Compact Directed Percolation

Random Walk and CDP's critical exponents

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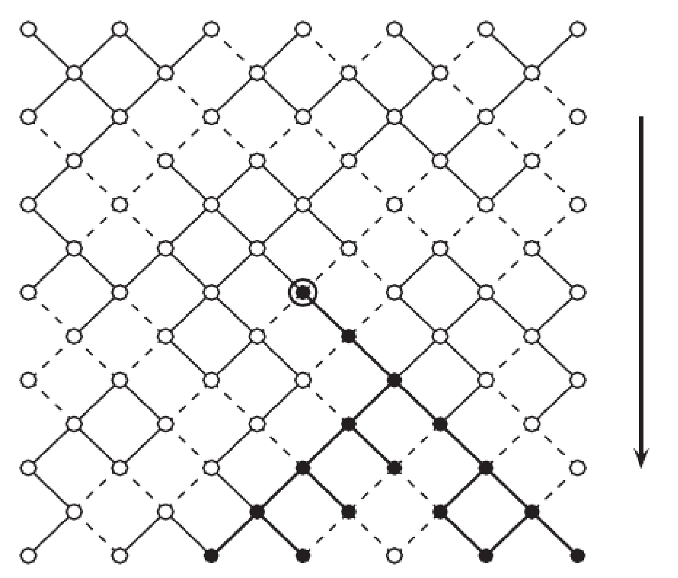
Introduction Directed (Bond) Percolation



Transition probabilities:

$$
\begin{aligned}
&P(1|0,0) = 0 \\
&P(1|0,1) = P(1|1,0) = p \\
&P(1|1,1) = q
\end{aligned}
$$

DP: $q_{DP} = p(2-p)$

Properties:

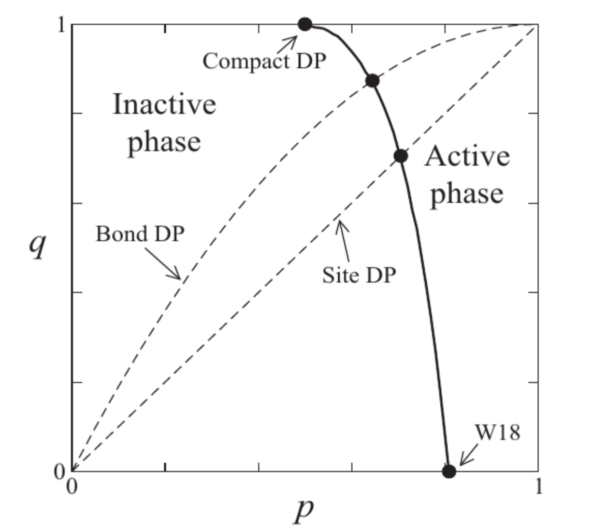
- One absorbing state $\longrightarrow$ Detail Balance violated
- Critical phase transition between *active* and *inactive* phases

Order parameters:

- $N(t) = \langle \sum_i s_i(t) \rangle$ = number of active sites
- $P(t) = \langle 1 - \prod_i (1 - s_i(t)) \rangle$ = survival probability

$q_{DP} = p(2 - p)$

$q_{CDP} = 1$



CDP Properties

Transition probabilities:

$$
\begin{aligned}
&P(1|0,0) = 0 \\
&P(1|0,1) = P(1|1,0) = p \\
&P(1|1,1) = q
\end{aligned}
$$

CDP: $q_{CDP} = 1$

Properties:

- Two Absorbing states $\longrightarrow$ Different universality class
- Symmetry between active and inactive sites $\longrightarrow p_c = 1/2$
- Denser clusters

The evolution can be studied through the dynamics of *domain walls*

Observations:

- Domanin walls can't be created thanks to $P(0|1,1) = P(1|0,0) = 0$, but can disappear
- They can "diffuse"

