

Quantum Field Theory 1

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Part I

Field Theory

Classical Field Theory

1.1 Continuous limit

1.1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x -axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a . Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n , one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2} \dot{\eta}_n^2 - \frac{\lambda}{2} (\eta_n - \eta_{n-1})^2 \right] \quad (1.1)$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda (\eta_{n+1} - 2\eta_n + \eta_{n-1}) \quad (1.2)$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \quad (1.3)$$

where the dispersion relation is:

$$\omega^2 = 2\lambda (1 - \cos k) \approx \lambda k^2 \quad (1.4)$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \quad \Rightarrow \quad k_m = \frac{2\pi m}{N}, \quad m = 0, 1, \dots, N-1 \quad (1.5)$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} [\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}^* e^{-i(k_m n - \omega_m t)}] \quad (1.6)$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjugated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \quad (1.7)$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*¹:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \quad (1.8)$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators*² $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} [e^{i(k_m n - \omega_m t)} \hat{a}_m + e^{-i(k_m n - \omega_m t)} \hat{a}_m^\dagger] \quad (1.9)$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \quad (1.10)$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2} \hat{\pi}_n^2 + \frac{\lambda}{2} (\hat{\eta}_n - \hat{\eta}_{n-1})^2 \right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) \quad (1.11)$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m |0\rangle \quad \forall m = 0, 1, \dots, N-1 \quad (1.12)$$

$$|n_0, n_1, \dots, n_{N-1}\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} |0\rangle \quad (1.13)$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2} \sum_{m=0}^{N-1} \hbar\omega_m \quad (1.14)$$

$$E_{n_0, n_1, \dots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \quad (1.15)$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as “particles” and n_m is the number of phonons in the m^{th} normal mode.

¹Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t} \hat{\mathcal{O}}(0) e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{d\hat{\mathcal{O}}}{dt} = \frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

²For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{d\hat{x}}{dt} = \hat{p}(t)$ and $\frac{d\hat{p}}{dt} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) + \hat{a}^\dagger(t)] \quad \hat{p}(t) = -i\omega \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) - \hat{a}^\dagger(t)]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega (\hat{a}^\dagger(t)\hat{a} + \frac{1}{2})$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.

1.1.2 One-dimensional harmonic string

Taking the continuum limit, the crystal becomes a string: to achieve this, one takes the limits $a \rightarrow 0$ and $N \rightarrow \infty$ while keeping the total length $R \equiv Na$ fixed. In this context, the displacement becomes a field $\eta(x, t)$ dependent on the continuous real coordinate $x \in [0, R]$, therefore:

$$(\eta_{n+1} - \eta_n)^2 \longrightarrow a^2 \left(\frac{\partial \eta}{\partial x} \right)^2 \quad \sum_n \longrightarrow \frac{1}{a} \int_0^R dx$$

Proposition 1.1.1

In the continuous limit:

$$\frac{\delta_{nn'}}{a} \longrightarrow \delta(x - x') = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-x')}$$

Proof. By direct calculation:

$$a \sum_n f(an) \frac{\delta_{nm}}{a} = f(ma) \longrightarrow f(y) = \int_0^R dx f(x) \delta(x - y)$$

Recalling Eq. 1.10, since $k_m n = \frac{k_m}{a} na \rightarrow kx$, symmetrizing $k_m \in [-\pi, \pi]$ (instead of $[0, 2\pi]$) one finds:

$$\delta(x - x') \longleftarrow \frac{\delta_{nn'}}{a} = \frac{1}{Na} \sum_m e^{ik_m(n-n')} \longrightarrow \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-x')}$$

where integration limits are $\pm \frac{\pi}{a} \rightarrow \pm \infty$. □

Proposition 1.1.2

The inverse Fourier transform of the Dirac Delta reads:

$$\int_0^R dx e^{i(k-k')x} = 2\pi \delta(k - k')$$

By these relations, it can be seen that $\frac{dk}{2\pi}$ has the physical meaning of the number of normal modes per unit spatial volume with wavenumber between k and $k + dk$, while the interpretation of the divergent $\delta(0)$ varies: in x space, it is the reciprocal of the lattice spacing, i.e. the number of normal modes per unit spatial volume, but in k space $2\pi\delta(0)$ is the (hyper-)volume of the system.

In the continuous limit, the Lagrangian of the harmonic string becomes:

$$L = \int_0^R dx \left[\frac{1}{2} \rho_0 (\partial_t \eta)^2 - \frac{\kappa}{2} (\partial_x \eta)^2 \right]$$

where ρ_0 is the equilibrium mass density of the string. It is customary to absorb constants in the fields, thus, setting $\phi(x, t) \equiv \sqrt{\rho_0} \eta(x, t)$ and $\kappa = c^2 \rho_0$ and adding a pinning term $\propto \varphi^2$, the Lagrangian can be written as:

$$L = \int_0^R dx \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{c^2}{2} (\partial_x \phi)^2 - \frac{m^2 c^4}{2} \phi^2 \right] \quad (1.16)$$

The classical equation of motion of this field yields:

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi - m^2 c^4 \phi \quad (1.17)$$

The solutions of this wave equation can be written as:

$$\phi(x, t) = e^{i(kx - \omega_k t)} \quad (1.18)$$

with dispersion relation:

$$\omega_k^2 = c^2 k^2 + m^2 c^4 \quad (1.19)$$

To quantize this system, one needs to compute the Hamiltonian. The canonical momentum field is:

$$\Pi(x, t) := \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi(x, t) \quad (1.20)$$

The classical Hamiltonian can then be found as:

$$\hat{\mathcal{H}} = \int_0^R dx \left[\frac{1}{2} \Pi^2 + \frac{c^2}{2} (\partial_x \phi)^2 + \frac{m^2 c^4}{2} \phi^2 \right] \quad (1.21)$$

The quantum field is analogous to Eq. 1.9:

$$\hat{\phi}(x, t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \sqrt{\frac{\hbar}{2\omega_k}} \left[e^{i(kx - \omega_k t)} \hat{a}_k + e^{-i(kx - \omega_k t)} \hat{a}_k^\dagger \right] \quad (1.22)$$

with commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\pi \delta(k - k') \quad (1.23)$$

$$[\hat{\phi}(x, t), \hat{\Pi}(x', t)] = i\hbar \delta(x - x') \quad (1.24)$$

The quantum Hamiltonian can be written as:

$$\hat{\mathcal{H}} = \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{2} \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) = E_0 + \int_{\mathbb{R}} \frac{dk}{2\pi} \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \quad (1.25)$$

The ground-state energy can be computed from Eq. 1.14, defining $\text{Vol} := 2\pi \delta(k = 0)$:

$$E_0 = \text{Vol} \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{2} \hbar \omega_k \quad (1.26)$$

For a strictly continuous system there is no cut-off in the k integral, thus the zero-point energy diverges: however, this is not necessarily a problem, as often only changes in E_0 are relevant (and experimentally accessible), and in this case it is known as *Casimir energy*.

