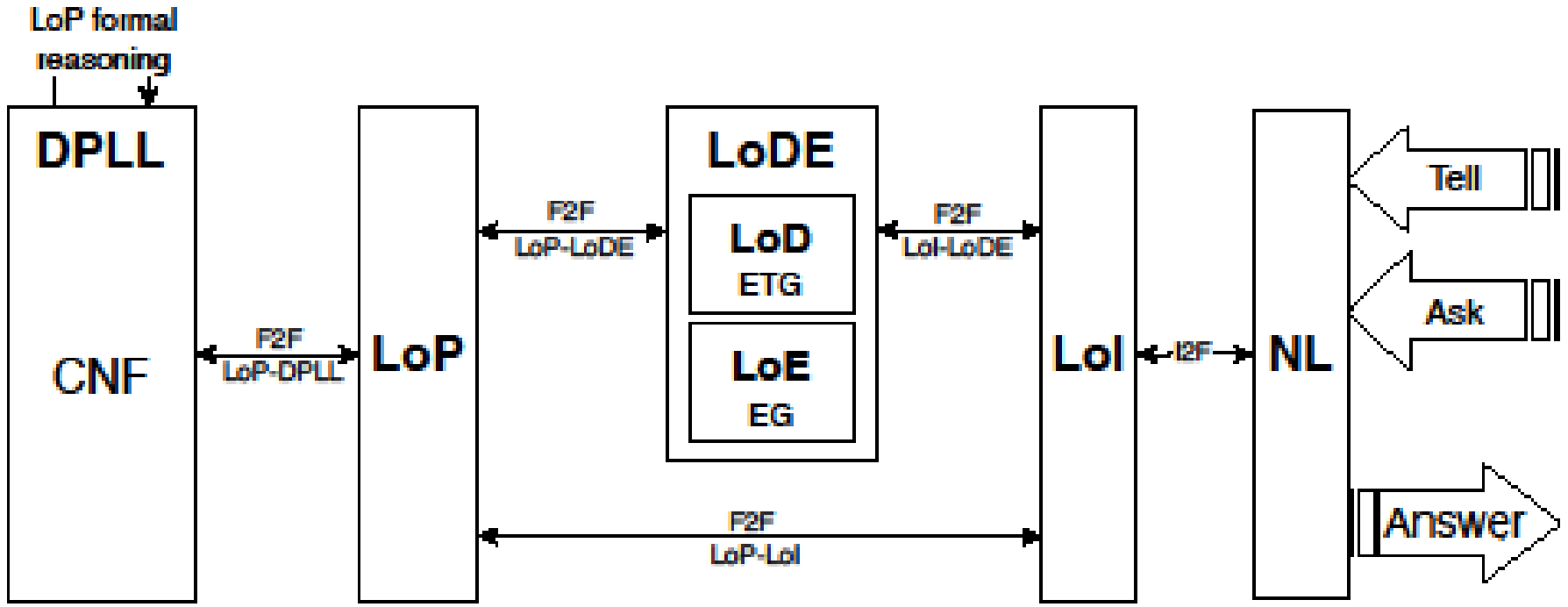
Reasoning about Propositions

The big picture (HP2T)

Reasoning about propositions

- **Intuition**
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

Reasoning about propositions – the big picture



Notation. NL: Natural (informal) Language. I2F: Informal to Formal. F2F: Formal to Formal

Reasoning about propositions

- Intuition
- **Normal Forms**
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

Normal Forms

Definition (Literal) A literal is either a proposition or the negation of a proposition, two examples being the formulas p, $\neg q$

Definition (Negative Normal Form (NNF). Any formula involving only disjunctions and conjunctions of literals. A CNF formula has the following generic shape:

$$(L_{<1,1>} \wedge \ldots \vee L_{<1,n_1>}) \wedge \ldots \vee (L_{<m,1>} \vee \ldots \vee L_{<m,n_m>})$$

Definition (Conjunctive Normal Form (CNF)). A conjunction of disjunctions of literals. A CNF formula has the following generic shape:

$$(L_{<1,1>} \vee \ldots \vee L_{<1,n_1>}) \wedge \ldots \wedge (L_{<m,1>} \vee \ldots \vee L_{<m,n_m>})$$

Definition (Disjunctive Normal Form (DNF). A disjunction of conjunctions of literals. A DNF formula has the following generic shape:

$$(L_{<1,1>} \wedge \ldots \wedge L_{<1,n_1>}) \vee \ldots \vee (L_{<m,1>} \wedge \ldots \wedge L_{<m,n_m>})$$

Addendum* - Normal forms

Observation (Complexity of NNF conversion) Transformation into NNF can increase the size of a formula only linearly. The number of occurrences of atomic formulas remains the same, the total number of occurrences of ∧ and ∨ is unchanged, and the number of occurrences of ¬ in the normal form is bounded by the length of the original formula**.**

Observation (Complexity of CNF / DNF conversion) Transformation into CNF / DNF can increase the size of a formula exponentially. The distributivity ∧ over ∨, and vice versa, of ∨ over ∧ doubles, respectively, the number of occurrences of ∧ and ∨**.**

Observation (Complexity of NNF reasoning). The same as LoP reasoning, We have NP complete and Co-NP complete problems.