$L(t) =$ number of contiguous active sites at time t with evolution rule:

$$L(t+1) = \begin{cases} L(t)+1, & \text{with probability } p^2 \\ L(t)-1, & \text{with probability } (1-p)^2 \\ L(t), & \text{with probability } 2p(1-p) \end{cases}$$

Mapping:

Compact directed percolation dynamics $\iff$ $L(t)$ performs a RW with absorbing barriers at $L(t) = 0$ and $L(t) = N$

Transition probabilities:

$$W_{i+1,i} = \frac{p^2}{p^2 + (1-p)^2} = r$$
$$W_{i-1,i} = 1 - r$$

For a single site IC, the probability to have at least one active site at time t is:

$$P(t) = \langle 1 - \prod_{i=0}^{\infty}(1 - s_i(t))\rangle$$

For $t \to \infty$ and $p \geq p_c = 1/2$:

$$P(\infty) \sim (p - p_c)^{\beta'}$$

What is the probability to grow an infinite cluster in time?

or

What is the probability to not be absorbed by the state $L(t) = 0$?

Transition matrix:

$$W = \begin{pmatrix} 1 & 1-r & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1-r & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r & 0 & 1-r & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 1-r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & r & 1 \end{pmatrix}$$

$p_j =$ probability to be trapped at 1 given that the RW started in j at $t = 0$

Equation to solve[1]:

$$\begin{aligned} p_j &= W_{j+1,j}\, p_{j+1} + W_{j-1,j}\, p_{j-1} \\ &= r\, p_{j+1} + (1-r)\, p_{j-1} \end{aligned}$$

[1] Formally equal to the one for the probability distribution of a RW on a ring.

Assuming $p_k = \psi^k$:

$$\psi^j = r\psi^{j+1} + (1-r)\psi^{j-1} \quad \Rightarrow \quad 0 = r\psi^2 - \psi + (1-r)$$

Asymmetric case $r \neq 1/2$:

$$\boxed{p_j^{as} = \frac{s^{j-1} - s^{N-1}}{1 - s^{N-1}}} \tag{1}$$

where $(1-r)/r := s$

Symmetric case $r = 1/2$:

$$\boxed{p_j^{s} = \frac{N-j}{N-1}} \tag{2}$$

For $r > 1/2$ (i.e. $s < 1$):[2]

$$p_j^{\text{as}} \xrightarrow{N\to\infty} s^{j-1}$$

Going back to $P(\infty)$:

$$P(\infty) = 1 - p_2^{as} = 1 - s^1 = 1 - \left(\frac{1-r}{r}\right) = \frac{2}{p^2}\left(p - \frac{1}{2}\right) = \frac{2}{p^2}\,(p - p_c)$$

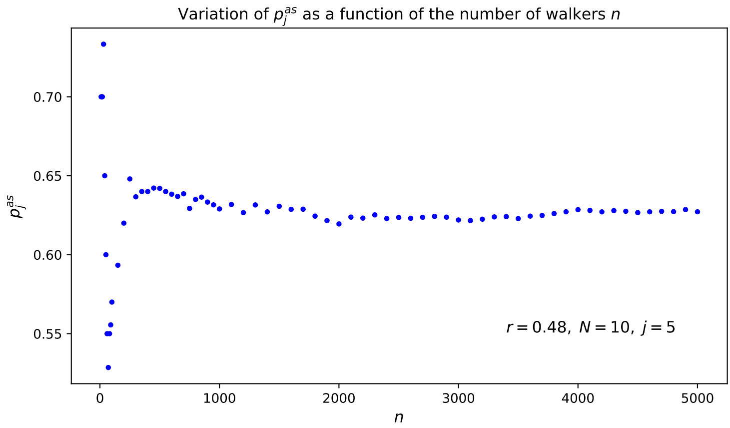
Hence

$$\beta' = 1$$

[2] For $r \leq 1/2$, the thermodynamic limit gives $p_j = 1\ \forall j$.

