Appendix B
Stability and Asymptotic Behavior of Nonlinear Systems

B.1 The theorems of Lyapunov for nonlinear systems

We assume in what follows that the reader is familiar with basic concepts concerning the stability of an equilibrium in a nonlinear system. In this section we provide a sketchy summary of some fundamental results, mainly to the purpose of introducing notations and results that are currently used throughout the book. [1]

Comparison functions. A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathscr{K}_\infty$. A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathscr{K}\mathscr{L}$ if, for each fixed t, the function

$$\begin{aligned} \alpha : [0,a) &\to [0,\infty) \\ r &\mapsto \beta(r,t) \end{aligned}$$

belongs to class $\mathscr{K}$ and, for each fixed r, the function

$$\begin{aligned} \varphi : [0,\infty) &\to [0,\infty) \\ t &\mapsto \beta(r,t) \end{aligned}$$

is decreasing and $\lim_{t\to\infty} \varphi(t) = 0$.

The composition of two class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathscr{K}$ function, defined on $[0,a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique function, $\alpha^{-1} : [0,b) \to [0,a)$, such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \text{ for all } r \in [0,a) \\ \alpha(\alpha^{-1}(r)) &= r, \text{ for all } r \in [0,b). \end{aligned}$$

[1] For further reading, see [2], [4],[6], [5].

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathscr{K}$ function. If $\alpha(\cdot)$ is a class $\mathscr{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$. If $\beta(\cdot,\cdot)$ is a class $\mathscr{KL}$ function and $\alpha_1(\cdot), \alpha_2(\cdot)$ are class $\mathscr{K}$ functions, the function thus defined

$$\begin{aligned} \gamma\colon\ [0,a)\times[0,\infty) &\to [0,\infty) \\ (r,t) \qquad &\mapsto \alpha_1(\beta(\alpha_2(r),t)) \end{aligned}$$

is a class $\mathscr{KL}$ function.

The Theorems of Lyapunov. Consider an autonomous nonlinear system

$$\dot{x} = f(x) \tag{B.1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well known criterion of Lyapunov, which, using comparison functions, can be expressed as follows. Let B_d denote the open ball of radius d in $\mathbb{R}^n$, i.e.

$$B_d = \{x \in \mathbb{R}^n : \|x\| < d\}\,.$$

Theorem B.1. [Direct Theorem] *Let $V : B_d \to \mathbb{R}$ be a C^1 function such that, for some class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, defined on $[0,d)$,*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \text{for all } x \in B_d. \tag{B.2}$$

If

$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \text{for all } x \in B_d, \tag{B.3}$$

the equilibrium $x = 0$ of (B.1) is stable.

If, for some class $\mathscr{K}$ function $\alpha(\cdot)$, defined on $[0,d)$,

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|) \qquad \text{for all } x \in B_d, \tag{B.4}$$

the equilibrium $x = 0$ of (B.1) is locally asymptotically stable.

If $d = \infty$ and, in the above inequalities, $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathscr{K}_\infty$ functions, the equilibrium $x = 0$ of (B.1) is globally asymptotically stable.

Remark B.1. The usefulness of the comparison functions, in the statement of the Theorem, is motivated by the following simple arguments. Suppose (B.3) holds. Then, so long as $x(t) \in B_d$, $V(x(t))$ is non-increasing, i.e. $V(x(t)) \le V(x(0))$. Pick $\varepsilon < d$ and define $\delta = \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)$. Then, using (B.2), it is seen that, if $\|x(0)\| < \delta$,

$$\underline{\alpha}(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \overline{\alpha}(\|x(0)\|) \le \overline{\alpha}(\delta) = \underline{\alpha}(\varepsilon)$$

which implies $\|x(t)\| \le \varepsilon$. This shows that $x(t)$ exists for all t and the equilibrium $x = 0$ is stable.

Suppose now that (B.4) holds. Define $\gamma(r) = \alpha(\overline{\alpha}^{-1}(r))$, which is a class $\mathscr{K}$ function. Using the estimate on the right of (B.2), it is seen that $\alpha(\|x\|) \ge \gamma(V(x))$

and hence
$$\frac{\partial V}{\partial x} f(x) \le -\gamma(V(x))\,.$$
Since $V(x(t))$ is a continuous function of t, non-increasing and non-negative for each t, there exists a number $V^* \ge 0$ such that $\lim_{t\to\infty} V(x(t)) = V^*$. Suppose V^* is strictly positive. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \le -\gamma(V(x(t)) \le -\gamma(V^*) < 0\,.$$

Integration with respect to time yields

$$V(x(t)) \le V(x(0)) - \gamma(V^*)t$$

for all t. This cannot be the case, because for large t the right-hand side is negative, while the left-hand side is non-negative. From this it follows that $V^* = 0$ and therefore, using the fact that $V(x)$ vanishes only at $x = 0$, it is concluded that $\lim_{t\to\infty} x(t) = 0$. Note also that identical arguments hold for the analysis of the asymptotic properties of a time-dependent system

$$\dot{x} = f(x,t)$$

so long $f(0,t) = 0$ for all $t \ge 0$ and $V(x)$ is independent of t. $\triangleleft$

Sometimes, in the design of feedback laws, while it is difficult to obtain a system whose equilibrium $x = 0$ is globally asymptotically stable, it is relatively more easy to obtain a system in which trajectories are bounded (maybe for a specific set of initial conditions) and have suitable decay properties. Instrumental, in such context, is the notion of *sublevel set* of a Lyapunov function $V(x)$ which, for a fixed non-negative real number c, is defined as

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}.$$

The function $V(x)$, which is *positive definite* (i.e. is positive for all nonzero x and zero at $x = 0$) is said to be *proper* if, for each $c \in \mathbb{R}$, the sublevel set Ω_c is a *compact* set. Now, it is easy to check that the function $V(x)$ is proper if and only the inequality on the left-hand side of (B.2) holds for all $x \in \mathbb{R}^n$, with a function $\underline{\alpha}(\cdot)$ which is of class $\mathscr{K}_\infty$. Note also that, if $V(x)$ is proper, for any $c > 0$ it is possible to find a numbers $c_1 > 0$ and $c_2 > 0$ such that

$$B_{c_1} \subset \Omega_c \subset B_{c_2}\,.$$

An typical example of how sublevel sets can be used to analyze boundedness and decay of trajectories is the following one. Let r_1 and r_2 be two positive numbers, with $r_2 > r_1$. Suppose $V(x)$ is a function satisfying (B.2), with $\underline{\alpha}(\cdot)$ a class $\mathscr{K}_\infty$ function. Pick any pair of positive numbers c_1, c_2, such that

$$\Omega_{c_1} \subset B_{r_1} \subset B_{r_2} \subset \Omega_{c_2}\,.$$

and let $S_{c_1}^{c_2}$ denote the “annular” compact set

$$S_{c_1}^{c_2} = \{x \in \mathbb{R}^n : c_1 \leq V(x) \leq c_2\}.$$

Suppose that, for some $a > 0$,

$$\frac{\partial V}{\partial x} f(x) \leq -a \qquad \text{for all } x \in S_{c_1}^{c_2}.$$

Then, for each initial condition $x(0) \in B_{r_2}$, the trajectory $x(t)$ of (B.1) is defined for all t and there exists a finite time T such that $x(t) \in B_{r_1}$ for all $t \geq T$. In fact, take any $x(0) \in B_{r_2} \setminus \Omega_{c_1}$. Such $x(0)$ is in $S_{c_1}^{c_2}$. So long as $x(t) \in S_{c_1}^{c_2}$, the function $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t) \leq -a$$

and hence

$$V(x(t)) \leq V(x(0)) - at \leq c_2 - at.$$

Thus, at a time $T \leq (c_2 - c_1)/a$, $x(T)$ is on the boundary of the set Ω_{c_1}. On the boundary of Ω_{c_1} the derivative of $V(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_{c_1} and remains there for all $t \geq T$.

