Quantum Field Theory 1

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Classical Field Theory

1.1 Continuous limit

1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x-axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a.
Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n, one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2}\dot{\eta}_n^2 - \frac{\lambda}{2} \left(\eta_n - \eta_{n-1} \right)^2 \right] \tag{1.1}$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda \left(\eta_{n+1} - 2\eta_n + \eta_{n-1} \right) \tag{1.2}$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \tag{1.3}$$

where the dispersion relation is:

$$\omega^2 = 2\lambda \left(1 - \cos k \right) \approx \lambda k^2 \tag{1.4}$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \qquad \Rightarrow \qquad k_m = \frac{2\pi m}{N}\,,\; m = 0, 1, \ldots, N-1 \tag{1.5}$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} \left[\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)} \right] \tag{1.6}$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjucated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \tag{1.7}$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*[1]:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \tag{1.8}$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators* [2]$\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} \left[e^{i(k_m n - \omega_m t)} \hat{a}_m + e^{-i(k_m n - \omega_m t)} \hat{a}_m^\dagger\right] \tag{1.9}$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \tag{1.10}$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2}\hat{\pi}_n^2 + \frac{\lambda}{2}\left(\hat{\eta}_n - \hat{\eta}_{n-1}\right)^2\right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}\right) \tag{1.11}$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m \left|0\right\rangle \quad \forall m = 0, 1, \ldots, N-1 \tag{1.12}$$

$$\left|n_0, n_1, \ldots, n_{N-1}\right\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} \left|0\right\rangle \tag{1.13}$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2}\sum_{m=0}^{N-1} \hbar\omega_m \tag{1.14}$$

$$E_{n_0, n_1, \ldots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \tag{1.15}$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as "particles" and n_m is the number of phonons in the m^{th} normal mode.

[1] Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t}\hat{\mathcal{O}}(0)e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{\mathrm{d}\hat{\mathcal{O}}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

[2] For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{p}(t)$ and $\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) + \hat{a}^\dagger(t)\right] \qquad \hat{p}(t) = -i\omega\sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}(t) - \hat{a}^\dagger(t)\right]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger(t)\hat{a} + \frac{1}{2}\right)$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.

1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \to 0$ and $N \to \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x,t)$ dependent on the continuous real coordinate $x \in [0, R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x}\right)^2 \qquad \sum_n \longrightarrow \frac{1}{a}\int_0^R \mathrm{d}x$$

Proposition 1.1.1. *In the continuous limit:*

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x-x') = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

Proof. By direct calculation:

$$a\sum_n f(an)\frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R \mathrm{d}x\, f(x)\delta(x-y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \to kx$, symmetrizing $k_m \in [-\pi, \pi]$ (instead of $[0, 2\pi]$) one finds:

$$\delta(x-x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na}\sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} e^{ik(x-x')}$$

where integration limits are $\pm\frac{\pi}{a} \to \pm\infty$. □

Proof. The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R \mathrm{d}x\, e^{i(k-k')x} = 2\pi\delta(k-k')$$

□

By these relations, it can be seen that $\frac{\mathrm{d}k}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k + \mathrm{d}k$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.
In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}\rho_0(\partial_t\eta)^2 - \frac{\kappa}{2}(\partial_x\eta)^2\right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x,t) \equiv \sqrt{\rho_0}\eta(x,t)$ and $\kappa = c^2\rho_0$ and adding a pinning term $\propto \varphi^2$, the Lagrangian can be written as:

$$L = \int_0^R \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \frac{m^2c^4}{2}\phi^2\right] \tag{1.16}$$

The classical equation of motion of this field yields:

$$\partial_t^2\phi = c^2\partial_x^2\phi - m^2c^4\phi \tag{1.17}$$

The solutions of this wave equation can be written as:

$$\phi(x,t) = e^{i(kx-\omega_k t)} \tag{1.18}$$

with dispersion relation:

$$\omega_k^2 = c^2k^2 + m^2c^4 \tag{1.19}$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x,t) := \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi(x,t) \tag{1.20}$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R \mathrm{d}x \left[\frac{1}{2}\Pi^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2c^4}{2}\phi^2\right] \tag{1.21}$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\sqrt{\frac{\hbar}{2\omega_k}}\left[e^{i(kx-\omega_k t)}\hat{a}_k + e^{-i(kx-\omega_k t)}\hat{a}_k^\dagger\right] \tag{1.22}$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi\delta(k-k') \tag{1.23}$$

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = i\hbar\delta(x-x') \tag{1.24}$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\frac{1}{2}\hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger\right) = E_0 + \int_{\mathbb{R}}\frac{\mathrm{d}k}{2\pi}\hbar\omega_k\hat{a}_k^\dagger\hat{a}_k \tag{1.25}$$

The ground-state energy can be computed from Eq. 1.14, defining $\mathrm{Vol} := 2\pi\delta(k=0)$:

$$E_0 = \mathrm{Vol}\int_{\mathbb{R}}\frac{\mathrm{d}k}{2\pi}\frac{1}{2}\hbar\omega_k \tag{1.26}$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

1.2 Spacetime symmetries

1.2.1 Lie groups

Definition 1.2.1. A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d-dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0,\dots,0) = e$.

