LoP- The Logic of Propositions

Reasoning about what is True and what is False

(HP2T)

LoP – The Logic of Propositions

- **Intuition**
- Definition
- Domain
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

LoP – Why a logic of propositions?

- The mainstream name for LoP is **PL** (for **Propositional Logic**);
- The **Logic of Propositions** (LoP) is the core (language) logic which allows to reason about propositions.
- It satisfies all the properties of logical entailment described above. In particular, it implements monotonic reasoning
- All forms of monotonic reasoning on finite domains can be reduced to LoP reasoning
- In particular, reasoning in all the world logics can be encoded in polynomial time into LoP reasoning
- In particular, LoI reasoning on finite domains can be encoded into LoP reasoning, paying the price of a worst case exponential blow-up of the reasoning time.
- All NP complete problems can be written in polynomial time as LoP problems

LoP – Highlights

- The version of LoP introduced here allows for all the main stream propositional connectives, that is:
  - Negation (not)
  - Conjunction (and)
  - Disjunction (or)
  - Implication (implies, if P then, →).
  - Equivalence (iff)
  - Disjointness (exor)
- Many of this connectives could be eliminated by encoding them into others. The most common minimal sets of connectives (bases) are: and/or/not, implication/not

LoP – The Logic of Propositions

- Intuition
- **Definition**
- Domain
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

LoP – Definition

We formally define LoP as follows

$$\mathbf{L_{LoP} = \langle\, L_{LoP},\ |\!=_{LoP}\, \rangle}$$

where

$$\text{if M } |\!=_{LoDE}\ a \ \text{ then M } |\!=_{LoP}\ a'$$

with a and a' being in a one-to-one relation and

$$\mathbf{L_{LoDE}} = \langle\, \mathbf{EG_{LoDE}},\ |\!=_{\mathbf{LoDE}}\, \rangle$$

$$\mathbf{EG_{LoDE} = \langle\, L_{LoDE},\ D_{LoDE},\ I_{LoDE} \rangle}$$

Below, any time no confusion arises, we drop the subscripts.

Observation (LoDE, LoP). LoDE is the Logic of Entity Bases, that is, of Entities together with their Definitions and Descriptions. LoP is the Logic of propositions (propositional logic).

LoP – The Logic of Propositions

- Intuition
- Definition
- **Domain**
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

LoDE domain (the same as LoE)

Definition (Domain, percepts)

$$D = < U, \{C\}, \{R\} >$$

where:

- $U = \{u\}$ is called the **universe (of interpretation)** of D.
- $\{u\}$ is a set of **units** $u_1, \ldots, u_n$, for some n
- $\{C\}$ is a set of **classes** $C_1, \ldots, C_m$ of units, for some m, with $C_i \subseteq U$
- $\{R\}$ is a set of **binary relations** $R_1, \ldots, R_p$ between units, for some p, with $R_i \subseteq U \times U$

Observation (EG, Binary relations). To comply to the graph notation we restrict ourselves to binary relations.

LoDE assertions – (same as LoDE facts)

Observation. LoDE allows for the following assertions:

- Every etype (**primitive, defined** or **described**) / dtype and its argument is a fact.
- Every relation R and its two arguments is a fact.

Facts only have one of four possible forms:

- $E_T(e)$, meaning that the entity e is of etype E_T ,
- $D_T(v)$, meaning that the value *v* is of dtype D_T
- $O(e_1, e_2)$, meaning that the object property O holds between e_1 and e_2
- A(e, v), meaning that the data property A of entity e has value v

Observation (LoE vs. LoDE). Differently from LoE, in LoDE we have not only primitive etypes, but also defined etypes and descriptions of etypes (in **bold**).

LoP Domain

Definition (Domain – facts).

$$D_{LoP} = < T, F >$$

Definition (Domain – percepts). Let

$$D_{LoDE} = < E, \{C\}, \{R\} >$$

be a LoDE domain of interpretation. Let $L_{LoDE} = \{a\}$ be a LoDE language for D_{LoP} where $\{a\}$ is the set of assertions in L_{LoDE}. Let $a_i \in L \subseteq L_{LoDE}$ be an assertion. Then

$$D_{LoP} = \{a^+, a^-\} = \{a_1^+, a_1^-, \ldots, a_N^+, a_N^-\} = \{T(a_1), F(a_1), \ldots, T(a_N), F(a_N)\}$$

where a^+, a^- are values of atomic propositions such that:

- $a^+ = T(a) = T$ if the LoDE assertion a is True
- $a^- = F(a) = T$ if the LoDE assertion a is False

Definition (Model). M is a set of atomic propositions $\{a^+, a^-\}$ such that, for each a, M contains one and only one between $a^+$and a^-.

$$M = \{f\} = \{a^+, a^-\} = \{\ldots, a_i^+, \ldots, a_j^-, \ldots\} \subseteq D_{LoP}.$$

Terminology (Model, atomic proposition). From now on, when no confusion arises, we talk of propositions meaning atomic propositions, the only propositions which belong to models.