It is well-known that the criterion for asymptotic stability provided by the previous Theorem has a *converse*, namely, the existence of a function $V(x)$ having the properties indicated in Theorem B.1 is *implied* by the property of asymptotic stability of the equilibrium $x = 0$ of (B.1). In particular, the following result holds.

Theorem B.2. [Converse Theorem] *Suppose the equilibrium $x = 0$ of (B.1) is locally asymptotically stable. Then, there exist $d > 0$, a C^1 function $V : B_d \to \mathbb{R}$, and class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold. If the equilibrium $x = 0$ of (B.1) is globally asymptotically stable, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold with $d = \infty$.*

It is well-known that, for a nonlinear system, the property of asymptotic stability of the equilibrium $x = 0$ does not necessarily imply *exponential* decay to zero of $\|x(t)\|$. If the equilibrium $x = 0$ of system (B.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$ and $\lambda > 0$ such that

$$x(0) \in B_d \quad \Rightarrow \quad \|x(t)\| \leq M e^{-\lambda t} \|x(0)\| \quad \text{for all } t \geq 0$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. In this context, the following criterion is useful.

Lemma B.1. *The equilibrium $x = 0$ of nonlinear system (B.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function $V(x) : \mathbb{R}^n \to \mathbb{R}$, three class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and three real numbers $\delta > 0, \underline{a} > 0, a > 0$, such that*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|)$$

for all $x \in \mathbb{R}^n$ and

$$\underline{\alpha}(s) = \underline{a}s^2, \qquad \alpha(s) = as^2$$

for all $s \in B_\delta$.

B.2 Input-to-state stability and the theorems of Sontag

In the analysis of *forced* nonlinear systems, the property of *input-to-state stability*, introduced and thoroughly studied by E.D. Sontag, plays a role of paramount importance.[2] Consider a forced nonlinear system

$$\dot{x} = f(x,u) \tag{B.5}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0,\infty) \to \mathbb{R}^m$ of (B.5) can be any piecewise continuous bounded function. The space of all such functions is endowed with the so-called supremum norm $\|u(\cdot)\|_\infty$, which is defined as

$$\|u(\cdot)\|_\infty = \sup_{t \ge 0} \|u(t)\| \,.$$

Definition B.1. System (B.5) is said to be input-to-state stable if there exist a class $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$, called a *gain function*, such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \beta(\|x(0)\|,t) + \gamma(\|u(\cdot)\|_\infty) \tag{B.6}$$

for all $t \ge 0$.

Since, for any pair $\beta > 0, \gamma > 0$, $\max\{\beta,\gamma\} \le \beta + \gamma \le \max\{2\beta, 2\gamma\}$, an alternative way to say that a system is input-to-state stable is to say that there exists a class $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \max\{\beta(\|x(0)\|,t), \gamma(\|u(\cdot)\|_\infty)\} \tag{B.7}$$

for all $t \ge 0$. Note also that, letting $\|u(\cdot)\|_{[0,t]}$ denote the supremum norm of the restriction of $u(\cdot)$ to the interval $[0,t]$, namely

[2] The concept of input-to-state stability, its properties and applications have been introduced in the sequence of papers [10, 11, 14]. []A summary of the most relevant aspect of the theory can also be found in [5, p. 17-31]

$$\|u(\cdot)\|_{[0,t]} = \sup_{s \in [0,t]} \|u(s)\| ,$$

the bound (B.6) can be also expressed in the alternative form [3]

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_{[0,t]}) , \tag{B.8}$$

and the bound (B.7) in the alternative form

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_{[0,t]})\} ,$$

both holding for all $t \geq 0$.

The property of input-to-state stability can be given a characterization which extends the well known criterion of Lyapunov for asymptotic stability. The key tool for such characterization is the notion of *ISS-Lyapunov function*.

Definition B.2. A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (B.5) if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \tag{B.9}$$

and

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|) \quad \text{for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfying } \|x\| \geq \chi(\|u\|). \tag{B.10}$$

An equivalent form in which the notion of an ISS-Lyapunov function can be described is the following one.

Lemma B.2. *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (B.5) if and only if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\sigma(\cdot)$ such that (B.9) holds and*

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \qquad \textit{for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m . \tag{B.11}$$

The existence of an ISS-Lyapunov function turns out to be a necessary and sufficient condition for input-to-state stability.

Theorem B.3. *System (B.5) is input-to-state stable if and only if there exists an ISS-Lyapunov function. In particular, if such function exists, then an estimate of the form (B.6) holds with $\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)))$.*

The following elementary examples describe how the property of input-to-state stability can be checked and an estimate of the gain function can be evaluated.

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[3] In fact, since $\|u(\cdot)\|_{[0,t]} \leq \|u(\cdot)\|_\infty$ and $\gamma(\cdot)$ is increasing, (B.8) implies (B.6). On the other hand, since $x(t)$ depends only on the restriction of $u(\cdot)$ to the interval $[0,t]$, one could in (B.6) replace $u(\cdot)$ with an input $\bar{u}(\cdot)$ defined as $\bar{u}(s) = u(s)$ for $0 \leq s \leq t$ and $\bar{u}(s) = 0$ for $s > t$, in which case $\|\bar{u}(\cdot)\|_\infty = \|u(\cdot)\|_{[0,t]}$, and observe that (B.6) implies (B.8).

Example B.1. A stable linear system

$$\dot{x} = Ax + Bu$$

is input-to-state stable, with a linear gain function. In fact, let P denote the unique positive definite solution of the Lyapunov equation $PA + A^{\mathrm{T}}P = -I$ and observe that $V(x) = x^{\mathrm{T}}Px$ satisfies

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\|x\|^2 + c\|x\|\,\|u\|$$

for some $c > 0$. Pick $0 < \varepsilon < 1$ and set $\ell = c/(1-\varepsilon)$. Then, it is easy to see that

$$\|x\| \geq \ell\|u\| \qquad \Rightarrow \qquad \frac{\partial V}{\partial x}(Ax + Bu) \leq -\varepsilon\|x\|^2 .$$

The system is input-to-state, with $\chi(r) = \ell r$. Since $\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$, we obtain following the estimate for the (linear) gain function

$$\gamma(r) = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\ell r . \qquad \triangleleft$$

Example B.2. Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k + bx^p u ,$$

in which $k \in \mathbb{N}$ is *odd*, $a > 0$ and $p \in \mathbb{N}$ is such that $p < k$. Pick $V(x) = \frac{1}{2}x^2$ and note that, since $k+1$ is even,

$$\frac{\partial V}{\partial x}(-ax^k + bx^p u) \leq -a|x|^{k+1} + |b||x|^{p+1}|u|$$

Pick $0 < \varepsilon < a$ and define (recall that $k > p$)

$$\chi(r) = \left(\frac{|b|r}{a-\varepsilon}\right)^{\frac{1}{k-p}} .$$

Then, it is easy to see that

$$\|x\| \geq \chi(\|u\|) \qquad \Rightarrow \qquad \frac{\partial V}{\partial x}(-ax^k + bx^p u) \leq -\varepsilon|x|^{k+1} .$$

The system is input-to-state stable, with $\gamma(r) = \chi(r)$.

