**3.2.1 *Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions.***

***Using basic definition of $\Theta-$notation, prove that max$\{f(n), g(n)\}$***

*Since $f(n)$ and $g(n)$ are asymptotically nonnegative functions $\exists\ n_0$ and $n_1$ such that*

$n_2 = \max\{n_0, n_1\} \to f(n) \geq 0$ *and* $g(n) \geq 0;\ \forall n > n_2$

$$let\ h(n) = \max\{f(n), g(n)\} \to \begin{cases} h(n) = f(n), if\ f(n) \geq g(n) \\ h(n) = g(n), if\ g(n) \geq f(n) \end{cases}$$

$\Theta\big(f(n) + g(n)\big) = \{t(n); \exists\ c_1,\ c_2, and\ n_0 > 0\ such\ that\ 0 \leq c_1[f(n) + g(n)] \leq t(n)$

$\leq c_2[f(n) + g(n)]\ \forall\ \ n \geq n_0\}$

$h(n) \in \Theta\big(f(n) + g(n)\big)$ *if we find $c_1, c_2$ and $n_0$ such that:*

$c_1[f(n) + g(n)] \leq h(n) \leq c_2[f(n) + g(n)]$

*if we take $n_0$ as $n_2$ we know that $h(n) = f(n)$ or $g(n)$. So $h(n) \leq f(n) + g(n)\ \forall n \geq n_0$,*

*so take $c_2 = 1$*

*we know that* $0 \leq f(n) \leq h(n)$ *and* $0 \leq g(n) \leq h(n) \to 0 \leq \frac{1}{2}\big(f(n) + g(n)\big)$

$\leq h(n)$ *so take* $c_1 = \frac{1}{2}\ \therefore$ ***QED***

---

**3.2.2 *Explain why the statement, The running time of algorithm A is at least***

***$O(n^2)$ is meaningless***

*We have* : $O\big(g(n)\big) = \{f(n), \exists\ n_0\ and\ c_1 \geq 0\ such\ that\ 0 \leq\ f(n) \leq c_1 g(n)\ \forall\ n \geq n_0\}$.

*We have an Upper Bound, so the statment would mean*

"The running time of Algorithm A is at least at maximun $\mathrm{O}(\mathrm{n}^2)$ *so meaningless*"

---

**3.2.3 *Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?***

*case* $2^{n+1}$:

*if so, there $\exists\ c_1$ and $n_0$ such that $\forall n \geq\ n_0$ we have* $0 \leq 2 \cdot 2^n \leq c_1 2^n$ *. Take $c_1$*

$= 2$ *and* $n_0 = a$, *where* $a > 0$. ***TRUE***

*case* $2^{2n}$:

*if so, there $\exists\ c_1$ and $n_0$ such that $\forall n \geq\ n_0$ we have* $0 \leq 2^n \cdot 2^n \leq c_1 2^n \to 0 \leq\ 2^n$

$\leq c_1$. ***False*** *since*: $\lim\limits_{n\to\infty} 2^n = \infty$ . ***FALSE***

---

**3.2.4 *Prove the following***

"*For any two functions* $f(n)$ *and* $g(n)$, *we have* $f(n) = \Theta(g(n))$ *if and only if* $f(n) = O(g(n))$ *and* $f(n) = \Omega(g(n))$"

*case* 1:

*If* $f(n) = \Theta(g(n))$ *then* $\exists\, c_1, c_2$ *and* $n_0 > 0$ *such that* $\forall\, n \geq n_0$ *we have* $0 \leq c_1 g(n) \leq f(n) \leq c_2(g(n))$.

*case* 2:

*If* $f(n) = O(g(n))$ *and* $f(n) = \Omega(g(n))$ *then* $\exists\, c_1$ *and* $n_0$ *such that* $0 \leq c_1 g(n) \leq f(n); \forall n \geq n_0$ *and* $\exists c_2$ *and* $n_1$ *such that* $0 \leq f(n) \leq c_2 g(n)\ \forall\, n \geq n_1$. *Choose now* $n_0^* = \max\{n_0, n_1\}$ *and* $c_1^* = c_1$ *and* $c_2^* = c_2$

---

**3.2.5 *Prove that the running time of al algorithm is* $\Theta(g(n))$ *if and only if its worst case running time is* $O(g(n))$ *and its best case running time is* $\Omega(g(n))$**

*Case* 1:

*Let* $T(n)$ *be the running time of an algorithm and* $W(n)$ *is the worst running case and* $B(n)$ *is the best running case. We know* $W(n) = O(g(n))$ *and* $B(n) = \Omega(g(n))$ *so there* $\exists c_1, n_0, n_1,$ *and* $c_2 > 0$ *such that* $0 \leq W(n) \leq c_1 g(n)\ \forall n \geq n_0$ *and* $0 \leq c_2 g(n) \leq \mathrm{B(n)};$ $\forall n \geq n_1$. *Take* $n_2 = \max\{n_1, n_0\} \rightarrow 0 \leq c_2 g(n) \leq \mathrm{B(n)} \leq \mathrm{T(n)} \leq W(n) \leq c_1 g(n)$ $\forall n \geq n_2 \rightarrow \boldsymbol{T(n) = \Theta(g(n))}$

*Case* 2:

*The worst case and best case running time are part of the running time function, if it* $\in \Theta(g(n))$ *then we know that* $W(n) = O(g(n))$ *and* $B(n) = \Omega(g(n))$