Simulation p_j vs number of walkers

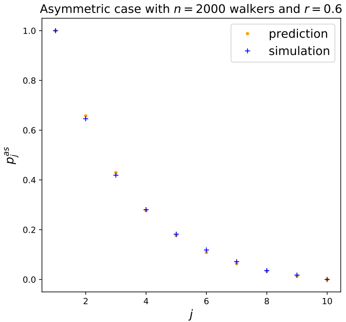
$$p_j^{as/s} = \frac{\text{Number of absorbed walkers in 1}}{\text{Total number of walkers}}$$



Simulation Results

$$p_j^{as} = \frac{s^{j-1} - s^{N-1}}{1 - s^{N-1}}$$

$$p_j^{s} = \frac{N - j}{N - 1}$$



Simulation Results

$$p_j^{as} = \frac{s^{j-1} - s^{N-1}}{1 - s^{N-1}}$$

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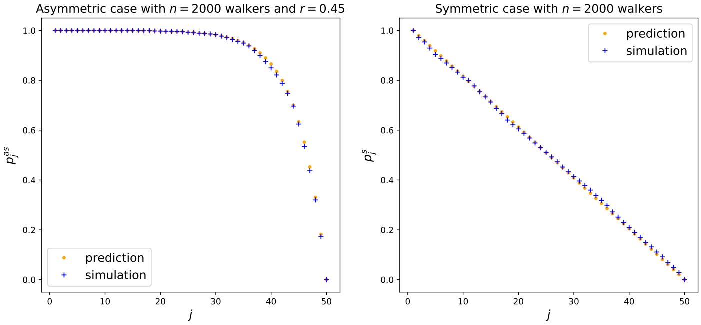
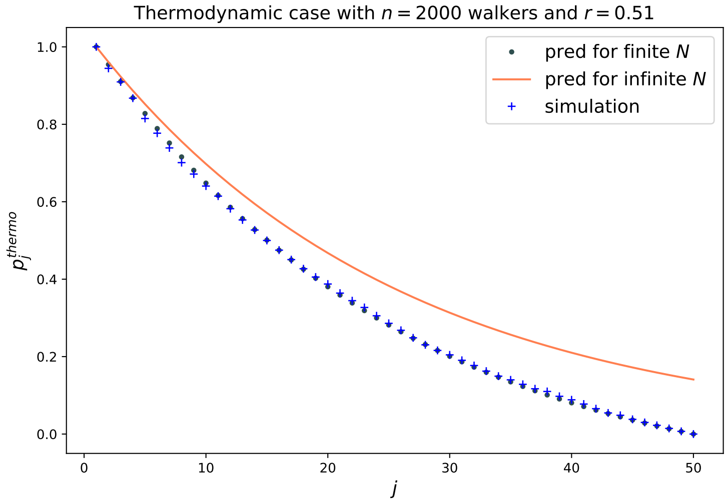
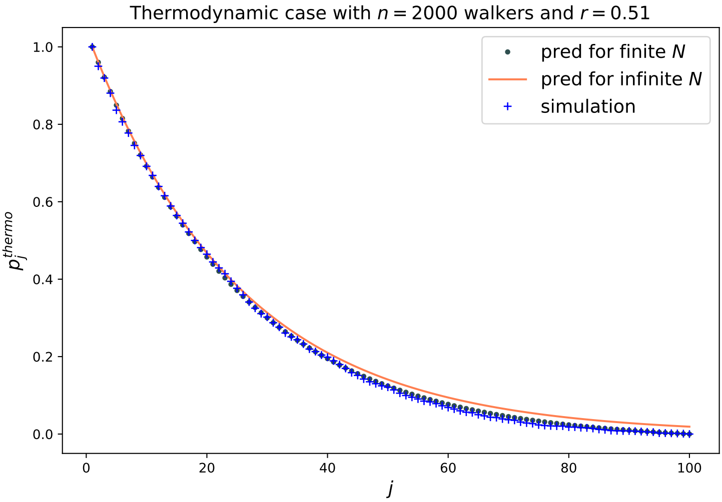