Note that the condition $k > p$ is essential. In fact, the following system, in which $k = p = 1$,

$$\dot{x} = -x + xu$$

is not input-to-state stable. Under the bounded (constant) input $u(t) = 2$ the state $x(t)$ evolves as a solution of $\dot{x} = x$ and hence diverges to infinity. $\triangleleft$

The notion of input-to-state stability lends itself to a number of alternative (equivalent) characterizations, among which one of the most useful can be expressed as follows.

Theorem B.4. *System (B.5) is input-to-state stable if and only if there exists class $\mathscr{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any bounded input and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\le \max\{\gamma_0(\|x(0)\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\le \gamma(\limsup_{t\to\infty} \|u(t)\|)\ . \end{aligned}$$

B.3 Cascade-connected systems

In this section we investigate the *asymptotic* stability of the equilibrium $(z, \xi) = (0,0)$ of a pair of cascade connected subsystems of the form

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi)\ , \end{aligned} \tag{B.12}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0) = 0$, and $f(z, \xi)$, $g(\xi)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ (see Fig. B.1).

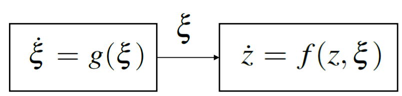

Fig. B.1 A cascade-connection of systems.

Since similar cascade connections occur quite often in the analysis (and feedback design) of nonlinear systems, it is important to understand under what conditions the stability properties of the two components subsystems determine the stability of the cascade. If both systems were linear systems, the cascade would be a system modeled as

$$\begin{aligned} \dot{z} &= Fz + G\xi \\ \dot{\xi} &= A\xi\ , \end{aligned}$$

and it is trivially seen that if both F and G have all eigenvalues in $\mathbb{C}^-$, the cascade is an asymptotically stable system. The nonlinear counterpart of such property, though, requires some extra care.

The simplest scenario, in this respect, is one in which one is interested in seeking only local stability. In this case, the following result holds. [4]

Lemma B.3. *Suppose the equilibrium $z = 0$ of*

$$\dot{z} = f(z, 0) \tag{B.13}$$

is locally asymptotically stable and the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is stable. Then the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is stable. If the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is locally asymptotically stable, then the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is locally asymptotically stable.

It must be stressed, though, that in this Lemma only the property of *local* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ is considered. In fact, by means of a simple counterexample, it can be shown that the *global* asymptotic stability of $z = 0$ as an equilibrium of (B.13) and the *global* asymptotic stability of $\xi = 0$ as an equilibrium of $\dot{\xi} = g(\xi)$ *do not* imply, in general, *global* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ of the cascade. As a matter of fact, the cascade connection of two such systems may even have finite escape times. To infer global asymptotic stability of the cascade, a (strong) extra condition is needed, as shown below.

Example B.3. Consider the case in which

$$\begin{aligned} f(z, \xi) &= -z + z^2 \xi \\ g(\xi) &= -\xi \,. \end{aligned}$$

Clearly $z = 0$ is a globally asymptotically equilibrium of $\dot{z} = f(z, 0)$ and $\xi = 0$ is a globally asymptotically equilibrium of $\dot{\xi} = g(\xi)$. However, this system has finite escape times. To show that this is the case, consider the differential equation

$$\dot{\tilde{z}} = -\tilde{z} + \tilde{z}^2 \tag{B.14}$$

with initial condition $\tilde{z}(0) = z_0$. Its solution is

$$\tilde{z}(t) = \frac{-z_0}{z_0 - 1 - z_0 \exp(-t)} \exp(-t) \,.$$

Suppose $z_0 > 1$. Then, the $\tilde{z}(t)$ escapes to infinity in finite time. In particular, the maximal (positive) time interval on which $\tilde{z}(t)$ is defined is the interval $[0, t_{\max}(z_0))$ with

$$t_{\max}(z_0) = \ln\left(\frac{z_0}{z_0 - 1}\right).$$

Now, return to system (B.12), with initial condition (z_0, ξ_0) and let ξ_0 be such that

$$\xi(t) = \exp(-t)\xi_0 \geq 1 \qquad \text{for all } t \in [0, t_{\max}(z_0))\,.$$

[4] More details and proofs of the results stated in this section can be found in [5, p. 11-17 and 31-36].

Clearly, on the time interval $[0, t_{\max}(z_0))$, we have

$$\dot{z} = -z + z^2\xi \geq -z + z^2 .$$

By comparison with (B.14), it follows that

$$z(t) \geq \tilde{z}(t) .$$

Hence $z(t)$ escapes to infinity, at a time $t^* \leq t_{\max}(z_0)$. The lesson learned from this example is that, even if $\xi(t)$ exponentially decreases to 0, this may not suffice to prevent finite escape time in the upper system. The state $z(t)$ escapes to infinity at a time in which the effect of $\xi(t)$ on the upper equation is still not negligible. $\triangleleft$.

The following results provide the extra condition needed to ensure global asymptotic stability in the cascade.

Lemma B.4. *Suppose the equilibrium $z = 0$ of (B.13) is asymptotically stable, and let S be a subset of the domain of attraction of such equilibrium. Consider the system*

$$\dot{z} = f(z, \xi(t)) . \tag{B.15}$$

in which $\xi(t)$ is a continuous function, defined for all $t \geq 0$ and suppose that $\lim_{t\to\infty} \xi(t) = 0$. Pick $z_0 \in S$, and suppose that the integral curve $z(t)$ of (B.15) satisfying $z(0) = z_0$ is defined for all $t \geq 0$, bounded, and such that $z(t) \in S$ for all $t \geq 0$. Then $\lim_{t\to\infty} z(t) = 0$.

This last result implies, in conjunction with Lemma B.3, that if the equilibrium $z = 0$ of (B.13) is globally asymptotically stable, if the equilibrium $\xi = 0$ of the lower subsystem of (B.12) is globally asymptotically stable, and all trajectories of the composite system (B.12) *are bounded*, the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is globally asymptotically stable.

To be in a position to use this result in practice, one needs to determine conditions under which the boundedness property holds. This is indeed the case if the upper subsystem of the cascade, viewed as a system with state z and input ξ, is input-to-state stable. In view of this, it can be claimed that if the upper subsystem of the cascade is input-to-state stable and the lower subsystem is globally asymptotically stable (at the equilibrium $\xi = 0$), the cascade is globally asymptotically stable (at the equilibrium $(z, \xi) = (0, 0)$).

As a matter fact, a more general result holds, which is stated as follows.