---

**3.2.6 *Prove that* $o(g(n)) \cap \omega(g(n))$ *is the empty set***

$o(g(n)) = \{f(n); \textit{for any } c > 0\ \exists\, n_0 > 0 \textit{ such that } 0 \leq f(n) < cg(n)\ \forall\, n \geq n_0\}$

$\omega(g(n)) = \{t(n); \textit{for any } c_1 > 0\ \exists\, n_1 > 0 \textit{ such that } 0 \leq c_1 g(n) < f(n)\ \forall\, n \geq n_1\}$

*Supose Exists a function* $J(n)$ *such that* $J(n) \in o(g(n))$ *and* $\in \omega(g(n))$ *so for any* $c$ *and* $c_1$[*so if we take* $c_1 = c$ *for our choices* $> 0$ *we would have*: $0 \leq cg(n) < f(n) < cg(n)\ \forall\, n \geq n_2$, *such that*: $n_2 = \max\{n_0, n_1\}$. *What is Impossible*

---

$\mathbf{3.2.7}$ ***We can extend our notation to the case of two parameters n and m that can go to $\infty$ independlty at different rates. For a given function $g(n,m)$, we deone by $O(g(n,m))$ the set of functions $O(g(n,m)) = \{f(n,m); \exists\, c, n_o$ and $m_0 > 0$ such $0 \le f(n,m) \le cg(n,m)\ \forall n \ge n_0$ or $m \ge m_0\}$ Give corresponding definitions for $\Omega(g(n,m))$ and $\Theta(g(n,m))$***

*Similar definitions would be*:

$\Omega(g(n,m)) = \{f(n,m)$: *there exists positive constants*, $c, n_0$ *and* $m_0$ *such that* $0 \le cg(n,m) \le f(n,m))\forall\, n \ge 0$ *or* $m \ge 0\}$

$\Theta(g(n,m)) = \{f(n,m)$: *there exists positive constants*, $c, n_0$ *and* $m_0$ *such that* $0 \le cg(n,m) \le f(n,m)) \le cg(n,m)\forall\, n \ge 0$ *or* $m \ge 0$

---

$\mathbf{3.3.1}$ ***Show that if $f(n)$and $g(n)$are monotonically increasing functions, then so are the functions $f(n) + g(n)$and $f(g(n))$and if f and g are also, nonnegative $f(n) \cdot g(n)$is monotonically increasing***

*Let* $h(n) = f(n) + g(n)$*and* $L(n) = f(g(n))$

*if* $f(n)$ *and* $g(n)$*are monotonically increasing, then if*

$n_1 \ge n_2 \to f(n_1) \ge f(n_2)$ *and* $g(n_1) \ge g(n_2) \to f(n_1) + g(n_1) \ge f(n_2) + g(n_2) \to h(n_2) \ge h(n_1)$. *Hence* $f(n) + g(n)$ *is monotonically increasing*

*if* $n_1 \ge n_2 \to g(n_1) \ge g(n_2) \to f(g(n_2)) \ge f(g(n_1))$.

*Hence* $f(g(n))$*is monotonically increasing*

*if* $n_1 \ge n_2 \to g(n_1) \ge g(n_2) \ge 0$ *and* $f(n_1) \ge f(n_2) \ge 0 \to fg(n_1) \ge fg(n_2)$;

*Hence* $f(g(n))$*is monotonically increasing*

---

$\mathbf{3.3.2}$ ***Prove that $\lfloor \alpha n \rfloor + \lceil 1 - \alpha n \rceil = n$ for any integer n and real number $\alpha$***

*We just need to use the following properties of floor and ceiling functions*:

$-\lfloor x \rfloor = \lceil x \rceil$ *and* $\lceil n + x \rceil = n + \lceil x \rceil$, *where* $x \in R$ *and* $n \in N$

$\lfloor \alpha n \rfloor + \lceil (1-\alpha)n \rceil = \lfloor \alpha n \rfloor + \lceil n - \alpha n \rceil = \lfloor \alpha n \rfloor + n + \lceil -\alpha n \rceil = \lfloor \alpha n \rfloor + n - \lfloor \alpha n \rfloor = n$

---

$3.3.3$ ***Show that*** $\left(n+o(n)^{k}\right)=\Theta\left(n^{k}\right)$ ***for any real constant k. Concluse that*** $\lceil n\rceil^{k}=\Theta\left(n^{k}\right)$ ***and*** $\lfloor n\rfloor^{k}=\Theta\left(n^{k}\right)$

*let be* $f(n)\in o(n)$*, so we have that* $\forall\ c>0\ \exists\ n_0>0$ *such that* $0\le f(n)<cn\ \forall\ n>n_0$

$n\le f(n)+n<cn+n=(c+1)n\rightarrow$

*We have that for* $c=c_1>0\ \exists\ n_1>0$ *such that* $n^k\le(f(n)+n)^k<(c_1+1)^k n^k\ \forall\ n\ge n_1$*, so choose* $c_2=1, c_3=(c_1+1)^k$ *and* $n_0=n_1$ *and so we have that:*

$\left(n+o(n)\right)^{k}=\Theta\left(n^{k}\right)$ *since* $\exists\ c_2, c_3$ *and* $n_0$ *that* $0\le c_2n^k\le\left(n+o(n)\right)^k\le c_3n^k\ \forall n>n_0$