Definition 1.2.2. Given a group G and a vector space $V(\mathbb{K})$, a *representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Given the isomorphism $\mathrm{GL}(V) \to \mathbb{K}^{n\times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to *de facto* represent G as matrices acting on elements of V, i.e. $\rho : G \to \mathbb{K}^{n\times n}$.

Proposition 1.2.1. *Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ as:*

$$\rho(g(\theta)) = e^{i\theta_a T^a} \tag{1.27}$$

where $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ are the generators of G on V.

Definition 1.2.3. Given a Lie group G with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, its *Lie algebra* is:

$$[T^a, T^b] = i f^{ab}{}_c T^c \tag{1.28}$$

where $f^{ab}{}_c$ are called the *structure constants*.

Proposition 1.2.2. *The Lie algebra of a Lie group is independent of the representation.*

Proposition 1.2.3. *Any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.*

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

Definition 1.2.4. Given a Lie group with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n\times n}$ on $V(\mathbb{C})$, a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation, Casimir operators are operators proportional to id_V, and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

Proposition 1.2.4. *A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.*

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

1.2.2 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu}x^\mu x^\nu$, i.e. the orthogonal group $\mathrm{O}(1,3)$ (with signature $(+,-,-,-)$). The condition that $\Lambda^\mu{}_\nu$ must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \tag{1.29}$$

This implies that $\det\Lambda = \pm 1$: a transformation with $\det\Lambda = -1$ can always be written as the product of a transformation with $\det\Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines $\mathrm{SO}(1,3) \coloneqq \{\Lambda \in \mathrm{O}(1,3) : \det\Lambda = 1\}$.
Writing explicitly the temporal component $1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$, it is clear that $(\Lambda^0{}_0)^2 \geq 1$. Therefore, $\mathrm{O}(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0{}_0 \geq 1$ and the non-orthochronous component with $\Lambda^0{}_0 \leq 1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a dircrete transformation which reverses the sign of the temporal component.

Definition 1.2.5. The *Lorentz group* is the orthochronous component of SO(1, 3).

The discrete transformations are factored out of the Lorentz group. Considering the infinitesimal transformation and applying Eq. 1.29:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \qquad \Rightarrow \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x, y), (y, z) and (z, x) planes and the 3 rapidities of hyperbolic rotations in the (t, x), (t, y) and (t, z) planes.

Proposition 1.2.5. *The Lorentz group is a non-compact Lie group.*

Proof. Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in $[0, 2\pi)$, rapidities vary in $\mathbb{R}$, so the differentiable manifold associated to SO(1, 3) is not compact. □

1.2.2.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \mathrm{SO}(1, 3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.30}$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a n-dimensional representation of SO(1, 3), both $[J^{\mu\nu}]^{i}{}_{j}$ and $[\Lambda]^{i}{}_{j}$ are $\mathbb{C}^{n\times n}$ matrices (Λ is real): for example, the $n = 1$ representation acts on *scalars*, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \, \forall \mu, \nu = 0, \ldots, 3$.

4-vectors The $n = 4$ representation acts on *contravariant 4-vectors* v^{μ}, which transform according to $v^{\mu} \mapsto \Lambda^{\mu}{}_{\nu}v^{\nu}$, and *covariant 4-vectors* v_{μ}, which transform according to $v_{\mu} \mapsto \Lambda_{\mu}{}^{\nu}v_{\nu}$. In this representation, the generators are represented as $\mathbb{C}^{4\times 4}$ matrices:

$$[J^{\mu\nu}]^{\rho}{}_{\sigma} = i\left(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}\right) \tag{1.31}$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}(1, 3)$, called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma}\right) \tag{1.32}$$

It is convenient to rearrange the 6 components of $J^{\mu\nu}$ into two spatial vectors:

$$J^{i} := \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad\qquad K^{i} := J^{i0} \tag{1.33}$$

The $\mathfrak{so}(1, 3)$ can then be rewritten as:

$$[J^{i}, J^{j}] = i\epsilon^{ijk}J^{k} \qquad\qquad [J^{i}, K^{j}] = i\epsilon^{ijk}K^{k} \qquad\qquad [K^{i}, K^{j}] = -i\epsilon^{ijk}J^{k} \tag{1.34}$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^{i} are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^{i} := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \qquad\qquad \eta^{i} := \omega^{i0} \tag{1.35}$$

so that:

$$\Lambda = \exp\left[-i\boldsymbol{\theta}\cdot\mathbf{J} + i\boldsymbol{\eta}\cdot\mathbf{K}\right] \tag{1.36}$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts wich increase velocities with respect to said frame.