Theorem B.5. *Suppose that system*

$$\dot{z} = f(z, \xi) , \tag{B.16}$$

viewed as a system with input ξ and state z is input-to-state stable and that system

$$\dot{\xi} = g(\xi, u) , \tag{B.17}$$

viewed as a system with input u and state ξ is input-to-state stable as well. Then, system

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi, u) \end{aligned}$$

is input-to-state stable.

Example B.4. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \xi_1 \xi_2 \\ \dot{x}_2 &= -x_2 + \xi_1^2 - x_1 \xi_1 \xi_2 \\ \dot{\xi}_1 &= -\xi_1^3 + \xi_1 u_1 \\ \dot{\xi}_2 &= -\xi_2 + u_2 \,. \end{aligned}$$

The subsystem consisting of the two top equations, seen as a system with state $x = (x_1, x_2)$ and input $\xi = (\xi_1, \xi_2)$ is input-to-state stable. In fact, let this system be written as

$$\dot{x} = f(x, \xi)$$

and consider the candidate ISS-Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

for which we have

$$\frac{\partial V}{\partial x} f(x, \xi) = -(x_1^2 + x_2^2) + x_2 \xi_1^2 \le -x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{2}\xi_1^4 \le -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|\xi\|^4 \,.$$

Thus, the function $V(x)$ satisfies the condition indicated in Lemma B.2, with

$$\alpha(r) = \frac{1}{2}r^2 \,, \qquad \sigma(r) = \frac{1}{2}r^4 \,.$$

The subsystem consisting of the two bottom equations is composed of two separate subsystems, both of which are input-to-state stable, as seen in examples B.2 and B.1. Thus, the overall system is input-to-state stable. ◁

B.4 Limit sets

Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{B.18}$$

with $x \in \mathbb{R}^n$. It is well known that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for all $x_0 \in \mathbb{R}^n$ the solution of (B.18) with initial condition $x(0) = x_0$, denoted by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique.

Definition B.3. Let $x_0 \in \mathbb{R}^n$ be fixed. Suppose that $x(t,x_0)$ is defined for all $t \geq 0$. A point x is said to be an ω-limit *point* of the motion $x(t,x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k,x_0) = x\,.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is the union of all ω-limit points of the motion $x(t,x_0)$.

It is obvious from this definition that an ω-limit point *is not* necessarily a limit of $x(t,x_0)$ as $t \to \infty$, because the solution in question may not admit any limit as $t \to \infty$ (see for instance fig. B.2). [5]

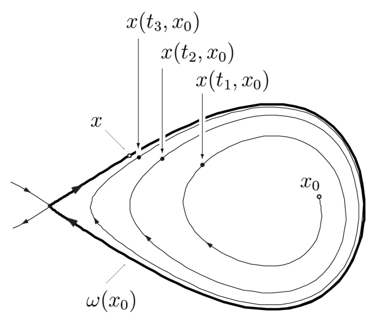

Fig. B.2 The ω-limit set of the point x_0.

However, it is known that, if the motion $x(t,x_0)$ is *bounded*, then $x(t,x_0)$ asymptotically approaches the set $\omega(x_0)$, as specified in the Lemma that follows.[6] In this respect, recall that a set $S \subset \mathbb{R}^n$ is said to be *invariant* under (B.18) if for all initial conditions $x_0 \in S$ the solution $x(t,x_0)$ of (B.18) exists for all $t \in (-\infty,+\infty)$ and $x(t,x_0) \in S$ for all such t. [7]. Moreover, the *distance* of a point $x \in \mathbb{R}$ from a set $S \subset \mathbb{R}^n$, denoted $\mathrm{dist}(x,S)$, is the non-negative real number defined as

$$\mathrm{dist}(x,S) = \inf_{z\in S} \|x - z\|\,.$$

[5] Figures B.2, B.3, B.4 are reprinted from *Annual Reviews in Control*, Vol. 32, A.Isidori and C.I.Byrnes, Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection, Pages 1-16, Copyright (2008), with permission from Elsevier.

[6] See [1, page 198].

[7] We recall, for the sake of completeness, that a set S is said to be *positively invariant*, or *invariant in positive time* (respectively, *negatively invariant* or *invariant in negative time*) if for all initial conditions $x_0 \in X$, the solution $x(t,x_0)$ exists for all $t \geq 0$ and $x(t,x_0) \in X$ for all $t \geq 0$ (respectively exists for all $t \leq 0$ and $x(t,x_0) \in X$ for all $t \leq 0$). Thus, a set is invariant if it is both positively invariant and negatively invariant.

Lemma B.5. *Suppose there is a number M such that $\|x(t,x_0)\| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty connected compact set, invariant under (B.18). Moreover,*

$$\lim_{t\to\infty} \operatorname{dist}(x(t,x_0), \omega(x_0)) = 0\,.$$

Example B.5. Consider the classical (stable) Van der Pol oscillator, written in state-space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1-x_1^2)x_2 \end{aligned} \tag{B.19}$$

in which, as it is well known, the damping term $\mu(1-x_1^2)y$ can be seen as a model of a nonlinear resistor, negative for small x_1 and positive for large x_1 (see [6]). From the phase portrait of this system (depicted in fig. B.3 for $\mu = 1$) it is seen that all motions except the trivial motion occurring for $x_0 = 0$ are bounded in positive time and approach, as $t \to \infty$, the limit cycle $\mathscr{L}$. As consequence, $\omega(x_0) = \mathscr{L}$ for any $x_0 \neq 0$, while $\omega(0) = \{0\}$. $\triangleleft$

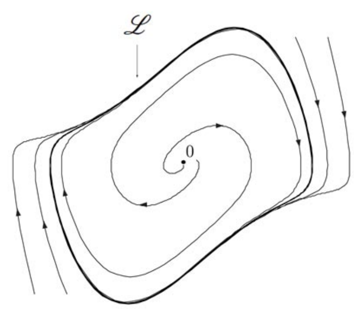

Fig. B.3 The phase portrait of the Van der Pol oscillator.

An important useful application of the notion of ω-limit set of a point is found in the proof of the following result, commonly known as LaSalle's invariance principle.[8]

Theorem B.6. *Consider system (B.18). Suppose there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \qquad \text{for all } x \in \mathbb{R}^n,$$

for some pair of class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and such that

$$\frac{\partial V}{\partial x} f(x) \leq 0 \qquad \text{for all } x \in \mathbb{R}^n\,. \tag{B.20}$$

[8] See e.g. [6]

Let $\mathscr{E}$ denote the set

$$\mathscr{E} = \{x \in \mathbb{R}^n : \frac{\partial V}{\partial x} f(x) = 0\}\,. \tag{B.21}$$