Observation (Complexity of CNF reasoning). Deciding satisfiability of a CNF formula takes exponential in the worst case. It is an NP-complete proble(it could not be different because of the equivalence above). Deciding validity / unsatisfiability takes polynomial time. It is sufficient to parse the formulas. If there is one disjunct which does not contain $P \vee \neg P$ then the formula is not valid, it is valid otherwise.

Observation (Complexity of DNF reasoning). Deciding validity/ unsatisfiability of a DNF formula is exponential in the worst case. Deciding satisfiability takes polynomial time. It is sufficient to parse the formula. If there is one conjunct which does not contain $P \wedge \neg P$ then the formula is satisfiable, it is unsatisfiable otherwise.

Question (SAT reasoning). Why people translate in CNF and not in DNF being DNF exponentially simpler?

Reasoning about propositions

- Intuition
- Normal Forms
- **Conjunctive Normal Form (CNF)**
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

Conjunctive normal form (CNF)

Definition (Clause). A clause is a disjunction of literals, an example being the formula $(p \vee \neg q \vee r)$

Definition 3 (Conjunctive normal form (CNF)). A formula is in Conjunctive Normal Form (CNF), if it is a conjunction of clauses, an example being the formula

$$(p \vee \neg q \vee r) \wedge (q \vee r) \wedge (\neg p \vee \neg q) \wedge r$$

Conjunctive Normal Form (CNF) (Observations)

Observation (CNF). A CNF formula has the following shape:

$$(L_{<1,1>} \vee \ \ldots \ \vee L_{<1,n_1>}) \ \wedge \ \ldots \ \ \wedge \ (L_{<m,1>} \ \vee \ldots \vee \ L_{<m,n_m>})$$

Example (CNF, special cases). We have the following limit cases:

- $\{\}$
- p
- $\neg p$
- $p \wedge q \wedge r$
- $p \vee q \vee r$

Properties of clauses 1

Observation (Order of literals does not matter). If a clause is obtained by reordering the literals of a another clause, then the two clauses are equivalent.

Example (Order of literals does not matter).

$$(p \vee q \vee r \vee \neg r) \equiv (\neg r \vee q \vee p \vee r)$$

Observation (Order of literals does not matter). The order does not matter because of the commutativity of disjunction, that is:

$$\phi \vee \psi \equiv \psi \vee \phi$$

Properties of clauses 2

Observation (Multiple literals can be merged). If a clause contains more than one occurrence of the same literal then it is equivalent to the clause obtained by deleting all but one of these occurrences

Example (Multiple literals can be merged).

$$(p \vee q \vee r \vee q \vee \neg r) \equiv (p \vee q \vee r \vee \neg r)$$

Observation(Multiple literals can be merged). Multiple literals can be merged because of the absorption of disjunction, that is:

$$\phi \vee \phi \equiv \phi$$

Properties of clauses 3

Observation (Clauses as sets of literals). From these properties we can represent a clause as a set of literals, by leaving disjunction implicit and by ignoring replication and order of literals

Example (Clauses as sets of literals). The clause

$$(p \vee q \vee r \vee \neg r)$$

can be represented by the set

$$\{p, q, r, \neg r\}$$

Properties of CNF formulas 1

Observation (Order of clauses does not matter). If a CNF formula ϕ is obtained by reordering the clauses of a CNF formula ϕ' then ϕ and ϕ' are equivalent

Example (Order of clauses does not matter).

$$(p \vee q) \wedge (r \vee \neg q) \wedge (\neg q) \equiv (r \vee \neg q) \wedge (\neg q) \wedge (p \vee q)$$

Observation (Order of clauses does not matter). The order does not matter because of the commutativity of conjunction, that is:

$$\phi \wedge \psi \equiv \psi \wedge \phi$$

Properties of CNF formulas 2

Observation (Multiple clauses can be merged). If a CNF formula contains more than one occurrence of the same clause then it is equivalent to the formula obtained by deleting all but one of the duplicated occurrences

Observation (Multiple clauses can be merged).

$$(p \vee q) \wedge (r \vee \neg q) \wedge (p \vee q) \equiv (p \vee q) \wedge (r \vee \neg q)$$

Observation (Multiple clauses can be merged). Multiple clauses can be merged because of the absorption of conjunction, that is:

$$\phi \wedge \phi \equiv \phi$$

Properties of CNF formulas 3

Observation (A CNF formula can be seen as a set of clauses). A CNF formula can be represented as a set of sets of literals.

Observation (A CNF formula can be seen as a set of clauses). The following CNF formula

$$(p \vee q) \wedge (r \vee \neg q) \wedge (\neg q)$$

can be represented as

$$\{\{p, q\}, \{r, \neg q\}, \{\neg q\}\}$$

CNF formulas (main properties)

Proposition (Existence). Every LoP formula ϕ can be rewritten into Conjunctive Normal Form, written $CNF(\phi)$.

