Chapter 3
The Small-gain Theorem for Linear Systems and its Applications to Robust Stability

Abstract In a system consisting of the interconnection of several component subsystems, some of which could be only poorly modeled, stability analysis and feedback design might not be easy tasks. Thus, methods allowing to understand the influence of interconnections on stability and asymptotic behavior are important. The methods in question are based on the use of a concept of *gain*, which can take alternative forms and can be evaluated by means of a number of alternative methods. This Chapter describes the various alternative forms of such concept of gain, and shows why this is useful in the analysis of stability of interconnected systems. A major consequence is the development of a systematic method for stabilization in the presence of a (general class of) model uncertainties.

3.1 The $\mathscr{L}_2$ gain of a stable linear system

In this section, we analyze some properties of the forced response of a linear system to piecewise continuous input functions – defined on the time interval $[0,\infty)$ – which have the following property

$$\lim_{T\to\infty}\int_0^T \|u(t)\|^2 dt < \infty\,.$$

The space of all such functions, endowed with the so-called $\mathscr{L}_2$ norm, which is defined as

$$\|u(\cdot)\|_{\mathscr{L}_2} := \Big(\int_0^\infty \|u(t)\|^2 dt\Big)^{\frac{1}{2}},$$

is denoted by $\mathscr{U}_{\mathscr{L}_2}$. The main purpose of the analysis is to show that, if the system is *stable*, the forced output response from the initial state $x(0) = 0$ has a similar property, i.e.

$$\lim_{T\to\infty}\int_0^T \|y(t)\|^2 dt < \infty\,. \tag{3.1}$$

This makes it possible to compare the $\mathscr{L}_2$ norms of input and output functions and define a concept of "gain" accordingly.

Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.2}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$. Suppose the matrix A has all eigenvalues with negative real part. Let α be a positive number. According to the direct criterion of Lyapunov, the equation

$$PA + A^{\mathrm{T}}P = -\alpha I \tag{3.3}$$

has a unique solution P, which is a symmetric and positive definite matrix.

Let $V(x) = x^{\mathrm{T}}Px$ be the associated positive definite quadratic form. If $x(t)$ is any trajectory of (2.2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}(t)Px(t)] &= \frac{\partial V}{\partial x}\Big|_{x=x(t)} \dot{x}(t) = 2x^{\mathrm{T}}(t)P(Ax(t)+Bu(t)) \\ &= x^{\mathrm{T}}(t)(PA + A^{\mathrm{T}}P)x(t) + 2x^{\mathrm{T}}(t)PBu(t)\,. \end{aligned}$$

Adding and subtracting $d^2u(t)^{\mathrm{T}}u(t)$ to the right-hand side, and dropping – for convenience – the dependence on t, we obtain successively

$$\begin{aligned} \frac{\partial V}{\partial x}(Ax+Bu) &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + d^2u^{\mathrm{T}}u - d^2u^{\mathrm{T}}u + 2x^{\mathrm{T}}PBu \\ &= -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u - d^2(u - \tfrac{1}{d^2}B^{\mathrm{T}}Px)^{\mathrm{T}}(u - \tfrac{1}{d^2}B^{\mathrm{T}}Px) + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px \\ &\leq -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px\,. \end{aligned}$$

Subtracting and adding to the right-hand side the quantity $y^{\mathrm{T}}y$ and using the inequality

$$y^{\mathrm{T}}y = (Cx+Du)^{\mathrm{T}}(Cx+Du) \leq 2x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2u^{\mathrm{T}}D^{\mathrm{T}}Du$$

it is seen that

$$\begin{aligned} \frac{\partial V}{\partial x}(Ax+Bu) &\leq -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px - y^{\mathrm{T}}y + 2x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2u^{\mathrm{T}}D^{\mathrm{T}}Du \\ &= x^{\mathrm{T}}(-\alpha I + \tfrac{1}{d^2}PBB^{\mathrm{T}}P + 2C^{\mathrm{T}}C)x + u^{\mathrm{T}}(d^2I + 2D^{\mathrm{T}}D)u - y^{\mathrm{T}}y\,. \end{aligned}$$

Clearly, for any choice of $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$-\alpha I + 2C^{\mathrm{T}}C \leq -2\varepsilon I\,.$$

Pick one of such α and let P be determined accordingly as a solution of (3.3). With P fixed in this way, it is seen that, if the number d is sufficiently large, the inequality

$$\frac{1}{d^2}PBB^{\mathrm{T}}P \leq \varepsilon I$$

holds, and hence

$$x^{\mathrm{T}}(-\alpha I+\frac{1}{d^2}PBB^{\mathrm{T}}P+2C^{\mathrm{T}}C)x\le -\varepsilon x^{\mathrm{T}}x\,.$$

Finally, let $\bar\gamma>0$ be any number satisfying

$$d^2I+2D^{\mathrm{T}}D\le \bar\gamma^2 I\,.$$

Using the last two inequalities, it is concluded that the derivative of $V(x(t))$ *along the trajectories of* (3.2) satisfies an inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))=\frac{\partial V}{\partial x}\Big|_{x=x(t)}(Ax(t)+Bu(t))\le -\varepsilon\|x(t)\|^2+\bar\gamma^2\|u(t)\|^2-\|y(t)\|^2\,. \quad (3.4)$$

This inequality is called a *dissipation inequality*. [1] Summarizing the discussion up to this point, one can claim that for *any stable linear system*, given any positive real number ε, it is always possible to find a positive definite symmetric matrix P and a coefficient $\bar\gamma$ such that a dissipation inequality of the form (3.4), which – dropping for convenience the dependence on t – can be simply written as

$$\frac{\partial V(x)}{\partial x}(Ax+Bu)\le -\varepsilon\|x\|^2+\bar\gamma^2\|u\|^2-\|y\|^2\,, \quad (3.5)$$

is satisfied.

The inequality thus established plays a fundamental role in characterizing a parameter, associated with a stable linear system, which is called the $\mathscr{L}_2$ *gain*. As a matter of fact, suppose the input $u(\cdot)$ of (3.2) is a function in $\mathscr{U}_{\mathscr{L}_2}$. Integration of the inequality (3.4) on the interval $[0,t]$ yields, for any initial state $x(0)$,

$$\begin{aligned}V(x(t)) &\le V(x(0))+\bar\gamma^2\int_0^t\|u(\tau)\|^2d\tau-\int_0^t\|y(\tau)\|^2d\tau\\ &\le V(x(0))+\bar\gamma^2\int_0^\infty\|u(\tau)\|^2d\tau=V(x(0))+\bar\gamma^2\Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2\,,\end{aligned}$$

from which it is deduced that the response $x(t)$ of the system is defined for all $t\in[0,\infty)$ and bounded. Now, suppose $x(0)=0$ and observe that the previous inequality yields

$$V(x(t))\le \bar\gamma^2\int_0^t\|u(\tau)\|^2d\tau-\int_0^t\|y(\tau)\|^2d\tau$$

for any $t>0$. Since $V(x(t))\ge 0$, it is seen that

$$\int_0^t\|y(\tau)\|^2d\tau\le\bar\gamma^2\int_0^t\|u(\tau)\|^2dt\le\bar\gamma^2\Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2$$

[1] The concept of dissipation inequality was introduced by J.C. Willems in [1], to which the reader is referred for more details.

for any $t > 0$ and therefore the property (3.1) holds. In particular,

$$\left[\|y(\cdot)\|_{\mathscr{L}_2}\right]^2 \le \bar{\gamma}^2 \left[\|u(\cdot)\|_{\mathscr{L}_2}\right]^2 ,$$

i.e.

$$\|y(\cdot)\|_{\mathscr{L}_2} \le \bar{\gamma}\|u(\cdot)\|_{\mathscr{L}_2} .$$

In summary, for any $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$, the response of a stable linear system from the initial state $x(0) = 0$ is defined for all $t \ge 0$ and produces an output $y(\cdot)$ which has the property (3.1). Moreover the *ratio* between the $\mathscr{L}_2$ norm of the output and the $\mathscr{L}_2$ norm of the input is bounded by the number $\bar{\gamma}$ which appears in the dissipation inequality (3.5).

Having seen that, in stable linear system, an input having finite $\mathscr{L}_2$ norm produces, from the initial state $x(0) = 0$, an output response which also has a finite $\mathscr{L}_2$ norm, suggests to look at the ratios between such norms, for all possible $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$, and to seek the least upper bound of such ratios. The quantity thus defined is called the $\mathscr{L}_2$ *gain* of the (stable) linear system. Formally the gain in question is defined as follows: pick any $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$ and let $y_{0,u}(\cdot)$ be the resulting response from the initial state $x(0) = 0$; the $\mathscr{L}_2$ gain of the system is the quantity

$$\mathscr{L}_2 \; gain = \sup_{\|u(\cdot)\|_{\mathscr{L}_2} \neq 0} \frac{\|y_{0,u}(\cdot)\|_{\mathscr{L}_2}}{\|u(\cdot)\|_{\mathscr{L}_2}} .$$

With this definition in mind, return to the dissipation inequality (3.5). Suppose that a system of the form (3.2) is given and that an inequality of the form (3.5) holds for a positive definite matrix P. We have in particular (set $u = 0$)

$$\frac{\partial V(x)}{\partial x} Ax \le -\varepsilon \|x\|^2 ,$$

from which it is seen that the system is stable. Since the system is stable, the $\mathscr{L}_2$ gain can be defined and, as a consequence of the previous discussion, it is seen that

$$\mathscr{L}_2 \; gain \le \bar{\gamma}. \tag{3.6}$$

Thus, in summary, if a inequality of the form (3.5) holds for a positive definite matrix P, the system is stable and *its* $\mathscr{L}_2$ *gain is bounded from above by the number* $\bar{\gamma}$.

We will see in the next section that the fulfillment of an inequality of the form (3.5) is equivalent to the fulfillment of a *linear matrix inequality* involving the system data A, B, C, D.

3.2 An LMI characterization of the $\mathscr{L}_2$ gain

In this section, we derive alternative characterizations of the inequality (3.5).

Lemma 3.1. *Let $V(x) = x^{\mathrm{T}}Px$, with P a positive definite symmetric matrix. Suppose that, for some $\varepsilon > 0$ and some $\bar{\gamma} > 0$, the inequality*

$$\frac{\partial V}{\partial x}(Ax+Bu) \leq -\varepsilon\|x\|^2 + \bar{\gamma}^2\|u\|^2 - \|Cx+Du\|^2 \tag{3.7}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Then, for any $\gamma > \bar{\gamma}$,

$$D^{\mathrm{T}}D - \gamma^2 I < 0 \tag{3.8}$$

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0\,. \tag{3.9}$$

Conversely, suppose (3.8) and (3.9) hold for some γ. Then there exists $\varepsilon > 0$ such that, for all $\bar{\gamma}$ satisfying $0 < \gamma - \varepsilon < \bar{\gamma} < \gamma$, the inequality (3.7) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Proof. Suppose that the inequality (3.7), that can be written as

$$2x^{\mathrm{T}}P[Ax+Bu] + \varepsilon x^{\mathrm{T}}x - \bar{\gamma}^2 u^{\mathrm{T}}u + x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2x^{\mathrm{T}}C^{\mathrm{T}}Du + u^{\mathrm{T}}D^{\mathrm{T}}Du \leq 0\,, \tag{3.10}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. For $x = 0$ this implies, in particular,

$$-\bar{\gamma}^2 I + D^{\mathrm{T}}D \leq 0\,.$$

Since $\gamma > \bar{\gamma}$, it follows that $D^{\mathrm{T}}D < \gamma^2 I$, which is precisely condition (3.8). Moreover, since $\gamma > \bar{\gamma}$, (3.7) implies

$$2x^{\mathrm{T}}P[Ax+Bu] + \varepsilon x^{\mathrm{T}}x - \gamma^2 u^{\mathrm{T}}u + x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2x^{\mathrm{T}}C^{\mathrm{T}}Du + u^{\mathrm{T}}D^{\mathrm{T}}Du \leq 0 \tag{3.11}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Now, observe that, for each fixed x, the left-hand side of (3.11) is a quadratic form in u, expressible as

$$M(x) + N(x)u - u^{\mathrm{T}}Wu\,. \tag{3.12}$$

in which

$$W = \gamma^2 I - D^{\mathrm{T}}D$$

and

$$M(x) = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon I)x\,, \qquad N(x) = 2x^{\mathrm{T}}(PB + C^{\mathrm{T}}D)\,.$$

Since W is positive definite, this form has a unique maximum point, at

$$u_{\max}(x) = \frac{1}{2}W^{-1}N^{\mathrm{T}}(x)\,.$$

Hence, (3.11) holds if and only if the value of the form (3.12) at $u = u_{\max}(x)$ is non positive, that is if and only if

$$M(x) + \frac{1}{4}N(x)W^{-1}N(x)^{\mathrm{T}} \leq 0\,.$$

Using the expressions of $M(x), N(x), W$, the latter reads as

$$x^{\mathrm{T}}(PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon I + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}})x \leq 0$$

and this, since $\varepsilon > 0$, implies condition (3.9).

