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Applications

Determination of electrical potential distribution in grounding systems

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A rigorous determination of the electric field in the ground of an earthed electrical installation is presented. Results of laboratory tests show that the Poisson equation could be used to represent the divergence of the electric field in a sample of sand from the river Sanaga basin. The results of these tests and of the numerical solution of the Poisson equation and boundary conditions by finite differences are shown to be in good agreement.

Keywords: Poisson equation, Finite differences, Simulation, Grounding systems, Earthed electrical installation, Soil resistivity.

1. Introduction

To be able to optimize conditions for protecting an electrical installation by the search for an optimum dimensioning of an earth network (an earth rod), a good understanding of the behaviour of the resistivity of the soil at the site of the installation is necessary. Nowadays the procedure for doing this, for big installations such as a transformer station, a power station, and industrial plant, etc., relies on the geological examination of the site as the impedance of an earth installation and the electric field in the immediate vicinity depend essentially on the electrical characteristics of the terrain, its resistivity in particular. By this method, the measurements usually take a long period, at least one year, before one is able to determine the range within which one can make a good estimate [1].

The aim of our approach is to develop a reliable method for the determination of the resistivity in a laboratory, which would enable companies anywhere in Cameroon to do away with time-consuming measurements or to do only a few elementary measurements. One of the ways of achieving this aim is to obtain the electric field in a given soil sample by simulation. In this paper we present the state equations, their solution and finally the results and their interpretation.

2. Derivation of the state equations

2.1. Poisson equation

Consider a box of a parallelepiped filled with Sanaga sand with the top side open. Two of the remaining sides, which are opposite, are of sheet metal with one at a potential U and the other at 0 volt (Fig. 1). An element of volume $\Delta v = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ (Fig. 2) in the interior of the box has a suface S. If $\boldsymbol{D}$ and $\boldsymbol{E}$ are respectively the electric induction and electric field caused by the potential U and if $Q = Q_1 + Q_2 + \cdots + Q_i$ is the sum of all charges in the volume, v, then Gauss' theorem gives:

$$Q = \int_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{s}. \tag{1}$$

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The divergence of the field vector A at a point P is given by:

$$\nabla \cdot A = \lim_{\Delta v \to 0} \frac{\int_S A \cdot \mathrm{d}s}{\Delta v}. \tag{2}$$

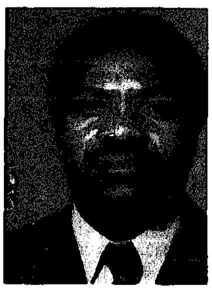
The integration is carried out on the surface of an infinitesimal volume Δv around P that tends to zero.

By analogy with the element Δv (Fig. 2) taken in the box of sand, where ρ is the density of the charge, the field vector D is:

$$\nabla \cdot D = \lim_{\Delta v \to 0} \frac{\int_S D \cdot \mathrm{d}s}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho \tag{3}$$

We thus obtain one of the Maxwell equations:

$$\nabla \cdot D = \rho \quad \text{and} \quad \nabla \cdot E = \rho / \epsilon. \tag{4}$$



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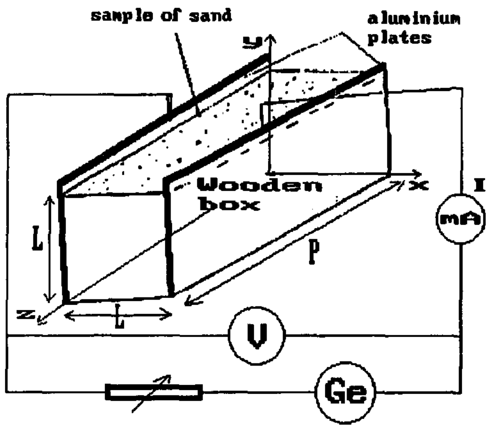

Fig. 1. Dry wooden box of sides 10 mm thick and of the form of a cube, with the upper side open. Two of its faces which are parallel are covered inside with an alluminium sheet of 2 mm thickness to serve as electrodes. The cube is filled with sifted Sanaga sand of similar granules. This constitutes an element for measuring the current and potential difference at the metallic terminals. The source G is that of a variable direct current.

Equation (4) is right only if the dielectric constant ϵ does not vary in the area considered. In our model we suppose ϵ constant.

