Instituto Superior Técnico

Integrated Masters in Aerospace Engineering

Digital Signal Processing

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Laboratory 5

Image Segmentation

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1 R1. Maximum likelihood estimators

1.1 R1.a) Maximum likelihood estimates

First, the maximum likelihood estimates of the exponential, beta, and normal distributions are derived.

1.2 a) Exponential distribution

Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a sequence of N independent realizations of a random variable with exponential distribution

$$p(x_i; \beta) = \frac{1}{\beta}\exp(-x_i/\beta)\; x_i \geq 0\,.$$

The log-likelihood function is, because of the independence among realizations, given by

$$\begin{aligned} l(1/\beta) &= \prod_{i=1}^{N} \log(p(x_i|\beta)) \\ &= N\log(1/\beta) - \frac{1}{\beta}\sum_{i=1}^{N} x_i\,. \end{aligned}$$

The maximum likelihood estimator, $\hat{\beta}_{ML}$, verifies the constraint

$$\begin{aligned} \left.\frac{dl}{d(1/\beta)}\right|_{\beta=\hat{\beta}_{ML}} &= 0 \\ N\hat{\beta}_{ML} - \sum_{i=1}^{N} x_i = 0\,, \end{aligned}$$

thus the maximum likelihood estimate is given by

$$\hat{\beta}_{ML} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Computing the second derivative of it is possible to confirm that it is, in fact, a maximum of the log-likelihood function.

1.3 b) Rayleigh distribution

Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a sequence of N independent realizations of a random variable with Rayleigh distribution

$$p(x_i; f) = \frac{x_i}{f}\exp(-x_i^2/(2f))\; x_i \geq 0\,.$$

The log-likelihood function is, because of the independence among realizations, given by

$$\begin{aligned} l(f) &= \prod_{i=1}^{N} \log(p(x_i|f)) \\ &= -N\log(f) + \sum_{i=1}^{N}\log(x_i) - \frac{1}{2f}\sum_{i=1}^{N} x_i^2\,. \end{aligned}$$

The maximum likelihood estimator, $\hat{l}_{ML}$, verifies the constraint

$$\begin{aligned} \left.\frac{dl}{df}\right|_{f=\hat{f}_{ML}} &= 0 \\ -N/\hat{f}_{ML} - \frac{1}{\hat{f}_{ML}^2}\sum_{i=1}^{N} x_i^2 = 0\,, \end{aligned}$$

thus the maximum likelihood estimate is given by

$$\hat{f}_{ML} = \frac{1}{2N}\sum_{i=1}^{N} x_i^2$$

Computing the second derivative of it is possible to confirm that it is, in fact, a maximum of the log-likelihood function.

1.4 c) Normal distribution

Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a sequence of N independent realizations of a random variable with normal distribution

$$p(x_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp((x_i - \mu)^2/(2\sigma^2)) .$$

The log-likelihood function is, because of the independence among realizations, given by

$$\begin{aligned} l(\mu, \sigma^2) &= \prod_{i=1}^{N} \log(p(x_i|\mu, \sigma^2)) \\ &= -\frac{N}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{N}(x_i - \mu)^2/(2\sigma^2) . \end{aligned}$$

The maximum likelihood estimators, $\hat{\mu}_{ML}$, $\hat{\sigma}^2_{ML}$, verify the constraints

$$\left.\frac{dl}{d\mu}\right|_{\mu=\hat{\mu}_{ML},\sigma^2=\hat{\sigma}^2_{ML}} = 0$$

$$-\frac{1}{2\hat{\sigma}^2_{ML}}\sum_{i=1}^{N}(x_i - \hat{\mu}_{ML})^2 = 0 ,$$

and

$$\left.\frac{dl}{d\sigma^2}\right|_{\mu=\hat{\mu}_{ML},\sigma^2=\hat{\sigma}^2_{ML}} = 0$$

$$-N + \frac{1}{\hat{\sigma}^2_{ML}}\sum_{i=1}^{N}(x_i - \hat{\mu}_{ML})^2 = 0 ,$$

thus the maximum likelihood estimates are given by

$$\hat{\mu}_{ML} = \frac{1}{N}\sum_{i=1}^{N} x_i \quad \text{and} \quad \hat{\sigma}^2_{ML} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu}_{ML})^2$$

Computing the second derivative of it is possible to confirm that it is, in fact, a maximum of the log-likelihood function.

1.5 R1.b) Synthetic Aperture Radar Image

The Synthetic Aperture Radar (SAR) Image was loaded and it is displayed in Fig. 1.