LoP domain - observations

Observation (Fact). As from the original intuition, a fact is what is the case in the world. In LoP, an analogical representation (model) is a set of judgements about what is true and what is false in the world. Therefore, in LoP the only two facts which can be observed are whether a certain judgement is true or whether a certain judgement is true false.

Observation (Percept). As from the original intuition, a percept is an element of the domain which can be perceived as distinct from other percepts. Despite that we only have two facts (Truth and Falsity) we have multiple percepts denoting one of the two facts. The distinguishing element of a percepts is the assertion being judged. One can think of propositions as being two disjoint sets of synonyms of one fact or the other.

Observation (A domain of propositions). D = $\{a^+, a^-\}$ as, if it allows for a judgement about an assertion, D also allows for the opposite judgement. This is because, as from the original intuition, a domain must allow for any fact to be case or not to be the case.

Observation (A domain of propositions). $a \in L \subseteq L_{LoDE}$ because LoP may be focused only on a subset of the assertions defined in LoDE.

Observation (Model). A LoP model has as many elements as there are propositions, that is, it is exactly half the size of the domain. For each proposition a model records its being true or false.

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- **Language**
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

Language (the same as LoE)

Definition (Assertional language)

$$L = < A , FR > = \{P\}$$

where:

- L is a **propositional language**, where $P \in \{P\}$ is a **proposition**.
- A = is an **alphabet** of atomic propositions
- FR is a **set of formation rules**

Formation rules – BNF

$< P >$ ::= <atomic proposition> |

$\neg < P >$ |

$< P > \wedge < P >$ |

$< P > \vee < P >$ |

$< P > \supset < P >$ |

$< P > \equiv < P >$ |

$< P > \oplus < P >$

<atomic proposition> ::= $P_1 \ldots P_n \in \{P\}$

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- **Interpretation function**
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

Interpretation function

Definition (LOP Interpretation function). Let $\{P\}$ the set of atomic proposition of a LoP language. Then

$$I_{LoP}: \{P\} \rightarrow \{T,F\}.$$

Observation (Truth value of a proposition). In LoP how the truth value of an atomic proposition is computed is irrelevant. However this relevant from an application point of view as it is constrained by the world as described by a LoDE EG and constrains (when a truth value is imposed on an proposition whose LoDE truth value is undefined).

Definition (From I_{LoP} to I_{LoDE}). Let D = < E, {C}, {R} > be a LoDE domain of interpretation. Let L_W = $\langle$W, $|=_{LoDE}$ $\rangle$ be a LoDE Logic and L_{LoDE} = $\{a\}$ be a LoDE language for D where a is an assertion and $\{a\}$ is the set of assertions in L_{LoDE}. Let $L_{LoP} = \{P\}$ be LoP language, where P ∈ $\{P\}$ is a proposition. Let I_{LoP} be a LoP interpretation function, with I_{LoP}: $\{P\} \rightarrow$ {T,F}. Then we have

$$\text{If } |=_{LoDE}\, a \text{ then } I_{LOP}(a^{+}) = T \text{ and } I_{LOP}(a^{-}) = F \qquad (*)$$

Observation (From I_{LoP} to I_{LoDE}). If (*) does not hold then I_{LoP} is not constrained and it will assign a truth value depending on the LoP theory.

Propositions

Intuition (Simple, articulative and nested propositions). In the description above of how to compute the truth value of (atomic) propositions we have assumed that propositions are judgements about LoDE assertions a, with $L_{LoDE} = \{a\}$. The implicit assumptions, not made explicitly is that these LoE assertions were LoE assertions, that is description of facts. We call these (atomic) propositions, **simple propositions**. However, this is not necessary the case. In fact we can also have propositions that express judgements about complex assertions made using the full power of the LoDE language, and also propositions about propositions. We call the first type of (atomic) propositions, **articulative propositions**, and the second **nested propositions.**

Intuition (Articulative propositions). These are propositions that make judgements about any LoDE assertion.

