Optimal Control

DEPARTMENT OF COMPUTER, CONTROL, AND MANAGEMENT ENGINEERING ANTONIO RUBERTI

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Lecture - exercises

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Examples taken from Bruni et al. 2005

Example 1 (exercise 3.3)

Consider a vehicle of mass m=1 moving along a straight street.

Assume null initial position and velocity.

Indicate with u the force acting on the vehicle having the same direction of the street.

The aim is to transfer the vehicle to the **maximum distance** with **final null velocity** in a **fixed time interval [0, t_f],** with **a fixed energy**:

$$\int_0^{t_f} u^2(t)dt = K > 0$$

Indicate with:

$x_1(t)$ the position of the vehicle
$x_2(t)$ the velocity of the vehicle

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t) \\
&\dot{x}_2(t) = u(t) \\
&x_1(0) = x_2(0) = 0 \quad x_2(t_f) = 0 \\
&\int_0^{t_f} u^2(t)dt = K > 0
\end{aligned}$$

Cost Index

$$J(x) = -x_1(t_f) = -\int_0^{t_f} x_2(t)dt$$

The problem is not convex since the isoperimetric constraint is not linear

→ only necessary conditions

Define the Hamiltonian in the normal case:

$$H(x,u,1,\lambda,\rho) = -x_2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t) + \rho u^2(t)$$

→

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$\dot{\lambda}_1(t) = 0$$
$$\dot{\lambda}_2(t) = 1 - \lambda_1(t)$$
$$0 = 2\rho u(t) + \lambda_2(t)$$
$$\lambda_1(t_f) = 0$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$\dot{\lambda}_1(t) = 0$$
$$\dot{\lambda}_2(t) = 1 - \lambda_1(t)$$

$$\lambda_1(t) = 0$$
$$\lambda_2(t) = t + C$$

$$\lambda_1(t_f) = -\frac{\partial \aleph}{\partial x(t_f)}\bigg|^{*T} \zeta = 0$$

$$0 = 2\rho u(t) + \lambda_2(t)$$

u doesn't have discontinuity points

ρ can't be zero

$$u(t) = -\frac{\lambda_2(t)}{2\rho} = -\frac{t+C}{2\rho}$$

Substitute in the state equation

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$$u(t) = -\frac{\lambda_2(t)}{2\rho} = -\frac{t+C}{2\rho}$$

$$x_1(t) = -\frac{t^3}{12\rho} - \frac{C}{4\rho}t^2$$
$$x_2(t) = -\frac{t^3}{4\rho} - \frac{C}{2\rho}t$$

$$x_2(t_f) = 0$$

$$C = -\frac{t_f}{2}$$

$$K = \int_0^{t_f} u^2(t)dt = \frac{1}{4\rho^2}\int_0^{t_f}\left[t^2 - t_f t + \frac{t_f^2}{4}\right]dt = \frac{t_f^3}{48\rho^2}$$

$$K=\int_0^{t_f} u^2(t)dt=\frac{1}{4\rho^2}\int_0^{t_f}\left[t^2-t_f t+\frac{t_f^2}{4}\right]dt=\frac{t_f^3}{48\rho^2}$$

$$\Rightarrow \quad \rho=\pm\frac{t_f}{4}\sqrt{\frac{t_f}{3K}}K$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} x_1(t)&=-\frac{t^3}{12\rho}-\frac{t_f}{8\rho}t^2 \\ x_2(t)&=-\frac{t^3}{4\rho}-\frac{t_f}{4\rho}t \qquad x_1(t_f)=\frac{t_f^3}{24\rho}=\pm\frac{t_f}{2}\sqrt{\frac{t_f}{3}K} \\ u(t)&=\frac{1}{2\rho}\left(\frac{t_f}{2}-t\right)\end{aligned}}$$

No abnormal extremum exists.

