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General Relativity and Cosmology

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Notation

Conventions

In these notes, the Lorentz–Minkowski metric $\eta_{\mu\nu}$ has signature $(-,+,+,+)$ and Greek indices generally run over spacetime coordinates, while Latin indices are general $\mathbb{N}_0$-indices defined in each context. Repeated indices are generally summed over, unless otherwise specified. The n-dimensional Levi–Civita symbol $\epsilon^{i_1\cdots i_n}$ is defined with the convention $\epsilon^{01\dots n} = +1$.
Given $\alpha \in \mathbb{C}^{n\times n}$, with $n \in \mathbb{N}_0$, its complex conjugate is denoted as α^*, its transpose as $\alpha^\intercal$ and its Hermitian conjugate as $\alpha^\dagger \coloneqq (\alpha^*)^\intercal$. Given a Dirac spinor $\Psi \in \mathbb{C}^4$, its Dirac dual (or Dirac adjoint) is defined as $\bar{\Psi} \coloneqq \Psi^\dagger\gamma^0$.
The Landau symbol for a function $f : D \subseteq \mathbb{R} \to \mathbb{C}$ is defined by the condition $\exists M \in \mathbb{R} : |o(f(x))| \leq M\,|f(x)| \;\; \forall x \in D$.

Mathematical notation

The empty set is denoted by $\varnothing$ and the power set of a set A by $\wp(A) \coloneqq \{B : B \subseteq A\}$. The counting numbers are $\mathbb{N} \equiv \{1, 2, 3, \dots\}$, and the natural numbers are defined by $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$.
The imaginary unit is denoted by i and the unit quaternions by $\mathrm{i}, \mathrm{j}, \mathrm{k}$, so that $\mathbb{C}(\mathbb{R}) = \mathrm{span}(1, \mathrm{i})$ and $\mathbb{H}(\mathbb{R}) = \mathrm{span}(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$.
The n-dimensional sphere is denoted by $\mathbb{S}^n$, while the n-dimensional disk by $\mathbb{D}^n$, so that $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$, where in general $\partial\Omega$ denotes the boundary of Ω.
The permutation group of n objects, i.e. the n^{th} symmetric group, is denoted by S_n.
Given two $\mathbb{K}$-vector spaces V and W, with $\mathbb{K}$ a generic field, the space of all $\mathbb{K}$-linear applications $f : V \to W$ is denoted by $\mathrm{Hom}_{\mathbb{K}}(V, W)$: in particular, $\mathrm{Hom}_{\mathbb{K}}(V) \equiv \mathrm{End}(V)$. The subset of $\mathrm{End}(V)$ of all automorphisms of V is the automorphism group $\mathrm{Aut}(V)$, which is a group under composition of morphisms.
Given a manifold $\mathcal{M}$, the space of all smooth scalar functions on $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$, the space of all vector fields on $\mathcal{M}$ by $\mathfrak{X}(\mathcal{M})$, the space of all p-forms on $\mathcal{M}$ by $\bigwedge^p(\mathcal{M})$ and the Grassmann algebra of $\mathcal{M}$ by $\bigwedge(\mathcal{M}) \coloneqq \bigoplus_{k=0}^n \bigwedge^k(\mathcal{M})$.
The exterior derivative is denoted by d, the partial derivative by ∂, the nabla operator by $\boldsymbol{\nabla} \equiv (\partial_1, \partial_2, \partial_3)$, the Laplacian operator by $\triangle \equiv \boldsymbol{\nabla}^2$ and the D'Alembert operator by $\Box \equiv \partial_0^2 - \triangle$.
A list of “important” Lie groups:
$\mathrm{GL}(n, \mathbb{K}) \coloneqq \{\mathrm{A} \in \mathbb{K}^{n\times n} : \det \mathrm{A} \neq 0\}$ general linear group (Lie group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$)
$\mathrm{SL}(n, \mathbb{K}) \coloneqq \{\mathrm{A} \in \mathrm{GL}(n, \mathbb{K}) : \det \mathrm{A} = 1\}$ special linear group (Lie group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$)
$\mathrm{O}(n) \coloneqq \{\mathrm{A} \in \mathbb{R}^{n\times n} : \mathrm{A}\mathrm{A}^\intercal = \mathrm{A}^\intercal\mathrm{A} = \mathrm{I}_n\}$ orthogonal group
$\mathrm{SO}(n) \coloneqq \{\mathrm{A} \in \mathrm{O}(n) : \det \mathrm{A} = 1$ special orthogonal group
$\mathrm{U}(n) \coloneqq \{\mathrm{A} \in \mathbb{C}^{n\times n} : \mathrm{A}\mathrm{A}^\dagger = \mathrm{A}^\dagger\mathrm{A} = \mathrm{I}_n\}$ unitary group
$\mathrm{SU}(n) \coloneqq \{\mathrm{A} \in \mathrm{U}(n) : \det \mathrm{A} = 1$ special unitary group
Given a Lie group G, its associated Lie algebra is denoted by $\mathfrak{g}$.

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Introduction

Part I

Differential Geometry

Chapter 1

Manifolds

§1.1 Differentiable manifolds

Definition 1.1.1 (Topological space)

The **topology** $\mathcal{T}$ of a set X is a family of subsets of X, i.e. $\mathcal{T} \subseteq \wp(X)$, defined as **open sets**, with the following properties:

1. $\varnothing, X \in \mathcal{T}$;
2. $O_\alpha, O_\beta \in \mathcal{T} \implies O_\alpha \cap O_\beta \in \mathcal{T}$;
3. $\{O_\alpha\}_{\alpha\in\mathcal{I}} \subset \mathcal{T} \implies \bigcup_{\alpha\in\mathcal{I}} O_\alpha \in \mathcal{T}$.

A **topological space** M is a set of points, endowed with a topology $\mathcal{T}$.

Given a topological space $(M, \mathcal{T})$, $O \in \mathcal{T}$ is a **neighbourhood** of a point $p \in M$ if $p \in O$: then, $(M, \mathcal{T})$ is **Hausdorff** if $\forall p, q \in M\ \exists O_1, O_2 \in \mathcal{T}$ neighbourhoods of p and q respectively such that $O_1 \cap O_2 = \varnothing$.
Topological spaces allow to introduce the concept of continuity: given two topological spaces $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$, a map $f : M_1 \to M_2$ is **continuous** if $O \in \mathcal{T}_2 \Rightarrow f^{-1}(O) \in \mathcal{T}_1$.

Definition 1.1.2 (Homeomorphism)

Given two topological spaces $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$, a map $f : M_1 \to M_2$ is a **homeomorphism** if it is bijective and bicontinuous, i.e. both f and f^{-1} are continuous.

§1.1.1 Definitions

Definition 1.1.3 (Differentiable manifold)

An n-dimensional **differentiable manifold** $\mathcal{M}$ is a Hausdorff topological space such that:

1. $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall p \in \mathcal{M}\ \exists O \in \mathcal{T}(\mathcal{M}) : p \in O \land \exists \varphi : O \to U \in \mathcal{T}(\mathbb{R}^n)$ homeomorphism;
2. given $O_\alpha, O_\beta \in \mathcal{T}(\mathcal{M}) : O_\alpha \cap O_\beta \neq \varnothing$, the corresponding maps $\varphi_\alpha : O_\alpha \to U_\alpha$ and $\varphi_\beta : O_\beta \to U_\beta$ must be *compatible*, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(O_\alpha \cap O_\beta) \to \varphi_\beta(O_\alpha \cap O_\beta)$ and its inverse must be smooth (of $\mathcal{C}^\infty$ class).

The maps φ_α are called **charts** and a collection of compatible charts is called an **atlas**: a maximal atlas $\mathcal{A}$ is an atlas such that $\bigcup_{\alpha\in\mathcal{I}} O_\alpha = \mathcal{M}$. Two atlases are compatible if each chart of one atlas is compatible with every chart of the other: they define the same differentiable structure on the manifold.
Each chart φ_α provides a coordinate system on O_α: $\varphi_\alpha(p) = (x^1(p), \dots, x^\mu(p), \dots, x^n(p))$. The transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are therefore coordinate transformations on overlapping regions.

Example 1.1.1 (Spheres)

$\mathbb{S}^n$ is a differentiable manifold for $n \in \mathbb{N}$. In particular, to define a differentiable structure on $\mathbb{S}^1$, an atlas of two charts is needed: the standard parametrization $\vartheta \in [0, 2\pi)$ is not a well-defined chart because $[0, 2\pi)$ is not an open set in the Euclidean topology of $\mathbb{R}$, therefore the elimination of a point is necessary; usually, the two charts of the atlas are defined by $\vartheta_1 \in (0, 2\pi)$, excluding $(1, 0)$ (in the embedding space $\mathbb{R}^2$), and $\vartheta_2 \in (-\pi, \pi)$, excluding $(-1, 0)$: they are evidently compatible, thus they form a maximal atlas.

In the remainder of these notes, $\mathcal{M}$ is always taken to be an n-dimensional differentiable manifold.

§1.1.2 Maps between manifolds

Locally mapping $\mathcal{M}$ to $\mathbb{R}^n$ allows for the extension of the concepts of Analysis from $\mathbb{R}^n$ to $\mathcal{M}$.

Definition 1.1.4 (Smooth maps)

function $f : \mathcal{M} \to \mathbb{R}$ on a differentiable manifold $(\mathcal{M}, \mathcal{A})$ is **smooth** if $f \circ \varphi_\alpha^{-1} : U_\alpha \to \mathbb{R}$ is smooth for all charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$.
A map $f : \mathcal{M} \to \mathcal{N}$ between two differentiable manifolds $(\mathcal{M}, \mathcal{A}_1), (\mathcal{N}, \mathcal{A}_2)$ is **smooth** if $\psi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1} : U_{\alpha_1} \to V_{\alpha_2}$ is smooth for all charts $(U_{\alpha_1}, \varphi_{\alpha_1}) \in \mathcal{A}_1, (V_{\alpha_2}, \varphi_{\alpha_2}) \in \mathcal{A}_2$.

A smooth homeomorphism $f : \mathcal{M} \to \mathcal{N}$ between two differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$ is called a **diffeomorphism**.

Proposition 1.1.1 (Diffeomorphic manifolds)

If $\mathcal{M}$ and $\mathcal{N}$ are **diffeomorphic**, then $\dim_{\mathbb{R}} \mathcal{M} = \dim_{\mathbb{R}} \mathcal{N}$.

Example 1.1.2 (Diffenetiable structures)

$\mathbb{S}^7$ can be covered by multiple incompatible atlases: the resulting manifolds are homeomorphic but not diffeomorphic.
$\mathbb{R}^n$ has a unique differentiable structure for all $n \in \mathbb{N}$, except for $n = 4$: $\mathbb{R}^4$ can be covered by infinitely-many incompatible atlases.

§1.2 Tangent spaces

The notions of calculus can be defined on a differential manifold $(\mathcal{M}, \mathcal{A})$ with the notion of tangent spaces. Indeed, the derivative of a function $f : \mathcal{M} \to \mathbb{R}$ at a point $p \in \mathcal{M}$, covered by the chart (φ, U), is defined as:

$$\left.\frac{\partial f}{\partial x^\mu}\right|_p := \left.\frac{\partial(f \circ \varphi^{-1})}{\partial x^\mu}\right|_{\varphi(p)} \tag{1.1}$$

Evidently, this definition depends on the choise of coordinates x^μ, thus it depends on the chart.

§1.2.1 Tangent vectors

Definition 1.2.1 (Space of smooth functions)

The set of all smooth functions on $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$.

Definition 1.2.2 (Tangent vector)

A **tangent vector** to $\mathcal{M}$ in $p \in \mathcal{M}$ is an operator $X_p : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ such that:

1. $X_p(f + g) = X_p(f) + X_p(g)\ \forall f, g \in \mathcal{C}^\infty(\mathcal{M})$;
2. $X_p(f) = 0$ for all constant functions $f \in \mathcal{C}^\infty(\mathcal{M})$;
3. $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)\ \forall f, g \in \mathcal{C}^\infty(\mathcal{M})$.

Conditions 2. and 3. trivially imply that $X_p(\alpha f) = \alpha X_p(f)\ \forall \alpha \in \mathbb{R}$, which means that X_p is a linear operator, i.e. $X_p \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{C}^\infty(\mathcal{M}), \mathbb{R})$. Moreover, it is simple to check that $\partial_\mu|_p$ satisfies the conditions of Def. 1.2.2.

Theorem 1.2.1 (Tangent space)

The set $T_p\mathcal{M}$ of all tangent vectors at a point $p \in \mathcal{M}$ forms an n-dimensional space, called **tangent space**, and $\{\partial_\mu|_p\}_{\mu=1,\dots,n}$ is a basis of such space.

Proof. Defining $f \circ \varphi^{-1} \equiv F : U \subset \mathcal{M} \to \mathbb{R}$, with $f : \mathcal{M} \to \mathcal{M}$ and $(\varphi, U) \in \mathcal{A}$, it can be shown that, in some neighbourhood of p (not necessarily U), F cal always be written as:

$$F(x) = F(x^\mu(p)) + (x^\mu - x^\mu(p))\, F_\mu(x)$$

for some functions $\{F_\mu\}_{\mu=1,\dots,n}$ (e.g. $F(x) = F(0) + x\int_0^1 \mathrm{d}t\, F(xt)$). Applying $\partial_\mu|_{x(p)}$:

$$\left.\frac{\partial F}{\partial x^\mu}\right|_{x(p)} = F_\mu(x(p))$$

Defining $f_\mu \equiv F_\mu \circ \varphi$, for any $q \in \mathcal{M}$ in an appropriate neighbourhood of p:

$$f(q) = f(p) + (x^\mu(q) - x^\mu(p))\, f_\mu(q)$$

Moreover, remembering Eq. Eq. 1.1:

$$f_\mu(p) = F_\mu \circ \varphi(p) = F_\mu(x(p)) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x(p)} = \left.\frac{\partial f}{\partial x^\mu}\right|_p$$

Using these facts, the action of a tangent vector can be written explicitly:

$$\begin{aligned} X_p(f) &= X_p\left(f(p) + \left(x^\mu - x^\mu(p)\right) f_\mu\right) \\ &= X_p\left(f(p)\right) + X_p\left(\left(x^\mu - x^\mu(p)\right)\right) f_\mu(p) + \left(x^\mu - x^\mu(p)\right)(p) X_p\left(f_\mu\right) \\ &= X_p\left(x^\mu\right) f_\mu(p) \end{aligned}$$

because $f(p)$ is a constant and $\left(x^\mu - x^\mu(p)\right)(p) = x^\mu(p) - x^\mu(p) = 0$. Therefore, remembering the expression for $f_\mu(p)$:

$$X_p = X_p(x^\mu) \left.\frac{\partial}{\partial x^\mu}\right|_p \equiv X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p$$

Thus, $T_p\mathcal{M} = \langle \{\partial_\mu|_p\}_{\mu=1,\dots,n} \rangle$. To check for linear independence, suppose $\alpha = \alpha^\mu \partial_\mu|_p \equiv 0$: acting on $f = x^\nu$, it gives $\alpha(f) = \alpha_\mu \partial_\mu(x^\nu)|_p = \alpha_\nu = 0$. This concludes the proof. □

§1.2.1.1 Changing coordinates

Although $\partial_\mu|_p$ depends on the choice of coordinates (it is a **coordinate basis**), the existence of X_p is independent of that choice.
If two different charts (φ, U) and $(\tilde{\varphi}, V)$ intersect in a neighbourhood of $p \in U \cap V$, the transition from x^μ to y^μ can be expressed as:

$$X_p(f) = X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p = X^\mu \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\varphi(p)} \left.\frac{\partial f}{\partial y^\nu}\right|_p \tag{1.2}$$

This equation can have two interpretations, namely the alibi interpretation:

$$\left.\frac{\partial}{\partial x^\mu}\right|_p = \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\varphi(p)} \left.\frac{\partial}{\partial y^\nu}\right|_p \tag{1.3}$$

and the alias interpretation:

$$\tilde{X}^\nu = X^\mu \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\varphi(p)} \tag{1.4}$$

Components of vectors which transform this way are called **contravariant**.

§1.2.1.2 Curves

Consider a smooth curve on $\mathcal{M}$, i.e. a smooth map $\sigma : I \in \mathcal{T}(\mathbb{R}) \to \mathcal{M}$, WLOG parametrized as $\sigma(t) : \sigma(0) = p \in \mathcal{M}$; with a given chart (φ, U), this curve becomes $\varphi \circ \sigma : I \to \mathbb{R}^n$, parametrized by $x^\mu(t)$. The tangent vector to the curve in p is:

$$X_p = \left.\frac{dx^\mu(t)}{dt}\right|_{t=0} \left.\frac{\partial}{\partial x^\mu}\right|_p \tag{1.5}$$

This operator, applied to a function $f \in \mathcal{C}^\infty(\mathcal{M})$, computes the directional derivative of f along the curve. It can be showed that every tangent vector can be written as in Eq. Eq. 1.5, therefore the tangent space can be seen as the space of all possible tangent vectors to curves passing through p.
It must be noted that tangent spaces at different points are entirely different spaces: there is no way to directly compare vectors between them.

§1.2.2 Vector fields

Definition 1.2.3 (Vector fields)

A **vector field** X is a smooth map $X : p \in \mathcal{M} \mapsto X_p \in T_p\mathcal{M}$. The space of all vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$.

Note that a vector field can also be viewed as a smooth map $X : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, since $(X(f))(p) = X_p(f) \in \mathbb{R}$. Given a chart (φ, U), a vector field X can be expressed as:

$$X = X^\mu \frac{\partial}{\partial x^\mu} \tag{1.6}$$

with $X^\mu \in \mathcal{C}^\infty(\mathcal{M})$. This expression is only defined on U.

§1.2.2.1 Lie brakets

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, their product is clearly not a vector field, as it does not satisfy Leibniz' rule:

$$XY(fg) = XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) \neq XY(f)g + fXY(g)$$

where $XY(f) \equiv X(Y(f))$.

Definition 1.2.4 (Lie brackets)

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, their commutator (or **Lie bracket**) is defined as:

$$[X, Y](f) = XY(f) - YX(f) \tag{1.7}$$

With a given chart:

$$\begin{aligned} [X, Y](f) &= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial f}{\partial x^\nu}\right) - Y^\mu \frac{\partial}{\partial x^\mu}\left(X^\nu \frac{\partial f}{\partial x^\nu}\right) \\ &= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial f}{\partial x^\nu} \end{aligned}$$

therefore:

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial x^\nu} \tag{1.8}$$

Theorem 1.2.2 (Jacobi identity)

Given $X, Y, Z \in \mathfrak{X}(\mathcal{M})$, then:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.9}$$

With Lie brackets, $\mathfrak{X}(\mathcal{M})$ can be given the structure of a Lie algebra.

§1.2.2.2 Integral curves

Definition 1.2.5 (Flow)

A **flow** on $\mathcal{M}$ is a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$, labelled by $t \in \mathbb{R}$, with group structure: $\sigma_0 = \mathbb{1}_\mathcal{M}$ and $\sigma_s \circ \sigma_t = \sigma_{s+t}$, thus $\sigma_{-t} = \sigma_t^{-1}$.

Such flows give rise to streamlines on the manifold: these streamlines are required to be smooth. Defining $x^\mu(\sigma_t) \equiv x^\mu(t)$, a vector field can be defined by the tangent to the streamlines at each point on the manifold:

$$X^\mu(x^\mu(t)) = \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} \tag{1.10}$$

The inverse reasoning is also possible: given $X \in \mathfrak{X}(\mathcal{M})$, the streamlines defined by Eq. 1.10 are the **integral curves** of X.

Proposition 1.2.1 (Infinitesimal flow)

The **infinitesimal flow** generated by $X \in \mathfrak{X}(\mathcal{M})$ is:

$$x^\mu(t) = x^\mu(0) + tX^\mu(x(t)) + o(t) \tag{1.11}$$

A vector field which generates a flow defined for all $t \in \mathbb{R}$ is called **complete**.

Theorem 1.2.3

f $\mathcal{M}$ is compact, then all $X \in \mathfrak{X}(\mathcal{M})$ are complete.

Example 1.2.1 (Integral curves on the 2-sphere)

On $\mathbb{S}^2$, the flow generated by $X = \partial_\phi$ is described by $\dot{\phi} = 1, \dot{\theta} = 0$, thus $\theta(t) = \theta_0$ and $\phi(t) = \phi_0 + t$: the flow lines are lines of constant latitude.

§1.2.3 Lie derivative

Defining calculus for vector fields requires a way to compare vectors of different tangent spaces.

Definition 1.2.6 (Pull-back of functions)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a function $f : \mathcal{N} \to \mathbb{R}$, the **pull-back** of f is the function $\varphi^* f : \mathcal{M} \to \mathbb{R}$ such that $\varphi^* f(p) = f(\varphi(p))$.

Definition 1.2.7 (Push-forward of vector fields)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$, the **push-forward** of X is the vector field $\varphi_* X \in \mathfrak{X}(\mathcal{N})$ such that $\varphi_* X(f) = X(\varphi^* f)$.

This last equality must be evaluated at the appropriate points: $[\varphi_* X(f)](\varphi(p)) = [X(\varphi^* f)](p)$. With the appropriate charts on $\mathcal{M}$ and $\mathcal{N}$, the definitions above can be rewritten with coordinates:

$$\varphi^* f(x) = f(y(x)) \tag{1.12}$$

$$\varphi_* X(f) = X^\mu \frac{\partial f(y(x))}{\partial x^\mu} = X^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial f(y)}{\partial y^\alpha} \tag{1.13}$$

The notions of pull-back and push-forward allow to compare tangent vectors at neighbouring points and, in particular, to define the derivative along a vector field.

Definition 1.2.8 (Lie derivative of functions)

Given a function $f : \mathcal{M} \to \mathbb{R}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$, the derivative of f along X, called **Lie derivative**, is defined as:

$$\mathcal{L}_X f(x) := \lim_{t\to 0} \frac{f(\sigma_t(x)) - f(x)}{t} = \left.\frac{df(\sigma_t(x))}{dt}\right|_{t=0} \tag{1.14}$$

where σ_t is the flow generated by X.

Lemma 1.2.1

$$\mathcal{L}_X f = X(f) \tag{1.15}$$

Proof. $\mathcal{L}_X f = \frac{df(\sigma_t)}{dt} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu(t)}{dt} = X^\mu \frac{\partial f}{\partial x^\mu} = X(f)$. □

The Lie derivative can be extended to vector fields.

Definition 1.2.9 (Lie derivative of vector fields)

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, the **Lie derivative** of Y along X is defined as:

$$\mathcal{L}_X Y_p := \lim_{t\to 0} \frac{((\sigma_{-t})_* Y)_p - Y_p}{t} \tag{1.16}$$

where σ_t is the flow generated by X.

The use of the inverse flow σ_{-t} is necessary because to evaluate the vector field $\mathcal{L}_X Y$ at the point $p \in \mathcal{M}$, the tangent vector $Y_{\sigma_t(p)} \in T_{\sigma_t(p)}\mathcal{M}$ must be "pushed-back" to $T_p\mathcal{M} = T_{\sigma_0(p)}\mathcal{M}$. With $t \to 0$, the infinitesimal flow σ_{-t} is, according to Eq. 1.11, $x^\mu(t) = x^\mu(0) - tX^\mu + o(t)$, therefore the Lie derivative of basis tangent vectors can be expressed as:

$$(\sigma_{-t})_*\partial_\mu = \frac{\partial x^\nu(t)}{\partial x^\mu}\frac{\partial}{\partial x^\nu(t)} = \left(\delta^\nu_\mu - t\frac{\partial X^\nu}{\partial x^\mu} + o(t)\right)\partial_\nu(t) \implies \mathcal{L}_X\partial_\mu = -\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu \tag{1.17}$$

Moreover, by the Jacobi identity it follows that:

$$\mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z = \mathcal{L}_{[X,Y]}Z \tag{1.18}$$

Lemma 1.2.2

$$\mathcal{L}_X Y = [X, Y] \tag{1.19}$$

Proof. $\mathcal{L}_X Y = \mathcal{L}_X(Y^\mu\partial_\mu) = (\mathcal{L}_X Y^\mu)\,\partial_\mu + Y^\mu\,(\mathcal{L}_X\partial_\mu) = X^\nu\frac{\partial Y^\mu}{\partial x^\nu}\partial_\mu - Y^\mu\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu = [X, Y]$. □

§1.3 Tensors

§1.3.1 Dual Spaces

Definition 1.3.1 (Dual space)
Given a vector space V, its **dual space** V^* is the space of all linear maps $f : V \to \mathbb{R}$.

Given a basis $\{\mathbf{e}_\mu\}_{\mu=1,\dots,n}$ of V, its **dual basis** $\{\mathbf{f}^\mu\}_{\mu=1,\dots,n}$ of V^* can be defined by:

$$\mathbf{f}^\nu(\mathbf{e}_\mu) = \delta^\nu_\mu \tag{1.20}$$

A general vector in V can be written as $X = X^\mu \mathbf{e}_\mu$, thus, according to Eq. 1.20, $X^\mu = \mathbf{f}^\mu(X)$. Clearly, the map $f : \mathbf{e}_\mu \mapsto \mathbf{f}^\mu$ is a non-canonical isomorphism between V and V^*, hence $\dim_\mathbb{R} V = \dim_\mathbb{R} V^*$.

Proposition 1.3.1 (Dual of the dual)

$$(V^*)^* \cong V \tag{1.21}$$

Proof. The natural isomorphism between $(V^*)^*$ and V is basis-independent: suppose $X \in V$ and $\omega \in V^*$, so that $\omega(X) \in \mathbb{R}$; X can be viewed as $X \in (V^*)^*$ by setting $X(\omega) \equiv \omega(X)$. □

§1.3.2 Cotangent vectors

Definition 1.3.2 (Cotangent space)
Given a differentiable manifold $(\mathcal{M}, \mathcal{A})$ and a point $p \in \mathcal{M}$, the **cotangent space** to $\mathcal{M}$ at p is defined as $T^*_p\mathcal{M} := (T_p\mathcal{M})^*$.

Elements of $T^*_p\mathcal{M}$ are called cotangent vectors (or **covectors**).

Definition 1.3.3 (Covector field)
A covector field (or **1-form**) is a smooth map $\omega : p \in \mathcal{M} \mapsto \omega_p \in T^*_p\mathcal{M}$. The space of all 1-forms on $\mathcal{M}$ is denoted by $\bigwedge^1(\mathcal{M})$.

Note that a 1-form can also be viewed as a smooth map $\omega : \mathfrak{X}(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, since $(\omega(X))(p) = \omega_p(X_p) \in \mathbb{R}$.

Proposition 1.3.2
$\{\mathrm{d}x^\mu\}_{\mu=1,\dots,n}$ is a basis of $\bigwedge^1(\mathcal{M})$, dual to the basis $\{\partial_\mu\}_{\mu=1,\dots,n}$ of $\mathfrak{X}(\mathcal{M})$.

Proof. Consider $f \in \mathcal{C}^\infty(\mathcal{M})$ and define $\mathrm{d}f \in \bigwedge^1(\mathcal{M})$ by $\mathrm{d}f(X) = X(f)$: taking $f = x^\mu$ and $X = \partial_\mu$, $\mathrm{d}f(X) = \partial_\nu(x^\mu) = \delta^\mu_\nu$, therefore $\{\mathrm{d}x^\mu\}_{\mu=1,\dots,n}$ is the dual basis of $\bigwedge^1(\mathcal{M})$. □

This is also confirmed by $\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu$. These are coordinate bases: in fact, given two different charts $(\varphi, U), (\tilde{\varphi}, V)$:

$$\mathrm{d}y^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}x^\nu}\mathrm{d}x^\nu \tag{1.22}$$

which is the inverse of Eq. 1.3 (not evaluated at a specific point). This ensures that:

$$\mathrm{d}y^\mu\left(\frac{\partial}{\partial y^\nu}\right) = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\nu}\mathrm{d}x^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^\nu} = \delta^\mu_\nu$$

A 1-form $\omega \in \bigwedge^1(\mathcal{M})$ can thus be expressed both as $\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\mu \mathrm{d}y^\mu$, with:

$$\tilde{\omega}_\omega = \frac{\partial x^\nu}{\partial y^\mu}\omega_\nu \tag{1.23}$$

Components of 1-forms which transform this way are called **covariant**.