Then, for each x_0, the integral curve $x(t,x_0)$ of (B.18) passing through x_0 at time $t = 0$ is bounded, and

$$\omega(x_0) \subset \mathscr{E}\,.$$

Proof. A direct consequence of (B.20) is that, for any x_0, the motion $x(t,x_0)$ is bounded in positive time. In fact, this property yields $V(x(t,x_0)) \le V(x_0)$ for all $t \ge 0$ and this in turn implies (see Remark B.1)

$$\|x(t,x_0)\| \le \underline{\alpha}^{-1}(\overline{\alpha}(\|x_0\|)\,.$$

Thus, the limit set $\omega(x_0)$ is nonempty, compact and invariant. The non-negative-valued function $V(x(t,x_0))$ is non-increasing for $t \ge 0$. Thus, there is a number $V_0 \ge 0$, possibly dependent on x_0, such that

$$\lim_{t\to\infty} V(x(t,x_0)) = V_0\,.$$

By definition of limit set, for each point $x \in \omega(x_0)$, there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that $\lim_{k\to\infty} x(t_k,x_0) = x$. Thus, since $V(x)$ is continuous,

$$V(x) = \lim_{k\to\infty} V(x(t_k,x_0)) = V_0\,.$$

In other words, the function $V(x)$ takes the same value V_0 at any point $x \in \omega(x_0)$. Now, pick any initial condition $\bar{x}_0 \in \omega(x_0)$. Since the latter is invariant, we have $x(t,\bar{x}_0) \in \omega(x_0)$ for all $t \in \mathbb{R}$. Thus, along this particular motion, $V(x(t,\bar{x}_0)) = V_0$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} V(x(t,\bar{x}_0)) = \frac{\partial V}{\partial x} f(x)\Big|_{x=x(t,\bar{x}_0)}\,.$$

This, implies

$$x(t,\bar{x}_0) \in \mathscr{E}\,, \qquad \text{for all } t \in \mathbb{R}$$

and, since $\bar{x}_0$ is any point in $\omega(x_0)$, proves the Theorem. ◁

This Theorem is often used to determine the asymptotic properties of the integral curves of (B.18). In fact, in view of Lemma B.5, it is seen that if a function $V(x)$ can be found such that (B.20) holds, any trajectory of (B.18) is bounded and converges, asymptotically, to an invariant set that is entirely contained in the set $\mathscr{E}$ defined by (B.21). In particular, if system (B.18) has an equilibrium at $x = 0$ and it can be determined that, in the set $\mathscr{E}$, the only possible invariant set is the point $x = 0$, then the equilibrium in question is globally asymptotically stable.

Example B.6. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1(1 + x_1 x_2)\,, \end{aligned}$$

pick $V(x) = x_1^2 + x_2^2$, and observe that

$$\frac{\partial V}{\partial x} f(x) = -2(x_1 x_2)^2 .$$

The function on the right-hand side is not negative definite, but it is negative semi-definite, i.e. satisfies (B.20). Thus, trajectories converge to bounded sets that are invariant and contained in the set

$$\mathscr{E} = \{x \in \mathbb{R}^2 : x_1 x_2 = 0\} .$$

Now, it is easy to see that no invariant set may exist, other than the equilibrium, entirely contained in the set $\mathscr{E}$. In fact, if a trajectory of the system is contained in $\mathscr{E}$ for all $t \in \mathbb{R}$, this trajectory must be a solution of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 . \end{aligned}$$

This system, an harmonic oscillator, has only one trajectory entirely contained in $\mathscr{E}$, the trivial trajectory $x(t) = 0$. Thus, the equilibrium point $x = 0$ is the only possible invariant set contained in $\mathscr{E}$ and therefore this equilibrium is globally asymptotically stable. $\triangleleft$

Returning to the analysis of the properties of limit sets, let $B \subset \mathbb{R}^n$ be a fixed bounded set and suppose that *all* motions with initial condition $x_0 \in B$ are bounded in positive time. Since any motion $x(t, x_0)$ asymptotically approaches the limit set $\omega(x_0)$ as $t \to \infty$, it seems reasonable to look at the set

$$\Omega = \bigcup_{x_0 \in B} \omega(x_0) , \tag{B.22}$$

as to a "target" set that is asymptotically approached by motions of (B.18) with initial conditions in B. However, while it is true that the distance of $x(t, x_0)$ from the set (B.22) tends to 0 as $t \to \infty$ for any x_0, the convergence to such set may fail be *uniform* in x_0, even if the set B is compact. In this respect, recall that – by definition – the distance of $x(t, x_0)$ from a set S tends to 0 as $t \to \infty$ if for every ε there exists T such that

$$\operatorname{dist}(x(t, x_0), S) \le \varepsilon , \qquad \text{for all } t \ge T . \tag{B.23}$$

The number T in this expression depends on ε but also on x_0. [9] The distance of $x(t, x_0)$ from S is said to tend to 0, as $t \to \infty$, *uniformly* in x_0 on B, if for every ε there exists T, which depends on ε *but not* on x_0, such that (B.23) holds for all $x_0 \in B$.

Example B.7. Consider again the example B.5, in which the set Ω defined by (B.22) consists of the union of the equilibrium point $\{0\}$ and of the limit cycle $\mathscr{L}$ and let

[9] In fact it is likely that, the more is x_0 distant from S, the longer one has to wait until $x(t, x_0)$ becomes ε-distant from S.

B be a compact set satisfying $B \supset \mathscr{L}$. All $x_0 \in B$ are such that $\text{dist}(x(t,x_0),\Omega) \to 0$ as $t \to \infty$. However, the convergence is not uniform in x_0. In fact, observe that, if $x_0 \neq 0$ is inside $\mathscr{L}$, the motion $x(t,x_0)$ is bounded in negative time and remains inside $\mathscr{L}$ for all $t \leq 0$ (as a matter of fact, it converges to 0 as $t \to -\infty$). Pick any $x_1 \neq 0$ inside $\mathscr{L}$ such that $\text{dist}(x_1,\mathscr{L}) > \varepsilon$ and let T_1 be the minimal time needed to have $\text{dist}(x(t,x_1),\mathscr{L}) \leq \varepsilon$ for all $t \geq T_1$. Let $T_0 > 0$ be fixed and define $x_0 = x(-T_0,x_1)$. If T_0 is large, x_0 is close to 0, and the minimal time T needed to have $\text{dist}(x(t,x_0),\Omega) \leq \varepsilon$ for all $t \geq T$ is $T = T_0 + T_1$. Since the time T_0 can be taken arbitrarily large, it follows that the time $T > 0$ needed to have $\text{dist}(x(t,x_0),\Omega) \leq \varepsilon$ for all $t \geq T$ can be made arbitrarily large, even if x_0 is taken within a compact set. $\triangleleft$

Uniform convergence to the target set is important for various reason. On one side, for practical purposes it is important to have a fixed bound on the time needed to get within an ε-distance of that set. On another side, uniform convergence plays a relevant role in the existence Lyapunov functions, an indispensable tool in analysis and design of feedback systems. While convergence to the set (B.22) is not guaranteed to be uniform, there is a larger set – though – for which such property holds.

Definition B.4. Let B be a bounded subset of $\mathbb{R}^n$ and suppose $x(t,x_0)$ is defined for all $t \geq 0$ and all $x_0 \in B$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k,t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k,x_k) = x\,.$$

It is clear from the definition that, if B consists of only one single point x_0, all x_k's in the definition above are necessarily equal to x_0 and the definition in question returns the definition of ω-limit set of a point. It is also clear that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, with all the x_k's being taken equal to x_0. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B)\,.$$

However, the converse inclusion is not true in general.