---

$3.3.4$ ***Prove the following:***

$\boldsymbol{a)\ a^{\log_b c}=c^{Log_b(a)}}$

$\boldsymbol{b)\ n!\ =\ o\left(n^{k}\right), n!=\omega\ (2^{n})}$ ***and*** $\boldsymbol{lg(n!)=\Theta(nlg(n))}$

$\boldsymbol{c)\ lg\left(\Theta(n)\right)=\Theta(lg(n))}$

$a)\ \log_b c\ \cdot\log_b a=\log_b a\ \cdot\ \log_b c\rightarrow\log_b a^{\log_b c}=\ \log_b c^{\log_b a}\rightarrow a^{\log_b c}=c^{Log_b(a)}$

$b.1)$ *We know that* $\dfrac{n!}{n^n}=\dfrac{n\cdot(n-1)\cdot(n-2)\ldots1}{n\cdot n\cdot n\ldots n}$ *. Each term is* $\le 1$*. We have that* $n!\le n^n$*. It is true for* $n=1$*. Suppose it holds for* $n$*, lets prove for* $n+1$:

$(n+1)\cdot n!\le(n+1)n^n=n^{n+1}+n^n\le(n+1)^{n+1}$

$=(n)^{n+1}+(n+1)\cdot n^n+$ *positive number. So, it holds for* $n\ge1$*. We have*

$0\le\dfrac{n!}{n^n}\le\dfrac{1}{n}$*. Now since the* $\lim\limits_{n\to\infty}\dfrac{1}{n}=0\rightarrow\lim\limits_{n\to\infty}\dfrac{n!}{n^n}=0\ \therefore$ *for any* $c\ \exists n_0$ *such that* $0\le n!<cn^n\ \forall n\ge n_0$

$b.2)$ *for any* $c\ \exists n_0$ *such that* $0\le c2^n<n!\ \forall n\ge n_0$. $\dfrac{2^n}{n!}=\dfrac{2\cdot2\cdot2..2}{n\cdot(n-1)..2.1}\le\dfrac{4}{n}$*. So* $\lim\limits_{n\to\infty}\dfrac{2^n}{n!}=0\rightarrow\forall\ \epsilon>0\ \exists n_0>0$ *that if,* $n>n_0$

$\dfrac{2^n}{n!}<\mathrm{E}$*, take* $\mathrm{E}=\dfrac{1}{c}$ *and the proof holds.*

$b.3)$ *We want to prove that* $\exists c_1, c_2$ *and* $n_0>0$*, such that* $0\le c_1\mathrm{nlg(n)}\le\mathrm{lg(n!)}\le\mathrm{nc_2lg}\ (n)$

*We know that* $n^n>n!\rightarrow nlog(n)>\lg(n!)\ .\ \forall\ n\ge0$ *(We can prove it by induction ).*

*Now if we can prove that* $\exists c_1$ *such that* $n^{c_1n}\le n!\ \forall n\ge n_0$ *we are done. If we take* $c_1=0.5$ *it holds since:*

$n^{\frac{n}{2}}\le n!$ *if* $n=4$ *. Now lets see the ratio of the terms* $\dfrac{(n+1)!}{n!}=(n+1)$

$$\frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}}} = \left((n+1)\left(\frac{n+1}{n}\right)^n\right)^{\frac{1}{2}}. Now\ this\ ratio\ is < (n+1)\ for\ n \geq 4 \leftrightarrow$$

$$\left((n+1)\left(\frac{n+1}{n}\right)^n\right)^{\frac{1}{2}} \leq (n+1) \leftrightarrow (n+1)\left(\frac{n+1}{n}\right)^n \leq (n+1)^2 \leftrightarrow \left(\frac{n+1}{n}\right)^n \leq (n+1) \rightarrow$$

$since\ n+1 = 5\ and\ \left(\frac{n+1}{n}\right)^n\ is\ bounded\ by\ euler$

$There\ exists\ c_1 = 0.5, such\ that\ if\ n > 4\ we\ know\ that\ c_1\ \mathrm{n}\log(n) \leq \lg(n!).$

$c)\ let\ g(n) = \Theta(n) \rightarrow \exists c_1, c_2\ and\ n_0\ such\ that\ 0 \leq c_1 n \leq g(n) \leq c_2 n\ \forall n$

$> n_0. Since\ logarithm\ is\ a\ monotonically\ increasing, we\ know\ that \rightarrow$

$0 \leq \lg(c_1 n) \leq \lg\big(g(n)\big) \leq \lg(c_2 n) \rightarrow$

$0 \leq \lg(n) \leq \lg(c_1) + \lg(n) \leq \lg\big(g(n)\big) \leq \lg(c_2) + \lg(n) \rightarrow So\ we\ know\ by\ far\ that$

$\exists c_1, c_2\ and\ n_0\ such\ that\ 0 \leq c_1 n \leq g(n) \leq c_2 n\ \forall n > n_0$

$0 \leq \lg(n) \leq \lg\big(g(n)\big) \leq \lg(c_2) + \lg(n).\ if\ c_2 \leq n_0\ then\ \lg(c_2) \leq \lg(n)\ \forall n > n_0\ and\ so:$

$0 \leq \lg(n) \leq \lg\big(g(n)\big) \leq \lg(c_2) + \lg(n) \leq 2\lg(n). If\ c_2 > n_0, take\ a\ new\ n_0\ such\ n_0^{new} is\ c_2$

