IR-Safe NLO QCD with Massive Quarks

An extension of the NSC subtraction scheme

Bachelor Degree in Physics

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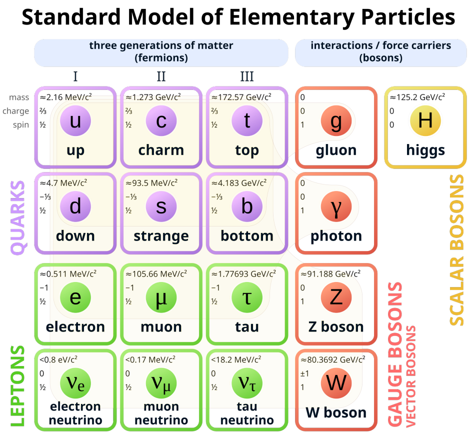
DEGLI STUDI
DI MILANO

Precision estimates at the LHC

Standard Model of fundamental interactions

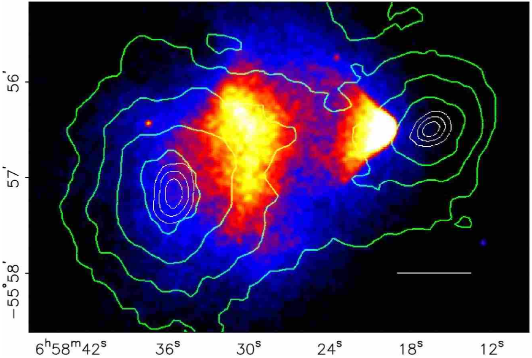
Gauge theory: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$

- $\mathrm{SU}(3)$: strong interaction, with 8 gluons
- $\mathrm{SU}(2)$: weak interaction, with Z, $W^{\pm}$ bosons
- $\mathrm{U}(1)$: electromagnetic interaction, with the photon
- EW symmetry breaking: the Higgs boson



Precision estimates at the LHC

Evidence for Beyond-Standard-Model physics



Main BSM evidence

- dark matter and dark energy
- matter-antimatter asymmetry
- neutrino masses

Figure from Clowe et al. 2006.
Offset between the observed baryonic mass distribution and the gravitational potential in the Bullet Cluster (1E 0657-56).

Precision estimates at the LHC

BSM constraints and shift in research paradigm

Main BSM proposals

- supersymmetric models (MSSM, …)
- dark matter models (WIMPs, axions, …)
- extended gauge sectors ($SO(10)$, …)
- SM Effective Field Theory (SMEFT)

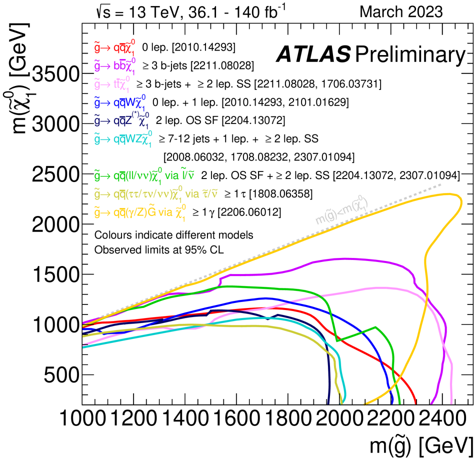
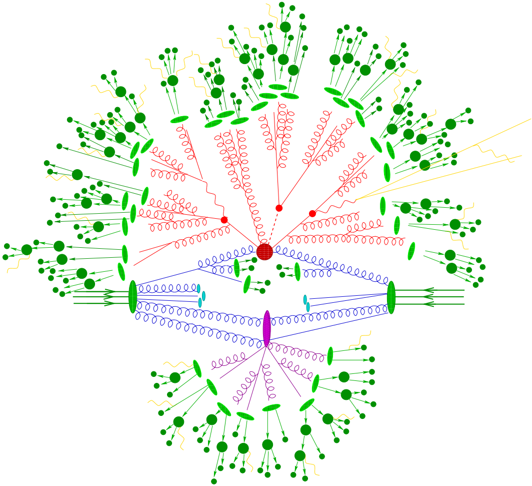


Figure from ATLAS PUB Note 2023-025. Exclusion limits in the $\tilde{g} - \tilde{\chi}_1^0$ mass plane for various models for the decay of the gluino to the lightest supersymmetric particle.