To prove the converse claim, observe that (3.9) implies

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon_1 I + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0 \qquad (3.13)$$

provided that $\varepsilon_1 > 0$ is small enough. The left-hand sides of (3.8) and (3.13), which are negative definite, by continuity remain negative definite if γ is replaced by any $\bar{\gamma}$ satisfying $\gamma - \varepsilon_2 < \bar{\gamma} < \gamma$, provided that $\varepsilon_2 > 0$ is small enough. Take now $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. The resulting form $M(x) + N(x)u - u^{\mathrm{T}}\bar{W}u$, in which $\bar{W} = \bar{\gamma}^2 I - D^{\mathrm{T}}D$, is non positive and (3.7) holds.

Remark 3.1. Note that the inequalities (3.8) and (3.9) take a substantially simpler form in the case of a system in which $D = 0$ (i.e. systems with no direct feed-through between input and output). In this case, in fact, (3.8) becomes irrelevant and (3.9) reduces to

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \frac{1}{\gamma^2}PBB^{\mathrm{T}}P < 0\,.$$

Lemma 3.2. *Let γ be a fixed positive number. The inequality (3.8) holds* and *there exists a positive definite symmetric matrix P satisfying (3.9) if and only if there exists a positive definite symmetric matrix X satisfying*

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{pmatrix} < 0\,. \qquad (3.14)$$

Proof. Consider the matrix inequality

$$\begin{pmatrix} A^{\mathrm{T}}X + XA + \frac{1}{\gamma}C^{\mathrm{T}}C & XB + \frac{1}{\gamma}C^{\mathrm{T}}D \\ B^{\mathrm{T}}X + \frac{1}{\gamma}D^{\mathrm{T}}C & -\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D \end{pmatrix} < 0\,. \qquad (3.15)$$

This inequality holds if and only if the lower right block

$$-\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D$$

is negative definite, which is equivalent to condition (3.8), and so is its the Schur's complement

$$A^{\mathrm T}X+XA+\frac{1}{\gamma}C^{\mathrm T}C-[XB+\frac{1}{\gamma}C^{\mathrm T}D][-\gamma I+\frac{1}{\gamma}D^{\mathrm T}D]^{-1}[XB+\frac{1}{\gamma}C^{\mathrm T}D]^{\mathrm T}\,.$$

This, having replaced X by $\frac{1}{\gamma}P$, is identical to condition (3.9).

Rewrite now (3.15) as

$$\begin{pmatrix} A^{\mathrm T}X+XA & XB \\ B^{\mathrm T}X & -\gamma I\end{pmatrix}+\begin{pmatrix} C^{\mathrm T} \\ D^{\mathrm T}\end{pmatrix}\frac{1}{\gamma}\begin{pmatrix} C & D\end{pmatrix}<0$$

and use again (backward) Schur's complement to arrive at (3.14). $\triangleleft$

3.3 The H_∞ norm of a transfer function

Functions having finite $\mathscr{L}_2$ norm may be seen as signals having *finite energy* [2] over the infinite time interval $[0,\infty)$, and therefore the $\mathscr{L}_2$ gain can be given the interpretation of (an upper bound of the) ratio between energies of output and input. Another similar interpretation, in terms of energies associated with input and output, is possible, which does not necessarily require the consideration of the case of finite energy over the infinite time interval $[0,\infty)$. Suppose the input is a *periodic* function of time, with period T, i.e. that

$$u(t+kT)=\bar{u}(t),\quad \text{for all } t\in[0,T),\text{ all integer } k$$

for some piecewise continuous function $\bar{u}(t)$, defined on $[0,T)$. Also, suppose that, for some suitable initial state $x(0)=\bar{x}$, the state response $x(t)$ of the system is defined for all $t\in[0,T]$ and satisfies

$$x(T)=\bar{x}\,.$$

Then, it is obvious that $x(t)$ exists for all $t\geq 0$, and is a *periodic* function, having the same period T of the input, namely

$$x(t+kT)=x(t),\quad \text{for all } t\in[0,T),\ k\geq 0$$

and so is the corresponding output response $y(t)$.

For the triplet $\{u(t),x(t),y(t)\}$ thus defined, integration of the inequality (3.4) over an interval $[t_0,t_0+T]$, with arbitrary $t_0\geq 0$, yields

$$V(x(t_0+T))-V(x(t_0))\leq\bar{\gamma}^2\int_{t_0}^{t_0+T}\|u(\tau)\|^2d\tau-\int_{t_0}^{t_0+T}\|y(\tau)\|^2d\tau\,,$$

i.e., since $V(x(t_0+T))=V(x(t_0))$,

[2] If, in the actual physical system, the components of the input $u(t)$ are *voltages* (or *currents*), the quantity $\|u(t)\|^2$ can be seen as instantaneous *power*, at time t, associated with such input and its integral over a time interval $[t_0,t_1]$ as *energy* associated with such input over this time interval.

$$\int_{t_0}^{t_0+T} \|y(\tau)\|^2 d\tau \leq \bar{\gamma}^2 \int_{t_0}^{t_0+T} \|u(\tau)\|^2 d\tau \,. \tag{3.16}$$

Observe that the integrals on both sides of this inequality are independent of t_0, because the integrands are periodic functions having period T, and recall that the *root mean square* value of any (possibly vector-valued) periodic function $f(t)$ (which is usually abbreviated as "r.m.s." and characterizes the *average power* of the signal represented by $f(t)$ is defined as

$$\|f(\cdot)\|_{\text{r.m.s.}} = \Big(\frac{1}{T}\int_{t_0}^{t_0+T} \|f(\tau)\|^2 d\tau\Big)^{\frac{1}{2}} .$$

With this in mind, (3.16) yields

$$\|y(\cdot)\|_{\text{r.m.s.}} \leq \bar{\gamma}\|u(\cdot)\|_{\text{r.m.s.}} \,. \tag{3.17}$$

In other words, the number $\bar{\gamma}$ which appears in the inequality (3.5) can be seen also an upper bound for the ratio between the r.m.s. value of the output and the r.m.s value of the input, whenever a periodic input is producing (from an appropriate initial state) a periodic (state and output) response.

Consider now the special case in which the input signal is a *harmonic function* of time, i.e.

$$u(t) = u_0 \cos(\omega_0 t)$$

It is known that, if the state $x(0)$ is appropriately chosen, [3] the output response of the system coincides with the so-called *steady-state response*, which is the harmonic function

$$y_{\text{ss}}(t) = \text{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \text{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t) \,,$$

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D \,.$$

Recall that

$$\int_0^{\frac{2\pi}{\omega_0}} \|u(t)\|^2 dt = \frac{\pi}{\omega_0}\|u_0\|^2$$

and therefore

$$\int_0^{\frac{2\pi}{\omega_0}} \|y_{\text{ss}}(t)\|^2 dt = \frac{\pi}{\omega_0}\|T(j\omega_0)u_0\|^2 \,.$$

In other words

$$\begin{aligned} \|u(\cdot)\|^2_{\text{r.m.s.}} &= \tfrac{1}{2}\|u_0\|^2 \\ \|y_{\text{ss}}(\cdot)\|^2_{\text{r.m.s.}} &= \tfrac{1}{2}\|T(j\omega_0)u_0\|^2 \,. \end{aligned}$$

Thus, from the interpretation illustrated above one can conclude that, if the system satisfies (3.5), then

[3] See Section A.5 in Appendix A. This is the case if $x(0) = \Pi_1$, in which Π_1 is the first column of the solution Π of the Sylvester equation (A.29).

$$\|T(j\omega_0)u_0\|^2 = 2\|y_{\rm ss}(\cdot)\|^2_{\rm r.m.s.} \le \bar\gamma^2 2\|u(\cdot)\|^2_{\rm r.m.s.} = \bar\gamma^2\|u_0\|^2$$

i.e.

$$\|T(j\omega_0)u_0\| \le \bar\gamma\|u_0\|\,,$$

or, bearing in mind the definition of norm of a matrix, [4]

$$\|T(j\omega_0)\| \le \bar\gamma.$$

Define now the quantity

$$\|T\|_{H_\infty} := \sup_{\omega\in\mathbb{R}} \|T(j\omega)\|\,,$$

which is called the H_∞ *norm* of the matrix $T(j\omega)$. Observing that ω_0 in the above inequality is arbitrary, it is concluded

$$\|T\|_{H_\infty} \le \bar\gamma\,. \tag{3.18}$$

Therefore, a linear system that satisfies (3.5) is stable and *the H_∞ norm of its frequency response matrix is bounded from above by the number $\bar\gamma$.*

3.4 The bounded real lemma

We have seen in the previous sections that, for a system of the form (3.2), if there exists a number $\gamma > 0$ and a symmetric positive definite matrix X satisfying (3.14), then there exists a number $\bar\gamma < \gamma$ and a positive definite quadratic form $V(x)$ satisfying (3.5) for some $\varepsilon > 0$. This, in view of the interpretation provided above, proves that the fulfillment of (3.14) for some γ (with X positive definite) *implies* that

(i) the system is asymptotically stable,

(ii) its $\mathscr{L}_2$ gain is strictly less than γ,

(iii) the H_∞ norm of its transfer function is strictly less than γ.

However, put in these terms, we have only learned that (3.5) implies both (ii) and (iii) and we have not investigated yet whether converse implications might hold. In this section, we complete the analysis, by showing that the two properties (ii) and (iii) are, in fact, two different manifestations of the same property and both imply (3.14).

This will be done by means of a circular proof involving another equivalent version of the property that the number γ is an upper bound for the H_∞ norm of the

[4] Recall that the norm of a matrix $T \in \mathbb{R}^{p\times m}$ is defined as

$$\|T\| = \sup_{\|u\|\neq 0} \frac{\|Tu\|}{\|u\|} = \max_{\|u\|=1} \|Tu\|\,.$$

transfer function matrix of the system, which is very useful for practical purposes, since it can be easily checked. More precisely, the fact that γ is an upper bound for the H_∞ norm of the transfer function matrix of the system can be checked by looking at the spectrum of a matrix of the form

$$H = \begin{pmatrix} A_0 & R_0 \\ -Q_0 & -A_0^{\mathrm{T}} \end{pmatrix}, \tag{3.19}$$

in which R_0 and Q_0 are *symmetric* matrices which, together with A_0, depend on the matrices A, B, C, D which characterize the system and on the number γ.[5]

As a matter of fact the following result, known in the literature as *Bounded Real Lemma*, holds.