Example (Articulative propositions). (it is False that) "Silvia is a friend of paolo and she is 2 meters tall".

Intuition (Nested propositions). We can have any level of nesting of propositions, that is propositions of propositions of … assertions. This is extensively used in Natural languages. In logic, nested propositions are (sometimes) formalized using metatheories.

Example (Nested propositions). Propositions can be nested, assertions cannot. The nesting can also be "multiagent".

- It is true that what Fausto said about the weather is false
- It is true that Mario said that what Fausto said about the weather is false

Observation (Problematic / interesting nested propositions).

- (It is true that) this sentence is false **(the Liar paradox)**
- (It is true that) this sentence is unprovable **(Goedel unprovable incompleteness formula)**

Note: **In the following we only consider simple propositions.**

Model and interpretation

Proposition (Interpretation and model). An interpretation assigns a truth value to all the atomic formulas, in this case, the atomic propositions, of a language, in this case a LoP language. As from above, a model also assigns a truth value to all atomic propositions. That is, in LoP, models and interpretations are the same formal object.

Proposition (Interpretation and model). A model is an interpretation which entails a formula.

Notation (Model and interpretation). Being models interpretations, we write and say that interpretations entail formulas.

Model as set of true propositions

Terminology (A Model as a set of true propositions). A model, can be represented set theoretically as the set of atomic propositions it defines as true.

Example (Model). See the table on the right.

Terminology(model). A model M can be thought as a subset S of $\{P\}$ where I, an interpretation function, is the characteristic function of S, i.e.

$$A \in S \text{ if and only if } I(A) = True.$$

	p	q	r	Set Theoretic Representation
I_1	True	True	True	$\{p, q, r\}$
I_2	True	True	False	$\{p, q\}$
I_3	True	False	True	$\{p, r\}$
I_4	True	False	False	$\{p\}$
I_5	False	True	True	$\{q, r\}$
I_6	False	True	False	$\{q\}$
I_7	False	False	True	$\{r\}$
I_8	False	False	False	$\{\}$

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- Interpretation function
- **Entailment**
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

Entailment

$I \vDash P$,	if $I(P) = T$,	with	$P \in \{P\}$
$I \vDash \neg P$,	if	not	$I \vDash P$
$I \vDash P_1 \wedge P_2$,	if $I \vDash P_1$	and	$I \vDash P_2$
$I \vDash P_1 \vee P_2$,	if $I \vDash P_1$	or	$I \vDash P_2$
$I \vDash P_1 \supset P_2$,	if when $I \vDash P_1$,	then	$I \vDash P_2$
$I \vDash P_1 \equiv P_2$,	if $I \vDash P_1$	if and only if	$I \vDash P_2$
$I \vDash P_1 \oplus P_2$,	if $I \vDash P_1$	if and only if not	$I \vDash P_2$

ì

Observation: How Connectives Operate

Negation	
¬ True	False
¬ False	True

Conjunction	
True ∧ True	True
True ∧ False	False
False ∧ True	False
False ∧ False	False

Disjunction	
True V True	True
True V False	True
False V True	True
False V False	False

Disjointness (exor)	
True ⊕ True	False
True ⊕ False	True
False ⊕ True	True
False ⊕ False	False

Equivalence	
True ≡ True	True
True ≡ False	False
False ≡ True	False
False ≡ False	True

Consequence	
True ⊃ True	True
True ⊃ False	False
False ⊃ True	True
False ⊃ False	True

Observations

Observation (Proposition). We have four cases where a proposition P can assert the truth or falsity of LoDE fact, which in turn can be true /false. A proposition $\boldsymbol{P}$**,** if true, depending on its content, asserts the falsity of a false fact or the truth or a truth fact

Observation (Negation). ¬P asserts the opposite of the proposition P**.**

Observation (Disjointness). $P_1 \oplus P_2$, if true, asserts that the truth / falsity of one of the two propositions excludes the truth / falsity of the other.

Observation (Conjunction). $P_1 \wedge P_2$, if true, asserts that both propositions are true

Observation (Equivalence). $P_1 \equiv P_2$**,** if true, asserts that the truth / falsity of one of the two propositions garantees the truth / falsity of the other.

Observation (Disjunction). $P_1 \vee P_2$, if true, asserts one or both propositions are true.

