Power-law statistics and universal scaling in the absence of criticality

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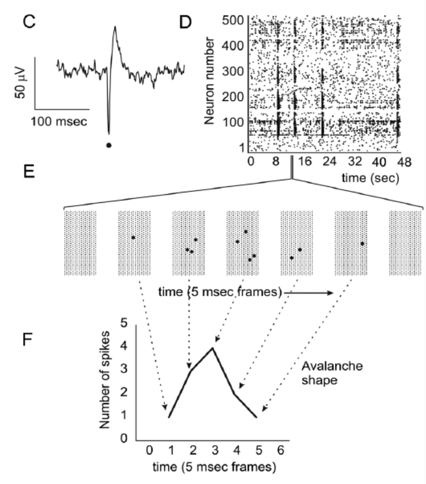
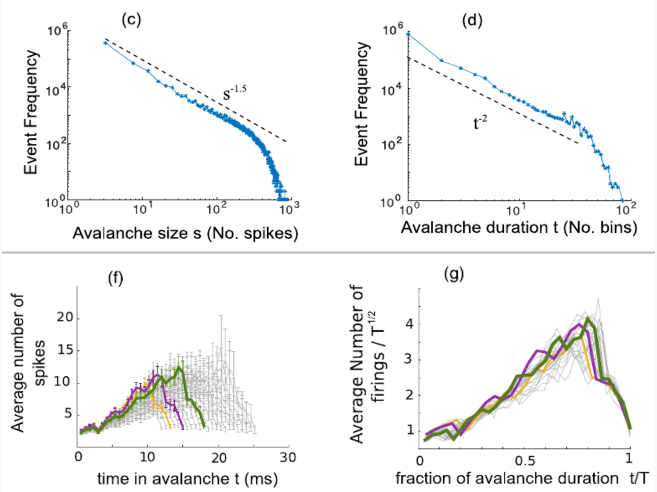


Friedman2012, Fig 1



Touboul2017, Fig 1

- Criticality
  E.g. SOC

- Purely stochastic mechanism
  E.g. random typewriter[3]

[3] Miller1957

Idea:
In the limit of large networks, correlation between neurons vanishes $\rightarrow$ *propagation of chaos*

Network simulated:
N identical and independent non homogeneous Poisson processes with rate $\rho(t)$ given by the positive part of a Ornstein-Uhlenbeck process

Procedure:
OU process $\Rightarrow$ non homogeneous Poisson ST statistics $\Rightarrow$ Avalanche statistics

Analytic results Duration distribution

Assumptions:

1. Stationarity
2. $q(t)$ slowing varying compared to avalanches
3. $N \gg 1$

$$
\begin{aligned}
p(d) &\overset{1}{=} \int_{[0,1]^{d+2}} q^{2N}(1-q^N)^d \, d\rho^{d+2}(q_0, \dots, q_{d+1}) \\
&\overset{2}{\simeq} \int_0^1 q^{2N}(1-q^N)^d \, d\rho(q) \\
&= \frac{1}{N} \int_0^1 x^{\frac{1}{N}+1}(1-x)^d \rho(x^{\frac{1}{N}}) \, dx \overset{3}{\simeq} \\
&= \frac{\rho(1)}{N} \frac{1}{(d+1)(d+2)} \sim d^{-2}
\end{aligned}
$$

Prediction: $\alpha = 2$

Analytic results Sizes distribution

Probability to have an avalanche of size s and duration d:

$$p(s,d) \simeq \int_0^1 \exp\left[-\frac{[s-d-Nd(1-q)]^2}{2Ndq(1-q)}\right] \frac{q^{2N}(1-q^N)^d}{\sqrt{2\pi Nqd(1-q)}}\rho(q)\,dq$$
$$\simeq \int_0^1 \frac{\rho(1)}{N}\exp\left[-\frac{[s-d(1+\ln x)]^2}{2d\ln x}\right] \frac{x(1-x)^d}{\sqrt{2\pi d\ln x}}\,dx$$