1.2 Spacetime symmetries

1.2.1 Lie groups

Definition 1.2.1: Lie group

A *Lie group* is a group whose elements depend in a continuous and differentiable way on a set of real parameters $\{\theta_a\}_{a=1,\dots,d} \subset \mathbb{R}^d$.

A Lie group can be seen both as a group and as a d -dimensional differentiable manifold (with coordinates θ_a). WLOG it is always possible to choose $g(0, \dots, 0) = e$.

Definition 1.2.2: Representation

Given a group G and a vector space $V(\mathbb{K})$, a *representation* of G on V is a homomorphism $\rho : G \rightarrow \text{GL}(V)$.

Given the isomorphism $\text{GL}(V) \rightarrow \mathbb{K}^{n \times n}$, with $n \equiv \dim_{\mathbb{K}} V$, it is usual to *de facto* represent G as matrices acting on elements of V , i.e. $\rho : G \rightarrow \mathbb{K}^{n \times n}$.

Theorem 1.2.1

Given a Lie group G and $g \in G$ connected with the identity, a representation of degree n on $V(\mathbb{C})$ as:

$$\rho(g(\theta)) = e^{i\theta_a T^a} \quad (1.27)$$

where $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ are the *generators* of G on V .

Definition 1.2.3: Lie algebra

Given a Lie group G with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ on $V(\mathbb{C})$, its *Lie algebra* is:

$$[T^a, T^b] = i f^{ab}_c T^c \quad (1.28)$$

where f^{ab}_c are called the *structure constants*.

Proposition 1.2.1

The Lie algebra of a Lie group is independent of the representation.

Proposition 1.2.2

Any d -dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional Lie algebras.

As a consequence, all irreducible representations of an abelian Lie group are of degree $n = 1$.

Definition 1.2.4: Casimir operator

Given a Lie group with generators $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ on $V(\mathbb{C})$, a *Casimir operator* is an operator which commutes with each generator.

Given an irreducible representation, Casimir operators are operators proportional to id_V , and the proportionality constants can be used to label the representation: they correspond to conserved physical quantities.

Proposition 1.2.3

A non-compact group cannot have finite unitary representations, except for those with trivial non-compact generators.

This means that the non-compact component of a group cannot be represented with unitary operators of finite dimension.

1.2.2 Lorentz group

Consider the group of linear transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on $\mathbb{R}^{1,3}$ which leave invariant the quantity $\eta_{\mu\nu} x^\mu x^\nu$, i.e. the orthogonal group $O(1,3)$ (with signature $(+, -, -, -)$). The condition that $\Lambda^\mu{}_\nu$ must satisfy reads:

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \quad (1.29)$$

This implies that $\det \Lambda = \pm 1$: a transformation with $\det \Lambda = -1$ can always be written as the product of a transformation with $\det \Lambda = 1$ and a discrete transformation which reverses the sign of an odd number of coordinates. One further defines $SO(1,3) := \{\Lambda \in O(1,3) : \det \Lambda = 1\}$.

Writing explicitly the temporal component $1 = (\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2$, it is clear that $(\Lambda^0_0)^2 \geq 1$. Therefore, $O(1,3)$ has two disconnected components: the orthochronous component with $\Lambda^0_0 \geq 1$ and the non-orthochronous component with $\Lambda^0_0 \leq -1$. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a discrete transformation which reverses the sign of the temporal component.

Definition 1.2.5: Lorentz group

The *Lorentz group* $SO^+(1,3)$ is the orthochronous component of $SO(1,3)$.

The discrete transformations are factored out of the Lorentz group: these are *parity* and *time reversal*, which can be represented as $\mathcal{P}^\mu{}_\nu = \text{diag}(+1, -1, -1, -1)$ and $\mathcal{T}^\mu{}_\nu = \text{diag}(-1, +1, +1, +1)$. Applying these discrete transformations in all combinations (id, \mathcal{P} , \mathcal{T} and \mathcal{PT}) one gets the four disconnected components of $SO(1,3)$, which are not simply connected. This means that Lorentz invariance does not include parity and time reversal invariance.

Considering the infinitesimal transformation and applying Eq. 1.29:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad \Rightarrow \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Anti-symmetry means that $\omega_{\mu\nu}$ has only 6 parameters, which define the Lorentz group: these can be identified by the 3 angles of spherical rotations in the (x, y) , (y, z) and (z, x) planes and the 3 rapidities of hyperbolic rotations in the (t, x) , (t, y) and (t, z) planes.

Theorem 1.2.2

The Lorentz group is a non-compact Lie group.

Proof. Spherical and hyperbolic rotations are continuous and differential w.r.t. angles and rapidities, but while angles vary in $[0, 2\pi)$, rapidities vary in \mathbb{R} , so the differentiable manifold associated to $\text{SO}^+(1, 3)$ is not compact. \square

1.2.2.1 Lorentz algebra

The 6 parameters of the Lorentz group correspond to 6 generators of the associated Lorentz algebra. Labelling these generators as $J^{\mu\nu} : J^{\mu\nu} = -J^{\nu\mu}$, the generic element $\Lambda \in \text{SO}^+(1, 3)$ can be written as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \quad (1.30)$$

The $\frac{1}{2}$ factor arises from each generator being counted twice (product of two anti-symmetric objects). Given a n -dimensional representation of $\text{SO}^+(1, 3)$, both $[J^{\mu\nu}]^i_j$ and $[\Lambda]^i_j$ are $\mathbb{C}^{n \times n}$ matrices (Λ is real): for example, the $n = 1$ representation acts on *scalars*, which are invariant under Lorentz transformations, so $J^{\mu\nu} \equiv 0 \forall \mu, \nu = 0, \dots, 3$.

