BLEICHENBACHER'S ORACLE

LEONARDO TAMIANO

TABLE OF CONTENTS

- Overview
- TLS Handshake
- Textbook RSA
  - Malleability
  - Why textbooks better remain on the shelves
- PKCS #1 v1.5
- Oracles on TLS (RSA + PKCS #1 v1.5)
- Bleichenbacher's Attack
  - Consequences of PKCS #1 v1.5
  - Decryption Algorithm In ϵ Minutes
- References

OVERVIEW

I'm **Leonardo Tamiano**, and currently I'm studying a bunch of **cryptographic attacks** for my master's thesis.

In this video I will try to explain a famous cryptographic attack which can be done to old vulnerable **SSL** and **TLS** servers that still support **RSA** with padding scheme **PKCS #1 v1.5** as the **key exchange method**.

The attack is called **Bleichenbacher's Oracle**, and it was discovered by **Daniel Bleichenbacher** in '98.

Chosen Ciphertext Attacks Against Protocols Based on the RSA Encryption Standard PKCS #1

Daniel Bleichenbacher

Bell Laboratories
700 Mountain Ave., Murray Hill, NJ 07974
bleichen@research.bell-labs.com

The mathematics of the paper can be a bit hard to read, especially for those who struggle with **mathematical formalism**.

Step 3: Narrowing the set of solutions. After s_i has been found, the set M_i is computed as

$$M_i \leftarrow \bigcup_{(a,b,r)} \left\{ \left[\max\left(a, \left\lceil \frac{2B + rn}{s_i} \right\rceil\right), \min\left(b, \left\lfloor \frac{3B - 1 + rn}{s_i} \right\rfloor\right) \right] \right\} \tag{3}$$

$$\text{for all } [a, b] \in M_{i-1} \text{ and } \frac{as_i - 3B + 1}{n} \leq r \leq \frac{bs_i - 2B}{n}.$$

Anyhow, the **computational idea** behind the attack is simple and beautiful, and the attack itself can be implemented in a few lines of code.

```python
s = ceil(N, B3)
M = set([ (B2, B3 - 1) ])
while True:
    if len(M) > 1 or TOTAL_REQUESTS == 0:
        s = bleichenbacher_step_1(s)
    else:
        interval = M.pop()
        if interval[0] == interval[1]:
            print(f"Found result: {interval[0]}")
            break
        else:
            M.add(interval)
            s = bleichenbacher_opt_1(s, M)
    M = bleichenbacher_step_2(s, M)
```

Let's try to understand something…

TLS HANDSHAKE

The **TLS** protocol suite is used to create a **secure communication channel** on top of a typical **TCP socket** between a client and a server.

The creation of the cryptographic session which takes care of the security is delegated to the

TLS handshake phase

In this phase client and server send each-others messages in order to:

1. Decide on **what kind of crypto** to use.
2. Transfer the session secret (**pre-master-key**).

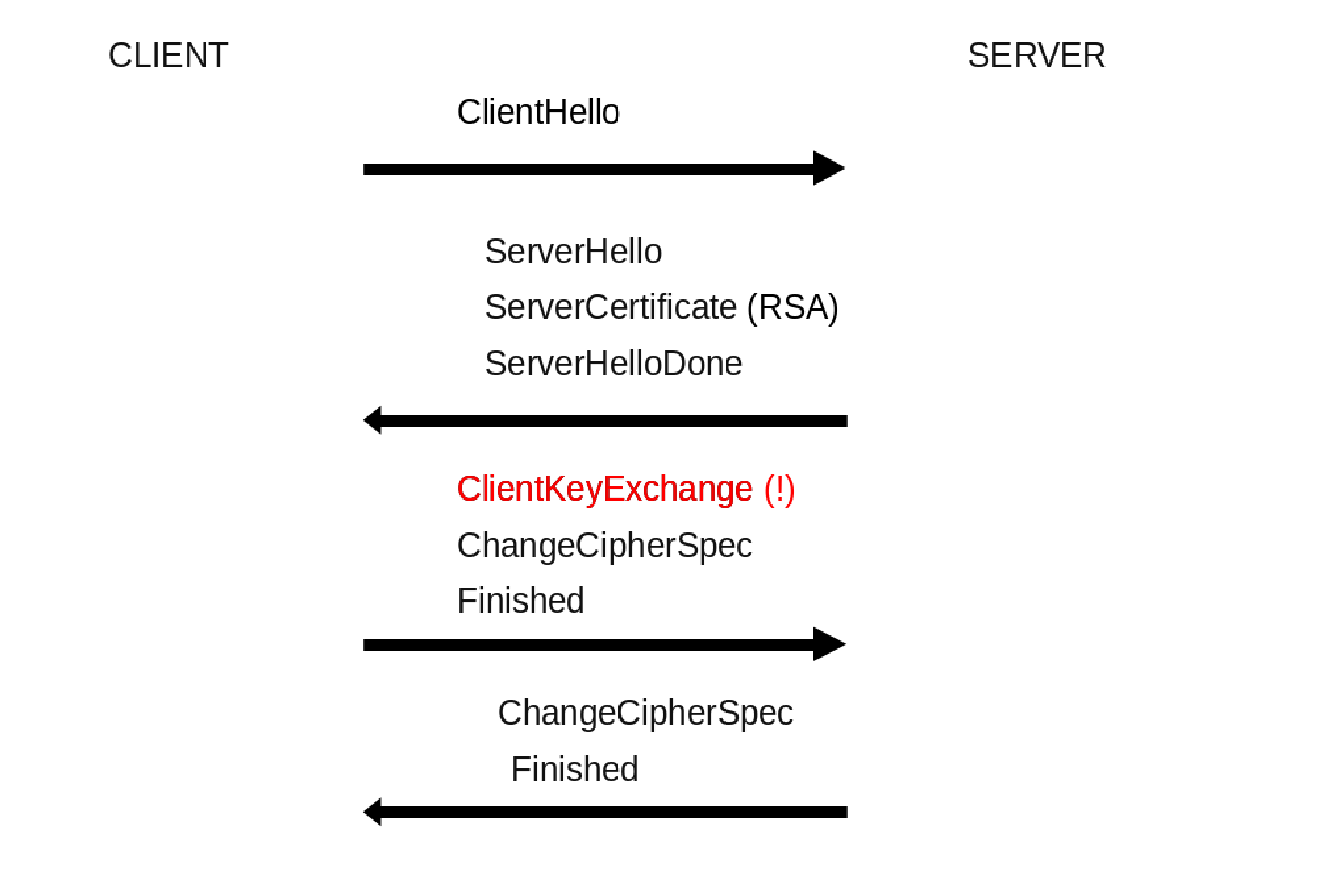
In order to transfer the session secret the protocol relies on **public key cryptography schemes**.