Example B.8. Consider again the system in the example B.5, and let B be a compact set satisfying $B \supset \mathscr{L}$. We know that $\{0\}$ and $\mathscr{L}$, being ω-limit sets of points of B, are in $\omega(B)$. But it is also easy to see that any other point inside $\mathscr{L}$ is a point of $\omega(B)$. In fact, let $\bar{x}$ be any of such points and pick any sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$. Since $x(t,\bar{x})$ remains inside $\mathscr{L}$ (and hence in B) for all negative values of t, it is seen that $x_k := x(-t_k,\bar{x})$ is a point in B for all k. The sequence $\{x_k,t_k\}$ is such that $x(t_k,x_k) = \bar{x}$ and therefore the property required for $\bar{x}$ to be in $\omega(B)$ is trivially satisfied. This shows that $\omega(B)$ includes not just $\{0\}$ and $\mathscr{L}$, but also all points of the open region surrounded by $\mathscr{L}$. $\triangleleft$

The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma B.5, can be summarized as follows.[10]

Lemma B.6. *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t,x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Moreover, the distance of $x(t,x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

Thus, as it is the case for the ω-limit set of a point, the ω-limit set *of a bounded set B* is compact and invariant. Being invariant, the set $\omega(B)$ is filled with motions which exist for all $t \in (-\infty, +\infty)$ and all such motions, since this set is compact, are bounded in positive and in negative time. Moreover, this set is uniformly approached by motions with initial conditions $x_0 \in B$. We conclude the section with another property, that will be used later to define the concept of *steady-state behavior* of a system. [11]

Lemma B.7. *If B is a compact set invariant for (B.18), then $\omega(B) = B$.*

B.5 Limit sets and stability

It is well known that, in a nonlinear system, an equilibrium point which attracts all motions with initial conditions in some open neighborhood of this point is not necessarily stable in the sense of Lyapunov. A classical example showing that convergence to an equilibrium does not imply stability is provided by the following 2-dimensional system. [12]

Example B.9. Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(x_1,x_2) \\ g(x_1,x_2) \end{pmatrix} \tag{B.24}$$

in which $f(0,0) = g(0,0) = 0$ and

$$\begin{pmatrix} f(x_1,x_2) \\ g(x_1,x_2) \end{pmatrix} = \frac{1}{(x_1^2+x_2^2)(1+(x_1^2+x_2^2)^2)} \begin{pmatrix} x_1^2(x_2-x_1)+x_2^5 \\ x_2^2(x_2-2x_1) \end{pmatrix}$$

for $(x_1,x_2) \neq (0,0)$. The phase portrait of this system is the one depicted in fig. B.4. This system has only one equilibrium at $(x,y) = (0,0)$ and any initial condition $(x_1(0),x_2(0))$ in the plane produces a motion that asymptotically tends to this point. However, it is not possible to find, for every $\varepsilon > 0$, a number $\delta > 0$ such that every initial condition in a disc of radius δ produces a motion which remains in a disc of radius ε for all $t \geq 0$. $\triangleleft$

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[10] For a proof see, e.g. , [3], [7] and [8].

[11] For a proof, see [17].

[12] See [9] and [2, p. 191-194].

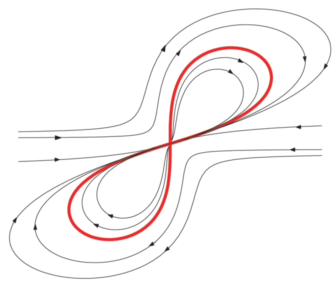

Fig. B.4 The phase portrait of system (B.24).

It is also known – though – that if the convergence to the equilibrium is *uniform*, then the equilibrium in question *is stable*, in the sense of Lyapunov. This property is a consequence of the fact that $x(t,x_0)$ depends continuously on x_0 (see for example [2, p. 181]).

We have seen before that bounded motions of (B.18) with initial conditions in a bounded set B asymptotically approach the compact invariant set $\omega(B)$. Thus, the question naturally arises to determine whether or not this set is also stable in the sense of Lyapunov. In this respect, we recall that the notion of *asymptotic stability of a closed invariant set* $\mathscr{A}$ is defined as follows. The set $\mathscr{A}$ is asymptotically stable if:

(i) for every $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\mathrm{dist}(x_0,\mathscr{A}) \leq \delta \qquad \Rightarrow \qquad \mathrm{dist}(x(t,x_0),\mathscr{A}) \leq \varepsilon \quad \text{for all } t \geq 0.$$

(ii) there exists a number $d > 0$ such that

$$\mathrm{dist}(x_0,\mathscr{A}) \leq d \qquad \Rightarrow \qquad \lim_{t\to\infty} \mathrm{dist}(x(t,x_0),\mathscr{A}) = 0\,.$$

It is not difficult to show (see [12] or [15]) that if the set $\mathscr{A}$ is also bounded and hence compact, and the convergence in (ii) is *uniform* in x_0, then property (ii) implies property (i). This yields the following important property of the set $\omega(B)$.

Lemma B.8. *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t,x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Suppose also that $\omega(B)$ is contained in the interior of B. Then, $\omega(B)$ is asymptotically stable, with a domain of attraction that contains B.*

B.6 The steady state behavior of a nonlinear system

We use the concepts introduced in the previous section to define a notion of *steady state* for a nonlinear system.

Definition B.5. Consider system (B.18) with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose that X is positively invariant. The motions of this system are said to be *ultimately bounded* if there is a bounded subset $B \subset X$ with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \geq T$ and all $x_0 \in X_0$.

Motions with initial conditions in a set B having the property indicated in the previous definition are indeed bounded and hence it makes sense to consider the limit set $\omega(B)$, which – according to Lemma B.6 – is nonempty and has all the properties indicated in that Lemma. What it is more interesting, though, is that – while a set B having the property indicated in the previous definition is clearly not unique – the set $\omega(B)$ is a unique well-defined set.

Lemma B.9. [13] *Let the motions of (B.18) be ultimately bounded and let B' be any other bounded subset of X with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B'$ for all $t \geq T$ and all $x_0 \in X_0$. Then, $\omega(B) = \omega(B')$.*

It is seen from this that, in any system whose motions are ultimately bounded, all motions asymptotically converge to a well-defined compact invariant set, which is filled with trajectories that are bounded in positive and negative time. This motivates the following definition.

Definition B.6. Suppose the motions of system (B.18), with initial conditions in a closed and positively invariant set X, are ultimately bounded. A *steady state motion* is any motion with initial condition in $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady state locus* of (B.18) and the restriction of (B.18) to $\omega(B)$ is the *steady state behavior* of (B.18). $\triangleleft$

This definition characterizes the steady state *behavior* of a nonlinear *autonomous* system, such as system (B.18). It can be used to characterized the steady state *response* of a *forced* nonlinear system

$$\dot{z} = f(z, u) \tag{B.25}$$

so long as the input u can be seen as the output of an *autonomous* "input generator"

$$\begin{aligned} \dot{w} &= s(w) \\ u &= q(w)\,. \end{aligned} \tag{B.26}$$

[13] See [17] for a proof.

In this way, the concept of steady state response (to specific classes of inputs) can be extended to nonlinear systems.