4-vectors The $n = 4$ representation acts on *contravariant 4-vectors* v^μ , which transform according to $v^\mu \mapsto \Lambda^\mu_\nu v^\nu$, and *covariant 4-vectors* v_μ , which transform according to $v_\mu \mapsto \Lambda_\mu^\nu v_\nu$. In this representation, the generators are represented as $\mathbb{C}^{4 \times 4}$ matrices:

$$[J^{\mu\nu}]^\rho_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma) \quad (1.31)$$

This is an irreducible representation, and the associated Lie algebra $\mathfrak{so}^+(1, 3)$, called the *Lorentz algebra*, is:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \quad (1.32)$$

It is convenient to rearrange the 6 components of $J^{\mu\nu}$ into two spatial vectors:

$$J^i := \frac{1}{2}\epsilon^{ijk}J^{jk} \quad K^i := J^{i0} \quad (1.33)$$

The $\mathfrak{so}^+(1, 3)$ can then be rewritten as:

$$[J^i, J^j] = i\epsilon^{ijk}J^k \quad [J^i, K^j] = i\epsilon^{ijk}K^k \quad [K^i, K^j] = -i\epsilon^{ijk}J^k \quad (1.34)$$

The first equation defines a $\mathfrak{su}(2)$ sub-algebra, thus showing that J^i are the generators of angular momentum. Angles and rapidities are then defined as:

$$\theta^i := \frac{1}{2}\epsilon^{ijk}\omega^{jk} \quad \eta^i := \omega^{i0} \quad (1.35)$$

so that:

$$\Lambda = \exp[-i\boldsymbol{\theta} \cdot \mathbf{J} + i\boldsymbol{\eta} \cdot \mathbf{K}] \quad (1.36)$$

This definition reflect the *alias* interpretation: the angles define counterclockwise rotations of vectors with respect to a fixed reference frame, while rapidities define boosts which increase velocities with respect to said frame.

1.2.2.2 Tensor Representations

A generic (p, q) -tensor transforms as:

$$T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mapsto \Lambda^{\mu_1}_{\alpha_1} \dots \Lambda^{\mu_p}_{\alpha_p} \Lambda_{\nu_1}^{\beta_1} \dots \Lambda_{\nu_q}^{\beta_q} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \quad (1.37)$$

The representation of the Lorentz group which acts on (p, q) -tensors is of degree $n = 4^{p+q}$, however it is reducible into the direct product of $p + q$ 4-dimensional representations as of Eq. 1.38.

Moreover, consider the action of the Lorentz group on $(2, 0)$ -tensors: being $T^{\mu\nu} \mapsto \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$, if $T^{\mu\nu}$ is (anti-)symmetric it will remain so under a Lorentz transformation. Therefore, the 16-dimensional representation reduces to a 6-dimensional representation on anti-symmetric tensors and a 10-dimensional representation of symmetric tensors. Furthermore, the trace of a symmetric tensor is invariant, as $T \equiv \eta_{\mu\nu} T^{\mu\nu} \mapsto \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta} = T$, so the latter representation further reduces into a 9-dimensional representation on symmetric traceless tensors and a 1-dimensional representation on scalars. This means that:

$$4 \otimes 4 = 1 \oplus 6 \oplus 9 \quad (1.38)$$

These are irreducible representations which, given a generic tensor $T^{\mu\nu}$, act on S , $A^{\mu\nu}$ and $S^{\mu\nu} - \frac{1}{4} \eta^{\mu\nu} S$ respectively, with $A^{\mu\nu} \equiv \frac{1}{2} (T^{\mu\nu} - T^{\nu\mu})$ and $S^{\mu\nu} \equiv \frac{1}{2} (T^{\mu\nu} + T^{\nu\mu})$.

Decomposition under rotations Restricting the action to the $\text{SO}(3)$ sub-group of $\text{SO}^+(1, 3)$, tensors can be decomposed according to irreducible representations of $\text{SO}(3)$, which are labelled by the angular momentum $j \in \mathbb{N}_0$ and are of degree $n = 2j + 1$. Also recall the Clebsh-Gordan composition of angular momenta:

$$\mathbf{j}_1 \otimes \mathbf{j}_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{j} \quad (1.39)$$

A Lorentz scalar α is a scalar under rotations too, so $\alpha \in \mathbf{0}$. A 4-vector v^μ is irreducible under the action of $\text{SO}^+(1, 3)$, but under $\text{SO}(3)$ it is decomposed into v^0 and \mathbf{v} , so $v^\mu \in \mathbf{0} \oplus \mathbf{1}$. A $(2, 0)$ -tensor then is:

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \end{aligned}$$

This is equivalent to Eq. 1.38: the trace is a scalar, so $S \in \mathbf{0}$, while the anti-symmetric part can be written as two spatial vectors A^{0i} and $\frac{1}{2} \epsilon^{ijk} A^{jk}$, so $A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. The traceless symmetric part then decomposes as $\bar{S}^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ under spatial rotations.

Equivalently, $T^{\mu\nu}$ can be decomposed into $T^{00} \in (\mathbf{0} \otimes \mathbf{0})$, $T^{0i} \in (\mathbf{0} \otimes \mathbf{1})$, $T^{i0} \in (\mathbf{1} \otimes \mathbf{0})$ and $T^{ij} \in (\mathbf{1} \otimes \mathbf{1})$: the formers are a scalar and two spatial vectors associated to $\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1}$, while the latter can be decomposed into the trace, which is $\mathbf{0}$, the anti-symmetric part, which is $\mathbf{1}$ ($\epsilon^{ijk} A^{jk}$), and the traceless symmetric part, which is $\mathbf{2}$.

Example 1.2.1

Gravitational waves in de Donder gauge are described by a traceless symmetric matrix, therefore they have $j = 2$ (spin of the graviton).

There are two *invariant tensors* under $\text{SO}^+(1, 3)$: the metric $\eta_{\mu\nu}$, by Eq. 1.29, and the Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$:

$$\epsilon^{\mu\nu\sigma\rho} \mapsto \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\sigma_\gamma \Lambda^\rho_\delta \epsilon^{\alpha\beta\gamma\delta} = (\det \Lambda) \epsilon^{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho}$$

1.2.2.3 Spinorial representations

The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same, which means that $SU(2)$ and $SO(3)$ are indistinguishable by infinitesimal transformations; however, they are globally different, as $SO(3)$ rotations are periodic by 2π , while $SU(2)$ rotations are periodic by 4π : in particular, it can be shown that $SO(3) \cong SU(2)/\mathbb{Z}_2$, i.e. $SU(2)$ is the universal cover of $SO(3)$. This means that $SU(2)$ representations can be labelled by $j \in \frac{1}{2}\mathbb{N}_0$, where half-integer spin representations are known as *spinorial representations*: they act on spinors, i.e. objects which change sign under rotations of 2π (thus not suitable to represent $SO(3)$).