Theorem 3.1. *Consider the linear system (3.2) and let $\gamma > 0$ be a fixed number. The following are equivalent:*

(i) *there exists $\bar{\gamma} < \gamma$, $\varepsilon > 0$ and a symmetric positive definite matrix P such that (3.7) holds for $V(x) = x^{\mathrm{T}}Px$,*

(ii) *all the eigenvalues of A have negative real part and the frequency response matrix of the system $T(j\omega) = C(j\omega I - A)^{-1}B + D$ satisfies*

$$\|T\|_{H_\infty} := \sup_{\omega\in\mathbb{R}} \|T(j\omega)\| < \gamma\,, \tag{3.20}$$

(iii) *all the eigenvalues of A have negative real part, the matrix $W = \gamma^2 I - D^{\mathrm{T}}D$ is positive definite, and the Hamiltonian matrix*

$$H = \begin{pmatrix} A + BW^{-1}D^{\mathrm{T}}C & BW^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}C - C^{\mathrm{T}}DW^{-1}D^{\mathrm{T}}C & -A^{\mathrm{T}} - C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}} \end{pmatrix} \tag{3.21}$$

has no eigenvalues on the imaginary axis,

(iv) *there exists a positive definite symmetric matrix X satisfying*

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{pmatrix} < 0\,. \tag{3.22}$$

Proof. We have already shown, in the previous sections, that if (i) holds, then (3.2) is an asymptotically stable system, with a frequency response matrix satisfying

$$\|T\|_{H_\infty} \le \bar{\gamma}\,.$$

Thus, (i) $\Rightarrow$ (ii).

To show that (ii) $\Rightarrow$ (iii), first of all that note, since

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[5] A matrix (of real numbers) with this structure is called an *Hamiltonian matrix* and has the property that its spectrum is symmetric with respect to the imaginary axis (see Lemma A.5 in Appendix A).

$$\lim_{\omega\to\infty} T(j\omega) = D$$

it necessarily follows that $\|Du\| < \gamma$ for all u with $\|u\| = 1$ and this implies $\gamma^2 I > D^{\mathrm{T}} D$, i.e. the matrix W is positive definite.

Now observe that the Hamiltonian matrix (3.21) can be expressed in the form

$$H = L + MN$$

for

$$L = \begin{pmatrix} A & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{pmatrix}, \quad M = \begin{pmatrix} B \\ -C^{\mathrm{T}}D \end{pmatrix},$$

$$N = \begin{pmatrix} W^{-1}D^{\mathrm{T}}C & W^{-1}B^{\mathrm{T}} \end{pmatrix}.$$

Suppose, by contradiction, that the matrix H has eigenvalues on the imaginary axis. By definition, there exist a $2n$-dimensional vector x_0 and a number $\omega_0 \in \mathbb{R}$ such that

$$(j\omega_0 I - L)x_0 = MNx_0.$$

Observe now that the matrix L has no eigenvalues on the imaginary axis, because its eigenvalues coincide with those of A and $-A^{\mathrm{T}}$, and A is by hypothesis stable. Thus $(j\omega_0 I - L)$ is nonsingular. Observe also that the vector $u_0 = Nx_0$ is nonzero because otherwise x_0 would be an eigenvector of L associated with an eigenvalue at $j\omega_0$, which is a contradiction. A simple manipulation yields

$$u_0 = N(j\omega_0 I - L)^{-1} M u_0. \tag{3.23}$$

It is easy to check that

$$N(j\omega_0 I - L)^{-1} M = W^{-1}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D] \tag{3.24}$$

where $T(s) = C(sI - A)^{-1}B + D$. In fact, it suffices to compute the transfer function of

$$\begin{aligned} \dot{x} &= Lx + Mu \\ y &= Nx \end{aligned}$$

and observe that $N(sI-L)^{-1}M = W^{-1}[T^{\mathrm{T}}(-s)T(s) - D^{\mathrm{T}}D]$.

Multiply (3.24) on the left by $u_0^{\mathrm{T}} W$ and on the right by u_0, and use (3.23), to obtain

$$u_0^{\mathrm{T}} W u_0 = u_0^{\mathrm{T}}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D]u_0 ,$$

which in turn, in view of the definition of W, yields

$$\gamma^2 \|u_0\|^2 = \|T(j\omega_0)u_0\|^2 .$$

This implies

$$\|T(j\omega_0)\| = \sup_{\|u_0\| \neq 0} \frac{\|T(j\omega_0)u_0\|}{\|u_0\|} \geq \gamma$$

and this contradicts (ii), thus completing the proof.

To show that (iii) $\Rightarrow$ (iv), set

$$\begin{aligned} F &= (A+BW^{-1}D^{\mathrm{T}}C)^{\mathrm{T}} \\ Q &= -BW^{-1}B^{\mathrm{T}} \\ GG^{\mathrm{T}} &= C^{\mathrm{T}}(I+DW^{-1}D^{\mathrm{T}})C \end{aligned}$$

(the latter is indeed possible because $I+DW^{-1}D^{\mathrm{T}}$ is a positive definite matrix: in fact, it is a sum of the positive definite matrix I and of the positive semidefinite matrix $DW^{-1}D^{\mathrm{T}}$; hence there exists a nonsingular matrix M such that $I+DW^{-1}D^{\mathrm{T}} = M^{\mathrm{T}}M$; in view of this, the previous expression holds with $G^{\mathrm{T}} = MC$).

It is easy to check that

$$H^{\mathrm{T}} = \begin{pmatrix} F & -GG^{\mathrm{T}} \\ -Q & -F^{\mathrm{T}} \end{pmatrix}$$

and this matrix by hypothesis has no eigenvalues on the imaginary axis. Moreover, it is also possible to show that the pair $(F, -GG^{\mathrm{T}})$ thus defined is stabilizable. In fact, suppose that this is not the case. Then, there is a vector $x \neq 0$ such that

$$x^{\mathrm{T}} \begin{pmatrix} F-\lambda I & -GG^{\mathrm{T}} \end{pmatrix} = 0$$

for some λ with non-negative real part. Then,

$$0 = \begin{pmatrix} A+BW^{-1}D^{\mathrm{T}}C-\lambda I \\ -C^{\mathrm{T}}M^{\mathrm{T}}MC \end{pmatrix} x\,.$$

This implies in particular $0 = x^{\mathrm{T}}C^{\mathrm{T}}M^{\mathrm{T}}MCx = \|MCx\|^2$ and hence $Cx = 0$, because M is nonsingular. This in turn implies $Ax = \lambda x$, and this is a contradiction because all the eigenvalues of A have negative real part.

Thus [6] there is a unique solution Y^- of the Riccati equation

$$Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q = 0\,, \tag{3.25}$$

satisfying $\sigma(F-GG^{\mathrm{T}}Y^-) \subset \mathbb{C}^-$. Moreover [7], the set of solutions Y of the inequality

$$YF + F^{\mathrm{T}}Y - YGG^{\mathrm{T}}Y + Q > 0 \tag{3.26}$$

is nonempty and any Y in this set is such that $Y < Y^-$.

Observe now that

$$\begin{aligned} &Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q \\ &= Y^-(A^{\mathrm{T}} + C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}}) + (A+BW^{-1}D^{\mathrm{T}}C)Y^- - Y^-C^{\mathrm{T}}(I+DW^{-1}D^{\mathrm{T}})CY^- - BW^{-1}B^{\mathrm{T}} \\ &= Y^-A^{\mathrm{T}} + AY^- - [Y^-C^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY^- - B^{\mathrm{T}}] - Y^-C^{\mathrm{T}}CY^-, \end{aligned}$$

[6] See Proposition A.1 in Appendix A

[7] See Proposition A.2 in Appendix A

and therefore (3.25) yields

$$Y^{-}A^{\mathrm{T}} + AY^{-} \geq 0\,.$$

Set now $U(z) = z^{\mathrm{T}}Y^{-}z$, let $z(t)$ denote (any) integral curve of

$$\dot{z} = A^{\mathrm{T}}z\,, \tag{3.27}$$

and observe that the function $U(z(t))$ satisfies

$$\frac{\partial U(z(t))}{\partial t} = 2z^{\mathrm{T}}(t)Y^{-}A^{\mathrm{T}}z(t) = z^{\mathrm{T}}(t)[Y^{-}A^{\mathrm{T}} + AY^{-}]z(t) \geq 0\,.$$

This inequality shows that the function $V(z(t))$ is non-decreasing, i.e. $V(z(t)) \geq V(z(0))$ for any $z(0)$ and any $t \geq 0$. On the other hand, system (3.27) is by hypothesis asymptotically stable, i.e. $\lim_{t\to\infty} z(t) = 0$. Therefore, necessarily, $V(z(0)) \leq 0$, i.e the matrix Y^{-} is negative semi-definite. From this, it is concluded that any solution Y of (3.26), that is of the inequality

$$YA^{\mathrm{T}} + AY - [YC^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY - B^{\mathrm{T}}] - YC^{\mathrm{T}}CY > 0\,, \tag{3.28}$$

which necessarily satisfies $Y < Y^{-} \leq 0$, is a negative definite matrix.

Take any of the solutions Y of (3.28) and consider $P = -Y^{-1}$. By construction, this matrix is a positive definite solution of the inequality in (3.9). Thus, by Lemma 3.2, (iv) holds.

The proof that (iv) $\Rightarrow$ (i) is provided by Lemma 3.2 and 3.1. $\triangleleft$

Example 3.1. As an elementary example of what the criterion described in (iii) means, consider the case of the single-input single-output linear system

$$\begin{aligned} \dot{x} &= -ax + bu \\ y &= x + du \end{aligned} \tag{3.29}$$

in which it is assumed that $a > 0$, so that the system is stable. The transfer function of this system is

$$T(s) = \frac{ds + (ad + b)}{s + a}\,.$$

This function has a pole at $p = -a$ and a zero at $z = -(ad + b)/d$. Thus, bearing in mind the possible Bode plots of a function having one pole and one zero (see Fig. 3.1), it is seen that

$$\begin{aligned} |d| < |\frac{ad+b}{a}| \qquad &\Rightarrow \qquad \|T\|_{H_\infty} = |T(0)| = |\frac{ad+b}{a}| \\ |d| > |\frac{ad+b}{a}| \qquad &\Rightarrow \qquad \|T\|_{H_\infty} = \lim_{\omega\to\infty} |T(j\omega)| = |d|\,. \end{aligned}$$

Thus,

$$\|T\|_{H_\infty} = \max\{|d|, |\frac{ad+b}{a}|\}\,. \tag{3.30}$$

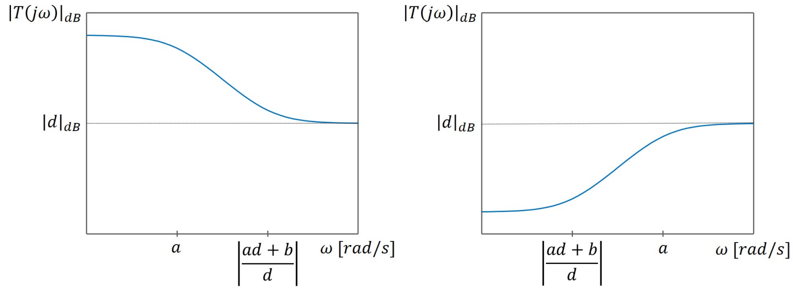

Fig. 3.1 The Bode plots of the transfer function of (3.29). Left: the case $|T(0)| > |d|$. Right: the case $|T(0)| < |d|$