In fact: define the Hamiltonian in the abnormal case:

$$H(x,u,1,\lambda,\rho) = \lambda_1(t)x_2(t) + \lambda_2(t)u(t) + \rho u^2(t)$$

$\Rightarrow$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$\dot{\lambda}_1(t) = 0$$
$$\dot{\lambda}_2(t) = -\lambda_1(t)$$
$$0 = 2\rho u(t) + \lambda_2(t)$$
$$\lambda_1(t_f) = 0$$

$\Rightarrow$

$$\lambda_1(t) = 0$$
$$\lambda_2(t) = C$$

$\Downarrow$

$\Rightarrow$ $C \neq 0$ $\Rightarrow$ $u(t) = -\dfrac{C}{2\rho}$

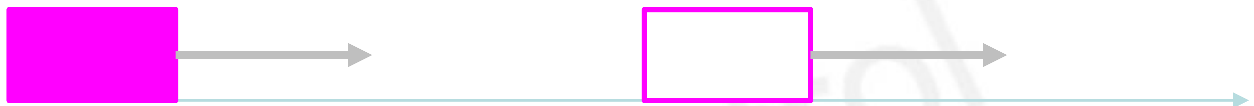
$$u(t) = -\frac{C}{2\rho}$$

$$\Rightarrow \quad x_2(t) = -\frac{C}{2\rho}t$$

That is not compatible with $x_2(t_f) = 0$

Example 2- same as example 1 with a change

Consider a vehicle of mass m=1 moving along a straight street.



Assume **null initial position and velocity**.
Indicate with u the force acting on the vehicle having the same direction of the street.
<u>Assume **NOT fixed the final instant**.</u>

The aim is to **perform an automatic coupling of a second vehicle moving with a uniform move (with velocity v) along the same street,**
assuming as cost index:

$$J(u,t_f) = \frac{1}{2}\int_0^{t_f} u^2(t)dt$$

$$\aleph\left(x(t_f),t_f\right)=\begin{pmatrix} x_1(t_f)-vt_f-r \\ x_2(t_f)-v \end{pmatrix}=0$$

The final instant is not fixed, therefore the problem is not convex.

$$H(x,u,1,\lambda,\rho)=\frac{1}{2}u^2(t)+\lambda_1(t)x_2(t)+\lambda_2(t)u(t)$$

$$\dot{\lambda}_1(t)=0$$
$$\dot{\lambda}_2(t)=-\lambda_1(t)$$
$$0=u(t)+\lambda_2(t)$$

$$\lambda_1(t)=C_1$$
$$\lambda_2(t)=-C_1t+C_2$$
$$u(t)=C_1t-C_2$$

$$x_1(t) = C_1 \frac{t^3}{6} - C_2 \frac{t^2}{2}$$

Integrate the state equations with null initial conditions

$$x_2(t) = C_1 \frac{t^2}{2} - C_2 t$$

From the transversality conditions and taking into account the stationarity of the problem:

$$\lambda(t_f) = -\frac{\partial \aleph}{\partial x(t_f)}\bigg|^{*T} \zeta = -\varsigma \ , \quad \varsigma = \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \end{pmatrix},$$

$$H\Big|_{t_f} = \frac{\partial \aleph}{\partial t_f}\bigg|^{*T} \varsigma = -\varsigma_1 v$$

$$\lambda_1(t_f)v = H\Big|_0 \quad \Rightarrow \quad 2vC_1 = -C_2^2$$

$$\aleph\left(x(t_f),t_f\right)=\begin{pmatrix} x_1(t_f)-vt_f-r \\ x_2(t_f)-v \end{pmatrix}=0$$

$$2vC_1=-C_2^2$$

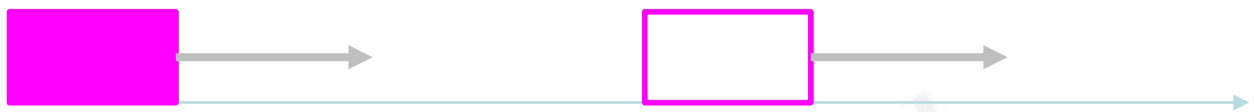
$$-C_2^2\frac{t_f^3}{12v}-C_2\frac{t_f^2}{2}=vt_f+r$$

$$-C_2^2\frac{t_f^2}{4v}-C_2t_f=v$$

$$t_f=-\frac{3r}{v} \qquad C_2=\frac{2v^2}{3r} \Rightarrow \qquad C_1=\frac{2v^3}{9r^2}$$

Example 3- same as example 2 with a change

Consider a vehicle of mass m=1 moving along a straight street.



