If, in addition, the vector fields (9.2) commute, the equations (9.10) assume the special form

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= v \\ y &= \xi_1 \ . \end{aligned} \tag{9.11}$$

Of course, if $r = n$, the system in question is linear, controllable and observable.

Note also that, if $r < n$, the submanifold $Z^\star$ is the largest (with respect to inclusion) smooth submanifold of $h^{-1}(0)$ with the property that, at each $x \in Z^\star$, there is $u^\star(x)$ such that $f^\star(x) = f(x) + g(x)u^\star(x)$ is tangent to $Z^\star$. Actually, for each $x \in Z^\star$ there is only one $u^\star(x)$ rendering this condition satisfied, namely,

$$u^\star(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \ .$$

In particular, the vector field $f^\star(x)|_{Z^\star}$ which characterizes the zero dynamics of the system can be identified with the vector field

$$f_0(z, 0, \ldots, 0) \frac{\partial}{\partial z}$$

of $Z^\star$.

Remark 9.1.1. In the previous analysis, the $(n-r)$-dimensional sumbanifold $Z^\star$ is *not* required to be diffeomorphic to $\mathbb{R}^{n-r}$. However, in all subsequent sections, we will – almost always – consider the case in which the vector field $f^\star(x)|_{Z^\star}$ has a globally asymptotically stable equilibrium at $x = 0$. If this the case, then necessarily $Z^\star$ *is* diffeomorphic to $\mathbb{R}^{n-r}$ (see section B.2). Thus, for the sake of simplicity, we will assume throughout that, in the equations (9.10) and (9.11),

$$(z, \xi) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \ . \triangleleft$$

9.2 Examples of Global Asymptotic Stabilization

In this section we discuss a number of cases in which it is possible to design a feedback law which globally asymptotically stabilizes the equilibrium $x = 0$ of system (9.1). We restrict our attention to those system which have (some) uniform relative degree r, in which the submanifold $Z^\star$ is diffeomorphic to $\mathbb{R}^{n-r}$ and in which the vector fields (9.2) are complete. Thus, without loss of generality, in view of the results established in the previous section, we can assume that the system in question is modeled by equations of the form (9.10) or, more particularly, of the form (9.11) if the vector fields (9.2) also commute.

The results which follows describe a simple "modular" property which is instrumental in proving an important stabilizability result about the system in question. Recall that a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, and *proper* if, for any $a > 0$, the set $V^{-1}([0,a]) = \{x \in \mathbb{R}^n : 0 \leq V(x) \leq a\}$ is compact.

Lemma 9.2.1. *Consider a system described by equations of the form*

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= u \end{aligned} \tag{9.12}$$

in which $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$, *and* $f(0,0) = 0$. *Suppose there exists a smooth real-valued function* $V(z)$, *which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f(z,0) < 0$$

for all nonzero z. *Then, there exists a smooth static feedback law* $u = u(z,\xi)$ *with* $u(0,0) = 0$, *and a smooth real-valued function* $W(z,\xi)$, *which is positive definite and proper, such that*

$$\begin{pmatrix} \dfrac{\partial W}{\partial z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z,\xi) \\ u(z,\xi) \end{pmatrix} < 0$$

for all nonzero (z,ξ).

Proof. Observe that the function $f(z,\xi)$ can be put in the form

$$f(z,\xi) = f(z,0) + p(z,\xi)\xi \tag{9.13}$$

where $p(z,\xi)$ is a smooth function. For, it suffices to observe that the difference

$$\bar{f}(z,\xi) = f(z,\xi) - f(z,0)$$

is a smooth function vanishing at $\xi = 0$, and express $\bar{f}(z,\xi)$ as

$$\bar{f}(z,\xi) = \int_0^1 \frac{\partial \bar{f}(z,s\xi)}{\partial s} ds = \int_0^1 \left[\frac{\partial \bar{f}(z,\zeta)}{\partial \zeta}\right]_{\zeta = s\xi} \xi ds \ .$$

Now, consider the positive definite and proper function

$$W(z,\xi) = V(z) + \frac{1}{2}\xi^2 \ , \tag{9.14}$$

and observe that

$$\begin{pmatrix} \dfrac{\partial W}{\partial z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z,\xi) \\ u \end{pmatrix} = \frac{\partial V}{\partial z} f(z,0) + \frac{\partial V}{\partial z} p(z,\xi)\xi + \xi u \ .$$

Choosing

$$u = u(z,\xi) = -\xi - \frac{\partial V}{\partial z} p(z,\xi) \tag{9.15}$$

yields the required result. $\triangleleft$

In view of the converse Lyapunov Theorem (see section B.2), the hypothesis of this Lemma (namely the hypothesis of the existence of a smooth positive definite and proper function $V(z)$ such that $\frac{\partial V}{\partial z} f(z,0)$ is negative for each nonzero z) is implied by the hypothesis that the *subsystem*

$$\dot{z} = f(z,0)$$

has a globally asymptotically stable equilibrium at $z = 0$. On the other hand, now by the direct Lyapunov Theorem, the conclusion of the Lemma implies that system

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= u(z,\xi) \end{aligned}$$

has a globally asymptotically stable equilibrium at $(z,\xi) = (0,0)$. Thus, the result indicated in this Lemma simply says that, if $\dot{z} = f(z,0)$ a globally asymptotically stable equilibrium at $z = 0$, then the equilibrium $(z,\xi) = (0,0)$ of system (9.12) can be rendered globally asymptotically stable by means of a smooth feedback law $u = u(z,\xi)$.