It emerges from these results that the fields E and D will both have zero divergence in areas devoid of any charge. Based on this result Gauss' theorem [2] can be written in the next form:

$$\int_S E \cdot \mathrm{d}s = Q/\epsilon. \tag{5}$$

If the charge density function were known in the volume being considered, the field could be obtained by integrating ρ throughout this volume [3]; thus:

$$\int_S E \cdot \mathrm{d}s = \frac{1}{\epsilon} \int_v \rho \, \mathrm{d}v. \tag{6}$$

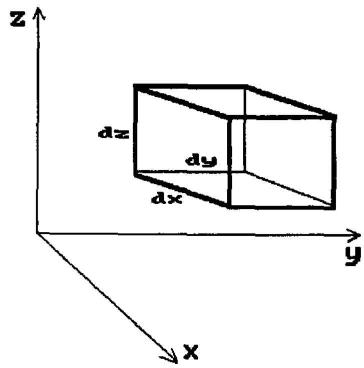

Fig. 2. Isolated physical model of the sample of Fig. 1 used to establish the equations of the sample.

According to the divergence or Ostrogradsky theorem we have:

$$\int_S \boldsymbol{E} \cdot \mathrm{d}s = \int_v \nabla \cdot \boldsymbol{E} \, \mathrm{d}v, \tag{7}$$

where v represents the volume limited by the surface S. However,

$$\boldsymbol{E} = \rho/\epsilon \quad \text{and} \quad \boldsymbol{E} = -\nabla \cdot (\nabla U) = -\Delta U.$$

Therefore

$$\Delta U = -\rho/\epsilon. \tag{8}$$

Equation (8)is known as Poisson's equation.

If the region under study has a known charge distribution $\rho(U)$, Poisson's equation permits us to determine the potential function. Most often the area does not contain any charges (even if its permittivity is uniform); Poisson's equation now becomes:

$$\Delta U = 0. \tag{9}$$

This is also known as the Laplace equation.

2.2. Determination of the expression of the charge distribution

Experimental studies were conducted on samples of soil through which electric current was channeled. The impedances of the samples measured by varying moisture content (Fig. 3) show that:

- sand acts like an insulator with a moisture content below 2%;
- it is quasi-conducting when the moisture content is between 2.5% and 20%;

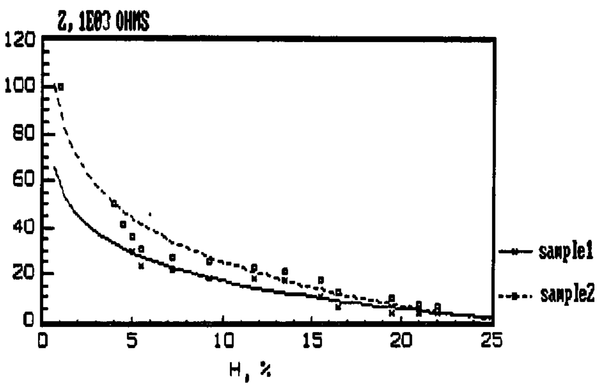

Fig. 3. Variation of the impedance and resistance of the sample as a function of its moisture content. The upper curve represents $R(H)$ and the lower $Z(H)$. In order to obtain $R(H)$, the current supply is from a direct current source, and the curve $Z(H)$ is obtained by supplying alternating current.

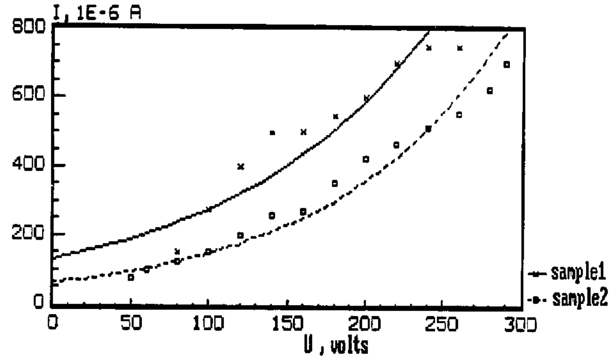

Fig. 4. Current curves as functions of the voltage applied to the terminals of sample. The upper curve is obtained from direct current and the lower curve from alternating current application. This experiment us to model our sample as a semi-conductor.

- it is a perfect conductor for a moisture content which is greater than 20%.