Precision estimates at the LHC

Hard hadronic scattering processes



Hard scattering processes

Characterized by a large momentum transfer, which allows for a perturbative description thanks to *asymptotic freedom*.
Individual partons treated as free particles: hadronic scattering cross-sections studied in terms of partonic scattering cross-section.

Figure from Höche 2015.
Hadronization of jets in hadronic scattering.

Precision estimates at the LHC

Factorization theorem and perturbative QCD

Factorization theorem for hard hadronic scattering processes (Collins et al. 1989):

$$\begin{aligned}
&\mathrm{d}\sigma_{h_1,h_2}(P_1,P_2) \\
&= \sum_{a,b} \int_{[0,1]^2} \frac{\mathrm{d}\xi_1}{\xi_1}\frac{\mathrm{d}\xi_2}{\xi_2} f_a^{(h_1)}(\xi_1,\mu_\mathsf{F}^2) f_b^{(h_2)}(\xi_2,\mu_\mathsf{F}^2) \times \\
&\times \mathrm{d}\hat{\sigma}_{a,b}(\xi_1 P_1, \xi_2 P_2, \alpha_\mathsf{s}, \mu_\mathsf{R}^2, \mu_\mathsf{F}^2) \left[1 + o\left(\frac{\Lambda_\mathsf{QCD}^n}{Q^n}\right)\right]
\end{aligned}$$

Clear separation of energy scales:

- hard hadronic scattering: $Q \sim 100\,\mathrm{GeV} - 1\,\mathrm{TeV}$
- hadronization: $\Lambda_\mathsf{QCD} \approx 200\,\mathrm{MeV}$

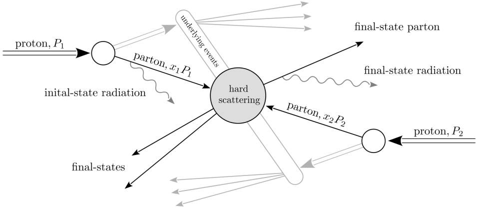


Figure from Asteriadis 2021.
Detail of the partonic scattering in a hard hadronic scattering.

Precision estimates at the LHC

Factorization theorem and perturbative QCD

Asymptotic freedom allows for a perturbative analysis of the underlying partonic scattering:

$$\mathrm{d}\hat{\sigma}_{a,b}(p_1, p_2) = \sum_{n \in \mathbb{N}_0} \mathrm{d}\hat{\sigma}^{(n)}_{a,b}(p_1, p_2)$$

where $\mathrm{d}\hat{\sigma}^{(n)} \sim \alpha_\mathrm{s}^{n_0+n}$.

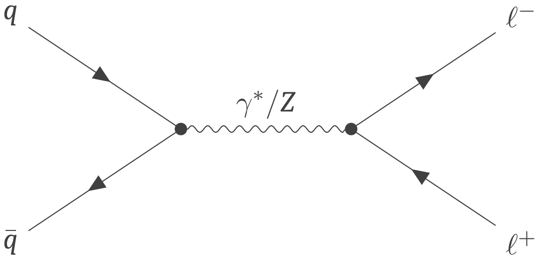
$n \geq 1$ are denoted by N^{n}LO QCD corrections:

- real corrections: additional initial- or final-state radiation
- virtual corrections: additional partonic loops

IR-pole structure of QCD

Radiative corrections to partonic processes

Our focus is on NLO QCD corrections. Consider e.g. the Drell-Yan process:



LO process

IR-pole structure of QCD

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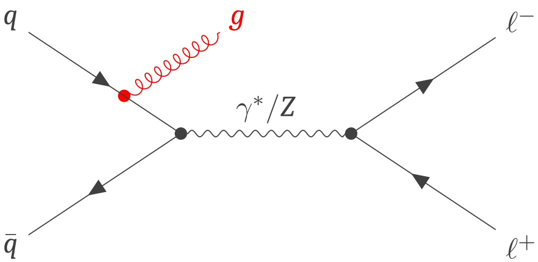