Summing over durations and at the leading order:

$$p(s) \sim e^s \sum_{d=1}^{s} \int_0^1 e^{\frac{(s-d)^2}{2d}} \frac{x[\sqrt{x}(1-x)]^d}{\sqrt{2\pi d\ln x}}\,dx$$

Prediction[4]: $\tau = 3/2$

[4] Touboul2017, Fig 3

Firing rate $\rho(t)$ generated through a stochastic process

Constrains:

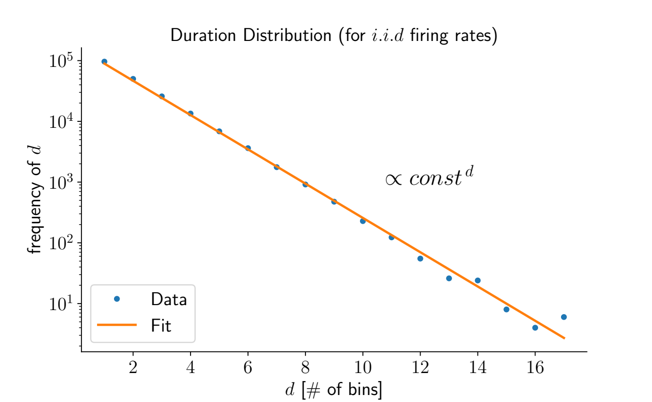
- Time correlation $\rightarrow$ Markov chain
- Periods of silence occur aperiodically

Simulation *i.i.d.* firing rate

If firing rate $\lambda \sim \gamma e^{-\gamma}$, i.i.d. at each timestep,

$$p_N(d) \propto \left(\frac{a}{\gamma + a}\right)^d$$

where $a = Ndt$



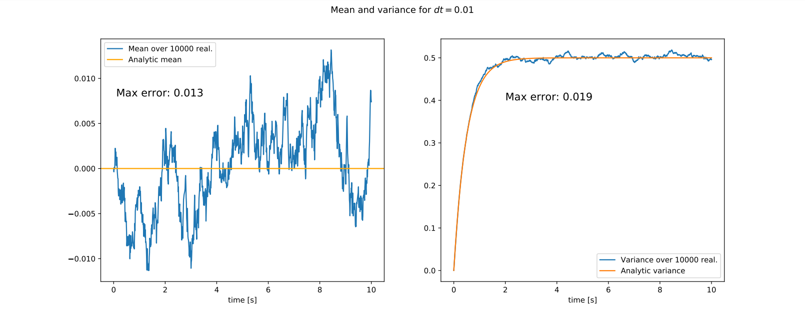
Euler-Mayurama algorithm[5]:

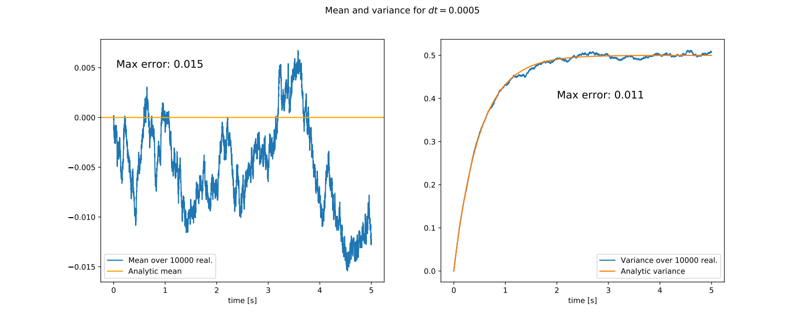
$$\rho_{i+1} = \rho_t - \alpha \rho_t dt + \sigma dW_t$$
$$dW_t \sim \sqrt{dt}\mathcal{N}(0,1), \quad i.i.d.$$

Choice of *dt*:

- Affects convergence of the algorithm
- Affects final exponents

[5] Higham2000 and Gardiner2004


Mean and variance for $dt = 0.01$
Mean over 10000 real.
Analytic mean
Max error: 0.013
0.010
0.005
0.000
−0.005
−0.010
0
2
4
6
8
10
time [s]
Variance over 10000 real.
Analytic variance
Max error: 0.019
0.5
0.4
0.3
0.2
0.1
0.0
0
2
4
6
8
10
time [s]


Mean and variance for $dt = 0.0005$
Max error: 0.015
Mean over 10000 real.
Analytic mean
0.005
0.000
−0.005
−0.010
−0.015
0
1
2
3
4
5
time [s]
Max error: 0.011
Variance over 10000 real.
Analytic variance
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
time [s]

Spike trains:
Thinning method[6]

Binning:
$\Delta t =$ mean time between any two spikes

Remark:

$dt \longrightarrow$ used for OU process and for generating spike trains

$\Delta t \longrightarrow$ used for time binning and avalanches' definition; $\Delta t = m dt, m \in \mathbb{N}_+$

[6] Higham2000, Algorithm 6

Power law **fit** in range $[x_{min}, x_{max}]$:

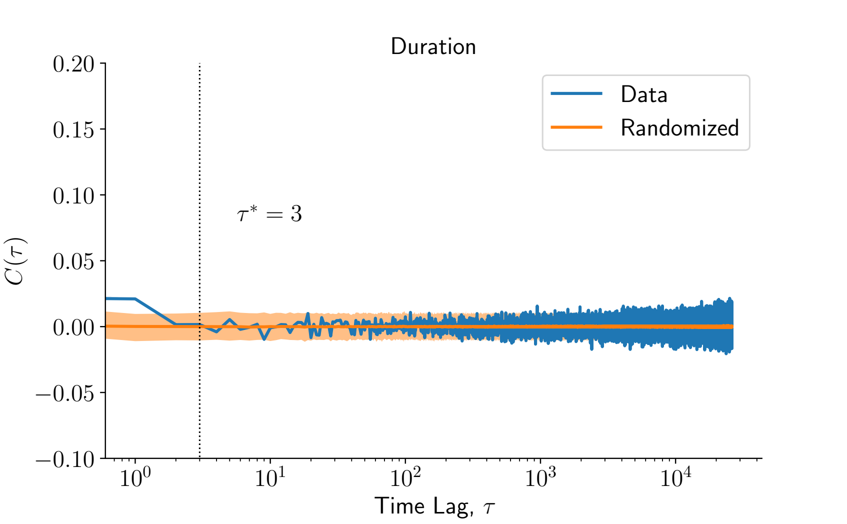
1. Chose x_{max} visually
2. Find x_{min} by minimizing the Kolmogorov-Smirnov distance
3. Maximum likelihood estimators for the scaling exponent, *discrete case*[7]

Goodness-of-fit:
Generation of synthetic data and compute the *p-value*

Null model:

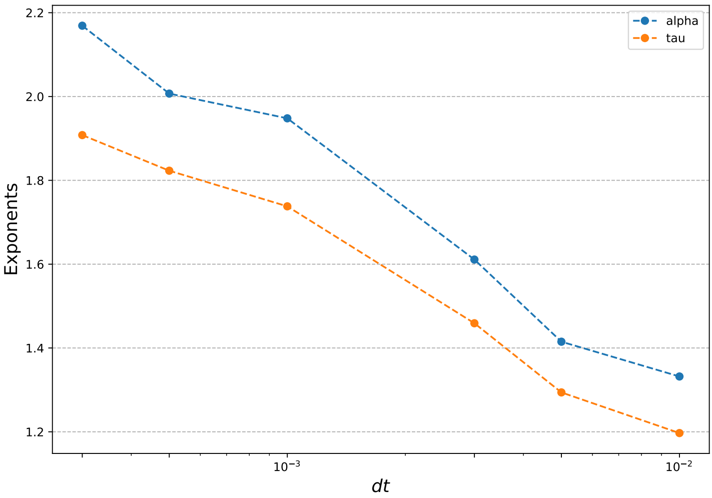
- The distribution follows a power law $p(s) = Cs^{-\alpha}$
- Empirical observations are independent[8]