Observation (Consequence). $P_1 \supset P_2$, if true, asserts that the truth of $\boldsymbol{P_1}$ guarantees the truth of $\boldsymbol{P_2}$. It does not say anything about $\boldsymbol{P_2}$ in case $\boldsymbol{P_1}$ is false.

Interpretation equivalence wrt. a formula

Proposition (Interpretation equivalence wrt a formula). Let A be a formula where P is the set of all the atomic propositions which occur in it. If

$$I(P) = I'(P),$$

then

$$I \vDash A \text{ iff } I' \vDash A.$$

That is:

- The truth value of atomic propositions which occur in A fully determines the truth value of A
- The truth value of the atomic propositions which do not occur in A play no role in the computation of the truth value of A;

Observation (Interpretation equivalence and model generation). The above proposition allows us to focus only on the propositions which occur in a formula. The interpretation of the other atomic propositions are irrelevant to deciding whether an interpretation is a model of a formula.

Model and theory, observations

Observation (Maximum number of models for a LOP language). If $|\{P\}|$ is the cardinality of $\{P\}$, then there are $2^{|\{P\}|}$ different models, corresponding to all the different subsets of $\{P\}$.

Observation (Number of theories for a model). A LOP model can be described by multiple theories, as long as these theories assign the same truth value as the model to any subset of the model propositions.

Observation (Number of models of a theory). A theory T has usually multiple models. T can have any number of models between 0 (when it contains a contradiction) and $2^{|\{P\}|}$ when all its formulas are tautologies.

Minimal models, maximal theories

Observation 14 (Partiality of LoP theories). The more partial a LoP theory T is, in terms of truth values assigned to propositions, the more models. For instance, assume $\{P\} = \{P_1, P_2\}$.

- T = $\{P_1 \vee \neg P_1, P_2 \vee \neg P_2\}$, has four models
- T = $\{P_1 \vee \neg P_2\}$, has three models
- T = $\{P_1\}$ has two models
- T = $\{P_1 \wedge P_2\}$ has one model
- T = $\{P_1 \wedge \neg P_1\}$ has no models

Definition (Maximal theory - reprise). A maximal theory is a theory which has only one model

Observation (Maximal theory - reprise). A model has multiple maximal theories. For instance, T1= $\{P_1, P_2\}$, and T2= $\{P_1 \wedge P_2\}$, with $\{P\} = \{P_1, P_2\}$ are two maximal theories for the same model.

Definition (Minimal model). Given a theory, a minimal model is a model which is the intersection of all the models for that theory.

Observation (Minimal model). Minimal models do not necessarily exist. For instance, assume $\{P\} = \{P_1, P_2\}$. T= $\{P_1 \vee P_2\}$ has three models and no minimal model.

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- Interpretation function
- Entailment
- **The meaning of logical connectives**
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- Key notions

Negation

Principle of non-contradiction

$$\neg (P \land \neg P)$$

- Never true, in no model.
- It imposes the key constraint for a model to be a possible picture of reality
- All formulas of the form $P \land \neg P$, independently of the shape of P, are called ***contradictions***.
- Sometimes $P \land \neg P$ iswritten as ⊥ (for falsity, as represented in the language). Not to be confused with ⊥ (Bottom) in LoDE!
- The interpretation of ⊥ is F

Principle of the excluded middle

$$P \lor \neg P$$

- True in all models.
- It imposes the key constraint for a model to be complete. P or $\neg P$ must be true in a model, but not both, for the law of non-contradiction
- All formulas of the above form, independently of the shape of P, are called ***tautologies***.
- Sometimes $P \lor \neg P$ is written as **T** (for truth, as represented in the language). Not to be confused with T (Top) in LoDE!
- The interpretation of **T** is T

Observation (Negation). The meaning of negation is given by the two laws above. They are theorems of LoP. The first enforces in the language the fact that a model cannot contain two facts (and therefore two propositions) which contradict one another. The second enforces in the language that (1) a proposition is either True or False, and (2) that statement (1) is true.

Observations

Proposition 1 (Principle of non contradiction): $\neg(P \wedge \neg P)$ is a LoP tautology.

Proposition 2 (Disjointness): $P \oplus \neg P$ is a LoP tautology

Proposition 3 (Consequence): $P \supset \neg\neg P$ is a LoP tautology

Proposition 4 (Consequence): $\neg\neg P \supset P$ is a LoP tautology

Proposition 5 (Equivalence): $\neg\neg P \equiv P$ is a LoP tautology

Proposition 6 (Principle of the excluded middle): $P \vee \neg P$ is a LoP tautology

Observation (Representing negation). Tautologies (1), (2) are «more primitive» than tautologies (3) – (6) as they only depend on fact (1) in the previous page (that is, that a proposition is either true or false), while the others exploit also fact (2) in the previous page (namely that fact (1) is itself a fact.

