Optimal Control

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Lecture 3

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THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

Course outline

- Introduction to optimal control
- Nonlinear optimization
- Dynamic programming
- Calculus of variations
- Calculus of variations and optimal control
- LQ problem
- Minimum time problem

Problem 1

Let us consider the **dynamical system** described by:

$$\dot{x} = f(x,u,t)$$

f function of C^2 class

x(t) state vector in R^n

u(t) control vector in R^p

$$x(t_i) = x_i$$

known

$$\chi\left(x(t_f), t_f\right) = 0$$

$$q(x,u,t) \le 0$$

Vectorial function of C[1] class of dimension $\sigma_f \le n+1$

Vectorial function of C[2] class of dimension β

Assume the norm:

$$\left\|\left(x,u,t_f\right)\right\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^{t} u(\tau)d\tau \right\| + \sup_t \|u(t)\| + |t_f|$$

Define the cost index $J(x,u,t_f) = \int_{t_i}^{t_f} L\left(x,u,t\right)dt$

with *L* function of C[2] class

AIM: Find (if it exists)

- the instant t_f^o
- the control $u^o \in \overline{C}^0(R)$
- the state $x^o \in \overline{C}^1(R)$

that satisfy the previous equations and minimize the cost index

DEFINE the scalar function

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 L(x,u,t) + \lambda^T(t) f(x,u,t)$$

the Hamiltonian function

Theorem 1

Let (x^*, u^*, t_f^*) be an admissible solution such that

dimension of

$$rk\left\{ \frac{\partial \chi}{\partial (x(t_f), t_f)} \Bigg|^* \right\} = \sigma_f \qquad rk\left\{ \frac{\partial q_{active}}{\partial u} \Bigg|^* \right\} = \beta_a(t), \ \ \forall t \in \left[t_i, t_f^*\right]$$

IF (x^*, u^*, t_f^*) is a local minimum

there exist $\lambda_0^* \geq 0, \ \ \lambda^* \in \overline{C}^1[t_i, t_f^*], \ \eta^* \in \overline{C}^0[t_i, t_f^*]$

<u>not simultaneously null i</u>n $[t_i, t_f^*]$ such that:

- $$\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T} - \left.\frac{\partial q}{\partial x}\right|^{*T} \eta^*$$
- $$0 = \left.\frac{\partial H}{\partial u}\right|^{*T} + \left.\frac{\partial q}{\partial u}\right|^{*T} \eta^*$$
- $$\eta_j^*(t) q_j(x^*, u^*, t) = 0, \quad \eta_j^*(t) \geq 0, \quad j = 1, 2, \ldots, \beta$$
- $$\lambda^*(t_f^*) = -\left.\frac{\partial \chi}{\partial (x(t_f))}\right|_{t_f^*}^{*T} \varsigma, \ \varsigma \in R^{\sigma_f}$$
- $$H\Big|_{t_f^*}^{*} = \left.\frac{\partial \chi}{\partial t_f}\right|_{t_f^*}^{*T} \varsigma$$

The discontinuity of $\dot{\lambda}^*$ *and* η^* may occurr only in the points $\bar{t}$

where u^* has a discontinuity and $H\Big|_{\bar{t}^-}^{*} = H\Big|_{\bar{t}^+}^{*}$ ♦

Problem 2

Consider Problem 1 assuming L, f, q not depending on t

Theorem 2

Let (x^*, u^*, t_f^*) be an admissible solution

for any continuity point of u^*

$$\left.\frac{dH}{dt}\right|^* = \left.\frac{\partial H}{\partial t}\right|^* + \eta^{*T} \left.\frac{\partial q}{\partial t}\right|^*$$