Example 1.2.2

The $\frac{1}{2}$ representation of $SU(2)$ is a 2-dimensional representation where $J^i = \frac{\sigma^i}{2}$: Pauli matrices satisfy $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$, thus the $\mathfrak{su}(2)$ algebra is satisfied. Denoting the $m = \pm\frac{1}{2}$ states in the $\frac{1}{2}$ representation as $|\uparrow\rangle$ and $|\downarrow\rangle$, the Clebsch-Gordan decomposition $\frac{1}{2} \otimes \frac{1}{2} = \mathbf{0} \oplus \mathbf{1}$ yields the triplet ($j = 1$) $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$ and the singlet ($j = 0$) $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Proposition 1.2.4

The Lorentz algebra $\mathfrak{so}^+(1, 3)$ can be decomposed as $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

Proof. Given the $\mathfrak{so}^+(1, 3)$ algebra in Eq. 1.33, it is possible to define:

$$\mathbf{J}_\pm := \frac{1}{2} (\mathbf{J} \pm i\mathbf{K})$$

The Lie algebra then becomes:

$$[\mathbf{J}_\pm^i, \mathbf{J}_\pm^j] = i\epsilon^{ijk} \mathbf{J}_\pm^k \quad [\mathbf{J}_\pm^i, \mathbf{J}_\mp^j] = 0$$

These are two commuting $\mathfrak{so}(2)$ algebras, thus proving the thesis. \square

As observed before, this does not imply that $SO^+(1, 3)$ is globally equivalent to $SU(2) \times SU(2)$: in fact, $SU(2) \times SU(2)/\mathbb{Z}_2 \cong SO(4)$, while the universal cover of $SO^+(1, 3)$ is $SL(2, \mathbb{C})$, as it can be shown that $SO^+(1, 3) \cong SL(2, \mathbb{C})/\mathbb{Z}_2$.

By Prop. 1.2.4, representations of $SO^+(1, 3)$ can be labelled by $(j_-, j_+) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$, with each index labelling a representation of $SU(2)$: as $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, the (j_-, j_+) representation contains states with all possible spins $|j_+ - j_-| \leq j \leq j_+ + j_-$, and it is a representation of degree $n = (2j_- + 1)(2j_+ + 1)$. $(\mathbf{0}, \mathbf{0})$ is the trivial (scalar) representation, as both $\mathbf{J}_\pm = \mathbf{0}$ and $\mathbf{J} = \mathbf{K} = \mathbf{0}$.

$(\frac{1}{2}, \mathbf{0})$ and $(\mathbf{0}, \frac{1}{2})$ are 2-dimensional spinorial representations. These representations act on different spinors $(\psi_L)_\alpha \in (\frac{1}{2}, \mathbf{0})$ and $(\psi_R)_\alpha \in (\mathbf{0}, \frac{1}{2})$, with $\alpha = 1, 2$, which are called *left-* and *right-handed Weyl spinors*. In $(\frac{1}{2}, \mathbf{0})$ the generators are $\mathbf{J}_- = \frac{\sigma}{2}$ and $\mathbf{J}_+ = \mathbf{0}$, while in $(\mathbf{0}, \frac{1}{2})$ they are $\mathbf{J}_- = \mathbf{0}$ and $\mathbf{J}_+ = \frac{\sigma}{2}$, thus one finds $\mathbf{J}_L = \mathbf{J}_R = \frac{\sigma}{2}$ and $\mathbf{K}_L = -\mathbf{K}_R = i\frac{\sigma}{2}$, so that:

$$\psi_L \mapsto \Lambda_L \psi_L = \exp \left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2} \right] \psi_L \quad (1.40)$$

$$\psi_R \mapsto \Lambda_R \psi_R = \exp \left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2} \right] \psi_R \quad (1.41)$$

Note that the generators K^i are not hermitian, as expected from Prop. 1.2.3. Furthermore, note that $\Lambda_{L,L} \in \mathbb{C}^{2 \times 2}$, therefore $\psi_{L,R} \in \mathbb{C}^2$.

Proposition 1.2.5

Given $\psi_L \in (\frac{1}{2}, \mathbf{0})$ and $\psi_R \in (\mathbf{0}, \frac{1}{2})$, then $\sigma^2 \psi_L^* \in (\mathbf{0}, \frac{1}{2})$ and $\sigma^2 \psi_R^* \in (\frac{1}{2}, \mathbf{0})$.

Proof. Recall that for Pauli matrices $\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$, so $\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$ and:

$$\sigma^2 \psi_L^* \mapsto \sigma^2 (\Lambda_L \psi_L)^* = (\sigma^2 \Lambda_L^* \sigma^2) \sigma^2 \psi_L^* = \Lambda_R \sigma^2 \psi_L^* \Rightarrow \sigma^2 \psi_L^* \in (\mathbf{0}, \frac{1}{2})$$

where $\sigma^2 \sigma^2 = I_2$ was used. The proof for $\sigma^2 \psi_R^*$ is analogous. \square

Definition 1.2.6: Charge conjugation

On Weyl spinors, the *charge conjugation operator* is defined as:

$$\psi_L^c := i\sigma^2 \psi_L^* \quad \psi_R^c := -i\sigma^2 \psi_R^* \quad (1.42)$$

By Prop. 1.2.5, charge conjugation changes transforms a left-handed Weyl spinor into a right-handed one and vice versa. Moreover, the i factor ensures that applying this operator twice yields the identity operator.

$(\frac{1}{2}, \frac{1}{2})$ is a 4-dimensional complex representation: as $j = 0, 1$, this representation acts on complex 4-vectors of the form $((\psi_L)_\alpha, (\xi_R)_\beta) \in \mathbb{C}^4$, called *Dirac spinors*, and $\Lambda = \text{diag}(\Lambda_L, \Lambda_R) \in \mathbb{C}^{4 \times 4}$. To explicit this relation, set $\psi_R \equiv i\sigma^2 \psi_L^*$, $\xi_L \equiv -i\sigma^2 \xi_R^*$ and $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$: it can be shown, then, that $\xi_R^\dagger \sigma^\mu \psi_R$ and $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ are contravariant 4-vectors. Although these 4-vectors are complex by construction, being the matrix $\Lambda^\mu{}_\nu$ which represents the Lorentz transformation of a 4-vector real, a reality condition $v_\mu^* = v_\mu$ is Lorentz invariant.