The increase in current flow with increase of the applied voltage (Fig. 4) was also studied. Since the increase of I as function of U parallel, the increase of the charge density in the area, we adopted the following expression for the charge density:

$$\rho = \rho_0 [\exp(U) - 1]. \tag{10}$$

Substituting eqn. (10) onto (8):

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\rho_0}{\epsilon} [\exp(U) - 1]. \tag{11}$$

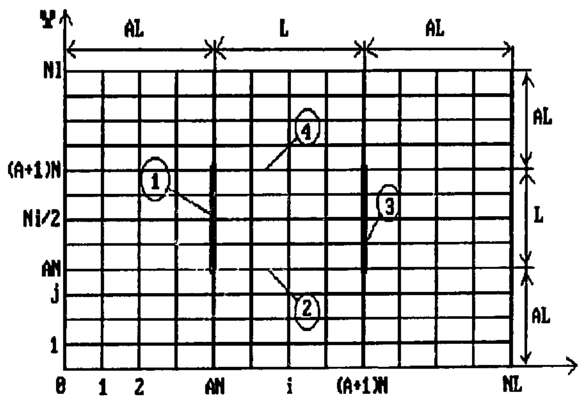

Fig. 5. Illustrating the principle of shaping the sample into a closed mesh. L represents the length of sand, A its length extension, h the discrete step size, $L/h = N$, where N is defined as in paragraph 3.3.1. Each face is increased by AL in order to satisfy the condition $U = 0$ at infinity. Sand, which is the soil sample studied, is bounded by the sides 1–4.

Assuming U constant along z, (Fig. 1), we deduce the following system equations to be solved:

$$\begin{array}{ll} \text{(a)} & \Delta U = -\rho/\epsilon \quad \text{between the plates,} \\ \text{(b)} & \Delta U = 0 \quad \text{in air,} \\ \text{(c)} & U = 0 \quad \text{on side 1,} \\ \text{(d)} & U = U_0 \quad \text{on side 3,} \\ \text{(e)} & \epsilon \left.\frac{\partial U}{\partial Y}\right|_{+} = \epsilon_0 \left.\frac{\partial U}{\partial Y}\right|_{-} \quad \text{on side 2,} \\ \text{(f)} & \epsilon \left.\frac{\partial U}{\partial y}\right|_{-} = \epsilon_0 \left.\frac{\partial U}{\partial y}\right|_{+} \quad \text{on the side 4,} \\ \text{(g)} & U = 0 \quad \text{on the edges of the large plate.} \end{array} \tag{12}$$

The annotation used in eqn. (12) refers to Fig. 5.

3. Solution of the state equation

3.1. Choice of method

The ideal method to use would be analytical. It would give an exact solution that at the same time would be continous. However, this can be applied only to equations that are less complex. An alternative would be to use numerical methods, among which are the finite element and finite differences methods. But the configuration of the lines of force of a field in the soil is often a function of the form of the earthing electrodes (cylindrical, disk, parallelepiped, rectangular, etc.). As a result of these spatial forms we have decided to use the finite difference method because these regular forms are taken into account by the above method [4].

3.2. Theory of the method

Suppose that a given domain G of boundary Γ is limited by $\mathbb{R}^2$ (Fig. 6). We wish to solve the set (12) in this domain. The general form is:

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = f, \\ \text{boundary conditions,} \end{cases} \tag{13}$$

where

$$f = K[\exp(U) - 1]. \tag{14}$$

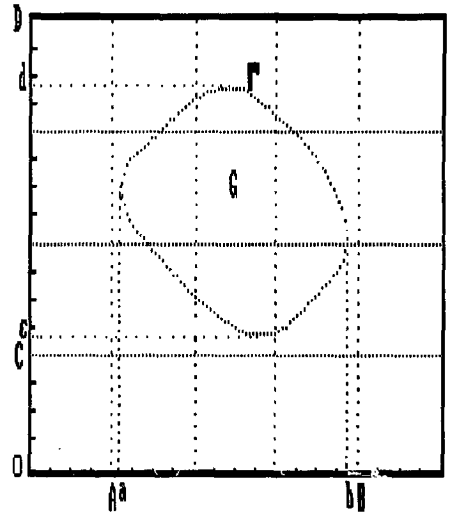

Fig. 6. Model illustrating the mesh principle of the two-dimensional domain G and its boundary Γ. The domain G is subdivided into rectangles ($N \times K$). We consider the set of rectangles $[A, B]\times[C, D]$ such that $[a, b]\subset[A, B]$ and $[c, d] \subset[C, D]$; $h = (B - A)/Nx$ and $k = (D - c)/Ny$.

