Project part 2

Task 1.

$$\underbrace{f(a,c)}$$

$$\text{minimize} \quad \frac{1}{k}\sum_{k=1}^{k}\left[\log\left(1+\exp\left(a^T x_k - c\right)\right) - y_k\left(a^T x_k - c\right)\right]$$

Let $x = [a^T \; c^T]^T$, $g = \begin{bmatrix} x_k \\ -1 \end{bmatrix}$,

$$f(a,c) = f(x) = \frac{1}{k}\sum \log\left(1+\exp\left(\left[x_k^T \; -1\right]^T x\right)\right) - \sum \frac{y_k}{k}\left[x_k^T \; -1\right]^T x$$

$\log\left(1+\exp\left(\cdot\right)\right)$: constant + exp function → is convex

$\exp\left(g^T x\right)$: affine

composition of convex + affine map → is convex

$\log\left(1+\exp\left(g^T x\right)\right)$: composition of convex non-decreasing function and convex function → is convex

$\frac{1}{k} \geq 0$: $\log\left(1+\exp\left(g^T x\right)\right)$: sum of convex function weighted by nonnegative scalars → is convex

$\frac{y_k}{k} \geq 0$: $(-g)^T x$: affine map → is convex

$\frac{y_k}{k}(-g)^T x$: sum of convex functions weighted by nonnegative scalars → is convex

$f(a,c)$: sum of 2 convex function is CVX → is CVX

$$\underset{(a,c)\in\mathbb{R}^n\times\mathbb{R}}{\text{maximize}} \; h(x_1 \cdots x_k, y_1 \cdots y_k \mid a, c) = \underset{(a,c)\in\mathbb{R}^n\times\mathbb{R}}{\text{maximize}} \; \prod_{k=1}^{k} h(x_k, y_k \mid a, c)$$

$$h(x_k, y_k \mid a, c) = \begin{cases} \dfrac{\exp\left(a^T x_k - c\right)}{1+\exp\left(a^T x_k - c\right)}, & y_k = 1 \\ \dfrac{1}{1+\exp\left(a^T x_k - c\right)}, & y_k = 0 \end{cases}$$

$$= \underset{(a,c)\in\mathbb{R}^n\times\mathbb{R}}{\text{maximize}} \; \prod_{k=1}^{k} \frac{\exp\left(y_k\left(a^T x_k - c\right)\right)}{1+\exp\left(a^T x_k - c\right)}$$

$$\Rightarrow h(x_k y_k \mid a, c) = \frac{\exp\left(y_k\left(a^T x_k - c\right)\right)}{1+\exp\left(a^T x_k - c\right)}$$

$$= \underset{(a,c)\in\mathbb{R}^n\times\mathbb{R}}{\text{maximize}} \; \frac{1}{k}\prod_{k=1}^{k} \log \frac{\exp\left(y_k\left(a^T x_k - c\right)\right)}{1+\exp\left(a^T x_k - c\right)}$$

$$= \underset{(a,c)\in\mathbb{R}^n\times\mathbb{R}}{\text{maximize}} \; \frac{1}{k}\sum_{k=1}^{k}\left[y_k\left(a^T x_k - c\right) - \log\left(1+\exp\left(a^T x_k - c\right)\right)\right]$$

$$= \underset{(a,c)\in\mathbb{R}^n\times\mathbb{R}}{\text{minimize}} \; \frac{1}{k}\sum_{k=1}^{k}\left[\log\left(1+\exp\left(a^T x_k - c\right)\right) - y_k\left(a^T x_k - c\right)\right]$$