Assume null initial position and velocity.
Indicate with u the force acting on the vehicle having the same direction of the street.
Assume NOT fixed the final instant.
The aim is to perform **an automatic coupling** of a second vehicle moving with a uniform move (with velocity v) along the same street,
assuming fixed the energy for the control

$$\int_0^{t_f} u^2(t)dt = K > 0$$

and **minimizing the cost index:**

$$J(t_f) = t_f$$

Define the Hamiltonian in the normal case:

$$H(x,u,1,\lambda,\rho)=1+\lambda_1(t)x_2(t)+\lambda_2(t)u(t)+\rho u^2(t)$$

$$\dot{x}_1(t)=x_2(t)$$
$$\dot{x}_2(t)=u(t)$$
$$\dot{\lambda}_1(t)=0$$
$$\dot{\lambda}_2(t)=-\lambda_1(t)$$
$$0=2\rho u(t)+\lambda_2(t)$$

$$\lambda_1(t)=C_1$$
$$\lambda_2(t)=-C_1t+C_2$$

$$C_1 v=1+C_2u(0)+\rho u^2(0)$$

$$u(t)=\frac{C_1t}{2\rho}-\frac{C_2}{2\rho}$$

$$x_1(t) = \frac{C_1 t^3}{12\rho} - \frac{C_2}{4\rho} t^2$$

$$x_2(t) = \frac{C_1 t^2}{4\rho} - \frac{C_2}{2\rho} t$$

$$C_1 v = 1 + C_2 u(0) + \rho u^2(0)$$

$$C_1 = \frac{1}{v} - \frac{C_2^2}{4v\rho}$$

$$\left(\frac{1}{v} - \frac{C_2^2}{4v\rho}\right)^2 \frac{t_f^3}{12\rho^2} - \frac{C_2 t_f^2}{4\rho^2} = v t_f + r$$

$$\left(\frac{1}{v} - \frac{C_2^2}{4v\rho}\right)^2 \frac{t_f^2}{4\rho^2} - \frac{C_2 t_f}{2\rho} = v$$

$$K=\int_0^{t_f}u^2(t)dt=$$

$$=\left(\frac{1}{v}-\frac{C_2^2}{4v\rho}\right)^2\frac{t_f^3}{12\rho^2}-\left(\frac{1}{v}-\frac{C_2^2}{4v\rho}\right)^2\frac{C_2t_f^2}{4\rho^2}+\frac{C_2t_f}{4\rho^2}$$

$$\left(\frac{1}{v}-\frac{C_2^2}{4v\rho}\right)^2\frac{t_f^3}{12\rho^2}-\frac{C_2t_f^2}{4\rho^2}=vt_f+r$$

$$\left(\frac{1}{v}-\frac{C_2^2}{4v\rho}\right)^2\frac{t_f^2}{4\rho^2}-\frac{C_2t_f}{2\rho}=v \quad\Rightarrow\quad \left(\frac{1}{v}-\frac{C_2^2}{4v\rho}\right)^2\frac{t_f^2}{4\rho^2}=\frac{C_2t_f}{2\rho}+v$$

$$\frac{C_2 t_f}{\rho} = -\frac{8vt_f + 12r}{t_f}$$

$$K = \int_0^{t_f} u^2(t)dt =$$

$$= \left(\frac{1}{v} - \frac{C_2^2}{4v\rho}\right)^2 \frac{t_f^3}{12\rho^2} - \left(\frac{1}{v} - \frac{C_2^2}{4v\rho}\right)^2 \frac{C_2 t_f^2}{4\rho^2} + \frac{C_2 t_f}{4\rho^2}$$

$$12(vt_f + 2r)^2 + 4(2vt_f + 3r)^2 - 12(vt_f + 2r)(2vt_f + 3r) = Kt_f^3$$

Assume

$$r = \pm 1 \quad v = 3 \quad K = 12$$

If $r = 1$ $\Rightarrow$ $t_f = 3.85$ $\Rightarrow$

$$C_1 = -0.74$$
$$C_2 = -1.83$$
$$\rho = 0.26$$

If $r = -1$ $\Rightarrow$ $t_f = 1$ $\Rightarrow$

The two equations

$$\left(\frac{1}{v} - \frac{C_2^2}{4v\rho}\right)^2 \frac{t_f^2}{4\rho^2} = \frac{C_2 t_f}{2\rho} + v$$

$$\frac{C_2 t_f}{\rho} = -\frac{8vt_f + 12r}{t_f}$$

don't have solution

Define the Hamiltonian in the **abnormal case:**

$$H(x,u,1,\lambda,\rho)=\lambda_1(t)x_2(t)+\lambda_2(t)u(t)+\rho u^2(t)$$

$$C_1 v = C_2 u(0)+\rho u^2(0) \quad \Longrightarrow \quad C_1=-\frac{C_2^2}{4v\rho}$$

