Math 451, Numerical Methods: Solutions Homework #4

Problem 4.1: By hand, showing your work, use six intervals to approximate the integral

$$\int_0^\pi \sin(x)\,dx$$

a) using the Trapezoidal rule

solution
$h = (\pi - 0)/6 = \frac{\pi}{6}$. Therefore,

$$\begin{aligned}
\textstyle\int_0^\pi \sin(x)\,dx \approx\ & \tfrac{\pi/6}{2}(\sin(0) + \sin(\pi)) \\
& + \tfrac{\pi}{6}\left(\sin\left(\tfrac{\pi}{6}\right) + \sin\left(\tfrac{2\pi}{6}\right) + \sin\left(\tfrac{3\pi}{6}\right) + \sin\left(\tfrac{4\pi}{6}\right) + \sin\left(\tfrac{5\pi}{6}\right)\right) \\
\approx\ & 1.954097
\end{aligned}$$

b) using Simpson's rule

solution

$$\begin{aligned}
\textstyle\int_0^\pi \sin(x)\,dx \approx\ & \tfrac{\pi}{18}(\sin(0) + \sin(\pi)) \\
& + 4 \cdot \tfrac{\pi}{18}\left(\sin\left(\tfrac{\pi}{6}\right) + \sin\left(\tfrac{3\pi}{6}\right) + \sin\left(\tfrac{5\pi}{6}\right)\right) \\
& + 2 \cdot \tfrac{\pi}{18}\left(\sin\left(\tfrac{2\pi}{6}\right) + \sin\left(\tfrac{4\pi}{6}\right)\right) \\
\approx\ & 2.000863
\end{aligned}$$

c) using the Improved Trapezoidal rule

solution
For the improved rule we need the constant

$$c = \frac{\sin'(0) - \sin'(\pi)}{12} = \frac{1}{6}$$

Then,

$$\begin{aligned}\int_0^{\pi} \sin(x)\,dx \approx\ & \frac{\pi/6}{2}(\sin(0)+\sin(\pi)) \\ & +\frac{\pi}{6}\left(\sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{2\pi}{6}\right)+\sin\left(\frac{3\pi}{6}\right)+\sin\left(\frac{4\pi}{6}\right)+\sin\left(\frac{5\pi}{6}\right)\right) \\ & +\frac{1}{6}\left(\frac{\pi}{6}\right)^2 \\ \approx\ & 1.999790\end{aligned}$$

d) Compare the errors of these methods.

solution

Method	Error
Trapezoid Rule	-4.59×10^{-2}
Simpson's Rule	8.63×10^{-4}
Improved Trapezoid Rule	-2.10×10^{-4}

The $O(h^4)$ methods both had much smaller errors. The Improved Trapezoid rule had about one fourth the error of Simpson's Rule.

Problem 4.2: Add to your Python file `integral.py` by writing a short python program called `imtrap` which takes `a,b,` and `n` as arguments and returns the Improved Trapezoid Rule approximation to $\int_a^b f(t)\,dt$ using n intervals. It should also print the error in exponential notation with two significant figures. Check your program by comparing your results to those of Example 4.11.

solution

```
#Improved Trapezoid Rule
def imtrap(a,b,n):
    h = abs(a-b)/n
    #Exterior endpoints only count once
    area = (f(a) + f(b))/2
    #Interior endpoints count twice
    x = a
    for k in range(1,n):
        x = x + h
        area = area + f(x)
    area = area*h
    #Correction for h^2 error term
    corr = (df(a)-df(b))*h**2/12
    area = corr + area
    ex = F(b) - F(a)
    print('Error = %3.2e' % (area-ex))
    return area
```

Problem 4.3: Consider the integral

$$\int_0^2 5x^6 - 12x^5 + 30x^3 - 90x^2 + 4x + 1\,dx$$

a) Use `trap` to apply the Trapezoid Rule to estimate this integral for $n = 10, n = 100$, and $n = 1000$.

solution

`n`	Value	Error
10	-146.57568	-4.25×10^{-3}
100	-146.57143	-4.27×10^{-7}
1000	-146.57143	-4.25×10^{-11}

b) What is the order of the estimate? How do you know?

solution
The order of the estimate is **four**. Each time we raise `n` by an order of magnitude, h decreases by an order of magnitude, and the error decreases by four orders of magnitude.

c) Is this behavior surprising? Do you have an explanation? (*Hint:* Consider the Improved Trapezoid Rule)

solution

Yes, it's very surprising because the Trapezoid Rule is only supposed to be a $O(h^2)$ method.

