Quantum Field Theory 1

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Contents

Part I

Field Theory

Chapter 1

Classical Field Theory

1.1 Continuous limit

1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x-axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a.
Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n, one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2} \dot{\eta}_n^2 - \frac{\lambda}{2} \left(\eta_n - \eta_{n-1} \right)^2 \right] \tag{1.1}$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda \left(\eta_{n+1} - 2\eta_n + \eta_{n-1} \right) \tag{1.2}$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \tag{1.3}$$

where the dispersion relation is:

$$\omega^2 = 2\lambda \left(1 - \cos k \right) \approx \lambda k^2 \tag{1.4}$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \qquad \Rightarrow \qquad k_m = \frac{2\pi m}{N}\,,\; m = 0, 1, \ldots, N-1 \tag{1.5}$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} \left[\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)} \right] \tag{1.6}$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjucated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \tag{1.7}$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*[1]:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \tag{1.8}$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators* [2] $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} \left[e^{i(k_m n - \omega_m t)}\hat{a}_m + e^{-i(k_m n - \omega_m t)}\hat{a}_m^\dagger\right] \tag{1.9}$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \tag{1.10}$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2}\hat{\pi}_n^2 + \frac{\lambda}{2}\left(\hat{\eta}_n - \hat{\eta}_{n-1}\right)^2\right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}\right) \tag{1.11}$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m \left|0\right\rangle \quad \forall m = 0, 1, \dots, N-1 \tag{1.12}$$

$$\left|n_0, n_1, \dots, n_{N-1}\right\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} \left|0\right\rangle \tag{1.13}$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2}\sum_{m=0}^{N-1} \hbar\omega_m \tag{1.14}$$

$$E_{n_0, n_1, \dots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \tag{1.15}$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as "particles" and n_m is the number of phonons in the m^{th} normal mode.

[1] Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t}\hat{\mathcal{O}}(0)e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{\mathrm{d}\hat{\mathcal{O}}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

[2] For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{p}(t)$ and $\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) + \hat{a}^\dagger(t)\right] \qquad \hat{p}(t) = -i\omega\sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) - \hat{a}^\dagger(t)\right]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger(t)\hat{a} + \frac{1}{2}\right)$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.

1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \to 0$ and $N \to \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x,t)$ dependent on the continuous real coordinate $x \in [0, R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x}\right)^2 \qquad\qquad \sum_n \longrightarrow \frac{1}{a}\int_0^R \mathrm{d}x$$

Proposition 1.1.1

In the continuous limit:

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x - x') = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

Proof. By direct calculation:

$$a \sum_n f(an)\frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R \mathrm{d}x\, f(x)\delta(x-y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \to kx$, symmetrizing $k_m \in [-\pi, \pi]$ (instead of $[0, 2\pi]$) one finds:

$$\delta(x-x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na}\sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

where integration limits are $\pm\frac{\pi}{a} \to \pm\infty$. □

Proposition 1.1.2

The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R \mathrm{d}x\, e^{i(k-k')x} = 2\pi\delta(k-k')$$

By these relations, it can be seen that $\frac{\mathrm{d}k}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k + \mathrm{d}k$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.
In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}\rho_0(\partial_t\eta)^2 - \frac{\kappa}{2}(\partial_x\eta)^2\right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x,t) \equiv \sqrt{\rho_0}\eta(x,t)$ and $\kappa = c^2\rho_0$ and adding a pinning term $\propto \varphi^2$, the Lagrangian can be written as:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \frac{m^2c^4}{2}\phi^2\right] \tag{1.16}$$

The classical equation of motion of this field yields:

$$\partial_t^2\phi = c^2\partial_x^2\phi - m^2c^4\phi \tag{1.17}$$

The solutions of this wave equation can be written as:

$$\phi(x,t) = e^{i(kx-\omega_k t)} \tag{1.18}$$

with dispersion relation:

$$\omega_k^2 = c^2k^2 + m^2c^4 \tag{1.19}$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x,t) \coloneqq \frac{\partial L}{\partial(\partial_t\phi)} = \partial_t\phi(x,t) \tag{1.20}$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R \mathrm{d}x \left[\frac{1}{2}\Pi^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2c^4}{2}\phi^2\right] \tag{1.21}$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\sqrt{\frac{\hbar}{2\omega_k}}\left[e^{i(kx-\omega_k t)}\hat{a}_k + e^{-i(kx-\omega_k t)}\hat{a}_k^\dagger\right] \tag{1.22}$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi\delta(k-k') \tag{1.23}$$

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = i\hbar\delta(x-x') \tag{1.24}$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\frac{1}{2}\hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger\right) = E_0 + \int_{\mathbb{R}}\frac{\mathrm{d}k}{2\pi}\hbar\omega_k\hat{a}_k^\dagger\hat{a}_k \tag{1.25}$$

The ground-state energy can be computed from Eq. 1.14, defining $\mathrm{Vol} \coloneqq 2\pi\delta(k=0)$:

$$E_0 = \mathrm{Vol}\int_{\mathbb{R}}\frac{\mathrm{d}k}{2\pi}\frac{1}{2}\hbar\omega_k \tag{1.26}$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

1.2 Spacetime symmetries

1.2.1 Lie groups

Definition 1.2.1: Lie group

A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d-dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0,\dots,0) = e$.

Definition 1.2.2: Representation

Given a group G and a vector space $V(\mathbb{K})$, a *representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Given the isomorphism $\mathrm{GL}(V) \to \mathbb{K}^{n\times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to *de facto* represent G as matrices acting on elements of V, i.e. $\rho : G \to \mathbb{K}^{n\times n}$.

Theorem 1.2.1

Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ as:

$$\rho(g(\theta)) = e^{i\theta_a T^a} \tag{1.27}$$

where $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ are the *generators* of G on V.