Proposition (Equivalence) $\models CNF(\phi) \equiv \phi$

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- **CNF reasoning**
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

Why DPLL SAT

- **LoP SAT**: SAT is a key property, as it amounts to checking whether a certain theory can be instantiated in practice (think, e.g., of scheduling);
- **NP-complete of LoP SAT**: all the NP-complete problems can be encoded in PL SAT;
- **Deduction theorem**: If $\Gamma, \phi \models \psi$ then $\Gamma \models \phi \supset \psi$, (with Γ possibly empty). This allows to reduce checking logical consequence to a PL SAT problem;
- **CNN SAT**: SAT can be reduced to CNF SAT (for Conjunctive Normal form, see later, i.e., conjunctions of disjunctions of possibly negated atomic propositions);
- **Efficiency**: CNF SAT can be implemented very efficiently by exploiting smart **heuristics** (e.g. strategies for selecting the "best" truth assignment);
- **Implementation**: State of the art SAT solvers, called CDCL (for Conflict-Driven Clause-Learning) solve industrial problem up to a few million atomic propositions;
- **Innovation**: Last year, IBM and Google announced a quantum computer solving SAT problems.

DPLL Highlights

- *Davis, Putnam. A Computing Procedure for Quantification Theory. Journal of the ACM, 7(3), 1960.*
- *Davis, Logemann, Loveland. A Machine Program for Theorem-Proving. Communications of the ACM, 1962*.
- Huge amount of work based on these ideas. In the literature you often find the acronym **DPLL** (for Davis, Putnam, Longemann, Loveland). Lots of work still on going, with applications in virtually all the domains where there is a need of automated reasoning (e.g., hardware and software verification, scheduling, planning, space, ..).
- **DPLL** is the *de-facto* reference standard for the implementation of SAT-based reasoning.
- http://minisat.se/: here you can find binaries, sources, documentation and projects related to **MiniSat**. MiniSat is a minimalistic, open-source DPLL based SAT solver. Released under the MIT licence

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - **CNF Satisfiability**
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

CNF satisfiability – Base

Proposition (CNF Satisfiability). Let

$$CNF(\phi) = C_0, \ldots C_n$$

be a formula in CNF,where $C_0, \ldots C_n$ are the clauses in CNF(ϕ). Then we have the following:

- $I \models \phi$ if and only if $I = C_i$ for all $i = 0 \ldots n$
- $I \models C_i$ if and only if for some literal $L \in C_i$, $I = L$

where I is an intepretation and L is a literal.

Observation (CNF Satisfiability). The above proposition says that a formula ϕ is satisfiable is all the clauses which occur in $CNF(\phi)$ are satisfiable. In turn a clause is satisfiable if the interpretation function (that is, the model) makes it true.

Partial evaluation

Observation (Satisfiability of a formula). To check if a model I satisfies a formula ϕ we do not need to know the truth values that I assigns to all the literals appearing in ϕ.

Example (Satisfiability of a formula). For instance, if $I(p) = \mathrm{T}$ and $I(q) = \mathrm{F}$, we can say that

$$I \models \{\{p, q, \neg r\}, \{\neg q, s\}\}$$

Definition (Partial evaluation). A partial evaluation is a partial function that associates to some propositions a truth value, but not to all of them.

Observation (Partial evaluation). Let I be a partial evaluation. Then the literals and clauses occurring in I can be true, false or undefined. We have the following four cases:

- A clause is true under I if at least one of its literals is true;
- A clause is false (or conflicting) if all literals are false;
- In all the other cases, a clause C is undefined (or unresolved).

A clause is left undefined when the truth value of its literals is irrelevant to the formula satisfiability.

Literal evaluation

Definition (Formula simplification by positive literal). For any CNF formula ϕ and proposition p, $\phi|_p$ stands for the formula obtained from ϕ by

- replacing all occurrences of p by the truth value $\top$ and
- by simplifying the result by removing:
  - the clauses containing the disjunctive term $\top$;
  - the literals $\neg\top = \bot$ in all remaining clauses.

Definition (Formula simplification by negative literal). For any CNF formula ϕ and proposition p, $\phi|_{\neg p}$ stands for the formula obtained from ϕ by

- replacing all occurrences of p by the truth value $\bot$ and
- by simplifying the result by removing:
  - the clauses containing the disjunctive term $\neg\bot = \top$;
  - the literals $\top$ in all remaining clauses.

Literal evaluation (example)

Example (Simplification of a formula by an evaluated literal). Consider the formula below

$$\{\{p, q, \neg r\}, \{\neg p, \neg r\}\}|_{\neg p} = \{\{q, \neg r\}\}$$

The second clause is verified because it contains ¬p which we assume to be true.

The first clause which contains p which can be eliminated. So we keep this clause, in a simplified form, and we try to verify it via the evaluation of the remaining literals.

CNF satisfiability – Final

Proposition (CNF satisfiability). Let ϕ be a formula with

$$CNF(\phi) = C_0, \ldots C_n$$

where $C_0, \ldots, C_n$ are the clauses in CNF(ϕ). Let us assume that we iterate the process of literal evaluation.
Then, the process will terminate with one of two possible situations:

- {}, that is, with an **empty set of clauses**, in which case ϕ is satisfiable;
- {…{}…}, that is, with a **non empty set of clauses containing one empty clause**, in which case ϕ is unsatisfiable.