The criterion in the Bounded Real Lemma says that the H_∞ norm of $T(s)$ is strictly less than γ if and only if $\gamma^2 I - D^{\mathrm{T}} D > 0$, which in this case becomes

$$\gamma^2 > d^2 \tag{3.31}$$

and the Hamiltonian matrix (3.21), which in this case becomes

$$H = \begin{pmatrix} -a + \dfrac{bd}{\gamma^2 - d^2} & \dfrac{b^2}{\gamma^2 - d^2} \\ -1 - \dfrac{d^2}{\gamma^2 - d^2} & a - \dfrac{bd}{\gamma^2 - d^2} \end{pmatrix},$$

has no eigenvalues with zero real part. A simple calculation shows that the characteristic polynomial of H is

$$p(\lambda) = \lambda^2 - a^2 + \frac{2abd}{\gamma^2 - d^2} + \frac{b^2}{\gamma^2 - d^2}\,.$$

This polynomial has no roots with zero real part if and only if

$$a^2 - \frac{2abd}{\gamma^2 - d^2} - \frac{b^2}{\gamma^2 - d^2} > 0$$

which, since $\gamma^2 - d^2 > 0$, is the case if and only if

$$\gamma^2 a^2 > (ad + b)^2\,. \tag{3.32}$$

Using both (3.31) and (3.32), it is concluded that, according to the Bounded Real Lemma, $\|T\|_{H_\infty} < \gamma$ if and only if

$$\gamma > \max\{|d|, |\frac{ad+b}{a}|\}$$

which is exactly what (3.30) shows. ◁

3.5 Small gain theorem and robust stability

The various characterizations of the $\mathscr{L}_2$ gain of a system given in the previous sections provide a powerful tool for the study of the stability properties of feedback interconnected systems.To see why this is the case, consider two systems Σ_1 and Σ_2, described by equations of the form

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i + D_i u_i \end{aligned} \tag{3.33}$$

with $i = 1, 2$, in which we assume that

$$\begin{aligned} \dim(u_2) &= \dim(y_1) \\ \dim(u_1) &= \dim(y_2)\ . \end{aligned}$$

Suppose that the matrices D_1 and D_2 are such that the *interconnections*

$$\begin{aligned} u_2 &= y_1 \\ u_1 &= y_2 \end{aligned} \tag{3.34}$$

makes sense (see Figure 3.2).

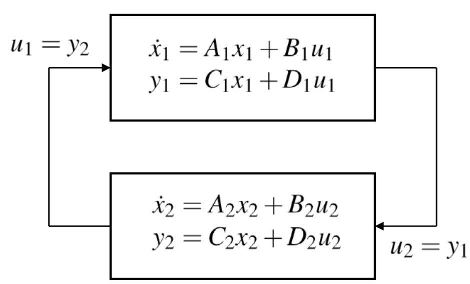

Fig. 3.2 A pure feedback interconnection of two systems Σ_1 and Σ_2.

This will be the case if, for each x_1, x_2, there is a unique pair u_1, u_2 satisfying

$$\begin{aligned} u_1 &= C_2 x_2 + D_2 u_2 \\ u_2 &= C_1 x_1 + D_1 u_1\ , \end{aligned}$$

i.e. if the system of equations

$$\begin{aligned} u_1 - D_2 u_2 &= C_2 x_2 \\ -D_1 u_1 + u_2 &= C_1 x_1 \end{aligned}$$

has a *unique* solution u_1, u_2. This occurs if and only if the matrix

$$\begin{pmatrix} I & -D_2 \\ -D_1 & I \end{pmatrix}$$

is invertible, i.e. the matrix $I - D_2 D_1$ is nonsingular. [8] The (autonomous) system defined by (3.33) together with (3.34) is the *pure feedback interconnection* of Σ_1 and Σ_2.

Suppose now that both systems Σ_1 and Σ_2 are stable. As shown in section 3.1, there exist two positive definite matrices P_1, P_2, two positive numbers $\varepsilon_1, \varepsilon_2$ and two real numbers $\bar{\gamma}_1, \bar{\gamma}_2$ such that Σ_1 and Σ_2 satisfy inequalities of the form (3.5), namely

$$\frac{\partial V_i}{\partial x_i}(A_i x_i + B_i u_i) \leq -\varepsilon_i \|x_i\|^2 + \bar{\gamma}_i^2 \|u_i\|^2 - \|y_i\|^2 \tag{3.35}$$

in which $V_i(x_i) = x_i^{\mathrm{T}} P_i x_i$.

Consider now the quadratic form

$$W(x_1, x_2) = V(x_1) + aV(x_2) = (\,x_1^{\mathrm{T}} \quad x_2^{\mathrm{T}}\,) \begin{pmatrix} P_1 & 0 \\ 0 & aP_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

which, if $a > 0$, is positive definite. A simple calculation shows that, for any trajectory $(x_1(t), x_2(t))$ of the pure feedback interconnection of Σ_1 and Σ_2, the function $W(x_1(t), x_2(t))$ satisfies

$$\begin{aligned} \frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 &\leq -\varepsilon_1 \|x_1\|^2 - a\varepsilon_2 \|x_1\|^2 + \bar{\gamma}_1^2 \|u_1\|^2 - \|y_1\|^2 + a\bar{\gamma}_2^2 \|u_2\|^2 - a\|y_2\|^2 \\ &\leq -\varepsilon_1 \|x_1\|^2 - a\varepsilon_2 \|x_1\|^2 + \bar{\gamma}_1^2 \|y_2\|^2 - \|y_1\|^2 + a\bar{\gamma}_2^2 \|y_1\|^2 - a\|y_2\|^2 \\ &= -\varepsilon_1 \|x_1\|^2 - a\varepsilon_2 \|x_1\|^2 + (\,y_1^{\mathrm{T}} \quad y_2^{\mathrm{T}}\,) \begin{pmatrix} (-1 + a\bar{\gamma}_2^2)I & 0 \\ 0 & (\bar{\gamma}_1^2 - a)I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \end{aligned}$$

If

$$\begin{pmatrix} (-1 + a\bar{\gamma}_2^2)I & 0 \\ 0 & (\bar{\gamma}_1^2 - a)I \end{pmatrix} \leq 0, \tag{3.36}$$

we have

$$\frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 \leq -\varepsilon_1 \|x_1\| - a\varepsilon_2 \|x_2\| = (\,x_1^{\mathrm{T}} \quad x_2^{\mathrm{T}}\,) \begin{pmatrix} -\varepsilon_1 I & 0 \\ 0 & -a\varepsilon_2 I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The quadratic form on the right-hand side is negative definite and hence, according to the criterion of Lyapunov, the interconnected system is asymptotically stable.

[8] Note that an equivalent condition is that the matrix $I - D_1 D_2$ is nonsingular.

Condition (3.36), on the other hand, can be fulfilled for some $a > 0$ if (and only if)

$$-1 + a\bar{\gamma}_2^2 \le 0, \qquad \bar{\gamma}_1^2 - a \le 0$$

i.e. if

$$\bar{\gamma}_1^2 \le a \le \frac{1}{\bar{\gamma}_2^2}.$$

A number $a > 0$ satisfying these inequalities exists if and only if $\bar{\gamma}_2^2\bar{\gamma}_1^2 \le 1$.

All of the above yields the following important result.

Theorem 3.2. *Consider a pair of systems (3.33) and suppose that the matrix $I - D_2D_1$ is nonsingular. Suppose (3.33) satisfy inequalities of the form (3.35), with P_1, P_2 positive definite and*

$$\bar{\gamma}_1\bar{\gamma}_2 \le 1.$$

Then, the pure feedback interconnection of Σ_1 and Σ_2 is asymptotically stable.

Remark 3.2. In the proof of the above statement, we have considered the general case in which both component subsystems have an internal dynamics. To cover the special case in which one of the two subsystem is a memoryless system, a slightly different (actually simpler) argument is needed. Suppose the second component subsystem is modeled as

$$y_2 = D_2u_2$$

and set $\bar{\gamma}_2 = \|D_2\|$. Then,

$$\|u_1\|^2 = \|y_2\|^2 \le \bar{\gamma}_2^2\|u_2\|^2 = \bar{\gamma}_2^2\|y_1\|^2.$$

As a consequence, for the interconnection, we obtain

$$\begin{aligned}\frac{\partial V_1}{\partial x_1}(A_1x_1 + B_1u_1) &\le -\varepsilon_1\|x_1\|^2 + \bar{\gamma}_1^2\|u_1\|^2 - \|y_1\|^2 \\ &\le -\varepsilon_1\|x_1\|^2 + (\bar{\gamma}_1^2\bar{\gamma}_2^2 - 1)\|y_1\|^2.\end{aligned}$$

If $\bar{\gamma}_1\bar{\gamma}_2 \le 1$, the quantity above is negative definite and the interconnected system is stable. ◁

Note also that, in view of the Bounded Real Lemma, the Theorem above can be rephrased in terms of H_∞ norms of the transfer functions of the two component subsystems, as follows.

Corollary 3.1. *Consider a pair of systems (3.33) and suppose that the matrix $I - D_2D_1$ is nonsingular. Suppose both systems are asymptotically stable. Let*

$$T_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$$

denote the respective transfer functions. If

$$\|T_1\|_{H_\infty} \cdot \|T_2\|_{H_\infty} < 1 \tag{3.37}$$

the pure feedback interconnection of Σ_1 and Σ_2 is asymptotically stable.

Proof. Suppose that condition (3.37) holds, i.e. that

$$\|T_1\|_{H_\infty} < \frac{1}{\|T_2\|_{H_\infty}}.$$

Then, it is possible to find a positive number γ_1 satisfying

$$\|T_1\|_{H_\infty} < \gamma_1 < \frac{1}{\|T_2\|_{H_\infty}}.$$

Set $\gamma_2 = 1/\gamma_1$. Then

$$\|T_i\|_{H_\infty} < \gamma_i\,, \qquad i = 1,2\,.$$

From the Bounded Real Lemma, it is seen that – for both $i = 1,2$ – there exists $\varepsilon_i > 0$, $\bar{\gamma}_i < \gamma_i$ and positive definite P_i such that (3.33) satisfy inequalities of the form (3.35), with $V_i(x) = x_i^{\mathrm{T}} P_i x_i$. Moreover, $\bar{\gamma}_1\bar{\gamma}_2 < \gamma_1\gamma_2 = 1$. Hence, from Theorem 3.2 and Remark 3.2, the result follows. ◁

The result just proven is known as the *Small Gain Theorem* (of linear systems). In a nutshell, it says that if both component systems are stable, a *sufficient* condition for the stability of their (pure feedback) interconnection is that the product of the H_∞ norms of the transfer functions of the two component subsystems is *strictly less than 1*. This theorem is the point of departure for the study of robust stability via H_∞ methods.

It should be stressed that the "small gain condition" (3.37) provided by this Corollary is only sufficient for stability of the interconnection and that the strict inequality in (3.37) cannot be replaced, in general, by a loose inequality. Both these facts are explained in the example which follows.

Example 3.2. Let the two component subsystems be modeled by

$$\begin{aligned} \dot{x}_1 &= -ax_1 + u_1 \\ y_1 &= x_1 \end{aligned}$$

and

$$y_2 = du_2\,.$$

Suppose $a > 0$ so that the first subsystem is stable. The stability of the interconnection can be trivially analyzed by direct computation. In fact, the interconnection is modeled as

$$\dot{x}_1 = -ax_1 + dx_1$$

from which is it is seen that a necessary and sufficient condition for the interconnection to be stable is that $d < a$.