Definition 1.2.3: Lie algebra

Given a Lie group G with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, its *Lie algebra* is:

$$[T^a, T^b] = i f^{ab}{}_c T^c \tag{1.28}$$

where $f^{ab}{}_c$ are called the *structure constants*.

Proposition 1.2.1

The Lie algebra of a Lie group is independent of the representation.

Proposition 1.2.2

Any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

Definition 1.2.4: Casimir operator

Given a Lie group with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation, Casimir operators are operators proportional to id_V, and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

Proposition 1.2.3

A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

1.2.2 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu}x^\mu x^\nu$, i.e. the orthogonal group $\mathrm{O}(1,3)$ (with signature $(+,-,-,-)$). The condition that $\Lambda^\mu{}_\nu$ must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \tag{1.29}$$

This implies that $\det\Lambda = \pm 1$: a transformation with $\det\Lambda = -1$ can always be written as the product of a transformation with $\det\Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines $\mathrm{SO}(1,3) \coloneqq \{\Lambda \in \mathrm{O}(1,3) : \det\Lambda = 1\}$.
Writing explicitly the temporal component $1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$, it is clear that $(\Lambda^0{}_0)^2 \geq 1$. Therefore, $\mathrm{O}(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0{}_0 \geq 1$ and the non-orthochronous component with $\Lambda^0{}_0 \leq 1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a dircrete transformation which reverses the sign of the temporal component.

Definition 1.2.5: Lorentz group

The *Lorentz group* $\mathrm{SO}^+(1,3)$ is the orthochronous component of $\mathrm{SO}(1,3)$.

The discrete transformations are factored out of the Lorentz group: these are *parity* and *time reversal*, which can be represented as $\mathcal{P}^\mu{}_\nu = \mathrm{diag}\,(+1,-1,-1,-1)$ and $\mathcal{T}^\mu_\nu = \mathrm{diag}\,(-1,+1,+1,+1)$. Applying these discrete transformations in all combinations (id, $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{PT}$) one gets the four disconnected components of $\mathrm{SO}(1,3)$, which are not simply connected. This means that Lorentz invariance does not include parity and time reversal invariance.
Considering the infinitesimal transformation and applying Eq. 1.29:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \qquad \Rightarrow \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x,y), (y,z) and (z,x) planes and the 3 rapidities of hyperbolic rotations in the (t,x), (t,y) and (t,z) planes.

Theorem 1.2.2

The Lorentz group is a non-compact Lie group.

Proof. Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in $[0, 2\pi)$, rapidities vary in $\mathbb{R}$, so the differentiable manifold associated to $\mathrm{SO}^+(1,3)$ is not compact. □

1.2.2.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \mathrm{SO}^+(1,3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.30}$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a n-dimensional representation of $\mathrm{SO}^+(1,3)$, both $[J^{\mu\nu}]^i{}_j$ and $[\Lambda]^i{}_j$ are $\mathbb{C}^{n\times n}$ matrices (Λ is real): for example, the $n = 1$ representation acts on *scalars*, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \, \forall \mu, \nu = 0, \ldots, 3$.

4-vectors The $n = 4$ representation acts on *contravariant 4-vectors* v^μ, which transform according to $v^\mu \mapsto \Lambda^\mu{}_\nu v^\nu$, and *covariant 4-vectors* v_μ, which transform according to $v_\mu \mapsto \Lambda_\mu{}^\nu v_\nu$. In this representation, the generators are represented as $\mathbb{C}^{4\times 4}$ matrices:

$$[J^{\mu\nu}]^\rho{}_\sigma = i\left(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma\right) \tag{1.31}$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}^+(1,3)$, called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \tag{1.32}$$

It is convenient to rearrange the 6 components of $J^{\mu\nu}$ into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad K^i := J^{i0} \tag{1.33}$$

The $\mathfrak{so}^+(1,3)$ can then be rewritten as:

$$[J^i, J^j] = i\epsilon^{ijk}J^k \qquad [J^i, K^j] = i\epsilon^{ijk}K^k \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k \tag{1.34}$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^i are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \qquad \eta^i := \omega^{i0} \tag{1.35}$$

so that:

$$\Lambda = \exp\left[-i\boldsymbol{\theta}\cdot\mathbf{J} + i\boldsymbol{\eta}\cdot\mathbf{K}\right] \tag{1.36}$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts wich increase velocities with respect to said frame.

1.2.2.2 Tensor Representations

A generic (p, q)-tensor transforms as:

$$T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \mapsto \Lambda^{\mu_1}{}_{\alpha_1} \ldots \Lambda^{\mu_p}{}_{\alpha_p} \Lambda_{\nu_1}{}^{\beta_1} \ldots \Lambda_{\nu_q}{}^{\beta_q} T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} \tag{1.37}$$

The representation of the Lorentz group which acts on (p, q)-tensors is of degree $n = 4^{p+q}$, however it is reducible into the direct product of $p + q$ 4-dimensional representations as of Eq. 1.38.
Moreover, consider the action of the Lorentz group on $(2, 0)$-tensors: being $T^{\mu\nu} \mapsto \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu} T^{\mu\nu} \mapsto \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$\mathsf{4} \otimes \mathsf{4} = \mathsf{1} \oplus \mathsf{6} \oplus \mathsf{9} \tag{1.38}$$

These are irreducible representations which, given a generic tensor $T^{\mu\nu}$, act on S, $A^{\mu\nu}$ and $S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu} S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right)$ and $S^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right)$.