CNF satisfiability – Observations

Observation (case {}). This situation arises when, within CNF(ϕ), all the clauses have been progressively eliminated because of multiple occurrences of $\top$. Clauses are eliminated when they are satisfied. Since no more clauses are left to satisfy, then ϕ is satisfiable;

Observation (case {…{}…}). This situation arises when, within one clause in CNF(ϕ), all the literals have been progressively eliminated because of multiple occurrences of $\bot$. If one clause becomes empty then there is no literal which can be used to make it satisfiable. There the clause and ϕ are unsatisfiable;

Observation (Termination). The process is guarnteed to terminate, independently of the order of selection of the literals being evaluated. In the worst case all the possible assignments will have to be tested

CNF satisfiability (example)

Example. Check the satisfiability of the following formula

$$(\neg p \vee q) \wedge (\neg r \vee q)$$

1. $(\neg p \vee q) \wedge (\neg r \vee q)$
2. $\{\{\neg p, q\}, \{\neg r, q\}\}$
3. $\{\{\neg p, \top\}, \{\neg r, \top\}\}|_q$
4. $\{\}$

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - **DPLL**
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

DPLL decision procedure – base

Algorithm DPLL

Input: $\varphi = \{c_1, \ldots, c_n\}$.

Output: I.

Call DPLL(φ, {})

DPLL(φ, I)

if {} $\in \varphi$

then exit-return {} **end**

if φ = {}

then return I **end**

L← select-literal(φ);

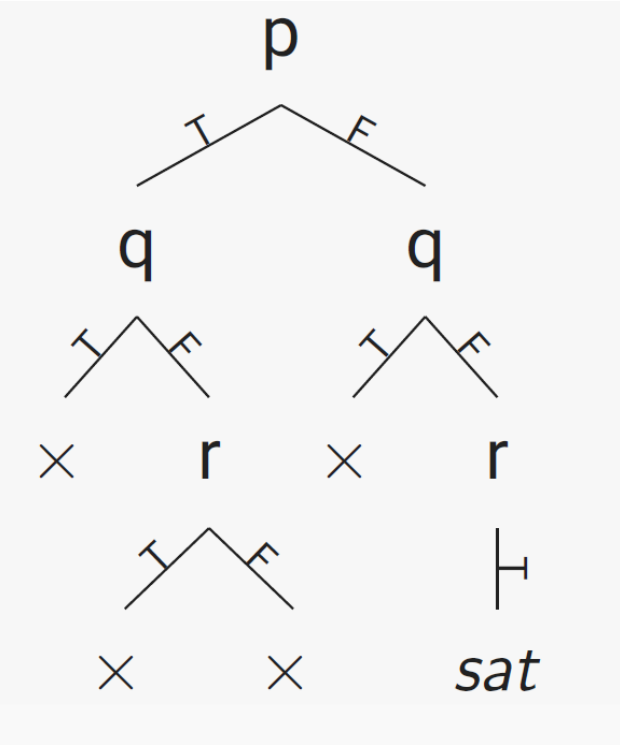
DPLL($\varphi|_L$, $I \cup \{L\}$) **or DPLL**($\varphi|_{\neg L}$, $I \cup \{\neg L\}$)

Backtracking in DPLL (example)

Observation (backtracking). The selection of the literal is heuristic. As such, it may lead to a dead-end and generate a need for **backtracking**. Lots of research is focused on advanced heuristics.

Example (Backtracking). Consider the following formula:

$$(p \vee \neg q) \wedge (p \vee r) \wedge (\neg p).$$



The search for an assignment can be represented by the following tree

Unit propagation – enhancement 1

Definition (Unit clause). A clause $C = \{L\}$ that consists of a single literal L, is a unit clause.

Proposition (Unit clause). A formula ϕ containing a unit clause {L} is satisfiable only if L is evaluated to T.

Observation (Unit clause). Testing the satisfiability of a Unit clause require testing one, instead of two, truth assignments to L.

Unit propagation – example

Example (Unit propagation). Consider the CNF formula

$$\varphi = \{\{p\}, \{\neg p, \neg q\}, \{\neg q, r\}\}.$$

Check whether φ is satisfiable and find an interpretation I such that $I \models \varphi$.

1. $\{\{p\}, \{\neg p, \neg q\}, \{\neg q, r\}\}$
2. $\{\{p\}, \{\neg p, \neg q\}, \{\neg q, r\}\} \;|_p$
3. $\{\{\top\}, \{\bot, \neg q\}, \{\neg q, r\}\}$
4. $\{\{\neg q\}, \{\neg q, r\}\}$
5. $\{\{\neg q\}, \{\neg q, r\}\} \;|_{\neg q}$
6. $\{\{\top\}, \{\top, r\}\}$
7. $\{\}$

φ is satisfiable, $I = \{p, \neg q\}$. The literal r is left undefined, an example of partial evaluation.

Unit propagation – enhancement 1

Observation (Unit propagation). Assume that we have a unit clause in the input formula. How would you modify the algorithm produced in the previous step to take into account this situation. When do you check this information?

Observation (Unit propagation). Consider the following examples

1. $(p \supset q \supset r) \wedge p \wedge \neg q$
2. $(p \wedge q) \vee \neg p \supset r$
3. $(p \wedge r) \vee (\neg q \wedge p) \vee (\neg r \wedge \neg p)$

Execute DPLL first without and then with your modification. Then compute how many iterations you saved

Pure literal – enhancement 2

Observation (Pure literal). Assume that a literal occurs only positively or only negatively. How would you modify the algorithm produced in the previous step to take into account this situation. When do you check this information?