On the other hand, the Small Gain Theorem yields a more conservative estimate. In fact

$$T_1(s) = \frac{1}{s+a} \qquad \text{and} \qquad T_2(s) = d$$

and hence

$$\|T_1\|_{H_\infty} = T_1(0) = \frac{1}{a} \qquad \text{and} \qquad \|T_2\|_{H_\infty} = |d|$$

The (sufficient) condition (3.37) becomes $|d| < a$, i.e. $-a < d < a$.

This example also shows that the inequality in (3.37) has to be strict. In fact, the condition $\|T_1\|_{H_\infty} \cdot \|T_2\|_{H_\infty} = 1$ would yield $|d| = a$, which is not admissible because, if $d = a$ the interconnection is not (asymptotically) stable (while it would be asymptotically stable if $d = -a$).

Example 3.3. It is worth observing that a special version of such sufficient condition was implicit in the arguments used in Chapter 2 to prove asymptotic stability. Take, for instance, the case studied in Section 2.3. System (2.19) can be seen as pure feedback interconnection of a subsystem modeled by

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)u_1 \\ y_1 &= A_{10}(\mu)z \end{aligned} \tag{3.38}$$

and of a subsystem modeled by

$$\begin{aligned} \dot{\xi} &= [A_{11}(\mu) - b(\mu)k]\xi + u_2 \\ y_2 &= \xi\,. \end{aligned} \tag{3.39}$$

The first of two such subsystems is a stable system, because $A_{00}(\mu)$ has all eigenvalues in $\mathbb{C}^-$. Let

$$T_1(s) = A_{10}(\mu)[sI - A_{00}(\mu)]^{-1}A_{01}(\mu)\,.$$

denote its transfer function. Its H_∞ norm depends on μ but, since μ ranges over a compact set $\mathbb{M}$, it is possible to find a number $\gamma_1 > 0$ such that

$$\max_{\mu\in\mathbb{M}} \|T_1\|_{H_\infty} < \gamma_1\,.$$

Note that this number γ_1 depends only on the data that characterize the controlled system (2.17) and not on the coefficient k that characterizes the feedback law (2.18).

Consider now the second subsystem. If $b(\mu)k - A_{11}(\mu) > 0$, this is a stable system with transfer function

$$T_2(s) = \frac{1}{s + (b(\mu)k - A_{11}(\mu))}$$

Clearly,

$$\|T_2\|_{H_\infty} = T_2(0) = \frac{1}{b(\mu)k - A_{11}(\mu)}\,.$$

It is seen from this that $\|T_2\|_{H_\infty}$ can be arbitrarily *decreased* by *increasing* the coefficient k. In other words: a *large* value of the output feedback gain coefficient k in (2.18) forces a *small* value of $\|T_2\|_{H_\infty}$.

As a consequence of the small-gain Theorem, the interconnected system (namely, system (2.19)) is stable if

$$\frac{\gamma_1}{b(\mu)k - A_{11}(\mu)} < 1$$

which is indeed the case if

$$k > \max_{\mu \in \mathbb{M}} \frac{\gamma_1 + A_{11}(\mu)}{b(\mu)} .$$

In summary, the result established in Section 2.3 can be re-interpreted in the following terms. Subsystem (3.38) is a stable system, with a transfer function whose H_∞ norm has some fixed bound γ_1 (on which the control has no influence, though). By increasing the value of the gain coefficient k in (2.18), the subsystem (3.39) can be rendered stable, with a H_∞ norm that can be made arbitrarily small, in particular smaller than the (fixed) number $1/\gamma_1$. This makes the small gain condition (3.37) fulfilled and guarantees the (robust) stability of system (2.19). $\triangleleft$

We turn now to the discussion of the problem of robust stabilization. The problem in question can be cast, in rather general terms, as follows. A plant with *control* input u and *measurement* output y whose model is uncertain can be, from a rather general viewpoint, thought of as the interconnection of a *nominal system* modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v \end{aligned} \tag{3.40}$$

in which the "additional" input v and the "additional" output z are seen as output and, respectively, input of a system

$$\begin{aligned} \dot{x}_{\mathrm{p}} &= A_{\mathrm{p}} x_{\mathrm{p}} + B_{\mathrm{p}} z \\ v &= C_{\mathrm{p}} x_{\mathrm{p}} + D_{\mathrm{p}} z \end{aligned} \tag{3.41}$$

whose parameters are *uncertain* .[9]

This setup includes the special case of a plant of fixed dimension whose parameters are uncertain, that is a plant modeled as

$$\begin{aligned} \dot{x} &= (A_0 + \delta A)x + (B_0 + \delta B)u \\ y &= (C_0 + \delta C)x \end{aligned} \tag{3.42}$$

in which A_0, B_0, C_0 represent nominal values and $\delta A, \delta B, \delta C$ uncertain perturbations. In fact, the latter can be seen as interconnection of a system of the form (3.40) in which

[9] Note that, for the interconnection (3.40) – (3.41) to be well-defined, the matrix $I - D_{\mathrm{p}} D_{11}$ is required to be invertible.

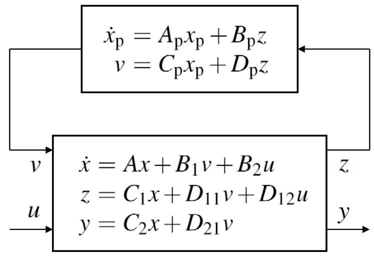

Fig. 3.3 A controlled system seen as interconnection of an accurate model and of a lousy model.

$$
\begin{aligned}
&A = A_0 \qquad B_1 = \begin{pmatrix} I & 0 \end{pmatrix} \quad B_2 = B_0 \\
&C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix} D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix} \\
&C_2 = C_0 \qquad D_{21} = \begin{pmatrix} 0 & I \end{pmatrix}
\end{aligned}
$$

with a system of the form

$$
v = \begin{pmatrix} \delta A & \delta B \\ \delta C & 0 \end{pmatrix} z
$$

which is indeed a special case of a system of the form (3.41). The interconnection (3.40) – (3.41) of a *nominal model* and of a *dynamic perturbation* is more general, though, because it accommodates for perturbations which possibly include *unmodeled dynamics* (see Example at the end of the section). In this setup, all modeling uncertainties are confined in the model (3.41), including the dimension itself of x_p.

Suppose now that, in (3.41), A_p is a Hurwitz matrix and that the transfer function

$$
T_p(s) = C_p(sI - A_p)^{-1} B_p + D_p
$$

has an H_∞ norm which is bounded by a known number γ_p. That is, *assume* that, no matter what the perturbations are, the perturbing system (3.41) is a stable system satisfying

$$
\|T_p\|_{H_\infty} < \gamma_p \,. \tag{3.43}
$$

for some γ_p.

Let

$$
\begin{aligned}
\dot{x}_c &= A_c x_c + B_c y \\
u &= C_c x_c + D_c y
\end{aligned} \tag{3.44}
$$

be a controller for the nominal plant (3.40) yielding a closed loop system

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{pmatrix} v$$

$$z = \begin{pmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + (D_{11} + D_{12} D_c D_{21}) v$$

which is asymptotically stable and whose transfer function, between the input v and output z has an H_∞ norm bounded by a number γ satisfying

$$\gamma \gamma_p < 1 . \tag{3.45}$$

If this is the case, thanks to the Small Gain Theorem, it can be concluded that the controller (3.44) stabilizes any of the perturbed plants (3.40) – (3.41), so long as the perturbation is such that (3.41) is asymptotically stable and the bound (3.43) holds.

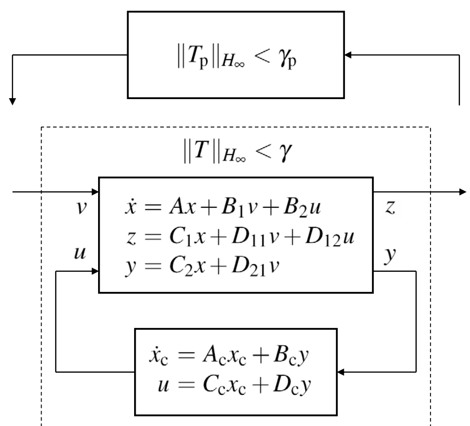

Fig. 3.4 The H_∞ approach to robust stabilization.

In this way, the problem of robust stabilization is reduced to a problem of stabilizing a nominal plant and to simultaneously enforce a bound on the H_∞ norm of its transfer function.

Example 3.4. A simplified model describing the motion of a vertical takeoff and landing (VTOL) aircraft in the vertical-lateral plane can be obtained in the following way. Let y, h and θ denote, respectively, the horizontal and vertical position of the center of mass C and the roll angle of the aircraft with respect to the horizon, as in 3.5. The control inputs are the thrust directed out the bottom of the aircraft, denoted by T, and the rolling moment produced by a couple of equal forces, denoted by F, acting at the wingtips. Their direction is not perpendicular to the horizontal body axis, but tilted by some fixed angle α. Letting M denote the mass of the aircraft, J the moment of inertia about the center of mass, ℓ the distance between the wingtips and

g the gravitational acceleration, it is seen that the motion of the aircraft is modeled by the equations

$$\begin{aligned} M\ddot{y} &= -\sin(\theta)T + 2\cos(\theta)F\sin\alpha \\ M\ddot{h} &= \cos(\theta)T + 2\sin(\theta)F\sin\alpha - gM \\ J\ddot{\theta} &= 2\ell F\cos\alpha\ . \end{aligned}$$

The purpose of the control input T is that of moving the center of mass of the airplane in the vertical-lateral plane, while that of the input F is to control the airplane's attitude, i.e. the roll angle θ.

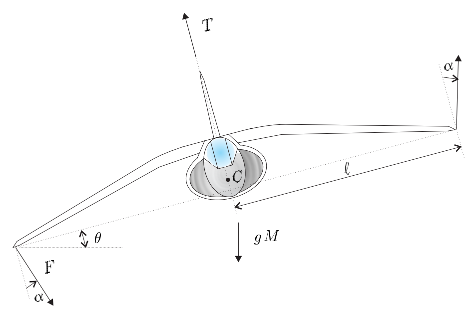

Fig. 3.5 A VTOL moving in the lateral-vertical plane.

In the hovering maneuver, the thrust T is expected to compensate exactly the force of gravity. Thus, T can be expressed as $T = gM + \delta T$ in which δT is a residual input use to control the attitude. If the angle θ is sufficiently small one can use the approximations $\sin(\theta) \simeq \theta$, $\cos(\theta) \simeq 1$ and neglect the nonlinear terms, so as to obtain a linear simplified model

$$\begin{aligned} M\ddot{y} &= -gM\theta + 2F\sin\alpha \\ M\ddot{h} &= \delta T \\ J\ddot{\theta} &= 2\ell F\cos\alpha\ . \end{aligned}$$

In this simplified model, the motion in the vertical direction is totally decoupled, and controlled by δT. On the other hand, the motion in the lateral direction and the roll angle are coupled, and controlled by F. We concentrate on the analysis of the latter.