1.6 R1.c) Fit cropped images

The SAR image is cropped into two regions, one of water and another of ice, chosen to be as large as possible, as depicted in Fig. 2. The maximum likelihood estimators derived in Section 1.1 are applied to both regions using the Matlab functions `MLEexponential`,`MLErayleigh`, and `MLEnormal` that we developed. The maximum likelihood estimates are presented in Table 1 for both ice and water regions.

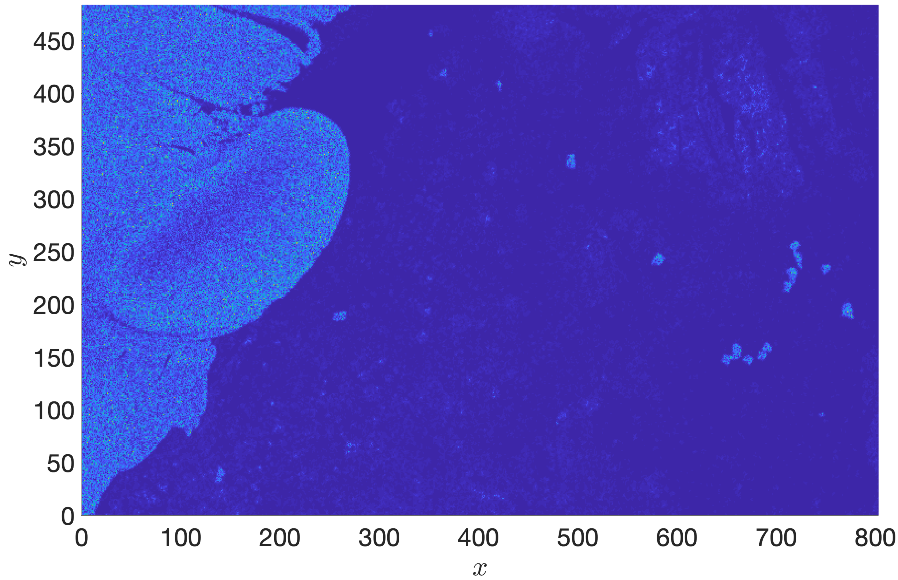

Figure 1: Synthetic Aperture Radar (SAR) Image.

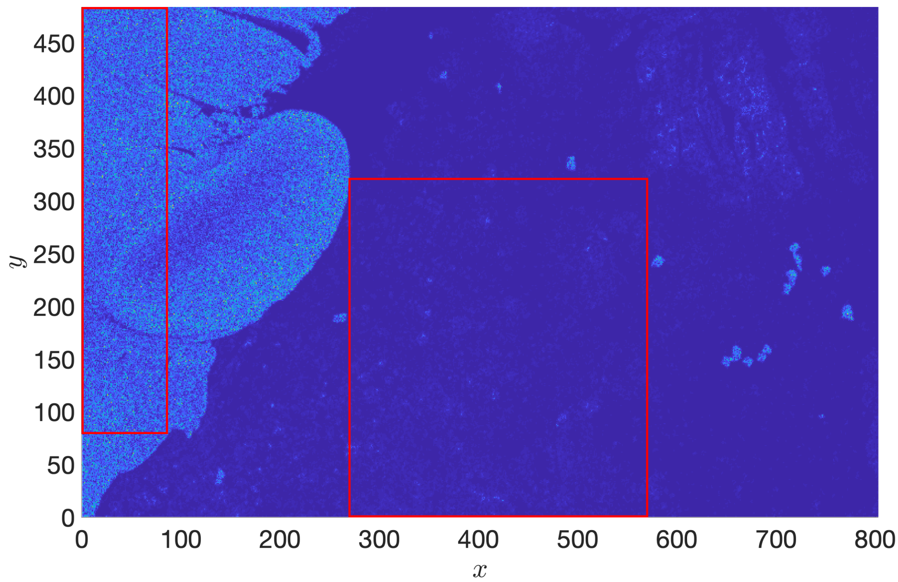

Figure 2: Cropped Synthetic Aperture Radar Image.

Table 1: Maximum likelihood estimators for the ice and water regions.