Example (Pure literal). Consider the following examples

1. $(p \supset q \supset r) \wedge p \wedge q$
2. $((p \wedge q) \supset r) \wedge (p \supset r)$
3. $(p \wedge \neg r) \vee (q \wedge p) \vee (\neg r \wedge q)$

Execute DPLL first without and then with your modification. Then compute how many iterations you saved

Literal counting – enhancement 3

Observation (Literal counting). Assume that you count the number of time each single literal occurs in a formula. How would you modify the algorithm produced in the previous step to take into account this additional information? When do you compute this information? Is it guaranteed to improve performance?

Exercise (Literal counting). Consider the following examples

1. $(p \supset q \supset r) \wedge p \wedge \neg q$
2. $(p \wedge q) \vee \neg p \supset r$
3. $(p \wedge r) \vee (\neg q \wedge p) \vee (\neg r \wedge \neg p)$

Execute DPLL first without and then with your modification. Then compute how much iterations you saved

DPLL decision procedure – final

DPLL(φ, I**)**

if {} ∈ φ **then exit** {} **end**;

if φ = {} **then exit-return** I **end**;

while {L} ∈ φ

  do φ ← **DPLL**(unit-propagate (φ|$_L$, I∪{L}) **end**;

while pure(L) **and** {L} ∈ φ

  do φ ← **DPLL**(pure-literal-assign (φ|$_L$, I∪{L}) **end**;

L← select-literal(φ);

DPLL(φ|$_L$, I∪{L} or **DPLL**(φ|$_{¬L}$,I∪{¬L})

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- **LoP to CNF**
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

Conversion to CNF (1)

Definition 4 (The CNF function) Given a PL formula ϕ the function CNF, which transforms ϕ in its CNF form, called CNF(ϕ) is recursively defined as follows:

CNF(p)	=	p if p ∈ **PROP**
CNF(¬ p)	=	¬ p if p ∈ **PROP**
CNF($\phi \supset \psi$)	=	CNF(¬ϕ) ⊗ CNF(ψ)
CNF($\phi \wedge \psi$)	=	CNF(ϕ) ∧CNF(ψ)
CNF($\phi \vee \psi$)	=	CNF(ϕ) ⊗ CNF(ψ)
CNF($\phi \equiv \psi$)	=	CNF($\phi \supset \psi$) ∧ CNF($\psi \supset \phi$)
CNF(¬¬ϕ)	=	CNF(ϕ)
CNF(¬($\phi \supset \psi$))	=	CNF(ϕ) ∧ CNF(¬ψ)
CNF(¬($\phi \wedge \psi$))	=	CNF(¬ϕ) ⊗ CNF(¬ψ)
CNF(¬ ($\phi \vee \psi$))	=	CNF(¬ϕ) ∧ CNF(¬ψ)
CNF(¬ ($\phi \equiv \psi$))	=	CNF($\phi \wedge \neg\psi$) ⊗ CNF($\psi \wedge \neg\phi$)

... see next page

Conversion to CNF (2 – continued)

… where

$$(C_1 \wedge \ldots \wedge C_n) \otimes (D_1 \wedge \ldots \wedge D_m) \qquad (*)$$

is defined as:

$$(C_1 \vee D_1) \wedge \ldots \wedge (C_1 \vee D_m) \wedge \ldots \wedge (C_n \vee D_1) \wedge \ldots \wedge (C_n \vee D_m) \qquad (**)$$

with C_i being a conjunction (possibly a single formula) and D_j being a disjunction (possibly a single formula).

Example (special cases). Rewrite the following special cases of (*) into their corresponding formulas (**)

- Single formula conjuncts: $(a \wedge b) \otimes (D_1 \wedge \ldots \wedge D_m)$
- Single formula disjuncts: $(C_1 \wedge \ldots \wedge C_n) \otimes (a \wedge b)$
- Single formula conjuncts and disjuncts: $(a \wedge b) \otimes (c \wedge d)$

Conversion to CNF (example)

Example (CNF conversion). Compute the CNF of $(q \wedge p) \vee \neg p$

$$
\begin{aligned}
&CNF\,((q \wedge p) \vee \neg p) = \\
&CNF\,((q \wedge p)) \otimes CNF\,(\neg p) = \\
&(CNF\,(q) \wedge CNF\,(p)) \otimes \neg p = \\
&(q \wedge p) \otimes \neg p = \\
&(q \vee \neg p) \wedge (p \vee \neg p)
\end{aligned}
$$

CNF conversion (example)

Example (Exponential explosion of a CNF conversion). Try computing the CNF of the following formula

$$p1 \equiv (p2 \equiv (p3 \equiv (p4 \equiv (p5 \equiv p6)))).$$

The formula resulting from the first conversion step is:

$$CNF(p1 \supset (p2 \equiv (p3 \equiv (p4 \equiv (p5 \equiv p6))))) \wedge CNF((p2 \equiv (p3 \equiv (p4(p5 \equiv p6)))) \supset p1)$$

This formula is double the length of the previous formula. Continuing the expansion, the formula will keep growing exponentially.