There are different types of public key crypto, such as:

- Diffie-Hellman (on finite groups).
- Diffie-Hellman (on elliptic curves).
- RSA.

The bleichenbacher's attack can be applied only when **RSA** is used along with a specific padding scheme known as **PKCS #1 v1.5**.

TLS < 1.3 Handshake (RSA Transport)



As an attacker, the message we're interested in is the **ClientKeyExchange**.

As an attacker, the message we're interested in is the **ClientKeyExchange**.

This message contains the **pre-master-key** which is encrypted with the **public RSA key** of the server the client is connecting to.

By decrypting this message we get access to the entire session.

TEXTBOOK RSA

RSA is a public cryptography system which can be used, among other things, for:

1. Encryption of messages, granting **confidentiality**.
2. Signing of messages, granting **authenticity**.

To give these properties **RSA** makes use of certain results taken from **classical number theory**.

When working with RSA, the following holds:

When working with RSA, the following holds:

- Messages are seen as simple **numbers**.

When working with RSA, the following holds:

- Messages are seen as simple **numbers**.
- All work is done in a **modular arithmetic**.

When working with RSA, the following holds:

- Messages are seen as simple **numbers**.
- All work is done in a **modular arithmetic**.
- Encryption and decryption are implemented through **modular exponentiation**.

Process of generating a public/private key in RSA

Process of generating a public/private key in RSA

1. We **choose** p and q, two **big primes** distant from eachothers.

Process of generating a public/private key in RSA

1. We **choose** p and q, two **big primes** distant from eachothers.
2. We **compute** N and $\Phi(N)$ as

$$N = p \cdot q$$

$$\Phi(N) = (p - 1) \cdot (q - 1)$$

Process of generating a public/private key in RSA

1. We **choose** p and q, two **big primes** distant from eachothers.

2. We **compute** N and $\Phi(N)$ as

$$N = p \cdot q$$

$$\Phi(N) = (p - 1) \cdot (q - 1)$$

3. We **choose** $e < \Phi(N)$ **coprime** with $\Phi(N)$.

Process of generating a public/private key in RSA

1. We **choose** p and q, two **big primes** distant from eachothers.

2. We **compute** N and $\Phi(N)$ as

$$N = p \cdot q$$

$$\Phi(N) = (p - 1) \cdot (q - 1)$$

3. We **choose** $e < \Phi(N)$ **coprime** with $\Phi(N)$.

4. We **compute** d by **solving**

$$d \equiv e^{-1} \mod \Phi(N)$$

To encrypt a message $m \in [0, N)$ we use **modular exponentiation**

$$c = m^e \mod N$$

To encrypt a message $m \in [0, N)$ we use **modular exponentiation**

$$c = m^e \mod N$$

NOTE: Everyone can encrypt messages, as (e, N) is the public key.

To decrypt an encrypted message $c \in [0, N)$ we proceed once again with **modular exponentiation**

$$m = c^d \mod N$$

To decrypt an encrypted message $c \in [0, N)$ we proceed once again with **modular exponentiation**

$$m = c^d \mod N$$

NOTE: Only the owner of the private key d can decrypt messages.

The **correctness** of RSA relies on the famous **Euler's Theorem**, which states that

$$a \equiv 1 \mod \Phi(N) \implies m^a \equiv m \mod N$$

In particular, given that

In particular, given that

- $e \equiv d^{-1} \mod \Phi(N) \implies e \cdot d \equiv 1 \mod \Phi(N)$

In particular, given that

- $e \equiv d^{-1} \mod \Phi(N) \implies e \cdot d \equiv 1 \mod \Phi(N)$
- $c^d \mod N \equiv (m^e)^d \mod N \equiv m^{e \cdot d} \mod N$

In particular, given that

- $e \equiv d^{-1} \mod \Phi(N) \implies e \cdot d \equiv 1 \mod \Phi(N)$
- $c^d \mod N \equiv (m^e)^d \mod N \equiv m^{e \cdot d} \mod N$

we have

In particular, given that

- $e \equiv d^{-1} \mod \Phi(N) \implies e \cdot d \equiv 1 \mod \Phi(N)$
- $c^d \mod N \equiv (m^e)^d \mod N \equiv m^{e \cdot d} \mod N$

we have

$$c^d \mod N \equiv m^{e \cdot d} \mod N \equiv m \mod N$$

The **security** of RSA on the other hand is based on the **computational intractability of the factorization problem**.

By knowing only (N, e), we're not able to compute d, because for computing d we have to solve the congruence

$$d \equiv e^{-1} \mod \Phi(N)$$

that is, we have to compute the **inverse** of e in $\mathbb{Z}_{\Phi(N)}$.

This can be done in a fast way only by knowing

$$\Phi(N) = (P-1) \cdot (Q-1)$$

which requires being able to **factorize** N into its **prime factors**.

MALLEABILITY

The RSA cryptosystem is said to be **malleable**.

Given an encrypted text and a public key

$$c = m^e \mod N, \qquad (e, N)$$

we can compute a new value

$$s^e \cdot c \mod N$$

and this new value can be seen as a **new ciphertext**

$$\begin{aligned} s^e \cdot c \mod N &= s^e \cdot m^e \mod N \\ &= (s \cdot m)^e \mod N \end{aligned}$$

in particular we can know the **exact relationship** between the plaintext of our new crafted ciphertext and the plaintext of the original ciphertext.