Using meshes we define a network in $\mathbb{R}^2$ with the points:

$$R = \{M_{ij} \in \mathbb{R}^2 \mid M_{ij} = (A + ih, C + jk),\quad i = 0, 1 \ldots N_x,\ j = 0, 1 \ldots N_y\}. \tag{15}$$

Suppose that the potential function $U(x, y)$ being determined is sufficiently differentiable, it can be developed by Taylor series at the point $(x + h, y + h)$:

$$\begin{aligned} U(x+h,\ y+k) &= U(x, y) + h\frac{\partial U(x, y)}{\partial y} + \frac{h^2}{2!}\frac{\partial U^2(x, y)}{\partial x^2} \\ &\quad + hk\frac{\partial^2 U(x, y)}{\partial x\,\partial y} + \frac{k^2}{2!}\frac{\partial^2 u(x, y)}{\partial y^2} + \cdots \\ &\quad + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(n-1)} U(x, y). \end{aligned} \tag{16}$$

If we state that

$$\begin{aligned} &U_{ij} = U(x, y), \\ &U_{i+1,j} = U(x+h, y), \qquad U_{i,j+1} = U(x, y+h), \\ &U_{i-1,j} = U(x-h, y), \qquad U_{i,j-1} = U(x, y-h), \end{aligned} \tag{17}$$

where

$$x = A + ik, \qquad y = C + jk, \tag{18}$$

the Taylor series expansion gives

$$\left.\frac{\partial U}{\partial x}\right|_{+} = \frac{U_{i+1,j} - U_{ij}}{h},$$
$$\left.\frac{\partial U}{\partial y}\right|_{+} = \frac{U_{i,j+1} - U_{ij}}{h},$$
$$\left.\frac{\partial U}{\partial x}\right|_{-} = \frac{U_{ij} - U_{i-1,j}}{h},$$
$$\left.\frac{\partial U}{\partial y}\right|_{-} = \frac{U_{ij} - U_{i,j-1}}{h},$$
$$\frac{\partial U}{\partial x} = \frac{U_{i+1,j} - U_{i-1,j}}{2h}, \tag{18}$$
$$\frac{\partial U}{\partial y} = \frac{U_{i,j+1} - U_{i,j-1}}{2h},$$
$$\frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2},$$
$$\frac{\partial^2 U}{\partial y^2} = \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{k^2}.$$

3.3. Application to the box of sand

3.3.1. Mesh parameters

Since the potential is zero ($U = 0$) at infinity, we lengthen the plate on each side by $A \cdot L$, in order to simulate this aspect. (L is the length of each side of the plate.) The value is real positive. This is a discrete step size chosen equally for the x- and y-axes in such a way that $L/h = N$, N being a positive even number. The value A is also chosen in such a way that $A \cdot N$ is an integer that will be made as large as possible (Fig. 6). The total length taken for each side is:

$$L_T = AL + L + AL = (2A + 1)L = (2A + 1)Nh = h\mathrm{NI},$$

where $\mathrm{NI} = (2A + 1)N$. Also $i = 0, 1, \ldots, \mathrm{NI}$ and $j = 0, 1, \ldots, \mathrm{NI}$.

We remark that the problem is symmetrical with $j = \mathrm{NI}/2$. The equation could be solved in half the area by taking $i = 0, 1, \ldots, \mathrm{NI}$ and $j = 0, 1, \ldots, \mathrm{NI}/2$.

3.3.2. System of equations to be solved

The system of equations to be solved is deduced from Section 3.2:

(1) For $AN + 1 \leqslant i \leqslant (A + 1)N - 1$ and $AN + 1 \leqslant j \leqslant \mathrm{NI}/2 - 1$:

$$-4U_{ij} + U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} = K(\mathrm{e}^{U_{ij}} - 1),$$

where

$$K = \frac{h^2 qNd}{U_T \epsilon}, \qquad U_T = \frac{kq}{T},$$

q is the charge of an electron, N is the donor number density, k is Boltzmann's constant, and T is the absolute temperature.

(2) For $AN + 1 \leqslant i \leqslant (A + 1)N - 1$ and $j = \mathrm{NI}/2$:

$$-4U_{ij} + U_{i-1,j} + U_{i+1,j} + 2U_{i,j-1} = K(\mathrm{e}^{U_{ij}} - 1).$$

(3) For $1 \leqslant i \leqslant AN - 1$ and $1 \leqslant j \leqslant \mathrm{NI}/2 - 1$, or $(A + 1)N + 1 \leqslant i \leqslant \mathrm{NI} - 1$ and $1 \leqslant j \leqslant \mathrm{NI}/2 - 1$, or $AN \leqslant i \leqslant (A + 1)N$ and $1 \leqslant j \leqslant AN - 1$:

$$-4U_{ij} + U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} = 0.$$

(4) For $1 \leqslant i \leqslant AN - 1$ and $j = \mathrm{NI}/2$, or $(A + 1)N + 1 \leqslant i \leqslant \mathrm{NI} - 1$ and $j = \mathrm{NI}/2$:

$$-4U_{ij} + U_{i-1,j} + U_{i+1,j} + 2U_{i,j-1} = 0.$$

(5) For $AN + 1 \leqslant i \leqslant (A + 1)N - 1$:

$$U_{i,AN+1} - (1 + \eta)U_i, AN + \eta U_{i,AN-1} = 0,$$

where $\eta = \epsilon_0/\epsilon = 1/\epsilon_r$.