$So\ we\ know\ that\ c_2 = 2, c_1 = 1\ and \exists\ n_0\ that\ guarantees\ that\ \lg(\Theta(n) = \Theta(\lg(n)))$

---

**$3.3.5\ \boldsymbol{Is\ the\ function}\ \lceil\mathbf{lg}(\boldsymbol{n})\rceil!\ \boldsymbol{polynomially\ bounded}?\ \boldsymbol{Is\ the\ function} \lceil\mathbf{lglg}(\boldsymbol{n})\rceil!$**

**$\boldsymbol{polynomially\ bounded}?$**

$\lceil\lg(n)\rceil!\ is\ not\ polynomially\ bounded: supose\ \exists n_0\ and\ c_1 > 0\ such\ that\ \forall n \geq n_0\ we\ have$

$0 \leq \lceil\lg(n)\rceil! \leq c_1 n^k. for\ any\ n_0\ we\ will\ always\ have\ numbers\ 2^b > n_0\ \forall\ b \geq log\ (n_0)$

$And\ so\ 0 \leq b! \leq c_1 2^{bk}. This\ would\ imply\ that\ \lim_{b\to\infty} \frac{b!}{2^{bk}} = 0, but\ this\ limit = \infty$

---

**$3.3.6\ \boldsymbol{Which\ is\ asymptotically\ larger}?\ \mathbf{lg}(\boldsymbol{lg^*\ n})\ \boldsymbol{or}\ \mathbf{lg^*\ lg}\ (\boldsymbol{n})\ ?$**

$\lg^* \lg(n) = lg^* n - 1.$ So $\lg^* \lg(n)\ is\ asymptotially\ larger$

---

**$3.3.7\ \boldsymbol{Show\ that\ the\ goldn\ ratio\ \phi\ and\ its\ conjugate\ \overline{\phi}\ both\ satisfy\ the\ equation}$**

**$\boldsymbol{x^2 = x + 1}$**

$Just\ subtitute\ the\ golden\ ratio\ value\ in\ the\ equation$

---

$3.3.8$ ***Prove by induction that the ith Fibonacci number satisfies the equation***:

$$F_i = \frac{\phi^i - \overline{\phi^\iota}}{\sqrt{5}}, \textit{where } \phi \textit{ is the golden ratio and } \phi^i \textit{ its conjugate}$$

$F_0 = 0$, *and* $F_1 = 1$. *Base cases verified. if* $F_i$ *and* $F_{i-1}$ *holds, lets prove it holds for* $F_{i+1}$

$$F_{i+1} = \frac{\phi^i - \overline{\phi^\iota}}{\sqrt{5}} + \frac{\phi^{i-1} - \overline{\phi^{\iota-1}}}{\sqrt{5}} = \frac{\phi^{i+1}\left(\frac{1}{\phi^1} + \frac{1}{\phi^2}\right) - \overline{\phi}^{i+1}\left(\frac{1}{\overline{\phi}^1} + \frac{1}{\overline{\phi}^2}\right)}{\sqrt{5}} = \frac{\phi^{i+1} - \overline{\phi}^{i+1}}{\sqrt{5}}$$

---

$3.3.9$ ***Show that*** $\boldsymbol{klgk = \Theta(n)}$ ***implies*** $\boldsymbol{k = \Theta(n/\lg(n))}$

$\exists n_0, c_1$ *and* $c_2$ *such that* $\forall n \geq n_0\ 0 \leq c_1 n \leq klg(k) \leq c_2 n \rightarrow$

$0 \leq n \leq c_3 kLg(k)\ \forall n \geq n_0; \left[c_3 = \frac{1}{c_1}\right]$ *and* $0 \leq c_4 klg(k) \leq n \left[c_4 = \frac{1}{c_2}\right] \rightarrow$

$0 \leq c_4 klg(k) \leq n \leq c_3 kLg(k) \rightarrow$ *applying* log *we have*

$0 \leq \lg(k) \leq \lg(c_4) + \lg(k) + \lg(\lg(k)) = \lg(c_4 klg(k)) \leq \lg(n) \leq$

$\lg(c_3 kLg(k)) = \lg(c_3) + \lg(k) + \lg(\lg(k)) \leq \lg(c_3) + 2\lg(k)$

*So we know that* $0 \leq c_1 \lg(k) \leq \lg(n) \leq c_2 \lg(k)$ *and* $0 \leq c_1 k \lg(k) \leq \mathrm{n} \leq k \lg(k)$

*Dividing we have* $0 \leq c_1 k \leq \frac{n}{\lg(n)} \leq c_2 k$. *By the same argument in the beginning*

*We prove that* $k = \Theta\left(\frac{n}{\lg(n)}\right)$

---

$3.1$ ***Asynptotic behaviour of polynomials***

$$\textbf{\textit{let }} \boldsymbol{p(n) = \sum_{i=0}^{d} a_i n^i}\ , \boldsymbol{a_d > 0}, \textbf{\textit{be a degree d polynomial in n and let k be a constant}}$$

***Prove the following***:

$\boldsymbol{a)}$ ***if*** $\boldsymbol{k \geq d}$ ***then*** $\boldsymbol{p(n) = \mathrm{O}(n^k)}$

*Apply L'hopital Rule in* $\lim_{n\to\infty} \frac{p(n)}{n^k} = 0$ *or constant. So its clearly upper Bounded by* $n^k$

$\boldsymbol{b)}$ ***if*** $\boldsymbol{k \leq d}$ ***then*** $\boldsymbol{p(n) = \Omega(n^k)}$