Observation (Representing negation). Tautologies (1), (2) are useful in that they allow to infer, once one knows that one of the two elements is true (false) then the other is false (true). That is, they allow to distinguish negative knowledge from partial knowledge (that is, «knowing not» from «not knowing»). Tautologies (3)-6) allow additionally to prove the equivalence of the various tautologies, for instance, all the ones listed above. They enforce the use of truth tables.

Conjunction / disjunction / negation

Same proposition

$$A \wedge A \equiv A$$
$$A \vee A = A$$

Commutativity

$$A \wedge B \equiv B \wedge A$$
$$A \vee B \equiv B \vee A$$

De Morgan laws

$$\neg (A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg (A \wedge B) \equiv \neg A \vee \neg B$$

Associativity

$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$
$$(A \vee B) \vee C \equiv A \vee (B \vee C)$$

Distributivity

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

Observation (¬, ∧, ∨ in set theory). The above formulas are LoP tautologies. They are mapped one-to-one with the fundamental operations and properties of sets, that is, complement, conjunction and negation, and the De Morgan laws. This is a key property which enforces the fact that reasoning about truth in LoP maps one-to-one with the set operations in set-theory.

Observation (¬, ∧, ∨ as a LoP base). ¬, ∧, ∨ are a base for LoP, meaning by this that all the other connectives can be defined from these three. This is a key property which enforces the facat that all the reasoning which can be done in LoP can be explained in terms of set-theoretic operations. Namely that, ultimately, we can reason linguistically about what we perceive in analogic representations.

Implication / equivalence

Implication and disjunction (1)
$(A \supset B) \equiv (\neg A \vee B)$

Implication and negation (2)
$A \supset B \equiv \neg B \supset \neg A$

Implication and contradiction (3)
$\bot \supset A$, for any A

Implication and equivalence (4)
$(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$

Equivalence and disjointness (5)
$(A \equiv B) \equiv \neg(A \oplus B)$

Disjointness and disjunction (6)
$(A \oplus B) \equiv (\neg A \wedge B) \vee (A \wedge \neg B)$

Observation (Implication). The above formulas are LoP tautologies. Implication is the LoP symbol which allows to describe how reasoning works. (1) says that if A is true then B must be true. (2) says that if B is false then A must be false (the counterpositive). (3) says that a contradiction (since it does not have models) allows to derive everything . (4) defines equivalence as computing the implication in both directions. (5) says the exor (disjointness) is the equivalence to two propositions which negate one another.

Implication / conjunction / disjunction / negation

Implication and conjunction (1)

$$(A \land B) \supset C \equiv (A \supset C) \lor (B \supset C)$$

Implication and conjunction (2)

$$(A \land B) \supset C \equiv A \supset (B \supset C)$$

Implication and conjunction (3)

$$(A \land B) \supset C \equiv A \supset (C \lor \neg B)$$

Implication and conjunction (4)

$$A \supset (B \land C) \equiv (A \supset B) \land (A \supset C)$$

Implication and disjunction (5)

$$(A \lor B) \supset C \equiv (A \supset C) \land (B \supset C)$$

Implication and disjunction (6)

$$(\neg A \lor B) \supset C \equiv (A \supset B) \supset C$$

Implication and disjunction (7)

$$(A \supset (B \lor C) \equiv (A \land \neg B) \supset C)$$

Implication and disjunction (8)

$$(A \supset (B \lor C) \equiv (A \supset B) \lor (A \supset C)$$

LoP connectives – observations

Observation (Conjunction and implication). The above formulas are LoP tautologies.

- Look at equation (1): conjunction in the premise of an implication weakens the implication (more requirements for the conclusion to be true).
- Look at equation (4): conjunction in the conclusion of an implication strengthens the implication when in the conclusion (more truths for the same output).

Observation (Disjunction and implication). The above formulas are LoP tautologies.

- Look at equation (5): disjunction in the premise of an implication strengthens the implication (more truths for the same output)
- Look at equation (8): disjunction in the conclusion of an implication weakens the implication when in the conclusion (more requirements for the conclusion to be true).

Observation (Conjunction vs Disjunction). Conjunction and disjunction show an opposite behavior. Beyond what written above, the pairs of tautologies (2), (6) and (3), (7) show their opposite behaviour with respect to nested implications.