Definition 1.3.4 (Pull-back of 1-forms)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a 1-form $\omega \in \Lambda^1(\mathcal{N})$, the **pull-back** of ω is the 1-form $\varphi^*\omega \in \bigwedge^1(\mathcal{M})$ such that $\varphi^*\omega(X) = \omega(\varphi_* X)$.

With the appropriate charts on $\mathcal{M}$ and $\mathcal{N}$, the definition above can be rewritten with coordinates:

$$\varphi^*\omega = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}\mathrm{d}x^\mu \tag{1.24}$$

Definition 1.3.5 (Lie derivative of 1-forms)

Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a 1-form $\omega \in \bigwedge^1(\mathcal{M})$, the **Lie derivative** of ω along X is defined as:

$$\mathcal{L}_X\omega_p := \lim_{t\to 0}\frac{(\sigma_t^*\omega)_p - \omega_p}{t} \tag{1.25}$$

where σ_t is the flow generated by X.

In contrast with the Lie derivative of a vector field, which pushes forward with σ_{-t} (i.e. pushes back), the Lie derivative of a 1-form pulls back with σ_t: this results in the difference of a minus sign with respect to Eq. 1.17, giving:

$$\mathcal{L}_X \mathrm{d}x^\mu = \frac{\partial X^\mu}{\partial x^\nu}\mathrm{d}x^\nu \tag{1.26}$$

Therefore, on a general 1-form $\omega = \omega_\mu \mathrm{d}x^\mu$:

$$\mathcal{L}_X\omega = \left(X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu\right)\mathrm{d}x^\mu \tag{1.27}$$

§1.3.3 Tensor fields

Definition 1.3.6 (Tensor)

A **tensor** of rank (r, s) at a $p \in \mathcal{M}$ of a differentiable manifold $(\mathcal{M}, \mathcal{A})$ is a multilinear map defined as:

$$T_p : \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r} \times \overbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}^{s} \to \mathbb{R} \tag{1.28}$$

For example, a cotangent vector $\omega_p \in T_p^*\mathcal{M}$ is a tensor of rank $(1,0)$, while a tangent vector $X_p \in T_p\mathcal{M}$ is a tensor of rank $(0,1)$.

Definition 1.3.7 (Tensor field)

A **tensor field** of rank (r,s) is a smooth map $T : p \in \mathcal{M} \mapsto T_p$ tensor of rank (r,s) at p. It can also be viewed as a smooth map $T : [\bigwedge^1(\mathcal{M})]^r \times [\mathfrak{X}(\mathcal{M})]^s \to \mathcal{C}^\infty(\mathcal{M})$.

Given appropriate bases for vector fields $\{\mathbf{e}_\mu\}_{\mu=1,\dots,n}$ and 1-forms $\{\mathbf{f}^\mu\}_{\mu=1,\dots,n}$, the components of a tensor field are defined as:

$$T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} := T(\mathbf{f}^{\mu_1},\dots,\mathbf{f}^{\mu_r},\mathbf{e}_{\nu_1},\dots,\mathbf{e}_{\nu_s}) \tag{1.29}$$

Proposition 1.3.3 (Components of tensor fields)

On an n-dimensional manifold, a (r,s) tensor field has n^{r+s} components, each being an element of $\mathcal{C}^\infty(\mathcal{M})$.

Consider two general basis transformations, for vector fields and 1-forms, described by invertible matrices A and B such that $\tilde{\mathbf{e}}_\mu = A^\nu{}_\mu \mathbf{e}_\nu$ and $\tilde{\mathbf{f}}^\mu = B^\mu{}_\nu \mathbf{f}^\nu$, with necessary condition $A^\mu{}_\nu B^\rho{}_\mu = \delta^\rho_\nu$ to ensure duality: this implies $\mathrm{B} = \mathrm{A}^{-1}$, i.e. covectors transform inversely with respect to vectors. Thus:

$$\tilde{T}^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = B^{\mu_1}{}_{\rho_1}\dots B^{\mu_r}{}_{\rho_r} A^{k_1}{}_{\nu_1}\dots A^{k_s}{}_{\nu_s} T^{\rho_1\dots\rho_r}{}_{k_1\dots k_s} \tag{1.30}$$

If the considered basis are coordinate basis, then $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}$ and $B^\mu{}_\nu = \frac{\partial y^\mu}{\partial x^\nu}$.

§1.3.4 Operations on tensors

Algebraic addition and multiplication by functions are trivially defined on tensors of the same rank, and in fact the space of all (r,s) tensors at a point $p \in \mathcal{M}$ is a vector space.

Definition 1.3.8 (Tensor product)

Given two tensor fields S of rank (p,q) and T of rank (r,s), their **tensor product** is defined as:

$$\begin{aligned} S\otimes T(\omega_1,\dots,\omega_p,\eta_1,\dots,\eta_r,X_1,\dots,X_q,Y_1,\dots,Y_s)& \\ = S(\omega_1,\dots,\omega_p,X_1,\dots,X_q)T(\eta_1,\dots,\eta_r,Y_1,\dots,Y_s)& \end{aligned} \tag{1.31}$$

or, in components:

$$(S\otimes T)^{\mu_1\dots\mu_p\nu_1\dots\nu_r}{}_{\rho_1\dots\rho_q\dots\sigma_1\dots\sigma_s} = S^{\mu_1\dots\mu_p}{}_{\rho_1\dots\rho_q} T^{\nu_1\dots\nu_r}{}_{\sigma_1\dots\sigma_s} \tag{1.32}$$

It is also possible to contract tensor ($(r,s) \mapsto (r-1,s-1)$): for example, given a rank $(2,1)$ tensor, a rank $(1,0)$ tensor can be defined as $S(\omega) = T(\omega,\mathbf{f}^\mu,\mathbf{e}_\mu)$, with components $S^\mu = T^{\nu\mu}{}_\mu$; obviously, in general, $T^{\nu\mu}{}_\mu \neq T^{\mu\nu}{}_\mu$.

It is convenient to introduce some notation: given an object $T_{\mu_1\dots\mu_n}$ dependent on some indices, its symmetric and antisymmetric parts are respectively defined as:

$$T_{(\mu_1\dots\mu_n)} := \frac{1}{n!}\sum_{\sigma\in S^n} T_{\sigma(\mu_1)\dots\sigma(\mu_n)} \tag{1.33}$$

$$T_{[\mu_1 \ldots \mu_n]} := \frac{1}{n!} \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) T_{\sigma(\mu_1) \ldots \sigma(\mu_n)} \tag{1.34}$$

Conventionally, indices surrounded by $|\cdot|$ are ignored, e.g. $T_{[\mu|\nu|\rho]} = \frac{1}{2}(T_{\mu\nu\rho} - T_{\rho\nu\mu})$.
As previously seen, vector fields are pushed forward and 1-form are pulled back: tensors will thus behave in a mixed way.

Definition 1.3.9 (Push-forward of tensors)

Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and a (r, s) tensor field T on $\mathcal{M}$, the **push-forward** of T is the (r, s) tensor field $\varphi_* T$ on $\mathcal{N}$ such that, for $\{\omega_j\}_{j=1,\ldots,r} \subseteq \Lambda^1(\mathcal{N})$ and $\{X_j\}_{j=1,\ldots,s} \subseteq \mathfrak{X}(\mathcal{N})$:

$$\varphi_* T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) = T(\varphi^* \omega_1, \ldots, \varphi^* \omega_r, \varphi_*^{-1} X_1, \ldots, \varphi_*^{-1} X_s) \tag{1.35}$$

Definition 1.3.10 (Lie derivative of tensors)

Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a (r, s) tensor field T on $\mathcal{M}$, the **Lie derivative** of T along X is defined as:

$$\mathcal{L}_X T_p := \lim_{t \to 0} \frac{((\sigma_{-t})_* T)_p - T_p}{t} \tag{1.36}$$

where σ_t is the flow generated by X.

§1.4 Differential forms

Definition 1.4.1 (p-forms)
A p-**form** totally anti-symmetric $(0,p)$ tensor. The set of all p-forms over a manifold $\mathcal{M}$ is denoted as $\bigwedge^p(\mathcal{M})$.

Lemma 1.4.1 (Independent components of p-forms)

$$\dim_{\mathbb{R}} \bigwedge^p(\mathcal{M}) = \binom{n}{p} \tag{1.37}$$

The maximum degree of differential forms thus is $p = n \equiv \dim_{\mathbb{R}} \mathcal{M}$: forms in $\bigwedge^n(\mathcal{M})$ are called **top forms**.

Definition 1.4.2 (Wedge product)
Given $\omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, their **wedge product** is a $(p+q)$-form defined as:

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\nu_1 \dots \nu_q]} \tag{1.38}$$

Example 1.4.1 (1-forms to 2-form)
Given $\omega, \eta \in \bigwedge^1(\mathcal{M})$, their wedge product is $(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu$.

The wedge product is essentially a linear map $\bigwedge^p(\mathcal{M}) \times \bigwedge^q(\mathcal{M}) \to \bigwedge^{p+q}(\mathcal{M})$. Moreover, it can be shown to be associative.

Proposition 1.4.1 (Skew-symmetry)
Given $\omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, then:

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \tag{1.39}$$

Then, clearly, $\omega \wedge \omega = 0 \ \forall \omega \in \bigwedge^p(\mathcal{M}) : p$ is odd.

Proposition 1.4.2 (Bases for p-forms)
If $\{\mathbf{f}^\mu\}_{\mu=1,\dots,n}$ is a basis of $\bigwedge^1(\mathcal{M})$, then $\{\mathbf{f}^{\mu_1} \wedge \dots \wedge \mathbf{f}^{\mu_p}\}_{\mu_1,\dots,\mu_p=1,\dots,n}$ is a basis of $\bigwedge^p(\mathcal{M})$.

Locally $\{\mathrm{d}x^\mu\}_{\mu=1,\dots,n}$ is a basis of $T_p^*\mathcal{M}$, thus a general p-form can be locally written as:

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \tag{1.40}$$

Definition 1.4.3 (Exterior derivative)
The **exterior derivative** is a map $\mathrm{d} : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p+1}(\mathcal{M})$ defined as:

$$(\mathrm{d}\omega)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]} \tag{1.41}$$

In local coordinates:

$$\mathrm{d}\omega = \frac{1}{p!}\frac{\partial \omega_{\mu_1 \dots \mu_p}}{\partial x^\nu}\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \tag{1.42}$$

Theorem 1.4.1 (Poincaré's theorem)

$$\mathrm{d}^2 = 0 \tag{1.43}$$

Proof. Consequence of Schwarz lemma. □

Proposition 1.4.3 (Product rule)
Given $\omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, then:

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta \tag{1.44}$$

Proposition 1.4.4 (Pull-back and exterior derivative)
Given a diffeomorphism between two manifolds $\varphi : \mathcal{M} \to \mathcal{N}$ and $\omega \in \bigwedge^p(\mathcal{M})$, then $\mathrm{d}(\varphi^*\omega) = \varphi^*(\mathrm{d}\omega)$.

Corollary 1.4.4.1 (Lie derivative and exterior derivative)
Given $X \in \mathfrak{X}(\mathcal{M}), \omega \in \bigwedge^p(\mathcal{M})$, then $\mathrm{d}(\mathcal{L}_X\omega) = \mathcal{L}_X(\mathrm{d}\omega)$.

Definition 1.4.4 (Closed and exact forms)
Given $\omega \in \bigwedge^p(\mathcal{M})$, it is:

- **closed** if $\mathrm{d}\omega = 0$;
- **exact** if $\exists \eta \in \bigwedge^{p-1}(\mathcal{M}) : \omega = \mathrm{d}\eta$.

By Poincaré's theorem, clearly $\omega \in \bigwedge^p(\mathcal{M})$ exact $\implies \omega$ closed, while the converse is not true in general, but depends on the topology of the manifold: a full treatment of this matter led to the development of cohomology, as outlined in §1.4.1.

Definition 1.4.5 (Interior product)
Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, the **interior product** determined by X is the linear map $\iota_X : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p-1}(\mathcal{M})$ defined as:

$$\iota_X\omega(Y_1, \dots, Y_{p-1}) \coloneqq \omega(X, Y_1, \dots, Y_{p-1}) \tag{1.45}$$

On 0-forms (i.e. scalar functions), the interior product is defined as $\iota_X f \equiv 0 \; \forall X \in \mathfrak{X}(\mathcal{M})$.

Proposition 1.4.5 (Anti-commutation)
Given $X, Y \in \mathfrak{X}(\mathcal{M})$, then $\iota_X\iota_Y = -\iota_Y\iota_X$.

Proof. Consequence of the total anti-symmetry of p-forms. □

Proposition 1.4.6 (Distributivity)
Given $X \in \mathfrak{X}(\mathcal{M}), \omega \in \bigwedge^p(\mathcal{M}), \eta \in \bigwedge^q(\mathcal{M})$, then:

$$\iota_X(\omega \wedge \eta) = \iota_X\omega \wedge \eta + (-1)^p \omega \wedge \iota_X \eta \tag{1.46}$$

Theorem 1.4.2 (Cartan's theorem)
Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, then:

$$\mathcal{L}_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d} \tag{1.47}$$

Proof. Consider $\omega \in \bigwedge^1(\mathcal{M})$:

$$\iota_X(\mathrm{d}\omega) = \iota_X \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = X^\mu \partial_\mu \omega_\nu \mathrm{d}x^\nu - X^\nu \partial_\mu \omega_\nu \mathrm{d}x^\mu$$

$$\mathrm{d}(\iota_X \omega) = \mathrm{d}(\omega_\mu X^\mu) = X^\mu \partial_\nu \omega_\mu \mathrm{d}x^\nu + \omega_\mu \partial_\nu X^\mu \mathrm{d}x^\nu$$

Thus, adding these expressions and recalling Eq. 1.27:

$$(\mathrm{d}\iota_X + \iota_X \mathrm{d})\omega = \left(X^\mu \partial_\mu \omega_\nu + \omega_\mu \partial_\nu X^\mu\right) \mathrm{d}x^\nu = \mathcal{L}_X \omega$$

The $p > 1$ case is more complex. □

§1.4.1 de Rham cohomology

While exact $\implies$ closed, the converse is not true, in general: it depends on the topological properties of the manifold.

Lemma 1.4.2 (Poincaré's lemma)
If $\mathcal{M}$ is simply connected, then $\omega \in \bigwedge^p(\mathcal{M})$ closed $\implies$ ω exact.

In general, it is always possible to choose a simply connected neighbourhood of a point $p \in \mathcal{M}$, in which every closed form is exact, but that may not always be possible globally.
It is convenient to set the notation $d_p \equiv d : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p+1}(\mathcal{M})$, so that the set of all closed p-forms on $\mathcal{M}$ is denoted by $Z^p(\mathcal{M}) := \ker d_p$, while the set of all exact p-forms by $B^p(\mathcal{M}) := \operatorname{ran} d_{p-1}$.

Definition 1.4.6 (Equivalent forms)
Two closed p-forms $\omega, \omega' \in Z^P(\mathcal{M})$ are said to be **equivalent** if $\omega = \omega' + \eta$ for some $\eta \in B^p(\mathcal{M})$.

Definition 1.4.7 (de Rahm cohomology)
The p^{th} **de Rham cohomolgy group** of a manifold $\mathcal{M}$ is defined to be:

$$H^p(\mathcal{M}) := Z^p(\mathcal{M}) / B^p(\mathcal{M}) \tag{1.48}$$

and its dimension is the p^{th} **Betti number** of $\mathcal{M}$:

$$B_p \coloneqq \dim_{\mathbb{R}} H^p(\mathcal{M}) \tag{1.49}$$

Theorem 1.4.3 (Betti numbers)

Given a differentiable manifold, its Betti numbers are always finite.

$B_0 = 1$ for any connected manifold: there exist constant functions, which are manifestly closed and not exact, due to the non-existence of "(-1)-forms". Higher Betti numbers are non-zero only if the manifold has some non-trivial topology.

Definition 1.4.8 (Eulerv's character)

The **Euler's character** of a manifold $\mathcal{M}$ is defined as:

$$\chi(\mathcal{M}) \coloneqq \sum_{p\in\mathbb{N}_0} (-1)^p B_p \tag{1.50}$$

Example 1.4.2 (Spheres and tori)

The n-sphere $\mathbb{S}^n$ has only non-vanishing $B_0 = B_n = 1$, thus $\chi(\mathbb{S}^n) = 1 + (-1)^n$, while the n-torus $\mathbb{T}^n$ has $B_p = \binom{n}{p}$, hence $\chi(\mathbb{T}^n) = 0$.

§1.4.2 Integration

Definition 1.4.9 (Volume form)

A **volume form** on an n-dimensional differentiable manifold $\mathcal{M}$ is a nowhere-vanishing top form dV, i.e. locally $\mathrm{dV} = v(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n : v(x) \neq 0$.
If such a form exists, the manifold is said to be **orientable**, and its orientation is right/left-handed if respectively $v(x) > 0$ or $v(x) < 0$ locally on every subset of $\mathcal{M}$.

To ensure that the handedness of the manifold doesn't change on overlapping charts:

$$\mathrm{dV} = v(x)\frac{\partial x^1}{\partial y^{\mu_1}}\mathrm{d}y^{\mu_1} \wedge \cdots \wedge \frac{\partial x^n}{\partial x^{\mu_n}}\mathrm{d}x^{\mu_n} = v(x)\det\left[\frac{\partial x^\mu}{\partial y^\nu}\right]\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$$

It is therefore necessary that the two sets of coordinates on the overlapping region satisfy:

$$\det\left[\frac{\partial x^\mu}{\partial y^\nu}\right] > 0 \tag{1.51}$$

Non-orientable manifolds cannot be covered by overlapping charts satisfying this condition.

Example 1.4.3 (Projective spaces)

The real projective space $\mathbb{RP}^n$ is orientable for odd n and non-orientable for even n, while the complex projective space $\mathbb{CP}^n$ is orientable for all $n \in \mathbb{N}$.

Definition 1.4.10 (Integration over manifolds)

Given a function $f : \mathcal{M} \to \mathbb{R}$ on an orientable manifold $\mathcal{M}$ with volume form dV and a chart (φ, U) on $\mathcal{M}$ with coordinates $\{x^\mu\}_{\mu=1,\ldots,n}$, the **integral** of f on $O = \varphi^{-1}(U) \subset \mathcal{M}$ is defined as:

$$\int_O f\,\mathrm{dV} := \int_U \mathrm{d}x_1 \ldots \mathrm{d}x_n\, f(x)v(x) \tag{1.52}$$

It is clear that the volume form acts like a measure on the manifold. To integrate over the whole manifold, it must be divided up into different regions, each covered by a single chart.

Definition 1.4.11 (Submanifold)

A k-dimensional manifold Σ is a **submanifold** of an n-dimensional manifold $\mathcal{M}$, with $n > k$, if there exists an injective map $\varphi : \Sigma \to \mathcal{M}$ such that $\varphi_* : T_p(\Sigma) \to T_{\varphi(p)}(\mathcal{M})$ is injective.

Definition 1.4.12 (Integration over submanifolds)

Given a k-form $\omega \in \bigwedge^k(\mathcal{M})$, its integral over a k-dimensional submanifold Σ of $\mathcal{M}$ is defined as:

$$\int_{\varphi(\Sigma)} \omega := \int_\Sigma \varphi^*\omega \tag{1.53}$$

Example 1.4.4 (Integration over a curve)

Consider a 1-form $\omega \in \bigwedge^1(\mathcal{M})$ and a 1-dimensional submanifold γ of $\mathcal{M}$ described by a curve $\sigma : \gamma \to \mathcal{M} : x^\mu = \sigma^\mu(t)$: locally $\omega = \omega_\mu(x)\mathrm{d}x^\mu$, thus the integral of ω on γ can be calculated as $\int_{\sigma(\gamma)} \omega = \int_\gamma \sigma^*\omega = \int_\gamma \mathrm{d}\tau\, \omega_\mu(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$.

§1.4.2.1 Stokes' theorem

Integration can be generalized beyond smooth (i.e. differentiable) manifolds.

Definition 1.4.13 (Manifold with boundary)

An n-dimensional **manifold with boundary** is a Hausdorff topological space, equipped with a compatible maximal atlas, which is locally homeomorphic to $\mathbb{R}^{n-1} \times [a, \infty) : a \in \mathbb{R}$.
The **boundary** $\partial\mathcal{M}$ is the 1-dimensional submanifold determined parametrically by $x^n = a$.

Theorem 1.4.4 (Stokes' theorem)

Given an n-dimensional manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, then for any $\omega \in \bigwedge^{n-1}(\mathcal{M})$:

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega \tag{1.54}$$

This important theorem unifies many different results.

Fundamental theorem of calculus Given the 1-dimensional manifold $I = [a, b] \subset \mathbb{R}$, then for any 0-form (i.e. scalar function) $\omega = \omega(x)$:

$$\int_I d\omega = \int_a^b \frac{\mathrm{d}\omega}{\mathrm{d}x}\mathrm{d}x = \int_{\partial I} \omega = \omega(b) - \omega(a)$$

Green's theorem Given a 2-dimensional manifold with boundary $S \subset \mathbb{R}^2$ and a 1-form $\omega = \omega_1 \mathrm{d}x^1 + \omega_2 \mathrm{d}x^2$, then $\mathrm{d}\omega = (\partial_1\omega_2 - \partial_2\omega_1)\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and:

$$\int_S \mathrm{d}\omega = \int_S \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right)\mathrm{d}x^1\mathrm{d}x^2 = \int_{\partial S} \omega = \int_{\partial S} \omega_1\mathrm{d}x^1 + \omega_2\mathrm{d}x^2$$

Gauss' theorem Given a 3-dimensional manifold with boundary $V \subset \mathbb{R}^3$ and a 2-form $\omega = \omega_1\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \omega_2\mathrm{d}x^3 \wedge \mathrm{d}x^1 + \omega_3\mathrm{d}x^1 \wedge \mathrm{d}x^2$, then $\mathrm{d}\omega = (\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ and:

$$\int_V \mathrm{d}\omega = \int_V \left(\frac{\partial\omega_1}{\partial x^1} + \frac{\partial\omega_2}{\partial x^2} + \frac{\partial\omega_3}{\partial x^3}\right) = \int_{\partial V} \omega = \int_{\partial V} \omega_1\mathrm{d}x^2\mathrm{d}x^3 + \omega_2\mathrm{d}x^3\mathrm{d}x^1 + \omega_3\mathrm{d}x^1\mathrm{d}x^2$$

Chapter 2

Riemannian Geometry

§2.1 Metric manifolds

Definition 2.1.1 (Metric)

A **metric** g is a (0,2) tensor field on a manifold $\mathcal{M}$ that is:

1. symmetric: $g(X, Y) = g(Y, X)$;
2. non-degenerate: $\exists p \in \mathcal{M} : g(X, Y)|_p = 0 \ \forall Y \in T_p\mathcal{M} \implies X_p = 0$.

A manifold equipped with a metric $(\mathcal{M}, g)$ is called a **metric manifold**.

With a choice of coordinates, the metric can be written as:

$$g = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{2.1}$$

where:

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \tag{2.2}$$

It is often written also as $ds^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu$. The matrix $g_{\mu\nu}(x) \in \mathbb{R}^{n\times n}$ is symmetric, and there's always a choice of basis on each tangent space such that this matrix is diagonal: the non-degeneracy condition implies that none of the diagonal elements vanish.

Proposition 2.1.1 (Signature)

The **signature** of a metric, i.e. the number of negative entries when diagonalized, is independent on the choice of basis.

Proof. From Sylvester's theorem of inertia. □

§2.1.1 Riemannian manifolds

Definition 2.1.2 (Riemannian manifold)

A **Riemannian manifold** $(\mathcal{M}, g)$ is a manifold equipped with a metric with totally-positive signature.

Example 2.1.1 (Euclidean space)

The Euclidean space $\mathbb{R}^n$, equipped with the metric $g_{\mu\nu} = \delta_{\mu\nu}$ (in Cartesian coordinates), is a Riemannian manifold.

In a Riemannian manifold $(\mathcal{M}, g)$ and $X \in \mathfrak{X}(\mathcal{M})$, the **length** of X at $p \in \mathcal{M}$ is defined as:

$$|X_p| := \sqrt{g(X,X)|_p} \tag{2.3}$$

Given $Y \in \mathfrak{X}(\mathcal{M})$, the **angle** between X and Y at $p \in \mathcal{M}$ is defined as:

$$\cos\theta := \frac{g(X,Y)|_p}{|X_p|\,|Y_p|} \tag{2.4}$$

This can be generalized to distances between points on a curve $\sigma : \mathbb{R} \to \mathcal{M}$:

$$d(p,q) = \int_a^b dt \sqrt{g(X,X)|_{\sigma(t)}} \tag{2.5}$$

where $\sigma(a) = p$, $\sigma(b) = q$ and X is the tangent vector field of the curve. With parametrization $x^\mu(t)$, the tangent vector has components $X^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$, thus:

$$d(p,q) = \int_a^b \mathrm{d}t \sqrt{g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \tag{2.6}$$

It is important to note that this distance is independent of the parametrization.

§2.1.2 Lorentzian manifolds

Definition 2.1.3 (Lorentzian manifold)

A **Lorentzian manifold** $(\mathcal{M}, g)$ is a manifold equipped with a metric which has a signature with a single negative sign.

Example 2.1.2 (Lorentz–Minkowski metric)

The simplest Lorentzian manifold is $\mathbb{R}^n$ with the **Lorentz–Minkowski metric**:

$$\eta = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^{n-1} \otimes \mathrm{d}x^{n-1} \tag{2.7}$$

Its components are $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$, thus this is a Lorentzian manifold.

On a general Lorentzian manifold, at any point $p \in \mathcal{M}$ it is always possible to choose an orthonormal basis $\{e_\mu\}_{\mu=0,\ldots,n-1}$ of $T_p\mathcal{M}$ such that $g_{\mu\nu}|_p = \eta_{\mu\nu}$: this fact is closely related to the equivalence principle. Consider a different basis $\tilde{e}_\mu = \Lambda^\nu{}_\mu e_\nu$: the condition for it to leave the Lorentz–Minkowski metric unchanged is:

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} \tag{2.8}$$

This is the defining equation of a Lorentz transformation: on a Lorentzian manifold, the basic features of special relativity are locally recovered. Thus, other ideas from special relativity can be imported.

Definition 2.1.4 (Curves in a Lorentzian manifold)

Given a Lorentzian manifold $(\mathcal{M}, g)$ and $X \in \mathfrak{X}(\mathcal{M})$, at $p \in \mathcal{M}$ the vector field is said to be **timelike** if $g(X_p, X_p) < 0$, **null** if $g(X_p, X_p) = 0$ or **spacelike** if $g(X_p, X_p) > 0$.

At each point $p \in \mathcal{M}$ it is possible to draw **lightcones**, i.e. the null tangent vectors at that point, which are past-directed or future-directed: these lightcones vary smoothly as the point is varied smoothly on the manifold, elucidating the causal structure of spacetime.
The distance between two points on a curve depends on the nature of the tangent vector field of the curve: a *timelike curve* is a curve whose tangent vector field is everywhere timelike, and analogously for the other cases. The distance on a spacelike curve is defined as in Eq. 2.5, while that on a timelike curve gets a negative sign in the square root. With parametrization $x^\mu(t)$, it is possible to define the **proper time** on a timelike curve as:

$$\tau = \int_a^b dt \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \tag{2.9}$$

This is precisely the action of a free particle moving in spacetime.

§2.1.3 Metric properties

The metric defines a natural isomorphism between vectors and covectors.

Proposition 2.1.2

Given a metric manifold $(\mathcal{M}, g)$, the metric defines for each $p \in \mathcal{M}$ a natural isomorphism $g : X_p \in T_p\mathcal{M} \to \omega_p \in T_p^*\mathcal{M} : \omega_p(Y_p) = g(X_p, Y_p)\ \forall Y_p \in T_p\mathcal{M}$.