1.2.2.2 Tensor Representations

A generic (p, q)-tensor transforms as:

$$T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} \mapsto \Lambda^{\mu_1}{}_{\alpha_1}\ldots\Lambda^{\mu_p}{}_{\alpha_p}\Lambda_{\nu_1}{}^{\beta_1}\ldots\Lambda_{\nu_q}{}^{\beta_q}T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} \tag{1.37}$$

The representation of the Lorentz group which acts on (p, q)-tensors is of degree $n = 4^{p+q}$, however it is reducible into the direct product of $p + q$ 4-dimensional representations as of Eq. 1.38.
Moreover, consider the action of the Lorentz group on $(2, 0)$-tensors: being $T^{\mu\nu} \mapsto \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu}T^{\mu\nu} \mapsto \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$\mathsf{4}\otimes\mathsf{4} = \mathsf{1}\oplus\mathsf{6}\oplus\mathsf{9} \tag{1.38}$$

These are irreducible representations which, given a generic tensor $T^{\mu\nu}$, act on S, $A^{\mu\nu}$ and $S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right)$ and $S^{\mu\nu} \equiv \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right)$.

Decomposition under rotations Restricting the action to the SO(3) sub-group of SO(1, 3), tensors can be decomposed according to irreducible representations of SO(3), which are labelled by the angular momentum $j \in \mathbb{N}_0$ and are of degree $n = 2j + 1$. Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1\otimes\mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2}\mathbf{j} \tag{1.39}$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$. A 4-vector v^μ is irreducible under the action of SO(1, 3), but under SO(3) it is decomposed into v^0 and $\mathbf{v}$, so $v^\mu \in \mathbf{0}\oplus\mathbf{1}$. A $(2, 0)$-tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0}\oplus\mathbf{1})\otimes(\mathbf{0}\oplus\mathbf{1}) &= (\mathbf{0}\otimes\mathbf{0})\oplus(\mathbf{0}\otimes\mathbf{1})\oplus(\mathbf{1}\otimes\mathbf{0})\oplus(\mathbf{1}\otimes\mathbf{1}) \\ &= \mathbf{0}\oplus\mathbf{1}\oplus\mathbf{1}\oplus(\mathbf{0}\oplus\mathbf{1}\oplus\mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.38: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2}\epsilon^{ijk}A^{jk}$, so $A^{\mu\nu} \in \mathbf{1}\oplus\mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0}\oplus\mathbf{1}\oplus\mathbf{2}$ under spatial rotations.
Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0}\otimes\mathbf{0})$, $T^{0i} \in (\mathbf{0}\otimes\mathbf{1})$, $T^{i0} \in (\mathbf{1}\otimes\mathbf{0})$ and $T^{ij} \in (\mathbf{1}\otimes\mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0}\oplus\mathbf{1}\oplus\mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk}A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

Example 1.2.1. Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two *invariant tensors* under $\mathrm{SO}(1,3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.29, and the Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\sigma}{}_{\gamma}\Lambda^{\rho}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} = (\det\Lambda)\,\epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

1.2.2.3 Spinor representations

The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same, which means that $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are indistinguishable by infinitesimal transformations; however, they are globally different, as $\mathrm{SO}(3)$ rotations are periodic by 2π, while $\mathrm{SU}(2)$ rotations are periodic by 4π: in particular, it can be shown that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$, i.e. $\mathrm{SU}(2)$ is the universal cover of $\mathrm{SO}(3)$. This means that $\mathrm{SU}(2)$ representations can be labelled by $j \in \frac{1}{2}\mathbb{N}_0$, where half-integer spin representations are known as *spinorial representations*: they act on spinors, i.e. objects which change sign under rotations of 2π (thus not suitable to represent $\mathrm{SO}(3)$).