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- **LoI to LoP**
  - **Finite Predicate extension**
  - Quantifier elimination
- LoI to ALC / LoDE
- Key notions

Finite predicate extension

Definition (Finite predicate extension). A predicate has a finite extension when it holds is true only for a finite set of constants, that is,

$$\forall x.(P(x) \equiv (x = c_1 \vee \ldots \vee x = c_n))$$

Example (Finite predicate extension). The days of the week are: Monday, Tuesday, ..., Sunday. This can be formalized as the following formula.

$$\forall x.(WeekDay(x) \equiv x = Mon \vee x = Tue \vee \ldots \vee x = Sun))$$

Observation (Finite predicate extension). This property allows to eliminate a universal quantifier reducing a quantified formula into a ground formulas which can be then reasoned about in LoP. It applies always, also with infinite domains.

Observation (Finite predicate extension in AI). When defining data properties, many predicates have this property, all those which enumerate possible values, for instance: phoneNumber, Age, height, distance. Some approximation may be needed.

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - **Quantifier elimination**
- LoI to ALC / LoDE
- Key notions

Finite domains

Definition (Finite domain). A domain is said to be **finite** if it contains a finite amount of units.

Observation (Finite domains, in AI). Most AI applications have finite domains. The only issue is a careful management of data types. The trick is, in the translation from LoDE to LoP, to translate the single values of data properties. As before, some approximation maybe necessary. This applies in particular to the modeling of space and time.

Observation (Finite domains, in CS). It is a property which is quite extensively exploited in formal methods, as it allows to validate systems using model checking and satisfiability. It does not generalize to programs with loops and recursion.

Quantifier expansion

Proposition (Quantifier expansion). With finite domains, a quantified LoI formula can be grounded to an **equivalent** LoP formula based on the following equivalence:

$$\phi_\Delta = \{c_1, \ldots, c_n\} \models \forall x.\, \phi(x) \quad \equiv \phi(c_1) \wedge \ldots \wedge \phi(c_n)$$

$$\phi_\Delta = \{c_1, \ldots, c_n\} \models \exists x.\, \phi(x) \quad \equiv \phi(c_1) \vee \ldots \vee \phi(c_n)$$

Observation (Quantifier expansion). A universally (existentially) quantified formula can be substituted with a conjunction (disjunction) with as many conjuncts (disjuncts) as there are constants in the language.

Observation (Quantifier expansion). The LoP search space grows exponentially with the size of the domain.

Quantifier expansion – example

Example (Quantifier expansion). The Natural Language formula

"Everybody living in Toulouse speaks French or Spanish"

Is translated in FOL as the sentence

$$\forall x.(Lives(x, Toulouse) \supset (Speaks(x, French) \vee Speaks(x, Spanish)))$$

Assuming that the people living in Toulose a3e three

$$D = \{Robert, Luis, Naji\}$$

we can ground the above formula as:

$$(Lives(Robert,Toulouse) \supset Speaks(Robert, French) \vee Speaks(Robert, Spanish)) \wedge$$
$$(Lives(Luis,Toulouse) \supset Speaks(Luis, French) \vee Speaks(Luis, Spanish)) \wedge$$
$$(Lives(Naji,Toulouse) \supset Speaks(Naji, French) \vee Speaks(Naji, Spanish))$$

Multi – quantifier expansion

Definition (Multi-quantifier expansion). With finite domains, a LoI formula with nested quantifiers can be grounded to an **equivalent** LoP formula based on the following equivalence:

$$\phi_D = \{c_1, \ldots, c_n\} \models \forall x_1, \ldots, x_k.\ \phi(x_1, \ldots, x_k) \equiv \bigwedge_{c_{i,1},\ldots,c_{i,k} \in \{c_1,\ldots,c_n\}} \phi(c_{i,1}, \ldots, c_{i,k})$$

$$\phi_D = \{c_1, \ldots, c_n\} \models \exists x_1, \ldots, x_k.\ \phi(x_1, \ldots, x_k) \equiv \bigvee_{c_{i,1},\ldots,c_{i,k} \in \{c_1,\ldots,c_n\}} \phi(c_{i,1}, \ldots, c_{i,k})$$

Observation (Multi-quantifier expansion). A universally (existentially) quantified formula can be substituted with a conjunction (disjunction) with as many conjuncts (disjuncts) **as there are combinations of constants in the language** (all possible combinations).

Observation (Explosion of quantifier expansion). Grounding formulas is very expensive because of the combinatorial explosion of all the possible configurations. There are n^m combinations with n the number of variables, and m the size of the domain.

Quantifier expansion – example

Example (Quantifier expansion) The Natural Language formula

"If someone is noisy, everyone is annoyed"

is translated in LoI as the sentence

$$\exists x.(Noisy(x) \supset \forall y.(Annoyed(y))).$$

Assuming that the there are three people, that is

D={Marco, Francesco, Pierre}

we can ground the above formula as:

(Noisy(Marco) ⊃ (Annoyed(Marco) ∧ Annoyed(Francesco) ∧ Annoyed(Pierre))) **∨**
(Noisy(Francesco) ⊃ (Annoyed(Marco) ∧ Annoyed(Francesco) ∧ Annoyed(Pierre))) **∨**
(Noisy(Pierre) ⊃ (Annoyed(Marco) ∧ Annoyed(Francesco) ∧ Annoyed(Pierre)))