LoP vs. LoD connectives – observations

Observation (LoP Negation, conjunction, disjunction vs LoD complement, intersection, union). Because of the direct mapping between LoP and set-theory (see above), these three LoP propositional connectives are the main means for translation with the LoD connectives and composite etypes and LoP.

Observation (LoP vs LoD: equivalence, disjointness). LoP equivalence and disjointness map only partially to LoD equivalence and disjointness. In LoP, these symbols can be nested while this is not the case in LoD. This is a main advantage of moving from the LoD set-theoretic semantics to the LoP truth-theoretic semantics.

Observation (LoP implication vs LoD Subsumption). The same as for equivalence and disjointness.

Observation (LoP vs LoD nesting of operators). The LoD semantics do not allow for the nesting of subsumption, equivalence and disjointness, because, differently from negation (that is complement), conjunction (that is intersection) and disjunction (that is union), they do not construct a set but, rather, they return a truth value (as it is the case in LoP). LoP, since it only works on truth values, allows for the nesting of ALL logical connectives.

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- **Tell**
- Ask – Reasoning problems
- Entailment properties
- Key notions

Tell – Model building

Intuition (Model building). The model building is performed in three steps

- **(Step 1):** Define the LoP reference model, that is, the set of LoDE assertions which describe the facts which are true in the model
- **(Step 2):** Define the LoP language, that is, the set of atomic propositions and logical connectives which are used to judge what is true / false in the model
- **(Step 3):** Define the LoP theory, that is, the set of (atomic and complex) propositions which constrain what is the case in the model by:
  - (1) specifying the negative knowledge,
  - (2) completing the partial information encoded by the model, and
  - (3) putting further constraints on what is the case via complex propositions.

Tell – Model building (step 1)

Intuition (Define the LoP Reference Model). The first step is articulated in five phases:

- **(Phase 1a)** Define the set of LoE assertions of the EG
- **(Phase 1b)** Define the set of LoD language definitions
- **(Phase 1c)** Define the set of LoD knowledge descriptions
- **(Phase 1d)** Perform the LoD unfolding
- **(Phase 1e)** Perform the LoDe expansion

Observation (Define the LoP reference model). Any of the first three steps is optional. Step 1d and Step 1e are performed only when needed. The key observation is that LoP propositions can be built by espressing judgements on all three LoDE components: ground facts about entities, facts about defined etypes, facts about language concepts.

Tell – Model building (step 2)

Intuition (Define the LoP Language). The second step is articulated in three phases:

- **(Phase 2a)** Select which LoDE assertions are going to be judged
- **(Phase 2b)** Select a uniform method for encoding a LoDE assertion a into a LoP assertion a'^{+}, a'^{-}. This in turn is composed of two steps
  - (1) How to encode a structured formula into an atomic formula, e.g., from *HasFriend(Stefania#1,Paolo#1)* to *HF-S.P*
  - (2) which of the possible positive or negative encodings a'^{+}, a'^{-} select and how to encode them in the proposition name, e.g., from *HF-S.P to HF-S.P0* and *HF-S.P1*
- **(Phase 2c)** Select the logical connectives, not necessarily used to write complex propositions

Intuition (Phase 2b). There is a std encoding which performs a 1-to-1 mapping (see later).

Tell – Model building (step 3)

Intuition (Define the LoP Theory). The third step is articulated in three phases:

- **(Phase 3a)** Select the LoDE assertions which are going to be judged. This usually turns out to be a set of atomic or conjunctions of atomic propositions
- **(Phase 3b)** Select the negative knowledge, implicitly encoded in the LoDE theory, to be made explicit in the LoP theory. This usually turns out to be a set of negations, or disjointness or implication axioms.
- **(Phase 3c)** Select the partial knowledge, implicitly encoded in the LoDE theory, to be made explicit in the LoP theory. This usually turns out to be a set of disjunction axioms.

Observation (Define the LoP theory). Usually, not all the implicit negative and partial knowledge of a LoD theory is made explicit in a LoP theory, in particular when it takes, implicitly or explictly, the form of disjunctions. The reason being that the complexity of reasoning grows exponentially with the number of disjunctions.

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- **Ask – Reasoning problems**
- Reasoning problems – correlations
- Key notions

Reasoning as entailment

Reasoning Problem (Model checking). Given a theory T and a model M, check whether M |= T.