(6) For $0 \leqslant i \leqslant \mathrm{NI}$ and $j = 0$, or $i = \mathrm{NI}$ and $1 \leqslant j \leqslant \mathrm{NI}/2$, or $i = 0$ and $1 \leqslant j \leqslant \mathrm{NI}/2$, or $i = AN$ and $AN \leqslant j \leqslant (A + 1/2)N$:

$$U_{ij} = 0.$$

(7) For $AN \leqslant j \leqslant (A + 1/2)\, N$:

$$U_{(A+1)N,j} = U_0.$$

(8) For $0 \leqslant i \leqslant \mathrm{NI}$ and $\mathrm{NI}/2 + 1 \leqslant j \leqslant \mathrm{NI}$:

$$U_{ij} = U_{i,\mathrm{NI}-j}.$$

Since the problem is symmetrical the number of unknowns is

$$\mathrm{NM} = (\mathrm{NI} - 1)\mathrm{NI}/2 - 2(N/2 + 1).$$

There are NM non linear equations with NM unknowns. By changing the notation of the unknowns, we obtain the system of equations given by

$$[X]\{U\} = \{Y\},$$

where $[X]$ is a square matrix of order NM de-

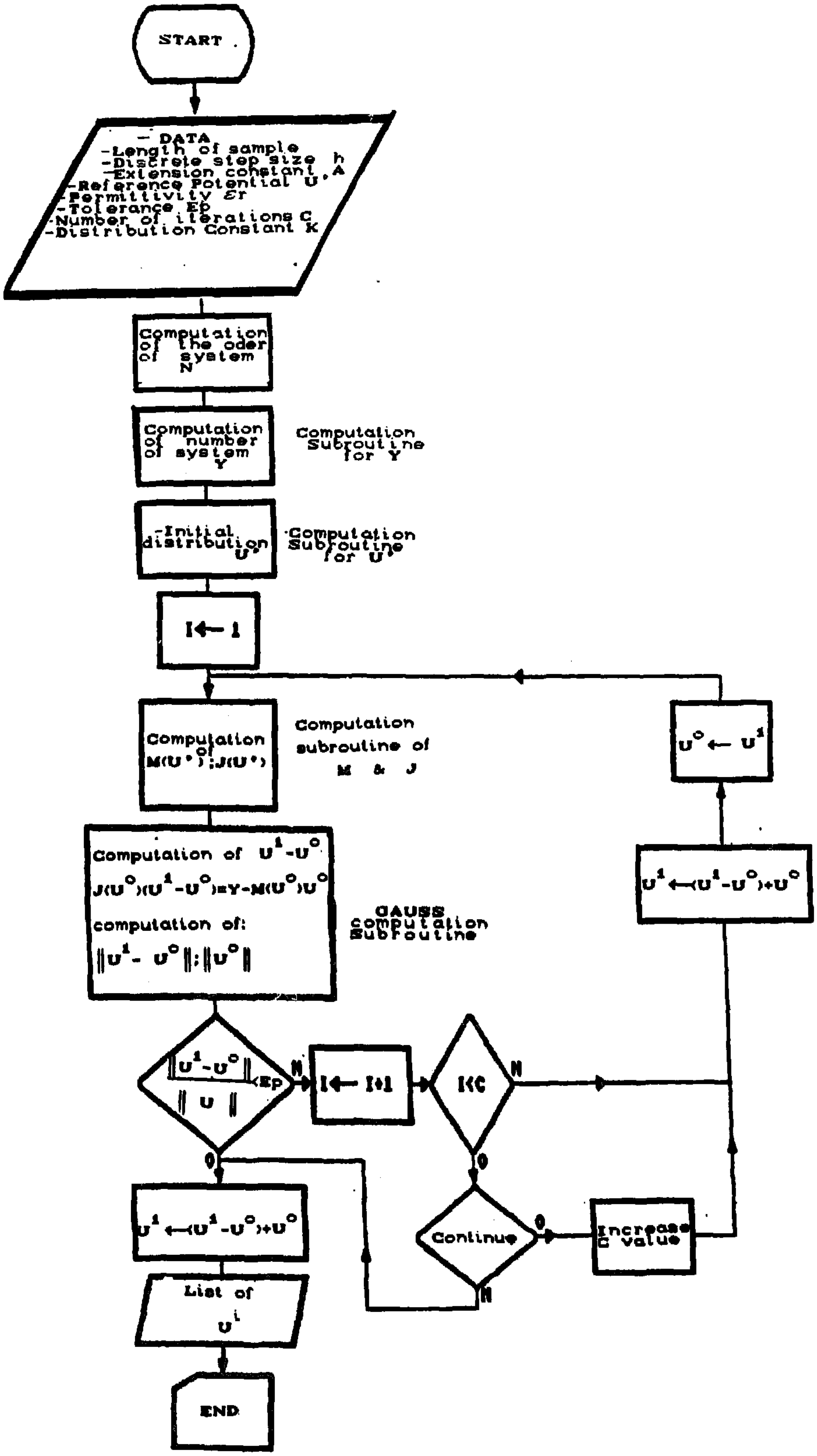

Fig. 7. Flowchart.

pending on $\{U\}$ and which is definable; $\{Y\}$ is a known column matrix; $\{U\}$ is an unknown column matrix.

3.3.3. Numerical analysis of the system of equations