Real correction

IR-pole structure of QCD

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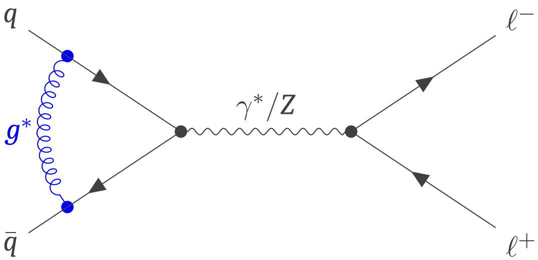
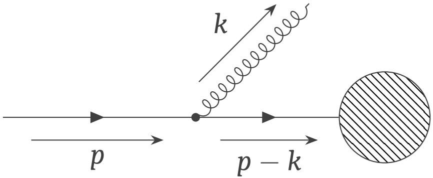


Virtual correction

Main difficulty: infrared singularities in particular kinematic regimes.
Example in real corrections:



$$\sim \quad \frac{1}{(p-k)^2} = -\frac{1}{2E_pE_k(1-\cos\theta)}$$

IR-pole structure of QCD

Infrared singularities of scattering amplitudes

Main difficulty: infrared singularities in particular kinematic regimes.

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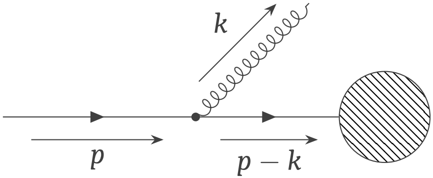

$$\sim \quad \frac{1}{(p-k)^2} = -\frac{1}{2E_p E_k (1-\cos\theta)}$$

Soft singularity: $E_k \to 0$

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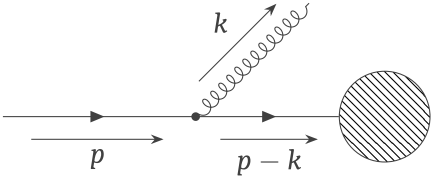

$$\sim \quad \frac{1}{(p-k)^2} = -\frac{1}{2E_pE_k(1-\cos\theta)}$$

Collinear singularity: $\theta \to 0$

IR-pole structure of QCD

Dimensional regularization

The key idea to regularize IR divergences is dimensional regularization:

$$d = 4 - 2\epsilon \quad , \quad \epsilon \in \mathbb{C} : \Re\epsilon < 0$$

Then, soft and collinear singularities are expressed as poles in ϵ:

$$\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k} \left|\mathcal{A}(p_k)\right|^2 \sim \int_0^{\mathcal{E}} \frac{\mathrm{d}E_k}{E_k^{5-d}} \int_0^{\pi} \mathrm{d}\theta \frac{\sin^{d-3}\theta}{1-\cos\theta} \left|\mathcal{A}_0\right|^2$$

IR-pole structure of QCD

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IR-pole structure of QCD
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soft singularity $= -\frac{\mathcal{E}^{-2\epsilon}}{2\epsilon}$

collinear singularity $= -\frac{2^{-2+\epsilon}}{\epsilon}$

IR-pole structure of QCD

Subtraction schemes

ϵ-poles can be extracted from partonic cross-sections via subtraction methods.
General idea using a regular function $f(x)$:

$$I = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} f(x) = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} \left[f(x) - f(0)\right] + f(0) \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}$$

- $\dfrac{f(x) - f(0)}{x^{1+\epsilon}}$ regular at $x = 0$, so it can be numerically integrated with $\epsilon \to 0$;
- $f(0) \displaystyle\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} = -\frac{f(0)}{\epsilon}$ contains the explicit ϵ-pole.

Our aim is finding the most general subtraction terms $f(0)$ for partonic scattering.

NSC subtraction scheme

Sequential extraction of singularities

The Nested Soft-Collinear (NSC) Subtraction Scheme (SS) at NLO is modular and local:

- singular limits extracted by operators: S_i soft limit $E_i \to 0$, C_{ij} collinear limit $\theta_{ij} \to 0$;

NSC subtraction scheme

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- potentially-unresolved partons isolated by clever partitions of unity Δ^i and ω^{ij};

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- potentially-unresolved partons isolated by clever partitions of unity Δ^i and ω^{ij};
- first soft singularities removed applying $\overline{\mathsf{S}}_{\mathfrak{m}} \equiv \mathrm{id} - \mathsf{S}_{\mathfrak{m}}$, then collinear ones removed using $\overline{\mathsf{C}}_{i\mathfrak{m}} \equiv \mathrm{id} - \mathsf{C}_{i\mathfrak{m}}$, with $\mathfrak{m}$ unresolved parton;