[7] Clauset2007
[8] Gerlach2017

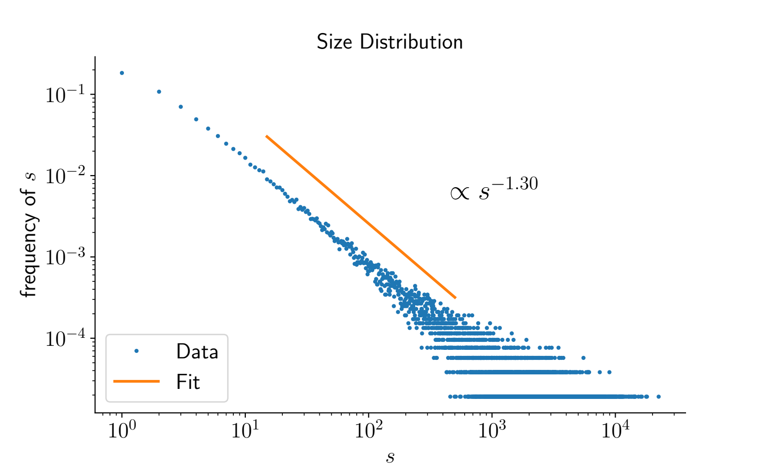


Duration
Data
Randomized
$\tau^* = 3$
$C(\tau)$
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
10^0
10^1
10^2
10^3
10^4
Time Lag, τ

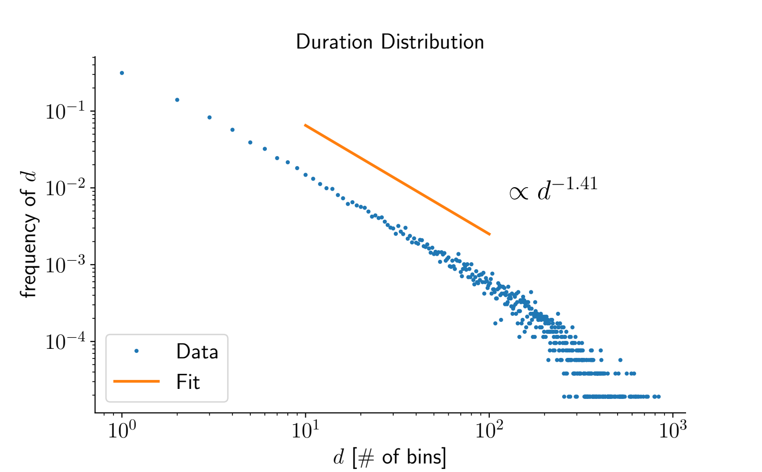
Results Exponents as a function of *dt*



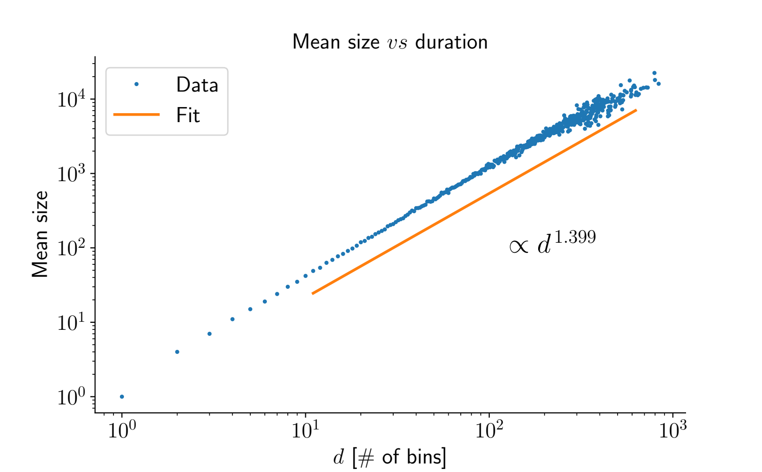
$dt = 0.005\,[sec]$


Size Distribution
frequency of s
s
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
$\propto s^{-1.30}$
Data
Fit