Plaintext	Ciphertext
m	$c = m^e \mod N$
$s \cdot m \mod N$	$s^e \cdot c \mod N$

WHY TEXTBOOKS BETTER REMAIN ON THE SHELVES

The crypto system we just described is known as **textbook RSA**.

This name comes from the fact that this system is only secure in the pages of a book.

In the real and scary world a system like this presents various problems…

For example, given that the system is **completely deterministic**, if we send two times the same message m, the resulting ciphertext will be the same as well.

For example, given that the system is **completely deterministic**, if we send two times the same message m, the resulting ciphertext will be the same as well.

- $m \longrightarrow c = m^e \mod N \quad , \quad \text{time } t_1$

For example, given that the system is **completely deterministic**, if we send two times the same message m, the resulting ciphertext will be the same as well.

- $m \longrightarrow c = m^e \mod N \quad , \quad \text{time } t_1$
- $m \longrightarrow c = m^e \mod N \quad , \quad \text{time } t_2 > t_1$

For example, given that the system is **completely deterministic**, if we send two times the same message m, the resulting ciphertext will be the same as well.

- $m \longrightarrow c = m^e \mod N \quad , \quad \text{time } t_1$
- $m \longrightarrow c = m^e \mod N \quad , \quad \text{time } t_2 > t_1$

This means we lose **semantic security**.

To solve this problem, in 1993 a padding scheme known as **PKCS #1 v1.5** was standardized.

It's possible to generate RSA keys with the `ssh-keygen` command

```
ssh-keygen -t rsa -b 1024 -N "" -f rsa_key
```

PKCS #1 V1.5

Consider we want to transmit a message.

The idea is to start from the message bytes and construct a particular number

$$m \in [0, N)$$

The number will be generated using randomness.

To this end let's assume that the **byte-length** of N is k.

$$\begin{aligned} \text{Message bytes} &\longrightarrow \text{Padding bytes} + \text{Message by} \\ &\longrightarrow m \in [0, N) \end{aligned}$$

Padding scheme **PKCS #1 v1.5** (1/5)

$$0\text{x } 00\ 02 \ \mid \ RB_1 \ \ldots \ RB_i \ \mid \ 00 \ \mid \ MB_1 \ \ldots \ MB_j$$

Padding scheme **PKCS #1 v1.5** (2/5)

1. First two bytes are set to `0x00` and `0x02`.
2. At least `8` random bytes different from `0x00`.
3. A single null byte `0x00`.
4. The remaining bytes are the message's byte.

Padding scheme **PKCS #1 v1.5** (3/5)

$$0\mathrm{x}\ 00\ 02 \ \mid\ \underbrace{RB_1\ \ldots\ RB_i}_{\geq 8}\ \mid\ 00\ \mid\ MB_1\ \ldots\ MB_j$$

Padding scheme **PKCS #1 v1.5** (4/5)

From the second constraint (2) it follows that

we can have at most $k - 11$ application data bytes per packet

In case our message is longer than that, the idea is to split the message in multiple packets.

Padding scheme **PKCS #1 v1.5** (5/5)

$$\overbrace{0\text{x } 00\ 02 \;|\; \underbrace{RB_1 \;\ldots\; RB_i}_{\geq 8} \;|\; 00 \;|\; \underbrace{MB_1 \;\ldots\; MB_j}_{\leq k-11}}^{k}$$

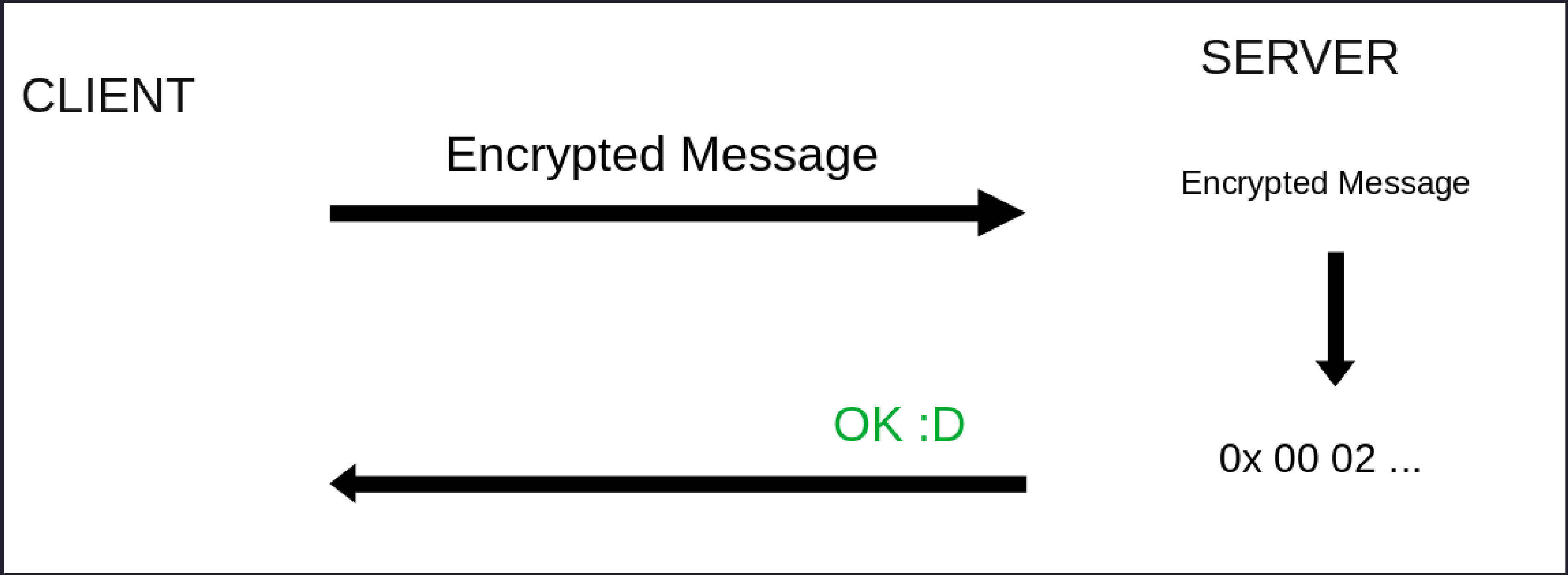
```
if __name__ == "__main__":
    if len(sys.argv) != 3:
        print(f"Usage: {sys.argv[0]} <msg> <key_size (in bits)>")
        exit()
    else:
        msg = sys.argv[1]
        msg_length = len(msg)
        key_size = int(int(sys.argv[2]) / 8) # transform size in

        if msg_length > key_size - 11:
            print(f"[PKCS#1 v1.5 error] message length ({msg_leng
            exit()

        padding_length = key_size - len(msg) - 3
        padding = "\x01" * padding_length # NOTE: this should be
```

```
00000000  00 02 01 01 01 01 01 01  01 01 01 01 01 01 01 01  |.........
00000010  01 01 01 01 01 01 01 01  01 01 01 01 01 01 01 01  |.........
...
00000070  01 01 01 01 00 68 65 6c  6c 6f 20 77 6f 72 6c 64  |.....hell
00000080
```