*We can apply L'hopital Rule in the* $\lim_{n\to\infty} \frac{n^k}{p(n)}$ *and verify that this limit is* $0$ *or a constant.*

*So* $n^k$ *is clearly Upper Bounded by* $p(n)$ *and so* $n^k$ *lower bound* $p(n)$

$\boldsymbol{c)}$***if*** $\boldsymbol{k = d}$ ***then*** $\boldsymbol{p(n) = \Theta(n)}$

$If\ a\ and\ b\ occur, so\ if\ k = d, then\ p(n) is\ O(n) and\ \Omega(n) e\ \therefore\ p(n) = \Theta(n)$

$\boldsymbol{d)\ if\ k > d\ then\ p(n) = o(n^k)}$

$We\ can\ apply\ L'hopital\ Rule\ in\ the\ \lim_{n\to\infty} \frac{p(n)}{n^k}\ and\ verify\ that\ this\ limit\ is\ 0\ .So\ its\ clearly$

$upper\ Bounded\ by\ n^k\ .The\ formal\ definition\ of\ Limits\ in\ the\ infinity\ prove\ it$

$\boldsymbol{e)\ if\ k < d\ then\ p(n) = \omega(n)}$

$We\ can\ apply\ L'hopital\ Rule\ in\ the\ \lim_{n\to\infty} \frac{n^k}{p(n)}\ and\ verify\ that\ this\ limit\ is\ 0\ .$
$So\ n^k\ is\ clearly\ Upper\ Bounded\ by\ p(n)\ and\ so\ n^k\ lower\ bound\ p(n).\ The\ formal\ definition\ of$

$limits\ in\ the\ infinity\ prove\ it\ more\ rigorously$

---

$\boldsymbol{3.2\ Relative\ Asymptotic\ growths}$

$\boldsymbol{Indicate, for\ each\ pair\ of\ expressions\ (A, B) in\ the\ table\ below\ wheter\ A\ is\ O, o, \Omega}$

$\boldsymbol{\omega, or\ \Theta\ of\ B, write\ yes\ or\ no.\ k \geq 1, e > 0\ and\ c > 1, are\ all\ constants}$

| $A$ | $B$ | $O$ | $o$ | $\Omega$ | $\omega$ | $\Theta$ |
|---|---|---|---|---|---|---|
| $lg^k(n)$ | $n^e$ | yes | yes | | | |
| $n^k$ | $c^n$ | yes | yes | no | no | no |
| $\sqrt{n}$ | $n^{sin(n)}$ | no | no | no | no | no |
| $2^n$ | $2^{n/2}$ | no | no | yes | yes | no |
| $n^{lgc}$ | $c^{lgn}$ | yes | no | yes | no | yes |
| $lg(n!)$ | $lg(n^n)$ | yes | no | yes | no | yes |

$\boldsymbol{1^{st}\ row}$

$\exists c_1\ and\ n_0\ such\ that\ \forall n \geq n_0\ we\ have\ 0 \leq (lg(n))^k \leq c_1 n^e ? Yes, lets\ analyze\ the\ limits$

$\lim_{n\to\infty} \frac{\lg^k n}{n^e} \to L'hôpital\ rule \to \frac{klog^{k-1}(n)}{en^e} \to well, we\ can\ see\ and\ prove\ by\ induction,$

$but\ the\ exponent\ of\ the\ denominator\ will\ remain\ constant$

$and\ the\ exponent\ of\ the\ numerator\ will\ decay\ until\ be\ negative, in\ this\ situation\ clearly\ the\ limit$

$will\ be\ 0.\ In\ this\ situation\ choose\ c_1$

$= 1\ and\ it\ holds.\ In\ addition\ for\ any\ c_1\ we\ have\ 0\ \leq (lg(n))^k < c_1 n^e$

$\exists c_1\ and\ n_0\ such\ that\ \forall n \geq n_0\ we\ have\ 0 \leq c_1 n^e \leq\ (lg(n))^k ? No, by\ the\ limit\ above$

$For\ any\ c\ \exists n_0, such\ that\ for\ all\ n \geq n_0\ we\ have\ have\ 0 \leq c_1 n^e <\ (lg(n))^k\ ? False, get\ c$

$= 1\ and\ by\ the\ limit\ above\ it\ doesn't\ hold$

**$2^{nd}$ row**

$\lim_{n\to\infty}\frac{n^k}{c^n} = 0\,.\,So\ \exists c_1\ and\ n_0\ such\ that\ 0 \le n^k \le c_1 c^n$

$\exists c_1\ and\ n_0\ such\ that\ 0 \le c_1 c^n \le n^k?\ No. reason\ is\ the\ limit\ above$

$for\ any\ c_1 > 0\ \exists n_0 > 0\ such\ that\ 0 \le c_1 c^n < n^k?\ False, get\ c_1 = 1.$

$Then, by\ the\ limit\ above, there\ is\ no\ n_0\ possible$

$for\ any\ c_1 > 0\ \exists n_0 > 0\ such\ that\ 0 \le n^k < c_1 c^n?\ Yes.$

$The\ formal\ definition\ of\ limit\ proves\ it\ directly$

**$3^{rd}$ row**

$\frac{1}{n} \le n^{\sin(n)} \le n \to verify\ constants{:}\ 0 \le \sqrt{n} \le c_1 n^{\sin(n)}\ for\ all\ n \ge n_0?\ Clearly\ no$