If we consider the constant in the Improved Trapezoid Rule,

$$c = \frac{f'(0) - f'(2)}{12}$$

we notice that

$$f'(x) = 30x^5 - 60x^4 + 90x^2 - 180x + 4$$

and that $f'(0) = f'(2) = 4$. Thus $c = 0$ for this problem, and we get the 'improvement' from the Improved Trapezoid Rule *for free*!

Problem 4.4: Use a table to find the $R_{3,3}$ Romberg approximation to the integral $\int_1^4 \sqrt{t}\,dt$. Compare with Simpson's rule applied using eight intervals (the same as $R_{3,3}$).

solution

n	2^n	R_{0n}	R_{1n}	R_{2n}	R_{3n}
0	1	4.5	-	-	-
1	2	4.621708	4.662277	-	-
2	4	4.655093	4.666221	4.666484	-
3	8	4.663747	4.666631	4.666659	4.666662

The actual solution is $2(4^{\frac{3}{2}} - 1)/3 = \frac{14}{3} \approx 4.666\ldots$. This tells us that the error for R_{33} is about -5.13×10^{-6}, while our `simp` program can be used to determine that Simpson's Rule with $n = 8$ intervals has an error of about -3.53×10^{-5}. This is about seven times the error for R_{33}.

Problem 4.5: There is a method for estimating a definite integral similar to Simpson's rule which we will call *Paul's Peculiar Rule.* It requires that we divide the interval into a number of subintervals divisible by **four**. Then,

$$\int_a^b f(x)\,dx \approx \frac{2h}{9}\Big(- f(a) - f(b) + 16 \sum_{k=0}^{n/4-1} f(x_{4k+1})$$
$$-12 \sum_{k=0}^{n/4-1} f(x_{4k+2}) + 16 \sum_{k=0}^{n/4-1} f(x_{4k+3}) - 2 \sum_{k=1}^{n/4-1} f(x_{4k})\Big)$$

a) Add to your Python file `integral.py` by writing a short python program called `PPR` which takes `a`, `b`, and `n` as arguments and estimates $\int_a^b f(x)dx$ according to Paul's Peculiar Rule using `n` subintervals.

solution

```
#Paul's Peculiar Rule
def PPR(a,b,n):
    if mod(n,4)!= 0:
        print('Number of intervals not div by 4!')
        return
    h = abs(a-b)/n
    #Exterior endpoints only count once
    area = -f(a)-f(b)
```

```
#Left endpoints
x = a+h
mid = 0
for k in range(0,n//4):
    mid += f(x)
    x += 4*h
area += 16*mid
#Left Interior endpoints
x = a+2*h
mid = 0
for k in range(0,n//4):
    mid += f(x)
    x += 4*h
area -= 12*mid
#Right Interior endpoints
x = a+3*h
mid = 0
for k in range(0,n//4):
    mid += f(x)
    x += 4*h
area += 16*mid
#Right endpoints
x = a+4*h
mid = 0
for k in range(1,n//4):
    mid += f(x)
    x += 4*h
area -= 2*mid
#Final bookkeeping
area *= 2*h/9
ex = F(b) - F(a)
print('Error = %3.2e' % (area-ex))
return area
```

b) Estimate $\int_1^4 \ln(x)\,dx$ using the Trapezoidal Rule, Simpson's Rule, and Paul's Peculiar Rule for $n = 12$, $n = 120$, and $n = 1200$. Compare the methods.

solution

n	Trap Rule	Error	Simp Rule	Error	PPR	Error
12	2.541282	-3.90×10^{-3}	2.545138	-3.94×10^{-5}	2.545954	7.77×10^{-4}
120	2.545138	-3.91×10^{-5}	2.545177	-4.27×10^{-9}	2.545178	1.02×10^{-7}
1200	2.545177	-3.91×10^{-7}	2.545177	-4.63×10^{-13}	2.545177	1.02×10^{-11}

c) What is the order of accuracy of Paul's Peculiar Rule? How do you know?

solution

Paul's Peculiar Rule is order 4, the error decreasing by four orders of magnitude for each order of magnitude n increases. Though Simpson's Rule is also order four, it is clearly better, giving errors which are about two orders of magnitude smaller than Paul's Peculiar for any given value of h.