left-handed and the right-handed compontents of the spinors, i.e. $u^s(p) \mapsto \gamma^0 u^s(p)$ and $v^s(p) \mapsto -\gamma^0 v^s(p)$:

$$\begin{aligned}
\mathcal{P}\Psi(x)\mathcal{P} &= \int \frac{\mathrm{d}^3 p'}{(2\pi)^3 \sqrt{E_{\mathbf{p}'}}} \sum_{s=1,2} \left[\eta_a a_{\mathbf{p}',s} \gamma^0 u^s(p') e^{-iE_{\mathbf{p}'} t + i\mathbf{p}'\cdot\mathbf{x}'} - \eta_b b^\dagger_{\mathbf{p}',s} \gamma^0 v^s(p') e^{iE_{\mathbf{p}'} t - i\mathbf{p}'\cdot\mathbf{x}'} \right]_{p'^0 = E_{\mathbf{p}'}} \\
&= \gamma^0 \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[\eta_a a_{\mathbf{p},s} u^s(p) e^{-ip_\mu x'^\mu} - \eta_b b^\dagger_{\mathbf{p},s} v^s(p) e^{ip_\mu x'^\mu} \right]_{p^0 = E_\mathbf{p}}
\end{aligned}$$

Requiring that Ψ is a representation of parity, up to a phase, means that:

$$\eta_a = -\eta_b \tag{2.31}$$

so that:

$$\mathcal{P}\Psi(t, \mathbf{x})\mathcal{P} = \eta_a \gamma^0 \Psi(t, -\mathbf{x}) \tag{2.32}$$

This, in the chiral representation, agrees with the fact that parity exchanges left-handed and right-handed Weyl spinors. The η_a factor cancels in any fermion bilinear involving only one type of particles; however, the relative phase factors of different particles can be observed: in particular, the opposite intrinsic parity of fermions and antifermions.

Spin-0 bosons As already noted, the scalar complex field is similar to the Dirac field, apart for the absence of spinors in the expansions of $\phi(x)$: in particular, this means that scalar fields have no relative negative signe between η_a and η_b, so a quantized complex scalar field gives a representation of parity if $\eta_a = \eta_b$, and the intrinsic parity of spin-0 particle and antiparticles is equal.

2.2.3.2 Charge conjugation

Recall Eq. 1.44: charge conjugation acts on the classical Dirac field as $\Psi \mapsto -i\gamma^2\Psi^*$.

Definition 2.2.1: Quantized charge conjugation

The *charge conjugation operator* is defined as:

$$\mathcal{C}a_{\mathbf{p},s}\mathcal{C} = \eta_c b_{\mathbf{p},s} \qquad \mathcal{C}b_{\mathbf{p},s}\mathcal{C} = \eta_c a_{\mathbf{p},s} \tag{2.33}$$

with $\eta_c = \pm 1$ for simplicity.

Thus, $\mathcal{C}^2 = \mathrm{id}$ too, and its physical interpretation is the exchange of particles and antiparticles, while leaving $\mathbf{p}$ and s unchanged: as $a_{\mathbf{p},s}$ and $b_{\mathbf{p},s}$ create particles with opposite spin, this means that charge conjugation reverses the helicity of particles.

Lemma 2.2.2: Charge conjugation on spinors

$$\mathcal{C}u^2(p)\mathcal{C} = -i\gamma^2\left[v^s(p)\right]^* \qquad \mathcal{C}v^2(p)\mathcal{C} = -i\gamma^2\left[u^s(p)\right]^* \tag{2.34}$$

Proposition 2.2.5: Charge conjugation on Dirac fields

Given a Dirac field $\Psi(x)$:

$$\mathcal{C}\Psi(x)\mathcal{C} = -i\eta_c\gamma^2[\Psi(x)]^* \tag{2.35}$$

Proof. Using Eqq. 2.33-2.34:

$$\begin{aligned}
\mathcal{C}\Psi(x)\mathcal{C} &= \eta_c \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left[b_{\mathbf{p},s}\mathcal{C}u^s(p)\mathcal{C}e^{-ip_\mu x^\mu} + a^\dagger_{\mathbf{p},s}\mathcal{C}v^s(p)\mathcal{C}e^{ip_\mu x^\mu}\right] \\
&= -i\eta_c\gamma^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left[b_{\mathbf{p},s}[v^s(p)]^*e^{-ip_\mu x^\mu} + a^\dagger_{\mathbf{p},s}[u^s(p)]^*e^{ip_\mu x^\mu}\right] = -i\eta_c\gamma^2[\Psi(x)]^*
\end{aligned}$$

□

As for parity, the transformation of quantized fields is analogous to that of classical fields, with an additional quantum phase factor which depends on the particle type.

Proposition 2.2.6: Charge conjugation of the vector current

$$\mathcal{C}\left(\bar{\Psi}\gamma^\mu\Psi\right)\mathcal{C} = -\bar{\Psi}\gamma^\mu\Psi \tag{2.36}$$

2.2.3.3 Time reversal

Theorem 2.2.1: Anti-unitary time reversal

Time reversal cannot be implemented as a linear unitary operator.