$0 \le c_1 n^{\sin(n)} \le \sqrt{n}, for\ all\ n \ge n_0?\ clearly\ no, since\ we\ will\ always\ have\ n\ such\ c_1 n > \sqrt{n}\ whatever\ n_0\ be$

$based\ on\ the\ two\ proofs\ above\ is\ Not\ \Theta(n)$

$for\ any\ c > 0\ \exists\ n_0 > 0\ such\ that\ 0 \le c n^{\sin(n)} < \sqrt{n}?\ no, choose\ c = 1, for\ all\ n_0 we\ choose\ will\ exist\ a\ n\ such\ n^{\sin(n)} > \sqrt{n}$

$for\ any\ c > 0\ \exists\ n_0 > 0\ such\ that\ 0 \le \sqrt{n} < c n^{\sin(n)}?\ no, choose\ c = 1, for\ all\ n_0 we\ choose\ will\ exist\ a\ n\ such\ \sqrt{n} > \frac{1}{n}$

**$4^{th}$ row**

$0 \le 2^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} = 2^n \le c_1 2^{\frac{n}{2}}?\ No!\ since\ this\ would\ imply\ that\ \exists c_1\ and\ n_0\ such\ that\ \forall n \ge n_0.\ 2^{\frac{n}{2}} \le c_1, which\ is\ clearly\ false.$

$0 \le c_1 2^{\frac{n}{2}} \le 2^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} \to c_1 \le 2^{\frac{n}{2}}\ \forall n \ge n_0?\ yes, choose\ n_0 = 2\ and\ c_1 = 1$

$Since\ its\ not\ O(B), its\ not\ \Theta\ (B)$

$Does, for\ any\ c > 0\ \exists n_0 > 0\ such\ that\ 0 \le c 2^{\frac{n}{2}} < 2^n\ \forall n \ge n_0?\ Yes. Choose\ n_0 = \lg_2 c^2\ \ and\ the\ property\ holds$

$Does, for\ any\ c > 0\ \exists n_0 > 0\ such\ that\ 0 \le 2^n < c 2^{\frac{n}{2}}\ \forall n \ge n_0?\ No. If\ we\ could, then\ for\ any\ c\ we\ have\ choosen\ 2^{n/2} < c\ \forall n > n_0.\ \ Which\ is\ clearly\ false$

**$5^{th}$ row**

$We\ know\ that\ x < x + k, where\ k > 0. So\ we\ know\ that\ \lg(c)\lg(n) \le \lg\,(c)\lg\,(n) + \lg(c_1)\,, where\ c_1 > 1$

$1\log\left(\mathrm{n}^{\log(\mathrm{c})}\right) \leq \lg\left(\mathrm{c}_1 \cdot \mathrm{c}^{\lg(\mathrm{n})}\right) \rightarrow 0 \leq \mathrm{n}^{\log(\mathrm{c})} \leq \mathrm{c}_1\mathrm{c}^{\log(\mathrm{n})}$

$0 \leq c_1 c^{\lg(n)} \leq n^{\lg(c)} \rightarrow just\ get\ c_1 < 1\ and > 0$

$For\ any\ c, \exists n_0\ such\ that\ 0 \leq c^{\lg(n)} < n^{\log(c)}\ ?\ False.\ Take\ c > 1\ and\ it\ won't\ hold$

$For\ any\ c, \exists n_0\ such\ that\ 0 \leq n^{\log(c)} < c^{\lg(n)}\ ?\ No\ take\ c < 1\ and\ it\ won't\ hold$

$\boldsymbol{6^{th}\ row}$

$We\ already\ know\ that\ \lg(n!) = \Theta(\lg(n^n))$

$For\ any\ c > 0\ \exists n_0\ such\ that\ \forall\ n \geq n_0\ we\ have\ 0 \leq \lg(n!) < clg(n^n)?$

$No. We\ already\ showed\ this\ doesn't\ hold\ for\ c = \frac{1}{2}$

$For\ any\ c > 0\ \exists n_0\ such\ that\ \forall\ n \geq n_0\ we\ have\ 0 \leq clg(n^n) < \lg(n!)\ ?\ choose\ c$

$1, doesn't\ hold.$

---

$\boldsymbol{3.4\ Asymptotic\ notation\ properties. Let\ f(n) and\ g(n) be\ asymptotically\ positive\ functions.}$

$\boldsymbol{Prove\ or\ Disprove\ each\ of\ the\ following\ conjectures}$

$\boldsymbol{a) f(n) = O(g(n)) implies\ g(n) = O(f(n))}$

$\boldsymbol{b) f(n) + g(n) = \Theta(\min\{f(n), g(n)\})}$

$\boldsymbol{c) f(n) = O(g(n))\ implies \lg(f(n)) = O(\lg(g(n)), where \lg(g(n)) \geq 1\ and\ f}$

$\boldsymbol{(n) \geq 1\ for\ all\ sufficiently\ large\ n}$

$\boldsymbol{d) f(n) = O(g(n)) implies\ 2^{f(n)} = O(2^{g(n)})}$

$\boldsymbol{e)\ f(n) = O(f(n)^2)}$

$\boldsymbol{f) f(n) = O(g(n))\ implies\ g(n) = \Omega(f(n))}$

$\boldsymbol{g)\ f(n) = \Theta(f(n/2))}$

$\boldsymbol{h) f(n) + o(f(n)) = \Theta(f(n))}$

$a)\ let\ f(n) = n\ and\ g(n) = n^2. \boldsymbol{FALSE}$

$b)\ let\ f(n) = n\ and\ g(n) = n^2. \boldsymbol{FALSE}$

$c)\ \exists\ c, n_0\ such\ that\ \forall\ n \geq n_0 \rightarrow\ 0 \leq f(n) \leq c_1 g(n) \rightarrow 0 \leq \log(f(n))$