Parity Note that $\mathcal{P}\mathbf{K} = -\mathbf{K}$, as the velocity of the boost gets reversed, while $\mathcal{P}\mathbf{J} = \mathbf{J}$: this means that $\mathcal{P}\mathbf{J}_\pm = \mathbf{J}_\mp$, i.e. parity exchanges a $(\mathbf{j}_-, \mathbf{j}_+)$ representation into a $(\mathbf{j}_+, \mathbf{j}_-)$ representation. Therefore, a $(\mathbf{j}_-, \mathbf{j}_+)$ representation of $\text{SO}^+(1, 3)$ is a basis for the representation of the parity transformation iff $j_- = j_+$.

Example 1.2.3

Weyl spinors (separately) are not a basis for a representation of the parity transformation, but Dirac spinors are.

1.2.2.4 Field representations

Given a field $\phi(x)$, under a Lorentz transformation $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ it transforms as $\phi(x) \mapsto \phi'(x')$.

Scalar fields A scalar field transforms as:

$$\phi'(x') = \phi(x) \quad (1.43)$$

Consider an infinitesimal transformation $x'^\rho = x^\rho + \delta x^\rho$, with $\delta x^\rho = -\frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}]^\rho{}_\sigma x^\sigma$ as of Eq. 1.30. Then, by definition, $\delta\phi \equiv \phi'(x') - \phi(x) = 0$, which corresponds to the fact that the scalar representation of $SO^+(1,3)$ is the trivial one ($J^{\mu\nu} = 0$).

However, one can consider the variation at fixed coordinate $\delta_0\phi \equiv \phi'(x) - \phi(x)$: while $\delta\phi$ studies only a single degree of freedom, as the point $P \in \mathbb{R}^{1,3}$ is kept constant and only $\phi(P)$ can vary (i.e. the base space is one-dimensional), $\delta_0\phi$ studies $\phi(P)$ with P varying over $\mathbb{R}^{1,3}$, thus the base space is now a space of functions, which is infinite-dimensional. Therefore, $\delta\phi$ provides a finite-dimensional representation of the generators, while $\delta_0\phi$ an infinite-dimensional one.

To explicit this representation:

$$\delta_0\phi = \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho \partial_\rho \phi = \frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}]^\rho{}_\sigma x^\sigma \partial_\rho \phi \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi$$

Recalling Eq. 1.31, the generators can be expressed as:

$$L^{\mu\nu} := i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (1.44)$$

This is an infinite-dimensional representation, as it acts on the space of scalar fields. As $p^\mu = i\partial^\mu$ (with signature $(+, -, -, -)$), it is clear that $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ is the orbital angular momentum.

Weyl fields A left-handed Weyl field transforms as:

$$\psi'_L(x') = \Lambda_L \psi_L(x) \quad (1.45)$$

with Λ_L defined in Eq. 1.40, and similarly for right-handed Weyl fields. The infinite-dimensional representation of the Lorentz generators determined by Weyl spinors can be found as:

$$\begin{aligned} \delta_0\psi_L &\equiv \psi'_L(x) - \psi_L(x) = \psi'_L(x' - \delta x) - \psi_L(x) \\ &= \psi'_L(x') - \delta x^\rho \partial_\rho \psi_L(x) - \psi_L(x) = (\Lambda_L - I_2) \psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x) \end{aligned}$$

The second term yields $L^{\mu\nu}$, while the first can be further elaborated by writing:

$$\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \quad (1.46)$$

Thus:

$$\delta_0\psi_L = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_L$$

where the angular momentum separates into the orbital and the spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \quad (1.47)$$

This separation is general: $L^{\mu\nu}$ is always expressed as in Eq. 1.44, while $S^{\mu\nu}$ depends on the specific representation. In the scalar representation $S^{\mu\nu} = 0$, while in the left/right-handed Weyl representation $S^{i0} = \pm i\frac{\sigma^i}{2}$.

Vector fields A (contravariant) vector field transforms as:

$$V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) \quad (1.48)$$

A general vector field has a spin-0 and a spin-1 component, and it is acted on by the $(\frac{1}{2}, \frac{1}{2})$ representation.

1.2.3 Poincaré group

Definition 1.2.7: Poincaré group

The *Poincaré group* is defined as $\text{ISO}^+(1, 3) := \text{T}^{1,3} \times \text{SO}^+(1, 3)$, where $\text{T}^{1,3} \cong \mathbb{R}^{1,3}$ is the group of translations of $\mathbb{R}^{1,3}$.

Given a translation $x^\mu \mapsto x^\mu + a^\mu$, the associated group element can be written as:

$$T = e^{-ia_\mu P^\mu} \quad (1.49)$$

where P^μ is the 4-momentum operator. Clearly translations commute, and so do their generators; on the other hand, as \mathbf{P} is a vector under rotations, while P^0 (energy) a scalar, one has:

$$[J^i, P^j] = ie^{ijk}P^k \quad [J^i, P^0] = 0$$

These equations uniquely determine the *Poincaré algebra* $\mathfrak{iso}^+(1, 3)$:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [J^{\mu\nu}, J^{\sigma\rho}] &= i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \\ [P^\mu, J^{\rho\sigma}] &= i(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho) \end{aligned} \quad (1.50)$$

It's easy to check that $[K^i, P^0] = iP^i$, while $[J^i, P^0] = [P^i, P^0] = 0$: given that P^0 generates time translations, linear and angular momentum are conserved quantities, while \mathbf{K} is not.

1.2.3.1 Field representations

Fields provide an infinite-dimensional representation of the Lorentz group as $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $S^{\mu\nu}$ does not depend on x^μ , but only on the spin of the field.

To represent P^μ on fields, their transformation law must be specified: all fields are required to be scalars under translations, independently of their spin. This means that, given a generic field $\phi(x)$, under a translation $x' = x + a$ it transforms as $\phi'(x') = \phi(x)$, so, under an infinitesimal translation $x' = x + \varepsilon$:

$$\begin{aligned} \delta_0\phi &\equiv \phi'(x) - \phi(x) = \phi'(x' - \varepsilon) - \phi(x) = -\varepsilon^\mu \partial_\mu \phi(x) \\ &= e^{-i(-\varepsilon_\mu)P^\mu} \phi'(x') - \phi(x) = (e^{i\varepsilon_\mu P^\mu} - \mathbf{I}) \phi(x) = i\varepsilon_\mu P^\mu \phi(x) \end{aligned}$$

It is clear then that:

$$P^\mu = +i\partial^\mu \quad (1.51)$$

Explicitly, $P^0 = i\partial_t$ and $\mathbf{P} = -i\nabla$. It is trivial to check that these generators obey the Poincaré algebra.