$dt = 0.005\,[sec]$


Duration Distribution
frequency of d
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\propto d^{-1.41}$
Data
Fit
10^{0}
10^{1}
10^{2}
10^{3}
d [# of bins]

$dt = 0.005\,[sec]$


Mean size vs duration
Data
Fit
$\propto d^{1.399}$
Mean size
10^0
10^1
10^2
10^3
10^4
d [# of bins]

Results Crackling noise relation

Introduce in different context when an avalanche dynamics is present, e.g. Crackling noise[9] or SOC models[10]

Crackling noise relation (CNR):

$$\gamma = \frac{\alpha - 1}{\tau - 1}$$

where

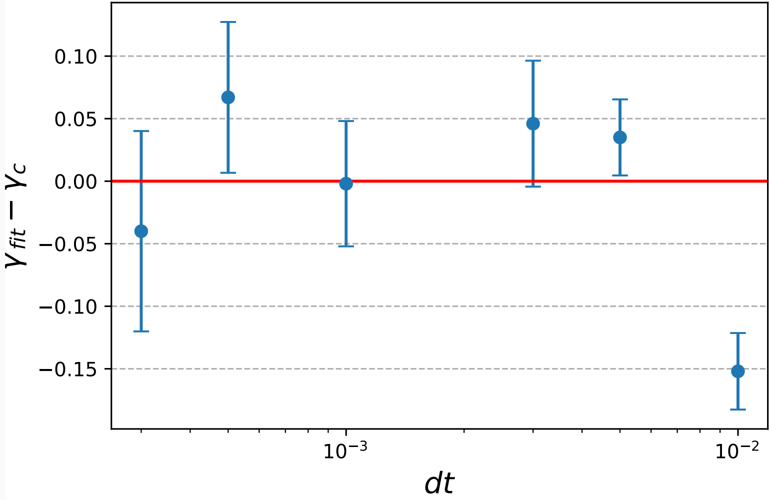
$$p(s) \sim s^{-\tau} \qquad p(d) \sim d^{-\alpha} \qquad \bar{s}(d) \sim d^{\gamma}$$

Remark:
CNR is derived using the avg temporal profile $\bar{V}(d,t)$ which satisfy the scaling relation:

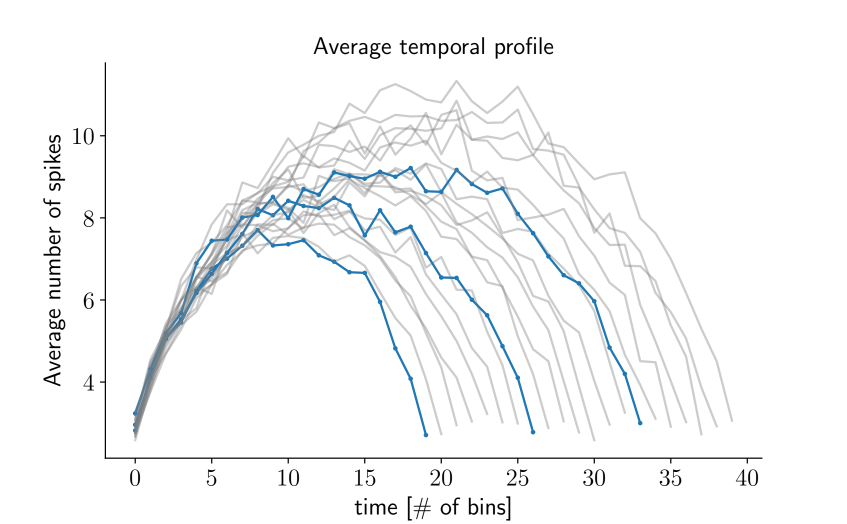
$$d^{-(\gamma-1)}\,\bar{V}(d,t) = v(t/d)$$