$\leq \log(c_1) + \log(g(n)) \leq \log(g(n))\log(c_1) + \log(g(n)) = (1 + \log(c_1))\log(g(n))$

$So\ Just\ take\ new\ c_1 = 1 + \log(c_1). \boldsymbol{TRUE}$

$d)\ Supose\ it's\ true\ and\ let\ f(n) = 2n\ and\ g(n) = n \rightarrow 2^n \cdot 2^n \leq c_1 2^n \rightarrow$

$(2^n < c_1)\ for\ certain\ c_1, which\ is\ clearly\ false(limit\ definition\ proves\ it)\ \boldsymbol{FALSE}$

$e) Suppose\ it's\ true\ and 0 \le f(n) \le c_1 f(n)^2 \to let\ f(n) = \frac{1}{n} \to n$

$\le c_1\ for\ certain\ c_1, which\ is\ clearly\ false.$ $\boldsymbol{FALSE}$

$f)\ 0 \le f(n) \le c_1 g(n) \to 0 \le \frac{1}{c_1} f(n) \le g(n).$ $\boldsymbol{TRUE}$

$g)\ 0 \le c_1 f\left(\frac{n}{2}\right) \le f(n) \le c_2 f\left(\frac{n}{2}\right) \to let\ f(n) be\ 2^n \to c_1 2^{\frac{n}{2}} \le 2^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} \le c_2 2^{\frac{n}{2}} \to 2^{\frac{n}{2}} \le c_2$

$\to which\ is\ clearly\ False, since\ 2^{\frac{n}{2}}\ approaches\ infinity.$ $\boldsymbol{FALSE}$

$h)\ let\ g(n) be\ a\ function\ such\ that\ for\ any\ c > 0\ \exists\ n_0 > 0\ such\ that\ if\ n \ge n_0\ 0 \le g(n)$

$< cf(n)\ \forall n \ge n_0$

$Now\ we\ wat\ to\ verify\ if\ \exists c_1, c_2, n_0 > 0\ such\ that\ 0 \le c_1 f(n) \le\ f(n) + g(n)$

$\le c_2 f(n). We\ know\ that, exist\ n_0\ such\ 0 \le g(n) < cf(n). So\ exist\ n_{01}\ such\ that\ g(n)$

$< f(n).\ If\ choose\ c_2\ to\ be\ 2, and\ n_0\ to\ be\ n_{01}.$ $\boldsymbol{TRUE}$

---

$\mathbf{3.5}\ \boldsymbol{Manipulating\ asymptotic\ notation.\ let\ f(n) and\ g(n) be\ asymptotically}$

$\boldsymbol{positive\ functions. Prove\ the\ following\ identities}$

$\boldsymbol{a)} \boldsymbol{\Theta}\left(\boldsymbol{\Theta}\big(\boldsymbol{f(n)}\big)\right) = \boldsymbol{\Theta}(\boldsymbol{f(n)})$

$\boldsymbol{b)} \boldsymbol{\Theta}\big(\boldsymbol{f(n)}\big) + \boldsymbol{O}\big(\boldsymbol{f(n)}\big) = \boldsymbol{\Theta}(\boldsymbol{f(n)})$

$\boldsymbol{c)}\ \boldsymbol{\Theta}\big(\boldsymbol{f(n)}\big) + \boldsymbol{\Theta}\big(\boldsymbol{g(n)}\big) = \boldsymbol{\Theta}(\boldsymbol{f(n)} + \boldsymbol{g(n)})$

$\boldsymbol{d)}\ \boldsymbol{\Theta}\big(\boldsymbol{f(n)}\big) \cdot \boldsymbol{\Theta}\big(\boldsymbol{g(n)}\big) =\ \boldsymbol{\Theta}(\boldsymbol{f(n)} \cdot \boldsymbol{g(n)})$

$\boldsymbol{e)\ argue\ that\ for\ any\ real\ constants\ a_1, a_2 > 0\ and\ integer\ constants\ k_1, k_2}$

$\boldsymbol{the\ following\ asymptotic\ bound\ holds}\ (\boldsymbol{a_1 n})^{\boldsymbol{k_1}} \boldsymbol{lg^{k2}}(\boldsymbol{a_2 n}) = \boldsymbol{\Theta}(\boldsymbol{n^{k_1} lg^{k2} n})$

$\boldsymbol{f)\ Prove\ that\ for\ S\ C\ Z, we\ have} \sum_{\boldsymbol{k \in S}} \boldsymbol{\Theta}\big(\boldsymbol{f(k)}\big) = \boldsymbol{\theta}\left(\sum_{\boldsymbol{k \in S}} \mathbf{f}(\boldsymbol{k})\right), \boldsymbol{assumming\ that\ both}$

$\boldsymbol{sums\ converge}$

$\boldsymbol{g)\ Show\ that\ for\ SC\ Z, the\ following\ asymptotic\ bound\ does\ not\ necessarily\ hold,}$