Substituting into the admissibility equations and the isoperimetric Constraint:

$$-\frac{C_2^2}{48v\rho^2}t_f^3-\frac{C_2}{4\rho}t_f^2=vt_f+r$$

$$-\frac{C_2^2}{16v\rho^2}t_f^2-\frac{C_2}{2\rho}t_f=v \quad \Longrightarrow \quad \frac{C_2}{\rho}=-\frac{4v}{t_f}$$

$$-\frac{C_2^4}{192v^2\rho^4}t_f^3+\frac{C_2^2}{4\rho^2}t_f+\frac{C_2^3}{16v\rho^3}t_f^2=K \quad \Longrightarrow \quad t_f=\frac{4v^2}{3K}$$

$$-\frac{C_2^2}{48v\rho^2}t_f^3-\frac{C_2}{4\rho}t_f^2=vt_f+r$$

$$-\frac{C_2^2}{16v\rho^2}t_f^2-\frac{C_2}{2\rho}t_f=v$$

$$-\frac{C_2^4}{192v^2\rho^4}t_f^3+\frac{C_2^2}{4\rho^2}t_f+\frac{C_2^3}{16v\rho^3}t_f^2=K$$

$$\boxed{K=-\frac{4v^3}{9r}}$$

Abnormal extremal exist only if the parameters satisfy this relation.

For the choice $r = \pm 1 \quad v = 3 \quad K = 12$

If $r = 1$ ⇒ no solution exists

If $r = -1$ ⇒ $t_f = 1$ ⇒ $C_1 = C_2 = -12\rho$

⇒ **there exist infinite number of abnormal solutions**

Example 4

Consider a simple model of phamacodynamic:

$$
\begin{aligned}
\dot{x}_1(t) &= -k_1 x_1(t) + u_1(t) \qquad && x_1(0) = 0 \\
\dot{x}_2(t) &= k_1 x_1(t) - k_2 x_2(t) + u_2(t) \qquad && x_2(0) = 0 \\
k_2 &> k_1
\end{aligned}
$$

x_1= quantity of drug in the gastrintestinal section
x_2= quantity of drug in the blood
r=reference value of the quantity of drug in the blood

u1 flow rate of the drug by oral administration
u2 flow rate of the drug by administered intravenously

Assume

$$
u_1(t,Q) = \begin{cases} Q & t \in [0, \alpha T] \\ 0 & t \in (\alpha T, T] \end{cases} \qquad \alpha \in (0,1)
$$

Minimize:

$$J(Q, x_2, u_2) = \frac{1}{2}\int_0^T \left\{[r - x_2(t)]^2 + \beta\left[u_1^2(t,Q) + u_2^2(t)\right]\right\} dt$$

$$= \frac{1}{2}\int_0^T \left\{[r - x_2(t)]^2 + \beta u_2^2(t)\right\} dt + \frac{1}{2}\alpha\beta T Q^2, \qquad \beta > 0$$

* * *

Solution of the state equation:

$$x_1(t,Q) = \begin{cases} \dfrac{Q}{k_1}\left(1 - e^{-k_1 t}\right) & t \in [0, \alpha T] \\ \overline{x}_1(Q) e^{-k_1(t-\alpha T)} & t \in [\alpha T, T] \end{cases}$$

$$\overline{x}_1(Q) = \frac{Q}{k_1}\left(1 - e^{-\alpha k_1 T}\right)$$

$$x_1(t,Q) \geq 0$$

Obviously:

$x_2 = x'_2 + x''_2$

intravenous Gastrintestinal

$$\begin{aligned}
\dot{x}'_2(t) &= -k_2 x'_2(t) + u_2(t), && x'_2(0) = 0 \\
\dot{x}''_2(t) &= -k_2 x''_2(t) + k_1 x_1(t, Q), && x''_2(0) = 0
\end{aligned}$$

$$x''_2(t, Q) = \begin{cases} Q\left(\dfrac{1}{k_2} + \dfrac{k_1 e^{-k_2 t} - k_2 e^{-k_1 t}}{k_2(k_2 - k_1)}\right) & t \in [0, \alpha T] \\ \bar{x}''_2(Q) e^{-k_2(t - \alpha T)} + \\ + \bar{x}_1(Q) \dfrac{k_1}{k_2 - k_1}\left(e^{k_1(\alpha T - t)} - e^{k_2(\alpha T - t)}\right) & t \in [\alpha T, T] \end{cases}$$

$$\bar{x}''_2(Q) = Q\left(\frac{1}{k_2} + \frac{k_1 e^{-k_2 \alpha T} - k_2 e^{-k_1 \alpha T}}{k_2(k_2 - k_1)}\right)$$

Since: $k_2 > k_1$ It could be verified that:

$$\frac{1}{k_2} \geq \frac{k_2 e^{-k_1 t} - k_1 e^{-k_2 t}}{k_2 (k_2 - k_1)}, \qquad \forall t \geq 0$$

$$x_2''(t,Q) \geq 0 \ \forall t \in [0, \alpha T] \ \forall Q > 0$$

$$x_2''(t,Q) \geq 0 \ \forall t \in [\alpha T, T] \ \forall Q > 0$$

Define: $\overline{r}(t,Q) = r - x_2''(t,Q)$

It can be defined the following tracking problem:

$$\dot{x}_2'(t) = -k_2 x_2'(t) + u_2(t), \qquad x_2'(0) = 0$$

$$J(Q, x_2', u_2) = \frac{1}{2} \int_0^T \left\{ \left[\overline{r}(t,Q) - x_2'(t) \right]^2 + \beta u_2^2(t) \right\} dt + \\ + \frac{1}{2} \alpha \beta T Q^2$$

The problem is convex and the Lagrangian is **strictly convex**

necessary and sufficient conditions and,
if the solution exists, it is unique

$$H(x_2'(t), u_2(t), 1, \lambda(t)) = \frac{1}{2}\left[\overline{r}(t,Q) - x_2'(t)\right]^2 + \frac{1}{2}\beta u_2^2(t) - \lambda(t)k_2 x_2'(t) + \\ +\lambda(t)u_2(t)$$

$$\dot{\lambda}^o(t) = k_2\lambda^o(t) - x_2'^o(t) + \overline{r}(t,Q)$$

$$\lambda^o(t) + \beta u_2^o(t) = 0$$

$$\lambda^o(T) = 0$$

By using the **theory of optimal tracking**:

$$u_2^o(t) = \frac{1}{\beta}\left[g(t) - K(t)x_2'^o(t)\right]$$

$$\dot{x}_2'^o(t) = -\left(k_2 + \frac{K(t)}{\beta}\right)x_2'^o(t) + \frac{1}{\beta}g(t), \qquad x_2'^o(0) = 0$$

where:

$$\dot{K}(t) = \frac{K^2(t)}{\beta} + 2k_2K(t) - 1, \qquad K(T) = 0$$

$$K(t) = \frac{\beta - \beta e^{-\sqrt{k_2^2+1/\beta}\,(T-t)}}{C_1 e^{-\sqrt{k_2^2+1/\beta}\,(T-t)} - C_2}$$

$$C_1 = -\beta k_2 + \beta\sqrt{k_2^2 + 1/\beta}$$

$$C_2 = -\beta k_2 - \beta\sqrt{k_2^2 + 1/\beta}$$

$$\dot{g}(t) = \left(k_2 + \frac{K(t)}{\beta}\right) g(t) - \overline{r}(t,Q), \quad g(T) = 0$$

$$g(t,Q) = \int_t^T e^{-\int_t^\tau \omega(\sigma)d\sigma} \overline{r}(\tau,Q)d\tau$$

where $\quad \omega(t) = \frac{K(t)}{\beta} + k_2$

If the admissibility condition $u_2^o(t) \geq 0$ is satisfied then the couple

$$u_2^o(t) = \frac{1}{\beta}\left[g(t) - K(t)x_2'^o(t)\right]$$

$$\dot{x}_2'^o(t) = -\left(k_2 + \frac{K(t)}{\beta}\right) x_2'^o(t) + \frac{1}{\beta} g(t), \quad x_2'^o(0) = 0$$