In a chosen coordinate basis, the vector $X = X^\mu \partial_\mu$ is mapped to the one-form $X = X_\mu \mathrm{d}x^\mu$, thus the following identity holds:

$$X_\mu = g_{\mu\nu} X^\nu \tag{2.10}$$

Being g non-degenerate, the matrix $g_{\mu\nu}$ is invertible, with inverse $g^{\mu\nu}$ such that:

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \tag{2.11}$$

Its elements are the components of a (2,0) symmetric tensor $\hat{g} := g^{\mu\nu}\partial_\mu \otimes \partial_\nu$, which defines the inverse of the natural isomorphism in Prop. 2.1.2:

$$X^\mu = g^{\mu\nu} X_\nu \tag{2.12}$$

The metric also defines a natural volume form on the manifold:

$$\mathrm{dV} := \sqrt{\mathrm{g}}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{2.13}$$

where $\mathrm{g} \equiv |\det g_{\mu\nu}|$.

Proposition 2.1.3

The volume form is basis-independent.

Proof. Consider a new set of coordinates y^μ such that $\mathrm{d}x^\mu = A^\mu{}_\nu \mathrm{d}y^\nu$, where $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}$. In general:

$$\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = A^1{}_{\mu_1} \ldots A^n{}_{\mu_n} \mathrm{d}y^{\mu_1} \wedge \cdots \wedge \mathrm{d}y^{\mu_n}$$

Recalling the anti-symmetry of the wedge product and the definition of determinant, this can be rewritten as:

$$\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \sum_{\pi \in S^n} \operatorname{sgn}(\pi) A^1{}_{\pi(1)} \ldots A^n{}_{\pi(n)} \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = \det \mathrm{A}\, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$$

Note the Jacobian factor which arises when changing the measure. On the other hand:

$$g_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma} = (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \tilde{g}_{\rho\sigma} \qquad \Longrightarrow \qquad \det g_{\mu\nu} = \frac{\det \tilde{g}_{\mu\nu}}{(\det \mathrm{A})^2}$$

The factors $\det \mathrm{A}$ and $(\det \mathrm{A})^{-1}$ cancel, thus yielding the thesis. □

The volume form can be rewritten as:

$$\mathrm{dV} = \frac{1}{n!} v_{\mu_1 \ldots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \equiv \frac{1}{n!}\sqrt{\mathrm{g}}\, \epsilon_{\mu_1 \ldots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \tag{2.14}$$

where $\epsilon_{\mu_1 \ldots \mu_n}$ is the totally-antisymmetric n-dimensional Levi–Civita symbol. $\epsilon_{\mu_1 \ldots \mu_n}$ cannot be considered a proper tensor, as its components are always $+1, -1, 0$ independently if the indices are covariant or contravariant: it is, in fact, a **tensor density**, i.e. a tensor divided by $\sqrt{\mathrm{g}}$. It can be shown that:

$$v^{\mu_1 \ldots \mu_n} = g^{\mu_1 \nu_1} \ldots g^{\mu_n \nu_n} v_{\mu_1 \ldots \mu_n} = \sigma \frac{1}{\sqrt{\mathrm{g}}} \epsilon^{\mu_1 \ldots \mu_n} \tag{2.15}$$

where σ is the sign of the signature, e.g. $\sigma = +1$ for Riemannian manifolds and $\sigma = -1$ for Lorentzian manifolds. As notation, the integral of a generic function f on $\mathcal{M}$ is denoted as:

$$\int_{\mathcal{M}} f\, \mathrm{dV} \equiv \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{\mathrm{g}} f \tag{2.16}$$

§2.1.3.1 Hodge theory

Definition 2.1.5 (Hodge dual)

Given an n-dimensional oriented metric manifold $(\mathcal{M}, g)$, the **Hodge dual** is defined as the map $\star : \bigwedge^p(\mathcal{M}) \to \bigwedge^{n-p}(\mathcal{M}) : \omega \mapsto \star\omega$ such that:

$$\star\omega_{\mu_1 \ldots \mu_{n-p}} := \frac{1}{(n-p)!}\sqrt{\mathrm{g}}\, \epsilon_{\mu_1 \ldots \mu_{n-p} \nu_1 \ldots \nu_p} \omega^{\nu_1 \ldots \nu_p} \tag{2.17}$$

In this section, the orientedness and n-dimensionality of the manifold $\mathcal{M}$ are implied.

Proposition 2.1.4 (Basis-independence)

The Hodge dual is basis-independent.

It is useful to state a lemma for future calculations.

Lemma 2.1.1 (Generalized Kronecker delta)
$v^{\mu_1\dots\mu_p\rho_1\dots\rho_{n-p}} v_{\nu_1\dots\nu_p\rho_1\dots\rho_{n-p}} = \sigma p!(n-p)!\delta^{\mu_1}_{[\nu_1}\dots\delta^{\mu_p}_{\nu_p]}$.

Proposition 2.1.5 (Nilpotence)
$\star(\star\omega) = \sigma\,(-1)^{p(n-p)}\,\omega$.

The Hodge dual defines an inner product on each $\bigwedge^p(\mathcal{M})$:

$$\langle\omega,\eta\rangle := \int_{\mathcal{M}} \omega\wedge\star\eta \tag{2.18}$$

This allows to define operators and their adjoints on form spaces.

Proposition 2.1.6 (Adjoint exterior derivative)
Given a metric manifold $(\mathcal{M}, g)$ and two forms $\omega\in\bigwedge^p(\mathcal{M}), \alpha\in\bigwedge^{p-1}(\mathcal{M})$, then:

$$\langle \mathrm{d}\alpha,\omega\rangle = \langle\alpha,\mathrm{d}^\dagger\omega\rangle \tag{2.19}$$

where the adjoint of the exterior derivative $\mathrm{d}^\dagger : \bigwedge^p(\mathcal{M}) \to \bigwedge^{p-1}(\mathcal{M})$ is defined as:

$$\mathrm{d}^\dagger := \sigma\,(-1)^{np+n-1}\star\mathrm{d}\star \tag{2.20}$$

Proof. To simplify the proof, consider a closed manifold; then, from Stokes' theorem and Eq. 1.44:

$$0 = \int_{\mathcal{M}} \mathrm{d}(\alpha\wedge\star\omega) = \langle\mathrm{d}\alpha,\omega\rangle + \int_{\mathcal{M}} (-1)^{p-1}\,\alpha\wedge\mathrm{d}\star\omega$$

The second term is proportional to $\langle\alpha,\star\mathrm{d}\star\omega\rangle$: to determine the relative sign, note that $\mathrm{d}\star\omega\in\bigwedge^{n-p+1}(\mathcal{M})$, thus, from Prop. 2.1.5, $\star\star\mathrm{d}\star\omega = \sigma\,(-1)^{(n-p+1)(p-1)}\,\mathrm{d}\star\omega$. In conclusion:

$$\langle\alpha,\star\mathrm{d}\star\omega\rangle = \sigma\,(-1)^{(n-p)(p-1)}\int_{\mathcal{M}}(-1)^{p-1}\,\alpha\wedge\mathrm{d}\star\omega$$
$$\implies \quad \langle d\alpha,\omega\rangle = \sigma\,(-1)^{(n-p)(p-1)+1}\,\langle\alpha,\star\mathrm{d}\star\omega\rangle$$

Noting that $(-1)^{(n-p)(p-1)+1} = (-1)^{np+n-1}$, as in general $(-1)^{-n} = (-1)^n$ and $(-1)^{-p^2+p+1} = (-1)^{-1}$ due to $p(p-1)$ being always even, concludes the proof. □

Definition 2.1.6 (Laplacian)
Given a metric manifold $(\mathcal{M}, g)$, the **Laplacian** $\triangle : \bigwedge^p(\mathcal{M}) \to \bigwedge^p(\mathcal{M})$ is defined as the operator:

$$\triangle := (\mathrm{d}+\mathrm{d}^\dagger)^2 \tag{2.21}$$

Clearly, since $\mathrm{d}^2 = \mathrm{d}^{\dagger 2} = 0$.

$$\triangle = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d} \equiv \{\mathrm{d},\mathrm{d}^\dagger\} \tag{2.22}$$

It is possible to compute an explicit expression for the Laplacian of functions.

Lemma 2.1.2

Given $f \in \mathcal{C}^\infty(\mathcal{M})$, then $\mathrm{d}^\dagger f = 0$.

Proof. Trivial noting that $\star f$ is a top-form. □

Proposition 2.1.7 (G)

ven $f \in \mathcal{C}^\infty(\mathcal{M})$, then:

$$\triangle f = -\frac{\sigma}{\sqrt{\mathrm{g}}}\partial_\nu\left(\sqrt{\mathrm{g}}\,g^{\mu\nu}\partial_\mu f\right) \tag{2.23}$$

Proof. Via direct calculation, using Lemma 2.1.2:

$$\begin{aligned}\triangle f &= \sigma\,(-1)^{n^2+n-1}\star\mathrm{d}\star(\partial_\mu f\,\mathrm{d}x^\mu) = -\sigma\star\mathrm{d}\,(\partial_\mu f\star\mathrm{d}x^\mu)\\ &= -\frac{\sigma}{(n-1)!}\star\mathrm{d}\left(\partial_\mu f g^{\mu\nu}\sqrt{\mathrm{g}}\,\epsilon_{\nu\rho_1\dots\rho_{n-1}}\mathrm{d}x^{\rho_1}\wedge\cdots\wedge\mathrm{d}x^{\rho_{n-1}}\right)\\ &= -\frac{\sigma}{(n-1)!}\star\partial_\alpha\left(\sqrt{\mathrm{g}}\,g^{\mu\nu}\partial_\mu f\right)\epsilon_{\nu\rho_1\dots\rho_{n-1}}\mathrm{d}x^\alpha\wedge\mathrm{d}x^{\rho_1}\wedge\cdots\wedge\mathrm{d}x^{\rho_{n-1}}\\ &= -\sigma\star\partial_\nu\left(\sqrt{\mathrm{g}}\,g^{\mu\nu}\partial_\mu f\right)\mathrm{d}x^1\wedge\dots\mathrm{d}x^n = -\frac{\sigma}{\sqrt{\mathrm{g}}}\partial_\nu\left(\sqrt{\mathrm{g}}\,g^{\mu\nu}\partial_\mu f\right)\end{aligned}$$

which is the thesis. □

The Laplacian operator is linked to the de Rham cohomology.

Definition 2.1.7 (Harmonic p-forms)

Given $\omega \in \bigwedge^p(\mathcal{M})$, it is said to be **harmonic** if $\triangle\omega = 0$. The space of harmonic p-forms on $(\mathcal{M}, g)$ is denoted as $\mathrm{Harm}^p(\mathcal{M})$.

Proposition 2.1.8

A harmonic form is both **closed** and **co-closed**.

Proof. $0 = \langle\omega, \triangle\omega\rangle = \langle\mathrm{d}\omega, \mathrm{d}\omega\rangle + \langle\mathrm{d}^\dagger\omega, \mathrm{d}^\dagger\omega\rangle$, thus $\mathrm{d}\omega = 0$ and $\mathrm{d}^\dagger\omega = 0$, for the inner product is positive-defined. □

Theorem 2.1.1 (Harmonic decomposition of forms)

Given a compact Riemannian manifold $(\mathcal{M}, g)$, any $\omega \in \bigwedge^p(\mathcal{M})$ can be uniquely decomposed as $\omega = \mathrm{d}\alpha + \mathrm{d}^\dagger\beta + \gamma$, with $\alpha \in \bigwedge^{p-1}(\mathcal{M})$, $\beta \in \bigwedge^{p+1}(\mathcal{M})$ and $\gamma \in \mathrm{Harm}^p(\mathcal{M})$.

Theorem 2.1.2 (Hodge's theorem)

Given a compact Riemannian manifold $(\mathcal{M}, g)$, then:

$$\mathrm{Harm}^p(\mathcal{M}) \cong H^p(\mathcal{M}) \tag{2.24}$$

Proof. From Prop. 2.1.8 $\mathrm{Harm}^p(\mathcal{M}) \subset Z^p(\mathcal{M})$, but the uniqueness of decomposition in Th. 2.1.1 implies $\forall \gamma \in \mathrm{Harm}^p(\mathcal{M})\, \exists \eta_\gamma \in \bigwedge^{p-1}(\mathcal{M}) : \gamma \neq \mathrm{d}\eta_\gamma$, thus $\mathrm{Harm}^p(\mathcal{M}) \subset H^p(\mathcal{M})$. WTS that any equivalence class $[\omega] \in H^p(\mathcal{M})$ can be represented by a harmonic form. By Th. 2.1.1 $\omega = \mathrm{d}\alpha + \mathrm{d}^\dagger\beta + \gamma$, but $\omega \in H^p(\mathcal{M})$ implies $\mathrm{d}\omega = 0$ by definition, so:

$$0 = \langle \mathrm{d}\omega, \beta \rangle = \langle \omega, \mathrm{d}^\dagger\beta \rangle = \langle \mathrm{d}\alpha + \mathrm{d}^\dagger\beta + \gamma, \mathrm{d}^\dagger\beta \rangle = \langle \mathrm{d}^\dagger\beta, \mathrm{d}^\dagger\beta \rangle$$

The inner product is positive-definite, thus $\mathrm{d}^\dagger\beta = 0$, hence $\omega = \gamma + \mathrm{d}\alpha$. By definition $H^p(\mathcal{M}) \coloneqq Z^p(\mathcal{M})/B^p(\mathcal{M})$, so $[\omega] = \gamma$. □

Betti numbers can then be computed as $B_p = \dim_{\mathbb{R}} \mathrm{Harm}^p(\mathcal{M})$.

§2.2 Connections

There is a different way to differentiate tensor fields, distinct from the Lie derivative, associated to a different way to map different vector spaces at different points: the covariant derivative. For the rest of this chapter, $\mathcal{M}$ is implied to be an n-dimensional metric manifold with metric tensor g.

§2.2.1 Covariant derivative

Definition 2.2.1 (Connection)

The **connection** is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, usually written as $\nabla(X, Y) \equiv \nabla_X Y$, where ∇_X is called the **covariant derivative**, satisfying the following properties for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;
2. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z \;\forall f, g \in \mathcal{C}^\infty(\mathcal{M})$;
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y \;\forall f \in \mathcal{C}^\infty(\mathcal{M})$.

Usually $X(f) \equiv \nabla_X f$. The covariant derivative endows the manifold with more structure: in particular, given a basis $\{e_\mu\}_{\mu=1,\dots,n}$ of $\mathfrak{X}(\mathcal{M})$, its covariant derivative is expressed as:

$$\nabla_{e_\rho} e_\nu \equiv \Gamma^\mu_{\rho\nu} e_\mu \tag{2.25}$$

The $\Gamma^\mu_{\rho\nu}$ are the components of the connection on that basis. Conventionally $\nabla_{e_\mu} \equiv \nabla_\mu$, thus resembling a partial derivative. To elucidate how the covariant derivative acts on vector fields:

$$\begin{aligned}
\nabla_X Y &= \nabla_X(Y^\mu e_\mu) \\
&= X(Y^\mu)e_\mu + Y^\mu \nabla_X e_\mu \\
&= X^\nu e_\nu(Y^\mu)e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu \\
&= X^\nu \left[e_\nu(Y^\mu) + \Gamma^\mu_{\nu\rho} Y^\rho\right] e_\mu \\
&= X^\nu \nabla_\nu Y = X^\nu (\nabla_\nu Y)^\mu e_\mu
\end{aligned}$$

The dependency on X can therefore be eliminated, and in components:

$$(\nabla_\nu Y)^\mu = e_\nu(Y^\mu) + \Gamma^\mu_{\nu\rho} Y^\rho \tag{2.26}$$

A sloppy notation is often used: $(\nabla_\nu Y)^\mu \equiv \nabla_\nu Y^\mu$. This must not be confused as the covariant derivative of Y^μ. Moreover $\nabla_\nu Y^\mu \equiv Y^\mu{}_{;\nu}$, while $\partial_\mu f \equiv f_{,\mu}$. On the coordinate basis $e_\mu = \partial_\mu$, then:

$$Y^\mu{}_{;\nu} = Y^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho} Y^\rho \tag{2.27}$$

Note that $Y^\mu{}_{;\nu}$ is the μ^{th} component of $\nabla_\nu Y$, while $Y^\mu{}_{,\nu}$ is the partial derivative of Y^μ along ∂_ν.

The covariant derivative coincides with other derivatives on $\mathcal{C}^\infty(\mathcal{M})$: it can be shown that $\nabla_X f = \mathcal{L}_X f = X(f)$ and $\nabla_\mu f = \partial_\mu f$. On $\mathfrak{X}(\mathcal{M})$, however, ∇_X and $\mathcal{L}_X$ are distinct: while $\nabla_X = X^\mu \nabla_\mu$, there is no way to write the same relation for $\mathcal{L}_X$, for it depends not only on X but on its first derivative too. The covariant derivative is thus the natural generalization of the partial derivative to curved manifolds.

Proposition 2.2.1

$\Gamma^\mu_{\rho\nu}$ are not components of a tensor.

Proof. Given the basis transformation $\tilde{e}_\nu = A^\mu{}_\nu e_\mu$, with A and invertible matrix (if they are both coordinate basis, then $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}$), the components of a (1,2) tensor must transform as:

$$\tilde{T}^\mu{}_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu T^\tau{}_{\sigma\lambda}$$

In the new basis:

$$\begin{aligned}\tilde{\Gamma}^\mu_{\rho\nu}\tilde{e}_\mu &= \nabla_{\tilde{e}_\rho}\tilde{e}_\nu = \nabla_{A^\sigma{}_\rho e_\sigma}(A^\lambda{}_\nu e_\nu) = A^\sigma{}_\rho \nabla_{e_\sigma}(A^\lambda{}_\nu e_\lambda)\\ &= A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} e_\tau + A^\sigma{}_\rho e_\lambda \partial_\sigma A^\lambda{}_\nu = \left[A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} + A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu\right] e_\tau\\ &= \left[A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} + A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu\right](A^{-1})^\mu{}_\tau \tilde{e}_\mu\end{aligned}$$

Thus, there is a second term proportional to ∂A which deviates from the transformation law:

$$\tilde{\Gamma}^\mu_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho \left[A^\lambda{}_\nu \Gamma^\tau_{\sigma\lambda} + \partial_\sigma A^\tau{}_\nu\right]$$

which means that $\Gamma^\mu_{\rho\nu}$ is not a tensor. □

§2.2.2 Covariant derivative of tensors

First of all, it is necessary to elucidate how the covariant derivative acts on 1-forms. Given a 1-form ω, the 1-form $\nabla_X\omega$ is defined by its action on vector fields. By Leibniz rule:

$$\nabla_X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)$$

Recalling that $\omega(Y)$ is a function, $\nabla_X(\omega(Y)) = X(\omega(Y))$, therefore:

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) \tag{2.28}$$

Expressing it in coordinates:

$$\begin{aligned}X^\mu(\nabla_\mu\omega)_\nu Y^\nu &= X^\mu\partial_\mu(\omega_\nu Y^\nu) - \omega_\nu X^\mu\left[\partial_\mu Y^\nu + \Gamma^\nu_{\mu\rho}Y^\rho\right]\\ &= X^\mu\left[\partial_\mu\omega_\rho - \Gamma^\nu_{\mu\rho}\omega_\nu\right]Y^\rho\end{aligned}$$

Crucially, the ∂Y terms cancel out, allowing to define $\nabla_X\omega$ without referencing Y:

$$(\nabla_\mu\omega)_\rho = \partial_\mu\omega_\rho - \Gamma^\nu_{\mu\rho}\omega_\nu \tag{2.29}$$

Using the same notation as for vector fields $(\nabla_\mu\omega)_\rho \equiv \nabla_\mu\omega_\rho \equiv \omega_{\rho;\mu}$:

$$\omega_{\rho;\mu} = \omega_{\rho,\mu} - \Gamma^\nu_{\mu\rho}\omega_\nu \tag{2.30}$$

This kind of argument can be extended to a general (p,q) tensor field:

$$\begin{aligned}\nabla_\rho T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\mu_q} = \partial_\rho T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q} &+ \Gamma^{\mu_1}_{\rho\sigma}T^{\sigma\mu_2\dots\mu_p}{}_{\nu_1\dots\nu_q} + \dots + \Gamma^{\mu_p}_{\rho\sigma}T^{\mu_1\dots\mu_{p-1}\sigma}{}_{\nu_1\dots\nu_q}\\ &- \Gamma^\sigma_{\rho\nu_1}T^{\mu_1\dots\mu_p}{}_{\sigma\nu_2\dots\nu_q} - \dots - \Gamma^\sigma_{\rho\nu_q}T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_{q-1}\sigma}\end{aligned} \tag{2.31}$$

The pattern is clear: for each upper index μ there is a $+\Gamma^\mu_{\rho\sigma}T^\sigma$ term, while for each lower index ν there is $-\Gamma^\sigma_{\rho\nu}T_\sigma$ term. Furthermore, it is necessary to generalize the comma-notation: for example, $X^\mu{}_{;\nu\rho} \equiv \nabla_\rho\nabla_\nu X^\mu$, so the rightmost index is the one acting first.

§2.2.2.1 Torsion and curvature

Even though the connection is not a tensor, it is used to construct two important tensors.

Definition 2.2.2 (Torsion tensor)

The **torsion** is a (1,2) tensor defined on $\bigwedge^1(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$ as:

$$\mathsf{T}(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \tag{2.32}$$

Alternatively, the torsion can be viewed as a map $\mathsf{T} : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ such that:

$$\mathsf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.33}$$

Definition 2.2.3 (Curvature tensor)

The **curvature** is a (1,3) tensor defined on $\bigwedge^1(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$ as:

$$\mathsf{R}(\omega, X, Y, Z) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{2.34}$$

Alternatively, the curvature can be viewed as a map from $\mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$ to the space of differential operators on $\mathfrak{X}(\mathcal{M})$ such that:

$$\mathsf{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{2.35}$$

The fact that these are indeed tensors, i.e. they are linear in each argument, can be shown by direct calculation, recalling that $[fX, Y] = f[X, Y] - Y(f)X$.

Proposition 2.2.2

On the coordinate basis $\{\partial_\mu\}$ and $\{\mathrm{d}x^\mu\}$ the torsion components are:

$$T^\rho{}_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{2.36}$$

Proof. By direct calculation:

$$\begin{aligned} T^\rho{}_{\mu\nu} &= T(\mathrm{d}x^\rho, \partial_\mu, \partial_\nu) = \mathrm{d}x^\mu(\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu - [\partial_\mu, \partial_\nu]) \\ &= \mathrm{d}x^\mu(\partial_\mu \partial_\nu - \Gamma^\sigma_{\mu\nu}\partial_\sigma - \partial_\nu \partial_\mu + \Gamma^\sigma_{\nu\mu}\partial_\sigma) \\ &= \left[\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right] \delta^\mu_\sigma = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \end{aligned}$$

which is the thesis. □

Interestingly, even though $\Gamma^\rho_{\mu\nu}$ is not a tensor, its anti-symmetric part $\Gamma^\rho_{[\mu\nu]} = \frac{1}{2}T^\sigma{}_{\mu\nu}$ is. Clearly, the torsion tensor is anti-symmetric in its lower indices, thus for connections which are symmetric in their lower indices the torsion is null: such connections are said to be **torsion-free**.

Proposition 2.2.3

On the coordinate basis $\{\partial_\mu\}$ and $\{\mathrm{d}x^\mu\}$ the curvature components are:

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda} \tag{2.37}$$

Proof. By direct calculation:

$$\begin{aligned}
\mathsf{R}(\mathrm{d}x^\sigma, \partial_\mu, \partial_\nu, \partial_\rho) &= \mathrm{d}x^\sigma(\nabla_\mu\nabla_\nu\partial_\rho - \nabla_\nu\nabla_\mu\partial_\rho - \nabla_{[\partial_\mu,\partial_\nu]}\partial_\rho) \\
&= \mathrm{d}x^\sigma(\nabla_\mu\nabla_\nu\partial_\rho - \nabla_\nu\nabla_\mu\partial_\rho) \\
&= \mathrm{d}x^\sigma(\nabla_\mu(\Gamma^\lambda_{\nu\rho}\partial_\lambda) - \nabla_\nu(\Gamma^\lambda_{\mu\rho}\partial_\lambda)) \\
&= \mathrm{d}x^\sigma((\partial_\mu\Gamma^\lambda_{\nu\rho})\partial_\lambda + \Gamma^\lambda_{\nu\rho}\Gamma^\tau_{\mu\lambda}\partial_\tau - (\partial_\nu\Gamma^\lambda_{\mu\rho})\partial_\lambda - \Gamma^\lambda_{\mu\rho}\Gamma^\tau_{\nu\lambda}\partial_\tau) \\
&= \partial_\mu\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}
\end{aligned}$$

which completes the proof. □

Clearly, the curvature tensor is anti-symmetric in its last two lower indices, i.e. $R^\sigma{}_{\rho\mu\nu} = R^\sigma{}_{\rho[\mu\nu]}$, and it can in fact be written as:

$$R^\sigma{}_{\rho\mu\nu} = 2\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + 2\Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho} \tag{2.38}$$

Theorem 2.2.1 (Ricci identity)

$$2\nabla_{[\mu}\nabla_{\nu]}Z^\sigma = R^\sigma{}_{\rho\mu\nu}Z^\rho - T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma \tag{2.39}$$

Proof. By direct calculation:

$$\begin{aligned}
\nabla_{[\mu}\nabla_{\nu]}Z^\sigma &= \partial_{[\mu}(\nabla_{\nu]}Z^\sigma) + \Gamma^\sigma_{[\mu|\lambda|}\nabla_{\nu]}Z^\lambda - \Gamma^\rho_{[\mu\nu]}\nabla_\rho Z^\sigma \\
&= \partial_{[\mu}\partial_{\nu]}Z^\sigma + (\partial_{[\mu}\Gamma^\sigma_{\nu]\rho})Z^\rho + (\partial_{[\mu}Z^\rho)\Gamma^\sigma_{\nu]\rho} + \Gamma^\sigma_{[\mu|\lambda|}\partial_{\nu]}Z\lambda + \Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho}Z^\rho - \frac{1}{2}T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma \\
&= \left(\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + \Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho}\right)Z^\rho - \frac{1}{2}T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma \\
&= \frac{1}{2}R^\sigma{}_{\rho\mu\nu}Z^\rho - \frac{1}{2}T^\sigma{}_{\mu\nu}\nabla_\rho Z^\sigma
\end{aligned}$$

which is the thesis. □

§2.2.2.2 Levi–Civita connection

The discussion on the connection has so far been independent of the metric. Starting to consider it, an important result is the fundamental theorem of Riemannian geometry.

Theorem 2.2.2 (Riemann's theorem)

On a metric manifold $(\mathcal{M}, g)$, there exists a unique torsion-free connection that is compatible with the metric, i.e. for all $X \in \mathfrak{X}(\mathcal{M})$:

$$\nabla_X g = 0 \tag{2.40}$$

This is called the **Levi–Civita connection**.

Proof. WTS uniqueness: suppose such a connection exists. Then, by Leibniz rule:

$$X(g(Y,Z)) = \nabla_X(g(Y,Z)) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X, Z)$$

Since $\nabla_X g = 0$, by cyclic permutations of X, Y and Z:

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

Since the connection is torsion-free, $\nabla_X Y - \nabla_Y X = [X,Y]$, thus these equations become:

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_Y X, Z) + g(\nabla_X Z, Y) + g([X,Y], Z) \\ Y(g(Z,X)) &= g(\nabla_Z Y, X) + g(\nabla_Y X, Z) + g([Y,Z], X) \\ Z(g(X,Y)) &= g(\nabla_X Z, Y) + g(\nabla_Z Y, X) + g([Z,X], y) \end{aligned}$$

Adding the first two and subtracting the third:

$$\begin{aligned} g(\nabla_Y X, Z) = \frac{1}{2}[&X(g(Y,Z)) + Y(g(Z,X)) + Z(g(X,Y)) \\ &- g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)] \end{aligned}$$

The metric is non-degenerate, thus this uniquely specifies the connection. By direct calculation it can be shown that it indeed satisfies all the properties of a connection. □

It is possible to explicit the components of the Levi–Civita connection in terms of the metric.