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- IR-safe part of cross-section extracted by:

$$\mathrm{id} = \mathsf{S}_{\mathfrak{m}} + \sum_{i \in \mathcal{H}^n_{m,0}(\mathfrak{m})} \overline{\mathsf{S}}_{\mathfrak{m}} \mathsf{C}_{i\mathfrak{m}} + \mathsf{O}^{\mathfrak{m}}_{\mathsf{NLO}} \qquad \mathsf{O}^{\mathfrak{m}}_{\mathsf{NLO}} := \sum_{i \in \mathcal{H}^n_{m,0}} \overline{\mathsf{S}}_{\mathfrak{m}} \overline{\mathsf{C}}_{i\mathfrak{m}} \omega^{\mathfrak{m}i}$$

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- counterterms contain the IR divergences.

NSC subtraction scheme

IR-safe cross-section

The total ϵ-finite NLO cross-section can be written as:

$$\mathrm{d}\hat{\sigma}^{(1)}_{a,b} = \mathrm{d}\hat{\sigma}^{\text{NLO,reg}}_{a,b} + \frac{1}{2\hat{s}} \sum_{n \in \mathcal{G}_m} \left[\frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}^2)}{2\pi} \left\langle I^{(0)}_{\mathrm{T}} \mathcal{F}_{a,b}[\mathcal{X}^n_m] \right\rangle + \left\langle \mathcal{F}^{\text{fin}}_{a,b}[\mathcal{X}^n_m] \right\rangle \right]$$

$$+ \frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}^2)}{2\pi} \sum_{c} \left[P^{\text{gen}}_{f_c, f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\text{gen}}_{f_c, f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]$$

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$$+ \frac{\alpha_s(\mu_R^2)}{2\pi} \sum_c \left[P^{\text{gen}}_{f_c,f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\text{gen}}_{f_c,f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]$$

- NLO final reminder, defined using $\mathrm{O}^{\mathfrak{m}}_{\text{NLO}}$;

NSC subtraction scheme

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NSC subtraction scheme

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- NLO final reminder, defined using $O^{\mathfrak{m}}_{\mathrm{NLO}}$;
- finite counterterm from virtual corrections;
- finite counterterm from PDF renormalization;

NSC subtraction scheme

IR-safe cross-section

The total ϵ-finite NLO cross-section can be written as:

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\mathrm{d}\hat{\sigma}^{(1)}_{a,b} = \mathrm{d}\hat{\sigma}^{\text{NLO,reg}}_{a,b} + \frac{1}{2\hat{s}} \sum_{n \in \mathcal{G}_m} \left[\frac{\alpha_s(\mu_R^2)}{2\pi} \left\langle I_T^{(0)} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \right\rangle + \left\langle \mathcal{F}^{\text{fin}}_{a,b}[\mathcal{X}_m^n] \right\rangle \right]
$$

$$
+ \frac{\alpha_s(\mu_R^2)}{2\pi} \sum_c \left[P^{\text{gen}}_{f_c, f_a} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{c,b} + P^{\text{gen}}_{f_c, f_b} \otimes \mathrm{d}\hat{\sigma}^{(0)}_{a,c} \right]
$$

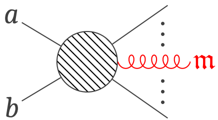
- NLO final reminder, defined using $O^{\mathfrak{m}}_{\text{NLO}}$;
- finite counterterm from virtual corrections;
- finite counterterm from PDF renormalization;
- counterterm we are interested in.

NSC subtraction scheme

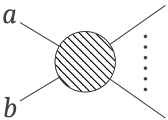
Pole extraction through operators

Soft singularities

$$\left\langle \mathsf{S}_{\mathfrak{m}} \Delta^{\mathfrak{m}} \left| \vphantom{\int} \right.\right.$$



$$\left.\left. \vphantom{\int} \right|^2 \right\rangle = [\alpha_s] \left\langle I_S(\epsilon) \cdot \left| \vphantom{\int} \right.\right.$$