Thermodynamic case with $n = 2000$ walkers and $r = 0.51$
pred for finite N
pred for infinite N
simulation
p_j^{thermo}
j
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50

Simulation Results: thermodynamic limit



Fokker-Planck equation:

$$\frac{\partial P(X,t)}{\partial t} = -\frac{\partial}{\partial X}(a(X,t)P(X,t)) + \frac{1}{2}\frac{\partial^2}{\partial X^2}\left(b^2(X,t)P(X,t)\right)$$

Continuity equation:

$$\frac{\partial P(X,t)}{\partial t} + \frac{\partial J(X,t)}{\partial X} = 0$$

where the probability current is:

$$J(X,t) = a(X,t)P(X,t) - \frac{1}{2}\frac{\partial}{\partial X}\left(b^2(X,t)P(X,t)\right)$$

Having a flux F of random walkers injected in X_0, the *probability that a walker is absorbed in* X_1 is[3]

$$\Pi\left(X_1|X_0\right) = \frac{J_-}{F} = \frac{X_2 - X_0}{X_2 - X_1}$$

[3] Isotropic and stationary diffusion.

β

For homogeneous random initial sites, the fraction of active sites at time t:

$$\rho(t) = \frac{N(t)}{M} = \frac{\sum_i \langle s_i(t) \rangle}{M}$$

For $t \to \infty$ and $p \geq p_c = 1/2$:

$$\rho(\infty) \sim (p - p_c)^\beta$$

For $p \geq p_c = 1/2$, we have $\rho(\infty) = 1$.

Hence:

$$\beta = 0$$

Remark: $\beta \neq \beta'$, hence different universality class respect to DP

θ Continuous isotropic RW with one trap

For a single site as IC and at $p = p_c = 1/2$:

$$L(t) \overset{t \gg 1}{\sim} t^{\theta}$$

$L(t)$ diffuses symmetrically following:

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}$$

with an absorbing trap at $x_t = 0$.

Def: x_t is an absorbing trap if $p(x_t, t) = 0 \; \forall t$

θ Continuous isotropic RW with one trap

Solution which satisfy $p(x_t, t) = 0\ \forall t$:

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[\exp\left(-\frac{(x-x_0)^2}{4Dt}\right) - \exp\left(-\frac{(x+x_0)^2}{4Dt}\right)\right]$$

with $p(x,0) = \delta(x - x_0)$

θ Continuous isotropic RW with one trap

$$
\begin{aligned}
L(t) \equiv \langle x(t) \rangle &= \int_0^{\infty} dx \frac{x}{\sqrt{4\pi D t}} \left[\exp\left(-\frac{(x-x_0)^2}{4Dt} \right) - \exp\left(-\frac{(x+x_0)^2}{4Dt} \right) \right] \\
&= \int_0^{\infty} dx \frac{x}{\sqrt{4\pi D t}} \exp\left(-\frac{(x-x_0)^2}{4Dt} \right) + \int_{-\infty}^{0} dx \frac{x}{\sqrt{4\pi D t}} \exp\left(-\frac{(x-x_0)^2}{4Dt} \right) \\
&= \int_{-\infty}^{\infty} dx \frac{x}{\sqrt{4\pi D t}} \exp\left(-\frac{(x-x_0)^2}{4Dt} \right) \\
&= x_0
\end{aligned}
$$

$L(t)$ is constant in time, thus

$$\theta = 0$$

Time correlation function:

$$c(t) = \left\langle \lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} \left(s_i(0) - \bar{s}\right)\left(s_i(t) - \bar{s}\right) \right\rangle \sim e^{-t/\xi_{\parallel}}$$

where $\xi_{\parallel}$ is the *time correlation length*

Near criticality and $p < p_c$:

$$\xi_{\parallel} \sim (p_c - p)^{-\nu_{\parallel}}$$

RW's context: $\xi_{\parallel} =$ average first passage time

Because $p < p_c$, we consider a FP equation with negative drift term $-v$ ($v > 0$):

$$\frac{\partial p}{\partial t} = v\frac{\partial p}{\partial x} + D\frac{\partial^2 p}{\partial x^2}$$