	Ice	Water
$\hat{\beta}_{ML}$	7.7559×10^4	5.6263×10^3
$\hat{f}_{ML}$	6.4909×10^4	4.9659×10^3
$\hat{\mu}_{ML}$	7.7559×10^4	5.6263×10^3
$\hat{\sigma}^2_{ML}$	2.4111×10^9	1.7665×10^7

1.7 R1.d) Choice of the best distribution

To choose the distribution that best fits the data in both selected regions, the distributions with the parameters in Table 1 were obtained with the MATLAB™ commands `raylpdf`, `exppdf`, and `normpdf` as represented in Figs. 3 and 4, as well as the histograms of the data obtained with the MATLAB™ command `histogram(data,'Normalization','pdf')`. Visually it is possible to observe that for both cases the Rayleigh distribution presents the best fitting to the data. In a more quantitative way, the RMSE between the edges values of the bins of the histogram and the distributions was computed and the conclusions taken were the same. The values of the RMSE are presented in Table 2.

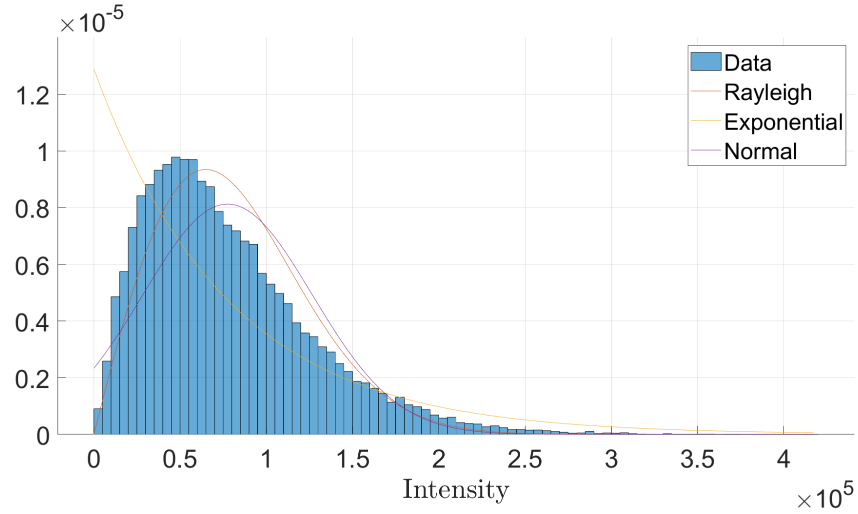

Figure 3: Histogram and probability distributions for the ice region.

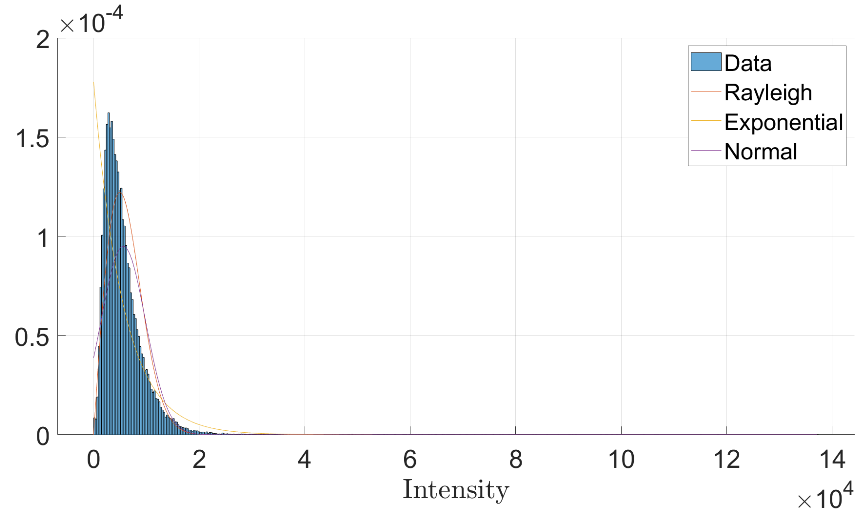

Figure 4: Histogram and probability distributions for the water region.

Table 2: RMSE values between the edges values of the bins of the histogram and the distributions.

	Water			Ice		
	Rayleigh	Exponential	Normal	Rayleigh	Exponential	Normal
RMSE	9.63×10^{-6}	1.67×10^{-5}	1.19×10^{-5}	1.00×10^{-6}	2.21×10^{-6}	1.29×10^{-6}

2 R2. Image Segmentation

2.1 R2.a) Segmentation based on the intensity of each pixel

In this section, the image in Fig. 1 is segmented according to the maximum likelihood method. In order to do this, the intensity of each pixel of the image is used to compute the probability of it corresponding to water or ice according to the Rayleight distributions with $\hat{\sigma}^2_{ML}$ in Table 1. The segmentation is done using the MATLAB™ command `imcontour` dividing the image using red borders as in Fig. 5.