Decomposition under rotations Restricting the action to the SO(3) sub-group of $\mathrm{SO}^+(1, 3)$, tensors can be decomposed according to irreducible representations of SO(3), which are labelled by the angular momentum $j \in \mathbb{N}_0$ and are of degree $n = 2j + 1$. Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \tag{1.39}$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$. A 4-vector v^μ is irreducible under the action of $\mathrm{SO}^+(1, 3)$, but under SO(3) it is decomposed into v^0 and $\mathbf{v}$, so $v^\mu \in \mathbf{0} \oplus \mathbf{1}$. A $(2, 0)$-tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.38: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2}\epsilon^{ijk} A^{jk}$, so $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ under spatial rotations.
Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$, $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$, $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$ and $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk} A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

Example 1.2.1

Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two *invariant tensors* under $\mathrm{SO}^+(1, 3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.29, and the Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\sigma{}_\gamma \Lambda^\rho{}_\delta \epsilon^{\alpha\beta\gamma\delta} = (\det \Lambda)\, \epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

1.2.2.3 Spinorial representations

The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same, which means that SU(2) and SO(3) are indistinguishable by infinitesimal transformations; however, they are globally different, as SO(3) rotations are periodic by 2π, while SU(2) rotations are periodic by 4π: in particular, it can be shown that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$, i.e. SU(2) is the universal cover of SO(3). This means that SU(2) representations can be labelled by $j \in \frac{1}{2}\mathbb{N}_0$, where half-integer spin representations are known as *spinorial representations*: they act on spinors, i.e. objects which change sign under rotations of 2π (thus not suitable to represent SO(3)).

Example 1.2.2

The $\mathbf{\frac{1}{2}}$ representation of SU(2) is a 2-dimensional representation where $J^i = \frac{\sigma^i}{2}$: Pauli matrices satisfy $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, thus the $\mathfrak{su}(2)$ algebra is satisfied. Denoting the $m = \pm\frac{1}{2}$ states in the $\mathbf{\frac{1}{2}}$ representation as $|\uparrow\rangle$ and $|\downarrow\rangle$, the Clebsch-Gordan decomposition $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$ yields the triplet $(j = 1)$ $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$ and the singlet $(j = 0)$ $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Proposition 1.2.4

The Lorentz algebra $\mathfrak{so}^+(1,3)$ can be decomposed as $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

Proof. Given the $\mathfrak{so}^+(1,3)$ algebra in Eq. 1.33, it is possible to define:

$$\mathbf{J}_\pm \coloneqq \frac{1}{2}(\mathbf{J} \pm i\mathbf{K})$$

The Lie algebra then becomes:

$$[\mathbf{J}^i_\pm, \mathbf{J}^j_\pm] = i\epsilon^{ijk}\mathbf{J}^k_\pm \qquad\qquad [\mathbf{J}^i_\pm, \mathbf{J}^j_\mp] = 0$$

These are two commuting $\mathfrak{so}(2)$ algebras, thus proving the thesis. □

As observed before, this does not imply that $\mathrm{SO}^+(1,3)$ is globally equivalent to $\mathrm{SU}(2) \times \mathrm{SU}(2)$: in fact, $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(4)$, while the universal cover of $\mathrm{SO}^+(1,3)$ is $\mathrm{SL}(2,\mathbb{C})$, as it can be shown that $\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.
By Prop. 1.2.4, representations of $\mathrm{SO}^+(1,3)$ can be labelled by $(j_-, j_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$, with each index labelling a representation of SU(2): as $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, the (j_-, j_+) representation contains states with all possible spins $|j_+ - j_-| \leq j \leq j_+ + j_-$, and it is a representation of degree $n = (2j_- + 1)(2j_+ + 1)$.
$(\mathbf{0}, \mathbf{0})$ is the trivial (scalar) representation, as both $\mathbf{J}_\pm = 0$ and $\mathbf{J} = \mathbf{K} = 0$.
$\left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ and $\left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$ are 2-dimensional spinorial representations. These representations act on different spinors $(\psi_\mathrm{L})_\alpha \in \left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ and $(\psi_\mathrm{R})_\alpha \in \left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$, with $\alpha = 1, 2$, which are called *left-* and *right-handed Weyl spinors*. In $\left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ the generators are $\mathbf{J}_- = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{J}_+ = \mathbf{0}$, while in $\left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$ they are $\mathbf{J}_- = \mathbf{0}$ and $\mathbf{J}_+ = \frac{\boldsymbol{\sigma}}{2}$, thus one finds $\mathbf{J}_\mathrm{L} = \mathbf{J}_\mathrm{R} = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{K}_\mathrm{L} = -\mathbf{K}_\mathrm{R} = i\frac{\boldsymbol{\sigma}}{2}$, so that:

$$\psi_\mathrm{L} \mapsto \Lambda_\mathrm{L}\psi_\mathrm{L} = \exp\left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_\mathrm{L} \tag{1.40}$$

$$\psi_\mathrm{R} \mapsto \Lambda_\mathrm{R}\psi_\mathrm{R} = \exp\left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_\mathrm{R} \tag{1.41}$$

Note that the generators K^i are not hermitian, as expected from Prop. 1.2.3. Furthermore, note that $\Lambda_{\mathrm{L,L}} \in \mathbb{C}^{2\times 2}$, therefore $\psi_{\mathrm{L,R}} \in \mathbb{C}^2$.

Proposition 1.2.5

Given $\psi_{\mathrm{L}} \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ and $\psi_{\mathrm{R}} \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$, then $\sigma^2\psi_{\mathrm{L}}^* \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ and $\sigma^2\psi_{\mathrm{R}}^* \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$.