Proposition 2.2.4 (Christoffel symbols)

On the coordinate basis $\{\partial_\mu\}$ and $\{dx^\mu\}$ the Levi–Civita connection's components, called **Christoffel symbols**, are:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{2.41}$$

Proof. Recalling that $[\partial_\mu, \partial_\nu] = 0$, $\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = g(\nabla_\mu \partial_\nu, \partial_\rho) = \frac{1}{2}(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu})$. □

Example 2.2.1 (Euclidean space)

In flat space $\mathbb{R}^n$, endowed with either Euclidean or Minkowski metric, it is always possible to choose Cartesian coordinates, in which case the Christoffel symbols vanish. Being the Riemann tensor a genuine tensor, it therefore will vanish in all possible coordinate systems on $\mathbb{R}^n$, even in those with $\Gamma^\rho_{\mu\nu} \neq 0$: this expresses the flatness of $\mathbb{R}^n$.

§2.2.2.3 Gauss' theorem

The divergence theorem (or Gauss' theorem) states that the integral of a total derivative is a boundary term. It is possible to express this theorem on curved manifolds in a convenient way.

Lemma 2.2.1

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \tag{2.42}$$

Proof. A useful identity for invertible matrices: $\operatorname{tr}\log \mathrm{A} = \log\det \mathrm{A}$. Thus (WLOG $\det g > 0$):

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2} g^{\mu\rho}\partial_\nu g_{\mu\rho} = \frac{1}{2}\operatorname{tr}(g^{-1}\partial_\nu g) = \frac{1}{2}\operatorname{tr}(\partial_\nu \log g) = \frac{1}{2}\partial_\nu \log\det g = \frac{1}{\sqrt{\det g}}\partial_\nu\sqrt{\det g}$$

which is the thesis. □

Theorem 2.2.3 (Gauss' theorem)

Given a Riemannian manifold $(\mathcal{M}, g)$, consider a region $M \subseteq \mathcal{M}$ with boundary ∂M and let n^μ be an outward-pointing unit vector orthogonal to ∂M. Then, for any vector field X^μ on M:

$$\int_M \mathrm{d}^n x \sqrt{\mathrm{g}}\, \nabla_\mu X^\mu = \int_{\partial M} \mathrm{d}^{n-1} x \sqrt{\gamma}\, n_\mu X^\mu \tag{2.43}$$

where γ_{ij} is the pull-back of the metric to ∂M and $\gamma \equiv \det \gamma_{ij}$.

Proof. From Lemma 2.2.1:

$$\sqrt{\mathrm{g}}\,\nabla_\mu X^\mu = \sqrt{\mathrm{g}}\left(\partial_\mu X^\mu + \Gamma^\mu_{\mu\nu} X^\nu\right) = \sqrt{\mathrm{g}}\left(\partial_\mu X^\mu + X^\nu \frac{1}{\sqrt{\mathrm{g}}}\partial_\nu \sqrt{\mathrm{g}}\right) = \partial_\mu(\sqrt{\mathrm{g}} X^\mu)$$

The integral becomes:

$$\int_M \mathrm{d}^n x \sqrt{\mathrm{g}}\,\nabla_\mu X^\mu = \int_M \mathrm{d}^n x\, \partial_\mu(\sqrt{\mathrm{g}} X^\mu)$$

This is the integral of an ordinary partial derivative, so the ordinary divergence theorem applies. To evaluate the integral on the boundary, it is convenient to pick coordinates so that ∂M is a surface at constant x^n. Moreover, to simplify the proof, the possible metrics will be restricted to $g_{\mu\nu} = \operatorname{diag}(\gamma_{ij}, N^2)$. By usual integration rules:

$$\int_M \mathrm{d}^n x\, \partial_\mu(\sqrt{\mathrm{g}} X^\mu) = \int_{\partial M} \mathrm{d}^{n-1} x \sqrt{\gamma N^2} X^n$$

The unit normal vector is $n^\mu = (0, \dots, 0, \frac{1}{N})$, so that $g_{\mu\nu} n^\mu n^\nu = 1$, therefore $n_\mu = g_{\mu\nu} n^\nu = (0, \dots, 0, N)$. The proof is then concluded because:

$$\int_{\partial M} \mathrm{d}^{n-1} x \sqrt{\gamma N^2} X^n = \int_{\partial M} \mathrm{d}^{n-1} x \sqrt{\gamma} n_\mu X^\mu$$

which is the thesis. □

Note that this theorem holds on Lorentzian manifolds too, with the condition that ∂M must be purely timelike or purely spacelike, ensuring that $\gamma \neq 0$ at any point.

§2.2.2.4 Maxwell action

Consider spacetime as a manifold $\mathcal{M}$. The electromagnetic field can be described by a form on this manifold: indeed, the electromagnetic gauge field $A_\mu = (\phi, \mathbf{A})$ is to be thought as the components of a 1-form $A = A_\mu(x)\mathrm{d}x^\mu$. The exterior derivative of this form is a 2-form $F = \mathrm{d}A$:

$$F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

The components $F_{\mu\nu}$ are in reality the components of a tensor, the Faraday tensor. By construction, a useful identity holds, sometimes called the **Bianchi identity**:

$$\mathrm{d}F = 0 \tag{2.44}$$

From this identity derive two Maxwell equations: $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$. Moreover, note that the gauge field is not unique: the gauge transformation $A \mapsto A + \mathrm{d}\alpha$, which equals $A_\mu \mapsto A_\mu + \partial_\mu \alpha$, leaves F unchanged.
To study the dynamics of these fields, an action is needed: Differential Geometry allows very few actions to be written down.
For example, suppose that on the considered manifold no metric is defined. To integrate over $\mathcal{M}$ a 4-form is needed, but F is a 2-form, thus the only possible action is:

$$\mathcal{S}_{\text{top}} = -\frac{1}{2}\int F \wedge F \tag{2.45}$$

The integrand become $\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\,\mathbf{E}\cdot\mathbf{B}$. Actions of this kind, independent of the metric, are called **topological actions** and are of no interest in classical physics: in fact, $F \wedge F = \mathrm{d}(A \wedge F)$, so the action is a total derivative and doesn't affect the equations of motion.
To construct an action of classical interest, a metric is needed. This allows to introduce a second 2-form, $\star F$, so to construct the Maxwell action:

$$\mathcal{S}_{\text{M}} = -\frac{1}{2}\int F \wedge \star F \tag{2.46}$$

The integrand can then be expanded as:

$$\mathcal{S}_{\text{M}} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{\mathrm{g}}\, g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho}F_{\nu\sigma} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{\mathrm{g}}F^{\mu\nu}F_{\mu\nu}$$

In flat spacetime $F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$. In a general curved spacetime, the equation of motion resulting from the variation of the Maxwell action is $\mathrm{d}\star F = 0$.
To complete the theory, consider a gauge field coupled to a current, described by a 1-form J. The Maxwell action then becomes:

$$\mathcal{S}_{\text{M}} = \int -\frac{1}{2}F \wedge \star F + A \wedge \star J \tag{2.47}$$

This action must retain its gauge invariance, but under $A \mapsto A + \mathrm{d}\alpha$ it transforms as $S_{\text{M}} \mapsto S_{\text{M}} + \int \mathrm{d}\alpha \wedge \star J$, therefore, after integrating by parts, the condition of gauge invariance translates to:

$$\mathrm{d}\star J = 0 \tag{2.48}$$

This is current conservation in the language of forms. Varying the action in Eq. 2.46 now leads to the Maxwell equations with source terms:

$$\mathrm{d}\star F = \star J \tag{2.49}$$

To define electric and magnetic charges, integrate over submanifolds. Consider a 3-dimensional spatial submanifold Σ: the electric charge in Σ is defined as:

$$Q_e(\Sigma) := \int_\Sigma \star J \tag{2.50}$$

This agrees with the usual definition in flat spacetime $Q_e = \int_\Sigma \mathrm{d}^3x J^0$. Using the equations of motion and Stokes' theorem, a general form of Gauss' law is obtained:

$$Q_e(\Sigma) = \int_{\partial\Sigma} \star F \tag{2.51}$$

Similarly, the magnetic charge in Σ is defined as:

$$Q_m(\Sigma) := \int_{\partial\Sigma} F \tag{2.52}$$

The non-existence of magnetic charges, following from Bianchi identity, can be evaded in topologically interesting manifolds.
From charge conservation in Eq. 2.48, it follows that the electric charge in a region cannot change, unless current flows in or out of that region. Consider a cylindrical region of spacetime V, ending in two spatial hypersurfaces Σ_1 and Σ_2: its boundary is $\partial V = \Sigma_1 \cup \Sigma_2 \cup B$, where B is a cylindrical timelike hypersurface. The statement that no current flows in or out of V means that $J|_B = 0$. Then:

$$Q_e(\Sigma_1) - Q_e(\Sigma_2) = \int_{\Sigma_1} \star J - \int_{\Sigma_2} \star J = \int_{\partial V} \star J - \int_B \star J = \int_{\partial V} \star J = \int_V \mathrm{d} \star J = 0$$

Thus, electric charge in remains constant in time.

Maxwell equations from connections First note that, given the gauge field $A \in \bigwedge^1(\mathcal{M})$, the field strength can be expressed via covariant derivatives:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

The Christoffel symbols cancel out due to anti-symmetry: this is what allows to define the exterior derivative without introducing connections first.

Proposition 2.2.5 (Current conservation)
Current conservation can be written as: $\mathrm{d} \star J = 0 \iff \nabla_\mu J^\mu = 0$.

Proof. Recalling Lemma 2.2.1:

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\rho} J^\rho = \partial_\mu J^\mu + \partial_\rho(\log\sqrt{\mathrm{g}}) J^\rho = \frac{1}{\sqrt{\mathrm{g}}}\partial_\mu(\sqrt{\mathrm{g}} J^\mu) \propto \mathrm{d} \star J$$

Hence, $\mathrm{d} \star J = 0 \iff \nabla_\mu J^\mu = 0$. □

As an aside, in general the divergence in different coordinate systems can be computed using the formula $\nabla_\mu J^\mu = \frac{1}{\sqrt{\mathrm{g}}}\partial_\mu(\sqrt{\mathrm{g}} J^\mu)$.

Proposition 2.2.6

$$d \star F = \star J \Leftrightarrow \nabla_\mu F^{\mu\nu} = J^\nu \tag{2.53}$$

Proof. Recalling Lemma 2.2.1:

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} = \frac{1}{\sqrt{\mathrm{g}}}\partial_\mu(\sqrt{\mathrm{g}} F^{\mu\nu})$$

where $\Gamma^\nu_{\mu\rho} F^{\mu\rho} = 0$ because $\Gamma^\nu_{\mu\rho}$ is symmetric in μ and ρ, while $F^{\mu\rho}$ is anti-symmetric. The proof follows recalling the definition of the Hodge dual in Eq. 2.17. □

§2.3 Parallel transport

The connection connects tangent spaces, or more generally any tensor vector space, at different points of the manifold: this map is called parallel transport and it's necessary for the definition of differentiation.

Definition 2.3.1 (Parallel transport)

Consider a vector field X and some associated integral curve γ, with coordinates $x^\mu(\tau)$ such that:

$$X^\mu\big|_\gamma = \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau}$$

A tensor field T is said to be **parallely transported** along γ if:

$$\nabla_X T = 0 \tag{2.54}$$

Suppose that γ connects two points $p, q \in \mathcal{M}$: Eq. 2.54 provides a map from the tensor vector space defined at p to that defined at q. To illustrate this, consider the parallel transport of a vector field Y:

$$X^\nu(\partial_\nu Y^\mu + \Gamma^\mu_{\nu\rho} Y^\rho) = 0$$

Evaluating this equation on γ, considering $Y^\mu = Y^\mu(x(\tau))$:

$$\frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu_{\nu\rho} Y^\rho = 0 \tag{2.55}$$

These are a set of coupled ODEs, thus, given an initial condition (e.g. at $\tau = 0$, i.e. at p), these equations can be solved to find a unique vector at each point along the curve.
Note that the parallel transport depends both on the path (characterized by the vector field X) and on the connection.

Definition 2.3.2 (Geodesic)

Given a vector field X, a **geodesic** is a curve tangent to X such that:

$$\nabla_X X = 0 \tag{2.56}$$

Proposition 2.3.1 (Geodesic equation)

A geodesic is described by:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0 \tag{2.57}$$

Proof. From the above calculations, along γ:

$$0 = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\rho} X^\nu X^\rho = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}$$

which is the geodesic equation. □

For the Levi–Civita connection $\nabla_X g = 0$. If $Y \in \mathfrak{X}(\mathcal{M})$ is parallely transported along a geodesic associated to $X \in \mathfrak{X}(\mathcal{M})$, then $\nabla_X Y = \nabla_X X = 0$, therefore $\frac{\mathrm{d}}{\mathrm{d}\tau} g(X, Y) = 0$: this ensures that the two tangent vectors always determine the same angles along the geodesic.

§2.3.1 Normal coordinates

Theorem 2.3.1 (Equivalence principle)

Given a Riemannian manifold $(\mathcal{M}, g)$ and $p \in \mathcal{M}$, in a neighbourhood of p it's always possible to find coordinates, called **normal coordinates**, such that $g_{\mu\nu}(p) = \delta_{\mu\nu}$ and $g_{\mu\nu,\rho}(p) = 0$.

Proof. By brute force, consider initial coordinates y^μ and find a change of coordinates x^μ which satisfy the requirements:

$$\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma} = g_{\mu\nu}$$

WLOG p is the origin of both sets of coordinates, so:

$$y^\rho = \left.\frac{\partial y^\rho}{\partial x^\mu}\right|_{x=0} x^\mu + \frac{1}{2}\left.\frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu}\right|_{x=0} x^\mu x^\nu + \dots$$

This, together with the Taylor expansion of $\tilde{g}_{\rho\sigma}$, can be inserted in the transformation equation of the metric, thus finding a set of PDEs for each power of x, which can be solved to characterize the normal coordinates. For example, the first condition is:

$$\left.\frac{\partial y^\rho}{\partial x^\mu}\right|_{x=0}\left.\frac{\partial y^\sigma}{\partial x^\nu}\right|_{x=0}\tilde{g}_{\rho\sigma}(p) = \delta_{\mu\nu}$$

Given any $\tilde{g}_{\rho\sigma}(p)$, it's always possible to find $\frac{\partial y}{\partial x}$ that satisfies this condition. In fact, if $\dim_{\mathbb{R}} \mathcal{M} = n$, the Jacobian of the transformation has n^2 independent elements and the equation above puts $\frac{1}{2}n(n+1)$ constraints: the remaining free parameters are $\frac{1}{2}n(n-1)$, which is precisely the dimension of $\mathrm{SO}(n)$, the symmetry group of the flat metric. A similar counting shows that $g_{\mu\nu,\rho}(p) = 0$ puts $\frac{1}{2}n^2(n+1)$ constraints, precisely the number of independent elements of the Hessian of the transformation. $\square$

This theorem holds for Lorentzian manifolds too, but the flat metric is now $\eta_{\mu\nu}$ and its symmetry group is $\mathrm{SO}(1, n-1)$. The condition $g_{\mu\nu,\rho}(p) = 0$ implies that $\Gamma^\mu_{\nu\rho}(p) = 0$, but generally the Christoffel symbols won't vanish away from p. Note, however, that it's not generally possible to ensure the vanishing of second derivatives too: indeed, $g_{\mu\nu,\rho\sigma}(p) = 0$ would put $\frac{1}{4}n^2(n+1)^2$ constraints, but the independent $\frac{\partial^3 y}{\partial^3 x}$ terms are $\frac{1}{6}n^2(n+1)(n+2)$, thus leaving $\frac{1}{12}n^2(n^2-1)$ free terms: this is precisely the number of indepedent components of the Riemann tensor, therefore in general it's not possible to pick coordinates as to make the Riemann tensor vanish too.

§2.3.1.1 Exponential map

A simple way to construct normal coordinates is the following: given a tangent vector $X_p \in T_p\mathcal{M}$, there is a unique affinely parametrized geodesic through p with tangent vector X_p at p; then, any point q in the neighbourhood of p is labelled by the coordinates of the geodesic that takes from p to q a fixed amount of time.

Analytically, introducing a coordinate system (not necessarily normal) $\tilde{x}^\mu$ in the neighbourhood of p, an affinely parametrized geodesic solves Eq. 2.57, with initial conditions $\frac{\partial \tilde{x}^\mu}{\partial \tau}|_{\tau=0} = \tilde{X}^\mu_p$ and $\tilde{x}^\mu(\tau = 0) = 0$ that make the solution unique. The uniqueness of the solution allows to define a map $\mathrm{Exp} : T_p\mathcal{M} \to \mathcal{M}$, called the **exponential map**, which acts as follows: given $X_p \in T_p\mathcal{M}$, construct the appropriate geodesic as above and follow it for a fixed affine distance,

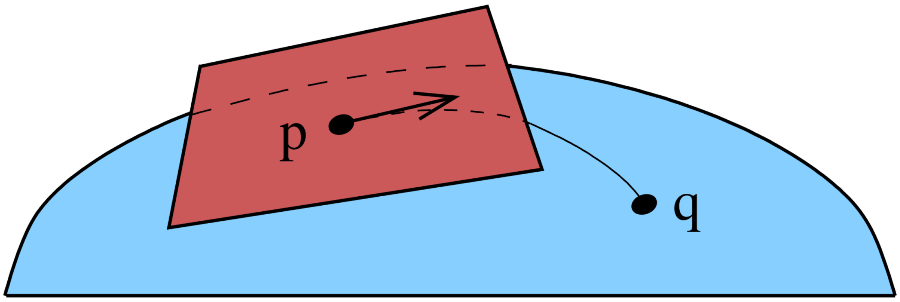

Figure 2.1: Visualization of the exponential map.

conventionally $\tau = 1$, to get a new point $q \in \mathcal{M}$. See Fig. Fig. 2.1 for visual aid. Obviously, there may be points which cannot be reached from p by geodesics, or there may be tangent vectors X_p for which Exp is ill-defined: in General Relativity, this occurs when spacetime has singularities, but these are not relevant issues.
Pick a basis $\{e_\mu\}$ of $T_p\mathcal{M}$. Then $\mathrm{Exp} : T_p\mathcal{M} \ni X^\mu e_\mu \mapsto q \in \mathcal{M}$, thus it is possible to assign coordinates in the neighbourhood of p such that $x^\mu(q) = X^\mu$: these are the normal coordinates. To show this, note that if $\{e_\mu\}$ is orthonormal, then the geodesics will point in orthogonal directions, ensuring that $g_{\mu\nu}(p) = \delta_{\mu\nu}$. Now, fix a point q associated to a given tangent vector $X_p \in T_p\mathcal{M}$: this means that q is at distance $\tau = 1$ from p along the given geodesic. Note that the geodesic equation is homogeneous in τ, thus in general $\mathrm{Exp} : \tau X_p \mapsto x^\mu(\tau) = \tau X^\mu$, which means that geodesics take a simple form in these coordinates:

$$x^\mu(\tau) = \tau X^\mu$$

Being these geodesics, they must solve Eq. 2.57, that is:

$$\Gamma^\mu_{\nu\rho}(x(\tau))X^\nu X^\rho = 0$$

which holds at any point along the geodesic, i.e. at any $\tau \in \mathbb{R}^+$. At most points $x(\tau)$, this equation only holds for those choices of X^μ tangent to the geodesics. However, at $x(0) = 0$, i.e. at p, it must hold for any tangent vector: this means that $\Gamma^\mu_{(\nu\rho)}(p) = 0$ which, for a torsion-free connection, ensures that $\Gamma^\mu_{\nu\rho}(p) = 0$. But vanishing Christoffel symbols imply a vanishing first derivative of the metric: for the Levi–Civita connection $2g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}$, thus symmetrizing $(\mu\nu)$ cancels the last two terms, leaving an identity that, evaluated at p, gives $g_{\mu\nu,\rho}(p) = 0$. Hence, these are indeed normal coordinates.

§2.3.1.2 Equivalence principle

Normal coordinates are conceptually important in General Relativity: an observer at point p who parametrizes their immediate surroundings using coordinates constructed by geodesics will experience a locally flat metric. This is precisely Einstein's equivalence principle: any free-falling observer, performing local experiments, will not experience a gravitational field. The formal definition of free-falling observer is an observer which follows geodesics, while the local lack of gravitational field means $g_{\mu\mu}(p) = \eta_{\mu\nu}$. In this context, normal coordinates are called **local inertial frame**.
To understand what "local" means, note that there is a way to distinguish whether a gravitational field is present at p: a non-vanishing Riemann tensor. This depends on the second derivatives of the metric, which in general will be non-vanishing. However, to measure the effects of the Riemann tensor, one needs to compare the results of experiments at p and at a nearby point q: this is a non-local observation.

§2.3.2 Curvature and torsion

With reference to Fig. 2.2, consider a tangent vector $Z_p \in T_p\mathcal{M}$ and two vector fields $X, Y \in \mathfrak{X}(\mathcal{M}) : [X, Y] = 0$, i.e. they are linearly independent. Construct two curves γ, γ' as in figure, both leading to a point $r \in \mathcal{M}$ which, for simplicity, is close to p. It is possible to impose normal coordinates centered at p such that $x^\mu = (\tau, \sigma, \dots)$, so that $X = \frac{\partial}{\partial\tau}$ and $Y = \frac{\partial}{\partial\sigma}$: then $x^\mu(p) = (0, 0, 0, \dots)$, $X^\mu(q) = (\delta\tau, 0, 0, \dots)$, $x^\mu(s) = (0, \delta\sigma, 0, \dots)$ and $x^\mu(r) = (\delta\tau, \delta\sigma, 0, \dots)$, with $\delta\tau, \delta\sigma$ small.
First, parallel transport Z_p along X to Z_q, so that Z^μ solves:

$$\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu_{\nu\rho} Z^\rho = 0$$

In normal coordinates $\Gamma^\mu_{\nu\rho}(p) = 0$, thus $\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\Big|_{\tau=0} = 0$ and the Taylor expansion is:

$$\begin{aligned}
Z^\mu_q &= Z^\mu_p + \frac{\delta\tau^2}{2}\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\bigg|_{\tau=0} + o(\delta\tau^3) \\
&= Z^\mu_p - \frac{\delta\tau^2}{s}\left[X^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\nu\rho}}{\mathrm{d}\tau} + \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} Z^\rho \Gamma^\mu_{\nu\rho} + X^\nu \frac{\mathrm{d}Z^\rho}{\mathrm{d}\tau}\Gamma^\mu_{\nu\rho}\right]_{\tau=0} + o(\delta\tau^3) \\
&= Z^\mu_p - \frac{\delta\tau^2}{2} X^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\nu\rho}}{\mathrm{d}\tau}\bigg|_{\tau=0} + o(\delta\tau^3) = Z^\mu_p - \frac{\delta\tau^2}{2}\left[X^\nu X^\sigma Z^\rho \Gamma^\mu_{\nu\rho,\sigma}\right]_p + o(\delta\tau^3)
\end{aligned}$$

where $\frac{\mathrm{d}}{\mathrm{d}\tau} = X^\sigma \partial_\sigma$. Now, Z_q needs to be parallely transported along Y to Z_r, but this time $\frac{\mathrm{d}Z^\mu}{\mathrm{d}\sigma}\Big|_{\sigma=0}$ doesn't vanish, in general. From the parallel transport equation:

$$\frac{\mathrm{d}Z^\mu}{\mathrm{d}\sigma}\bigg|_{\sigma=0} = -\left[Y^\nu Z^\rho \Gamma^\mu_{\nu\rho}\right]_q = -\left[Y^\nu Z^\rho X^\sigma \Gamma^\mu_{\nu\rho,\sigma}\right]_p \delta\tau + o(\delta\tau^2)$$

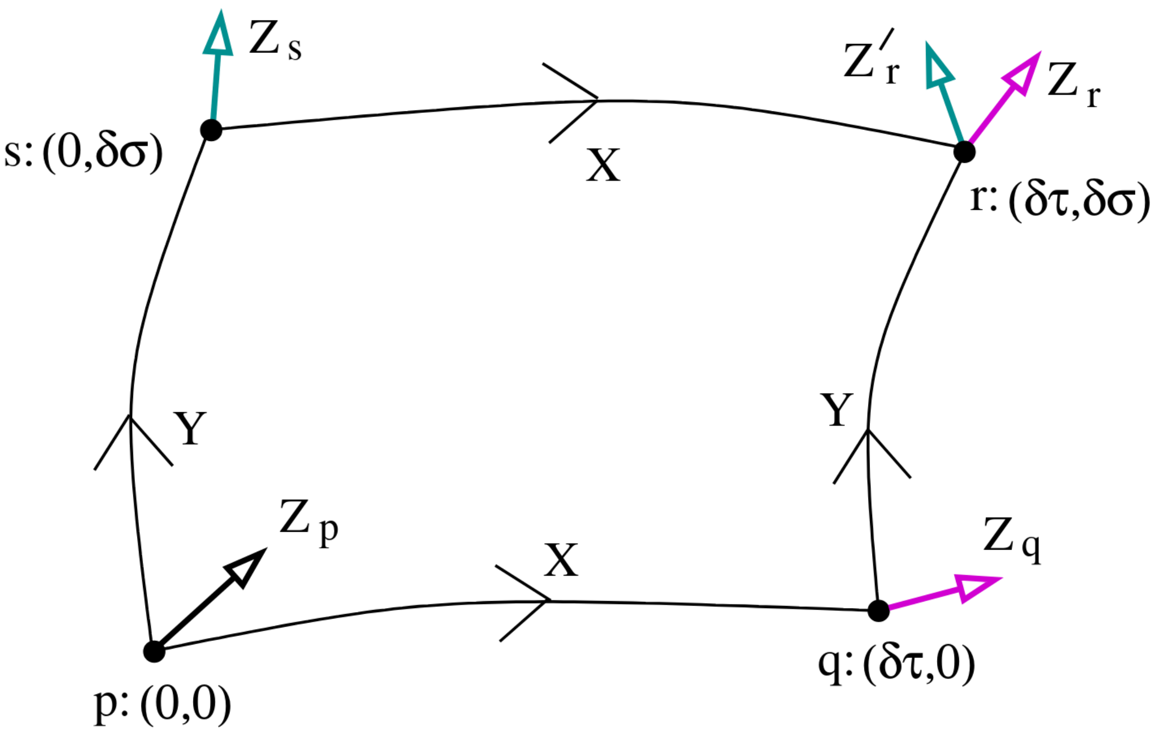

Figure 2.2: Parallel transport along different paths.

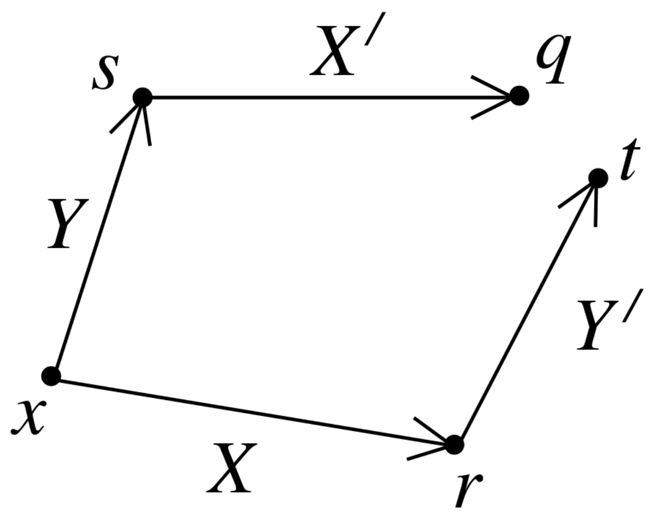

Figure 2.3: Visualization of torsion.