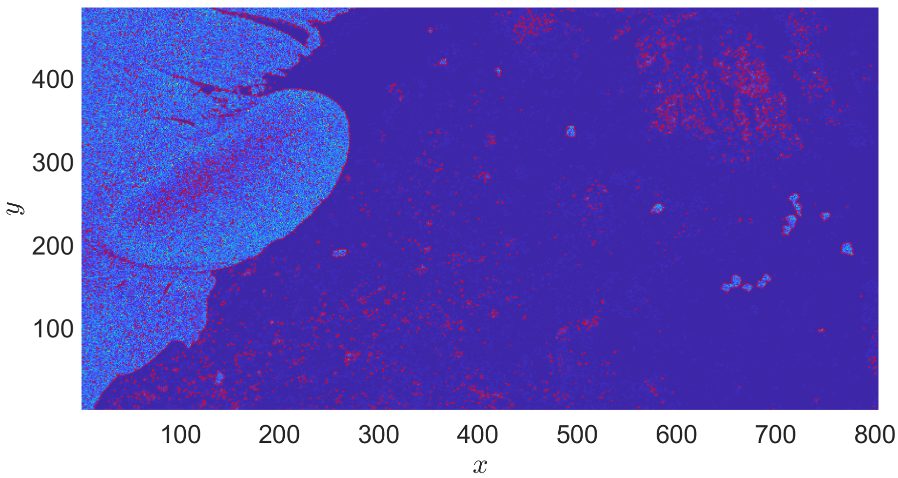

Figure 5: Segmentation based on the intensity of each pixel with maximum likelihood method and red borders.

2.2 R2.b) Segmentation based on a patch around each pixel

On the other hand, the same process can be applied to a patch of, for instance, 5×5 pixels around each pixel considering as the corresponding intensity of each pixel the average intensity of all pixels inside the patch. To do this, a moving average filter with a window of the corresponding size is applied to the matrix of intensities of the image using the command `conv2(data,kernel,'same')`. The results of this operation are shown in Fig. 6.

3 R2.c) Comments on the presented approaches

In the first place, it is possible to distinguish that the first approach with results in Fig. 5 is very corrupted by noise, *i.e.*, there are a lot of individual pixels or very small groups of them that may be affected noise and are, in this approach, mistakenly classified. In the second place, the results gotten with the second approach in Fig. 6 have an high robustness to noise since it is not possible to find regions in the middle of the water or ice where pixels are mistakenly classified. In fact, although the quality of the classifications were already adequate, after filtering it improved. In Table 3, the rate of correct decisions is presented for each case. Nevertheless, it is important to notice that it is important to find a balance between the size of the patch and the robustness to noise wanted. For instance, in the second approach it is possible to distinguish that the ice regions included sometimes small sections of water. Therefore, the size of the patch is a critical parameter and should be tuned for each application.

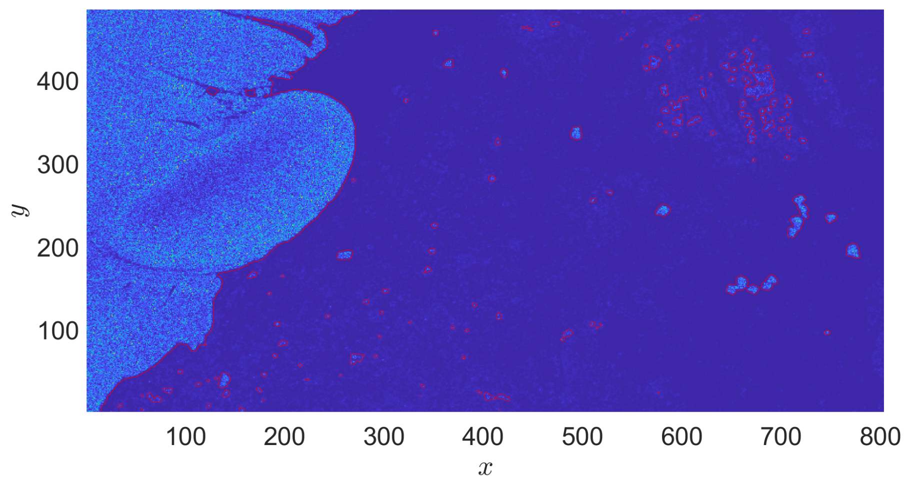

Figure 6: Segmentation based on the average intensity of a 5×5 patch around each pixel with maximum likelihood method and red borders.

Table 3: Rate of correct decisions for each approach and training region.

	Ice	Water	Ice with filtering	Water with filtering
Rate of correct decsions	0.9526	0.9811	1	0.9921