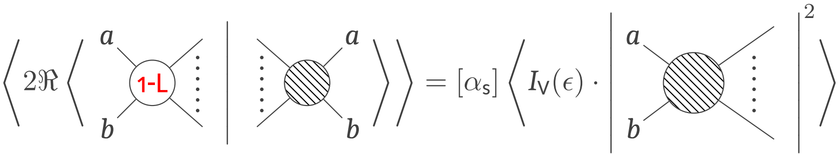
$$\left.\left. \vphantom{\int} \right|^2 \right\rangle$$

$$I_S(\epsilon) := -\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \eta_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j) \, {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij})$$

NSC subtraction scheme

Pole extraction through operators

Virtual singularities



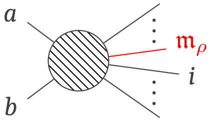
$$I_{\mathsf{V}}(\epsilon) \equiv \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{2 p_i \cdot p_j} \right)^{\epsilon} \cos\left(\lambda_{ij} \pi \epsilon\right) \left[\frac{1}{\epsilon^2} + \frac{1}{\mathbf{T}_i^2} \frac{\gamma_i}{\epsilon} \right]$$

NSC subtraction scheme

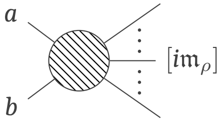
Pole extraction through operators

Hard-collinear singularities

$$\sum_{\rho=1}^{2n_f+1}\ \sum_{i\in\mathcal{H}^{n}_{m,0}(\mathfrak{m}_\rho)} \left\langle \bar{\mathsf{S}}_{\mathfrak{m}}\mathsf{C}_{i\mathfrak{m}_\rho}\Delta^{\mathfrak{m}_\rho} \left|\right.\right.$$



$$\left.\left.\right|^2 \right\rangle \sim [\alpha_s] \left\langle I_{\mathsf{C}}(\epsilon)\cdot \left|\right.\right.$$



$$\left.\left.\right|^2 \right\rangle$$

$$I_{\mathsf{C}}(\epsilon) := \sum_{i\in\mathcal{H}^{n}_{m,0}} \frac{\Gamma_{i,f_i}}{\epsilon} = \frac{1}{\epsilon}\sum_{i\in\mathcal{H}^{n}_{m,0}} \left[\gamma_i + 2\mathbf{T}_i^2 \log\frac{\mathcal{E}}{E_i}\right] + o(\epsilon^0)$$

NSC subtraction scheme

Pole cancellation: $I_{\mathsf{S}}(\epsilon) + I_{\mathsf{V}}(\epsilon)$

$$I_{\mathsf{S+V}} = \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[-\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} K_{ij} + \left(\frac{\mu^2}{2 p_i \cdot p_j} \right)^{\epsilon} \cos\left(\lambda_{ij} \pi \epsilon\right) \left(\frac{1}{\epsilon^2} + \frac{1}{\mathbf{T}_i^2} \frac{\gamma_i}{\epsilon} \right) \right]$$

$$\equiv \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij} \, {}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) = 1 - \epsilon \log \eta_{ij} + o(\epsilon^2)$$

$$= \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \left[-1 + \epsilon \log \eta_{ij} + \left(\frac{\mathcal{E}^2}{E_i E_j} \right)^{\epsilon} \eta_{ij}^{-\epsilon} \cos\left(\lambda_{ij} \pi \epsilon\right) \left(1 + \epsilon \frac{\gamma_i}{\mathbf{T}_i^2} \right) \right]$$

$$= \frac{1}{\epsilon} \sum_{i \neq j \in \mathcal{X}^n_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(L_i + L_j + \frac{\gamma_i}{\mathbf{T}_i^2} \right) + o(\epsilon^0) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) + o(\epsilon^0)$$

NSC subtraction scheme

Pole cancellation: $I_{\mathsf{T}}(\epsilon)$

Compare:

$$I_{\mathsf{S+V}}(\epsilon) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{S+V}} + o(\epsilon)$$

$$I_{\mathsf{C}}(\epsilon) = +\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^n_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{C}} + o(\epsilon)$$

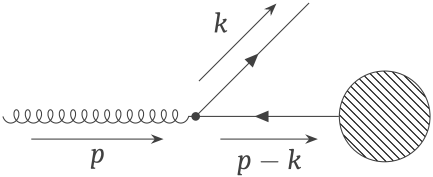
Hence:

$$I_{\mathsf{T}}(\epsilon) \equiv I_{\mathsf{S}}(\epsilon) + I_{\mathsf{C}}(\epsilon) + I_{\mathsf{V}}(\epsilon) = I^{(0)}_{\mathsf{T}} + o(\epsilon)$$

NSC SS with massive quarks

Mass-regualtion of soft and collinear limits

Massive partons do not determine soft or collinear singularities:



$$\sim \quad \frac{1}{(p-k)^2 - m^2} = -\frac{1}{2E_p\,(E_k - |\mathbf{k}|\cos\theta)}$$

Clearly, no more collinear singularities as $E_k \neq |\mathbf{k}|$ for $m \neq 0$.
The soft limit is non-singular even in the massless case.