After considering the methods available for solving the system of nonlinear equations, we have chosen the method of Newton–Raphson not only because it is one of the commonly used methods but because in our case particularly it converges faster than the substitution method for example. It consists of constructing a follow up of solutions such as: $\{U^0\}$ $\{U^1\}, \ldots, \{U^{i-1}$, $\{U^i\}$, each of which is calculated from $\{U^{i-1}\}$ by Taylor series expansion.

Our system of equations is in the form $\{Y\} = \{F(U)\}$, where

$$F(U) = \begin{cases} f_1(U) \\ f_2(U) \\ \vdots \\ f_{\mathrm{NM}}(U) \end{cases} \tag{19}$$

from where the Taylor series expansion is:

$$F(U) = F(U^0) + (U - U^0) \left.\frac{\partial F}{\partial U}\right|_{U=U^0} + \ldots \tag{20}$$

If we were to limit ourselves to the first order, the expression becomes:

$$Y = F(U) = F(U^0) + (U - U^0) \left.\frac{\partial F}{\partial U}\right|_{U=u^0}, \tag{21}$$

or

$$(U - U^0) \left.\frac{\partial F}{\partial U}\right|_{U=U^0} = Y - F(U^0), \tag{22}$$

an expression which at the ith iteration produces:

$$(U^i - U^{i-1}) \left.\frac{\partial F}{\partial U}\right|_{U=U^{i-1}} = Y - F(U^{i-1}).$$

The same expression in matrix form is given by:

$$\left| \left.\frac{\partial F}{\partial U}\right|_{U=U^{i-1}} \right| \{U^i - U^{i-1}\} = \{Y - F(U^{i-1})\}. \tag{23}$$