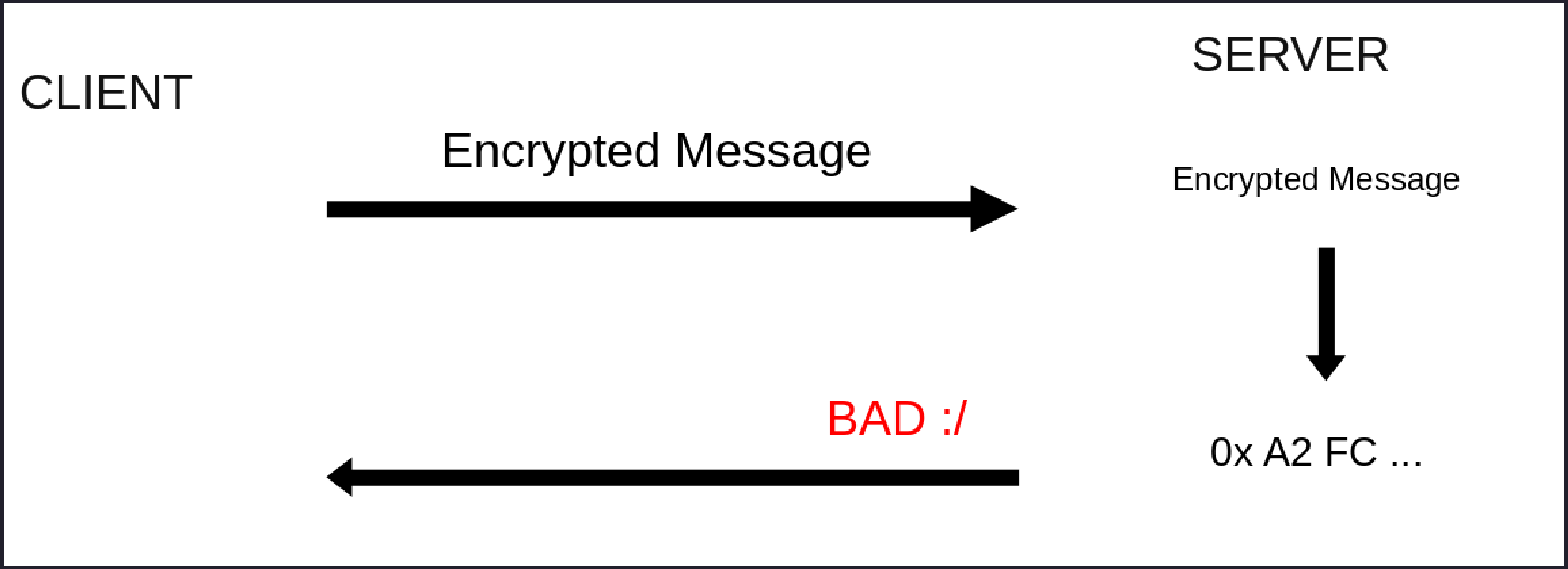
ORACLES ON TLS (RSA + PKCS #1 V1.5)

Let's assume now that we can interact with a `TLS` server in a way that when we send the message padded with `PKCS #1 v1.5` and encrypted with `RSA` the server let us distinguish the following two situations:

The encrypted message, once decrypted, **is correctly padded** according to `PKCS #1 v1.5`.



The encrypted message, once decrypted, **is NOT correctly padded** according to `PKCS #1 v1.5`.



In these cases it is said that the server offers a **cryptographic oracle**.

We can have various oracles, depending on what kind of things are checked with respect to the plaintext message's bytes.

The more checks are made, and the harder it is to produce messages that satisfy all constraints.

In this presentation we assume to have a single oracle that only checks the first two bytes of the message

```python
def oracle(msg_hex):
    global D, N

    # transform hex into number
    encrypted_msg = int("0x" + msg_hex, 0)

    # raw decrypt using RSA
    decrypted_msg = pow(encrypted_msg, D, N)
    decrypted_hex = f"%0{PADDING_VALUE}x" % decrypted_msg

    # check for padding
    if decrypted_hex[0:4] != "0002":
        return False
    else:
        return True
```

BLEICHENBACHER'S ATTACK

The **bleichenbacher** attack is an

adaptive choosen ciphertext attack

which uses a cryptographic oracle based on `RSA` and `PKCS #1 v1.5` to decrypt any message c which was encrypted using the public key of the TLS server we're trying to attack.

The attack is based on the following things:

The attack is based on the following things:

1. In RSA both plaintext and ciphertext are **numbers**.

The attack is based on the following things:

1. In RSA both plaintext and ciphertext are **numbers**.
2. Satisfying all padding rules of `PKCS #1 v1.5` puts heavy contraints on the **numerical range** the plaintext can belong to.