Proof. Assume that the time reversal operator $\mathcal{T}$ is linear and unitary; then, as it must be a symmetry of the free Dirac Lagrangian, $[\mathcal{T}, H] = 0$, so:

$$\begin{aligned}\mathcal{T}\Psi(t,\mathbf{x})\mathcal{T}\left|0\right\rangle &= \mathcal{T}e^{iHt}\Psi(\mathbf{x})e^{-iHt}\mathcal{T}\left|0\right\rangle = e^{iHt}\mathcal{T}\Psi(\mathbf{x})\mathcal{T}e^{-iHt}\left|0\right\rangle = e^{iHt}\mathcal{T}\Psi(\mathbf{x})\mathcal{T}\left|0\right\rangle \\ &= \Psi(-t,\mathbf{x})\left|0\right\rangle = e^{-iHt}\Psi(\mathbf{x})e^{iHt}\left|0\right\rangle = e^{-iHt}\Psi(\mathbf{x})\left|0\right\rangle\end{aligned}$$

assuming $H\left|0\right\rangle = 0$. But the first line is a sum of negative frequencies only, while the second line is a sum of positive frequencies only, yielding an absurdum. □

Time reversal is an example of operator which, according to Wigner's theorem (Th. 1.2.3), is represented as an anti-linear anti-unitary operator, i.e. an operator such that:

$$\langle\mathcal{T}a|\mathcal{T}b\rangle = \langle a|b\rangle^* \qquad \mathcal{T}\lambda\left|a\right\rangle = \lambda^*\mathcal{T}\left|a\right\rangle \qquad \forall\left|a\right\rangle,\left|b\right\rangle \in \mathscr{H},\ \forall\lambda\in\mathbb{C}$$

In particular, anti-linearity (second property) solves the absurdum in the proof of Th. 2.2.1.
Time reversal is expected to reverse the spin of particles, and this can be used to construct $\mathcal{T}$. First, define:

$$\xi^{-s} \equiv -i\sigma^2[\xi^s]^* \tag{2.37}$$

that is, $\xi^{-1} = \binom{0}{1}$ and $\xi^{-2} = \binom{-1}{0}$. This allows redefining $\eta^s \equiv \xi^{-s}$, and also $\xi^{-(-s)} = -\xi^s$.

Lemma 2.2.3: Reversed spinors

$$u^{-s}(-\mathbf{p}) = -\gamma^1\gamma^3[u^s(\mathbf{p})]^* \qquad v^{-s}(-\mathbf{p}) = -\gamma^1\gamma^3[v^s(\mathbf{p})]^* \tag{2.38}$$

Proof. Defining $\tilde{p}^\mu \equiv (p^0, -\mathbf{p})$ and recalling that $\boldsymbol{\sigma}\sigma^2 = -\sigma^2\boldsymbol{\sigma}^*$:

$$u^{-s}(-\mathbf{p}) = \begin{pmatrix}\sqrt{\tilde{p}^\mu\sigma_\mu}(-i\sigma^2[\xi^s]^*)\\ \sqrt{\tilde{p}^\mu\bar{\sigma}_\mu}(-i\sigma^2[\xi^s]^*)\end{pmatrix} = \begin{pmatrix}-i\sigma^2\sqrt{p^\mu\sigma^*_\mu}[\xi^s]^*\\ -i\sigma^2\sqrt{p^\mu\bar{\sigma}^*_\mu}[\xi^s]^*\end{pmatrix} = -i\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}[u^s(\mathbf{p})]^*$$

noting that $i\,\mathrm{diag}(\sigma^2,\sigma^2) = \gamma^1\gamma^3$ completes the proof. On the other hand:

$$\begin{aligned}v^{-s}(-\mathbf{p}) &= \begin{pmatrix}\sqrt{\tilde{p}^\mu\sigma_\mu}(-\xi^s)\\ -\sqrt{\tilde{p}^\mu\bar{\sigma}_\mu}(-\xi^s)\end{pmatrix} = \begin{pmatrix}\sigma^2\sqrt{p^\mu\sigma^*_\mu}\sigma^2(-\xi^s)\\ -\sigma^2\sqrt{p^\mu\bar{\sigma}^*_\mu}\sigma^2(-\xi^s)\end{pmatrix}\\ &= -i\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\sqrt{p^\mu\sigma^*_\mu}(-i\sigma^2[\xi^s]^*)\\ -\sqrt{p^\mu\bar{\sigma}^*_\mu}(-i\sigma^2[\xi^s]^*)\end{pmatrix} = -\gamma^1\gamma^3[v^s(-\mathbf{p})]^*\end{aligned}$$

□

It is useful to define the reversed ladder operators:

$$a_{\mathbf{p},-s} \equiv (a_{\mathbf{p},2}, -a_{\mathbf{p},1}) \qquad b_{\mathbf{p},-s} \equiv (b_{\mathbf{p},2}, -b_{\mathbf{p},1}) \tag{2.39}$$

Definition 2.2.2: Time reversal

The *time reversal operator* is defined as:

$$\mathcal{T} a_{\mathbf{p},s} \mathcal{T} = a_{-\mathbf{p},-s} \qquad \mathcal{T} b_{\mathbf{p},s} \mathcal{T} = b_{-\mathbf{p},-s} \tag{2.40}$$

An additional overall phase is irrelevant. Moreover, as other discrete symmetries, $\mathcal{T}^2 = \text{id}$.