NSC SS with massive quarks

Generalized soft operator

- In presence of final-state massive partons, $I_{\mathrm{S}}(\epsilon)$ becomes:

$$I_{\mathrm{S}}(\epsilon) = \frac{1}{2\epsilon}\left(\frac{2\mathcal{E}}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\sum_{i,j\in\mathcal{X}^n_{m,0}}(\mathbf{T}_i\cdot\mathbf{T}_j)I_{i,j}(\epsilon)$$

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Generalized virtual operator

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Pole cancellation: generalized pole terms in $I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon)$

It is convenient to group summations as follows:

$$
\begin{aligned}
I_{\mathrm{S+V}}(\epsilon) &= \sum_{i,j \in \mathcal{X}^n_{m,0}} I^{i,j}_{\mathrm{S}}(\epsilon) + \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \tilde{I}^{i,j}_{\mathrm{V}}(\epsilon) - \sum_{i \in \mathcal{X}^n_{m,0}} \Gamma_i(\epsilon) \\
&= \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \left[I^{i,j}_{\mathrm{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathrm{V}}(\epsilon) \right] + \sum_{i \in \mathcal{X}^n_{m,\mathrm{m}}} \left[I^{i,i}_{\mathrm{S}}(\epsilon) - \Gamma_i(\epsilon) \right] - \sum_{i \in \mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)
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&= \underline{\sum_{i \neq j \in \mathcal{X}^n_{m,0}} \left[I^{i,j}_{\mathsf{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon) \right]} + \sum_{i \in \mathcal{X}^n_{m,\mathrm{m}}} \left[I^{i,i}_{\mathsf{S}}(\epsilon) - \Gamma_i(\epsilon) \right] - \sum_{i \in \mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)
\end{aligned}
$$

Then, ϵ^{-2}-singularities are only present in the first sum:

$$
I^{i,j}_{\mathsf{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon) = \chi_{i,j} + Ж_{i,j} + o(\epsilon)
$$

Laurent expansion: pole terms and ϵ-finite finite reminder.

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Pole cancellation: quadratic pole terms

Quadratic pole terms have a complex general form:

$$\chi_{i,j} \equiv \frac{1}{\epsilon^2}\left(\frac{1}{2}I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)}\right) + \frac{1}{\epsilon}\left(\frac{1}{2}I_{i,j}^{(0)} + \mathcal{V}_{i,j}^{(-1)} - \mathscr{L}_{\mathrm{m}} I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)} \log\frac{\mu^2}{|s_{ij}|}\right)$$

However, with further manipulation, the explicit structure is simple:

$$\chi_{i,j} = \begin{cases} 0 & i,j \text{ massive} \\ \frac{1}{\epsilon}L_j & i \text{ massive}, j \text{ massless} \\ \frac{1}{\epsilon}L_i & i \text{ massless}, j \text{ massive} \\ \frac{1}{\epsilon}\left(L_i + L_j\right) & i,j \text{ massless} \end{cases} \qquad L_k \equiv \log\frac{\mathcal{E}}{E_k}$$

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It is convenient to group summations as follows:

$$I_{\mathsf{S+V}}(\epsilon) = \sum_{i,j \in \mathcal{X}^n_{m,0}} I^{i,j}_{\mathsf{S}}(\epsilon) + \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon) - \sum_{i \in \mathcal{X}^n_{m,0}} \Gamma_i(\epsilon)$$

$$= \sum_{i \neq j \in \mathcal{X}^n_{m,0}} \left[I^{i,j}_{\mathsf{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon) \right] + \underline{\sum_{i \in \mathcal{X}^n_{m,\mathrm{m}}} \left[I^{i,i}_{\mathsf{S}}(\epsilon) - \Gamma_i(\epsilon) \right]} - \sum_{i \in \mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)$$