The expansions of Y^ν and Z^ρ at leading order multiply $\Gamma^\mu_{\nu\rho}(p) = 0$, thus only contribute to higher order terms. Next order in $\delta\sigma$:

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\sigma^2}\right|_{\sigma=0} = -\left[\left(\frac{\mathrm{d}Y^\nu}{\mathrm{d}\sigma} Z^\rho + Y^\nu \frac{\mathrm{d}Z^\rho}{\mathrm{d}\sigma}\right)\Gamma^\mu_{\nu\rho} + Y^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\nu\rho}}{\mathrm{d}\sigma}\right]_q = -\left[Y^\nu Y^\sigma Z^\rho \Gamma^\mu_{\nu\rho,\sigma}\right]_p + o(\delta\tau)$$

The complete expansion thus is:

$$\begin{aligned} Z^\mu_r &= Z^\mu_q - \left[Y^\nu Z^\rho X^\sigma \Gamma^\mu_{\nu\rho,\sigma}\right]_p \delta\tau\delta\sigma - \frac{1}{2}\left[Y^\nu Y^\sigma Z^\rho \Gamma^\mu_{\nu\rho,\sigma}\right]_p \delta\sigma^2 + o(\delta^3) \\ &= Z^\mu_p - \frac{1}{2}\Gamma^\mu_{\nu\rho,\sigma}(p)\left[X^\nu X^\sigma Z^\rho \delta\tau^2 + 2Y^\nu Z^\rho X^\sigma \delta\tau\delta\sigma + Y^\nu Y^\sigma Z^\rho \delta\sigma^2\right] + o(\delta^3) \end{aligned}$$

Parallel transport along γ' leads to a similar expression (exchange of X and Y):

$$Z'^\mu_r = Z^\mu_p - \frac{1}{2}\Gamma^\mu_{\nu\rho,\sigma}(p)\left[Y^\nu Y^\sigma Z^\rho \delta\sigma^2 + 2X^\nu Z^\rho Y^\sigma \delta\sigma\delta\tau + X^\nu X^\sigma Z^\rho \delta\tau^2\right] + o(\delta^3)$$

The difference between the parallely transported tangent vectors to leading order is:

$$\Delta Z^\mu_r = Z^\mu_r - Z'^\mu_r = -\left[\Gamma^\mu_{\nu\rho,\sigma} - \Gamma^\mu_{\sigma\rho,\nu}\right]_p \left[Y^\nu Z^\rho X^\sigma\right]_p \delta\tau\delta\sigma + o(\delta^3)$$

Recalling that $\Gamma^\mu_{\nu\rho}(p) = 0$, it is possible to write:

$$\Delta Z^\mu_r = -\left[R^\mu{}_{\rho\sigma\nu} Y^\nu Z^\rho X^\sigma\right]_p \delta\sigma\delta\tau + o(\delta^3) \tag{2.58}$$

It would be possible to evaluate the expression at r too, as it would differ only by higher order terms. Although the calculation was carried in a particular choice of coordinates, Eq. 2.58 is a tensor relation, therefore it must hold in all coordinate systems: the Riemann tensor thus determines the path dependence of parallel transport.

§2.3.2.1 Torsion

Consider two tangent vectors $X_p, Y_p \in T_p\mathcal{M}$ and a coordinate system x^μ such that $X_p = X^\mu\partial_\mu$ and $Y_p = Y^\mu\partial_\mu$. If $p : x^\mu$, as in Fig. 2.3 construct $r, s \in \mathcal{M}$ such that $r : x^\mu + \varepsilon X^\mu$ and

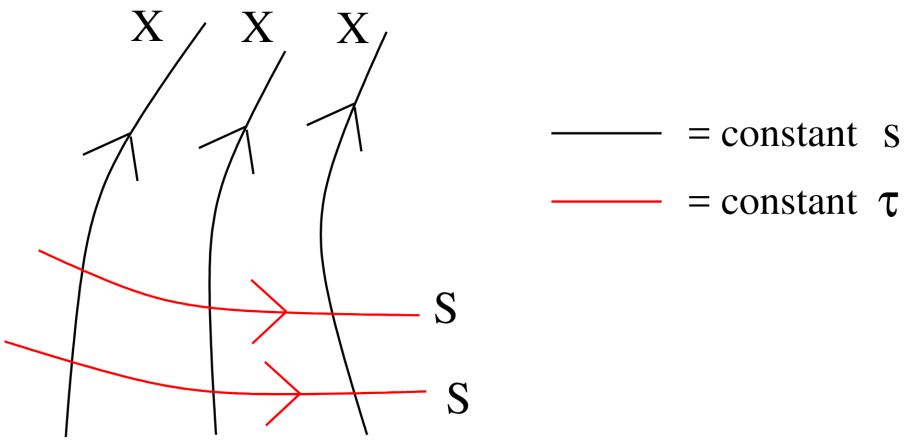

Figure 2.4: A one-parameter family of geodesics generated by X.

$s : x^\mu + \varepsilon Y^\mu$, with ε an infinitesimal parameter. Now, parallel transport $X_p \in T_p\mathcal{M}$ along the direction of Y_p to $X'_s \in T_s\mathcal{M}$ and $Y_p \in T_p\mathcal{M}$ along X_p to $Y'_r \in T_r\mathcal{M}$; their components will be:

$$X'_s = \left(X^\mu - \varepsilon\Gamma^\mu_{\nu\rho}Y^\nu X^\rho\right) \qquad\qquad Y'_r = \left(Y^\mu - \varepsilon\Gamma^\mu_{\nu\rho}X^\nu Y^\rho\right)$$

Repeating this process, starting from point s and moving along the direction of X'_s, a new point $q \in \mathcal{M}$ is determined, with coordinates:

$$q : x^\mu + \varepsilon\left(X^\mu + Y^\mu\right) - \varepsilon^2\Gamma^\mu_{\nu\rho}Y^\nu X^\rho$$

Analogously, starting at point r and moving along the direction of Y'_r, a new point $t \in \mathcal{M}$ is determined, with coordinates:

$$t : x^\mu + \varepsilon\left(X^\mu + Y^\mu\right) - \varepsilon^2\Gamma^\mu_{\nu\rho}X^\nu Y^\rho$$

If the connection is torsion-free, then $q \equiv t$. On the other hand, if $T^\mu_{\nu\rho} \neq 0$, the parallelogram fails to close, as in Fig. 2.3.

§2.3.3 Geodesic deviation

Definition 2.3.3 (Tangent and deviation vector fields)

Given a one-parameter family of geodesics $\{x^\mu(\tau; s)\}_{s\in\mathbb{R}}$ on a manifold $\mathcal{M}$, the **tangent vector field** and the **deviation vector field** are defined as:

$$X^\mu := \left.\frac{\partial x^\mu}{\partial\tau}\right|_s \qquad\qquad S^\mu := \left.\frac{\partial x^\mu}{\partial s}\right|_\tau \tag{2.59}$$

The meaning of these vector fields is evident: the tangent vector field fixes a particular geodesics (i.e. a particular s) and assigns at each point of the geodesic its tangent vector, while the deviation vector field fixes a particular value of the affine parameter τ and assigns at each point with this value a vector which takes to a nearby geodesic (at the same τ).
The family of geodesics sweeps a surface embedded in the manifold, so there is freedom in the choice of coordinates s and τ. In particular, it's always possible to pick them so that $X = \frac{\partial}{\partial\tau}$ and $S = \frac{\partial}{\partial s}$, in order for them to be linearly independent: $[X, S] = 0$, as in Fig. 2.4. Consider a torsion-free connection, so that:

$$\nabla_X S - \nabla_S X = [X, S] = 0 \quad\implies\quad \nabla_X\nabla_X S = \nabla_X\nabla_S X = \nabla_S\nabla_X X + \mathsf{R}(X, S)X$$

But X is tangent to geodesics, so from Eq. 2.56 $\nabla_X X = 0$ and:

$$\nabla_X \nabla_X S = \mathsf{R}(X, S)X \tag{2.60}$$

Restricting to an integral curve γ of the vector field X, i.e. $X^\mu|_\gamma = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, the covariant derivative along γ becomes:

$$\nabla_X\big|_\gamma = X^\mu\big|_\gamma \nabla_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\nabla_\mu \equiv \frac{D}{D\tau}$$

Hence, in index notation, the change of the deviation vector along the geodesic is expressed as:

$$\frac{D^2 S^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} X^\rho S^\sigma X^\nu \tag{2.61}$$

This can be interpreted as the relative acceleration of neighbouring geodesics, and it is determined by the Riemann tensor. Experimentally, such geodesic deviations are called **tidal forces**, which are the non-local manifestation of a gravitational field as a curvature of spacetime.

§2.4 Riemann tensor

Recall Eq. 2.37 for the components of the Riemann tensor $R^{\sigma}{}_{\rho\mu\nu}$: it is manifestly anti-symmetric in its last two indices, but there are other subtle symmetries when using the Levi–Civita connection.

Proposition 2.4.1 (Symmetries of the Riemann tensor)

On a metric manifold with a Levi–Civita connection:

$$R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu} = -R_{\rho\sigma\mu\nu} = R_{\mu\nu\sigma\rho} \qquad R_{\sigma[\rho\mu\nu]} = 0 \tag{2.62}$$

Proof. Set normal coordinates centered at a point p: then, $\Gamma^{\mu}_{\nu\rho} = 0$ and $\partial_\mu g^{\lambda\sigma} = 0$ at that point. At p, the Riemann tensor can be written as:

$$\begin{aligned} R_{\sigma\rho\mu\nu} = g_{\sigma\lambda} R^{\lambda}{}_{\rho\mu\nu} &= g_{\sigma\lambda}\left[\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho}\right] \\ &= \frac{1}{2}\left[\partial_\mu\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right) - \partial_\nu\left(\partial_\mu g_{\sigma\rho} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho}\right)\right] \\ &= \frac{1}{2}\left[\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\mu\rho}\right] \end{aligned}$$

The symmetries are then manifest, and being these tensor equations they are valid in all coordinate systems. □

An important computation tool is the Bianchi identity.

Theorem 2.4.1 (Bianchi identity)

On a metric manifold with a Levi–Civita connection:

$$\nabla_{[\lambda} R_{\sigma\rho]\mu\nu} = 0 \qquad \Longleftrightarrow \qquad R^{\sigma}{}_{\rho[\mu\nu;\lambda]} = 0 \tag{2.63}$$

Proof. The two equations are equivalent, so the proof is of the first one. In normal coordinates $\nabla_\mu = \partial_\mu$ at p, so schematically: $R = \partial\Gamma + \Gamma\Gamma$, thus $\nabla R = \partial R = \partial^2\Gamma + \Gamma\partial\Gamma = \partial^2\Gamma$. Explicitly:

$$\partial_\lambda R_{\sigma\rho\mu\nu} = \frac{1}{2}\partial_\lambda\left[\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\mu\rho}\right]$$

Anti-symmetrizing the appropriate indices yields the result. □

Note that Eq. 2.62-2.63 do not require that the connection is a Levi–Civita connection, but are valid for general torsion-free connections.

§2.4.1 Ricci and Einstein tensors

Definition 2.4.1 (Ricci tensor and scalar)

On a metric manifold, the **Ricci tensor** is defined as:

$$R_{\mu\nu} := R^{\rho}{}_{\mu\rho\nu} \tag{2.64}$$

The **Ricci scalar** is defined as:

$$R := g^{\mu\nu} R_{\mu\nu} \tag{2.65}$$

Proposition 2.4.2 (Symmetry)
On a metric manifold with a Levi–Civita connection:

$$R_{\mu\nu} = R_{\nu\mu} \tag{2.66}$$

Proof. Using Eq. 2.62: $R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu} = g^{\rho\sigma} R_{\rho\nu\sigma\mu} = R_{\nu\mu}$. □

Proposition 2.4.3
On a metric manifold:

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R \tag{2.67}$$

Proof. Writing explicitly Bianchi identity:

$$\nabla_\lambda R_{\sigma\rho\mu\nu} + \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\rho R_{\lambda\sigma\mu\nu} = 0$$

Contracting with $g^{\mu\lambda} g^{\rho\nu}$:

$$\nabla^\mu R_{\mu\sigma} - \nabla_\sigma R + \nabla^\nu R_{\nu\sigma} = 0$$

which yields the thesis. □

An important tensor obtained as a combination of the Ricci tensor and scalar is the **Einstein tensor**, which encodes information on the curvature of the metric manifold on which it is defined:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.68}$$

Most importantly, the Einstein tensor is covariantly conserved, as of Eq. 2.67:

$$\nabla^\mu G_{\mu\nu} = 0 \tag{2.69}$$

§2.4.2 Connection and curvature forms

This section focuses on a Lorentzian manifold, but the discussion is equivalent on a Riemannian one: it is sufficient to swap η_{ab} with δ_{ab} as the flat metric.

§2.4.2.1 Vielbeins

Although typically calculations are carried on a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, there are possible bases without such an interpretation. For example, a linear combination of a coordinate basis is not in general a coordinate basis itself:

$$\hat{e}_a = e_a{}^\mu \partial_\mu \tag{2.70}$$

A particularly useful non-coordinate basis is one such that:

$$g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab} \tag{2.71}$$

The components ${e_a}^\mu$ are called **vielbeins** or *tetrads*. In this non-coordinate system, the manifold looks flat (or, at least, its patch covered by the given chart). In the following computations, Greek indices are raised/lowered by the metric $g_{\mu\nu}$, while Latin indices by the metric η_{ab}.
The vielbeins are not unique, for given a set of vielbeins ${e_a}^\mu$ it is always possible to find a new one:

$$\tilde{e}_a{}^\mu = e_b{}^\mu (\Lambda^{-1})^b{}_a \tag{2.72}$$

The transformation matrix must satisfy the condition imposed by Eq. 2.71, i.e.:

$$\Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab} \tag{2.73}$$

These are **local Lorentz transformations**, because the condition is that of Lorentz transformation, but Λ is now allowed to vary over the manifold. The dual basis of one-forms $\{\hat{\theta}^a\}$ is defined by $\hat{\theta}^a(\hat{e}_b) = \delta^a_b$. The relation to the coordinate basis is:

$$\hat{\theta}^a = e^a{}_\mu \mathrm{d}x^\mu \tag{2.74}$$

where the coefficients satisfy $e^a{}_\mu e_b{}^\mu = \delta^a_b$ and $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$. The metric is a tensor, therefore $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$, thus it is related to the vielbeins by:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \tag{2.75}$$

§2.4.2.2 Connection 1-form

On a non-coordinate basis $\{\hat{e}_a\}$, connection components are computed in the usual way:

$$\nabla_{\hat{e}_c}\hat{e}_b = \Gamma^a_{cb}\hat{e}_a \tag{2.76}$$

However, these are not the same components $\Gamma^\mu_{\nu\rho}$ as in a coordinate basis.

Definition 2.4.2 (Connection 1-form)
On a metric manifold with vielbeins, the **connection 1-form** is defined as:

$$\omega^a{}_b := \Gamma^a_{cb}\hat{\theta}^c \tag{2.77}$$

Note that these are really n^2 1-forms, according to values of $a, b = 1, \ldots, n \equiv \dim_{\mathbb{R}} \mathcal{M}$. This is also known as the *spin connection*, due to its relationship to spinors in curved spacetime.

Proposition 2.4.4 (Transformation law)
Given a local Lorentzian transformation Λ:

$$\tilde{\omega}^a{}_b = \Lambda^a{}_c \omega^c{}_d (\Lambda^{-1})^d{}_b + \Lambda^a{}_c (d\Lambda^{-1})^c{}_b \tag{2.78}$$

The second term reflects the second term in Prop. 2.2.1, which involves the derivative of the coordinate transformation.
The curvature of a metric manifold can also be studied through Cartan's structure equations, which encode the curvature information into forms.

Theorem 2.4.2 (First Cartan structure equation)

On a metric manifold with a torsion-free connection:

$$d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0 \tag{2.79}$$

Proof. From Eq. 2.76 $\Gamma^a_{cb} = e^a{}_\rho e_c{}^\mu \nabla_\mu e_b{}^\rho$, thus, remembering $\Gamma^\rho_{[\mu\nu]} = 0$ (torsion-free):

$$\begin{aligned}
\omega^a{}_b \wedge \hat{\theta}^b &= \Gamma^a_{cb} \left(e^c{}_\mu dx^\mu\right) \wedge \left(e^b{}_\nu dx^\nu\right) \\
&= e^a{}_\rho e_c{}^\mu \left(\partial_\mu e_b{}^\rho + e_b{}^\nu \Gamma^\rho_{\mu\nu}\right) \left(e^c{}_\mu dx^\mu\right) \wedge \left(e^b{}_\nu dx^\nu\right) \\
&= e^a{}_\rho \underbrace{e_c{}^\lambda e^c{}_\mu}_{\delta^\lambda_\mu} e^b{}_\nu \left(\partial_\lambda e_b{}^\rho + e_b{}^\sigma \Gamma^\rho_{\lambda\sigma}\right) dx^\mu \wedge dx^\nu = e^a{}_\rho e^b{}_\nu \partial_\mu e_b{}^\rho dx^\mu \wedge dx^\nu
\end{aligned}$$

But $e^b{}_\nu e_b{}^\rho = \delta^\rho_\nu$, so $e^b{}_\nu \partial_\mu e_b{}^\rho = -e_b{}^\rho \partial_\mu e^n{}_\nu$, hence:

$$\omega^a{}_b \wedge \hat{\theta}^b = -e^a{}_\rho e_b{}^\rho \partial_\mu e^b{}_\nu dx^\mu \wedge dx^\nu = -\partial_\mu e^a{}_\nu dx^\mu \wedge dx^\nu = -d\hat{\theta}^a$$

which is the structure equation. □

For a Levi–Civita connection, a stronger result holds.

Proposition 2.4.5

On a metric manifold with a Levi–Civita connection:

$$\omega_{ab} = -\omega_{ba} \tag{2.80}$$

Proof. Being the Levi-Civita connection compatible with the metric:

$$\begin{aligned}
\Gamma_{abc} &= \eta_{ad} e^d{}_\rho e_b{}^\mu \nabla_\mu e_c{}^\rho = -\eta_{ad} e_c{}^\rho e_b{}^\mu \nabla_\mu e^d{}_\rho = -\eta_{cf} e^f{}_\sigma e_b{}^\mu \nabla_\mu (\eta_{ad} g^{\rho\sigma} e^d{}_\rho) \\
&= -\eta_{cf} e^f{}_\rho e_b{}^\mu \nabla_\mu e_a{}^\rho = -\Gamma_{cba}
\end{aligned}$$

From Eq. 2.77 $\omega_{ab} = \Gamma_{acb} \hat{\theta}^c$, thus completing the proof. □

Eq. 2.79-2.80 allow to quickly compute the spin connection, as they uniquely define it. Indeed, ω_{ab} being anti-symmetric means that there are $\frac{1}{2}n(n-1)$ independent 1-forms, i.e. $\frac{1}{2}n^2(n-1)$ independent components. The Cartan structure equation relates two 2-forms, each with $\frac{1}{2}n(n-1)$ independent components, thus posing $\frac{1}{2}n^2(n-1)$ constraints (as there are n equations) and uniquely fixing the spin connection.

§2.4.2.3 Curvature 2-form

Recall Eq. 2.62, which holds for Levi–Civita connections. Computing the Riemann tensor in the non-coordinate basis $R^a{}_{bcd} = \mathsf{R}(\hat{\theta}^a, \hat{e}_b, \hat{e}_c, \hat{e}_d)$, the anti-symmetry of the last two indices persists: $R^a{}_{bcd} = -R^a{}_{bdc}$.

Definition 2.4.3 (Curvature 2-form)

On a metric manifold with a Levi–Civita connection, the **curvature 2-form** is defined as:

$$\mathcal{R}^a{}_b := \frac{1}{2} R^a{}_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d \tag{2.81}$$

Again, these are really n^2 2-forms. A second Cartan structure relation holds.

Theorem 2.4.3 (Second Cartan structure equation)
On a metric manifold with a Levi–Civita connection:

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.82}$$

Connection and curvature forms make computing the Riemann tensor less tedious, as exterior derivatives take significant less effort than covariant derivatives.

Part II

General Relativity

Chapter 3

Geometrodynamics

The force of gravity is mediated by a gravitational field, which is identified with a metric $g_{\mu\nu}(x)$ on a 4-dimensional Lorentzian manifold called spacetime. This metric is a dynamical object, as all other fields in Nature, thus the laws governing its dynamics must be studied.

§3.1 Einstein-Hilbert action

Differential Geometry places strict limits to the possible actions that can be formulated, ensuring it is something intrinsic to the metric and independent on the choice of coordinates. Spacetime is a Lorentzian manifold $\mathcal{M}$, thus, recalling the canonical volume form in Eq. 2.13, there needs to be a metric-dependent scalar function on $\mathcal{M}$: the obvious non-trivial choice is the Ricci scalar. The resulting action is the **Einstein–Hilbert action**:

$$\mathcal{S} = \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\, R \tag{3.1}$$

where the negative sign makes it manifest that the metric has signature $(-,+,+,+)$. Schematically, the Riemann tensor is $R \sim \partial\Gamma + \Gamma\Gamma$, while $\Gamma \sim \partial g$, thus the Einstein-Hilbert action is second order in derivatives of the metric, like most other actions in physics.

§3.1.1 Equations of motion

To determine the Euler-Lagrange equations of the Einstein-Hilbert action, consider a fixed initial metric $g_{\mu\nu}(x)$ and a perturbation $g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$. For the inverse metric:

$$g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu \quad\implies\quad \delta g_{\mu\nu}g^{\nu\rho} + g_{\mu\nu}\delta g^{\nu\rho} = 0 \quad\implies\quad \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$$

Lemma 3.1.1

The variation of $\sqrt{-\mathrm{g}}$ is:

$$\delta\sqrt{-\mathrm{g}} = -\frac{1}{2}\sqrt{-\mathrm{g}}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{3.2}$$

Proof. Recall that for invertible matrices $\log\det \mathrm{A} = \mathrm{tr}\log \mathrm{A}$, i.e. $(\det \mathrm{A})^{-1}\delta(\det \mathrm{A}) = \mathrm{tr}(\mathrm{A}^{-1}\delta\mathrm{A})$. Applying this to the metric:

$$\delta\sqrt{-\mathrm{g}} = \frac{1}{2}\frac{1}{\sqrt{-\mathrm{g}}}\delta(-g) = \frac{1}{2}\frac{1}{\sqrt{-\mathrm{g}}}(-g)g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-\mathrm{g}}\, g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-\mathrm{g}}\, g_{\mu\nu}g^{\mu\nu}$$

which is the thesis. □

Lemma 3.1.2

The variation of the Christoffel symbols is:

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu}) \tag{3.3}$$

Proof. First note that, although $\Gamma^{\rho}_{\mu\nu}$ is not a tensor, $\delta\Gamma^{\rho}_{\mu\nu}$ is, for it is the difference of two Christoffel symbols, one computed with $g_{\mu\nu}$ and the other with $g_{\mu\nu} + \delta g_{\mu\nu}$, but the extra term in the transformation law of $\Gamma^{\rho}_{\mu\nu}$ is independent of the metric, thus cancels out.
This observation allows working in normal coordinates at $p \in \mathcal{M}$, so that $\partial_\rho g_{\mu\nu} = 0$ and $\Gamma^{\rho}_{\mu\nu} = 0$ at that point. Then, to linear order in the variation, at p:

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu \delta g_{\sigma\nu} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu}) = \frac{1}{2} g^{\rho\sigma}(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu})$$

This is a tensor equation, hence valid in all coordinate system and, being p arbitrary, on all $\mathcal{M}$. □

Lemma 3.1.3

The variation of the Ricci tensor is:

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}_{\rho\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\rho\mu} \tag{3.4}$$

Proof. Working in normal coordinates, the Riemann tensor becomes $R^{\sigma}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\sigma}_{\nu\mu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu}$, so:

$$\delta R^{\sigma}{}_{\mu\rho\nu} = \partial_\rho \delta\Gamma^{\sigma}_{\nu\mu} - \partial_\nu \delta\Gamma^{\sigma}_{\rho\mu} = \nabla_\rho \delta\Gamma^{\sigma}_{\nu\mu} - \nabla_\nu \delta\Gamma^{\sigma}_{\rho\mu}$$

This is a tensor equation, hence valid in all coordinates systems and on all $\mathcal{M}$. Contracting indices σ, ρ and working to leading order yields the result. □

Proposition 3.1.1 (Einstein field equations in vacuum)

The Euler-Lagrange equations of the Einstein-Hilbert action are:

$$G_{\mu\nu} = 0 \tag{3.5}$$

Proof. Varying the Einstein-Hilbert action:

$$\begin{aligned}
\delta\mathcal{S} &= \delta \int d^4x \sqrt{-\mathrm{g}}\, g^{\mu\nu} R_{\mu\nu} \\
&= \int d^4x \left[\delta(\sqrt{-\mathrm{g}}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-\mathrm{g}}(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-\mathrm{g}}\, g^{\mu\nu}(\delta R_{\mu\nu})\right] \\
&= \int d^4x \sqrt{-\mathrm{g}} \left[\left(-\frac{1}{2} R g_{\mu\nu} + R_{\mu\nu}\right) \delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_\rho \delta\Gamma^{\rho}_{\mu\nu} - \nabla_\nu \Gamma^{\rho}_{\rho\mu}\right)\right] \\
&= \int d^4x \sqrt{-\mathrm{g}} \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) \delta g^{\mu\nu} + \nabla_\mu \left(g^{\rho\nu} \nu \Gamma^{\mu}_{\rho\nu} - g^{\mu\nu} \delta\Gamma^{\rho}_{\rho\nu}\right)\right]
\end{aligned}$$

The last term is a total derivative, hence by the divergence theorem Eq. 2.43 it yields a boundary term which can be ignored (Gibbons-Hawking boundary term). The Euler-Lagrange equations are then found imposing $\delta\mathcal{S} = 0$ for arbitrary $\delta g_{\mu\nu}$, so recalling the definition of the Einstein tensor Eq. 2.68 the proof is completed. □

These equations are the Einstein field equations in vacuum, which govern the geometrodynamics of spacetime in the absence of any matter. For this reason, they can be further simplified: by contracting with $g^{\mu\nu}$ one finds $R = 0$, thus:

$$R_{\mu\nu} = 0 \tag{3.6}$$

§3.1.1.1 Dimensional analysis

The Einstein-Hilbert action in Eq. 3.1 does not have the right dimensions, which will be necessary when considering the presence of matter. If x^μ has dimension of length, then $g_{\mu\nu}$ is dimensionless and the Ricci scalar is $[R] = L^{-2}$. Including the integration measure, $[\mathcal{S}] = L^2$, but it must have the same dimension of $[\hbar] = ML^2T^{-1}$ (i.e. energy $\times$ times). Thus, the action with the right dimension is:

$$\mathcal{S} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\, R \tag{3.7}$$

In the following, natural units are adopted: $c = \hbar = 1$.

§3.1.1.2 Cosmological constant

In reality, Eq. 3.7 is not the simplest action allowed by Differential Geometry: in fact, a constant term could be added to the Ricci scalar. The action then becomes:

$$\mathcal{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\,(R - 2\Lambda) \tag{3.8}$$

Λ is a **cosmological constant** and has dimension $[\Lambda] = L^{-2}$. The resulting field equations are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{3.9}$$

Contracting with $g^{\mu\nu}$ yields $R = 4\Lambda$, thus in the absence of matter:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{3.10}$$

§3.1.2 Diffeomorphisms

Being the metric a symmetric $\mathbb{R}^{4\times4}$ matrix, it should have $\frac{1}{2}\times 4\times 5 = 10$ degrees of freedom. However, not all the components of $g_{\mu\nu}$ are physical. Indeed, two metrics related by a change of coordinates $x^\mu \mapsto \tilde{x}^\mu(x)$ describe the same physical spacetime, thus there is a redundancy in any given representation of the metric, which removes precisely 4 degrees of freedom, leaving only 6 physical degrees of freedom.
Mathematically, given a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, it maps all fields on $\mathcal{M}$ to a new set of fields on $\mathcal{M}$: the result is physically indistinguishable from the original, describing the same spacetime but in different coordinates. Such diffeomorphisms are analogous to gauge symmetries in Yang-Mills theory.
Consider a diffeomorphism $x^\mu \mapsto \tilde{x}^\mu = x^\mu + \delta x^\mu$: this can be viewed as an active change, where points in spacetime are mapped from one another, or as a passive change, in which only the

coordinates of each point are affected, but the two interpretations are equivalent. This change of coordinates can be thought as generated by an infinitesimal vector field $X^\mu : \delta x^\mu = -X^\mu(x)$, so that the metric transforms as:

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) = \left(\delta^\rho_\mu + \partial_\mu X^\rho\right)\left(\delta^\sigma_\nu + \partial_\nu X^\sigma\right) g_{\rho\sigma}(x) \\ &= g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g_{\nu\rho}(x)\partial_\mu X^\rho\end{aligned}$$

Meanwhile, the Taylor expansion around $\tilde{x} = x + \delta x$ is:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x + \delta x) = \tilde{g}_{\mu\nu}(x) - X^\lambda \partial_\lambda \tilde{g}_{\mu\nu}(x)$$

Comparing the metrics at the same point, it is understood that it undergoes an infinitesimal change:

$$\delta g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = X^\lambda \partial_\lambda g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g_{\nu\rho}(x)\partial_\mu X^\rho$$

By Eq. 1.27, this is the Lie derivative of the metric:

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu} \tag{3.11}$$

Thus, an infinitesimal diffeomorphism along $X \in \mathfrak{X}(\mathcal{M})$ makes the metric change by an infinitesimal amount given by its Lie derivative along X, which can be viewed as the leading term in the Taylor expansion along X. These equations can be rewritten in a simpler form:

$$\delta g_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \partial_\nu X_\mu - X^\rho \partial_\nu g_{\mu\rho} + \partial_\mu X_\nu - X^\rho \partial_\mu g_{\nu\rho} = \partial_\mu X_\nu + \partial_\nu X_\mu + 2 g_{\rho\sigma} \Gamma^\sigma_{\mu\nu} X^\rho$$

Therefore:

$$\delta g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu \tag{3.12}$$

This equation can be used in the path integral. In fact, insisting that $\delta\mathcal{S} = 0$ for any $\delta g_{\mu\nu}$ gives the equations of motion; on the other hand, those variations $\delta g_{\mu\nu}$ for which $\delta\mathcal{S} = 0$ for any metric are the *symmetries* of the action. From Prop. 3.1.1:

$$\delta\mathcal{S} = \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\, G^{\mu\nu} \delta g_{\mu\nu} = 2 \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\, G^{\mu\nu} \nabla_\mu X_\nu$$

Invariance for $x^\mu \mapsto x^\mu - X^\mu$ means that $\delta\mathcal{S} = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$, hence, integrating by parts:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{3.13}$$

This is the Bianchi identity: although it is a result from Differential Geometry, is follows from diffeomorphism invariance of the Einstein-Hilbert action. This identity contains 4 equations, which make the 10 Einstein equations not completely independent: in fact, only 6 of them are independent, the same number of independent components of the metric, thus ensuring that the field equations are not overdetermined.