Proposition 2.2.7: Time reversal on Dirac fields

Given a Dirac field $\Psi(x)$:

$$\mathcal{T}\Psi(t, \mathbf{x})\mathcal{T} = -\gamma^1\gamma^3\Psi(-t, \mathbf{x}) \tag{2.41}$$

Proof. Using Lemma 2.2.3 and defining $\tilde{p}^\mu \equiv (p^0, -\mathbf{p})$, $\tilde{x}^\mu \equiv (-t, \mathbf{x})$, so that $p^\mu x_\mu = -\tilde{p}^\mu \tilde{x}_\mu$:

$$\begin{aligned}
\mathcal{T}\Psi(x)\mathcal{T} &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \mathcal{T}\left[a_{\mathbf{p},s}u^s(p)e^{-ip_\mu x^\mu} + b^\dagger_{\mathbf{p},s}v^s(p)e^{ip_\mu x^\mu}\right]\mathcal{T} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{-\mathbf{p},-s}[u^s(p)]^* e^{ip_\mu x^\mu} + b^\dagger_{-\mathbf{p},-s}[v^s(p)]^* e^{-ip_\mu x^\mu}\right] \\
&= -\gamma^3\gamma^1 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_{s=1,2} \left[a_{\tilde{\mathbf{p}},-s}u^{-s}(\tilde{p})e^{-i\tilde{p}_\mu \tilde{x}^\mu} + b^\dagger_{\tilde{\mathbf{p}},-s}v^{-s}(\tilde{p})e^{i\tilde{p}_\mu \tilde{x}^\mu}\right] \\
&= \gamma^1\gamma^3 \int \frac{-\mathrm{d}^3\tilde{p}}{(2\pi)^3\sqrt{2E_{\tilde{\mathbf{p}}}}} \sum_{s=1,2} \left[a_{\tilde{\mathbf{p}},s}u^s(\tilde{p})e^{-i\tilde{p}_\mu \tilde{x}^\mu} + b^\dagger_{\tilde{\mathbf{p}},s}v^s(\tilde{p})e^{i\tilde{p}_\mu \tilde{x}^\mu}\right] = -\gamma^1\gamma^3\Psi(\tilde{x})
\end{aligned}$$

□

2.2.3.4 CPT symmetry

In order to study the invariance properties of fermionic Lagrangians, it is necessary to state the transformation relations of fermion bilinears (see Sec. 3.6 of [2] for details).

	$\bar{\Psi}\Psi$	$i\bar{\Psi}\gamma^5\Psi$	$\bar{\Psi}\gamma^\mu\Psi$	$\bar{\Psi}\gamma^\mu\gamma^5\Psi$	$\bar{\Psi}\sigma^{\mu\nu}\Psi$	∂_μ
$\mathcal{C}$	$+1$	$+1$	-1	$+1$	-1	$+1$
$\mathcal{P}$	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
$\mathcal{T}$	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
$\mathcal{CPT}$	$+1$	$+1$	-1	-1	$+1$	-1

Table 2.2: Eigenvalues of fermion bilinears and derivative operator, with $(-1)^\mu \equiv (+1, -1, -1, -1)$.

This shows that it is not possible to construct a Lorentz-invariant Lagrangian which violates CPT simmetry: this is an example of the *CPT theorem*, which states that, independently of the spin of the particle, a (locale) Lorentz-invariant field theory with a hermitian Hamiltonian cannot violate CPT symmetry.
A consequence of this theorem is that particles and antiparticles have exactly the same mass, which has been so far empirically confirmed.

Chapter 3

Quantum Electrodynamics

3.1 Maxwell theory

The electromagnetic field is described by a 4-vector A_μ, the *gauge potential*. From this, the field strength tensor is defined as:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.1}$$

which is related to the electric and magnetic fields as $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk}B^k$. The Lagrangian of the free electromagnetic field is:

$$\mathcal{L}_\mathrm{M} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} \tag{3.2}$$

The associated equations of motion are:

$$\partial_\mu F^{\mu\nu} = 0 \tag{3.3}$$

Moreover, defining $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ (the Hodge dual), it is trivial to check that, by Schwarz lemma:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{3.4}$$

Eqq. 3.3-3.4 are exactly Maxwell equations in the absence of sources: when written in terms of $\mathbf{E}$ and $\mathbf{B}$, Eq. 3.3 gives the equations for $\boldsymbol{\nabla}\cdot\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{B}$, while Eq. 3.4 those for $\boldsymbol{\nabla}\times\mathbf{E}$ and $\boldsymbol{\nabla}\cdot\mathbf{B}$.

3.1.1 Gauge invariance

A crucial local symmetry of the Maxwell Lagrangian is the symmetry under local gauge transformations like:

$$A_\mu(x) \mapsto A_\mu(x) - \partial_\mu\alpha(x) \tag{3.5}$$

with arbitrary $\alpha \in \mathcal{C}^\infty(\mathbb{R}^{1,3})$. Considering the free electromagnetic field, the global version of this transformation (that is, α independent of x) yields no conserved charge, as the associated Noether current vanishes identically.

Theorem 3.1.1: Radiation gauge

In the absence of sources, it is always possible to choose the *radiation gauge*:

$$A_0 = 0 \qquad\qquad \boldsymbol{\nabla}\cdot\mathbf{A} = 0 \tag{3.6}$$

Proof. Starting from a general gauge potential A_μ, the condition $A_0 = 0$ is achieved through:

$$A_\mu \mapsto A_\mu - \partial_\mu \int_{t_0}^{t} \mathrm{d}\tau\, A_0(\tau, \mathbf{x})$$

Then, $A_0 = 0$ will remain unchanged if another gauge transformation with $\alpha(x) = \alpha(\mathbf{x})$ is performed. Consider:

$$\alpha(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 y}{4\pi\,|\mathbf{x}-\mathbf{y}|} \partial_i A^i(t, \mathbf{y})$$

which is independent of t since $E^i = -\partial_0 A^i$, as $A_0 = 0$, so $\partial_i E^i = 0$ implies $\partial_0 \partial_i A^i = 0$. Recall the identity:

$$\triangle_{\mathbf{x}} \frac{1}{|\mathbf{x}-\mathbf{y}|} = -4\pi \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{3.7}$$