[9] Sathna2001
[10] Paczuski1996

Results Crackling noise relation

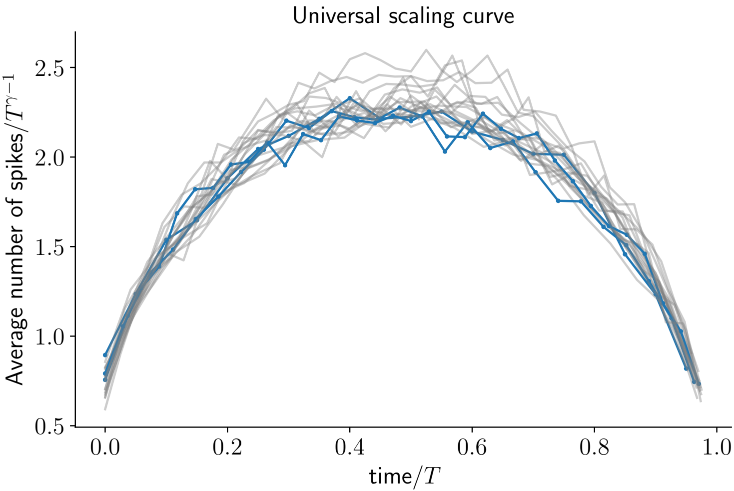


$dt = 0.005\,[sec]$


Average temporal profile
Average number of spikes
10
8
6
4
0
5
10
15
20
25
30
35
40
time [# of bins]

Results Universal scaling curve

$dt = 0.005\,[sec]$ $\qquad$ $d^{-(\gamma-1)}\,\bar{V}(d,t) = v(t/d)$



Recap and Discussion

Large neural networks may satisfy *propagation of caos*

Network of N independent neurons with a common firing rate $\rho(t)$ given by a OU process displays:

- Power laws (τ, α and γ)
- Universal scaling curve for avg temporal profile $\bar{V}(d, t)$

$\Rightarrow$ Power laws emerge from a non critical system

Recap and Discussion

Contributions:

- Scaling exponents varies as a functions of dt
- Crackling noise relations satisfied for almost all dt

Comments:

- Close relation between CNR and collapse of $\bar{V}(d, t)$
- Possible limits to the model

Appendix Connection between CNR and $\bar{V}(d,t)$

Defining[11] the avg. temporal profile $\bar{V}(d,t)$ and the change of variable $v(d,t/d) = \bar{V}(d,t)$ we can write the coarse-grained function as, assuming *self-similarity*:

$$
\begin{aligned}
v(d,t/d) &= Av(d/B,t/d) = (1+a\varepsilon)v((1-\varepsilon)d,t/d) \\
&\overset{\varepsilon \ll 1}{\simeq} v(d,t/d) - \varepsilon d\frac{\partial v}{\partial d} + a\varepsilon v(d,t/d)
\end{aligned}
$$

Which becomes

$$d\frac{\partial v}{\partial d} = av(d,t/d) \Rightarrow v(d,t/d) = d^a v_0(t/d)$$

Hence

$$v_0(t/d) = d^{-a}\bar{V}(d,t)$$

[11] Sathna2007

From its definition

$$\bar{S}(d) = \int \bar{V}(d,t)\,dt = \int d^a v_0(t/d)\,dt \sim d^{a+1} := d^\gamma$$

On the other hand, we know that

$$\bar{S}(d) \sim d^{d_f/z}$$

where d_f is the fractal dimention and z the exponent relating duration and typical largest extent of an avalanche

Hence

$$\gamma = \frac{d_f}{z}$$

Finally, the latter can be combined with the relation $\alpha = (\tau - 1)\frac{d_f}{z} + 1$ to obtain the Crackling noise relation