§3.2 Simple solutions

The Einstein equations in the absence of matter are $R_{\mu\nu} = \Lambda g_{\mu\nu}$, with $\Lambda \in \mathbb{R}$.

§3.2.1 Minkowski space

The simplest case is $\Lambda = 0$. The Einstein equations then reduce to $R_{\mu\nu} = 0$, with the condition of $g_{\mu\nu}$ being non-degenerate, since it is a metric and the field equations require the existence of the inverse metric $g^{\mu\nu}$. This restriction is physically unusual: it is not a holonomic constraint on the physical degrees of freedom, but an inequality $\det g < 0$ (with required signature $(-,+,+,+)$) which is not found for other fields of the Standard Model.
The simplest Ricci flat metric is Minkowski space $ds^2 = -dt^2 + d\mathbf{x}^2$, but it is not the only metric obeying $R_{\mu\nu} = 0$. Another example is Schwarzschild metric.

§3.2.2 de Sitter space

Consider $\Lambda > 0$. A possible ansatz is:

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

§3.2.2.1 Riemann tensor

First, one needs to compute the Ricci tensor for this metric: a simple way is using the curvature form. The non-coordinate 1-forms satisfy $ds^2 = \eta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$, thus:

$$\hat{\theta}^0 = f dt \qquad \hat{\theta}^1 = f^{-1} dr \qquad \hat{\theta}^2 = r d\vartheta \qquad \hat{\theta}^3 = r\sin\vartheta d\varphi$$

$$d\hat{\theta}^0 = f' dr \wedge dt \qquad d\hat{\theta}^1 = 0 \qquad d\hat{\theta}^2 = dr \wedge d\vartheta \qquad d\hat{\theta}^3 = \sin\vartheta dr \wedge d\varphi + r\cos\vartheta d\vartheta \wedge d\varphi$$

The first Cartan structure relation Eq. **??**, together with Eq. **??**, allow to determine the connection 1-form. For example, the first equation is ${\omega^0}_1 = f' f dt = f' d\hat{\theta}^0$, and then ${\omega^1}_0 = \omega_{10} = -\omega_{01} = {\omega^0}_1$. Using the other structure equation, the only non-vanishing components of the connection 1-form are:

$${\omega^0}_1 = {\omega^1}_0 = f'\hat{\theta}^0 \qquad {\omega^2}_1 = -{\omega^1}_2 = \frac{f}{r}\hat{\theta}^2 \qquad {\omega^3}_1 = -{\omega^1}_3 = \frac{f}{r}\hat{\theta}^3 \qquad {\omega^3}_2 = -{\omega^2}_3 = \frac{\cot\theta}{r}\hat{\theta}^3$$

The curvature 2-form can be computed from the second Cartan structure relation Eq. **??**. For example, ${\mathcal{R}^0}_1 = d{\omega^0}_1 + {\omega^0}_c \wedge {\omega^c}_1$, but ${\omega^0}_c \wedge {\omega^c}_1 = {\omega^0}_1 \wedge {\omega^1}_1 = 0$, thus ${\mathcal{R}^0}_1 = d{\omega^0}_1 = ((f')^2 + f'' f) dr \wedge dt$. From the curvature 2-form, the Riemann tensor can be calculated via Eq. **??** (recall the anti-symmetries of the Riemann tensor), finding the only non-vanishing components:

$$R_{0101} = f f'' + (f')^2 \qquad R_{0202} = R_{0303} = \frac{f f'}{r} \qquad R_{1212} = R_{1313} = -\frac{f f'}{r} \qquad R_{2323} = \frac{1 - f^2}{r^2}$$

To convert back to $x^\mu = (t, r, \vartheta, \varphi)$, use $R_{\mu\nu\rho\sigma} = {e^a}_\mu {e^b}_\nu {e^c}_\rho {e^d}_\sigma R_{abcd}$, which is particularly easy given that the matrices ${e_a}^\mu$ which define the non-coordiante 1-forms are diagonal:

$$R_{trtr} = f(r) f''(r) + f'(r)^2 \qquad R_{t\vartheta t\vartheta} = f(r)^3 f'(r) r \qquad R_{t\varphi t\varphi} = f(r)^3 f'(r) r \sin^2\vartheta$$

$$R_{r\vartheta r\vartheta} = -\frac{f'(r) r}{f(r)} \qquad R_{r\varphi r\varphi} = -\frac{f'(r) r}{f(r)} \sin^2\vartheta \qquad R_{\vartheta\varphi\vartheta\varphi} = (1 - f(r)^2) r^2 \sin^2\vartheta$$

§3.2.2.2 Ricci tensor

Given the Riemann tensor, it is easy to check that the Ricci tensor is diagonal with components:

$$R_{tt} = -f(r)^4 R_{rr} = f(r)^3 \left[f''(r) + \frac{2f'(r)}{r} + \frac{f'(r)^2}{f(r)} \right]$$

$$R_{\varphi\varphi} = \sin^2\vartheta R_{\vartheta\vartheta} = (1 - f(r)^2 - 2f(r)f'(r)r)\sin^2\vartheta$$

Imposing $R_{\mu\nu} = \Lambda g_{\mu\nu}$ determines two constraints on $f(r)$:

$$f''(r) + \frac{2f'(r)}{r} + \frac{f'(r)^2}{f(r)} = -\frac{\Lambda}{f(r)} \qquad 1 - 2f(r)f'(r)r - f(r)^2 = \Lambda r^2$$

A solution is:

$$f(r) = \sqrt{1 - \frac{r^2}{R^2}} \qquad R^2 \equiv \frac{3}{\Lambda}$$

This determines the metric of *de Sitter space*:

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{3.14}$$

More precisely, this is the *static patch* of de Sitter space.

§3.2.2.3 de Sitter geodesics

To interpret the metric, it's useful to study its geodesics. First, note that a non-trivial $g_{tt}(r)$ term means that a particle won't remain at rest at $r \neq 0$, but it will be pushed to smaller values of $g_{tt}(r)$, i.e. larger values of r. The action of a particle in the general $f(r)$ metric, parametrized by its proper time σ, is:

$$\mathcal{S} = \int d\sigma \left[-f(r)^2 \dot{t}^2 + f(r)^{-2}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta \varphi^2) \right] \tag{3.15}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\sigma}$. This Lagrangian has two ignorable degrees of freedom which lead to conserved quantities: $t(\sigma)$ and $\varphi(\sigma)$, as they appear only with time derivatives. The first one has energy as the conserved quantity, while the second has angular momentum:

$$E = -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} = f(r)^2 \dot{t} \tag{3.16}$$

$$\ell = \frac{1}{2}\frac{\partial L}{\partial \dot{\varphi}} = r^2 \sin^2\vartheta \dot{\varphi} \tag{3.17}$$

The $\frac{1}{2}$ are due to the absence of the usual factor in the kinetic term. The equations of motion from the action Eq. 3.15 should be supplemented with a constraint to distinguish whether a particle is massive or massless. For a massive particle, the trajectory must be timelike, so:

$$-f(r)^2 \dot{t}^2 + f(r)^{-2}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta \dot{\varphi}^2) = -1$$

WLOG consider geodeiscs that lie in the $\vartheta = \frac{\pi}{2}$ plane, so that $\dot{\theta} = 0$ and $\sin^2\vartheta = 1$. Replacing (t, φ) with (E, ℓ), the constraint becomes:

$$\dot{r}^2 + V_{\text{eff}}(r) = E^2 \qquad V_{\text{eff}}(r) = \left(1 + \frac{\ell^2}{r^2}\right) f(r)^2 \tag{3.18}$$

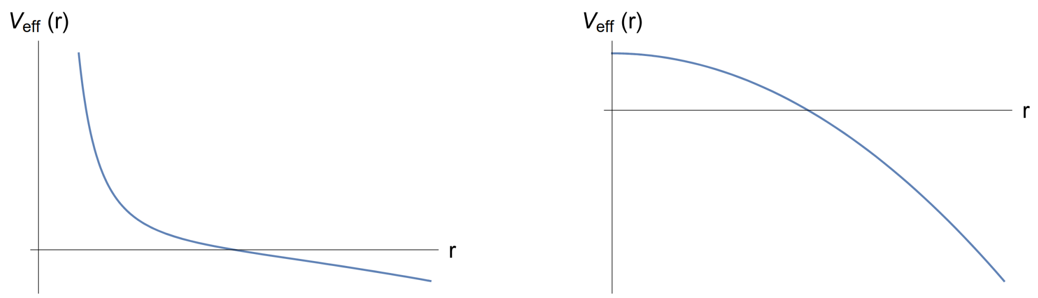

Figure 3.1: Effective potential for de Sitter geodesics, with $\ell \neq 0$ and $\ell = 0$ respectively.

For de Sitter space:

$$V_{\text{eff}}(r) = \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{r^2}{R^2}\right)$$

This effective potential is plotted in Fig. 3.1: immediately, one sees that the potential pushes the particle to larger values of r. Focusing on geodesics with $\ell = 0$, the potential is a harmonic repulsor: a particle stationary at $r = 0$ is an unstable geodesic, for if it has non-zero initial velocity it will follow the trajectory:

$$r(\sigma) = R\sqrt{E^2 - 1}\sinh\frac{\sigma}{R} \tag{3.19}$$

Note that the metric is singular at $r = R$, a fact not manifest in the geodesic Eq. 3.19 which shows that any observer reaches $r = R$ in finite proper time. Problems arise when studying the coordinate time t, which has the interpretation of the time experienced by someone stationary at $r = 0$; from Eq. 3.16:

$$\frac{dt}{d\sigma} = E\left(1 - \frac{r^2}{R^2}\right)^{-1}$$

This shows that $t(\sigma) \to \infty$ as $r(\sigma) \to R$: in fact, if $r(\sigma^*) = R$, then for $\sigma = \sigma^* - \varepsilon$ one has $\frac{dt}{d\sigma} = -\frac{\alpha}{\varepsilon}$, i.e. $t(\varepsilon) = -\log(\varepsilon/R)$, so $t(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This means that a particle moving along the geodesic Eq. 3.19 will reach $r = R$ in finite proper time, but a stationary observer at $r = 0$ will measure an infinite amount of time.
This strange behaviour is similar to what happens at the horizon of a black hole ($r = 2GM$): however, the Schwarzschild metric has a singularity at $r = 0$, while de Sitter metric looks flat at $r = 0$. de Sitter space seems like an inverted black hole in which particles are pushed outwars to $r = R$.

§3.2.2.4 de Sitter embeddings

de Sitter space can be nicely embedded as a submanifold of $\mathbb{R}^{1,4}$ with metric:

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{4}(dX^i)^2 \tag{3.20}$$

In particular, de Sitter metric Eq. 3.14 is a metric on the submanifold of $\mathbb{R}^{1,4}$ defined by the timelike hyperboloid:

$$-(X^0)^2 + \sum_{i=1}^{4}(X^i)^2 = R^2 \tag{3.21}$$

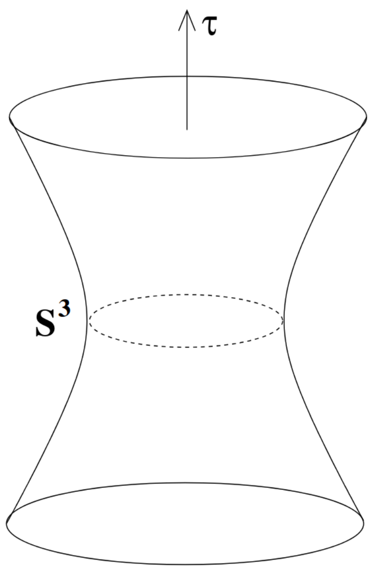

Figure 3.2: de Sitter space visualization with global coordinates.

A way of parametrizing this constraint is by imposing that $r^2 = (X^1)^2 + (X^2)^2 + (X^3)^2$, so that:

$$R^2 - r^2 = (X^4)^2 - (X^0)^2$$

The solutions are parametrized as:

$$X^0 = \sqrt{R^2 - r^2} \sinh \frac{t}{R} \qquad X^4 = \sqrt{R^2 - r^2} \cosh \frac{t}{R}$$

Computing the respective variations, along with $\sum_{i=1}^{3} (dX^i)^2 = dr^2 + r^2 d\Omega_2^2$, where $d\Omega_n^2$ is the metric element on $\mathbb{S}^n$, allows to show that the pull-back of the Minkowski metric Eq. 3.20 on the hyperboloid Eq. 3.21 so parametrized gives the de Sitter metric Eq. 3.14 in the static patch coordinates.

The coordinates $\{X^i\}_{i=0,\dots,4}$ so defined are not the most intuitive: they single out X^4 as special, when the constraint does no such thing, and they do not cover the whole hyperboloid, as they are limited to $X^4 \geq 0$. A better choice of parametrization is:

$$X^0 = R \sinh \frac{\tau}{T} \qquad X^i = y^i R \cosh \frac{\tau}{R} \; : \; \sum_{i=1}^{4} (y^i)^2 = 1$$

Given this constraint, $\{y^i\}_{i=1,,2,3,4}$ parametrize $\mathbb{S}^3$. These coordinates retain more of the symmetry of de Sitter space and cover the whole space, thus are a better parametrization. The pull-back of Minkowski metric, however, gives a rather different metric on de Sitter space:

$$ds^2 = -d\tau^2 + R^2 \cosh^2 \frac{\tau}{R} \, d\Omega_3^2 \tag{3.22}$$

These are known as *global coordinates*, as they cover the whole space (except for usual singularities at the poles for any choice of coordinates on $\mathbb{S}^3$). Since this metric is related to that in Eq. 3.14 by a change of coordinates, it too must obey the Einstein equations. Global coordinates also show that the singularity which happens at $X^4 = 0$, i.e. at $r = R$, is nothing but a coordinate singularity.

These coordinates provide a clearer intuition for the physics of de Sitter space: it is a time-dependent solution of the field equations in which a spatial $\mathbb{S}^3$ first shrinks to a minimal radius R and then expands, as shown in Fig. 3.2. The expansionary phase is a good approximation to our current universe on large scales.

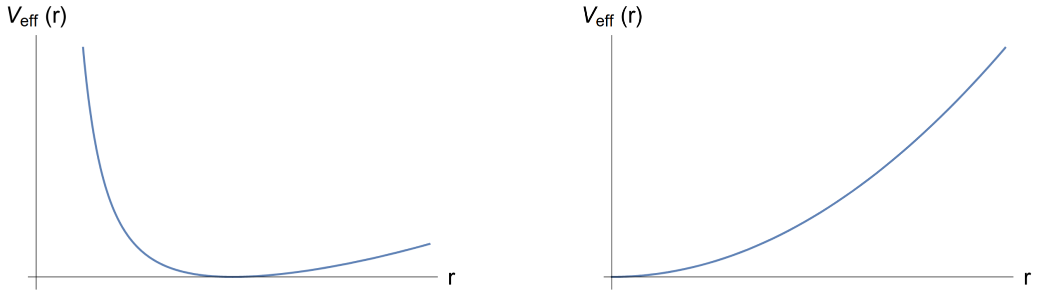

Figure 3.3: Effective potential for anti-de Sitter geodesics, with $\ell \neq 0$ and $\ell = 0$ respectively.

§3.2.3 Anti-de Sitter space

Consider $\Lambda < 0$. With the same ansatz as for de Sitter space, it's easy to find the metric for *anti-de Sitter space* (AdS):

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \qquad R^2 \equiv -\frac{3}{\Lambda} \tag{3.23}$$

Equivalent coordinates are found setting $r = R\sinh\rho$:

$$ds^2 = -\cosh^2\rho\, dt^2 + R^2 d\rho^2 + R^2\sinh^2\rho\, d\Omega_2^2 \tag{3.24}$$

In AdS there's no coordinate singularity in r and indeed now coordinates cover the whole space.

§3.2.3.1 Anti-de Sitter geodesics

AdS has the same action as in Eq. 3.15, thus the radial trajectory of a massive particle moving along a geodesic in the $\vartheta = \frac{\pi}{2}$ plane is $\dot{r}^2 + V_{\text{eff}}(r) = E^2$, with:

$$V_{\text{eff}}(r) = \left(1 + \frac{\ell^2}{R^2}\right)\left(1 + \frac{r^2}{R^2}\right) \tag{3.25}$$

Plotting it in Fig. 3.3, the geodesics' behavior is clear: with no angular momentum, anti-de Sitter space acts like a harmonic oscillator, pulling the particle towards the origin and making it oscillate around $r = 0$. If the particle has non-zero angular momentum, then the potential has a minimum at $r_*^2 = R\ell$, thus particles oscillate around r_*: importantly, particles with finite energy cannot escape to $r \to \infty$, but are constrained by the spacetime within some finite distance from the origin.

This picture of AdS as a harmonic trap which pulls particle to the origin clashes with the fact that AdS is a homogeneous space (roughly, all points are the same). To understand how these two facts are compatible, consider a stationary observer at $r = 0$: this is a geodesic and, from its perspective, other observers (with $\ell = 0$) will oscillate around $r = 0$ along geodesics. However, since these observers move along geodesics, in their reference frame they are stationary at $r = 0$, with all other observers oscillating. Thus, while in dS each observer views themselves at the center of the universe, with other observers moving away from them, in AdS each observer views themselves as the center of the universe, with other observer oscillating around them.

Its possible to study geodesics for a massless particle too. This time, the constraint to the action is:

$$-f(r)^2\dot{t}^2 + f(r)^{-2}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2) = 0$$

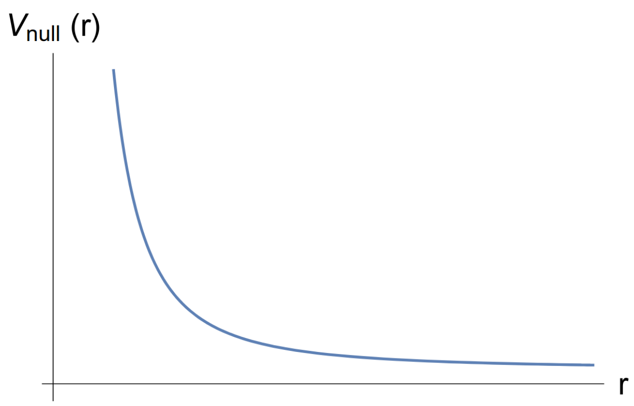

Figure 3.4: Potential experienced by massless particles with $\ell \neq 0$ in AdS.

This means that the particle follows a null geodesic. Its equation of motion is:

$$\dot{r}^2 + V_{\text{null}}(r) = E^2 \qquad V_{\text{null}}(r) = \frac{\ell^2}{r^2}\left(1 + \frac{r^2}{R^2}\right)$$

Its plot in Fig. 3.4 makes it clear that any massless particle can escape to $r \to \infty$, as the null potential is asymptotically constant, and it will only experience the usual gravitational redshift. AdS only confines massive particles. To solve for the trajectory, it's easier to work with $r = R\sinh\rho$ and $\ell = 0$, so that:

$$R\dot{\rho} = \pm\frac{E}{\cosh\rho} \qquad \Rightarrow \qquad R\sinh\rho(\sigma) = E(\sigma - \sigma_0)$$

One sees that $\rho \to \infty$ only in infinite affine time (i.e. $\sigma \to \infty$). However, in coordinate time, recalling Eq. 3.16, $E = \cosh^2\rho\,\dot{t}$, so:

$$R\tan\frac{t}{R} = E(\sigma - \sigma_0)$$

Hence, as affine time $\sigma \to \infty$, coordinate time $t \to \frac{\pi}{2}R$. This means that light rays escape to infinity in a finite amount of coordinate time: to make sense of dynamics in AdS, one must specify some boundary conditions at infinity to dictate how massless particles and field behave. AdS does not appear to be of any cosmological interest.

§3.2.3.2 Anti-de Sitter embeddings

AdS too has a natural embedding in a 5d spacetime: it is a submanifold of $\mathbb{R}^{2,3}$ with metric:

$$ds^2 = -(dX^0)^2 - (dx^4)^2 + \sum_{i=1}^{3}(dX^i)^2 \tag{3.26}$$

In particular, anti-de Sitter metric Eq. 3.23 is a metric on the hyperboloid:

$$-(X^0)^2 - (X^4)^2 + \sum_{i=1}^{3}(X^i)^2 = -R^2 \tag{3.27}$$

This constraint can be solved via the parametrization:

$$X^0 = R\cosh\rho\sin\frac{t}{R} \qquad X^4 = R\cosh\rho\cos\frac{t}{R} \qquad X^i = y^i R\sinh\rho\ :\ \sum_{i=1}^{3}(y^i)^2 = 1$$

Given this constraint, $\{y^i\}_{i=1,2,3}$ parametrize $\mathbb{S}^2$. The pull-back of the metric Eq. 3.26 on the hyperboloid so parametrized gives anti-de Sitter metric Eq. 3.24.
There's a small subtlety: the embedding hyperboloid has topology $\mathbb{S}^1 \times \mathbb{R}^3$, not $\mathbb{R}^4$: this corresponds to a compact time direction, as $t \in [0, 2\pi R)$. However, AdS metric does not have such restriction, with $t \in \mathbb{R}$: this is a universal covering of the hyperboloid.
Another useful parametrization of the hyperboloid is the following:

$$X^4 - X^3 = r \qquad X^4 + X^3 = \frac{R^2}{r} + \frac{r}{R^2}\eta_{ij}dx^i dx^j \qquad X^i = \frac{r}{R}x^i \,,\; i = 0,1,2$$

with $r \in [0, \infty)$. With these coordinates, the metric takes the form:

$$ds^2 = R^2 \frac{dr^2}{r^2} + \frac{r^2}{R^2}\eta_{ij}dx^i dx^j$$

These coordinates don't cover the whole AdS, but only one-half of the hyperboloid, restricted to $X^4 - X^3 > 0$: this is known as the *Poincaré patch* of AdS. Moreover, x^0 cannot be further extended beyond $x^0 \in (-\infty, +\infty)$, thus in global coordinates the restriction $t \in [0, 2\pi R)$ remains. In this case, massive particles fall towards $r = 0$.
Finally, there are two more possible coordinate systems on the Poincaré patch, obtained by setting $z = \frac{R^2}{r}$ and $r = Re^\rho$:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \eta_{ij}dx^i dx^j\right) \qquad\qquad ds^2 = R^2 d\rho^2 + e^{2\rho}\eta_{ij}dx^i dx^j$$

In each case, massive particles fall towards $z = \infty$ or $\rho = -\infty$.

§3.3 Symmetries

What makes Minkowski, dS and AdS special solutions to the Einstein equations are their symmetries.
The symmetries of Minkowski spacetime are familiar: translations and rotations of spacetime, with the latter splitting between proper rotations and Lorentz boosts. These symmetries are responsible for the existence of energy, momentum and angular momentum on a fixed Minkowski background.
It is important to characterize the symmetries of a general metric.

§3.3.1 Isometries

Recall Def. **??** of flow on a manifold: by Eq. **??**, it is possible to identify a flow with a vector field $K \in \mathfrak{X}(\mathcal{M})$ such that it is tangent to the flow at each point of the manifold: $K^\mu = \frac{dx^\mu}{dt}$, with $x^\mu(t) \equiv x^\mu(\sigma_t)$.

Definition 3.3.1 (G)

ven a flow associated to $K \in \mathfrak{X}(\mathcal{M})$, it is said to be an *isometry* if:

$$\mathcal{L}_K g = 0 \quad \Leftrightarrow \quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{3.28}$$

Recall Eqq. 3.11-3.12 for the equivalence. This condition means that the metric doesn't change along flow lines: this is called *Killing equation*, and a vector field which satisfied it is a *Killing* *vector field*. Prop. ?? can be generalized to hold for the Lie derivative of arbitrary tensor fields, thus Killing vectors too form a Lie algebra: it is the Lie algebra of the isometry group of the manifold.

Example 3.3.1 (G)

ven a metric $g_{\mu\nu}(x)$, if it doesn't depend on $x^1 \equiv y$, then $X = \partial_y$ is a Killing vector field, for $(\mathcal{L}_{\partial_y} g)_{\mu\nu} = \partial_y g_{\mu\nu} = 0$. This is the case of ignorable degrees of freedom, as in Eqq. 3.16-3.17.

§3.3.1.1 Minkowski spacetime

Consider Minkowski spacetime $\mathbb{R}^{1,3}$ with $g_{\mu\nu} = \eta_{\mu\nu}$. Killing equation is:

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 0$$

There two forms of solution. The first one is:

$$K_\mu = c_\mu$$

for any constant vector c_μ. These generate translations. Alternatively:

$$K_\mu = \omega_{\mu\nu} x^\nu \; : \; \omega_{\mu\nu} = -\omega_{\nu\mu}$$

These generate rotations and Lorentz boosts. The emergence of the Lie algebra structure can be elucidated defining the Killing vectors:

$$P_\mu := \partial_\mu \qquad M_{\mu\nu} := \eta_{\mu\rho} x^\rho \partial_\nu - \eta_{\nu\rho} x^\rho \partial_\mu$$

These are 10 Killing vectors: 4 for translations and 6 for rotations and boosts.

Proposition 3.3.1 (K)

lling vectors of Minkowski spacetime obbey:

$$[P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\sigma] = -\eta_{\mu\sigma} P_\nu + \eta_{\sigma\nu} P_\mu$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}$$

Proof. By direct calculation. □

This is precisely the Lie algebra of the Poincaré group $\mathbb{R}^4 \times \mathrm{SO}(1,3)$, i.e. the isometry group of Minkowski spacetime.

§3.3.1.2 de Sitter and anti-de Sitter space

The isometries of dS and AdS are best seen from their embeddings. The constraint Eq. 3.21 which defined de Sitter space is invariant under the rotations of $\mathbb{R}^{1,4}$, thus dS inherits the $\mathrm{SO}(1,4)$ isometry group. Similarly, the constraint Eq. 3.27 which defines anti-de Sitter space is invariant under rotations of $\mathbb{R}^{2,3}$, so AdS inherits the $\mathrm{SO}(2,3)$ isometry group. Note that both these groups are 10-dimensional, as $\mathbb{R}^4 \times \mathrm{SO}(1,3)$.
It's simple to determine the 10 Killing spinors of 5d spacetime:

$$M_{AB} = \eta_{AC} X^C \partial_B - \eta_{BC} X^C \partial_A$$

where $X^A, A = 0, 1, 2, 3, 4$ are 5d coordinates and η_{AB} is the appropriate Minkowski metric, with signature $(-, +, +, +, +)$ for dS and $(-, -, +, +, +)$ for AdS. In either case, the Lie algebra is that of the appropriate Lorentz group:

$$[M_{AB}, M_{CD}] = \eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}$$

The embedding hyperbolae are both invariant under these Killing vectors: flows generated by them map points on the hyperbolae to other points on the hyperbolae. Therefore, these Killing vectors are inherited by dS and AdS respectively.