Proof. Recall that for Pauli matrices $\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^*$, so $\sigma^2\Lambda_{\mathrm{L}}^*\sigma^2 = \Lambda_{\mathrm{R}}$ and:

$$\sigma^2\psi_{\mathrm{L}}^* \mapsto \sigma^2\left(\Lambda_{\mathrm{L}}\psi_{\mathrm{L}}\right)^* = \left(\sigma^2\Lambda_{\mathrm{L}}^*\sigma^2\right)\sigma^2\psi_{\mathrm{L}}^* = \Lambda_{\mathrm{R}}\sigma^2\psi_{\mathrm{L}}^* \quad \Rightarrow \quad \sigma^2\psi_{\mathrm{L}}^* \in \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right)$$

where $\sigma^2\sigma^2 = \mathrm{I}_2$ was used. The proof for $\sigma^2\psi_{\mathrm{R}}^*$ is analogous. □

Definition 1.2.6: Charge conjugation

On Weyl spinors, the *charge conjugation operator* is defined as:

$$\psi_{\mathrm{L}}^c := i\sigma^2\psi_{\mathrm{L}}^* \qquad \psi_{\mathrm{R}}^c := -i\sigma^2\psi_{\mathrm{R}}^* \tag{1.42}$$

By Prop. 1.2.5, charge conjugation changes transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the i factor ensures that applying this operator twice yields the identity operator.
$\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ is a 4-dimensional complex representation: as $j = 0, 1$, this representation acts on complex 4-vectors of the form $((\psi_{\mathrm{L}})_\alpha, (\xi_{\mathrm{R}})_\beta) \in \mathbb{C}^4$, called *Dirac spinors*, and $\Lambda = \mathrm{diag}\left(\Lambda_{\mathrm{L}}, \Lambda_{\mathrm{R}}\right) \in \mathbb{C}^{4\times 4}$. To explicit this relation, set $\psi_{\mathrm{R}} \equiv i\sigma^2\psi_{\mathrm{L}}^*$, $\xi_{\mathrm{L}} \equiv -i\sigma^2\xi_{\mathrm{R}}^*$ and $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$: it can be shown, then, that $\xi_{\mathrm{R}}^\dagger\sigma^\mu\psi_{\mathrm{R}}$ and $\xi_{\mathrm{L}}^\dagger\bar{\sigma}^\mu\psi_{\mathrm{L}}$ are contravariant 4-vectors. Although these 4-vectors are complex by construction, being the matrix $\Lambda^\mu{}_\nu$ which represents the Lorentz transformation of a 4-vector real, a reality condition $v_\mu^* = v_\mu$ is Lorentz invariant.

Parity Note that $\mathcal{P}\mathbf{K} = -\mathbf{K}$, as the velocity of the boost gets reversed, while $\mathcal{P}\mathbf{J} = \mathbf{J}$: this means that $\mathcal{P}\mathbf{J}_\pm = \mathbf{J}_\mp$, i.e. parity exchanges a $(\mathbf{j}_-, \mathbf{j}_+)$ representation into a $(\mathbf{j}_+, \mathbf{j}_-)$ representation. Therefore, a $(\mathbf{j}_-, \mathbf{j}_+)$ representation of $\mathrm{SO}^+(1,3)$ is a basis for the representation of the parity transformation iff $j_- = j_+$.

Example 1.2.3

Weyl spinors (separately) are not a basis for a representation of the parity transformation, but Dirac spinors are.

1.2.2.4 Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields A scalar field transforms as:

$$\phi'(x') = \phi(x) \tag{1.43}$$

Consider an infinitesimal transformation $x'^{\rho} = x^{\rho} + \delta x^{\rho}$, with $\delta x^{\rho} = -\frac{i}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^{\rho}{}_{\sigma}\, x^{\sigma}$ as of Eq. 1.30. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar representation of $\mathrm{SO}^{+}(1,3)$ is the trivial one ($J^{\mu\nu} = 0$).
However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$: while $\delta\phi$ studies only a single degree of freedom, as the point $\mathrm{P} \in \mathbb{R}^{1,3}$ is kept constant and only $\phi(\mathrm{P})$ can vary (i.e. the base space is one-dimensional), $\delta_0\phi$ studies $\phi(\mathrm{P})$ with P varying over $\mathbb{R}^{1,3}$, thus the base space is now a space of functions, which is infinite-dimensional. Therefore, $\delta\phi$ provides a finite-dimensional representation of the generators, while $\delta_0\phi$ an infinite-dimensional one.
To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^{\rho}\partial_{\rho}\phi = \frac{i}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^{\rho}{}_{\sigma}\, x^{\sigma}\partial_{\rho}\phi \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi$$

Recalling Eq. 1.31, the generators can be expressed as:

$$L^{\mu\nu} := i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right) \tag{1.44}$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As $p^{\mu} = i\partial^{\mu}$ (with signature $(+,-,-,-)$), it is clear that $L^{i} \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ is the orbital angular momentum.

Weyl fields A left-handed Weyl field transforms as:

$$\psi'_{\mathrm{L}}(x') = \Lambda_{\mathrm{L}}\psi_{\mathrm{L}}(x) \tag{1.45}$$

with Λ_{L} defined in Eq. 1.40, and similarly for right-handed Weyl fields. The infinite-dimensional representation of the Lorentz generators determined by Weyl spinors can be found as:

$$\begin{aligned}\delta_0\psi_{\mathrm{L}} &\equiv \psi'_{\mathrm{L}}(x) - \psi_{\mathrm{L}}(x) = \psi'_{\mathrm{L}}(x' - \delta x) - \psi_{\mathrm{L}}(x)\\ &= \psi'_{\mathrm{L}}(x') - \delta x^{\rho}\partial_{\rho}\psi_{\mathrm{L}}(x) - \psi_{\mathrm{L}}(x) = \left(\Lambda_{\mathrm{L}} - \mathrm{I}_2\right)\psi_{\mathrm{L}}(x) - \delta x^{\rho}\partial_{\rho}\psi_{\mathrm{L}}(x)\end{aligned}$$

The second term yields $L^{\mu\nu}$, while the first can be further elaborated by writing:

$$\Lambda_{\mathrm{L}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{1.46}$$

Thus:

$$\delta_0\psi_{\mathrm{L}} = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_{\mathrm{L}}$$

where the angular momentum separates into the orbital and the spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1.47}$$

This separation is general: $L^{\mu\nu}$ is always expressed as in Eq. 1.44, while $S^{\mu\nu}$ depends on the specific representation. In the scalar representation $S^{\mu\nu} = 0$, while in the left/right-handed Weyl representation $S^{i0} = \pm i\frac{\sigma^{i}}{2}$.