The system with input F and output y is a four-dimensional system, having relative degree 2. Setting $\zeta_1 = \theta$, $\zeta_2 = \dot{\theta}$, $\xi_1 = y$, $\xi_2 = \dot{y}$ and $u = F$, it can be expressed, in state-space form, as

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2\\
\dot{\zeta}_2 &= \tfrac{2\ell}{J}(\cos\alpha)u\\
\dot{\xi}_1 &= \xi_2\\
\dot{\xi}_2 &= -g\zeta_1 + \tfrac{2}{M}(\sin\alpha)u\,.
\end{aligned}$$

Note that this is not a strict normal form. It can be put in strict normal form, though, changing ζ_2 into

$$\tilde{\zeta}_2 = \zeta_2 - \frac{\ell M\cos\alpha}{J\sin\alpha}\xi_2\,,$$

as the reader can easily verify. The strict normal form in question is given by

$$\begin{aligned}
\begin{pmatrix}\dot{\zeta}_1\\ \dot{\tilde{\zeta}}_2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ \frac{\ell Mg\cos\alpha}{J\sin\alpha} & 0\end{pmatrix}\begin{pmatrix}\zeta_1\\ \tilde{\zeta}_2\end{pmatrix} + \begin{pmatrix}\frac{\ell M\cos\alpha}{J\sin\alpha}\\ 0\end{pmatrix}\xi_2\\
\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} + \begin{pmatrix}0\\ -g\end{pmatrix}\zeta_1 + \begin{pmatrix}0\\ 1\end{pmatrix}\tfrac{2}{M}(\sin\alpha)u\,.
\end{aligned} \tag{3.46}$$

It is seen from this that the zeros of the system are the roots of

$$N(\lambda) = \lambda^2 - \frac{\ell Mg\cos\alpha}{J\sin\alpha}\,,$$

and hence the system *is not* minimum phase, because one zero has positive real part. Therefore the (elementary) methods for robust stabilization described in the previous section cannot be applied.

However, the approach presented in this section is applicable. To this end, observe that the characteristic polynomial of the system is the polynomial $D(\lambda) = \lambda^4$ and consequently its transfer function has the form

$$T(s) = \frac{\frac{2}{M}(\sin\alpha)s^2 - \frac{2\ell g}{J}(\cos\alpha)}{s^4}\,.$$

From this – by known facts – it is seen that an equivalent realization of (3.46) is given by

$$\begin{aligned}
\dot{x} &= \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}x + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}u := A_0 + B_0 u\\
y &= \begin{pmatrix}-\frac{2\ell g}{J}(\cos\alpha) & 0 & \frac{2}{M}(\sin\alpha) & 0\end{pmatrix}x := Cx\,.
\end{aligned} \tag{3.47}$$

The matrices A_0 and B_0 are not subject to perturbations. Thus, the only uncertain parameters are the entries of the matrix C. We may write Cx in the form

$$Cx = C_0 x + D_{\mathrm{p}} z$$

in which $C_0 \in \mathbb{R}^4$ is a vector of nominal values, $z = \mathrm{col}(x_1, x_3)$ and $D_\mathrm{p} \in \mathbb{R}^2$ is an (uncertain) term whose elements represent the deviations, from the assumed nominal values, of the first and third entry of C. [10]

This being the case, it is concluded that the (perturbed) model can be expressed as interconnection of an accurately known system, and of a (memoryless) uncertain system, with the former modeled as

$$\begin{aligned} \dot{x} &= A_0 x + B_0 u \\ z &= \mathrm{col}(x_1, x_3) \\ y &= C_0 x + v\,, \end{aligned}$$

and the latter modeled as

$$v = D_\mathrm{p} z\,. \qquad \triangleleft$$

Example 3.5. Consider a d.c. motor, in which the stator field is kept constant, and the control is provided by the rotor voltage. The system in question can be characterized by equations expressing the mechanical balance and the electrical balance. The mechanical balance, in the hypothesis of viscous friction only (namely friction torque purely proportional to the angular velocity) has the form

$$J\dot{\Omega} + F\Omega = T$$

in which Ω denotes the angular velocity of the motor shaft, J the inertia of the rotor, F the viscous friction coefficient, and T the torque developed at the shaft. The torque T is proportional to the rotor current I, namely,

$$T = k_m I\,.$$

The rotor current, in turn, is determined by the electrical balance of the rotor circuit. This circuit, with reasonable approximation, can be modeled as in Fig. 3.6, in which R is the resistance of the rotor winding, L is the inductance of the rotor winding, R_b is the contact resistance of the brushes, C_s is a stray capacitance. The voltage V is the control input and the voltage E is the so-called "back electromotive force (e.m.f.)" which is proportional to the angular velocity of the motor shaft, namely

$$E = k_e \Omega\,.$$

The equations expressing the electrical balance are

---

[10] Since α is a small angle, it makes sense to take

$$C_0 = \begin{pmatrix} -\frac{2\ell_0 g}{J_0} & 0 & 0 & 0 \end{pmatrix},$$

where ℓ_0 and J_0 are the nominal values of ℓ and J, in which case

$$D_\mathrm{p} = \mathrm{row}(2g(\frac{\ell_0}{J_0} - \frac{\ell}{J}(\cos\alpha)), \frac{2}{M}(\sin\alpha))\,.$$

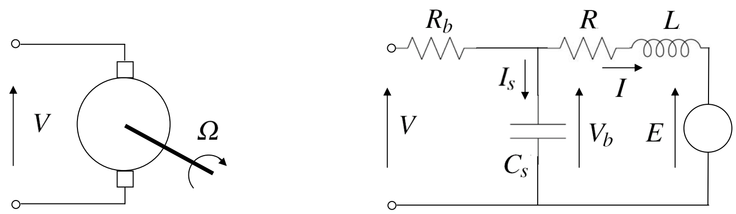

Fig. 3.6 A d.c. motor and its equivalent circuit.

$$
\begin{aligned}
R_b(I+I_s) &= V - V_b \\
C_s \frac{\mathrm{d}V_b}{\mathrm{d}t} &= I_s \\
L\frac{\mathrm{d}I}{\mathrm{d}t} + RI &= V_b - E\,.
\end{aligned}
$$

These can be put in state-space form by setting

$$\xi = \mathrm{col}(V_b, I)\,, \qquad z = \mathrm{col}(V, \Omega)$$

to obtain (recall that $E = k_e\Omega$)

$$
\begin{aligned}
\dot{\xi} &= F\xi + Gz \\
I &= H\xi
\end{aligned}
$$

in which

$$F = \begin{pmatrix} \frac{-1}{C_sR_b} & \frac{-1}{C_s} \\ \frac{1}{L} & \frac{-R}{L} \end{pmatrix}, \quad G = \begin{pmatrix} \frac{1}{C_sR_b} & 0 \\ 0 & \frac{-k_e}{L} \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

In this way, the rotor current I is seen as output of a system with input z. Note that this system is stable, because characteristic polynomial of the matrix F

$$d(\lambda) = \lambda^2 + (\frac{1}{C_sR_b} + \frac{R}{L})\lambda + \frac{1}{LC_s} + \frac{R}{LC_sR_b}$$

has roots with negative real part. A simple calculation shows that the two entries of the transfer function of this system

$$T(s) = \begin{pmatrix} T_1(s) & T_2(s) \end{pmatrix}$$

have the expressions

$$T_1(s) = \frac{1}{(C_s R_b s + 1)(Ls + R) + R_b}, \qquad T_2(s) = \frac{-(C_s R_b s + 1)k_e}{(C_s R_b s + 1)(Ls + R) + R_b}.$$

Note that, if R_b is negligible, the two functions can be approximated as

$$T_1(s) \cong \frac{1}{(Ls + R)}, \qquad T_2(s) \cong \frac{-k_e}{(Ls + R)}.$$

and also that, if $(L/R) \ll 1$, the functions can be further approximated, over a reasonable range of frequencies, as

$$T_1(s) \cong \frac{1}{R}, \qquad T_2(s) = \frac{-k_e}{R}.$$

This shows that, neglecting the dynamics of the rotor circuit, the rotor current can be approximately expressed as

$$I_0 \cong \frac{1}{R}(V - k_e \Omega) = Kz,$$

in which K is the row vector

$$K = \begin{pmatrix} \frac{1}{R} & \frac{-k_e}{R} \end{pmatrix}.$$

With this in mind, the (full) expression of the rotor current can be written as

$$I = I_0 + v$$

where v is the output of a system

$$\begin{aligned} \dot{\xi} &= F\xi + Gz \\ v &= H\xi - Kz. \end{aligned}$$

Replacing this expression into the equation of the mechanical balance, letting x_1 denote the angular position of the rotor shaft (in which case it is seen that $\dot{x}_1 = \Omega$), setting

$$x_2 = \Omega, \qquad u = V$$

one obtains a model of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\Big(\frac{F}{J} + \frac{k_m k_e}{JR}\Big)x_2 + \frac{k_m}{J}v + \frac{k_m}{JR}u. \end{aligned}$$

In summary, letting $y = x_1$ denote the measured output of the system, the (full) model of the system in question can be seen as a system of the form (3.40), with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\left(\dfrac{F}{J} + \dfrac{k_m k_e}{JR}\right) \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ \dfrac{k_m}{J} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 \\ \dfrac{k_m}{JR} \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad D_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C_2 = (1 \quad 0), \qquad D_{21} = 0,$$

interconnected with a system of the form (3.41) in which $A_\mathrm{p}, B_\mathrm{p}, C_\mathrm{p}, D_\mathrm{p}$ coincide, respectively, with the matrices $F, G, H, -K$ defined above. The system is modeled as the interconnection of a two subsystems: a low-dimensional subsystem, that models only the dominant dynamical phenomena (the dynamics of the motion of the rotor shaft) and whose parameters are assumed to be known with sufficient accuracy, and a subsystem which may include highly uncertain parameters (the parameters R_b and C_s) and whose dynamics are neglected when a model valid only on a low range of frequencies is sought. The design philosophy described above is that of seeking a feedback law, acting on the system that models the dominant dynamical phenomena, so as to obtain robust stability in spite of parameter uncertainties and un-modeled dynamics.

3.6 The coupled LMIs approach to the problem of γ-suboptimal H_∞ feedback design

Motivated by the results discussed at the end of the previous section, we consider now a design problem which goes under the name of *problem of γ-suboptimal H_∞ feedback design*.[11] Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v\,. \end{aligned} \tag{3.48}$$

Let $\gamma > 0$ be a fixed number. The problem of γ-suboptimal H_∞ feedback design consists in finding a controller

$$\begin{aligned} \dot{x}_\mathrm{c} &= A_\mathrm{c} x_\mathrm{c} + B_\mathrm{c} y \\ u &= C_\mathrm{c} x_\mathrm{c} + D_\mathrm{c} y \end{aligned} \tag{3.49}$$

yielding a closed loop system

[11] In this section, the exposition closely follows the approach of [9], [11] to the problem of γ-suboptimal H_∞ feedback design. For the numerical implementation of the design methods, see also [10].

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A+B_2D_cC_2 & B_2C_c \\ B_cC_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1+B_2D_cD_{21} \\ B_cD_{21} \end{pmatrix} v$$
$$z = \begin{pmatrix} C_1+D_{12}D_cC_2 & D_{12}C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + (D_{11}+D_{12}D_cD_{21})\, v \tag{3.50}$$

which is *asymptotically stable* and whose transfer function, between the input v and output z, has an H_∞ norm *strictly less* than γ.

The interest in such problem, in view of the results discussed earlier, is obvious. In fact, if this problem is solved, the controller (3.49) robustly stabilizes any perturbed system that can be seen as pure feedback interconnection of (3.48) and of an uncertain system of the form (3.41), so long as the latter is a stable system having a transfer function whose H_∞ norm is less than or equal to the inverse of γ. Of course, the smaller is the value of γ for which the problem is solvable, the "larger" is the set of perturbations against which robust stability can be achieved.