The attack is based on the following things:

1. In RSA both plaintext and ciphertext are **numbers**.
2. Satisfying all padding rules of `PKCS #1 v1.5` puts heavy contraints on the **numerical range** the plaintext can belong to.
3. The **malleability** of RSA allows an attacker to **create new ciphertexts** whos plaintexts can be related to the original plaintext.

Let m be a message padded with **PKCS #1 v1.5**, and let c be its encrypted form.

By using c and the crypto oracle our objective is to find the original m.

NOTE: The attack can still be done if m is any message (even not correctly padded), but it requires an additional starting phase which we'll briefly cover at the end.

CONSEQUENCES OF PKCS #1 V1.5

If we remember how the padding standard **PKCS #1 v1.5** was defined, we'll recall that all messages that satisfy this padding scheme start with the byte sequence `0x 00 02`.

Let k be the size in byte of the modulus N, where N is part of the public key of the server.

By defining

$$B = 2^{8 \cdot (k-2)}$$

we have that

$$\begin{aligned} B &\longrightarrow \text{0x } 00\ 01\ \overbrace{00\ 00 \ldots 00}^{k-2 \text{ bytes}} \\ 2B &\longrightarrow \text{0x } 00\ 02\ \ 00\ 00 \ldots 00 \\ 3B &\longrightarrow \text{0x } 00\ 03\ \ 00\ 00 \ldots 00 \end{aligned}$$

This means that,

$$2B \leq m \leq 3B - 1$$

DECRYPTION ALGORITHM IN ϵ MINUTES

The decryption algorithm works in various phases.

Each phase is indexed by a natural number $i \in \mathbb{N}$ and contains two different steps.

The step for each phase are described as follows:

The step for each phase are described as follows:

1. The first step tries to find a **value** s_i such that

$$s_i \cdot m \mod N \quad \text{is PKCS \#1 v1.5 compliant}$$

The step for each phase are described as follows:

1. The first step tries to find a **value** s_i such that

$$s_i \cdot m \mod N \quad \text{is PKCS \#1 v1.5 compliant}$$

2. The second step constructs, starting from the s_i discovered previously, a **set of intervals** M_i such that

$$\exists [a, b] \in M_i : \quad m \in [a, b]$$

INITIALIZATION PHASE: $i = 0$

At the start we **initialize** the following values

$$s_0 = 2$$

$$M_0 = \{\ [2B, 3B-1]\ \}$$

GENERIC PHASE: $i \in \mathbb{N}^+$

Step 1: Search for next s_i

Step 1: Search for next s_i

We start from $s_i = s_{i-1} + 1$ and find the next value such that

$$s_i \cdot m \mod N \quad \text{is PKCS \#1 v1.5 compliant}$$

if a given s_i does not work, we try the next $s_i = s_i + 1$

Step 1: Search for next s_i

To test a given s_i we send to the oracle the following value

$$s_i^e \cdot c \mod N$$

If the oracle replies with "YES", then we stop and go to the next step, otherwise we keep going and update s_i.

Step 1: Search for next s_i

The code for this phase is

```python
def bleichenbacher_step_1(s):
    global E, N, TOTAL_REQUESTS

    s = s + 1
    while True:
        new_ciphertext = (pow(s, E, N) * ENCRYPTED_FLAG) % N
        encrypted_hex = f"%0{PADDING_VALUE}x" % new_ciphertext
        if oracle(encrypted_hex) == True:
            return s
        s = s + 1
        TOTAL_REQUESTS += 1
```

Step 2: Construction of M_i

Step 2: Construction of M_i

Suppose we found a value s_i such that

$$s_i \cdot m \mod N \quad \text{è PKCS \#1 v1.5 compliant}$$

The idea now is to use the value s_i to **propagate the knowledge of M_{i-1} in the new set M_i**, and, while doing so, restrict the size of the new intervals so that we can understand better where m is located.

Step 2: Construction of M_i

We already saw that

$$s_i \cdot m \mod N \quad \text{is PKCS \#1 v1.5 compliant}$$

implies

$$2B \leq s_i \cdot m \mod N \leq 3B - 1$$

Step 2: Construction of M_i

For how we define the modulus, we have that

$$2B \leq s_i \cdot m \mod N \leq 3B - 1$$

implies that there exists a $k \in \mathbb{Z}$ such that

$$2B \leq s_i \cdot m - k \cdot N \leq 3B - 1$$

Step 2: Construction of M_i

Summarizing,

Step 2: Construction of M_i

Summarizing,

- $s^e \cdot c \mod N$ is accepted by the oracle.

Step 2: Construction of M_i

Summarizing,

- $s^e \cdot c \mod N$ is accepted by the oracle.
- $s \cdot m \mod N$ is PKCS #1 v1.5 compliant.

Step 2: Construction of M_i

Summarizing,

- $s^e \cdot c \mod N$ is accepted by the oracle.
- $s \cdot m \mod N$ is PKCS #1 v1.5 compliant.
- $2B \leq s \cdot m \mod N \leq 3B - 1$

Step 2: Construction of M_i

Summarizing,

- $s^e \cdot c \mod N$ is accepted by the oracle.
- $s \cdot m \mod N$ is PKCS #1 v1.5 compliant.
- $2B \leq s \cdot m \mod N \leq 3B - 1$
- $\exists k \in \mathbb{Z}: \; 2B \leq s \cdot m - k \cdot N \leq 3B - 1$