Reasoning Problem (Satisfiability). Given a theory T, check whether there exists a model M such that M |= T .

Reasoning Problem (Validity). Given a theory T, check whether for all models M, M |= T .

Reasoning Problem (Unsatisfiability). Given a theory T, check whether there is no model M such that M |= T .

Reasoning Problem (Logical consequence). Given T_1 and T_2, check whether $T_1 \models T_2$;

Reasoning Problem (Logical equivalence). Given T_1 and T_2, check whether $T_1 \models T_2$ and $T_2 \models T_1$.

Model Checking

Given a theory T and a model M, check whether $M \vDash T$.

For example we can determine whether

$$(\neg\, p \vee q) \wedge (q \supset \neg\, r \wedge \neg\, p) \wedge (p \vee r)$$

is a model for

p = T, q = F, r = T **or** p = F, q = F, r = F.

p	q	r	$\neg\, p \vee q$	$\neg\, r \wedge \neg\, p$	$q \supset \neg\, r \wedge \neg\, p$	$p \vee r$	Answer
T	F	T	F	F	T	T	F
F	F	F	F	T	T	F	F

Observation. This is useful for checking properties (the input theory T) of existing (artificial or natural) systems (the model M).

Satisfiability

Given a theory T, check whether there exists a model M such that $M \vDash T$.

For example, we can determine if

$$(\neg p \vee q) \wedge (q \supset \neg r \wedge \neg p) \wedge (p \vee r)$$

is satisfiable.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \supset \neg r \wedge \neg p$	$p \vee r$	Answer
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
…	…	…	…	…	…	…	…
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

Observation. The first reasoning problem by excellence! Given a set of requirements (the theory T) find a model which satisfies it (e.g. TSP, scheduling)

Validity

Given a theory T, check whether there for all models M we have $M \vDash T$.

For example, we can determine if

$$(p \supset q) \vee (p \supset \neg q)$$

is a valid formula or not.

p	q	$p \supset q$	$\neg q$	$p \supset \neg q$	$(p \supset q) \vee (p \supset \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Observation: Find whether a property is true in all models (of interest).
Useful for theory reformulation (using, e.g., equivalence)

Unsatisfiability

Given a theory T, check whether there is no model M such that $M \vDash T$.

For example, we can determine if

$$\neg((p \supset q) \vee (p \supset \neg q))$$

is unsatisfiable or not.

p	q	$p \supset q$	$\neg q$	$p \supset \neg q$	$\neg((p \supset q) \vee (p \supset \neg q))$
T	T	T	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

Observation: Find whether a property (the theory T) is not realisable. Useful to evaluate the suitability of a LoDE theory (e.g., AI, non monotonic resoning, planning).

Logical Consequence

Given two theories T_1 and T_2, check whether $T_1 \vDash T_2$.

For example, we can determine if

$\neg q \vee \neg p$ is a logical consequence of the formula $\neg q$.

p	q	$\neg p$	$\neg q$	$\neg q \vee \neg p$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Whenever $\neg q$ is True, $\neg q \vee \neg p$ is also True, making it a logical consequence of $\neg q$.

Observation: The second reasoning problem by excellence. Compute the consequences of a set of facts.

Logical Equivalence

Given two theories T_1 and T_2, check whether $T_1 \vDash T_2$ and $T_1 \vDash T_2$.

For example, using the truth table method we can determine whether $p \supset (q \land \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \land \neg q$	$p \supset (q \land \neg q)$	$\neg p$
T	T	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

The truth value is the same for every interpretation, therefore the formulas are logically equivalent.

Observation: Useful to substitute equivalents for equivalents (property reformulation).

Reasoning Problems - observations

Observation (Truth table method). Truth tables are an effective method for deciding any of the LoP reasoning problems. They are used, often manually, to check some simple problems. As presented before, they do not scale to complex problems. The DPLL decision procedure is their concrete implementation, the state of the art of LoP reasoning in real world problems.

Observation (Model generation). All the problems described above are based on the same steps: (1) generate all the interpretations, (2) for each interpretation compute whether it is a model for the input, possibly via the intermediate computation of some sub-formulas of the input, and (3) decide whether the answer to the imput problem is YES/NO.

Observation (Deciding SAT /UNSAT/ VAL). The process for these three problems is the same. The termination condition is different. With SAT you terminate as soon as you find an interpretation which is a model. With VAL and /UNSAT you need to exhaustively check all interpretations.