The idea of seeing the steady state response of a forced system as a particular response of an augmented autonomous system has been already exploited in section A.5, in the analysis of the steady state response of a stable linear system to harmonic inputs. In the present setting, the results of such analysis can be recast as follows. Let (B.25) be a stable linear system, written as

$$\dot{z} = Az + Bu\,, \tag{B.27}$$

in which $z \in \mathbb{R}^n$, and let (B.26) be the "input generator" defined in (A.26). The composition of (B.25) and (B.26) is the autonomous linear system (compare with (A.28))

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{B.28}$$

Pick a set W_c defined as

$$W_c = \{w \in \mathbb{R}^2 : \|w\| \leq c\}$$

in which c is a fixed number, and consider the set $X = W_c \times \mathbb{R}^n$. The set X is a closed set, positively invariant for the motion of (B.28). Moreover, since the lower subsystem of (B.28) is a linear asymptotically stable system driven by a bounded input, the motions of system (B.28), with initial conditions taken in X, are ultimately bounded. In fact, let Π be the solution of the Sylvester equation (A.29) and recall that the difference $z(t) - \Pi w(t)$ tends to zero as $t \to \infty$. Then, any bounded set B of the form

$$B = \{(w,z) \in W_c \times \mathbb{R}^n : \|z - \Pi w\| \leq d\}$$

in which d is any positive number, has the property requested in the definition of ultimate boundedness. It is easy to check that

$$\omega(B) = \{(w,z) \in W_c \times \mathbb{R}^n : z = \Pi w\}\,,$$

that is, $\omega(B)$ is the graph of the restriction of the linear map $x = \Pi w$ to the set W_c. The set $\omega(B)$ is invariant for (B.28), and the restriction of (B.28) to the set $\omega(B)$ characterizes the steady state response of (B.27) to harmonic inputs of fixed angular frequency ω, and amplitude not exceeding c.

A totally similar result holds if the input generator is a nonlinear system of the form (B.26), whose initial conditions are chosen in a *compact invariant* set W. The fact that W is invariant for the dynamics of (B.26) implies, as a consequence of Lemma B.8, that the steady state locus of (B.26) is the set W itself, i.e. that the input generator is "in steady state". [14] The composition of (B.26) and (B.27) yields an augmented system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= Az + Bq(w)\,, \end{aligned} \tag{B.29}$$

[14] Note that the set W_c considered in the previous example had exactly this property.

in which $(w,z) \in X := W \times \mathbb{R}^n$. Note that, since W is invariant for (B.26), the set X is invariant for (B.29).

Since the inputs generated by (B.26) are bounded and the lower subsystem of (B.29) is input-to-state stable, the motions of system (B.29), with initial conditions taken in X, are ultimately bounded. In fact, since W is compact and invariant, there exists a number U such that $\|q(w(t))\| \leq U$ for all $t \in \mathbb{R}$ and all $w(0) \in W$. Therefore, standard arguments can be invoked to deduce the existence of positive numbers K, λ and M such that

$$\|z(t)\| \leq Ke^{-\lambda t}\|z(0)\| + MU$$

for all $t \geq 0$. From this, it is immediate to check that any bounded set B of the form

$$B = \{(w,z) \in W \times \mathbb{R}^n : \|z\| \leq (1+d)MU\}$$

in which d is any positive number, has the property requested in the definition of ultimate boundedness. This being the case, it can be shown that the steady state locus of (B.29) is the graph of the (nonlinear) map [15]

$$\begin{aligned} \pi : W &\to \mathbb{R}^n \\ w &\mapsto \pi(w) = \int_{\infty}^{0} e^{-A\tau} Bq(\bar{w}(\tau, w)) d\tau , \end{aligned} \tag{B.30}$$

i.e. that

$$\omega(B) = \{(w,z) \in W \times \mathbb{R}^n : z = \pi(w)\} ,$$

To check that this is the case, observe first of all that – since $q(\bar{w}(t,w))$ is by hypothesis a bounded function of t and all eigenvalues of A have negative real part – the integral on the right-hand side of (B.30) is finite for every $w \in W$. Then, observe that the graph of the map $z = \pi(w)$ is invariant for (B.29). In fact, pick an initial state for (B.29) on the graph of this map, i.e. a pair (w_0, z_0) satisfying $z_0 = \pi(w_0)$ and compute the solution $z(t)$ of the lower equation of (B.29), via the classical variation of constants formula, to obtain

$$\begin{aligned} z(t) &= e^{At} \int_{-\infty}^{0} e^{-A\tau} Bq(\bar{w}(\tau, w_0)) d\tau + \int_{0}^{t} e^{A(t-\tau)} Bq(\bar{w}(\tau, w_0)) d\tau \\ &= \int_{\infty}^{0} e^{-A\theta} Bq(\bar{w}(\theta + t, w_0)) d\theta = \int_{\infty}^{0} e^{-A\theta} Bq(\bar{w}(\theta, w(t, w_0))) d\theta . \end{aligned}$$

This shows that $z(t) = \pi(w(t))$ and proves the invariance of the graph of $\pi(\cdot)$ for (B.29). Since the graph of $\pi(\cdot)$ is a compact set invariant for (B.29), this set is necessarily a subset of the steady state locus of (B.29). Finally, observe that, since the eigenvalues of A have negative real part, all motions of (B.29) whose initial conditions are not on the graph of $\pi(\cdot)$ are unbounded in negative time and therefore cannot be contained in the steady state locus, which by definition is a bounded in-

[15] In the following formula, $\bar{w}(t,w)$ denotes the integral curve of $\dot{w} = s(w)$ passing through w at time $t = 0$. Note that, as a consequence of the fact that W is closed and invariant, $\bar{w}(t,w)$ is defined for all $(t,w) \in \mathbb{R} \times W$.

variant set. Thus, the only points in the steady state locus are precisely the points of the graph of $\pi(\cdot)$.

This result shows that the steady state response of a stable linear system to an input generated by a nonlinear system of the form (B.26), with initial conditions $w(0)$ taken in a compact invariant set W, can be expressed in the form

$$z_{\mathrm{ss}}(t) = \pi(w(t))$$

in which $\pi(\cdot)$ is the map defined in (B.30).

Remark B.2. Note that the motions of the autonomous input generator (B.26) are not necessarily periodic motions, as it was the case for the input generator (A.26). For instance, the system in question could be a stable Van der Pol oscillator, with W defined as the set of all points inside and on the boundary of the limit cycle. In this case, it is possible to think of the steady state response of (B.27) not just as of the (single) periodic input obtained when the initial condition of (B.26) is taken on the limit cycle, but also as of all (non periodic) inputs obtained when the initial condition is taken in the interior of W. $\lhd$

Consider now the case of a general nonlinear system of the form (B.25), in which $z \in \mathbb{R}^n$, with input u supplied by a nonlinear input generator of the form (B.26). Suppose that system (B.25) is input-to-state stable and that the initial conditions of the input generator are taken in compact invariant set W. It is easy to see that the motions of the augmented system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, q(w))\,, \end{aligned} \tag{B.31}$$

with initial conditions in the set $X = W \times \mathbb{R}^n$, are ultimately bounded. In fact, since W is a compact set, there exists a number $U > 0$ such that

$$\|u(\cdot)\|_\infty = \|q(w(\cdot))\|_\infty \leq U$$

for all $w(0) \in W$. Since (B.25) is input-to-state stable, there exist a class $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that

$$\|z(t)\| \leq \beta(\|z(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \leq \beta(\|z(0)\|, t) + \gamma(U)$$

for all $t \geq 0$. Since $\beta(\cdot,\cdot)$ is a class $\mathscr{KL}$ function, for any compact set Z and any number $d > 0$, there exists a time T such that $\beta(\|z(0)\|, t) \leq d\gamma(U)$ for all $z(0) \in Z$ and all $t \geq T$. Thus, it follows that the set

$$B = \{(w,z) \in W \times \mathbb{R}^n : \|z\| \leq (1+d)\gamma(U)\}$$

has the property requested in the definition of ultimate boundedness.

