

Quantum Field Theory 1

Prof. S. Forte a.a. 2024-25

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Part I

Classical Field Theory

Discrete and Continuous Systems

1.1 One-dimensional harmonic crystal

Consider a simple one-dimensional model of a crystal where atoms of mass $m \equiv 1$ lie at rest on the x -axis, with equilibrium positions labelled by $n \in \mathbb{N}$ and equally spaced by a distance a .

Assuming these atoms are free to vibrate only in the x direction (longitudinal waves), and denoting the displacement of the atom at position n as η_n , one can write the Lagrangian for a *harmonic crystal* as:

$$L = \sum_n \left[\frac{1}{2} \dot{\eta}_n^2 - \frac{\lambda}{2} (\eta_n - \eta_{n-1})^2 \right] \quad (1.1)$$

where λ is the spring constant. From the Lagrange equations, the classical equations of motions are:

$$\ddot{\eta}_n = \lambda (\eta_{n+1} - 2\eta_n + \eta_{n-1}) \quad (1.2)$$

The solutions can be written as complex travelling waves:

$$\eta_n(t) = e^{i(kn - \omega t)} \quad (1.3)$$

where the dispersion relation is:

$$\omega^2 = 2\lambda (1 - \cos k) \approx \lambda k^2 \quad (1.4)$$

Therefore, in the long-wavelength limit $k \ll 1$ waves propagate with velocity $c = \sqrt{\lambda}$. To determine the normal modes, there need to be boundary conditions: imposing boundary conditions:

$$\eta_{n+N} = \eta_n \quad \Rightarrow \quad k_m = \frac{2\pi m}{N}, \quad m = 0, 1, \dots, N-1 \quad (1.5)$$

The normal-mode expansion can then be written as:

$$\eta(t) = \sum_{m=0}^{N-1} [\mathcal{A}_m e^{i(k_m n - \omega_m t)} + \mathcal{A}_m^* e^{-i(k_m n - \omega_m t)}] \quad (1.6)$$

where the complex conjugate is added to ensure that the total displacement is real. The momentum canonically-conjugated to the displacement is defined as:

$$\pi_n := \frac{\partial L}{\partial \dot{\eta}_n} = \dot{\eta}_n \quad (1.7)$$

In quantum mechanics, η_n and Π_n become operators with canonical commutator $[\hat{\eta}_j, \hat{\pi}_k] = i\hbar\delta_{jk}$. Implementing time evolution with the *Heisenberg picture*¹:

$$[\hat{\eta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \quad (1.8)$$

The commutator of operators evaluated at different times requires solving the dynamics of the system. It is useful to introduce *annihilation* and *creation operators* $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, so that Eq. 1.6 becomes:

$$\hat{\eta}_n(t) = \sum_{m=0}^{N-1} \sqrt{\frac{\hbar}{2\omega_m}} \frac{1}{\sqrt{N}} [e^{i(k_m n - \omega_m t)} \hat{a}_m + e^{-i(k_m n - \omega_m t)} \hat{a}_m^\dagger] \quad (1.9)$$

where $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and the $N^{-1/2}$ ensures the normalization of normal modes. The proof of Eq. 1.8 follows from the finite Fourier series identity (sum of a geometric progression):

$$\sum_{m=0}^{N-1} e^{ik_m(n-n')} = N\delta_{nn'} \quad (1.10)$$

The Hamiltonian of the system can be written as:

$$\hat{\mathcal{H}} = \sum_n \left[\frac{1}{2} \hat{\pi}_n^2 + \frac{\lambda}{2} (\hat{\eta}_n - \hat{\eta}_{n-1})^2 \right] = \sum_{m=0}^{N-1} \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) \quad (1.11)$$

Generalizing the harmonic oscillator operator algebra (proven unique by Von Neumann), one can construct the Hilbert space for the harmonic crystal as:

$$\hat{a}_m |0\rangle \quad \forall m = 0, 1, \dots, N-1 \quad (1.12)$$

$$|n_0, n_1, \dots, n_{N-1}\rangle = \prod_{m=0}^{N-1} \frac{(\hat{a}_m^\dagger)^{n_m}}{\sqrt{n_m!}} |0\rangle \quad (1.13)$$

These are normalized eigenstates of Eq. 1.1 with energy eigenvalues:

$$E_0 = \frac{1}{2} \sum_{m=0}^{N-1} \hbar\omega_m \quad (1.14)$$

$$E_{n_0, n_1, \dots, n_{N-1}} = E_0 + \sum_{m=0}^{N-1} n_m \hbar\omega_m \quad (1.15)$$

This Hilbert space is called *Fock space* and the excited states *phonons*: these can be thought as “particles” and n_m is the number of phonons in the m^{th} normal mode.

¹Recall that $\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}t} \hat{\mathcal{O}}(0) e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}$ and $\frac{d\hat{\mathcal{O}}}{dt} = \frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\mathcal{O}}]$.

²For a harmonic oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$, so $\frac{d\hat{x}}{dt} = \hat{p}(t)$ and $\frac{d\hat{p}}{dt} = -\omega^2\hat{x}(t)$ and the solution can be written as:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) + \hat{a}^\dagger(t)] \quad \hat{p}(t) = -i\omega \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t) - \hat{a}^\dagger(t)]$$

Inverting these expressions one finds $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $\hat{\mathcal{H}} = \hbar\omega (\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2})$. The time evolution $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$ ensures that $\hat{\mathcal{H}}$ is times-independent.