Observation (Deciding LC / LE). With LOC /LE you need to exhaustively check all interpretations. With LE you need always to compare the truth values of the two components of the equivalence. With LC you need to check the second component only when the first evaluates to True.

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- **Reasoning problems – correlations**
- Key notions

Reasoning problems - Correlations

Theorem. If a formula is valid, then it is also satisfiable, and it is also not unsatisfiable. That is:

Validity implies **Satisfiability** equivalent to **not Unsatisfiability**

Theorem. If a formula is unsatisfiable, then it is also not satisfiable, and also not valid. That is:

Unsatisfiability equivalent to **not Satisfiable** implies **not Valid**

Example: Valid, Satisfiable or Unsatisfiable?

Prove that

- Blue Fomulas are valid,
- Magenta Formulas are satisfiable but not valid
- Red Formulas are unsatisfiable.

Satisfiable / Valid:

$A \supset A$

$A \vee \neg A$

$\neg \neg A \equiv A$

$\neg (A \wedge \neg A)$

$A \wedge \mathrm{B} \supset A$

$A \supset A \vee \mathrm{B}$

Satisfiable / Not Valid:

$A \vee B$

$A \supset B$

$\neg (A \vee B) \supset C$

Unsatisfiable / Not Valid:

$A \wedge \neg A$

$\neg (A \supset A)$

$A \equiv \neg A$

$\neg (A \equiv A)$

Reasoning problems - Correlations

Theorem. The validity, satisfiability and unsatisfiability of a formula and of its negation correlate as follows:

If A is	then ¬ A is
Valid	**Unsatisfiable**
Satisfiable	**Not Valid**
Not Valid	**Satisfiable**
Unsatisfiable	**Valid**

Reasoning problems - Correlations

Model checking (= entailment) (MC) is the core decision problem

Satisfiability (SAT) reduces to success in proving MC in (at least) one model

Validity (VAL) reduces to success in proving MC for all models

Unsatisfiability (UNSAT) reduces to failure in proving MC in all models

Reasoning Problem correlations - observations

Observation 1. Differently from Satisfiability, testing the holding of Validity or Unsatisfiability requires checking all the 2^n interpretations for success. With satisfiability this is only a worst case analysis (only one model, which is also the last to be selected).

Observation 2. For any finite set of formulas T, (i.e., T = $A_1, \ldots, A_n$ for some n ≥ 1), Γ is valid (respectively, satisfiable and unsatisfiable) if and only if $A_1 \wedge \ldots \wedge A_n$ (respectively, satisfiable and unsatisfiable)

Observation 3. All mainstream reasoning algorithms implement SAT and, to a lesser extent, UNSAT.

Entailment properties (NEW!)

Deduction theorem (Logical consequence, validity):

$$\Gamma, \phi \vDash \psi \text{ if and only if } \Gamma \vDash \phi \supset \psi$$

Observation 1: The deduction theorem explains (left to right) the meaning of implication. Implication is how we express logical consequence in language.

Observation 2: It also says (right to left) that from absurdity (i.e, $P \wedge \neg P$),we can derive everything, any formula (and assertion) A.

Entailment properties (NEW!)

Refutation principle (Logical consequence, unsatisfiability):

$$\Gamma \vDash \phi \text{ if and only if } \Gamma \cup \{\neg \phi\} \text{ is unsatisfiable}$$

Observation 1: The refutation principle explains the meaning of negation. It captures the fact that absurdity (i.e, $P \wedge \neg P$) cannot be satisfied by any model depicting facts in the real world.

Observation 2: Algorithmitically, it suggests how to reason backwards from goals.

Reasoning problem correlations – observations

Logical Consequence (LC). Two possibilities

- Use the deduction theorem to reduce LC to a VAL problem
- Use the refutation principle to reduce to an UNSAT problem

Logical Equivalence (LE) reduces to LC.

LoP – The Logic of Propositions

- Intuition
- Definition
- Domain
- Language
- Interpretation function
- Entailment
- The meaning of logical connectives
- Tell
- Ask – Reasoning problems
- Reasoning problems – correlations
- **Key notions**

Key notions

- LoP facts, LoP percepts
- Negation, disjointness, equivalence, conjunction, disjunction, implication
- Complete model, minimal model
- Model as interpretation
- Partiality as set of models
- Three step model building
- Model checking, SAT, UNSAT, VAL
- Logical consequence, logical equivalence
- Deduction theorem
- Refutation principle

LoP- The Logic of Propositions

Reasoning about what is True and what is False

(HP2T)