is the unique optimal solution of the subproblem

To determine Q>0 it is possible to solve another quadratic problem:

$$J(Q)=\frac{1}{2}\int_0^T\left\{[\gamma(t)+Q\vartheta(t)]^2+\beta\,[\eta(t)+Q\zeta(t)]^2\right\}dt+\frac{1}{2}\alpha\beta TQ^2$$

where $\gamma,\ \vartheta,\ \eta,\ \zeta$ are suitable affine functions depending on the Problem; the cost index is strictly convex

$$\frac{d^2J}{dQ^2}=\int_0^T\left[\vartheta^2(t)+\zeta^2(t)\right]dt+\alpha\beta T>0$$

If a solution exists
it is unique and is given by
the condition:

$$\left.\frac{dJ}{dQ}\right|_{Q^o}=0 \quad\Rightarrow\quad \boxed{Q^o=-\frac{\displaystyle\int_0^T[\gamma(t)\vartheta(t)+\eta(t)\zeta(t)]\,dt}{\displaystyle\int_0^T\left[\vartheta^2(t)+\zeta^2(t)\right]dt+\alpha\beta T}}$$

Exercise

Consider the system: $\dot{x}(t) = u(t) \qquad x(0) = 0$

The aim is to have: $x(t_f) \geq 1, \quad t_f \geq 1$

with a bounded control $u(t) \in [0,1]$

minimizing

$$J(u, t_f) = \int_0^{t_f} [3 + u(t) - x(t)] dt$$

The final instant is not fixed $\Rightarrow$ the problem is not convex

The Hamiltonian is:

$$H(x(t), u(t), 1, \lambda(t)) = 3 + u(t) - x(t) + \lambda(t) u(t)$$

The costate equation yields $\lambda^*(t) = t + C$

The minimum condition is: $u^*(t)\left(1+\lambda^*(t)\right) \leq \omega^*(t)\left(1+\lambda^*(t)\right) \quad \forall \omega \in [0,\, 1]$

$$\Rightarrow \quad u^*(t) = \frac{1}{2}\left(1 - sign\{t + 1 + C\}\right)$$

he problem may be analyzed for four different cases:

$$
\begin{aligned}
S'_f &= \{t_f = 1, x(t_f) = 1\} \\
S''_f &= \{t_f = 1, x(t_f) > 1\} \\
S'''_f &= \{t_f > 1, x(t_f) = 1\} \\
S'^{v}_f &= \{t_f > 1, x(t_f) > 1\}
\end{aligned}
$$

and the transverality conditions will be particularized:

1) $t_f = 1, x(t_f) = 1$ Three alternatives are possible:

a) $C \geq -1$. From $u^*(t) = \frac{1}{2}\left(1 - sign\{t + 1 + C\}\right)$

$$t + 1 + C > 0$$

$$u^*(t) = 0 \qquad x^*(t) = 0$$ Not possible

b) $C \in (-2, -1)$ There exists a discontinuity instant $\bar{t} \in (0,1)$

$$\bar{t} = -1 - C$$

$$u^*(t) = \begin{cases} 1 & \text{per } t \in [0, \bar{t}) \\ 0 & \text{per } t \in (\bar{t}, 1] \end{cases}$$

$$x^*(t) = \begin{cases} t & \text{per } t \in [0, \bar{t}) \\ \bar{t} = -1 - C & \text{per } t \in (\bar{t}, 1] \end{cases}$$

$$x^*(1) = -1 - C = 1, \quad \Rightarrow \quad C = -2$$

Not possible

c) . $C \leq 2$ $\Rightarrow$ $t+1+C<0, \quad for \quad t \in [0,1)$

$\Rightarrow$ $u^*(t)=1$ $\Rightarrow$ $x^*(t)=t$ acceptable

$\Rightarrow$ In case 1) we have obtained one normal extremum:

$$t_f = 1 \qquad u^*(t)=1 \qquad x^*(t)=t$$

$$\lambda(t) = t + C, \ \forall C \leq 2$$

2) $t_f = 1, x(t_f) > 1.$ The transversality condition yields:

$$\lambda^*(1) = 1 + C = 0$$

$$\Rightarrow \quad C = -1$$

$$\Rightarrow \quad u^*(t) = 0 \quad \Rightarrow \quad x^*(t) = 0$$ Not possible

3) $t_f > 1, \quad x(t_f) = 1$ The transversality condition yields:

$$\Rightarrow \quad H\Big|_{t_f^*}^{*} = 0$$

From the stationarity of the problem:

$$H|_0^* = 3 + (1 + C)\frac{1}{2}(1 - sign\{1 + C\}) = 0$$

$$\Rightarrow \quad C = -4 \quad \Rightarrow \quad u^*(t) = \frac{1}{2}(1 - sign\{t - 3\})$$