1.2.3.2 Particle representations

The Poincaré group can also be represented using the Hilbert space \mathcal{H} of a free particle as a basis. Denoting a generic state as $|\mathbf{p}, s\rangle \in \mathcal{H}$, where \mathbf{p} is the particle's momentum and s collectively labels all other quantum numbers, it is clear that \mathcal{H} is infinite-dimensional, as \mathbf{p} is a continuous unbounded variable.

Theorem 1.2.3: Wigner's theorem

On the Hilbert space of physical states, any symmetry transformation can be represented by a linear and unitary or anti-linear and anti-unitary operator.

By this theorem, Poincaré transformations can be represented by unitary matrices, i.e. \mathbf{J} , \mathbf{K} , \mathbf{P} and P^0 can be represented by hermitian operators. These representations can be labeled by Casimir operators, which for $\text{ISO}^+(1, 3)$ are easily found as $P_\mu P^\mu$ and $W_\mu W^\mu$, where W^μ is the *Pauli-Lubanski operator*:

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}J_{\nu\sigma}P_\rho \quad (1.52)$$

On single-particle states $P_\mu P^\mu = m^2$, while $W_\mu W^\mu$ can be conveniently computed in a particular frame (due to Lorentz invariance). If $m \neq 0$, this frame is the rest-frame of the particle:

$$W^\mu = -\frac{m}{2}\epsilon^{\mu\nu\sigma 0}J_{\nu\sigma} = \frac{m}{2}\delta^{\mu i}\epsilon^{ijk}J^{jk} = \delta^{\mu i}mJ^i$$

Therefore, on single-particle states of mass m and spin j , the Casimir operator takes the form:

$$W_\mu W^\mu = -m^2 j(j+1) \quad (1.53)$$

If $m = 0$, the rest-frame does not exist, but it is possible to choose a frame where $P^\mu = (\omega, 0, 0, \omega)$, where $W^0 = W^3 = \omega J^3$, $W^1 = \omega(J^1 - K^2)$ and $W^2 = \omega(J^2 + K^1)$, so that:

$$W_\mu W^\mu = -\omega^2 [(K^2 - J^1)^2 + (K^1 + J^2)^2] \quad (1.54)$$

It is clear that the $m \rightarrow 0$ limit is not trivial, and massive and massless representation need to be studied separately.

Massive representations Restricting to $m \in \mathbb{R}^+$ ($m^2 < 0$ states, called tachyons, are excluded), the massive representations are labeled by mass m and spin j : in fact, after a Lorentz transformation such that $P^\mu = (m, \mathbf{0})$, spatial rotations can still be performed, i.e. the subspace of single-particle states with momentum $P^\mu = (m, \mathbf{0})$ is still a basis for the representation of $\text{SU}(2)$ (as spinors must be included too). The group of transformations which leaves invariant a certain choice of P^μ is called the *little group*, so $\text{SU}(2)$ is the little group of massive single-particle states: massive representations are labelled by m and j , which means that massive particles of spin j have $2j+1$ degrees of freedom.

Massless representations The little group for $P^\mu = (\omega, 0, 0, \omega)$ clearly is $\text{SO}(2)$, the group of rotations in the (x, y) plane generated by J^3 : as for any abelian group, its irreducible representations are one-dimensional, and they are labeled by the eigenvalue h of J^3 , which represents the angular momentum in the direction of propagation of the particle and is called *helicity*. Helicity can be shown to be quantized as $h \in \frac{1}{2}\mathbb{Z}_0$ (by topologic considerations on $\text{ISO}^+(1, 3) \equiv \mathbb{R}^4 \times \text{SL}(2, \mathbb{C})/\mathbb{Z}_2$).

As a consequence, massless particles only have one degree of freedom and are characterized by their helicity h . As $\text{SO}(2) \equiv \text{U}(1)$, on a state of helicity h the little group is represented as:

$$U(\theta) = e^{-ih\theta} \quad (1.55)$$

Although massless particles with opposite helicities are logically two different species of particles, it can be written as $h = \hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$ (unit vectors), so h is a pseudoscalar such that $h \mapsto -h$ under parity: this means that, if the interaction conserves parity, h and $-h$ must be symmetric.

Example 1.2.4

The electromagnetic and gravitational interactions conserve parity, thus photons and gravitons must be a basis for the representation of both $\text{ISO}^+(1, 3)$ and parity: photons can have $h = \pm 1$ (left- and right-handed), while gravitons have $h = \pm 2$.

Example 1.2.5

Neutrinos only interact via the weak interaction, which does not conserve parity, and in fact the two states $h = \pm \frac{1}{2}$ are different particles: neutrinos have $h = -\frac{1}{2}$, while antineutrinos have $h = +\frac{1}{2}$.

1.3 Classical equations of motion

Consider a *local field theory* of fields $\{\phi_i(x)\}_{i \in \mathcal{I}} \equiv \phi(x)$, where $x \in \mathbb{R}^{1,3}$ is a point in Minkowski spacetime. Its Lagrangian takes the form:

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.56)$$

where \mathcal{L} is the *Lagrangian density* of the theory (often referred to simply as the Lagrangian), which depends only on a finite number of derivatives. The action is then:

$$\mathcal{S} = \int dt L = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.57)$$

The integration is carried on the whole space-time, with usual boundary conditions that all fields decrease sufficiently fast at infinity; this also allows to drop all boundary terms.

Theorem 1.3.1

The *stationary action principle* $\delta \mathcal{S} = 0$ determines the classical equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} = 0 \quad (1.58)$$

Proof. Varying Eq. 1.57:

$$\delta \mathcal{S} = \int d^4x \sum_{i \in \mathcal{I}} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \delta (\partial_\mu \phi_i) \right] = \int d^4x \sum_{i \in \mathcal{I}} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right] \delta \phi_i = 0$$

□

Corollary 1.3.1.1

Two Lagrangians which differ by a total divergence $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu$ yield the same equations of motion.

Proof. This is a consequence of Stokes theorem:

$$\int_{\Sigma} d^4x \partial_{\mu} K^{\mu} = \int_{\partial\Sigma} dA n_{\mu} K^{\mu}$$

□

From the Lagrangian, it is possible to define the conjugate momenta and the Hamiltonian density:

$$\Pi_i(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \quad (1.59)$$

$$\mathcal{H} = \sum_{i \in \mathcal{I}} \Pi_i(x) \partial_0 \phi(x) - \mathcal{L} \quad (1.60)$$

Unlike the Hamiltonian formalism, the Lagrangian formalism keeps Lorentz covariance explicit.