Vector fields A (contravariant) vector field transforms as:

$$V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}V^{\nu}(x) \tag{1.48}$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the $\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)$ representation.

1.2.3 Poincaré group

Definition 1.2.7: Poincaré group

The *Poincaré group* is defined as $\mathrm{ISO}^+(1,3) \coloneqq \mathrm{T}^{1,3} \times \mathrm{SO}^+(1,3)$, where $\mathrm{T}^{1,3} \cong \mathbb{R}^{1,3}$ is the group of translations of $\mathbb{R}^{1,3}$.

Given a translation $x^\mu \mapsto x^\mu + a^\mu$, the associated group element can be written as:

$$T = e^{-ia_\mu P^\mu} \tag{1.49}$$

where P^μ is the 4-momentum operator. Clearly translations commute, and so do their generators; on the other hand, as $\mathbf{P}$ is a vector under rotations, while P^0 (energy) a scalar, one has:

$$[J^i, P^j] = i\epsilon^{ijk} P^k \qquad\qquad [J^i, P^0] = 0$$

These equations uniquely determine the *Poincaré algebra* $\mathfrak{iso}^+(1,3)$:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [J^{\mu\nu}, J^{\sigma\rho}] &= i\left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\rho} J^{\nu\sigma}\right) \\ [P^\mu, J^{\rho\sigma}] &= i\left(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho\right) \end{aligned} \tag{1.50}$$

It's easy to check that $[K^i, P^0] = iP^i$, while $[J^i, P^0] = [P^i, P^0] = 0$: given that P^0 generates time translations, linear and angular momentum are conserved quantities, while $\mathbf{K}$ is not.

1.2.3.1 Field representations

Fields provide an infinite-dimensional representation of the Lorentz group as $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $S^{\mu\nu}$ does not depend on x^μ, but only on the spin of the field.
To represent P^μ on fields, their transformation law must be specified: all fields are required to be scalars under translations, independently of their spin. This means that, given a generic field $\phi(x)$, under a translation $x' = x + a$ it transforms as $\phi'(x') = \phi(x)$, so, under an infinitesimal translation $x' = x + \varepsilon$:

$$\begin{aligned} \delta_0\phi \equiv \phi'(x) - \phi(x) = \phi'(x' - \varepsilon) - \phi(x) &= -\varepsilon^\mu \partial_\mu \phi(x) \\ &= e^{-i(-\varepsilon_\mu)P^\mu} \phi'(x') - \phi(x) = \left(e^{i\varepsilon_\mu P^\mu} - \mathrm{I}\right)\phi(x) = i\varepsilon_\mu P^\mu \phi(x) \end{aligned}$$

It is clear then that:

$$P^\mu = +i\partial^\mu \tag{1.51}$$

Explicitly, $P^0 = i\partial_t$ and $\mathbf{P} = -i\nabla$. It is trivial to check that these generators obey the Poincaré algebra.

1.2.3.2 Particle representations

The Poincaré group can also be represented using the Hilbert space $\mathscr{H}$ of a free particle as a basis. Denoting a generic state as $|\mathbf{p}, s\rangle \in \mathscr{H}$, where $\mathbf{p}$ is the particle's momentum and s collectively labels all other quantum numbers, it is clear that $\mathscr{H}$ is infinite-dimensional, as $\mathbf{p}$ is a continuous unbounded variable.

Theorem 1.2.3: Wigner's theorem

On the Hilbert space of physical states, any symmetry transformation can be represented by a linear and unitary or anti-linear and anti-unitary operator.

By this theorem, Poincaré transformations can be represented by unitary matrices, i.e. $\mathbf{J}$, $\mathbf{K}$, $\mathbf{P}$ and P^0 can be represented by hermitian operators. These representations can be labeled by Casimir operators, which for $\mathrm{ISO}^+(1,3)$ are easily found as $P_\mu P^\mu$ and $W_\mu W^\mu$, where W^μ is the *Pauli-Lubanski operator*:

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho} J_{\nu\sigma} P_\rho \tag{1.52}$$

On single-particle states $P_\mu P^\mu = m^2$, while $W_\mu W^\mu$ can be conveniently computed in a particular frame (due to Lorentz invariance). If $m \neq 0$, this frame is the rest-frame of the particle:

$$W^\mu = -\frac{m}{2}\epsilon^{\mu\nu\sigma 0} J_{\nu\sigma} = \frac{m}{2}\delta^{\mu i}\epsilon^{ijk} J^{jk} = \delta^{\mu i} m J^i$$

Therefore, on single-particle states of mass m and spin j, the Casimir operator takes the form:

$$W_\mu W^\mu = -m^2 j\,(j+1) \tag{1.53}$$

If $m = 0$, the rest-frame does not exist, but it is possible to choose a frame where $P^\mu = (\omega, 0, 0, \omega)$, where $W^0 = W^3 = \omega J^3$, $W^1 = \omega\,(J^1 - K^2)$ and $W^2 = \omega\,(J^2 + K^1)$, so that:

$$W_\mu W^\mu = -\omega^2 \left[\left(K^2 - J^1\right)^2 + \left(K^1 + J^2\right)^2\right] \tag{1.54}$$

It is clear that the $m \to 0$ limit is not trivial, and massive and massless representation need to be studied separately.