Moreover for stationary problems:

$$H\big|^* = c, \quad \forall t \in \left[t_i, t_f^*\right]$$

Problem 3

Let us consider the **dynamical system** described by:

$$\dot{x} = f(x,u,t)$$

f function of C^2 class

u(t) control vector in R^p

x(t) state vector in R^n

$$x(t_i) = x_i$$

known

$$\chi\left(x(t_f),t_f\right)=0$$

Vectorial function of C[1] class of dimension $\sigma_f \le n+1$

$$\int_{t_i}^{t_f} h\left(x(t),u(t),t\right)dt = K$$

Vectorial function of C[2] class of dimension σ

Assume the norm:

$$\left\|\left(x,u,t_f\right)\right\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^{t} u(\tau)d\tau \right\| + \sup_t \|u(t)\| + |t_f|$$

Define the cost index

$$J(x,u,t_f) = \int_{t_i}^{t_f} L\left(x,u,t\right)dt$$

with *L* function of C[2] class

AIM: Find

- the instant t_f^o
- the control $u^o \in \overline{C}^0(R)$
- the state $x^o \in \overline{C}^1(R)$

that satisfy the previous equations and minimize the cost index

DEFINE the scalar function

$$H(x,u,\lambda_0,\lambda,t)=\lambda_0 L(x,u,t)+\lambda^T(t)f(x,u,t)+\rho^T h\big(x(t),u(t),t\big)$$

the Hamiltonian function

Theorem 3

Let (x^*, u^*, t_f^*) be an admissible solution such that $rk\left\{ \left.\frac{\partial \chi}{\partial(x(t_f), t_f)}\right|^* \right\} = \sigma_f$

IF (x^*, u^*, t_f^*) is a local minimum

there exist $\lambda_0^* \in R,\ \lambda^* \in \overline{C}^1[t_i, t_f^*],\ \rho^* \in R$ not simultaneously null in $[t_i, t_f^*]$ such that:

- $\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T}$
- $0 = \left.\frac{\partial H}{\partial u}\right|^{*T}$
- $\lambda^*(t_f^*) = -\left.\frac{\partial \chi}{\partial(x(t_f))}\right|_{t_f^*}^{*T} \varsigma,\ \varsigma \in R^{\sigma_f}$
- $H\big|_{t_f^*}^{*} = \left.\frac{\partial \chi}{\partial t_f}\right|_{t_f^*}^{*T} \varsigma$

The discontinuity of $\dot{\lambda}^*$ may occurr only in the points $\bar{t}_k$ where u^* has a discontinuity

and

$$H\Big|_{\bar{t}_k^-}^{*} = H\Big|_{\bar{t}_k^+}^{*}$$

Problem 4:

Consider the linear system $\dot{x} = A(t)x + B(t)u$

Functions of C^2 class

Assume:

- $t_i \quad t_f \quad x(t_i) = x^i$ fixed
- $x(t_f) \in D_f$ being D_f a fixed point or R^n
- $q(x,u,t) \leq 0$ Vectorial function of C^2 class of dimension β **CONVEX**

Define the cost index

Functions of C^3 class- **CONVEX**

$$J(x,u) = \int_{t_i}^{t_f} L(x,u,t)dt + G(x(t_f))$$

Functions of C^2 class **CONVEX**

Theorem 4

Let (x^o, u^o) be an admissible solution such that

$$rk\left\{ \left. \frac{\partial q_{active}}{\partial u} \right|^o \right\} = \beta_a(t), \quad \forall t \in [t_i, t_f]$$

(x^o, u^o)

is a **normal** optimal solution

$\Leftrightarrow$

$$\dot{\lambda}^o = -\left.\frac{\partial H}{\partial x}\right|^{oT} - \left.\frac{\partial q}{\partial x}\right|^{oT} \eta^o$$

$$0 = \left.\frac{\partial H}{\partial u}\right|^{oT} + \left.\frac{\partial q}{\partial u}\right|^{oT} \eta^o$$

$$\eta_j^o(t) q_j(x^o, u^o, t) = 0,$$

$$j = 1, 2, \ldots, \beta$$

$$\eta^o(t) \geq 0$$

AND IF $D_f = R^n$ $\quad \lambda^o(t_f) = \left.\frac{dG}{dx(t_f)}\right|^{oT}$