Example 1.2.2. The $\mathbf{\frac{1}{2}}$ representation of $\mathrm{SU}(2)$ is a 2-dimensional representation where $J^i = \frac{\sigma^i}{2}$: Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, thus the $\mathfrak{su}(2)$ algebra is satisfied. Denoting the $m = \pm\frac{1}{2}$ states in the $\mathbf{\frac{1}{2}}$ representation as $|\uparrow\rangle$ and $|\downarrow\rangle$, the Clebsch-Gordan decomposition $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$ yields the triplet $(j = 1)$ $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$ and the singlet $(j = 0)$ $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Proposition 1.2.6. *The Lorentz algebra* $\mathfrak{so}(1,3)$ *can be decomposed as* $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

Proof. Given the $\mathfrak{so}(1,3)$ algebra in Eq. 1.33, it is possible to define:

$$\mathbf{J}_{\pm} \coloneqq \frac{1}{2}(\mathbf{J} \pm i\mathbf{K})$$

The Lie algebra then becomes:

$$[\mathbf{J}^i_{\pm}, \mathbf{J}^j_{\pm}] = i\epsilon^{ijk}\mathbf{J}^k_{\pm} \qquad\qquad [\mathbf{J}^i_{\pm}, \mathbf{J}^j_{\mp}] = 0$$

These are two commuting $\mathfrak{so}(2)$ algebras, thus proving the thesis. □

As observed before, this does not imply that $\mathrm{SO}(1,3)$ is globally equivalent to $\mathrm{SU}(2) \times \mathrm{SU}(2)$: in fact, $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(4)$, while the universal cover of $\mathrm{SO}(1,3)$ is $\mathrm{SL}(2,\mathbb{C})$, as it can be shown that $\mathrm{SO}(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

By Prop. 1.2.6, representations of $\mathrm{SO}(1,3)$ can be labelled by $(j_-, j_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$, with each index labelling a representation of $\mathrm{SU}(2)$: as $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, the (j_-, j_+) representation contains states with all possible spins $|j_+ - j_-| \le j \le j_+ + j_-$, and it is a representation of degree $n = (2j_- + 1)(2j_+ + 1)$.

$(\mathbf{0}, \mathbf{0})$ is the trivial (scalar) representation, as both $\mathbf{J}_{\pm} = 0$ and $\mathbf{J} = \mathbf{K} = 0$.

$\left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ and $\left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$ are 2-dimensional spinorial representations. These representations act on different spinors $(\psi_{\mathrm{L}})_\alpha \in \left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ and $(\psi_{\mathrm{R}})_\alpha \in \left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$, with $\alpha = 1, 2$, which are called *left-* and *right-handed Weyl spinors*. In $\left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ the generators are $\mathbf{J}_- = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{J}_+ = \mathbf{0}$, while in $\left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$ they are $\mathbf{J}_- = \mathbf{0}$ and $\mathbf{J}_+ = \frac{\boldsymbol{\sigma}}{2}$, thus one finds $\mathbf{J}_{\mathrm{L}} = \mathbf{J}_{\mathrm{R}} = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{K}_{\mathrm{L}} = -\mathbf{K}_{\mathrm{R}} = i\frac{\boldsymbol{\sigma}}{2}$, so that:

$$\psi_{\mathrm{L}} \mapsto \Lambda_{\mathrm{L}}\psi_{\mathrm{L}} = \exp\left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_{\mathrm{L}} \tag{1.40}$$

$$\psi_{\mathrm{R}} \mapsto \Lambda_{\mathrm{R}}\psi_{\mathrm{R}} = \exp\left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right]\psi_{\mathrm{R}} \tag{1.41}$$

Note that the generators K^i are not hermitian, as expected from Prop. 1.2.4. Furthermore, note that $\Lambda_{\mathrm{L,L}} \in \mathbb{C}^{2\times 2}$, therefore $\psi_{\mathrm{L,R}} \in \mathbb{C}^2$.

Proposition 1.2.7. *Given* $\psi_L \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ *and* $\psi_R \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$, *then* $\sigma^2\psi_L^* \in \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ *and* $\sigma^2\psi_R^* \in \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$.

Proof. Recall that for Pauli matrices $\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^*$, so $\sigma^2\Lambda_{\mathrm{L}}^*\sigma^2 = \Lambda_{\mathrm{R}}$ and:

$$\sigma^2\psi_{\mathrm{L}}^* \mapsto \sigma^2\left(\Lambda_{\mathrm{L}}\psi_{\mathrm{L}}\right)^* = \left(\sigma^2\Lambda_{\mathrm{L}}^*\sigma^2\right)\sigma^2\psi_{\mathrm{L}}^* = \Lambda_{\mathrm{R}}\sigma^2\psi_{\mathrm{L}}^* \quad \Rightarrow \quad \sigma^2\psi_{\mathrm{L}}^* \in \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right)$$

where $\sigma^2\sigma^2 = \mathrm{I}_2$ was used. The proof for $\sigma^2\psi_{\mathrm{R}}^*$ is analogous. □