Energy Energy is defined by timelike Killing vectors. In anti-de Sitter space there's no problem finding a timelike Killing vector, as the metric in global coordinates Eq. 3.24 is time-independent, so $K = \partial_t$. However, this changes in de Sitter space.
Considering dS in the static patch with coordinates $r^2 = (X^1)^2 + (X^2)^2 + (X^3)^2$, $X^0 = \sqrt{R^2 - r^2}\sinh\frac{t}{R}$ and $X^4 = \sqrt{R^2 - r^2}\cosh\frac{t}{R}$, the metric Eq. 3.14 is time-independent, thus $K = \partial_t$ is a Killing vector; pushing it forward to the 5d space:

$$\partial_t = \frac{\partial X^A}{\partial t}\partial_A = \frac{1}{R}(X^4\partial_0 + X^0\partial_4)$$

On the static patch, this Killing vector is timelike and the energy Eq. 3.16 follows from it. The problem is that the static patch is only one-half of the hyperboloid: when considering the whole AdS, one must account for the case $X^0 = 0, X^4 < 0$, in which the Killing vector points in the past direction, or $X^0 \neq 0, X^4 = 0$, in which it is spacelike. Therefore, the Killing vector can be both timelike (future-directed and past-directed) or spacelike when spanning the whole manifold: for this reason, it's not possible to define a global positive conserved energy on the total de Sitter space. The same conclusion follows by noting that the metric in global coordinates Eq. 3.22 is time-dependent.

§3.3.2 Conserved quantities

It is possible to reframe Noether's theorem in the context of Killing vectors.

Theorem 3.3.1 (G)

ven a massive particle moving on a geodesic $x^\mu(\tau)$ in a spacetime with metric g which admits a Killing vector field K, then the *Noether charge* is conserved along the geodesic:

$$Q := K_\mu \frac{dx^\mu}{d\tau} \tag{3.29}$$

Proof. First, show that Q is indeed conserved along the geodesic, recalling Eq. 3.28:

$$\frac{dQ}{d\tau} = \partial_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + K_\mu \frac{d^2x^\mu}{d\tau^2} = \partial_\nu K_\mu \frac{dx^\nu}{d\tau} - K_\mu \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \nabla_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

To show that Q follows from Noether's theorem, consider the action of the massive particle and introduce an infinitesimal transformation $\delta x^\mu = K^\mu(x)$:

$$\begin{aligned}\delta\mathcal{S} = \delta\int d\tau\, g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= \int d\tau\left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho + 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\right]\\ &= \int d\tau\left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}K^\rho + 2\frac{dx^\mu}{d\tau}\left(\frac{dK_\mu}{d\tau} - K^\nu\frac{dg_{\mu\nu}}{d\tau}\right)\right]\\ &= \int d\tau\left[\partial_\rho g_{\mu\nu}K^\rho - 2K^\rho\partial_\nu g_{\mu\rho} + 2\partial_\nu K_\mu\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \int d\tau\, 2\nabla_\nu K_\mu\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\end{aligned}$$

The transformation is a symmetry if $\delta\mathcal{S} = 0$, thus, by the symmetry of $\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$, the resulting equation is $\nabla_{(\nu}K_{\mu)} = 0$, i.e. Killing equation. □

Example 3.3.2 (E)

ergy and angular momentum defined on de Sitter geodesics by Eqq. 3.16-3.17 are Noether charges associated to Killing vectors ∂_t and ∂_φ respectively.

§3.3.3 Komar integrals

Definition 3.3.2 (G)

ven a Killing vector $K = K^\mu\partial_\mu$, defined the 1-form $K \equiv K_\mu dx^\mu$, the associated *Komar form* is the 2-form defined as:

$$F := dK \tag{3.30}$$

Proposition 3.3.2 (G)

ven a Killing vector $K = K^\mu\partial_\mu$, the associated Komar form is:

$$F_{\mu\nu} = \nabla_\mu K_\nu - \nabla_\nu K_\mu \tag{3.31}$$

Proof. $F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu = dK = \nabla_\mu K_\nu dx^\mu \wedge dx^\nu$. □

Theorem 3.3.2 (I)

the vacuum Einstein equations $R_{\mu\nu} = 0$ hold, then a Komar form obeys the vacuum Maxwell equations:

$$d\star F = 0 \quad\Leftrightarrow\quad \nabla^\mu F_{\mu\nu} = 0 \tag{3.32}$$

Proof. Recall Prop. **??** for the equivalence. From Ricci identity Eq. **??**:

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)K^\sigma = R^\sigma{}_{\rho\mu\nu}K^\rho \quad\Rightarrow\quad (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)K^\mu = R_{\rho\nu}K^\rho$$

From Killing equation $\nabla_{(\mu}K_{\nu)} = 0 \Rightarrow \nabla_\mu K^\mu = 0$, thus $\nabla_\mu\nabla_\nu K^\mu = R_{\rho\nu}K^\rho$ and:

$$\nabla^\mu F_{\mu\nu} = \nabla^\mu\nabla_\mu K_\nu - \nabla^\mu\nabla_\nu K_\mu = -2\nabla^\mu\nabla_\nu K_\mu = -2R_{\rho\nu}K^\rho$$

If $R_{\rho\nu} = 0$, then $\nabla^\mu F_{\mu\nu} = 0$. □

Definition 3.3.3 (G)

ven a Komar form F associated to a Killing vector K, the associated *Komar charge* (or Komar integral) on a spatial submanifold Σ is defined as:

$$Q_{\text{Komar}} := -\frac{1}{8\pi G}\int_{\Sigma} d\star F = -\frac{1}{8\pi G}\int_{\partial\Sigma} \star F = -\frac{1}{8\pi G}\int_{\partial\Sigma} \star dK \tag{3.33}$$

Proposition 3.3.3 (I)

Eq. 3.32 holds, then Q_{Komar} is conserved.

Proof. Recall Eq. **??**: the proof is identical. □

As for Noether integrals of a particle, Komar integrals of spacetime are interpreted based on the defining Killing vector. For example, if K^μ is everywhere timelike, i.e. $g_{\mu\nu}K^\mu K^\nu < 0$, then its Komar integral can be identified with the energy (or the mass) of spacetime; if the Killing vector is related to rotations, instead, the conserved charge is the angular momentum of spacetime.

§3.4 Asymptotics of spacetime

The three special solution (Minkowski, dS, AdS) not only have different spacetime curvature and symmetries, but, more fundamentally, they have different behavior at infinity. Their importance lies in the fact that, however complicated the metric might be, if fields are suitably localized, then they will asymptote to one of these three symmetric spaces.

§3.4.1 Conformal transformations

Definition 3.4.1 (G)

ven a metric manifold $(\mathcal{M}, g)$ and a non-vanishing $\Omega \in \mathcal{C}^\infty(\mathcal{M})$, a *conformal transformation* is defined as:

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{3.34}$$

Typically, g and $\tilde{g}$ describe different spacetime with considerably warped distances. However, conformal transformations preserve angles: in Lorentzian spacetime, the two metrics have the same causal structure, i.e. null/timelike/spacelike vector fields in one metric remain null/timelike/spacelike in the other too.

Proposition 3.4.1 (O)

ly conformal transformations of the metric preserve its causal structure.

Although timelike particle trajectories necessarily remain timelike under a conformal transformation, the same needs not be true for geodesics, as distances get warped.

Proposition 3.4.2 (N)

ll geodesics are mapped to null geodesics under a conformal transformation.

Proof. First compute the Christoffel symbols in the new metric:

$$\begin{aligned}\Gamma^{\mu}_{\rho\sigma}[\tilde{g}] &= \frac{1}{2}\tilde{g}^{\mu\nu}(\partial_\rho\tilde{g}_{\nu\sigma} + \partial_\sigma\tilde{g}_{\rho\nu} - \partial_\nu\tilde{g}_{\rho\sigma}) \\ &= \frac{1}{2}\Omega^{-2}g^{\mu\nu}(\partial_\rho(\Omega^2 g_{\nu\sigma}) + \partial_\sigma(\Omega^2 g_{\rho\nu}) - \partial_\nu(\Omega^2 g_{\rho\sigma})) \\ &= \Gamma^{\mu}_{\rho\sigma}[g] + \Omega^{-1}(\delta^\mu_\sigma\nabla_\rho\Omega + \delta^\mu_\rho\nabla_\sigma\Omega - g_{\rho\sigma}\nabla^\mu\Omega)\end{aligned}$$

In the last line, recall that $\nabla = \partial$ on scalar functions. Suppose an affinely parametrized geodesic in the metric g:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^{\mu}_{\rho\sigma}[g]\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad\Rightarrow\quad \frac{d^2x^\mu}{d\tau^2}+\Gamma^{\mu}_{\rho\sigma}[\tilde{g}]\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \Omega^{-1}(\delta^\mu_\sigma\nabla_\rho\Omega+\delta^\mu_\rho\nabla_\sigma\Omega-g_{\rho\sigma}\nabla^\nu\Omega)\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$$

For a null geodesic $g_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$, thus:

$$\frac{dx^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma}[\tilde{g}]\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 2\frac{dx^\mu}{d\tau}\frac{1}{\Omega}\frac{d\Omega}{d\tau}$$

This is the equation of non-affinely parametrized geodesic (like Eq. **??**), hence the thesis. □

Usual curvature tensors are not invariant under conformal transformations. A curvature tensor that is indeed invariant is the *Weyl tensor*:

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{2}{n-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu} \tag{3.35}$$

where $n \equiv \dim_{\mathbb{R}}\mathcal{M}$. The Weyl tensor has all the symmetries of the Riemann tensor, with the additional property that it vanishes when contracting any pair of indices with the metric: it can be viewed as the trace-free part of the Riemann tensor.

§3.4.2 Penrose diagrams

To study the asymptotic behavior of spacetime, one needs to perform a conformal transformation that maps infinity to a finite distance: the resulting causal structure can then be drawn on a finite area and the resulting picture is called a *Penrose diagram.*

§3.4.2.1 Minkowski spacetime

It is simplest to study $\mathbb{R}^{1,1}$, where the Minkowski metric is $ds^2 = -dt^2 + dx^2$. First, introduce *lightcone coordinates*:

$$u = t - x \qquad v = t + x \quad\Rightarrow\quad ds^2 = -du\,dv$$

These coordinates range in $u, v \in \mathbb{R}$. To work with finite quantities, define:

$$u = \tan\tilde{u} \qquad v = \tan\tilde{v} \quad\Rightarrow\quad ds^2 = -\frac{1}{\cos^2\tilde{u}\,\cos^2\tilde{v}}d\tilde{u}\,d\tilde{v}$$

where now $\tilde{u}, \tilde{v} \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$. Note that the metric diverges when approaching the boundary of Minkowski spacetime. However, a conformal transformation is possible with $\Omega(\tilde{u},\tilde{v}) \equiv \cos\tilde{u}\,\cos\tilde{v}$:

$$d\tilde{s}^2 = \left(\cos^2\tilde{u}\,\cos^2\tilde{v}\right)ds^2 = -d\tilde{u}\,d\tilde{v}$$

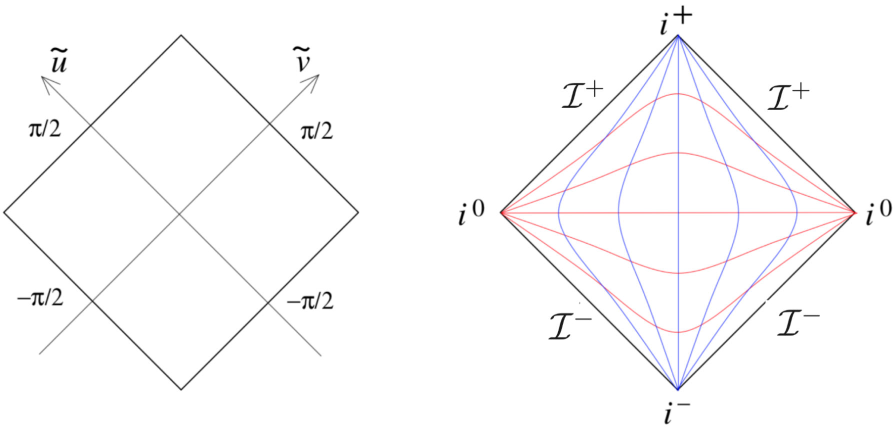

Figure 3.5: Penrose diagram for $d = 1 + 1$ Minkowski spacetime.

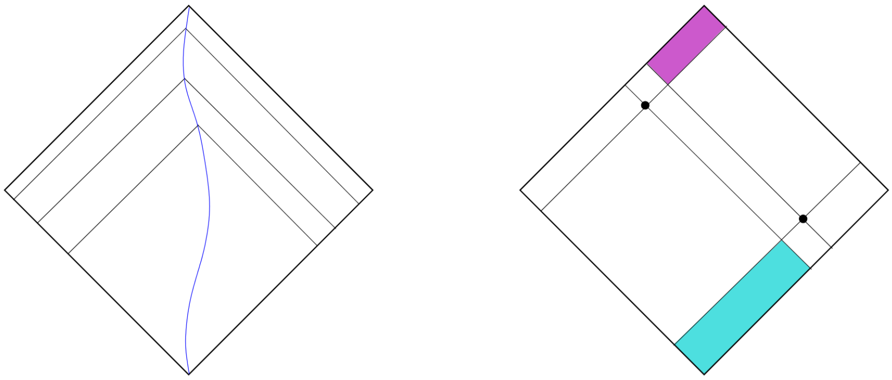

Figure 3.6: Lightcones in $d = 1 + 1$ Minkowski spacetime.

It is now possible to add $\tilde{u} = \pm\frac{\pi}{2}$ and $\tilde{v} = \pm\frac{\pi}{2}$: this operation is called *conformal compactification*.

Penrose diagrams, as other relativistic diagrams, present light-rays travelling at 45°, time in the vertical direction and space in the horizontal one. Fig. 3.5 shows the Penrose diagram for Minkowski spacetime $\mathbb{R}^{1,1}$ with $\tilde{u}, \tilde{v}$ coordinates, both with coordinate axis and geodesics. Moreover, the various infinities of Minkowski space are shown:

- timelike geodesics (blue) start at $i^-(\tilde{u}, \tilde{v}) = \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ and end at $i^+(\tilde{u}, \tilde{v}) = \left(+\frac{\pi}{2}, +\frac{\pi}{2}\right)$, called respectively *past* and *future timelike infinity*;
- spacelike geodesics (red) start and end at $i^0(\tilde{u}, \tilde{v}) = \left(\mp\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, both called *spacelike infinity*;
- all null curves (not shown) start at the boundary $\mathcal{I}^- \equiv \{\tilde{u} = -\frac{\pi}{2}\} \cup \{\tilde{v} = -\frac{\pi}{2}\}$ and end at the boundary $\mathcal{I}^+ \equiv \{\tilde{u} = +\frac{\pi}{2}\} \cup \{\tilde{v} = +\frac{\pi}{2}\}$, called respectively *past* and *future null infinity*.

It's clear that in Minkowski spacetime there are more ways reach infinity along a null direction than in a timelike or spacelike direction.

Penrose diagrams allow to immediately visualize the causal structure of spacetime. As shown in Fig. 3.6, given a particle moving along a timelike curve, as it approaches i^+ its past lightcone encompasses progressively more of spacetime: thus, an observer in Minkowski spacetime can in principle see everything, as long as they wait long enough. Relatedly, given any two points

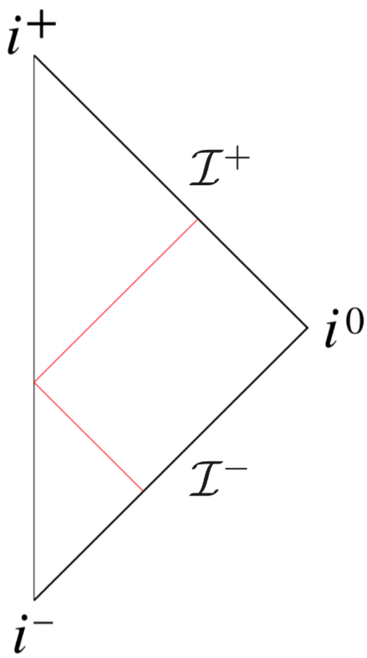

Figure 3.7: Penrose diagram for $d = 1 + 3$ Minkowski spacetime.

in Minkowski spacetime, they are causally connected both in the past and in the future, as both cones intersect (see Fig. 3.6): thus, there was always an event in the past that could have influenced both, and there will always be an event in the future that can be influenced by both.

4d Minkowski spacetime The analysis can be repeated for $\mathbb{R}^{1,3}$ with $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$. Lightcone coordinates are again:

$$u = t - r \qquad v = t + r \quad \Rightarrow \quad ds^2 = -du\,dv + \frac{1}{4}(u - v)^2\, d\Omega_2^2$$

Finite range coordinates are again:

$$u = \tan\tilde{u} \qquad v = \tan\tilde{v} \quad \Rightarrow \quad ds^2 = \frac{1}{4\cos^2\tilde{u}\,\cos^2\tilde{v}}\left(-4d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{u} - \tilde{v})d\Omega_2^2\right)$$

Finally, the conformal transformation with $\Omega(\tilde{u}, \tilde{v}) = 2\cos\tilde{u}\,\cos\tilde{v}$ leads to:

$$d\tilde{s}^2 = -4d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{u} - \tilde{v})d\Omega_2^2$$

In 4d Minkowski spacetime there's an additional constraint: $r \geq 0$, so $v \geq u$. Conformal compactification leads to:

$$-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq +\frac{\pi}{2}$$

The corresponding Penrose diagram is drawn in Fig. 3.7: the spatial $\mathbb{S}^2$ is not shows for simplicity, but every point on the diagram corresponds to an $\mathbb{S}^2$ of radius $|\sin(\tilde{u} - \tilde{v})|$. The line $\tilde{u} = \tilde{v}$ is not a boundary of Minkowski spacetime, but it's simply the origin $r = 0$ (at which $\mathbb{S}^2$ shrinks to a point): to illustrate this, a null geodesic is drawn.
In general, Penrose diagrams are only useful for spacetimes which contain an obvious $\mathbb{S}^2$, i.e. those with SO(3) isometry: however, these are the simplest and most important in physics.

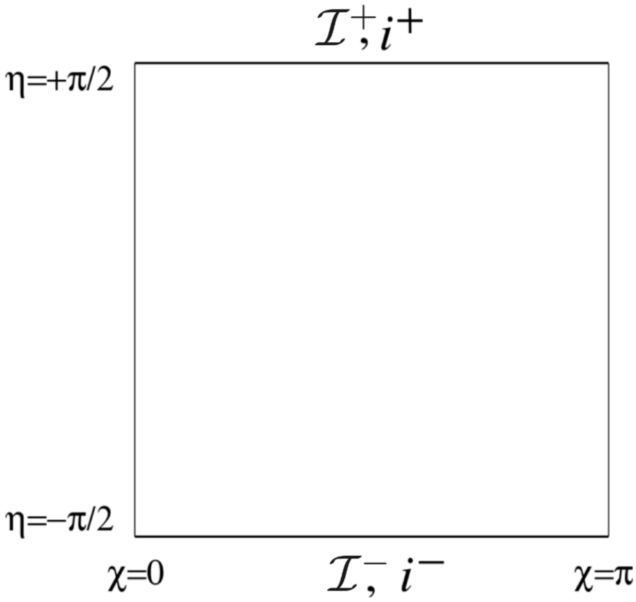

Figure 3.8: Penrose diagram for de Sitter space.

§3.4.2.2 de Sitter space

Recall global coordinates on dS: from Eq. 3.22, $ds^2 = -d\tau^2 + R^2\cosh^2\frac{\tau}{R}d\Omega_3^2$. To draw the Penrose diagram, define *conformal time* as:

$$\frac{d\eta}{d\tau} = \frac{1}{R\cosh(\tau/R)} \quad \Rightarrow \quad \cos\eta = \frac{1}{\cosh(\tau/R)}$$

Given that $\tau \in \mathbb{R}$, then $\eta \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$. In conformal time, de Sitter space has the metric:

$$ds^2 = \frac{R^2}{\cos^2\eta}\left(-d\eta^2 + d\Omega_3^2\right)$$

Writing $d\Omega_3^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2$, with $\chi \in [0,\pi]$, de Sitter metric is conformally equivalent to:

$$d\tilde{s}^2 = -d\eta^2 + d\chi^2 + \sin^2\chi\, d\Omega_2^2$$

After conformal compactification, $\eta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ and $\chi \in [0,\pi]$. The Penrose diagram is drawn in Fig. 3.8: the two vertical lines are not boundaries of dS, but simply the north and south poles of $\mathbb{S}^3$. The boundaries of the spacetime are the top and bottom lines, labelled both $i^\pm$ and $\mathcal{I}^\pm$ as they are where both timelike and null geodesics originate and terminate.

Proposition 3.4.3 (d)
Sitter spacetime has a *spacelike* $\mathbb{S}^3$ boundary with timelike normal vector.

The causal structure of dS is very different from that of Minkowski spacetime: it is no longer true that any observer can see everything by waiting long enough. For example, as shown in Fig. 3.9, an observer at the north pole will eventually see only exactly half the spacetime: the boundary of this half-space is the observer's *event horizon*, in the sense that signals from beyond the horizon cannot reach them. It is also clear that this event horizon is observer-dependent: in this context, these are referred to as *cosmological horizons*.

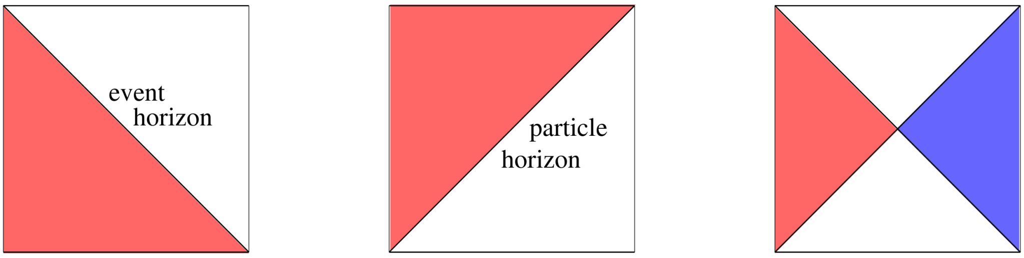

Figure 3.9: Event and particle horizons for an observer at the north pole in dS, and the causal diamonds for an observer at the north and south pole.

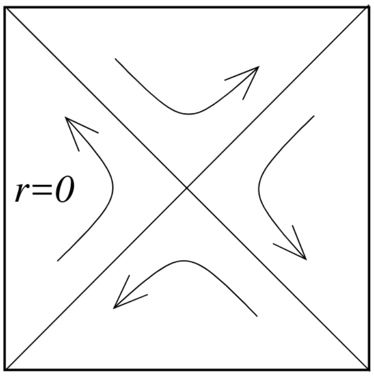

Figure 3.10: $K = \partial_t$ Killing vector field in de Sitter spacetime.

Furthermore, an observer at the north pole will only be able to communicate with another half of the spacetime, as shown in Fig. 3.9: the boundary of this region of influence is known as the *particle horizon* and it represents the furthest distance light can travel since the beginning of time. Its intersection with the event horizon determines the (nothern) *causal diamond*: usually, nothern and southern causal diamonds are causally disconnected.
Penrose diagrams can also be used to explain the divergence at $r = R$ of the metric Eq. 3.14 on the static patch of dS. Recalling the embedding of the static patch in $\mathbb{R}^{1,4}$, parametrized as $X^0 = \sqrt{R^2 - r^2}\sinh\frac{t}{R}$ and $X^4 = \sqrt{R^2 - r^2}\cosh\frac{t}{R}$. Naively, the surface $r = R$ corresponds to $X^0 = X^4 = 0$, but writing $r = R(1 - \varepsilon^2/2)$ yields that $X^0 \sim R\varepsilon\sinh\frac{t}{R}$ and $X^4 \sim R\varepsilon\cosh\frac{t}{R}$, so $\varepsilon \to 0$ can be obtained keeping $X^0, X^4 \neq 0$ and finite, provided that $t \to \pm\infty$: to do this, $\varepsilon\exp(\pm t/R)$ must be kept finite, thus the surface $r = R$ is identified with the lines $X^0 = \pm X^4$. Translation to polar coordinates is done by $X^0 = R\sinh\frac{\tau}{R}$ and $X^4 = R\cosh\frac{\tau}{R}\cos\chi$, with χ the polar angle on $\mathbb{S}^3$; after mapping to conformal time, one finds that:

$$X^0 = \pm X^4 \quad \Leftrightarrow \quad \sin\eta = \pm\cos\chi \quad \Leftrightarrow \quad \chi = \pm\left(\eta - \frac{\pi}{2}\right)$$

These are precisely the lines determining the polar causal diamonds. It can also be checked that $r = R$ on the static patch corresponds to the north pole $\chi = 0$ in global coordinates and that $t = \tau$ along this line. Therefore, the static patch of dS provides coordinates that cover only the nothern causal diamond of dS spacetime, with the coordinate singularity at $r = R$ corresponding to the past and future observer-dependent horizons.
Finally, Penrose diagrams help to understand the nature of the $K = \partial_t$ Killing vector field exhibited by the static patch metric. As shown in Fig. 3.10, there's no global timelike Killing vector field in dS spacetime: extending the Killing vector beyond the static patch, i.e. the nothern causal diamond, it is timelike but past-oriented on the souther causal diamond and spacelike on the upper and lower quadrants.

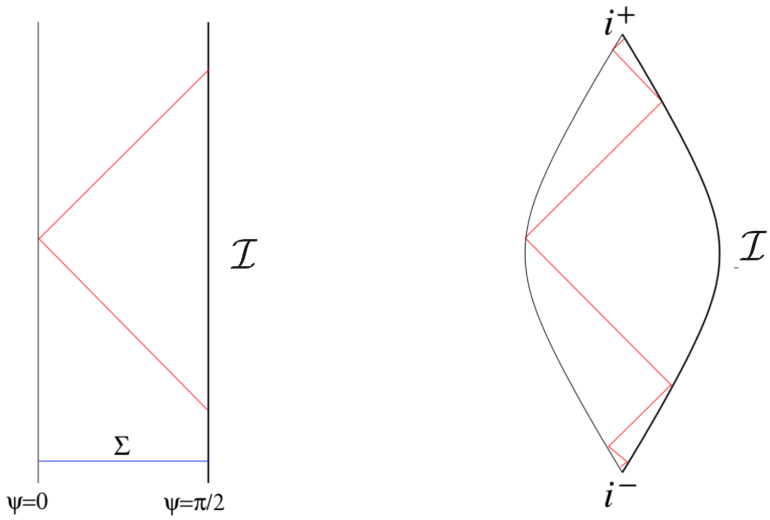

Figure 3.11: Penrose diagrams for AdS, without and with conformally compactified time coordinate.

§3.4.2.3 Anti-de Sitter space

The global coordinates on AdS are, from Eq. 3.24, $ds^2 = -\cosh^2\rho\, dt^2 + R^2 d\rho^2 + R^2\sinh^2\rho\, d\Omega_2^2$, with $\rho \in [0,\infty)$. Introduce a conformal radial coordinate:

$$\frac{d\psi}{d\rho} = \frac{1}{\cosh\rho} \quad \Rightarrow \quad \cos\psi = \frac{1}{\cosh\rho}$$

with $\psi \in [0, \frac{\pi}{2})$. With a dimensionless coordinate time $\tilde{t} = \frac{t}{R}$, the metric becomes:

$$ds^2 = \frac{R^2}{\cos^2\psi}\left(-d\tilde{t}^2 + d\psi^2 + \sin^2\psi\, d\Omega_2^2\right) = \frac{R^2}{\cos^2\psi}\left(-d\tilde{t}^2 + d\Omega_3^2\right)$$

AdS metric is conformally equivalent to:

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\psi^2 + \sin^2\psi\, d\Omega_2^2$$

After conformal compactification, $\tilde{t} \in \mathbb{R}$ and $\psi \in \left[0, \frac{\pi}{2}\right]$ and the resulting Penrose diagram is the infinite strip in Fig. 3.11. The edge at $\psi = 0$ is not a boundary, but the spatial origin where $\mathbb{S}^2$ shrinks to a point; in contrast, $\psi = \frac{\pi}{2}$ is a boundary of the spacetime, labelled $\mathcal{I}$, which should be viewed as a combination of $\mathcal{I}^-$, $\mathcal{I}^+$ and i^0, since null and spacelike geodesics begin and end there.