Explosion of quantifier expansion – example

Example (Explosion of quantifier expansion)

$$\phi = \forall x.\forall y.(Man(x) \supset Mortal(y))$$

$$D = \{Socrate, Plato, Senofonte\}$$

$$\begin{aligned}\phi' = {} & (Man(Socrate) \supset Mortal(Socrate)) \wedge (Man(Plato) \supset Mortal(Plato)) \wedge \\ & (Man(Socrate) \supset Mortal(Plato)) \wedge (Man(Plato) \supset Mortal(Socrate)) \wedge \\ & (Man(Socrate) \supset Mortal(Senofonte)) \wedge (Man(Plato) \supset Mortal(Senofonte)) \wedge \\ & (Man(Senofonte) \supset Mortal(Plato)) \wedge (Man(Senofonte) \supset Mortal(Socrate)) \wedge \\ & (Man(Senofonte) \supset Mortal(senofonte))\end{aligned}$$

Observation (Explosion of quantifier expansion). We have $3^2=9$ conjuncts. This explosion could have been limited substantially by pushing the two quantifiers to bind the smallest possible formula. This fact is usually exploited when reducing LoI formulas to LoP formulas.

UNIVERSITY OF TRENTO - Italy

Dipartimento di Ingegneria e Scienza dell'Informazione

Know dive

Quantifier expansion – exercise

Exercise Let {a,b,c} be three constants in a FOL language L. Let T be a theory defined as:

$$T = \{\forall x.y\,.P(x, y)\}$$

With reference to a model M of T with a finite domain containing only two elements, which of the following formulas are true in M if and only if T is true in M?

1. $P(a, a) \wedge P(b, b) \wedge P(c, c)$
2. $(P(a, a) \wedge P(b, b)) \vee (P(a, a) \wedge P(c, c)) \vee (P(b, b) \wedge P(c, c))$
3. $P(a, a) \wedge P(a, b) \wedge P(a, c) \wedge P(b, a) \wedge P(b, b) \wedge P(b, c) \wedge P(c, a) \wedge P(c, b) \wedge P(c, c)$
4. $P(a, a) \wedge P(a, c) \wedge P(c, a) \wedge P(c, c)$
5. $P(b, b) \wedge P(b, c) \wedge P(c, b) \wedge P(c, c)$
6. $P(b, b) \wedge P(b, c) \wedge P(c, b) \wedge P(c, c) \wedge \neg P(a, c)$

UNIVERSITY OF TRENTO - Italy

Dipartimento di Ingegneria e Scienza dell'Informazione

Know dive

Quantifier expansion – exercise

Exercise. Say which of the following interpretation functions are models of the formula

$$(\forall x.(P(x) \supset Q(x)) \wedge \forall x.P(x)) \supset (Q(b) \vee Q(c))$$

1. D={0,1,2} , I(c)={0}, I(b)={1}, I(P)={0,1,2}, I(Q)= {0,1,2}
2. D={0,1,2} , I(c)={0}, I(b)={0}, I(P)={0}, I(Q)= {0}
3. D={0,1,2} , I(c)={1}, I(b)={1}, I(P)={0}, I(Q)= {0}
4. D={0,1,2} , I(c)={0}, I(b)={1}, I(P)={2}, I(Q)= {0,1}
5. D={0,1,2} , I(c)={0}, I(b)={1}, I(P)={0,1,2}, I(Q)= {1}
6. D={} , I(c)={}, I(b)={}, I(P)={}, I(Q)= {}

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- **LoI to ALC / LoDE**
- Key notions

ALC translation to LoI – phase 1

Definition (LoI translation to LoE/LoD/LoDE). Consider the following two translation functions.

$$\Pi_x(A) = A(x) \qquad \Pi_y(A) = A(y)$$

$$\Pi_x(C \sqcap D) = \Pi_x(C) \wedge \Pi_x(D) \qquad \Pi_y(C \sqcap D) = \Pi_y(C) \wedge \Pi_y(D)$$

$$\Pi_x(C \sqcup D) = \Pi_x(C) \vee \Pi_x(D) \qquad \Pi_y(C \sqcup D) = \Pi_y(C) \vee \Pi_y(D)$$

$$\Pi_x(\exists r.C) = \exists y.r(x,y) \wedge \Pi_y(C) \qquad \Pi_y(\exists r.C) = \exists x.r(y,x) \wedge \Pi_x(C)$$

$$\Pi_x(\exists r.C) = \forall y.r(x,y) \supset \Pi_y(C) \qquad \Pi_y(\exists r.C) = \forall x.r(y,x) \supset \Pi_x(C)$$

Observation (ALC vs LoE/LoD/LoDE). ALC is an extension of LoE/LoD/LoD. It is the world logic which is deductiovaly equivalent to LoP.

Observation (LoI translation to LoE/LoD/LoDE). The intuition is to add the variable implicit in LoD formulas thus aligning it to the language of LoE/ LoI.

ALC translation to LoI – phase 2

Let T and A be a TBox and an Abox, respectively. Let φ[x/a] the formula obtained from φ by replacing all free occurrences of x with a. Then, the translation Π(T) and Π(A) of the TBox and the ABox are, respectively:

$$\Pi(T) = \forall x. \bigwedge_{C \sqsubseteq D \in T} (\Pi_x(C) \supset \Pi_x(D))$$

$$\Pi(A) = \bigwedge_{C(a) \in A} \Pi_x(C)\,[x/a] \quad \wedge \quad \bigwedge_{r(a,b) \in A} r(a,b)$$

Observation (mapping). The mapping preserves the intended semantics.