Thus:

$$\boldsymbol{\nabla} \cdot \mathbf{A} \mapsto \boldsymbol{\nabla} \cdot \mathbf{A} - \triangle_{\mathbf{x}} \alpha = \partial_i A^i(t, \mathbf{x}) - \partial_i A^i(t, \mathbf{x}) = 0$$

□

The radiation gauge clearly implies the *Lorentz gauge*:

$$\partial_\mu A^\mu = 0 \tag{3.8}$$

In this gauge, the equations of motions Eq. 3.3 become:

$$\Box A^\mu = 0 \tag{3.9}$$

which are massless KG equations for each component of the gauge potential. Plane-wave solutions take the form:

$$A_\mu(x) = \epsilon_\mu(k) e^{-ik_\mu x^\mu} + c.c. \tag{3.10}$$

where $\epsilon_\mu(x)$ is the *polarization vector*. Then, Eq. 3.9 gives $k^2 = 0$, while the chosen radiation gauge implies $\epsilon_0 = 0$ and $\boldsymbol{\epsilon} \cdot \mathbf{k} = 0$: therefore, an electromagnetic wave has only two degrees of freedom, represented by a polarization vector $\boldsymbol{\epsilon}$ perpendicular to the direction of propagation.
The advantage of the radiation gauge is that it exposes clearly the physical degrees of freedom of the electromagnetic field, while sacrificing explicit Lorentz covariance; on the other hand, the Lorentz gauge retains the explicit Lorentz covariance, at the cost of redundant degrees of freedom.

3.1.2 Energy-momentum tensor

By Eq. 1.78, writing Eq. 3.2 explicitly in terms of A_μ, the energy-momentum tensor of the electromagnetic field is:

$$\theta^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu} F^2 \tag{3.11}$$

with $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$. To show the gauge-invariance of this tensor, recall Eq. 3.3:

$$\theta^{\mu\nu} \mapsto \theta^{\mu\nu} + F^{\mu\rho}\partial^\nu\partial_\rho\alpha \qquad \Rightarrow \qquad P^\mu \mapsto P^\mu + \int \mathrm{d}^3x\, \partial_\rho(F^{0\rho}\partial^\mu\alpha) = P^\mu + \int \mathrm{d}^3x\, \partial_i(F^{0i}\partial^\mu\alpha) = P^\mu$$

where the last term is a total spatial derivative, hence vanishing by divergence theorem provided that the field decreases sufficiently fast at infinity. To improve the energy-momentum tensor, add $\partial_\rho(F^{\mu\rho}A^\nu)$, which is covariantly conserved by itself and whose $\mu = 0$ component is a total spatial derivative, so to obtain:

$$T^{\mu\nu} = F^{\mu\rho}F_\rho{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^2 \tag{3.12}$$

which is explicitly gauge-invariant and yields the usual expressions for the energy density $T^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$ and the momentum density $T^{0i} = (\mathbf{E}\times\mathbf{B})^i$.
In a general field theory, the observable quantities are the charges, not the currents: two Lagrangian densities which differ by a total 4-divergence are physically equivalent and give the same equations of motion, but the conserved currents obtained through Noether theorem are different, while the associated Noether charges are the same.

3.1.3 Matter coupling

In the presence of an external current j^μ, Eq. 3.4 is not modified, as it is a consequence of the definition of $F^{\mu\nu}$ (assuming regular gauge fields), while Eq. 3.3 becomes:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{3.13}$$

By Schwarz lemma, this equation is consistent only if $\partial_\mu j^\mu = 0$. This can be understood in light of gauge invariance, considering the action:

$$\mathcal{S}_\mathrm{M} = -\int \mathrm{d}^4x \left[\frac{1}{4}F^2 + j^\mu A_\mu\right] \tag{3.14}$$

A gauge transformation $A_\mu \mapsto A_\mu - \partial_\mu\alpha$ implies $\mathcal{S}_\mathrm{M} \mapsto \mathcal{S}_\mathrm{M} + \int \mathrm{d}^4x\, j^\mu\partial_\mu\alpha$: integrating by parts, it is clear that $\mathcal{S}_\mathrm{M}$ is gauge invariant only if $\partial_\mu j^\mu = 0$.

3.1.3.1 Dirac field

Appendices

Appendix A

Mathematical Reference

A.1 Lie groups

Definition A.1.1: Lie groups

A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d-dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0,\dots,0) = e$.

Definition A.1.2: Representations

Given a group G and a vector space $V(\mathbb{K})$, a *representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

A representation ρ which is a isomorphism is called *faithful*. As $\mathrm{GL}(V) \cong \mathbb{K}^{n\times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to represent G as matrices acting on elements of V, i.e. $\rho : G \to \mathbb{K}^{n\times n}$.

Theorem A.1.1

Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ is:

$$\rho(g(\theta)) = e^{i\theta_a T^a} \tag{A.1}$$

where $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ are the *generators* of G on V.

Definition A.1.3: Lie algebras

Given a Lie group G with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, its *Lie algebra* is:

$$[T^a, T^b] = i f^{ab}{}_c T^c \tag{A.2}$$

where $f^{ab}{}_c$ are called the *structure constants*.

The sum over repeated indices is understood.

Proposition A.1.1

The Lie algebra of a Lie group is independent of the representation.