Massive representations Restricting to $m \in \mathbb{R}^+$ ($m^2 < 0$ states, called tachyons, are excluded), the massive representations are labeled by mass m and spin j: in fact, after a Lorentz transformation such that $P^\mu = (m, \mathbf{0})$, spatial rotations can still be performed, i.e. the subspace of single-particle states with momentum $P^\mu = (m, \mathbf{0})$ is still a basis for the representation of SU(2) (as spinors must be included too). The group of transformations which leaves invariant a certain choice of P^μ is called the *little group*, so SU(2) is the little group of massive single-particle states: massive representations are labelled by m and j, which means that massive particles of spin j have $2j+1$ degrees of freedom.

Massless representations The little group for $P^\mu = (\omega, 0, 0, \omega)$ clearly is SO(2), the group of rotations in the (x, y) plane generated by J^3: as for any abelian group, its irreducible representations are one-dimensional, and they are labeled by the eigenvalue h of J^3, which represents the angular momentum in the direction of propagation of the particle and is called *helicity*. Helicity can be shown to be quantized as $h \in \frac{1}{2}\mathbb{Z}_0$ (by topologic considerations on $\mathrm{ISO}^+(1,3) \equiv \mathbb{R}^4 \times \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$).
As a consequence, massless particles only have one degree of freedom and are characterized by their helicity h. As $\mathrm{SO}(2) \equiv \mathrm{U}(1)$, on a state of helicity h the little group is represented as:

$$U(\theta) = e^{-ih\theta} \tag{1.55}$$

Although massless particles with opposite helicities are logically two different species of particles, it can be written as $h = \hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$ (unit vectors), so h is a pseudoscalar such that $h \mapsto -h$ under parity: this means that, if the interaction conserves parity, h and $-h$ must be symmetric.

Example 1.2.4

The electromagnetic and gravitational interactions conserve parity, thus photons and gravitons must be a basis for the representation of both $\mathrm{ISO}^+(1,3)$ and parity: photons can have $h = \pm 1$ (left- and right-handed), while gravitons have $h = \pm 2$.

Example 1.2.5

Neutrinos only interact via the weak interaction, which does not conserve parity, and in fact the two states $h = \pm\frac{1}{2}$ are different particles: neutrinos have $h = -\frac{1}{2}$, while antineutrinos have $h = +\frac{1}{2}$.

1.3 Classical equations of motion

Consider a *local field theory* of fields $\{\phi_i(x)\}_{i\in\mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkoski spacetime. Its Lagrangian takes the form:

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.56}$$

where $\mathcal{L}$ is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.57}$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

Theorem 1.3.1

The *stationary action principle* $\delta\mathcal{S} = 0$ determines the classical equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0 \tag{1.58}$$

Proof. Varying Eq. 1.57:

$$\delta\mathcal{S} = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi_i)\right] = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi_i = 0$$

□

Corollary 1.3.1.1

Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.

Proof. This is a consequence of Stokes theorem:

$$\int_\Sigma \mathrm{d}^4x\, \partial_\mu K^\mu = \int_{\partial\Sigma} \mathrm{d}A\, n_\mu K^\mu$$

□

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \tag{1.59}$$

$$\mathcal{H} = \sum_{i\in\mathcal{I}} \Pi_i(x)\partial_0\phi(x) - \mathcal{L} \tag{1.60}$$

Unlike the Hamiltonian formalism, the Lagrangian formalism keeps Lorentz covariance explicit.

1.3.1 Noether's theorem

Definition 1.3.1: Infinitesimal transformation

Given a field theory with fields $\{\phi_i\}_{i=1,\dots,k}$ and action $\mathcal{S}[\phi]$, an *infinitesimal transformation* parametrized by $\{\varepsilon^a\}_{a=1,\dots,N} : |\varepsilon^a| \ll 1$ is defined by two sets of functions $\{A^\mu_a(x)\}_{a=1,\dots,N}$ and $\{F_{i,a}(\phi, \partial\phi)\}_{i=1,\dots,k;\, a=1,\dots,N}$ such that:

$$\begin{aligned} x^\mu \mapsto x'^\mu &= x^\mu + \varepsilon^a A^\mu_a(x) \\ \phi_i(x) \mapsto \phi'_i(x) &= \phi_i(x) + \varepsilon^a F_{i,a}(\phi, \partial\phi) \end{aligned} \tag{1.61}$$

Definition 1.3.2: Symmetry transformation

An infinitesimal transformation is a *symmetry transformation* if it leaves $\mathcal{S}[\phi]$ invariant, regardless of ϕ being a solution of the equations of motion. It can further be classified as:

- *global symmetry*, if $\varepsilon^a \equiv \text{const.}$;
- *local symmetry*, if $\varepsilon^a = \varepsilon^a(x)$.

Symmetry transformations which leave spacetime unchanged, i.e. with $A^\mu_a(x) = 0$, are called *internal symmetries*.

Theorem 1.3.2: Noether's theorem

Given a global (but not local) symmetry parametrized by N generators, then there are N conserved currents $\{j^\mu_a(\phi)\}_{a=1,\dots,N}$ such that:

$$\partial_\mu j^\mu_a(\phi^{\text{cl}}) = 0 \tag{1.62}$$

where ϕ^{cl} is a classical solution of the equations of motion.