Step 2: Construction of M_i

From

$$2B \leq s_i \cdot m - k \cdot N \leq 3B - 1$$

we get

$$\frac{2B + k \cdot N}{s_i} \leq m \leq \frac{3B - 1 + k \cdot N}{s_i}$$

Step 2: Construction of M_i

PROBLEM: we do not know k

$$\frac{2B + k \cdot N}{s_i} \leq m \leq \frac{3B - 1 + k \cdot N}{s_i}$$

Step 2: Construction of M_i

SOLUTION: having fixed both s and m, we can **enumerate** all possible values of k

$$2B \leq s_i \cdot m - k \cdot N \leq 3B - 1$$

$$\Longrightarrow$$

$$\frac{-3B + 1 + s_i \cdot m}{N} \leq k \leq \frac{-2B + s_i \cdot m}{N}$$

Step 2: Construction of M_i

For example, from the bound $m \in [2B, 3B-1]$ we have that

Step 2: Construction of M_i

For example, from the bound $m \in [2B, 3B - 1]$ we have that

$$\frac{-3B + 1 + s_i \cdot 2B}{N} \leq \frac{-3B + 1 + s_i \cdot m}{N} \leq k$$

Step 2: Construction of M_i

For example, from the bound $m \in [2B, 3B-1]$ we have that

$$\frac{-3B+1+s_i \cdot 2B}{N} \leq \frac{-3B+1+s_i \cdot m}{N} \leq k$$

$$k \leq \frac{-2B+s_i \cdot m}{N} \leq \frac{-2B+s_i \cdot (3B-1)}{N}$$

Step 2: Construction of M_i

Thus, for each value of k taken from the interval

$$\frac{-3B + 1 + s_i \cdot 2B}{N} \leq k \leq \frac{-2B + s_i \cdot (3B - 1)}{N}$$

we have a new possible interval for m

$$\frac{2B + k \cdot N}{s_i} \leq m \leq \frac{3B - 1 + k \cdot N}{s_i}$$

Step 2: Construction of M_i

Out of all this intervals, m belongs to only **one of them**, associated to a specific value of k.

Given however that we don't know the particular k, we proceed by **saving all of valid the intervals in the new set M_i**.

Step 2: Construction of M_i

More specifically, given $[a, b] \in M_{i-1}$, we obtain the following range for k

$$\frac{-3B + 1 + s_i \cdot a}{N} \leq k \leq \frac{-2B + s_i \cdot b}{N}$$

and for every value in this range we get a new possible interval for m

$$\frac{2B + k \cdot N}{s_i} \leq m \leq \frac{3B - 1 + k \cdot N}{s_i}$$

Step 2: Construction of M_i

A particular interval is added to M_i only if **it has a non-empty intersection with** $[a, b] \in M_{i-1}$.

To compute these intersection the **max** and **min** functions are used.

Step 2: Construction of M_i

Graphically, for every $[a, b] \in M_{i-1}$ and for every k in the relative range we can have four possible valid cases with respect to the intersection of the new interval

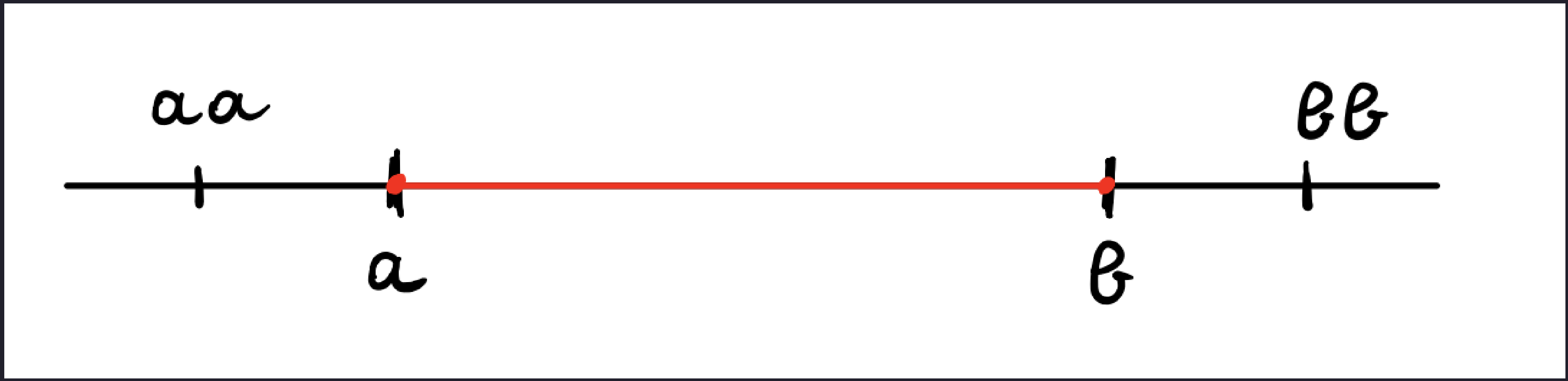
$$[aa, bb] = \left[\frac{2B + k \cdot N}{s_i}, \ \frac{3B - 1 + k \cdot N}{s_i}\right]$$

to the old one $[a, b]$.

Step 2: Construction of M_i

Case 1/4

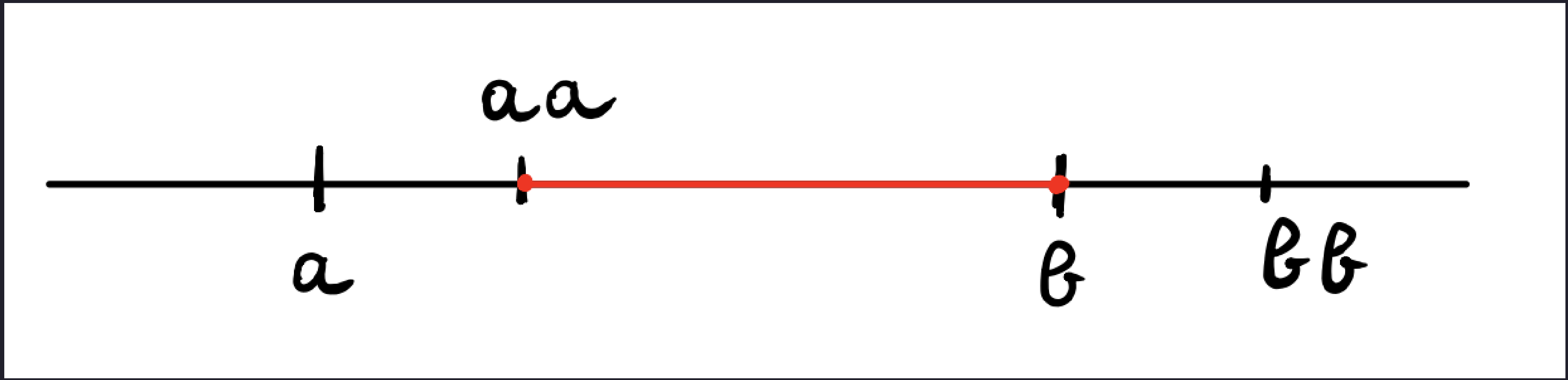
$$\begin{cases} \max(a, aa) & = a \\ \min(b, bb) & = b \end{cases}$$