Proposition A.1.2

Any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

Definition A.1.4: Casimir operators

Given a Lie group with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation on V, Casimir operators are operators proportional to id_V, and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

Proposition A.1.3

A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

A.1.1 Adjoint representation

To define the adjoint representation of a Lie group, it is necessary to give more formal definitions.

Definition A.1.5: Lie algrebas

An n-dimensional $\mathbb{K}$-*Lie algebra* $\mathfrak{g}$ is an n-dimensional $\mathbb{K}$-vector space equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

1. $[X, Y] = -[Y, X]\ \forall X, Y \in \mathfrak{g}$;
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ \forall X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

A Lie algebra $\mathfrak{g}$ is *commutative* if $[X, Y] = 0\ \forall X, Y \in \mathfrak{g}$.

Example A.1.1: $\mathbb{R}^3$ as a Lie algebra

Let $\mathfrak{g} = \mathbb{R}^3$ and $[\cdot,\cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}, \mathbf{y}] \equiv \mathbf{x} \times \mathbf{y} \in \mathbb{R}^3$. Then $\mathfrak{g}$ is a Lie algebra.

Definition A.1.6: Lie algebra morphisms

Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if:

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] \ \forall X, Y \in \mathfrak{g}$$

If φ is bijective, then it is a *Lie algebra isomorphism*.

A Lie algebra isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a *Lie algebra automorphism* $\mathfrak{g} \in \operatorname{Aut}\mathfrak{g}$, where $\operatorname{Aut}\mathfrak{g}$ is the *automorphism group* of $\mathfrak{g}$ (a group under the composition of morphisms).

Definition A.1.7: Adjoint map (pt. 1)

Let $\mathfrak{g}$ be a Lie group and, given $X \in \mathfrak{g}$, define $\operatorname{ad}_X : \mathfrak{g} \to \mathfrak{g} : Y \mapsto \operatorname{ad}_X(Y) := [X, Y]$. Then the *adjoint map* on $\mathfrak{g}$ is the map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}\mathfrak{g} : X \mapsto \operatorname{ad}_X$.

By Jacobi identity, the adjoint map is a *derivation* of the Lie bracket, as:

$$\operatorname{ad}_X([Y, Z]) = [\operatorname{ad}_X(Y), Z] + [Y, \operatorname{ad}_X(Z)]$$

Proposition A.1.4

Given a Lie algebra $\mathfrak{g}$, the adjoint map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}\mathfrak{g}$ is a Lie algebra homomorphism.

Proof. Note that, by Jacobi identity:

$$\operatorname{ad}_{[X,Y]}(Z) = [[X, Y], Z] = [X, [Y, Z]] + [Y, [Z, X]]$$

Moreover:

$$[\operatorname{ad}_X, \operatorname{ad}_Y](Z) = [X, [Y, Z]] - [Y, [X, Z] = [X, [Y, Z]] + [Y, [Z, X]]$$

Therefore, $\operatorname{ad}_{[X,Y]} = [\operatorname{ad}_X, \operatorname{ad}_Y]$, i.e. the thesis. □

Let $\mathfrak{g}$ be an n-dimensional $\mathbb{K}$-Lie algebra and $\{X_i\}_{i=1,\dots,n} \subset \mathfrak{g}$ a basis of $\mathfrak{g}$. Then there are unique constants $c_{ijk} \in \mathbb{K}$:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k$$

known as *structure constants*.

Theorem A.1.2: Lie algebras from Lie groups

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a Lie group. Then $\mathfrak{g} = \{X \in \mathbb{C}^{n\times n} : e^{tX} \in G \ \forall t \in \mathbb{R}\}$ is a Lie algebra.

Even if G is a complex Lie group, its Lie algebra can still be real.

Theorem A.1.3: Induced Lie algebra homomorphism

Let G, H be Lie groups, with Lie algebra $\mathfrak{g}, \mathfrak{h}$, and let $\varphi : G \to H$ be a Lie group homomorphism.

Then there exists a unique $\mathbb{R}$-linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ such that:

$$\varphi(e^X) = e^{\Phi(X)} \ \forall X \in \mathfrak{g}$$

The map Φ as additional properties:

1. $\Phi(gXg^{-1}) = \varphi(g)\Phi(X)\varphi(g)^{-1} \ \forall X \in \mathfrak{g}, \forall g \in G$;
2. $\Phi([X, Y]) = [\Phi(X), \Phi(Y)] \ \forall X, Y \in \mathfrak{g}$ (Lie algebra homomorphism);
3. $\Phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varphi(e^{tX}) \ \forall X \in \mathfrak{g}$.

To phrase Th. A.1.3 in the language of manifolds, Φ is the *derivative* of φ at the identity: $\Phi = d\varphi_e$.

Definition A.1.8: Adjoint map (pt. 2)

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. The *adjoint map* of $g \in G$ is the linear map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} : \mathrm{Ad}_g(X) := gXg^{-1}$.

As Ad_g is clearly invertible, with $\mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$, then $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g}) \ \forall g \in G$. Furthermore, it is clear that $\mathrm{Ad}_g([X, Y]) = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)] \ \forall X, Y \in \mathfrak{g}, \forall g \in G$, therefore each adjoint map is a Lie algebra homomorphism.

Proposition A.1.5: Adjoint representation

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{Ad}_g$ is a homomorphism.

Recalling Def. A.1.2, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$, called the *adjoint representation*. As $\mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}(n, \mathbb{K})$ (with $n \equiv \dim_{\mathbb{K}} \mathfrak{g}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), it can be viewed as a Lie group itself, and its Lie algebra is $\mathfrak{gl}(\mathfrak{g})$. Thus, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a Lie group homomorphism (as it can be shown to be continuous).