Since the motions of the augmented system (B.31) are ultimately bounded, its steady state locus $\omega(B)$ is well-defined. As a matter of fact, it is possible to prove that also in this case the set in question is the *graph of a map* defined on W.

Lemma B.10. *Consider a system of the form (B.31) with $(w,z) \in W \times \mathbb{R}^n$. Suppose its motions are ultimately bounded. If W is a compact set invariant for $\dot{w} = s(w)$, the steady state locus of (B.31) is the graph of a (possibly set-valued) map defined on W.*

Proof. Since W is compact and invariant for (B.26), $\omega(W) = W$. As a consequence, for all $\bar{w} \in W$ there is a sequence $\{w_k, t_k\}$ with w_k in W for all k such that $\bar{w} = \lim_{k\to\infty} w(t_k, w_k)$. Set $x = \mathrm{col}(w,z)$ and let $x(t,x_0)$ denote the integral curve of (B.31) passing through x_0 at time $t = 0$. Pick any point $z_0 \in \mathbb{R}^n$ and let $x_k = \mathrm{col}(w_k, z_0)$. All such x_k's are in a compact set. Hence, by definition of ultimate boundedness, there is a bounded set B and a integer $k^* > 0$ such that $x(t_{k^*+\ell}, x_k) \in B$ for all $\ell \geq 0$ and all k. Set $\bar{x}_\ell = x(t_{k^*}, x_\ell)$ and $\tau_\ell = t_{k^*+\ell} - t_{k^*}$, for $\ell \geq 0$, and observe that, by construction, $x(\tau_\ell, \bar{x}_\ell) = x(t_{k^*+\ell}, x_\ell)$, which shows that all $x(\tau_\ell, \bar{x}_\ell)$'s are in B, a bounded set. Hence, there exists a subsequence $\{x(\tau_h, \bar{x}_h)\}$ converging to a point $\hat{x} = \mathrm{col}(\hat{w}, \hat{z})$, which is a point of $\omega(B)$ because all $\bar{x}_h$'s are in B. Since system (B.31) is upper triangular, necessarily $\hat{w} = \bar{w}$. This shows that, for any point $\bar{w} \in W$, there is at least one point $\hat{z} \in \mathbb{R}^n$ such that $(\bar{w}, \hat{z}) \in \omega(B)$. $\triangleleft$

It should be stressed that the map whose graph characterizes the steady state locus of (B.31) may fail to be single-valued and, also, may fail to be continuously differentiable, as shown in the examples below.

Example B.10. Consider the system

$$\dot{z} = -z^3 + zu\,, \tag{B.32}$$

which is input-to-state stable, with input u provided by the input generator

$$\begin{aligned} \dot{w} &= 0 \\ u &= w \end{aligned}$$

for which we take $W = \{w \in \mathbb{R} : |w| \leq 1\}$. Thus, $u(t) = w(t) = w(0) := w_0$. If $w_0 \leq 0$, system (B.32) has a globally asymptotically stable equilibrium at $z = 0$. If $w_0 > 0$, system (B.32) has one unstable equilibrium at $z = 0$ and two locally asymptotically stable equilibria at $z = \pm\sqrt{w_0}$. For every fixed $w_0 > 0$, trajectories of (B.32) with initial conditions satisfying $|z_0| > \sqrt{w_0}$ asymptotically converge to either one of the two asymptotically stable equilibria, while the compact set

$$\{(w,z); w = w_0, |z| \leq \sqrt{w_0}\}$$

is invariant. As a consequence, the steady state locus of the augmented system

$$\begin{aligned} \dot{z} &= -z^3 + zw \\ \dot{w} &= 0 \end{aligned}$$

is the graph of the set-valued map

$$\pi\,:\, w \in W \quad \mapsto \quad \pi(w) \subset \mathbb{R}$$

defined as

$$\begin{array}{lll} -1 \le w \le 0 & \Rightarrow & \pi(w) = \{0\} \\ 0 < w \le 1 & \Rightarrow & \pi(w) = \{z \in \mathbb{R} : |z| \le \sqrt{w}\}\,. \end{array} \qquad \lhd$$

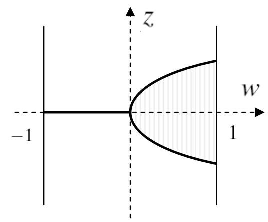

Fig. B.5 The steady state locus of system (B.32).

Example B.11. Consider the system

$$\dot{z} = -z^3 + u\,,$$

which is input to state stable, with input u provided by the harmonic oscillator

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 \\ u &= w_1 \end{aligned}$$

for which we take $W = \{w \in \mathbb{R}^2 : \|w\| \le 1\}$. It can be shown [16] that, for each $w(0) \in W$, there is one and only one value $z(0) \in \mathbb{R}$ from which the motion of the resulting augmented system (B.31) is bounded both in positive and negative time. The set of all such pairs identifies a single-valued map $\pi : W \to \mathbb{R}$, whose graph characterizes the steady state locus of the system. The map in question is continuously differentiable at any nonzero w, but it is only continuous at $w = 0$. $\lhd$

If the map whose graph characterizes the steady state locus of (B.31) *is single-valued*, the steady state response of an input-to-state stable system of the form (B.25) to an input generated by a system of the form (B.26) can be expressed as

$$z_{\rm ss}(t) = \pi(w(t))\,,$$

in which $\pi(\cdot)$ is a map defined on W. In general, it is not easy to give explicit expressions of this map (such as the one considered earlier in (B.30)). However, if $\pi(\cdot)$ is continuosly differentiable, a very expressive implicit characterization is possible. In fact, recall that the steady state locus of (B.31) is by definition an invariant set,

[16] See [16].

i.e. $z(t) = \pi(w(t))$ for all $t \in \mathbb{R}$ along any trajectory of (B.31) with initial condition satisfying $z(0) = \pi(w(0))$. Along all such trajectories,

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = f(z(t), q(w(t))) = f(\pi(w(t)), q(w(t)) .$$

If $\pi(w)$ is continuously differentiable, then

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\partial \pi}{\partial w}\Big|_{w=w(t)} \frac{\mathrm{d}w(t)}{\mathrm{d}t} = \frac{\partial \pi}{\partial w}\Big|_{w=w(t)} s(w(t))$$

and hence it is seen that $\pi(w)$ satisfies the partial differential equation

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), q(w)) \qquad \text{for all } w \in W. \tag{B.33}$$

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