If we impose $p(0,t) = 0$, $p(x,0) = \delta(x - x_0)$ and $L(0) = x_0 = 1$, the solution is

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[\exp\left(-\frac{(x - x_0 + vt)^2}{4Dt}\right) - \exp\left(\frac{vx_0}{D}\right)\exp\left(-\frac{(x + x_0 + vt)^2}{4Dt}\right)\right]$$

Probability current:

$$J(x,t) = -v\,p(x,t) - D\frac{\partial p}{\partial x}$$

Probability to be absorbed in $[t, t+dt]$:

$$\begin{aligned} f(t) &= \left| J(x,t)\Big|_{x=0} \right| = D\left|\frac{\partial p}{\partial x}\Big|_{x=0}\right| \\ &= \frac{x_0}{\sqrt{4\pi D t^3}} \exp\left(-\frac{(x_0 - vt)^2}{4Dt}\right) \end{aligned}$$

$$\xi_{\|} = \langle t_{\mathrm{rr}} \rangle = \int_0^\infty dt\; t\, f(t) = \frac{x_0}{\sqrt{4\pi D}} \int_0^\infty \frac{dt}{\sqrt{t}} \exp\left(-\frac{(x_0 - vt)^2}{4Dt} \right)$$
$$= \frac{x_0}{\sqrt{4\pi D}} \exp\left(\frac{x_0 v}{2D}\right) \int_0^\infty \frac{dt}{\sqrt{t}} \exp\left(-\left(\frac{x_0^2}{4D}\frac{1}{t} + \frac{v^2}{4D} t \right)\right)$$
$$= \sqrt{\frac{x_0^3}{\pi D v}} \exp\left(\frac{x_0 v}{2D}\right) K_{1/2}\left(\frac{x_0 v}{2D}\right)$$

where $K_{1/2}(x)$ is the modified Bessel function of the second kind

For small v, i.e. near criticality,

$$K_{1/2}(x) \simeq \sqrt{\pi/(2x)}$$

Hence:

$$\xi_{\|} \simeq \frac{x_0}{v} \exp\left(\frac{x_0 v}{2D}\right) \sim v^{-1} \qquad \text{for } v \to 0$$

In the isotropic case $t\, f(t) \sim 1/\sqrt{t}$ hence the average is divergent.

v is related to the *asymmetry* δ:

$$\delta = \frac{p^2 - (1-p)^2}{p^2 + (1-p)^2} = -\frac{1-2p}{1+2p(1-p)} \overset{p \to p_c}{\simeq} -(p_c - p)$$

Finally:

$$\xi_{\parallel} \sim v^{-1} \sim |\delta|^{-1} \simeq (p_c - p)^{-1}$$

$$\nu_{\parallel} = 1$$

Spatial correlation function:

$$c_{|i-j|} = \left\langle \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} \left(s_i(\tau) - \bar{s}\right)\left(s_j(\tau) - \bar{s}\right) \right\rangle \sim e^{-|i-j|/\xi_\perp}$$

where $\xi_\perp$ is the *spatial correlation length*

Near criticality and $p < p_c$:

$$\xi_\perp \sim (p_c - p)^{-\nu_\perp}$$

Interpretation: $\xi_\perp =$ average cluster's size at time $\xi_\parallel$

From the RW theory:

$$\xi_\perp \sim \xi_\parallel^{1/2} \sim (p - p_c)^{-1/2}$$

Hence:

$$\nu_\perp = \frac{1}{2}$$

Mapping:

Connection between CDP's dynamics and RW performed by $L(t)$

Exponents:

Exponent	Mean field DP	CDP
β'	1	1
β	1	0
θ	None	0
$\nu_{\parallel}$	1	1
$\nu_{\perp}$	1/2	1/2