Rewrite system (3.50) as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathscr{A}\mathbf{x} + \mathscr{B}v \\ z &= \mathscr{C}\mathbf{x} + \mathscr{D}v \end{aligned}$$

where

$$\mathscr{A} = \begin{pmatrix} A+B_2D_cC_2 & B_2C_c \\ B_cC_2 & A_c \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} B_1+B_2D_cD_{21} \\ B_cD_{21} \end{pmatrix}$$
$$\mathscr{C} = \begin{pmatrix} C_1+D_{12}D_cC_2 & D_{12}C_c \end{pmatrix}, \qquad \mathscr{D} = D_{11}+D_{12}D_cD_{21}\,.$$

In view of the Bounded Real Lemma, the closed loop system has the desired properties if and only if there exists a symmetric matrix $\mathscr{X} > 0$ satisfying

$$\mathscr{X} > 0 \tag{3.51}$$

$$\begin{pmatrix} \mathscr{A}^{\mathrm{T}}\mathscr{X} + \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} & \mathscr{C}^{\mathrm{T}} \\ \mathscr{B}^{\mathrm{T}}\mathscr{X} & -\gamma I & \mathscr{D}^{\mathrm{T}} \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{pmatrix} < 0\,. \tag{3.52}$$

Thus, the problem is to try to find a quadruplet $\{A_c, B_c, C_c, D_c\}$ such that (3.51) and (3.52) hold for some symmetric $\mathscr{X}$.

The basic inequality (3.52) will be now transformed as follows. Let

$$x \in \mathbb{R}^n, \quad x_c \in \mathbb{R}^k, \quad v \in \mathbb{R}^{m_1}, \quad u \in \mathbb{R}^{m_2}, \quad z \in \mathbb{R}^{p_1}, \quad y \in \mathbb{R}^{p_2}\,.$$

Set

$$\mathbf{A}_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad \mathbf{C}_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix}, \tag{3.53}$$

$$\Psi(\mathscr{X}) = \begin{pmatrix} \mathbf{A}_0^{\mathrm{T}}\mathscr{X} + \mathscr{X}\mathbf{A}_0 & \mathscr{X}\mathbf{B}_0 & \mathbf{C}_0^{\mathrm{T}} \\ \mathbf{B}_0^{\mathrm{T}}\mathscr{X} & -\gamma I & D_{11}^{\mathrm{T}} \\ \mathbf{C_0} & D_{11} & -\gamma I \end{pmatrix}, \tag{3.54}$$

$$\mathscr{P} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ B_2^{\mathrm{T}} & 0 & 0_{m_2 \times m_1} & D_{12}^{\mathrm{T}} \end{pmatrix},$$

$$\mathscr{Q} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ C_2 & 0 & D_{21} & 0_{p_2 \times p_1} \end{pmatrix},$$

and

$$\Xi(\mathscr{X}) = \begin{pmatrix} \mathscr{X} & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{p_1} \end{pmatrix}.$$

Collecting the parameters of the controller (3.49) in the $(n+m_2) \times (n+p_2)$ matrix

$$\mathbf{K} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \tag{3.55}$$

the inequality (3.52) can be rewritten as

$$\Psi(\mathscr{X}) + \mathscr{Q}^{\mathrm{T}} \mathbf{K}^{\mathrm{T}} [\mathscr{P} \Xi(\mathscr{X})] + [\mathscr{P} \Xi(\mathscr{X})]^{\mathrm{T}} \mathbf{K} \mathscr{Q} < 0. \tag{3.56}$$

Thus, the problem of γ-suboptimal H_∞ feedback design can be cast as the problem of finding a symmetric matrix $\mathscr{X} > 0$ and a matrix $\mathbf{K}$ such that (3.56) holds. Note that this inequality, is not a linear matrix inequality in the unknowns $\mathscr{X}$ and $\mathbf{K}$. Rather, it is a *bilinear* matrix inequality. [12] However, the problem of finding a matrix $\mathscr{X} > 0$ for which (3.56) is solved by some $\mathbf{K}$ can be cast in terms of linear matrix inequalities. In this context, the following result is very useful. [13]

Lemma 3.3. *Given a symmetric $m \times m$ matrix Ψ and two matrices P, Q having m columns, consider the problem of finding some matrix K of compatible dimensions such that*

$$\Psi + Q^{\mathrm{T}} K P + P^{\mathrm{T}} K^{\mathrm{T}} Q < 0. \tag{3.57}$$

Let W_P and W_Q be two matrices such that [14]

$$\begin{aligned} \mathrm{Im}(W_P) &= \mathrm{Ker}(P) \\ \mathrm{Im}(W_Q) &= \mathrm{Ker}(Q). \end{aligned}$$

Then (3.57) is solvable in K if and only if

$$\begin{aligned} W_Q^{\mathrm{T}} \Psi W_Q &< 0 \\ W_P^{\mathrm{T}} \Psi W_P &< 0. \end{aligned} \tag{3.58}$$

This Lemma can be used to eliminate $\mathbf{K}$ from (3.56) and obtain existence conditions depending only on $\mathscr{X}$ and on the parameters of the plant (3.48). Let $W_{\mathscr{P}\Xi(\mathscr{X})}$

[12] The inequality (3.56), for each fixed $\mathscr{X}$ is a linear matrix inequality in $\mathbf{K}$ and, for each fixed $\mathbf{K}$ is a linear matrix inequality in $\mathscr{X}$.

[13] The proof of Lemma 3.3 can be found in [9].

[14] Observe that, since $\mathrm{Ker}(P)$ and $\mathrm{Ker}(Q)$ are subspaces of $\mathbb{R}^m$, then W_P and W_Q are matrices having m rows.

be a matrix whose columns span $\mathrm{Ker}(\mathscr{P}\Xi(\mathscr{X}))$ and let $W_\mathscr{Q}$ be a matrix whose columns span $\mathrm{Ker}(\mathscr{Q})$. According to Lemma 3.3, there exists $\mathbf{K}$ for which (3.56) holds if and only if

$$\begin{aligned} W_\mathscr{Q}^\mathrm{T}\Psi(\mathscr{X})W_\mathscr{Q} &< 0 \\ W_{\mathscr{P}\Xi(\mathscr{X})}^\mathrm{T}\Psi(\mathscr{X})W_{\mathscr{P}\Xi(\mathscr{X})} &< 0\,. \end{aligned} \tag{3.59}$$

The second of these two inequalities can be further manipulated observing that if $W_\mathscr{P}$ is a matrix whose columns span $\mathrm{Ker}(\mathscr{P})$, the columns of the matrix $[\Xi(\mathscr{X})]^{-1}W_\mathscr{P}$ span $\mathrm{Ker}(\mathscr{P}\Xi(\mathscr{X}))$. Thus, having set

$$\Phi(\mathscr{X}) = \begin{pmatrix} \mathbf{A}_0\mathscr{X}^{-1} + \mathscr{X}^{-1}\mathbf{A}_0^\mathrm{T} & \mathbf{B}_0 & \mathscr{X}^{-1}\mathbf{C}_0^\mathrm{T} \\ \mathbf{B}_0^\mathrm{T} & -\gamma I & D_{11}^\mathrm{T} \\ \mathbf{C_0}\mathscr{X}^{-1} & D_{11} & -\gamma I \end{pmatrix}, \tag{3.60}$$

the second of (3.59) can be rewritten as

$$W_\mathscr{P}^\mathrm{T}[\Xi(\mathscr{X})]^{-1}\Psi(\mathscr{X})[\Xi(\mathscr{X})]^{-1}W_\mathscr{P} = W_\mathscr{P}^\mathrm{T}\Phi(\mathscr{X})W_\mathscr{P} < 0\,. \tag{3.61}$$

It therefore concluded that (3.56) is solvable for some $\mathbf{K}$ if and only if the matrix $\mathscr{X}$ satisfies the first of (3.59) and (3.61).

In view of this, the following (intermediate) conclusion can be drawn.

Proposition 3.1. *There exists a k-dimensional controller that solves the problem of γ-suboptimal H_∞ feedback design if and only if there exists a $(n+k)\times(n+k)$ symmetric matrix $\mathscr{X} > 0$ satisfying the following set of linear matrix inequalities*

$$\begin{aligned} W_\mathscr{Q}^\mathrm{T}\Psi(\mathscr{X})W_\mathscr{Q} &< 0 \\ W_\mathscr{P}^\mathrm{T}\Phi(\mathscr{X})W_\mathscr{P} &< 0\,, \end{aligned} \tag{3.62}$$

in which $W_\mathscr{P}$ is a matrix whose columns span $\mathrm{Ker}(\mathscr{P})$*, $W_\mathscr{Q}$ is a matrix whose columns span* $\mathrm{Ker}(\mathscr{Q})$*, and $\Psi(\mathscr{X})$ and $\Phi(\mathscr{X})$ are the matrices defined in (3.54) and (3.60). For any of such $\mathscr{X}$'s, a solution $\mathbf{K}$ of the resulting linear matrix inequality (3.56) exists and provides a solution of the problem of γ-suboptimal H_∞ feedback design.*

The two inequalities (3.62) thus found can be further simplified. To this end, it is convenient to partition $\mathscr{X}$ and $\mathscr{X}^{-1}$ as

$$\mathscr{X} = \begin{pmatrix} S & N \\ N^\mathrm{T} & * \end{pmatrix}, \qquad \mathscr{X}^{-1} = \begin{pmatrix} R & M \\ M^\mathrm{T} & * \end{pmatrix} \tag{3.63}$$

in which R and S are $n\times n$ and N and M are $n\times k$. With the partition (3.63), the matrix (3.54) becomes

$$\Psi(\mathscr{X}) = \begin{pmatrix} A^\mathrm{T}S + SA & A^\mathrm{T}N & SB_1 & C_1^\mathrm{T} \\ N^\mathrm{T}A & 0 & N^\mathrm{T}B_1 & 0 \\ B_1^\mathrm{T}S & B_1^\mathrm{T}N & -\gamma I & D_{11}^\mathrm{T} \\ C_1 & 0 & D_{11} & -\gamma I \end{pmatrix} \tag{3.64}$$

and the matrix (3.60) becomes

$$\Phi(\mathscr{X}) = \begin{pmatrix} AR + RA^{\mathrm{T}} & AM & B_1 & RC_1^{\mathrm{T}} \\ M^{\mathrm{T}}A^{\mathrm{T}} & 0 & 0 & M^{\mathrm{T}}C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 R & C_1 M & D_{11} & -\gamma I \end{pmatrix}. \tag{3.65}$$

From the definition of $\mathscr{Q}$, it is readily seen that a matrix $W_{\mathscr{Q}}$ whose columns span $\mathrm{Ker}(\mathscr{Q})$ can be expressed as

$$W_{\mathscr{Q}} = \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \\ Z_2 & 0 \\ 0 & I_{p_1} \end{pmatrix}$$

in which

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

is a matrix whose columns span

$$\mathrm{Ker}\begin{pmatrix} C_2 & D_{21} \end{pmatrix}.$$

Then, an easy calculation shows that the first inequality in (3.62) can be rewritten as

$$\begin{pmatrix} Z_1^{\mathrm{T}} & Z_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}}S + SA & SB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0.$$

Likewise, from the definition of $\mathscr{P}$, it is readily seen that a matrix $W_{\mathscr{P}}$ whose columns span $\mathrm{Ker}(\mathscr{P})$ can be expressed as

$$W_{\mathscr{P}} = \begin{pmatrix} V_1 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \\ V_2 & 0 \end{pmatrix}$$

in which

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

is a matrix whose columns span

$$\mathrm{Ker}\begin{pmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} \end{pmatrix}.$$

Then, an easy calculation shows that the second inequality in (3.62) can be rewritten as

$$\begin{pmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} AR + RA^{\mathrm{T}} & RC_1^{\mathrm{T}} & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{pmatrix} < 0.$$

In this way, the two inequalities (3.62) have been transformed into two inequalities involving the submatrix S of $\mathscr{X}$ and the submatrix R of $\mathscr{X}^{-1}$. To complete the analysis, it remains to connect these matrices to each other. This is achieved via the following Lemma.