Proposition A.1.6: Adjoint maps

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then, given the Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, the induced Lie algebra homomorphism is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $\mathrm{ad}_X(Y) = [X, Y]$.

Proof. By Th. A.1.3, the Lie algebra homomorphism induced by $\varphi \equiv \mathrm{Ad}$ is:

$$\Phi(X) \equiv \mathrm{ad}_X = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}_{e^{tX}}$$

Hence:

$$\mathrm{ad}_X(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{tX} Y e^{-tX} = [X, Y]$$

□

This result links the two adjoint maps in Deff. A.1.7-A.1.8.

A.1.2 SU(n) Lie group

The SU(n) group is the group of unitary transformations of n-dimensional complex vectors. Its (faithful) fundamental representation thus is:

$$\mathrm{SU}(n) = \{\mathrm{U} \in \mathbb{C}^{n\times n} : \mathrm{U}\mathrm{U}^\dagger = \mathrm{U}^\dagger\mathrm{U} = \mathrm{I}_n \wedge \det \mathrm{U} = +1\}$$

The generators of SU(n) can be found setting $\mathrm{U} = \exp\left(i\theta_a T^a\right) = \mathrm{I}_n + i\theta_a T^a$ and using $\mathrm{U}^\dagger\mathrm{U} = \mathrm{I}_n$:

$$T^a = T^{a\dagger} \tag{A.3}$$

Moreover, by the Jacobi formula $(\det A(t))\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \mathrm{tr}(A(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A(t))$ evaluated at $t = 0$:

$$\mathrm{tr}\, T^a = 0 \tag{A.4}$$

The traceless condition can be generalized to all semi-simple Lie algebras. Therefore, the generators of SU(n) are $\mathbb{C}^{n\times n}$ hermitian traceless matrices: the dimension of $\mathfrak{su}(n)$ then is $n^2 - 1$.
The adjoint representation can be is given by representing the generators of the Lie group (i.e. the basis of the Lie algebra) with the structure constants of the Lie algebra:

$$(T^b_{\mathrm{ad}})_{ac} = i f^{ab}{}_c \tag{A.5}$$

Proposition A.1.7: Structure constants

The structure constants of a Lie algebra satisfy the Lie algebra.

Proof. As $[T^a, T^b] = i f^{ab}{}_c T^c$, the Jacoby identity becomes (recalling that $f^{ab}{}_c$ is totally anti-symmetric):

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$$
$$\iff f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0$$

The condition $([T^a_{\mathrm{ad}}, T^c_{\mathrm{ad}}])_{be} = i f^{ac}{}_d (T^d_{\mathrm{ad}})_{be}$ then gives:

$$f^{ba}{}_d f^{dc}{}_e - f^{bc}{}_d f^{da}{}_e = f^{ac}{}_d f^{bd}{}_e$$

$$\iff f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0$$

These two expressions are equal, hence the thesis. □

Moreover, since the structure constant are real, the adjoint representation is always a real representation: the adjoint representation of SU(n) has degree $n^2 - 1$.
Representation are labelled by their Casimir operators. For any simple Lie algebra, given a representation $\mathtt{r}$, a Casimir operator is defined as:

$$T^a_{\mathtt{r}} T^a_{\mathtt{r}} = C_2(\mathtt{r})\mathrm{I}_{n_{\mathtt{r}}} \tag{A.6}$$

This is called the *quadratic Casimir operator*, as it is associated to $T^2 \equiv T^a T^a$ (a Casimir operator since $[T^b, T^2] = i f^{ba}{}_c \{T^c, T^a\} = 0$ by antisymmetry).

Proposition A.1.8: Quadratic Casimir operator

For the fundamental and the adjoint representations $\mathtt{n}$ and $\mathtt{g}$ of SU(n), the quadratic Casimir operator is:

$$C_2(\mathtt{n}) = \frac{n^2-1}{2n} \qquad\qquad C_2(\mathtt{g}) = n \tag{A.7}$$

Proof. First consider the fundamental representation. The $\mathtt{n}$ of SU(2) is the spinor representation, with generators $T_2^a = \frac{\sigma^a}{2}$, hence they satisfy $\mathrm{tr}(T_2^a T_2^b) = \frac{1}{2}\delta^{ab}$. It is possible to choose the generators of SU(n) such that, in $\mathtt{n}$, the first three are $T_{\mathtt{n}}^a = \frac{\sigma^a}{2}$, $a = 1, 2, 3$ (they act on the first three component of $\mathbb{C}^n$-vectors), so that the others can be chosen such that:

$$\mathrm{tr}(T_{\mathtt{n}}^a T_{\mathtt{n}}^b) = \frac{1}{2}\delta^{ab}$$

Contracting Eq. A.6 with δ^{ab} (with $a, b = 1, \ldots, n^2-1$, as they label the basis of $\mathfrak{su}(n)$) then:

$$C_2(\mathtt{n})n = \frac{1}{2}(n^2-1)$$

To compute the Casimir operator for the adjoint representation, first consider the decomposition of the direct product of two representations:

$$\mathtt{r}_1 \otimes \mathtt{r}_2 = \bigoplus_i \mathtt{r}_i$$

In this representation $T^a_{\mathtt{r}_1\otimes\mathtt{r}_2} = T^a_{\mathtt{r}_1} \otimes \mathrm{id}_{\mathtt{r}_2} + \mathrm{id}_{\mathtt{r}_1} \otimes T^a_{\mathtt{r}_2}$, and it acts on tensor objects Ξ_{pq} whose first index transforms according to $\mathtt{r}_1$ and the second index according to $\mathtt{r}_2$. Recalling that $\mathrm{tr}\, T^a = 0$:

$$\begin{aligned}\mathrm{tr}(T^a_{\mathtt{r}_1\otimes\mathtt{r}_2})^2 &= \mathrm{tr}((T^a_{\mathtt{r}_1})^2 \otimes \mathrm{id}_{\mathtt{r}_2} + 2T^a_{\mathtt{r}_1} \otimes T^a_{\mathtt{r}_2} + \mathrm{id}_{\mathtt{r}_1} \otimes (T^a_{\mathtt{r}_2})^2) \\ &= \mathrm{tr}(C_2(\mathtt{r}_1)\,\mathrm{id}_{\mathtt{r}_1} \otimes \mathrm{id}_{\mathtt{r}_2}) + \mathrm{tr}(C_2(\mathtt{r}_2)\,\mathrm{id}_{\mathtt{r}_1} \otimes \mathrm{id}_{\mathtt{r}_2}) = (C_2(\mathtt{r}_1) + C_2(\mathtt{r}_2))n_{\mathtt{r}_1}n_{\mathtt{r}_2}\end{aligned}$$

However, by the decomposition above:

$$\mathrm{tr}(T^a_{\mathtt{r}_1\otimes\mathtt{r}_2})^2 = \sum_i C_2(\mathtt{r}_i)n_{\mathtt{r}_i}$$

Consider $\mathtt{n} \otimes \mathtt{n}^*$, where $\mathtt{n}^*$ is the complex conjugate of the fundamental representation (for complex representations, $\mathtt{r}$ and $\mathtt{r}^*$ are generally inequivalent representations): then Ξ_{pq} contains a term proportional to the invariant δ_{pq}, while the other n^2-1 independent components transform as a general $n \times n$ traceless tensor, i.e. under the adjoint representation of SU(n) (as of Eqq. A.3-A.4), thus $\mathtt{n} \otimes \mathtt{n}^* = \mathbf{1} \oplus \mathtt{g}$ and the above identity becomes:

$$(C_2(\mathbf{1}) + C_2(\mathtt{g}))(n^2-1) = (C_2(\mathtt{n}) + C_2(\mathtt{n}^*))n^2$$

Using $C_2(\mathbf{1}) = 0$ (as all generators are trivially zero) and $C_2(\mathtt{n}^*) = C_2(\mathtt{n})$:

$$C_2(\mathtt{g})(n^2-1) = \frac{n^2-1}{n}n^2$$

which is completes the proof. □

A.1.2.1 SU(2) **Lie group**

The fundamental representation of SU(2) is $T^a_{\mathbf{2}} = \frac{\sigma^a}{2}$, while $\mathfrak{su}(2)$ is defined by commutators $[T^a_{\mathbf{2}}, T^b_{\mathbf{2}}] = i\epsilon^{abc}T^c_{\mathbf{2}}$ (as $\sigma^a\sigma^b = \delta^{ab}\mathrm{I}_2 + i\epsilon^{abc}\sigma^c$). The adjoint representation then is:

$$(T^a_{\mathfrak{g}})_{ij} = i\epsilon^{iaj} \tag{A.8}$$

Explicitely:

$$T^1_{\mathfrak{g}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad T^2_{\mathfrak{g}} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \qquad T^3_{\mathfrak{g}} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As an aside, these are exactly the generators of the fundamental representation of SO(3): this is due to the adjoint map of SU(2) being also the double-covering map on $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$.

A.1.3 Clifford algebras

Definition A.1.9: Associative algebras

An n-dimensional *associative* $\mathbb{K}$-*algebra* $\mathcal{A}$ is an n-dimensional vector space $V(\mathbb{K})$ equipped with a bilinear map $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto ab \in \mathcal{A}$ such that:

1. $(ab)c = a(bc)\ \forall a, b, c \in \mathcal{A}$ (associativity);
2. $a(b + c) = ab + ac \ \wedge\ (a + b)c = ac + bc\ \forall a, b, c \in \mathcal{A}$;
3. $\lambda(ab) = (\lambda a)b = a(\lambda b)\ \forall \lambda \in \mathbb{K}, \forall a, b \in \mathcal{A}$.

An algebra is said to be *unital* if $\exists \mathit{1} \in \mathcal{A} : \mathit{1}a = a\mathit{1} = a\ \forall a \in \mathcal{A}$, called *identity element*.

Definition A.1.10: Algebra morphisms

Given two associative $\mathbb{K}$-algebras $\mathcal{A}, \mathcal{B}$, a $\mathbb{K}$-linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is an *algebra morphism* if:

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in \mathcal{A}$$

If $\varphi(\mathit{1}_{\mathcal{A}}) = \mathit{1}_{\mathcal{B}}$, then φ is a *unital morphism*. An algebra morphism $\varphi : \mathcal{A} \to \mathcal{A}$ is an *endomorphism*, and if $\varphi^2 = \mathrm{id}_{\mathcal{A}}$ it is an *involution*.