Proof. First, consider an infinitesimal transformation with slowly-varying parameters, i.e. $l\,|\partial_\mu \varepsilon^a| \ll |\varepsilon^a|$ (l characteristic scale of the field theory): being it not a local symmetry, $\delta\mathcal{S} \neq 0$ at $o(\varepsilon)$ and:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x \, \left[\varepsilon^a(x) K_a(\phi) - (\partial_\mu \varepsilon^a(x))\, j^\mu_a(\phi) + o(\partial\partial\varepsilon)\right] + o(\varepsilon^2)$$

If $\varepsilon^a \equiv$ const. (global symmetry) then $\delta\mathcal{S}[\phi] = 0\,\forall\phi$, therefore $K_a(\phi) = 0\,\forall\phi$ (independent of ε). Assuming $\varepsilon^a(x) \to 0$ sufficiently fast as $x \to \infty$, then integration by parts yields:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int \mathrm{d}^4x \, \varepsilon^a(x) \partial_\mu j^\mu_a(\phi) + o(\partial\partial\varepsilon) + o(\varepsilon^2)$$

This expression is independent of the choice of ϕ. Moreover, note that Eq. 1.61 can be rewritten as an internal transformation by setting:

$$\phi_i(x) \mapsto \phi'_i(x) = \phi_i(x - \varepsilon^a A_a) + \varepsilon^a F_{i,a} = \phi_i(x) + \varepsilon^a F_{i,a} - \varepsilon^a A^\mu_a \partial_\mu \phi_i \equiv \phi_i(x) + \delta\phi_i(x)$$

$\delta\phi_i(x)$ vanishes at infinity, therefore it is the kind of variation used to derive the equations of motion: choosing $\phi \equiv \phi^{\mathrm{cl}}$ classical solution then implies $\delta\mathcal{S} = 0$ independently of ε, i.e. the thesis. □

These are often called *Noether currents*, and the associated *Noether charges* are defined as:

$$Q_a := \int \mathrm{d}^3x \, j^0_a(t, \mathbf{x}) \tag{1.63}$$

These are time-independent, as $\partial_0 Q_a = \int \mathrm{d}^3x \, \partial_0 j^0_a = -\int \mathrm{d}^3x \, \partial_i j^i_a$: on all spacetime it vanishes by divergence theorem (fields vanish at infinity), but on a finite volume it yields a boundary term interpreted as the incoming and outgoing flux.

Proposition 1.3.1: Noether currents

The explicit expression of Noether currents is:

$$j^\mu_a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \left[A^\nu_a(x)\partial_\nu\phi_i - F_{i,a}(\phi, \partial\phi)\right] - A^\mu_a(x)\mathcal{L} \tag{1.64}$$

Proof. Varying the action at $o(\partial\varepsilon)$:

$$\delta_\varepsilon\mathcal{S} = \delta_\varepsilon \int \mathrm{d}^4x \, \mathcal{L} = \int \left[\delta_\varepsilon(\mathrm{d}^4x)\mathcal{L} + \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta_\varepsilon\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta_\varepsilon(\partial_\mu\phi_i)\right)\right]$$

The Jacobian of Eq. 1.61 gives $\mathrm{d}^4x \mapsto \mathrm{d}^4x\,(1 + A^\mu_a\partial_\mu\varepsilon^a) + o(\varepsilon)$, while $\delta_\varepsilon\phi_i$ is not $o(\partial\varepsilon)$ and:

$$\delta_\varepsilon(\partial_\mu\phi_i) = \frac{\partial\phi'_i}{\partial x'^\mu} - \frac{\partial\phi_i}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\left(\phi_i + \varepsilon^a F_{i,a}\right) - \frac{\partial\phi_i}{\partial x^\mu} = -(\partial_\mu\varepsilon^a)\left(A^\nu_a\partial_\nu\phi_i - F_{i,a}\right) + o(\varepsilon)$$

The thesis follows from $\delta_\varepsilon\mathcal{S} = -\int \mathrm{d}^4x\,(\partial_\mu\varepsilon^a) j^\mu_a + o(\partial\partial\varepsilon) + o(\varepsilon^2)$. □

If the considered infinitesimal transformation is not a global symmetry, then $\delta_\varepsilon \mathcal{S}$ has a non-vanishing $o(\varepsilon)$ term which gives rise to a quasi-conserved current:

$$\partial_\mu j^a_\mu = -(\delta_a \mathcal{L})_{\text{global}} \tag{1.65}$$

1.3.1.1 Energy-momentum tensor

Consider spacetime translations: as all fields must be scalars under these transformations, they define a Noether current. In particular, translations have $A^\mu_\nu = \delta^\mu_\nu$ and $F_{i,\mu} = 0$ (the parameter index is a Lorentz index), so the conserved current is the *energy-momentum tensor* $j^\mu_\nu \equiv \theta^\mu{}_\nu$:

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \eta^{\mu\nu} \mathcal{L} \tag{1.66}$$

which is covariantly conserved on classical solutions of the equations of motion. The conserved Noether charge associated to the energy-momentum tensor is the *four-momentum*:

$$P^\mu := \int \mathrm{d}^3 x \, \theta^{0\mu} \tag{1.67}$$

The energy-momentum tensor so defined is not symmetric, however it can be made to via a tensor $A^{\rho\mu\nu}$ which is anti-symmetric w.r.t. (ρ, μ): $T^{\mu\nu} \equiv \theta^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$ is physically equivalent from $\theta^{\mu\nu}$, as the second term is a vanishing spatial divergence in the definition of P^μ and is contracted to 0 in the conservation law.