AS far as the value of $t_f{}^*$ is concerned two alternatives are possible:

i) $t_f^* \in (1, 3]$. From $u^*(t) = \frac{1}{2}\left(1 - sign\{t-3\}\right)$

$$\Rightarrow \quad u^*(t) = 1 \quad \Rightarrow \quad x^*(t) = t$$

$$\Rightarrow \quad x^*(t_f^*) = t^*_f - 1$$ Not possible

ii) $t_f^* > 3 \quad \Rightarrow \quad u^*(t) = \begin{cases} 1 & \text{per } t \in (0,3) \\ 0 & \text{per } t \in \left(3, t_f^*\right] \end{cases}$

$$\Rightarrow \quad x^*(t) = \begin{cases} t & \text{per } t \in [0,3] \\ 3 & \text{per } t \in \left[3, t_f^*\right] \end{cases}$$

$$x(t_f^*) = 3$$

4) $t_f > 1, \quad x(t_f) > 1$. The transversality condition yields:

$$\lambda^*(t_f^*) = t_f^* + C = 0 \qquad \text{From} \qquad u^*(t) = \frac{1}{2}\left(1 - sign\{t-3\}\right)$$

$$\Rightarrow \quad H\Big|_{t_f^*}^{*} = 3 - x^*(t_f^*) = 0$$

$\Rightarrow$ There exists an instant $\bar{t} = t_f^* - 1 \in \left(0, \, t_f^*\right)$ of discontinuity for the control such that:

$$u^*(t) = \begin{cases} 1 & \text{per } t \in \left[0, t_f^* - 1\right) \\ 0 & \text{per } t \in \left(t_f^* - 1, t_f^*\right] \end{cases}$$

$$x^*(t) = \begin{cases} t & \text{per } t \in \left[0, t_f^* - 1\right] \\ t_f^* - 1 & \text{per } t \in \left[t_f^* - 1, t_f^*\right] \end{cases}$$

From the transversality condition: $x(t_f^*) = 3 = t_f^* - 1$

In case 4) we have obtained one normal extremum:

$$t_f^* = 4 \quad u^*(t) = \begin{cases} 1 & \text{per } t \in [0,3) \\ 0 & \text{per } t \in (3,4] \end{cases} \qquad x^*(t) = \begin{cases} t & \text{per } t \in [0,3] \\ 3 & \text{per } t \in [3,4] \end{cases}$$

With: $\lambda(t) = t - 1$

The value of the cost index evaluated at point of case 4) is bigger than the value of the cost index evaluated at the solution of point found in case 1).

The normal extremum is:

$$t_f = 1 \qquad u^*(t) = 1 \qquad x^*(t) = t$$
$$\lambda(t) = t + C, \ \forall C \leq 2$$

As far as the abnormal solution is concerned, all the solutions with

$\lambda^*(t_f^*) = 0$ must be rejected .

For $x(t_f^*) = 1$ $\Rightarrow$ $\lambda^*(t) = C \neq 0$

$\Rightarrow$

$$u^*(t) = \begin{cases} 0 & \text{per } C > 0 \\ 1 & \text{per } C < 0 \end{cases}$$

$\Rightarrow$ $x(t_f^*) = 0$ Not acceptable:

$\Rightarrow$ Not possible with the condition $t_f > 1$

If $t_f = 1$ one obtains $t_f = 1$

$$u^*(t) = 1$$

with: $\lambda(t) = C, \ \forall C \leq 0$

$$x^*(t) = t$$

The unique possible solution is the extremum

$$t_f = 1 \qquad u^*(t) = 1 \qquad x^*(t) = t$$

That is both a normal and abnormal solution.