ALC vs. LoI – Main result

Theorem (LoE/LoD/LoDE vs. LoI). Let T be a TBox, A an ABox and EB= (T,A) be a LoDE theory. Let C, D possibly composite etypes and e an entity. Then we have the following:

- EB = (T,A) is satisfiable if and only if

$$\Pi(T) \wedge \Pi(A) \text{ is satisfiable;}$$

- $C \sqsubseteq_T D$ if and only if

$$(\Pi(T) \supset \forall x.(\Pi_x(C) \supset \Pi_x(D)) \text{ is valid;}$$

- b is an instance of C with respect to EB = (T,A) if and only if

$$(\Pi(T) \wedge \Pi(A)) \supset \Pi_x(C)\,[x/a] \text{ is valid;}$$

ALC vs LoI – Example

Example (LoDE to LoI). Consider the following LoDE theory

$$\exists \text{attends}.T \sqsubseteq \text{Person}$$

$$\text{Teacher} \equiv \text{Person} \sqcap \exists \text{teaches}.\text{Course}$$

$$\text{Teacher}(\text{Mary})$$

which makes the following assertions

«Whoever attends classes is a person»

«Teachers are persons who teach course»

«Mary is a teacher»

ALC vs LoI – Example

Example (LoDE to LoI). The following LoDE theory

$$\exists \text{attends}.T \sqsubseteq \text{Person}$$
$$\text{Teacher} \equiv \text{Person} \sqcap \exists \text{teaches}.\text{Course}$$
$$\text{Teacher}(\text{Mary})$$

can be translated into the following LoI theory.

$$\forall x.\left(\exists y.\,\text{attends}(y,x) \supset \text{Person}(x)\right)$$
$$\forall x.\,(\text{Teacher}(x) \equiv \text{Person}(x) \wedge \exists y.(\text{teaches}(x,y) \wedge \text{Course}(y)))$$
$$\text{Teacher}(\text{Mary})$$

ALC vs LoI – Observations

Example (LoDE to LoI). Consider the following translation. From the LODE formula

$$\exists \text{attends}.T \sqsubseteq \text{Person}$$

to the LoI formula.

$$\forall x.\left(\exists y.\,\text{attends}(y,x) \supset \text{Person}(x)\right).$$

Observation (LoDE to LoI). The above example is to be read as follows:

- Following their set-theoretic interpretation, etypes, possibly constructed via properties, are to be intended universally quantified formulas;
- Set subsumption maps to implication, in the opposite direction. That is, any member of a subset has all the properties associated to the superset. Therefore, makes true all the propositions made tru by the elements of the superset.

ALC vs LoI – Observations

Example (LoDE to LoI). Consider the following translation. From the LODE formula

$$\text{Teacher} \equiv \text{Person} \sqcap \exists \text{teaches.Course}$$

to the LoI formula.

$$\forall x.\,(\text{Teacher}(x) \equiv \text{Person}(x) \;\wedge \exists y.(\text{teaches}(x,y) \;\wedge\; \text{Course}(y))).$$

Observation (LoDE to LoI). The above example is to be read as follows:

- Set equivalence maps into equivalence. That is, the members of two sets which are coincident make true exactly the same propositions.
- Set intersection maps to conjunction. That is, the members of two sets satisfy the properties of both sets. The dual applies to set union
- A set theoretic existential relation maps to an existential statement which binds any element which satisfies the set on which the relation applies.

ALC vs LoI – Observations

Example (LoDE to LoI). Consider the following translation. From the LODE formula

Teacher(Mary)

to the LoI formula.

Teacher(Mary).

Observation (LoDE to LoI). The above example is to be read as follows:

- LoDE assertions encode exactly the same information which is encoded by LoI atomic formulas, where LoI allows for a much larger set of assertions, in particular moving from binary properties to n-ary relations and functions.

Reasoning about propositions

- Intuition
- Normal Forms
- Conjunctive Normal Form (CNF)
- CNF reasoning
  - CNF Satisfiability
  - DPLL
- LoP to CNF
- LoI to LoP
  - Finite Predicate extension
  - Quantifier elimination
- LoI to ALC / LoDE
- **Key notions**

DPLL – key notions

- Normal forms: NNF, CNF, DNF
- Literal, clause, CNF
- Conversion to CNF
- CNF satisfiability
- Partial evaluation
- Literal evaluation
- DPLL decision procedure
- Backtracking
- Unit propagation
- Pure literal
- Literal counting

- Finite predicate extension
- Finite domain
- Quantifier expansion
- Nested quantifier expansion
- Search space explosion of (nested) quantifier expansion
- Literal evaluation
- DPLL decision procedure
- Backtracking
- Unit propagation
- Pure literal
- Literal counting

Reasoning about Propositions

The big picture (HP2T)

Mapping Logics

1. Intuition
2. LoD definitions from / to LoP (* Language level translation *)
3. LoD defs+descrs from / to LoP (* Knowledge level translation *)
4. LoDE from / to LoP (* Data level translation *)
5. ALC from / to LoP (* Data level translation *)
6. LoI from / to LoDE (* Data level translation *)
7. LoI from / to LoD Full (* Knowledge level + prop. translation *)
8. LoI from / to ALC (* Data level + prop translation *)