The <u>second sum</u> is manifestly ϵ-finite:

$$\sum_{i \in \mathcal{X}^n_{m,\mathrm{m}}} \left[C_{\mathrm{F}} \left(\frac{1}{\epsilon} - \frac{1}{\kappa_i} \log \frac{1-\kappa_i}{1+\kappa_i} - 2\mathscr{L}_{\mathrm{m}} \right) - C_{\mathrm{F}} \left(\frac{1}{\epsilon} + \frac{1}{2} \log \frac{m^2_{Q_i}}{\mu^2} - 2 \right) \right] \equiv \sum_{i \in \mathcal{X}^n_{m,\mathrm{m}}} Д_i$$

NSC SS with massive quarks

Pole cancellation: generalized pole terms in $I_{\mathsf{S}}(\epsilon) + I_{\mathsf{V}}(\epsilon)$

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$$= \sum_{i\neq j\in\mathcal{X}^n_{m,0}} \left[I^{i,j}_{\mathsf{S}}(\epsilon) + \tilde{I}^{i,j}_{\mathsf{V}}(\epsilon)\right] + \sum_{i\in\mathcal{X}^n_{m,\mathrm{m}}} \left[I^{i,i}_{\mathsf{S}}(\epsilon) - \Gamma_i(\epsilon)\right] - \underline{\sum_{i\in\mathcal{H}^n_{m,0}} \Gamma_i(\epsilon)}$$

The third sum has isolated ϵ-poles by definition:

$$\sum_{i\in\mathcal{H}^n_{m,0}} \Gamma_i(\epsilon) = \sum_{i\in\mathcal{H}^n_{m,0}} \left(\frac{1}{\epsilon}\gamma_i + \text{Щ}_i\right) \qquad \text{Щ}_i \equiv -\delta_{f_i,g}\frac{2}{3}T_{\mathsf{R}}\sum_{\rho=1}^{n_F}\log\frac{m^2_{Q_\rho}}{\mu^2}$$

NSC SS with massive quarks

Pole cancellation: generalized integrated counterterms

Puttin geverything together, using colour-conservation we find:

$$I_{\mathsf{S+V}}(\epsilon) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^{n}_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{S+V}} + o(\epsilon)$$

$$I_{\mathsf{C}}(\epsilon) = +\frac{1}{\epsilon} \sum_{i \in \mathcal{H}^{n}_{m,0}} \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + I^{(0)}_{\mathsf{C}} + o(\epsilon)$$

The total operator $I_{\mathsf{T}} \equiv I_{\mathsf{S}} + I_{\mathsf{C}} + I_{\mathsf{V}}$ is the ϵ-finite, and the integrated counterterms read:

$$\begin{aligned} I^{(0)}_{\mathsf{T}} = \sum_{i \neq j \in \mathcal{X}^{n}_{m,0}} (\mathbf{T}_i \cdot \mathbf{T}_j) \text{Ж}_{i,j} + \sum_{i \in \mathcal{X}^{n}_{m,\mathrm{m}}} \text{Д}_i - \sum_{i \in \mathcal{H}^{n}_{m,0}} \text{Ц}_i - \sum_{i \in \mathcal{H}^{n}_{m,0}} \left[2\mathscr{L}_i \left(\gamma_i + 2\mathbf{T}_i^2 L_i\right) + 2\mathbf{T}_i^2 L_i^2\right] \\ + (N_q + N_{\bar{q}}) \frac{C_{\mathtt{F}}}{6} \left(39 - 4\pi^2\right) + N_g \left[\frac{C_{\mathtt{A}}}{9} \left(67 - 6\pi^2\right) - \frac{23}{9} T_{\mathsf{R}} n_f\right] \end{aligned}$$

Conclusions

Results and future developments

Main results

- proof of IR-poles cancellation even with generic massive final-state partons
- computation of generalized integrated counterterms

Future developments

- extension to NNLO in the NSC SS
- inclusion of initial-state massive partons
- resummation of massive logarithms
- application to heavy-quark processes

Figure from CMS Report CERN–EP 2025–025. Example Feynman diagram of non-resonant mono-t production at tree-level mediated by a spin-1 boson M, which decays directly into a dark-matter pair $\chi\bar{\chi}$.