In the next Lemma (which contains Lemma 9.2.1 as a particular case) this result is extended, by showing that, to the purpose of stabilizing the equilibrium $(z,\xi) = (0,0)$ of system (9.12), it suffices to assume that the equilibrium $z = 0$ of

$$\dot{z} = f(z,\xi)$$

is *stabilizable* by means of smooth law $\xi = v^\star(z)$.

Lemma 9.2.2. *Consider a system described by equations of the form (9.12). Suppose there exists a smooth real-valued function*

$$\xi = v^\star(z)\ ,$$

with $v^\star(0) = 0$, *and a smooth real-valued function* $V(z)$, *which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f(z, v^\star(z)) < 0$$

for all nonzero z. *Then, there exists a smooth static feedback law* $u = u(z,\xi)$ *with* $u(0,0) = 0$, *and a smooth real-valued function* $W(z,\xi)$, *which is positive definite and proper, such that*

$$\begin{pmatrix} \dfrac{\partial W}{\partial z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z,\xi) \\ u(z,\xi) \end{pmatrix} < 0$$

for all nonzero (z,ξ).

Proof. It suffices to consider the (globally defined) change of variables

$$y = \xi - v^\star(z) ,$$

which transforms (9.12) into

$$\begin{aligned} \dot{z} &= f(z, v^\star(z) + y) \\ \dot{y} &= -\frac{\partial v^\star}{\partial z} f(z, v^\star(z) + y) + u , \end{aligned} \tag{9.16}$$

and observe that the feedback law

$$u = \frac{\partial v^\star}{\partial z} f(z, v^\star(z) + y) + u'$$

changes the latter into a system satisfying the hypotheses of Lemma 9.2.1.◁

Using repeatedly the property indicated in Lemma 9.2.2 it is straightforward to derive the following stabilization result about a system in the form (9.11).

Theorem 9.2.3. *Consider a system of the form*

$$\begin{array}{rcl} \dot{z} &=& f_0(z, \xi_1) \\ \dot{\xi}_1 &=& \xi_2 \\ && \cdots \\ \dot{\xi}_{r-1} &=& \xi_r \\ \dot{\xi}_r &=& u . \end{array} \tag{9.17}$$

Suppose there exists a smooth real-valued function

$$\xi_1 = v^\star(z) ,$$

with $v^\star(0) = 0$, *and a smooth real-valued function* $V(z)$, *which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f_0(z, v^\star(z)) < 0$$

for all nonzero z. *Then, there exists a smooth static feedback law*

$$u = u(z, \xi_1, \ldots, \xi_r)$$

with $u(0, 0, \ldots, 0) = 0$, *which globally asymptotically stabilizes the equilibrium* $(z, \xi_1, \ldots, \xi_r) = (0, 0, \ldots, 0)$ *of the corresponding closed loop system.*

Of course, a special case in which the result of Theorem 9.2.3 holds is when $v^\star(z) = 0$ i.e. when $\dot{z} = f_0(z, 0)$ has a globally asymptotically stable equilibrium at $z = 0$. This is the case of a system of the form (9.11) whose *zero dynamics* have a globally asymptotically stable equilibrium at $z = 0$, which, for the sake of completeness, is described separately in the following (trivial) Corollary of Theorem 9.2.3.

Corollary 9.2.4. *Consider a system of the form (9.17). Suppose its zero dynamics have a globally asymptotically stable equilibrium at* $z = 0$*. Then, there exists a smooth static feedback law*

$$u = u(z, \xi_1, \dots, \xi_r)$$

with $u(0, 0, \dots, 0) = 0$*, which globally asymptotically stabilizes the equilibrium* $(z, \xi_1, \dots, \xi_r) = (0, 0, \dots, 0)$ *of the corresponding closed loop system.*

Remark 9.2.1. In analogy with the case of linear systems, which are traditionally said to be "minimum phase" when all their transmission zeros have negative real part, nonlinear systems (of the form (9.10)) whose zero dynamics have a globally asymptotically stable equilibrium at $z = 0$ are also called *minimum phase* systems. $\triangleleft$

We now present an extension of Lemma 9.2.1, in which the hypothesis that $\frac{\partial V}{\partial z} f(z, 0)$ is negative *definite* is replaced by the hypothesis that this function is just negative *semidefinite*, together with a "controllability"-like assumption.

Lemma 9.2.5. *Consider a system described by equations the form (9.12). Suppose there exists a smooth real-valued function* $V(z)$*, which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f(z, 0) \le 0$$

for all z*. Set*

$$f^\star(z) = f(z, 0) \qquad g^\star(z) = \frac{\partial f}{\partial \xi}(z, 0) \ ,$$

and

$$S^\star = \bigcap_{i \ge 0} \bigcap_{k \ge 0} \{z \in \mathbb{R}^n : L^i_{f^\star} L_{ad^k_{f^\star} g^\star} V(z) = 0\} \ .$$

Suppose $S^\star = \{0\}$*. Then, there exists a smooth static feedback law* $u = u(z, \xi)$ *with* $u(0, 0) = 0$*, and a smooth real-valued function* $W(z, \xi)$*, which is positive definite and proper, such that*

$$\begin{pmatrix} \dfrac{\partial W}{\partial z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z, \xi) \\ u(z, \xi) \end{pmatrix} < 0$$

for all nonzero (z, ξ)*.*

Proof. Consider again the expansion (9.13) and observe that, by definition

$$g^\star(z) = p(z, 0) \ .$$

Choosing the input (9.15), the positive definite function (9.14) satisfies