Proposition A.1.9: Even subalgebras

Given a unital associative $\mathbb{K}$-algebra $\mathcal{A}$ and an involution $\varphi \in \operatorname{End}\mathcal{A}$, then:

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$$

where, defining $\pi := \frac{1}{2}(\mathrm{id}_{\mathcal{A}} + \varphi)$:

$$\mathcal{A}^+ := \pi(\mathcal{A}) = \{a \in \mathcal{A} : \varphi(a) = a\}$$

$$\mathcal{A}^- := (\mathrm{id}_{\mathcal{A}} - \pi)(\mathcal{A}) = \{a \in \mathcal{A} : \varphi(a) = -a\}$$

As $\mathcal{A}^+\mathcal{A}^+, \mathcal{A}^-\mathcal{A}^- \subset \mathcal{A}^+$ and $\mathcal{A}^+\mathcal{A}^-, \mathcal{A}^-\mathcal{A}^+ \subset \mathcal{A}^-$, then $\mathcal{A}^+$ is a subalgebra of $\mathcal{A}$, the *even subalgebra*.

Definition A.1.11: Clifford algebras

Given an n-dimensional vector space $V(\mathbb{K})$ with a quadratic form q, associated linear form[1] ω and orthogonal basis $\{e_i\}_{i=1,\dots,n}$, and a unital associative $\mathbb{K}$-algebra $\mathcal{A}$, a *Clifford mapping* is an injective $\mathbb{K}$-linear map $\rho : V \to \mathcal{A} : \mathit{1} \notin \rho(V) \wedge \rho(x)^2 = -q(x)\mathit{1}\ \forall x \in V$.
If $\rho(V)$ generates $\mathcal{A}$, then $(\mathcal{A}, \rho)$ is a *Clifford algebra* for (V, q), and is denoted by $\mathfrak{cl}(V)$.

Lemma A.1.1

$$\{\rho(x), \rho(y)\} = -2\omega(x,y)\mathit{1} \quad \forall x, y \in V$$

Proof.

$$\rho(x)\rho(y) + \rho(y)\rho(x) = \rho(x+y)^2 - \rho(x)^2 - \rho(y)^2 = -\left(q(x+y) - q(x) - q(y)\right)\mathit{1} = -2\omega(x,y)\mathit{1}$$

□

Setting for ease of reading $\rho(x) \equiv x$, it is clear that $x \perp y \ \Rightarrow\ xy = -yx$.
More intuitively, the Clifford algebra $\mathfrak{cl}(V)$ can be seen as the associative algebra generated by V setting $xy = -\omega(x,y)\mathit{1}\ \forall x, y \in V$, so that:

$$\{x, y\} = 2\omega(x,y)\mathit{1} \quad \forall x, y \in V \tag{A.9}$$

In general, $\mathfrak{cl}_{m,n}(\mathbb{R})$ denotes the Clifford algebra associated to $\mathbb{R}^{m,n}$ with quadratic form:

$$q(x_1, \dots, x_m, y_1, \dots, y_n) = \sum_{i=1}^{m} x_i^2 - \sum_{j=1}^{n} y_j^2$$

Example A.1.2: Complex numbers

Via Clifford algebras, $\mathbb{C} \cong \mathfrak{cl}_{0,1}(\mathbb{R})$. In fact, $\mathbb{R}^{0,1}$ has orthonormal basis $\{e_1\}$ such that $q(e_1) = -1$, i.e. $e_1^2 = -\mathit{1}$, so elements of the Clifford algebra are generated by $\{\mathit{1}, e_1\}$: identifying $e_1 \equiv i$ gives the desired isomorphism.

Example A.1.3: Quaternions

$\mathbb{H} \cong \mathfrak{cl}_{0,2}(\mathbb{R})$. Indeed, $\mathbb{R}^{0,2}$ has orthonormal basis $\{e_1, e_2\} : e_1^2 = e_2^2 = -\mathit{1}$; moreover, as $e_1 \perp e_2$, then $e_1e_2 = -e_2e_1$ by Eq. A.9, so elements of $\mathfrak{cl}_{0,2}(\mathbb{R})$ are generated by $\{\mathit{1}, e_1, e_2, e_1e_2\}$: setting $e_1 \equiv i$, $e_2 \equiv j$ and $e_1e_2 = k$ yields the result.

[1] Given a vector space $V(\mathbb{K})$, a quadratic form is a map $q : V \to \mathbb{K}$ such that $q(\lambda x) = \lambda^2 q(x)\ \forall \lambda \in \mathbb{K}, \forall x \in V$, and the associated bilinear form is a $\mathbb{K}$-bilinear map $\omega : V \times V \to \mathbb{K}$ such that $\omega(x,y) = \frac{1}{2}\left(q(x+y) - q(x) - q(y)\right)$, which is manifestly symmetric and $q(x) = \omega(x,x)$.

A.1.3.1 Spin groups

Given an n-dimensional $\mathbb{K}$-vector space, the Clifford algebra $\mathfrak{cl}(V)$ is finite dimensional and is naturaly $\mathbb{N}$-graded[2]as:

$$\mathfrak{cl}(V) = \bigoplus_{i=1}^{n} \mathfrak{cl}^{(i)}(V) \tag{A.10}$$

where $\mathfrak{cl}^{(0)}(V) = \mathbb{K}$, $\mathfrak{cl}^{(1)}(V) = V$ and $\mathfrak{cl}^{(2)}(V) \equiv \mathrm{Spin}(V)$ is the *spin group* of V. The spin group is a Lie group and, via its natural action on V, can be shown to be $\mathrm{Spin}(V) \cong \mathfrak{so}(V)$.

[2]Given an index set I and a $\mathbb{K}$-vector space, the latter is *I-graded* if there exists a family of subspaces $\{V_i\}_{i\in I}$ of V such that:

$$V = \bigoplus_{i\in I} V_i$$

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