Lemma 3.4. *Let S and R be symmetric $n \times n$ matrices. There exist a $n \times k$ matrix N and $k \times k$ symmetric matrix Z a such that*

$$\begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix} > 0 \tag{3.66}$$

and

$$\begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix}^{-1} = \begin{pmatrix} R & * \\ * & * \end{pmatrix} \tag{3.67}$$

if and only if

$$\mathrm{rank}(I - SR) \le k \tag{3.68}$$

and

$$\begin{pmatrix} S & I \\ I & R \end{pmatrix} \ge 0\,. \tag{3.69}$$

Proof. To prove necessity, write

$$\mathscr{X} = \begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix} \qquad \text{and} \qquad \mathscr{X}^{-1} = \begin{pmatrix} R & M \\ M^{\mathrm{T}} & * \end{pmatrix}.$$

Then, by definition we have

$$\begin{aligned} SR + NM^{\mathrm{T}} &= I \\ N^{\mathrm{T}}R + ZM^{\mathrm{T}} &= 0\,. \end{aligned} \tag{3.70}$$

Thus $I - SR = NM^{\mathrm{T}}$ and this implies (3.68), because N has k columns. Now, set

$$\mathscr{T} = \begin{pmatrix} I & R \\ 0 & M^{\mathrm{T}} \end{pmatrix}.$$

Using (3.70), the first of which implies $MN^{\mathrm{T}} = I - RS$ because S and R are symmetric, observe that

$$\mathscr{T}^{\mathrm{T}} \mathscr{X} \mathscr{T} = \begin{pmatrix} S & I \\ I & R \end{pmatrix}.$$

Pick $y \in \mathbb{R}^{2n}$ and define $x \in \mathbb{R}^{n+k}$ as $x = \mathscr{T} y$. Then, it is seen that

$$y^{\mathrm{T}} \begin{pmatrix} S & I \\ I & R \end{pmatrix} y = x^{\mathrm{T}} \mathscr{X} x \ge 0$$

because the matrix $\mathscr{X}$ is positive definite by assumption. This concludes the proof of the necessity.

To prove sufficiency, let $\hat{k} \leq k$ be the rank of $I - SR$ and let $\hat{N}, \hat{M}$ two $n \times \hat{k}$ matrices of rank $\hat{k}$ such that

$$I - SR = \hat{N}\hat{M}^{\mathrm{T}}. \tag{3.71}$$

Using the property (3.71) it is possible to show that the equation

$$\hat{N}^{\mathrm{T}} R + \hat{Z}\hat{M}^{\mathrm{T}} = 0 \tag{3.72}$$

has a solution $\hat{Z}$. In fact, observe that

$$\hat{M}\hat{N}^{\mathrm{T}} R - R\hat{N}\hat{M}^{\mathrm{T}} = (I - RS)R - R(I - SR) = 0\,.$$

Let L be any matrix such that $\hat{M}^{\mathrm{T}} L = I$ (which exists because the $\hat{k}$ rows of $\hat{M}^{\mathrm{T}}$ are independent) and, from the identity above, obtain

$$\hat{M}\hat{N}^{\mathrm{T}} RL + R\hat{N} = 0\,.$$

This shows that the matrix $\hat{Z} = (\hat{N}^{\mathrm{T}} RL)^{\mathrm{T}}$ satisfies (3.72). The matrix $\hat{Z}$ is symmetric. In fact, note that

$$0 = \hat{M}(\hat{N}^{\mathrm{T}} R + \hat{Z}\hat{M}^{\mathrm{T}}) = (I - RS)R + \hat{M}\hat{Z}\hat{M}^{\mathrm{T}} = R - RSR + \hat{M}\hat{Z}\hat{M}^{\mathrm{T}}$$

and hence $\hat{M}\hat{Z}\hat{M}^{\mathrm{T}}$ is symmetric. This yields

$$0 = \hat{M}\hat{Z}\hat{M}^{\mathrm{T}} - \hat{M}\hat{Z}^{\mathrm{T}}\hat{M}^{\mathrm{T}} = \hat{M}(\hat{Z} - \hat{Z}^{\mathrm{T}})\hat{M}^{\mathrm{T}}$$

from which it is seen that $\hat{Z} = \hat{Z}^{\mathrm{T}}$ because $\hat{M}^{\mathrm{T}}$ has independent rows.

As a consequence of (3.71) and(3.72), the symmetric matrix

$$\hat{\mathscr{X}} = \begin{pmatrix} S & \hat{N} \\ \hat{N}^{\mathrm{T}} & \hat{Z} \end{pmatrix} \tag{3.73}$$

is a solution of

$$\begin{pmatrix} S & I \\ \hat{N}^{\mathrm{T}} & 0 \end{pmatrix} = \hat{\mathscr{X}} \begin{pmatrix} I & R \\ 0 & \hat{M}^{\mathrm{T}} \end{pmatrix}. \tag{3.74}$$

The symmetric matrix $\hat{\mathscr{X}}$ thus found is invertible because, otherwise, the independence of the rows of the matrix on the left-hand side of (3.74) would be contradicted. It is easily seen that

$$\hat{\mathscr{X}}^{-1} = \begin{pmatrix} R & * \\ \hat{M}^{\mathrm{T}} & * \end{pmatrix}. \tag{3.75}$$

Moreover, it is also possible to prove that $\hat{\mathscr{X}} > 0$. In fact, letting

$$\hat{\mathscr{T}} = \begin{pmatrix} I & R \\ 0 & \hat{M}^{\mathrm{T}} \end{pmatrix}$$

observe that

$$\hat{\mathscr{T}}^{\mathrm{T}} \hat{\mathscr{X}} \hat{\mathscr{T}} = \begin{pmatrix} S & I \\ I & R \end{pmatrix}.$$

Suppose $x^{\mathrm{T}} \hat{\mathscr{X}} x < 0$ for some $x \neq 0$. Using the fact that the rows of $\hat{\mathscr{T}}$ are independent, find y such that $x = \hat{\mathscr{T}} y$. This would make

$$y^{\mathrm{T}} \begin{pmatrix} S & I \\ I & R \end{pmatrix} y < 0,$$

which contradicts (3.69). Thus, $x^{\mathrm{T}} \hat{\mathscr{X}} x \geq 0$ for all x, i.e. $\hat{\mathscr{X}} \geq 0$. But since $\hat{\mathscr{X}}$ is nonsingular, we conclude that $\hat{\mathscr{X}}$ is positive definite.

If $\hat{k} = k$ the sufficiency is proven. Otherwise, set $\ell = k - \hat{k}$ and

$$\mathscr{X} = \begin{pmatrix} S & \hat{N} & 0 \\ \hat{N}^{\mathrm{T}} & \hat{Z} & 0 \\ 0 & 0 & I_{\ell \times \ell} \end{pmatrix},$$

observe that $\mathscr{X}$ is positive definite and that

$$\mathscr{X}^{-1} = \begin{pmatrix} R & * & 0 \\ \hat{M}^{\mathrm{T}} & * & 0 \\ 0 & 0 & I_{\ell \times \ell} \end{pmatrix}$$

has the required structure. $\triangleleft$

Remark 3.3. Note that condition (3.69) implies $S > 0$ and $R > 0$. This is the consequence of the arguments used in the proof of the previous Lemma, but can also be proven directly as follows. Condition (3.69) implies that the two diagonal blocks S and R are positive semidefinite. Consider the quadratic form associated with the matrix on left-hand side of (3.69),

$$V(x,z) = \begin{pmatrix} x^{\mathrm{T}} & z^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} S & I \\ I & R \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = x^{\mathrm{T}} S x + z^{\mathrm{T}} R z + 2 z^{\mathrm{T}} x.$$

We prove that R is nonsingular (and thus positive definite). By contradiction, suppose this is not the case. Then, there is a vector $\bar{z} \neq 0$ such that $\bar{z}^{\mathrm{T}} R \bar{z} = 0$. Pick any vector $\bar{x}$ such that $\bar{z}^{\mathrm{T}} \bar{x} \neq 0$. Then, for any $c \in \mathbb{R}$,

$$V(\bar{x}, c\bar{z}) \leq \lambda_{\max}(S) \|\bar{x}\|^2 + 2c\bar{z}^{\mathrm{T}} \bar{x}.$$

Clearly, by choosing an appropriate c, the right hand side can be made strictly negative. Thus, $V(x,z)$ is not positive semidefinite and this contradicts (3.69). The same arguments are used to show that also S is nonsingular. $\triangleleft$

This Lemma establishes a *coupling condition* between the two submatrices S and R identified in the previous analysis, that determines the positivity of the matrix $\mathscr{X}$. Using this Lemma it is therefore possible to arrive at the following conclusion.

Theorem 3.3. *Consider a plant of modeled by equations of the form (3.48). Let V_1, V_2, Z_1, Z_2 be matrices such that*

$$\mathrm{Im}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} C_2 & D_{21} \end{pmatrix}, \qquad \mathrm{Im}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} \end{pmatrix}.$$

The problem of γ-suboptimal H_∞ feedback design has a solution if and only if there exist symmetric matrices S and R satisfying the following system of linear matrix inequalities

$$\begin{pmatrix} Z_1^{\mathrm{T}} & Z_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}}S + SA & SB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{3.76}$$

$$\begin{pmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} AR + RA^{\mathrm{T}} & RC_1^{\mathrm{T}} & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{3.77}$$

$$\begin{pmatrix} S & I \\ I & R \end{pmatrix} \geq 0. \tag{3.78}$$

In particular, there exists a solution of dimension k if and only if there exist R and S satisfying (3.76), (3.77), (3.78) and, in addition,

$$\mathrm{rank}(I - RS) \leq k. \tag{3.79}$$

The result above describes necessary and sufficient conditions for the existence of a controller that solves the problem of γ-suboptimal H_∞ feedback design. For the actual *construction* of a controller, one may proceed as follows. Assuming that S and R are positive definite symmetric matrices satisfying the system of linear matrix inequalities (3.76), (3.77), (3.78), construct a matrix $\mathscr{X}$ as indicated in the proof of Lemma 3.4, that is, find two $n \times k$ matrices N and M such that

$$I - SR = NM^{\mathrm{T}}$$

with $k = \mathrm{rank}(I - SR)$, and solve for $\mathscr{X}$ the linear equation

$$\begin{pmatrix} S & I \\ N^{\mathrm{T}} & 0 \end{pmatrix} = \mathscr{X} \begin{pmatrix} I & R \\ 0 & M^{\mathrm{T}} \end{pmatrix}.$$

By construction, the matrix in question is positive definite, satisfies (3.62) and their equivalent versions (3.59). Thus, according to Lemma 3.3, there exists a matrix $\mathbf{K}$ satisfying (3.56). This is a linear matrix inequality in the only unknown $\mathbf{K}$. The solution of such inequality provides the required controller as

$$\mathbf{K} = \begin{pmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{pmatrix}.$$

Remark 3.4. It is worth observing that if the problem of γ-suboptimal H_∞ feedback design has *any* solution at all, it does have a solution in which the dimension of the controller, i.e. the dimension of the vector x_c in (3.49), does not exceed n, i.e. the dimension of the vector x in (3.48). This is an immediate consequence of the construction shown above, in view of the fact that the rank of $(I - SR)$ cannot exceed n.

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