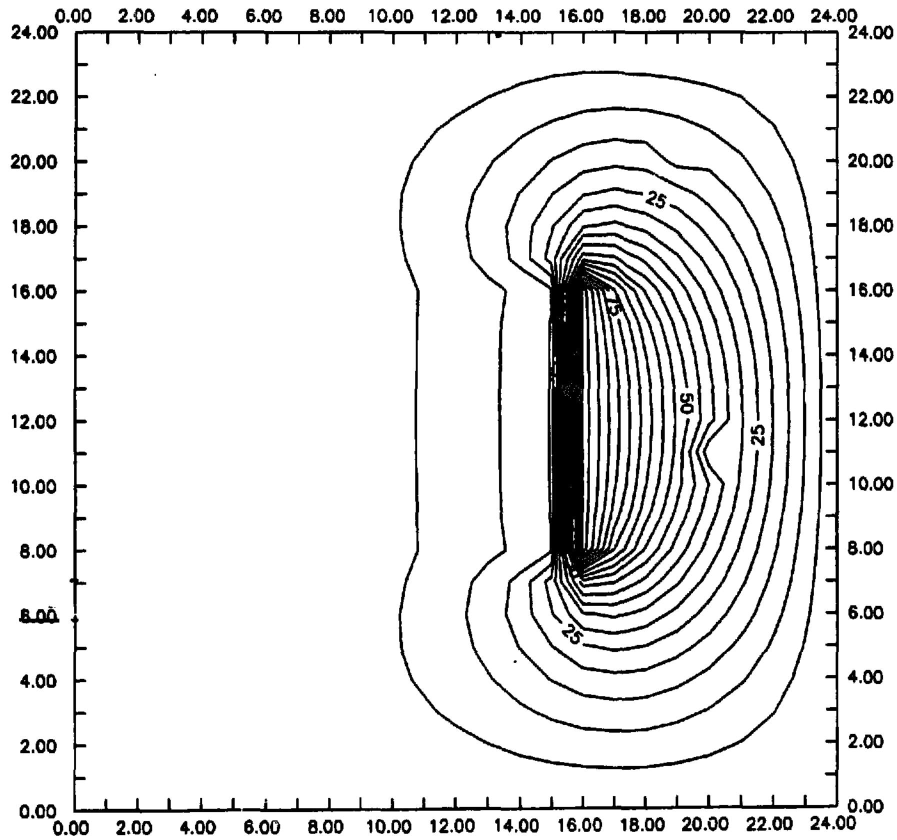

Fig. 8. Equipotential distribution around the plate. The potential decreases as one moves away from the plate.

Let $J^i = \partial F/\partial U\,|_{U=U^{i-1}}$ be a Jacobian matrix of the ith iteration, where:

$$\Delta U^i = U^i - U^{i-1}, \qquad DY^i = Y - F(U^{i-1}), \tag{24}$$

we will then have:

$$[J^i]\{\Delta U^i\} = \{\Delta Y^i\}, \tag{25}$$

which is a system of linear equations in $\{\Delta U^i\}$. The values $[J^i]$ and $\{\Delta Y^i\}$ are evaluated from $\{U^{i-1}\}$, which is known.

The computer program (Fig. 7) designed to calculate the potential distribution is made up of one main programme and subroutines in GW BASIC 100% compatible with IBM-PC.

4. Results and interpretation

The computer program to calculate the potential distribution in soil samples applies the finite difference method on a sample of the form of a parallelepiped. This programme requires that the following parameters are introduced

- number of iterations I;
- length of sample L;
- discrete step size h;
- the extension constant A defined in paragraph 3.3.1;
- extension of step PR defined in paragraph 3.3.1;
- relative permittivity ϵ_r;
- reference potential (fixed) U;
- distribution constant K defined in paragraph 3.3.1;
- tolerance EP defined in paragraph 3.3.1.

The solution is a table of the potential distribution in the sample. An interface helps to draw the corresponding equipotentials (Figs. 8 and 9) in two- and/or three-dimensional representation (Figs. 10–12).

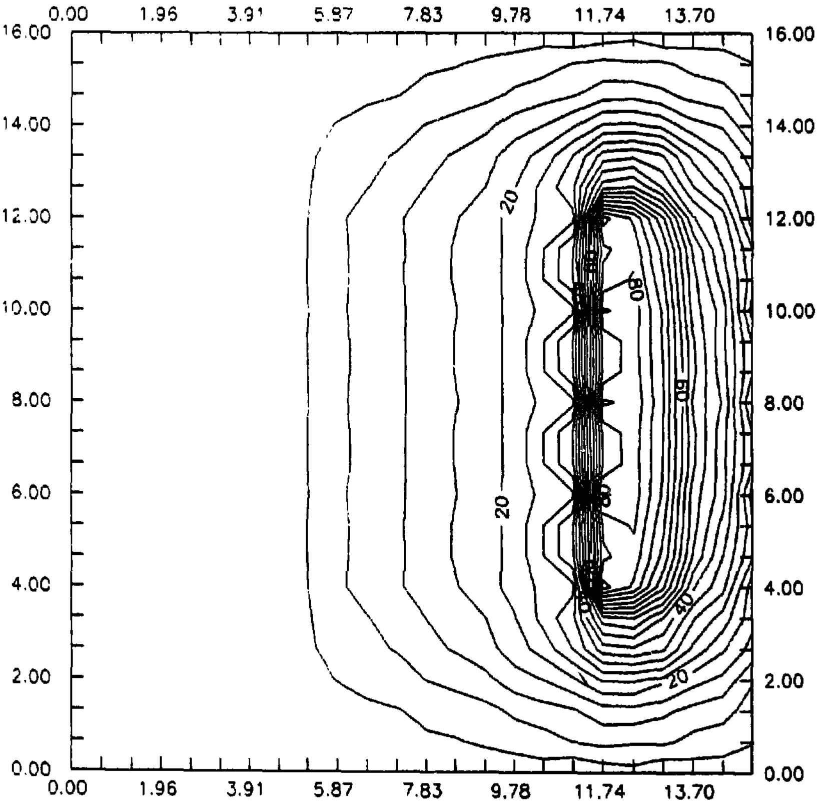

Fig. 9. The same illustration as in Fig. 8 but with a smaller discrete step size.