Step 2: Construction of M_i

Case 2/4

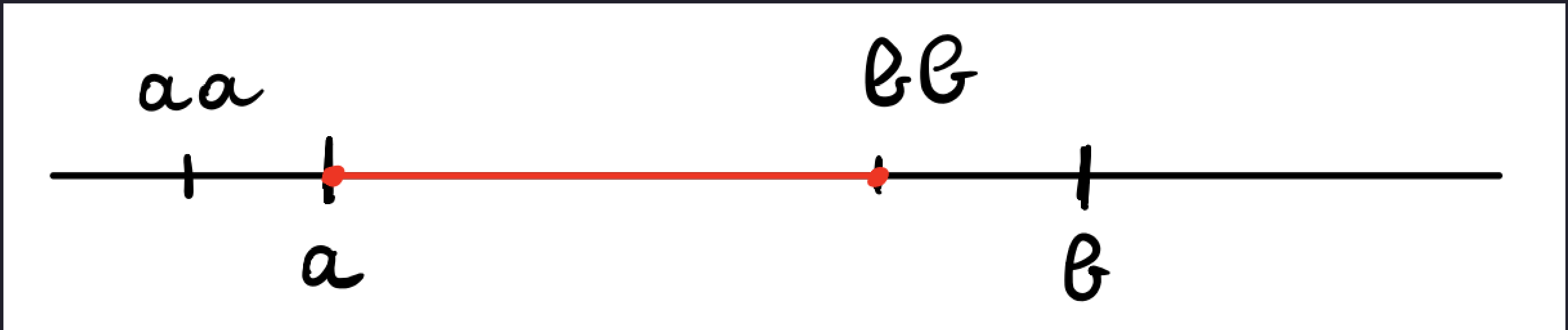
$$\begin{cases} \max(a, aa) & = aa \\ \min(b, bb) & = b \end{cases}$$



Step 2: Construction of M_i

Case 3/4

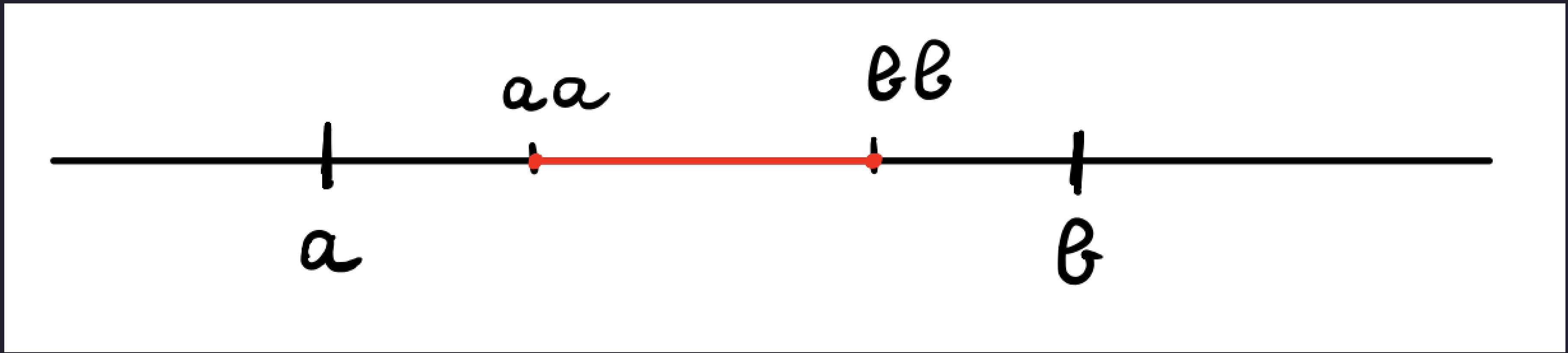
$$\begin{cases} \max(a, aa) & = a \\ \min(b, bb) & = bb \end{cases}$$



Step 2: Construction of M_i

Case 4/4

$$\begin{cases} \max(a, aa) & = aa \\ \min(b, bb) & = bb \end{cases}$$



Step 2: Construction of M_i

This is the idea behind the following formula

Step 3: Narrowing the set of solutions. After s_i has been found, the set M_i is computed as

$$M_i \leftarrow \bigcup_{(a,b,r)} \left\{ \left[\max\left(a, \left\lceil \frac{2B+rn}{s_i} \right\rceil\right), \min\left(b, \left\lfloor \frac{3B-1+rn}{s_i} \right\rfloor\right) \right] \right\} \qquad (3)$$

$$\text{for all } [a,b] \in M_{i-1} \text{ and } \frac{as_i - 3B + 1}{n} \le r \le \frac{bs_i - 2B}{n}.$$

Step 2: Construction of M_i

Note that at every iteration, the only thing we know for certain about the new set of intervals M_i is that **m belongs to a single interval of M_i**, even though we cannot tell which.

In formula,

$$\exists [a, b] \in M_i : \quad m \in [a, b]$$

Step 2: Construction of M_i

The code for this step is

```
def bleichenbacher_step_2(s, old_M):
    new_M = set([])
    for (a, b) in old_M:
        r1 = ceil((a * s - B3 + 1), N)
        r2 = floor((b * s - B2), N) + 1
        for r in range(r1, r2):
            aa = ceil(B2 + r*N, s)
            bb = floor(B3 - 1 + r*N, s)
            newa = max(a, aa)
            newb = min(b, bb)
            if newa <= newb:
                new_M |= set([ (newa, newb) ])
    return new_M
```

At the end of this second step for the phase i, if the set M_i contains a single interval, and if this interval is of the form $[a, a]$, then we can stop, since

$$m \in [a, a] \implies a \leq m \leq a \implies m = a$$

If instead M_i contains one or more intervals, or if it contains a single interval of the form $[a, b]$, with $a \neq b$, then the algorithm proceeds towards the next phase $i + 1$ as described previously.