Proposition 3.4.4 (A)

S spacetime has a timelike $\mathbb{R}\times\mathbb{S}^2$ boundary with spacelike normal vector.

Note that $\mathbb{R}$ is the time factor.
The Penrose diagram clearly shows that light rays reach the boundary in finite conformal time. To study physics in AdS, one needs to specify boundary conditions at $\mathcal{I}$: for example, reflecting boundary conditions make light rays bounce back and forth forever, rendering AdS a *box* spacetime in which massive particle are confined in the interior and massless particles bounce off the boundary.
Another characteristic of AdS space is that it isn't *globally hyperbolic*: there exists no Cauchy

surface on which initial data can be specified. Consider for example the 3d spacelike hypersurface Σ in Fig. 3.11. Specifying initial data on Σ it's not sufficient to solve for their time evolution: in AdS, there exist points in the future of Σ which are in causal contact with the boundary, thus the time evolution depends on boundary conditions too.
To make the Penrose diagram for AdS not stretch to infinity, the time coordinate can be restricted to finite values by:

$$\tilde{t} = \tan\tau \quad \Rightarrow \quad ds^2 = \frac{R^2}{\cos^2\psi\,\cos^4\tau}\left(-d\tau^2 + \cos^4\tau\, d\Omega_3^2\right)$$

This metric is conformally equivalent to:

$$d\tilde{s}^2 = -d\tau^2 + \cos^4\tau\left(d\psi^2 + \sin^2\psi\, d\Omega_2^2\right)$$

with conformally compactified $\tau \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$. Ignoring the spatial $\mathbb{S}^2$, the resulting Penrose diagram is drawn in Eq. 3.11: the spatial $\mathbb{S}^3$ grows and shrinks in time, the timelike boundary $\mathcal{I}$ is still present and now the past and future timelike infinities $i^\pm$ are also shown. This diagram makes it clear that a lightray bounces back and forth off the boundary of AdS an infinite number of times.

§3.5 Matter coupling

Spacetime is not merely the background on which matter exists, but it is dynamically influenced by the matter distribution on it. It is therefore necessary to study how matter couples to the spacetime metric.

§3.5.1 Field theories in curved spacetime

The simplest way to describe matter is by fields governed by a Lagrangian. Consider a scalar field $\phi(x)$. In flat Minkowski spacetime, its action is:

$$\mathcal{S}_{\text{scalar}} := \int d^4x\left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{3.36}$$

The negative sign of the kinetic term follows from the signature choice $(-,+,+,+)$. The generalization to curved spacetime is straightforward:

$$\mathcal{S}_{\text{scalar}} = \int d^4x\sqrt{-\mathrm{g}}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right]$$

Note the (useful) redundancy $\nabla_\mu\phi = \partial_\mu\phi$. Curved spacetime also introduces the possibility to add new terms to the Lagrangian. For example:

$$\mathcal{S}_{\text{scalar}} = \int d^4x\sqrt{-\mathrm{g}}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) - \frac{1}{2}\xi R\phi^2\right] \tag{3.37}$$

for some $\xi \in \mathbb{R}$. This theory correctly reduces to Eq. 3.36 on flat spacetime, as $R = 0$. To derive the equation of motion, vary the action keeping the metric fixed:

$$\begin{aligned}\delta\mathcal{S}_{\text{scalar}} &= \int d^4x\sqrt{-\mathrm{g}}\left[-g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi - \frac{\partial V}{\partial\phi}\delta\phi - \xi R\phi\delta\phi\right]\\ &= \int d^4x\sqrt{-\mathrm{g}}\left[\left(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\partial V}{\partial\phi} - \xi R\phi\right)\delta\phi - \nabla_\mu(\delta\phi\nabla^\mu\phi)\right]\end{aligned}$$

where integration by parts was possible due to $\nabla_\mu g_{\rho\sigma} = 0$ for the Levi-Civita connection. The last term is, by divergence theorem, a boundary term, thus the equation of motion for a scalar field theory in curved spacetime is:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\partial V}{\partial\phi} - \xi R\phi = 0 \tag{3.38}$$

Now covariant derivatives are necessary, as $\nabla_\mu\nabla_\nu \neq \partial_\mu\partial_\nu$.

§3.5.2 Einstein equations with matter

To understand how matter fields back-react on spacetime, consider the combined action:

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}\,(R - 2\Lambda) + \mathcal{S}_M \tag{3.39}$$

where $\mathcal{S}_M$ is the action for matter fields, which in general depends on both the fields and the metric.

Definition 3.5.1 (

Given a field theory for matter described by the action $\mathcal{S}_M$, the *energy-momentum tensor* is defined as:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}} \tag{3.40}$$

Proposition 3.5.1 (T)

e energy-momentum tensor is symmetric.

Proof. It inherits the symmetry of the metric. □

Proposition 3.5.2 (T)

e equations of motion derived from the action Eq. 3.39 are:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.41}$$

Proof. Varying the full metric, by Def. **??**:

$$\delta\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}\,[G_{\mu\nu} + \Lambda g_{\mu\nu}]\,\delta g^{\mu\nu} - \frac{1}{2}\int d^4x\sqrt{-\mathrm{g}}T_{\mu\nu}\delta g^{\mu\nu} = 0$$

□

These are the full *Einstein field equations*, describing gravity coupled to matter. It is possible to rewrite them by observing that the cosmological constant can be absorbed in the energy-momentum tensor as an additive component:

$$(T_\Lambda)_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}$$

This is justified by the fact that matter fields often mimic a cosmological constant. Contracting the remaining equation with $g^{\mu\nu}$ (i.e. taking the trace) then gives $-R = 8\pi G T$, where $T \equiv g^{\mu\nu}T_{\mu\nu}$, hence:

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) \tag{3.42}$$

Remember that the cosmological constant is present inside the energy-momentum tensor.

§3.5.3 Energy-momentum tensor

The action $\mathcal{S}_M$ is, by hypothesis, diffeomorphism-invariant, thus, recalling the argument which lead to Bianchi identity Eq. 3.13, given $\delta g_{\mu\nu} = (\mathcal{L}_X g_{\mu\nu}) = 2\nabla_{(\mu}X_{\nu)}$:

$$\delta\mathcal{S}_M = -\frac{1}{2}\int d^4x\sqrt{-\mathrm{g}}T_{\mu\nu}\delta g^{\mu\nu} = -2\int d^4x\sqrt{-\mathrm{g}}T_{\mu\nu}\nabla^\mu X^\nu$$

Diffeomorphism invariance means that $\delta\mathcal{S}_M = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$, hence, integrating by parts:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.43}$$

Of course, this was necessary to make Einstein equations consistent, as $\nabla_\mu G^{\mu\nu} = 0$. Anyway, this equation hints to the more profound nature of the energy-momentum tensor, which has nothing to do with gravity: it can be shown that the energy-momentum tensor is linked to Noether currents associated to translational invariance in space and time. Trivially, Eq. 3.43 reduces in flat spacetime to $\partial_\mu T^{\mu\nu} = 0$, which is the usual conservation law enjoyed by Noether currents.

Consider a translation $x^\mu \mapsto x^\mu + \delta x^\mu$, with $\delta x^\mu = X^\mu(x)$. The action restricted to flat spacetime is not invariant under such shift, but one which is invariant can be constructed coupling the matter fields to a background metric and allowing this to vary. The change of the action in flat space, where the metric is fixed, must be equal and opposite to the change of the action where the metric can vary but x^μ is fixed, thus:

$$\begin{aligned}\delta\mathcal{S}_{\text{flat}} &= -\int d^4x\left.\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\delta g^{\mu\nu} = -\int d^4x\left.\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\partial^{(\mu}X^{\nu)} \\ &= -2\int d^4x\left.\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\partial^\mu X^\nu = -2\int d^4x\,\partial^\mu\left[\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right]_{g_{\mu\nu}=\eta_{\mu\nu}}X^\nu\end{aligned}$$

But $\delta\mathcal{S}_{\text{flat}} = 0$ for all constant X^μ, as this is the definition of a translationally-invariant theory, hence the conserved Noether current in flat space is:

$$T_{\mu\nu} = -2\left.\frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

i.e. the flat version of Eq. **??**.

§3.5.3.1 Field theories

It is straightforward to compute $T_{\mu\nu}$ for a scalar field $\phi(x)$. Recall Eq. 3.37 (with $\xi = 0$) and Lemma **??**:

$$\delta\mathcal{S}_{\text{scalar}} = \int d^4x\sqrt{-\mathrm{g}}\left[\frac{1}{4}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi + \frac{1}{2}g_{\mu\nu}V(\phi) - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right]\delta g^{\mu\nu}$$

This gives the energy-momentum tensor:

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\rho \phi \nabla_\rho \phi + V(\phi)\right) \tag{3.44}$$

Restricting to flat Minkowski spacetime:

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

which is the energy density of a scalar field.

Maxwell theory Varying Maxwell action Eq. **??**:

$$\delta\mathcal{S}_{\text{Maxwell}} = -\frac{1}{4}\int dx^4 \sqrt{-\mathrm{g}}\left[-\frac{1}{2}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + 2g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right]\delta g^{\mu\nu}$$

So the energy-momentum tensor for Maxwell theory is:

$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{3.45}$$

In flat Minkowski spacetime:

$$T_{00} = \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{B}^2$$

which is the energy density of the electromagnetic field.

§3.5.3.2 Perfect fluids

A perfect fluid is described by its *energy density* $\rho(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$ and velocity 4-vector field $u^\mu(\mathbf{x}, t) : u^\mu u_\mu = -1$. Pressure and energy density are related by an *equation of state* $p = p(\rho)$.

Example 3.5.1 (D)

st is a fluid of massive particles floating around very slowly, so that the equation of state is $p = 0$.

Example 3.5.2 (R)

diation is a fluid of photons with $p = \rho/3$.

The energy-momentum tensor of a perfect fluid is:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu} \tag{3.46}$$

A fluid at rest ($u^\mu = \delta^{\mu,0}$) in flat Minkowski spacetime has $T^{\mu\nu} = \operatorname{diag}(\rho, p, p, p)$, thus T_{00} is yet again the energy density, as expected. Generally, $\rho = T_{\mu\nu}u^\mu u^\nu$ is the energy density measured by an observer co-moving with the fluid.
Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$ determines two constraints. The first is:

$$u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu = 0 \tag{3.47}$$

which is a generalization of mass conservation (where mass is identified with ρ). The first term calculates how fast ρ changes along u^μ, while the second expresses it depending on the rate of flow out of the region $\nabla_\mu u^\mu$. The second constraint is:

$$(\rho + p)u^\mu \nabla_\mu u^\nu = -(g^{\mu\nu} + u^\mu u^\nu)\nabla_\mu p \tag{3.48}$$

which is a generalization of Euler equation, i.e. the fluid equivalent of $F = ma$ (or rather $ma = F$).

§3.5.4 Energy conservation

There's a difference between the energy-momentum tensor and the current which arises from a global symmetry. Consider a conserved current $J^\mu : \nabla_\mu J^\mu = 0$. Invoking the divergence theorem Eq. **??**:

$$0 = \int_V d^4x \sqrt{-\mathrm{g}}\, \nabla_\mu J^\mu = \int_{\partial V} d^3x \sqrt{\gamma}\, n_\mu J^\mu$$

where V is a spatial volume with boundary $\partial V = \Sigma_1 \cup \Sigma_2 \cup B$, with Σ_1, Σ_2 past and future spacelike boundarires and B timelike (lateral) boundary. If no current flows out of the region, i.e. $n_\mu J^\mu|_B = 0$, then this expression becomes the conservation $Q(\Sigma_1) = Q(\Sigma)_2$ of the charge associated to the current:

$$Q(\Sigma) \equiv \int_\Sigma d^3x \sqrt{\gamma}\, n_\mu J^\mu$$

Thus, for a vector field, covariant conservation is equivalent to actual conservation.
The same argument doesn't apply to the energy-momentum tensor: the problem arises from generalizing $\nabla_\mu J^\mu = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}} J^\mu)$ to a higher-order tensor field, which is necessary in order to have a divergence theorem like Eq. **??**. Indeed:

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho} T^{\rho\nu} + \Gamma^\nu_{\mu\rho} T^{\mu\rho} = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}\, T^{\mu\nu}) + \Gamma^\nu_{\mu\rho} T^{\mu\rho}$$

The last term doesn't allow to convert the integral of $\nabla_\mu T^{\mu\nu}$ to a boundary term. Instead:

$$\partial_\mu(\sqrt{-\mathrm{g}}\, T^{\mu\nu}) = -\sqrt{-\mathrm{g}}\, \Gamma^\nu_{\mu\rho} T^{\mu\rho} \tag{3.49}$$

Therefore, for higher-order tensors, covariant conservation is not equivalent to actual conservation.

§3.5.4.1 Conserved energy from Killing vectors

Given a Killing vector K, it's possible to define a conserved current associated to the energy-momentum tensor as:

$$J^\nu_T := -K_\mu T^{\mu\nu} \tag{3.50}$$

This current is covariantly conserved, as:

$$\nabla_\nu J^\nu_T = -\left(T^{\mu\nu} \nabla_\nu K_\mu + K_\mu \nabla_\nu T^{\mu\nu}\right) = -T^{\mu\nu} \nabla_{(\nu} K_{\mu)} = 0$$

Its associated conserved charge on a spatial hypersurface Σ is defined as:

$$Q_T(\Sigma) := \int_\Sigma d^3x \sqrt{\gamma}\, n_\mu J^\mu_T \tag{3.51}$$

The interpretation of this charge depends on the properties of the Killing vector: if K is globally timelike, then the charge is the energy of matter $E = Q_T(\Sigma)$, meanwhile if K is globally spacelike, then it is the momentum of matter.

Absence of Killing vectors The problem of energy conservation becomes subtle when dealing with spacetimes which do not have any globally timelike Killing vector.
For example, a system comprised of two orbiting stars is modelled by a spacetime which doesn't have such a Killing vector: however, the problem of matter energy conservation does not arise in this case, as the stars, while orbiting each other, emit gravitational waves, thus losing energy and eventually spiraling towards each other. Nonetheless, a meaningful question is that of energy conservation of the total system, i.e. the two stars and the gravitational field. A guess would be to consider a total energy-momentum tensor defined similarly to Eq. **??**, but:

$$T^{\text{total}}_{\mu\nu} = -\frac{2}{\sqrt{-\text{g}}}\left[\frac{1}{16\pi G}\frac{\delta\mathcal{S}_{\text{EH}}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{S}_M}{\delta g^{\mu\nu}}\right] = -\frac{1}{8\pi G}G_{\mu\nu} + T_{\mu\nu} = 0$$

by Einstein field equations. This equation has no physical significance other than expressing the subtlety of energy conservation in Genera Relativity.
Clearly, one could try to understand the energy carried by the gravitational field alone. Unfortunately, there are compelling arguments that there exists no tensor which can be thought as the local energy density of the gravitational field: roughly speaking, the energy density of the Newtonian gravitational field Φ is proportional to $(\nabla\Phi)^2$, so the relativistic equivalent should be proportional to the first derivatives of the metric, which can be made locally vanishing by normal coordinates due to the equivalence principle, and a tensor which vanishes in one coordinate system does so in all of them.

§3.5.5 Energy conditions

To study the general properties of a spacetime without explicitly referencing the specific characteristics of matter, one needs to place certain restrictions on the kinds of energy-momentum tensor which are considered physical: these are the *energy conditions* and express the idea that energy should be positive.

§3.5.5.1 Weak energy condition

This condition states that, for any timelike vector field X:

$$T_{\mu\nu}X^\mu X^\nu \geq 0$$

This quantity is the energy measured by an observer moving along the timelike integral curves of X, and it should be non-negative. A timelike curve can be arbitrarily close to a null curve, thus by continuity:

$$T_{\mu\nu}X^\mu X^\nu \geq 0 \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}) : X_\mu X^\mu \leq 0 \tag{3.52}$$

Fluids Recall the energy-momentum tensor for a fluid Eq. 3.46 and impose the weak energy condition (WLOG $X \cdot X = -1$):

$$(\rho + p)(u \cdot X)^2 - p \geq 0$$

In the rest-frame $u^\mu = (1, 0, 0, 0)$, so considering a constant $X^\mu = (\cosh\varphi, \sinh\varphi, 0, 0)$, whose integral curves are world-lines of observers boosted with rapidity φ with respect to the fluid, then:

$$(\rho + p)\cosh^2\varphi - p \geq 0 \quad \Rightarrow \quad \begin{cases} \rho \geq 0 & \varphi = 0 \\ p \geq -\rho & \varphi \to \infty \end{cases}$$

The first condition ensures that the energy density is positive, the second allows for negative pressure limited from below.
Note that there are situations in which negative energy density makes physical sense: viewing the cosmological constant as part of the energy-momentum density, then any $\Lambda < 0$ violates the weak energy condition. In this sense, AdS space violates the weak energy condition.

Scalar fields The weak energy condition for the energy-momentum tensor of a scalar field theory Eq. 3.44 reads:

$$(X^\mu \partial_\mu \phi)^2 + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + V(\phi) \geq 0$$

The sum of the first two terms is always positive: define $Y_\mu = \partial_\mu \phi + X_\mu X^\nu \partial_\nu \phi : X_\mu Y^\mu = 0$, i.e. orthogonal to X_μ, so it must be spacelike or null, hence $Y_\mu Y^\mu \geq 0$. Rewriting the above condition:

$$\frac{1}{2}(X^\mu \partial_\mu \phi)^2 + \frac{1}{2} Y_\mu Y^\mu + V(\phi) \geq 0 \quad \Rightarrow \quad V(\phi) \geq 0$$

The weak energy condition is clearly violated by any classical field theory with $V(\phi) \leq 0$.

§3.5.5.2 Strong energy condition

This condition states that:

$$R_{\mu\nu} X^\mu X^\nu \geq 0 \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}) : X_\mu X^\mu \leq 0 \tag{3.53}$$

The strong energy condition ensures that timelike geodesics converge, i.e. that gravity is attractive. Using Eq. 3.42, this condition can be rewritten as:

$$\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) X^\mu X^\nu \geq 0$$

Taking yet again $X \cdot X = -1$, the strong energy condition for the energy-momentum tensor of a fluid Eq. 3.46 in the rest-frame reads:

$$(\rho+p)(u\cdot X)^2 - p + \frac{1}{2}(3p-\rho) \geq 0 \quad \Rightarrow \quad (\rho+p)\cosh\varphi^2 - p + \frac{1}{2}(3p-\rho) \geq 0 \quad \Rightarrow \quad \begin{cases} p \geq -\rho/3 & \varphi = 0 \\ p \geq -\rho & \varphi \to \infty \end{cases}$$

It's not difficult to show that $\Lambda > 0$ violates the strong energy condition: dS space is incompatible with it, as timelike geodesics are pulled apart by the expansion of space.
Moreover, any classical scalar field theory with $V(\phi) \geq 0$ violates the strong energy condition.

§3.5.5.3 Null energy condition

This condition states that:

$$T_{\mu\nu} X^\mu X^\nu \geq 0 \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}) : X_\mu X^\mu = 0 \tag{3.54}$$

This condition is implied by both the weak and strong energy condition, but the converse is not true: it is weaker than both conditions. However, it is satisfied by any classical field theory and any perfect fluid with $\rho + p \geq 0$.

§3.5.5.4 Dominant energy condition

Given a future-directed timelike vector field X, it's possible to define the energy density current measured by an observer moving along integral curves of X as:

$$J^\mu \equiv -T^{\mu\nu} X_\nu$$

The dominant energy condition requires that, in addition to the weak energy condition Eq. 3.52:

$$J_\mu J^\mu \leq 0 \tag{3.55}$$

This means that the energy density current is either timelike or null, so requiring that energy doesn't flow faster than time.
It's possible to check that this condition is always satisfied by a classical scalar field theory, while for a perfect fluid it imposes $\rho^2 \geq p^2$.

§3.6 Cosmology

Of the few situations in which Einstein field equations sourced by matter Eq. 3.41 need to be solved directly, the one where $T_{\mu\nu}$ plays a crucial role is Cosmology, the study of the universe as a whole.

§3.6.1 FLRW metric

The key assumption of cosmology is that the universe is spatially homogeneous and isotropic. These conditions restric the possible spatial geometries to only three:

- Euclidean space $\mathbb{R}^3$, with vanishing curvature and metric:

$$ds^2 = dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

- sphere $\mathbb{S}^3$, with uniform positive curvature and metric (implicit unitary radius):

$$ds^2 = \frac{1}{1-r^2} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

- hyperboloid $\mathbb{H}^3$, with uniform negative curvature and metric:

$$ds^2 = \frac{1}{1+r^2} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

The existence of these three symmetric spaces is analogous to the existance of three symmetric spacetimes as solutions of the vacuum field equations: dS and AdS have constant *spacetime curvature*, supplied by the cosmological constant, while $\mathbb{S}^3$ and $\mathbb{H}^3$ have constant *spatial curvature*. Indeed, the metric on $\mathbb{S}^3$ corresponds to the spatial part of the de Sitter metric Eq. 3.14, while the metric on $\mathbb{H}^3$ corresponds to the spatial part of the anti-de Sitter metric Eq. 3.23.
These spatial metrics are written in unified form as:

$$ds^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \tag{3.56}$$

with $k = +1, 0, -1$ on $\mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3$ respectively. Cosmology studies spacetimes in which space expands as the universe evolves, thus the metric takes the form:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \tag{3.57}$$

This is the *Friedmann-Lemaître-Robertson-Walker metric* and the dimensionless factor $a(t)$ can be viewed as the size of the spatial dimensions.

Example 3.6.1 (d)

Sitter metric in global coordinates Eq. 3.22 is a FLRW metric with $k = +1$.

§3.6.1.1 Curvature tensors

To solve the field equations for FLRW metrics, first compute the Ricci tensor. Christoffel symbols are straightforward:

$$\Gamma^{\mu}_{00} = \Gamma^{0}_{i0} = 0 \qquad \Gamma^{0}_{ij} = a\dot{a}\gamma_{ij} \qquad \Gamma^{i}_{0j} = \frac{\dot{a}}{a}\delta^i_j \qquad \Gamma^{i}_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right)$$

Proposition 3.6.1 (T)

e Ricci tensor for a FLRW metric has non-vanishing components:

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right)g_{ij} \tag{3.58}$$

Proof. First, $R_{0i} = 0$ because there's no covariant 3-vector that it could possibly equal to. Then, contracting Eq. **??** and recalling the only non-vanishing Christoffel symbols:

$$R_{00} = -\partial_0\Gamma^i_{i0} - \Gamma^j_{i0}\Gamma^i_{j0} = -3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

For the spatial components, consider the spatial metric in Cartesian coordinates:

$$\gamma_{ij} = \delta_{ij} + \frac{kx_ix_j}{1 - k\mathbf{x}\cdot\mathbf{x}}$$

The Christoffel symbols depend on $\partial\gamma$ and the Ricci tensor on $\partial^2\gamma$, thus to evaluate the latter at $\mathbf{x} = 0$ one only needs the metric up to quadratic order:

$$\gamma_{ij} = \delta_{ij} + kx_ix_j + o(x^4) \quad\Rightarrow\quad \gamma^{ij} = \delta^{ij} - kx^ix^j + o(x^4) \quad\Rightarrow\quad \Gamma^i_{jk} = kx^i\delta_{jk} + o(x^3)$$

where i, j indices are raised or lowered by δ^{ij}. The Ricci tensor is then computed as:

$$\begin{aligned} R_{ij} &= \partial_\rho\Gamma^\rho_{ij} - \partial_j\Gamma^\rho_{\rho i} + \Gamma^\lambda_{ij}\Gamma^\rho_{\rho\lambda} - \Gamma^\Gamma_{\rho i}\Gamma^\rho_{j\lambda} \\ &= (\partial_0\Gamma^0_{ij} + \partial_k\Gamma^k_{ij}) - \partial_j\Gamma^k_{ki} + (\Gamma^0_{ij}\Gamma^k_{k0} + \Gamma^k_{ij}\Gamma^l_{lk}) - (\Gamma^0_{ki}\Gamma^k_{j0} + \Gamma^k_{0i}\Gamma^0_{jk} + \Gamma^k_{li}\Gamma^l_{jk}) \end{aligned}$$

Evaluating this expression at $\mathbf{x} = \mathbf{0}$ allows to drop the $\Gamma^k_{ij}\Gamma^l_{lk}$ term and to replace any undifferentiated γ_{ij} with δ_{ij}, so that:

$$\begin{aligned} R_{ij}(\mathbf{x} = \mathbf{0}) &= \left(\partial_0(a\dot{a}) + 3k - k + 3\dot{a}^2 - \dot{a}^2 - \dot{a}^2\right)\delta_{ij} + o(x^2) \\ &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\delta_{ij} + o(x^2) \end{aligned}$$

Covariance implies that $R_{ij} \sim \gamma_{ij}$, thus the general result is:

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\gamma_{ij} = \frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) g_{ij}$$

□

The Ricci scalar is then easily computed:

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \tag{3.59}$$

Finally, the non-vanishing components of the Einstein tensor are:

$$G_{00} = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \qquad G_{ij} = -\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) g_{ij} \tag{3.60}$$

§3.6.2 Friedmann equations

There only remains to specify the matter content of the universe. By hypothesis, it is filled by a perfect fluid, so that the energy-momentum tensor is that in Eq. 3.46. Assuming that the fluid is at rest in the preferred frame of the universe, i.e. $u^\mu = (1,0,0,0)$ in the FLRW coordinates, the Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$ reads:

$$u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu = 0$$

Recalling that $\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\rho} u^\rho = \Gamma^i_{i0} u^0 = \dot{a}/a$, the *continuity equation* is obtained:

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0 \tag{3.61}$$

This equation expresses the conservation of energy in an expanding universe. The second constraint Eq. 3.48 is trivial for homogeneous isotropic fluids. There remains the equation of state, which for fluids of cosmological interest is simply:

$$p = w\rho \tag{3.62}$$

In particular, $w = 0$ describes pressureless dust, while $w = \frac{1}{3}$ radiation. The continuity equation thus becomes:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

This means that the energy density dillutes as the universe expands, for:

$$\rho = \frac{\rho_0}{a^{3(1+w)}} \tag{3.63}$$

For pressureless dust $\rho \sim a^{-3}$, which is the expected scaling of energy density with volume, while for radiation $\rho \sim a^{-4}$, accounting for an extra a^{-1} factor due to redshift.
With this setting, the temporal component of the Einstein field equations becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{3.64}$$

This is the *Friedmann equation*, and with Eq. 3.63 it describe how the universe expands. The spatial component of the field equations, computed using the Friedmann equation too, reads:

$$\frac{\ddot{a}}{a} - \frac{\Lambda}{3} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3.65}$$

This is the *Raychaudhuri equation*, and it describes the acceleration of the expansion of the universe: it isn't independent of the Friedmann equation, as a time derivative of Eq. 3.64 yields Eq. 3.65.

A particularly simple solution can be found by setting $k = \Lambda = 0$, i.e. considering a universe dominated only by a single homogeneous isotropic fluid with energy density Eq. 3.63:

$$\left(\frac{\dot{a}}{a}\right)^2 \sim \frac{1}{a^{3(1+w)}} \quad \Rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$$

The solution $w = \frac{1}{3}$, i.e. $a(t) \sim t^{1/2}$, describes a radiation-dominated universe, which is a model for roughly the first 50'000 years of our Universe, while $w = 0$, i.e. $a(t) \sim t^{2/3}$, describes a matter-dominated universe, which is a model for roughly the following 10 billion years of our Universe.