$\boldsymbol{even\ assuming\ that\ both\ products\ converge, by\ giving\ a\ counterexample}:$

$$\prod_{\boldsymbol{k \in S}} \boldsymbol{\Theta}(\boldsymbol{f(k)}) = \boldsymbol{\Theta}\left(\prod_{\boldsymbol{k \in S}} \boldsymbol{f(k)}\right)$$

$a)\ let\ g(n) be \in \Theta\big(f(n)\big). So, there\ exists\ c_1, c_2\ and\ n_0 > 0\ such\ that \to$

$0 \le c_1 f(n) \le g(n) \le c_2 f(n) \forall\ n \ge n_0$

$Now\ let\ j(n) be \in\ \Theta\big(g(n)\big)\ so\ o, there\ exists\ c_{10}, c_{20}\ and\ n_{00} > 0\ such\ that \rightarrow$

$0 \le c_{10} g(n) \le j(n) \le c_{20} f(n)\ \forall\ n \ge n_0 \rightarrow$

$0 \le c_1 c_{10} f(n) \le j(n) \le c_2 c_{20} f(n).$ $\boldsymbol{So\ \Theta\big(f(n)\big) =\ \Theta\Big(\Theta\big(f(n)\big)\Big)}$

$b)\ let\ g(n) be \in O\big(f(n)\big) \rightarrow So, there\ exists\ c_1\ and\ n_0 > 0\ such\ that \rightarrow$

$0 \le g(n) \le c_1 f(n) \forall\ n \ge n_0\ Now\ let\ J(n)$

$\in \Theta\big(f(n) + g(n)\big). So, there\ exists\ c_{10}, c_{20}\ and\ n_{00} > 0\ such\ that \rightarrow$

$0 \le c_{10}\big(f(n) + g(n)\big) \le j(n) \le c_2\big(f(n) + g(n)\big) \forall\ n \ge n_{00} \rightarrow$

$0 \le c_{10}\big(f(n) + g(n)\big) \le j(n) \le c_2 f(n) + c_1 c_2 f(n) = c_3 f(n)$

$And\ we\ know\ that\ \ 0 \le c_{10} f(n) \le c_{10}\big(f(n) + g(n)\big) \le j(n) \rightarrow$

$\boldsymbol{So\ \Theta(f(n) + O\big(f(n)\big) = \Theta(f(n)))}$

$c)\ let\ G(n) \in \Theta\big(g(n)\big)\ and\ F(n) \in \Theta\big(f(n)\big)\ So\ \exists\ c_1, c_2, c_3, c_4\ and\ n_0\ such\ that\ \forall\ n$

$\ge n_0\ we\ have\ 0 \le c_1 g(n) \le G(N) \le\ c_2 g(n)\ and\ 0 \le c_3 f(n) \le F(N) \le\ c_4 f(n) \rightarrow$

$0 \le c_1 g(n) + c_3 f(n) \le G(N) + F(N) \le\ c_2 g(n) + c_4 f(n). Now\ lets\ choose\ c_f$

$= \max\{c_2, c_4\}\ and\ c_i = \min\{c_1, c_3\}\ then\ 0 \le c_i\big(g(n) + f(n)\big) \le G(N) + F(N)$

$\le\ c_f(g(n) + f(n))$

$d)\ let\ G(n) \in \Theta\big(g(n)\big)\ and\ F(n) \in \Theta\big(f(n)\big)\ So\ \exists\ c_1, c_2, c_3, c_4\ and\ n_0\ such\ that\ \forall\ n$

$\ge n_0\ we\ have\ 0 \le c_1 g(n) \le G(N) \le\ c_2 g(n)\ and\ 0 \le c_3 f(n) \le F(N) \le\ c_4 f(n) \rightarrow$

$0 \le c_1 g(n) c_3 f(n) \le G(N) F(N) \le\ c_2 g(n) c_4 f(n). Now\ lets\ choose\ c_f = \max\{c_2, c_4\}\ and\ c_i$

$= \min\{c_1, c_3\}\ then\ 0 \le c_i(g(n) f(n)) \le G(N) F(N) \le\ c_f(g(n) f(n))$

$e)\ 0 \le c_1 n^{k_1} \lg(n)^{k_2} \le a_1^{k_1}\ n^{k_1} \lg(a_2 n)^{k_2}\ 0 \le c_2 n^{k_1} \lg(n)^{k_2} \rightarrow$

$0 \le c_1 \lg(n)^{k_2} \le a_1^{k_1}\ \lg(a_2 n)^{k_2} \le c_2 \lg(n)^{k_2} \rightarrow$

$$0 \le \left(\frac{c_1}{a_1^{k_1}}\right)^{\frac{1}{k_2}} \lg{(n)} \le (\lg(k_2) + \lg{(n)})\ \ \le \left(\frac{c_2}{a_1^{k_1}}\right)^{\frac{1}{k_2}} \lg{(n)}$$

$$0 \le \lg(n) \le \lg(a_2) + \lg(n) \le \left(\frac{c_2}{a_1^{k_1}}\right)^{\frac{1}{k_2}} \lg(n)\, ; let\ c_1 =\ a_1^{k_1}$$

$c_2 = a_1^{k_1}\, 2^{k_2}$

$\lg(a_2) + \lg(n) \le 2 \lg(n)\ \forall\ n \ge a_2$

$Just\ begin\ from\ this\ last\ equation\ and\ make\ the\ way\ up!$

$f)$

$g)$