1.3.1 Noether's theorem

Definition 1.3.1: Infinitesimal transformation

Given a field theory with fields $\{\phi_i\}_{i=1,\dots,k}$ and action $\mathcal{S}[\phi]$, an *infinitesimal transformation* parametrized by $\{\varepsilon^a\}_{a=1,\dots,N} : |\varepsilon^a| \ll 1$ is defined by two sets of functions $\{A_a^{\mu}(x)\}_{a=1,\dots,N}$ and $\{F_{i,a}(\phi, \partial\phi)\}_{i=1,\dots,k; a=1,\dots,N}$ such that:

$$\begin{aligned} x^{\mu} &\mapsto x'^{\mu} = x^{\mu} + \varepsilon^a A_a^{\mu}(x) \\ \phi_i(x) &\mapsto \phi'_i(x) = \phi_i(x) + \varepsilon^a F_{i,a}(\phi, \partial\phi) \end{aligned} \quad (1.61)$$

Definition 1.3.2: Symmetry transformation

An infinitesimal transformation is a *symmetry transformation* if it leaves $\mathcal{S}[\phi]$ invariant, regardless of ϕ being a solution of the equations of motion. It can further be classified as:

- *global symmetry*, if $\varepsilon^a \equiv \text{const.}$;
- *local symmetry*, if $\varepsilon^a = \varepsilon^a(x)$.

Symmetry transformations which leave spacetime unchanged, i.e. with $A_a^{\mu}(x) = 0$, are called *internal symmetries*.

Theorem 1.3.2: Noether's theorem

Given a global (but not local) symmetry parametrized by N generators, then there are N conserved currents $\{j_a^{\mu}(\phi)\}_{a=1,\dots,N}$ such that:

$$\partial_{\mu} j_a^{\mu}(\phi^{\text{cl}}) = 0 \quad (1.62)$$

where ϕ^{cl} is a classical solution of the equations of motion.

Proof. First, consider an infinitesimal transformation with slowly-varying parameters, i.e. $l |\partial_\mu \varepsilon^a| \ll |\varepsilon^a|$ (l characteristic scale of the field theory): being it not a local symmetry, $\delta \mathcal{S} \neq 0$ at $o(\varepsilon)$ and:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int d^4x [\varepsilon^a(x) K_a(\phi) - (\partial_\mu \varepsilon^a(x)) j_a^\mu(\phi) + o(\partial \partial \varepsilon)] + o(\varepsilon^2)$$

If $\varepsilon^a \equiv \text{const.}$ (global symmetry) then $\delta \mathcal{S}[\phi] = 0 \forall \phi$, therefore $K_a(\phi) = 0 \forall \phi$ (independent of ε). Assuming $\varepsilon^a(x) \rightarrow 0$ sufficiently fast as $x \rightarrow \infty$, then integration by parts yields:

$$\mathcal{S}[\phi'] = \mathcal{S}[\phi] + \int d^4x \varepsilon^a(x) \partial_\mu j_a^\mu(\phi) + o(\partial \partial \varepsilon) + o(\varepsilon^2)$$

This expression is independent of the choice of ϕ . Moreover, note that Eq. 1.61 can be rewritten as an internal transformation by setting:

$$\phi_i(x) \mapsto \phi'_i(x) = \phi_i(x - \varepsilon^a A_a) + \varepsilon^a F_{i,a} = \phi_i(x) + \varepsilon^a F_{i,a} - \varepsilon^a A_a^\mu \partial_\mu \phi_i \equiv \phi_i(x) + \delta \phi_i(x)$$

$\delta \phi_i(x)$ vanishes at infinity, therefore it is the kind of variation used to derive the equations of motion: choosing $\phi \equiv \phi^{\text{cl}}$ classical solution then implies $\delta \mathcal{S} = 0$ independently of ε , i.e. the thesis. \square

These are often called *Noether currents*, and the associated *Noether charges* are defined as:

$$Q_a := \int d^3x j_a^0(t, \mathbf{x}) \quad (1.63)$$

These are time-independent, as $\partial_0 Q_a = \int d^3x \partial_0 j_a^0 = - \int d^3x \partial_i j_a^i$: on all spacetime it vanishes by divergence theorem (fields vanish at infinity), but on a finite volume it yields a boundary term interpreted as the incoming and outgoing flux.

Proposition 1.3.1: Noether currents

The explicit expression of Noether currents is:

$$j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} [A_a^\nu(x) \partial_\nu \phi_i - F_{i,a}(\phi, \partial \phi)] - A_a^\mu(x) \mathcal{L} \quad (1.64)$$

Proof. Varying the action at $o(\partial \varepsilon)$:

$$\delta_\varepsilon \mathcal{S} = \delta_\varepsilon \int d^4x \mathcal{L} = \int \left[\delta_\varepsilon(d^4x) \mathcal{L} + d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta_\varepsilon \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta_\varepsilon(\partial_\mu \phi_i) \right) \right]$$

The Jacobian of Eq. 1.61 gives $d^4x \mapsto d^4x (1 + A_a^\mu \partial_\mu \varepsilon^a) + o(\varepsilon)$, while $\delta_\varepsilon \phi_i$ is not $o(\partial \varepsilon)$ and:

$$\delta_\varepsilon(\partial_\mu \phi_i) = \frac{\partial \phi'_i}{\partial x'^\mu} - \frac{\partial \phi_i}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} (\phi_i + \varepsilon^a F_{i,a}) - \frac{\partial \phi_i}{\partial x^\mu} = -(\partial_\mu \varepsilon^a) (A_a^\nu \partial_\nu \phi_i - F_{i,a}) + o(\varepsilon)$$

The thesis follows from $\delta_\varepsilon \mathcal{S} = - \int d^4x (\partial_\mu \varepsilon^a) j_a^\mu + o(\partial \partial \varepsilon) + o(\varepsilon^2)$. \square

If the considered infinitesimal transformation is not a global symmetry, then $\delta_\varepsilon \mathcal{S}$ has a non-vanishing $o(\varepsilon)$ term which gives rise to a quasi-conserved current:

$$\partial_\mu j_\mu^a = -(\delta_a \mathcal{L})_{\text{global}} \quad (1.65)$$

1.3.1.1 Energy-momentum tensor

Consider spacetime translations: as all fields must be scalars under these transformations, they define a Noether current. In particular, translations have $A_\nu^\mu = \delta_\nu^\mu$ and $F_{i,\mu} = 0$ (the parameter index is a Lorentz index), so the conserved current is the *energy-momentum tensor* $j_\nu^\mu \equiv \theta_\nu^\mu$:

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \eta^{\mu\nu} \mathcal{L} \quad (1.66)$$

which is covariantly conserved on classical solutions of the equations of motion. The conserved Noether charge associated to the energy-momentum tensor is the *four-momentum*:

$$P^\mu := \int d^3x \theta^{0\mu} \quad (1.67)$$

The energy-momentum tensor so defined is not symmetric, however it can be made to via a tensor $A^{\rho\mu\nu}$ which is anti-symmetric w.r.t. (ρ, μ) : $T^{\mu\nu} \equiv \theta^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$ is physically equivalent from $\theta^{\mu\nu}$, as the second term is a vanishing spatial divergence in the definition of P^μ and is contracted to 0 in the conservation law.