1.3.2 Real scalar fields

Consider a real scalar field ϕ: a non-trivial Poincaré-invariant action must contain $\partial_\mu \phi$ and must saturate each Lorentz index. For example:

$$\mathcal{S}[\phi] = \frac{1}{2} \int \mathrm{d}^4 x \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \tag{1.68}$$

The resulting equation of motion is the *Klein-Gordon equation*:

$$\left(\Box + m^2 \right) \phi = 0 \tag{1.69}$$

where $\Box \equiv \partial_\mu \partial^\mu$. A plane wave $e^{\pm i p_\mu x^\mu}$ is a solution if $p^2 = m^2$, so the KG equation imposes the relativistic dispersion relation and m can be interpreted as the mass. As ϕ must be real, the general solution is a superposition of waves:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2 E_\mathbf{p}}} \left[a_\mathbf{p} e^{-i p_\mu x^\mu} + a^*_\mathbf{p} e^{i p_\mu x^\mu} \right]_{p^0 = E_\mathbf{p}} \tag{1.70}$$

The positive energy solution has $E_\mathbf{p} = +\sqrt{\mathbf{p}^2 + m^2}$, but it contains both *positive* and *negative frequency modes* $e^{\mp i p_\mu x^\mu}$, while the $(2E_\mathbf{p})^{-1/2}$ factor is a convenient normalization of the $a_\mathbf{p}$ coefficients. The overall normalization of $\mathcal{S}[\phi]$ does not influence the equations of motion, however it is important for obtaining a positive-definite Hamiltonian. As the momentum conjugate to ϕ is $\Pi_\phi = \partial_0 \phi$, the Hamiltonian density is found as:

$$\mathcal{H} = \frac{1}{2} \left[\Pi^2_\phi + (\nabla \phi)^2 + m^2 \phi^2 \right] \tag{1.71}$$

The energy-momentum tensor is computed to be:

$$\theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{1.72}$$

It is trivial to see that $\theta^{00} = \mathcal{H}$: the Hamiltonian is the conserved charge related to the invariance under time translations.
To compute the conserved currents associated to Lorentz invariance, it is convenient to label the transformation parameters $\omega^{\mu\nu}$ by an anti-symmetric pair of Lorentz indices, so that Eq. 1.61 become:

$$x^\mu \mapsto x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}\left(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho\right) \equiv x^\mu + \frac{1}{2} A^\mu_{(\rho\sigma)} \omega^{\rho\sigma}$$

As $F_{i,a} = 0$, from Eq. 1.64 the conserved currents are:

$$j^{(\rho\sigma)\mu} = x^\rho \theta^{\mu\sigma} - x^\sigma \theta^{\mu\rho} \tag{1.73}$$

For spatial rotations, the conserved charge is:

$$M^{ij} = \int \mathrm{d}^3x \left(x^i \theta^{0j} - x^j \theta^{0i}\right) = \int \mathrm{d}^3x\, \partial_0 \phi \left(x^i \partial^j - x^j \partial^i\right) \phi = \frac{i}{2} \int \mathrm{d}^3x \left[\phi L^{ij} (\partial_0 \phi) - (\partial_0 \phi) L^{ij} \phi\right]$$

where integration by parts was carried and L^{ij} is defined by Eq. 1.44. This can be generalized.

Definition 1.3.3: Scalar product

Given two real scalar fields ϕ_1 and ϕ_2, their *scalar product* is defined as:

$$\langle \phi_1 | \phi_2 \rangle := \frac{i}{2} \int \mathrm{d}^3x\, \phi_1 \overleftrightarrow{\partial_0} \phi_2 \tag{1.74}$$

where $f \overleftrightarrow{\partial_\mu} g := f \partial_\mu g - (\partial_\mu f) g$.

Proposition 1.3.2

If ϕ_1 and ϕ_2 are KG solutions, then $\langle \phi_1 | \phi_2 \rangle$ is time-independent.

Proof. By the KG equation:

$$\partial_0 \left[\phi_1 \partial_0 \phi_2 - (\partial_0 \phi_1) \phi_2\right] = \phi_1 \partial_0^2 \phi_2 - (\partial_0^2 \phi_1) \phi_2 = \phi_1 \nabla^2 \phi_2 - (\nabla^2 \phi_1) \phi_2$$

which vanishes after integration by parts. □

Note that this scalar product is not positive-definite.

Theorem 1.3.3: Conserved charges

Given a symmetry represented by a Lie group and a representation $L^{\mu\nu}$ of its generators as operators acting on fields, the value of the associated conserved charges on a solution ϕ of the equations of motion is:

$$M^{\mu\nu} = \langle \phi | L^{\mu\nu} | \phi \rangle \tag{1.75}$$

Example 1.3.1: Four-momentum

Applying Th. 1.3.3 to four-momentum $P^\mu = \langle\phi|i\partial^\mu|\phi\rangle$; for example, the $\mu = 0$ component is:

$$\begin{aligned}
P^0 = \langle\phi|i\partial^0|\phi\rangle = \langle\phi|i\partial_0|\phi\rangle &= \frac{i}{2}\int \mathrm{d}^3x\,[\phi(i\partial_0)\partial_0\phi - (\partial_0\phi)i\partial_0\phi] \\
&= \frac{1}{2}\int \mathrm{d}^3x\,\left[-\phi\partial_0^2\phi + (\partial_0\phi)^2\right] = \frac{1}{2}\int \mathrm{d}^3x\,\left[-\phi\left(\nabla^2 - m^2\right)\phi + (\partial_0\phi)^2\right] \\
&= \frac{1}{2}\int \mathrm{d}^3x\,\left[(\nabla\phi)^2 + m^2\phi^2 + (\partial_0\phi)^2\right] = \frac{1}{2}\int \mathrm{d}^3x\,\theta^{00}
\end{aligned}$$

The free KG action can be generalized to a self-interacting real scalar field introducing a general potential:

$$\mathcal{S}[\phi] = \int \mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \tag{1.76}$$

The quadratic term in $V(\phi)$ is the mass term, while higher-order terms describe the self-interaction.