ORIGINAL OPTIMIZATION ('98)

The algorithm just described in theory already works.

In practice however **it is just too slow**.

The original ('98) paper thus introduces the following two optimizations:

The original ('98) paper thus introduces the following two optimizations:

1. At the start, instead of starting from $s_0 = 1$, we start with

$$s_0 = \left\lceil \frac{N}{3B} \right\rceil$$

The original ('98) paper thus introduces the following two optimizations:

1. At the start, instead of starting from $s_0 = 1$, we start with

$$s_0 = \left\lceil \frac{N}{3B} \right\rceil$$

2. If at the end of phase i, the set M_i contains a single interval $[a, b]$, then we optimize the search for the next s_{i+1} by using a **heuristic**.

The first optimization is a direct consequence of the fact that for value of $s_0 \leq \lceil \frac{N}{3B} \rceil$ it is simply not possible to have a **PKCS compliant plaintext**.

Indeed,

$$k \leq \frac{m_0 \cdot s_1 - 2B}{N} \leq \frac{(3B-1) \cdot s_1 - 2B}{N} < \frac{3Bs_1}{N}$$

Such value is 1 when $s_1 = \frac{N}{3B}$.

If $s_1 < \frac{N}{3B}$, then $k < 1 \implies k = 0$.

In these cases

$$m_1 = m_0 \cdot s_1 - k \cdot N = m_0 \cdot s_1 < N$$

and so we can't respect PKCS padding, since the value will never start with `00 02`.

The second optimization instead is based on a **heuristic**.

The general idea is to have that the next value of s_{i+1} is, approximately, double the previous value.

In formula,

$$s_{i+1} \approx 2 \cdot s_i$$

In detail, if $M_i = \{ \ [a, b] \ \}$, then we let

$$r_i \geq 2 \cdot \frac{b \cdot s_i - 2B}{n}$$

and we start to try all s_{i+1} in the following interval

$$\frac{2B + r_i \cdot n}{b} \leq s_{i+1} \leq \frac{3B + r_i n}{a}$$

until $s_{i+1}^e \cdot c \mod N$ is accepted by the oracle.

If the interval

$$\frac{2B + r_i \cdot n}{b} \leq s_{i+1} \leq \frac{3B + r_i n}{a}$$

finishes without finding any valid s_{i+1}, then we increment $r_i = r_i + 1$ and we start the search for the next s_{i+1} in the new interval.

Notice that if the value s_{i+1} works, then we have

$$\begin{aligned} s_{i+1} &\geq \frac{2B + r_i N}{b} \\ &= \frac{2B + 2bs_i - 4B}{b} \\ &= 2s_i - \frac{2B}{b} \end{aligned}$$

So we get,

$$s_{i+1} \approx 2s_i - \frac{2B}{b}$$

and since $b \geq 2B$, we have that $2B/b < 1$.

Thus in general $-2B/b$ **is not influent** in the final value and we can roughly say that

$$s_{i+1} \approx 2s_{i-1}$$

The code for this optimization is

```
def bleichenbacher_opt_1(s, old_M):
    global TOTAL_REQUESTS
    fst = old_M.pop()
    old_M.add(fst)
    a = fst[0]
    b = fst[1]
    r = ceil((b * s - B2) * 2, N)
    while True:
        low_bound = ceil((B2 + r * N), b)
        high_bound = ceil((B3-1 + r * N), a) + 1
        for s in range(low_bound, high_bound):
            new_ciphertext = (pow(s, E, N) * ENCRYPTED_FLAG) % N
            str_hex = f"%0{PADDING_VALUE}x" % new_ciphertext
            if oracle(str_hex):
                return s
            TOTAL_REQUESTS += 1
```

FINAL CODE

By putting all the pieces of the puzzle together, we get

```
s = ceil(N, B3)
M = set([ (B2, B3 - 1) ])
while True:
    if len(M) > 1 or TOTAL_REQUESTS == 0:
        s = bleichenbacher_step_1(s)
    else:
        interval = M.pop()
        if interval[0] == interval[1]:
            print(f"Found result: {interval[0]}")
            break
        else:
            M.add(interval)
            s = bleichenbacher_opt_1(s, M)
    M = bleichenbacher_step_2(s, M)
```

FURTHER OPTIMIZATIONS

Throughout the years various other optimizations have been suggested.

All of them try to tackle the following aspects of the algorithm:

1. Shorten the search-time for the next s_i.
2. Shorten the computation to construct the intervals in M_i.

Some names:

- Tigher Bounds.
- Beta Method.
- Parallel Threads Methods.
- Skipping Holes.
- Trimmers.

THE MISSING STEP

At the start we assumed that the original plaintext m was correctly padded according to `PKCS #1 v1.5`.

At the start we assumed that the original plaintext m was correctly padded according to `PKCS #1 v1.5`.

What if that's not the case?

At the start we assumed that the original plaintext m was correctly padded according to `PKCS #1 v1.5`.

What if that's not the case?

What if we catch a general encrypted message c and we want to decrypt it, even though the plaintext m is not PKCS compliant?

The attack can still be done, but before starting we need to find an s_0 value such that

$$c \cdot s_0^e \mod N$$

is accepted by the oracle.

To find that we just generate s_0 randomly and check the oracle.

If its valid, we let $c = c \cdot s_0^e \mod N$ and start to decrypt c as we did before.

If its not valid, we generate another s_0 and try again.

REFERENCES

The following resources were used for the realization of this video.

- Chosen Ciphertext Attacks Against Protocols Based on the RSA Encryption Standard PKCS #1, Daniel Bleichenbacher
- Experimenting with the Bleichenbacher Attack, Livia Capol
- Bleichenbacher Attack on RSA PKCS #1 v1.5 For Encryption, David Wong
- Practical Padding Oracle Attacks on RSA, Riccardo Focardi

I express my gratitude towards each and every author.

Thank you (♥)!