Definition 1.2.6. On Weyl spinors, the *charge conjugation operator* is defined as:

$$\psi_{\mathrm{L}}^c := i\sigma^2\psi_{\mathrm{L}}^* \qquad\qquad \psi_{\mathrm{R}}^c := -i\sigma^2\psi_{\mathrm{R}}^* \tag{1.42}$$

By Prop. 1.2.7, charge conjugation changes transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the i factor ensures that applying this operator twice yields the identity operator.
$\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ is a 4-dimensional complex representation: as $j = 0, 1$, this representation acts on complex 4-vectors of the form $((\psi_{\mathrm{L}})_\alpha, (\xi_{\mathrm{R}})_\beta) \in \mathbb{C}^4$, and $\Lambda = \mathrm{diag}\left(\Lambda_{\mathrm{L}}, \Lambda_{\mathrm{R}}\right) \in \mathbb{C}^{4\times 4}$. To explicit this relation, set $\psi_{\mathrm{R}} \equiv i\sigma^2\psi_{\mathrm{L}}^*$, $\xi_{\mathrm{L}} \equiv -i\sigma^2\xi_{\mathrm{R}}^*$ and $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$: it can be shown, then, that $\xi_{\mathrm{R}}^\dagger\sigma^\mu\psi_{\mathrm{R}}$ and $\xi_{\mathrm{L}}^\dagger\bar{\sigma}^\mu\psi_{\mathrm{L}}$ are contravariant 4-vectors. Although these 4-vectors are complex by construction, being the matrix $\Lambda^\mu{}_\nu$ which represents the Lorentz transformation of a 4-vector real, a reality condition $v_\mu^* = v_\mu$ is Lorentz invariant.

1.2.2.4 Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields A scalar field transforms as:

$$\phi'(x') = \phi(x) \tag{1.43}$$

Consider an infinitesimal transformation $x'^\rho = x^\rho + \delta x^\rho$, with $\delta x^\rho = -\frac{i}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^\rho{}_\sigma x^\sigma$ as of Eq. 1.30. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar representation of $\mathrm{SO}(1,3)$ is the trivial one ($J^{\mu\nu} = 0$).
However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$: while $\delta\phi$ studies only a single degree of freedom, as the point $\mathrm{P} \in \mathbb{R}^{1,3}$ is kept constant and only $\phi(\mathrm{P})$ can vary (i.e. the base space is one-dimensional), $\delta_0\phi$ studies $\phi(\mathrm{P})$ with P varying over $\mathbb{R}^{1,3}$, thus the base space is now a space of functions, which is infinite-dimensional. Therefore, $\delta\phi$ provides a finite-dimensional representation of the generators, while $\delta_0\phi$ an infinite-dimensional one.
To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho\partial_\rho\phi = \frac{i}{2}\omega_{\mu\nu}\left[J^{\mu\nu}\right]^\rho{}_\sigma x^\sigma\partial_\rho\phi \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi$$

Recalling Eq. 1.31, the generators can be expressed as:

$$L^{\mu\nu} := i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \tag{1.44}$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As $p^\mu = i\partial^\mu$ (with signature $(+,-,-,-)$), it is clear that $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ is the orbital angular momentum.

Vector fields A (contravariant) vector field transforms as:

$$V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} V^{\nu}(x) \tag{1.45}$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation.

1.2.3 Poincaré group

1.3 Classical equations of motion

Consider a *local field theory* of fields $\{\phi_i(x)\}_{i\in\mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkoski spacetime. Its Lagrangian takes the form:

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.46}$$

where $\mathcal{L}$ is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.47}$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

Proposition 1.3.1. *The stationary action principle $\delta\mathcal{S} = 0$ determines the classical equations of motion:*

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0 \tag{1.48}$$

Proof. Varying Eq. 1.47:

$$\delta\mathcal{S} = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi_i)\right] = \int \mathrm{d}^4x \sum_{i\in\mathcal{I}} \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi_i = 0$$

□

Proposition 1.3.2. *Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.*

Proof. This is a consequence of Stokes theorem:

$$\int_\Sigma \mathrm{d}^4x\, \partial_\mu K^\mu = \int_{\partial\Sigma} \mathrm{d}A\, n_\mu K^\mu$$

□

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)} \tag{1.49}$$

$$\mathcal{H} = \sum_{i\in\mathcal{I}} \Pi_i(x)\partial_0\phi(x) - \mathcal{L} \tag{1.50}$$