1.3.2 Real scalar fields

Consider a real scalar field ϕ : a non-trivial Poincaré-invariant action must contain $\partial_\mu \phi$ and must saturate each Lorentz index. For example:

$$\mathcal{S}[\phi] = \frac{1}{2} \int d^4x (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \quad (1.68)$$

The resulting equation of motion is the *Klein-Gordon equation*:

$$(\square + m^2) \phi = 0 \quad (1.69)$$

where $\square \equiv \partial_\mu \partial^\mu$. A plane wave $e^{\pm i p_\mu x^\mu}$ is a solution if $p^2 = m^2$, so the KG equation imposes the relativistic dispersion relation and m can be interpreted as the mass. As ϕ must be real, the general solution is a superposition of waves:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i p_\mu x^\mu} + a_{\mathbf{p}}^* e^{i p_\mu x^\mu}]_{p^0=E_{\mathbf{p}}} \quad (1.70)$$

The positive energy solution has $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$, but it contains both *positive* and *negative frequency modes* $e^{\mp i p_\mu x^\mu}$, while the $(2E_{\mathbf{p}})^{-1/2}$ factor is a convenient normalization of the $a_{\mathbf{p}}$ coefficients. The overall normalization of $\mathcal{S}[\phi]$ does not influence the equations of motion, however it is important for obtaining a positive-definite Hamiltonian. As the momentum conjugate to ϕ is $\Pi_\phi = \partial_0 \phi$, the Hamiltonian density is found as:

$$\mathcal{H} = \frac{1}{2} [\Pi_\phi^2 + (\nabla \phi)^2 + m^2 \phi^2] \quad (1.71)$$

The energy-momentum tensor is computed to be:

$$\theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \quad (1.72)$$

It is trivial to see that $\theta^{00} = \mathcal{H}$: the Hamiltonian is the conserved charge related to the invariance under time translations.

To compute the conserved currents associated to Lorentz invariance, it is convenient to label the transformation parameters $\omega^{\mu\nu}$ by an anti-symmetric pair of Lorentz indices, so that Eq. 1.61 become:

$$x^\mu \mapsto x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2} \omega^{\rho\sigma} (\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho) \equiv x^\mu + \frac{1}{2} A^\mu_{(\rho\sigma)} \omega^{\rho\sigma}$$

As $F_{i,a} = 0$, from Eq. 1.64 the conserved currents are:

$$j^{(\rho\sigma)\mu} = x^\rho \theta^{\mu\sigma} - x^\sigma \theta^{\mu\rho} \quad (1.73)$$

For spatial rotations, the conserved charge is:

$$M^{ij} = \int d^3x (x^i \theta^{0j} - x^j \theta^{0i}) = \int d^3x \partial_0 \phi (x^i \partial^j - x^j \partial^i) \phi = \frac{i}{2} \int d^3x [\phi L^{ij} (\partial_0 \phi) - (\partial_0 \phi) L^{ij} \phi]$$

where integration by parts was carried and L^{ij} is defined by Eq. 1.44. This can be generalized.

Definition 1.3.3: Scalar product

Given two real scalar fields ϕ_1 and ϕ_2 , their *scalar product* is defined as:

$$\langle \phi_1 | \phi_2 \rangle := \frac{i}{2} \int d^3x \phi_1 \overleftrightarrow{\partial}_0 \phi_2 \quad (1.74)$$

where $f \overleftrightarrow{\partial}_\mu g := f \partial_\mu g - (\partial_\mu f) g$.

Proposition 1.3.2

If ϕ_1 and ϕ_2 are KG solutions, then $\langle \phi_1 | \phi_2 \rangle$ is time-independent.

Proof. By the KG equation:

$$\partial_0 [\phi_1 \partial_0 \phi_2 - (\partial_0 \phi_1) \phi_2] = \phi_1 \partial_0^2 \phi_2 - (\partial_0^2 \phi_1) \phi_2 = \phi_1 \nabla^2 \phi_2 - (\nabla^2 \phi_1) \phi_2$$

which vanishes after integration by parts. □

Note that this scalar product is not positive-definite.

Theorem 1.3.3: Conserved charges

Given a symmetry represented by a Lie group and a representation $L^{\mu\nu}$ of its generators as operators acting on fields, the value of the associated conserved charges on a solution ϕ of the equations of motion is:

$$M^{\mu\nu} = \langle \phi | L^{\mu\nu} | \phi \rangle \quad (1.75)$$

Example 1.3.1: Four-momentum

Applying Th. 1.3.3 to four-momentum $P^\mu = \langle \phi | i\partial^\mu | \phi \rangle$; for example, the $\mu = 0$ component is:

$$\begin{aligned} P^0 &= \langle \phi | i\partial^0 | \phi \rangle = \langle \phi | i\partial_0 | \phi \rangle = \frac{i}{2} \int d^3x [\phi(i\partial_0)\partial_0\phi - (\partial_0\phi)i\partial_0\phi] \\ &= \frac{1}{2} \int d^3x [-\phi\partial_0^2\phi + (\partial_0\phi)^2] = \frac{1}{2} \int d^3x [-\phi(\nabla^2 - m^2)\phi + (\partial_0\phi)^2] \\ &= \frac{1}{2} \int d^3x [(\nabla\phi)^2 + m^2\phi^2 + (\partial_0\phi)^2] = \frac{1}{2} \int d^3x \theta^{00} \end{aligned}$$

The free KG action can be generalized to a self-interacting real scalar field introducing a general potential:

$$\mathcal{S}[\phi] = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \quad (1.76)$$

The quadratic term in $V(\phi)$ is the mass term, while higher-order terms describe the self-interaction.