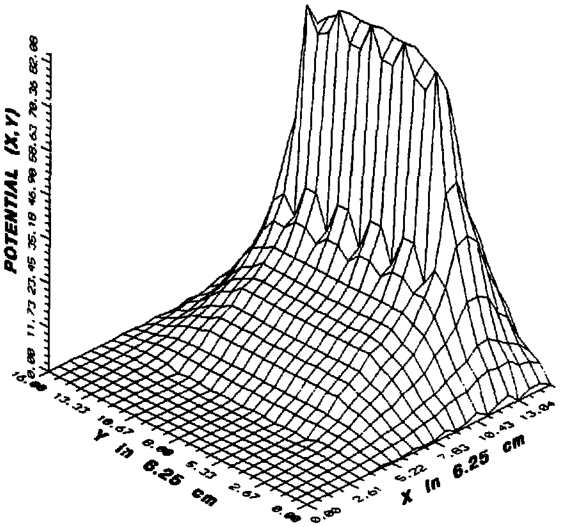

Fig. 10. Potential distribution in three dimensions for: $U = 28$, $L = 50$ cm, $H = 6.25$ cm, $A = 0.5$, $\mathrm{PR} = 4$, $\epsilon_r = 15$, $U^0 = 98$ V, $K = 0.00002$, $\mathrm{EP} = 0.00001$; the tolerance is given as $\mathrm{DE} = 1.903769 \times 10^{-7}$.

The variation of potential as a function of x and y conforms to the results of the potential distribution in a sample of the semi-conductor type [5–7].

The tolerance depends on:

- the step size of the mesh and therefore on the number of points;
- dimensions of the sample.

The speed of execution depends on:

- number of points;
- the tolerance assumed;
- the type of desk top computer.

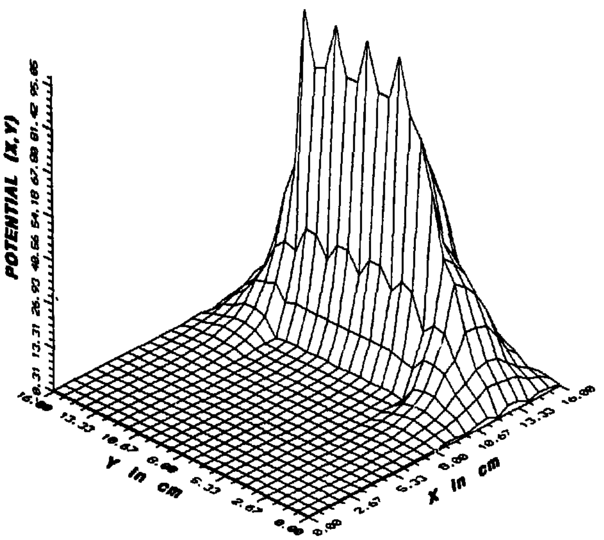

Fig. 11. Potential distribution as in Fig. 10, but with the following parameters: $U = 37$, $L = 50$ cm, $H = 6.5$ cm, $A = 0.625$ cm, $\mathrm{PR} = 5$, $\epsilon_r = 15$, $U^0 = 98$ V, $K = 0.1$, $\mathrm{EP} = 0.000002$, $\mathrm{DE} = 6.364716 \times 10^{-8}$.

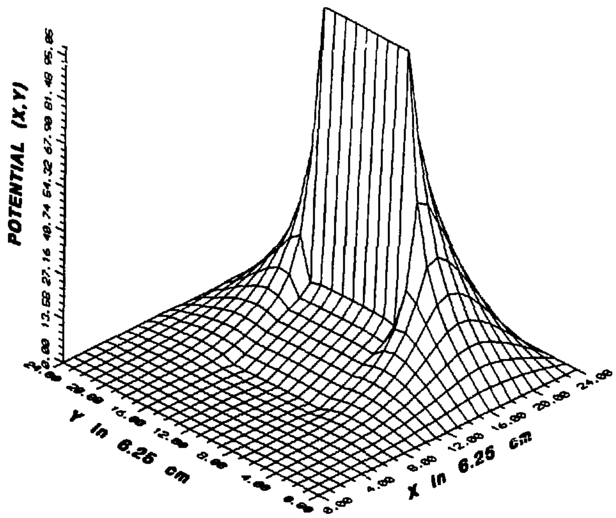

Fig. 12. Potential distribution as in Fig. 11, but with the following parameters: $U = 37$, $L = 50$ cm, $H = 6.25$ cm, $A = 0.5$, $\mathrm{PR} = 4$, $\epsilon_r = 15$, $U^0 = 98$ V, $K = 0.1$, $\mathrm{EP} = 0.0002$, $\mathrm{DE} = 3.328168 \times 10^{-6}$.

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