Chapter 2
Stabilization of Minimum-phase Linear Systems

Abstract It is well-known from the elementary theory of servomechanisms that a single-input single-output linear system whose transfer function has all zeros in the left-half complex plane can be stabilized via output feedback. If the transfer function of the system has n poles and m zeros, the feedback in question is a dynamical system of dimension $n-m-1$ whose eigenvalues (in case $n-m>1$) are far away in the left-half complex plane. In this chapter, this result is reviewed using a state-space approach. This makes it possible to systematically handle the case of systems whose coefficients depend on uncertain parameters and serves as a preparation to a similar set of results that will be presented in Chapter 6 for nonlinear systems.

2.1 Normal form and system zeroes

The purpose of this section is to show how single-input single-output linear systems can be given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which certain fundamental properties of the system are highlighted, that plays a relevant role in the method for robust stabilization discussed in the following sections.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally defined as follows. Given a single-input single-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,, \end{aligned} \tag{2.1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, consider the sequence of real numbers CB, CAB, $CA^2B, \ldots,$ $CA^kB, \ldots$ Let r denote the least integer for which $CA^{r-1}B \neq 0$. This integer is called the *relative degree* of system (2.1). In other words, r is the integer uniquely characterized by the conditions

$$\begin{aligned} &CB = CAB = \ldots = CA^{r-2}B = 0 \\ &CA^{r-1}B \neq 0\,. \end{aligned} \tag{2.2}$$

The relative degree of a system can be easily identified with an integer associated with the transfer function of the system. In fact, consider the transfer function of (2.1)

$$T(s) = C(sI - A)^{-1}B$$

and recall that $(sI - A)^{-1}$ can be expanded, in negative powers of s, as

$$(sI - A)^{-1} = \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots$$

This yields, for $T(s)$, the expansion

$$T(s) = \frac{1}{s}CB + \frac{1}{s^2}CAB + \frac{1}{s^3}CA^2B + \cdots$$

Using the definition of r, it is seen that the expansion in question actually reduces to

$$\begin{aligned} T(s) &= \frac{1}{s^r}CA^{r-1}B + \frac{1}{s^{r+1}}CA^rB + \frac{1}{s^{r+2}}CA^{r+1}B + \cdots \\ &= \frac{1}{s^r}[CA^{r-1}B + \frac{1}{s}CA^rB + \frac{1}{s^2}CA^{r+1}B + \cdots] \end{aligned}$$

Thus

$$s^rT(s) = CA^{r-1}B + \frac{1}{s}CA^rB + \frac{1}{s^2}CA^{r+1}B + \cdots$$

from which it is deduced that

$$\lim_{s\to\infty} s^rT(s) = CA^{r-1}B\,. \tag{2.3}$$

Recall now that $T(s)$ is a rational function of s, the ratio between a numerator polynomial $N(s)$ and a denominator polynomial $D(s)$

$$T(s) = \frac{N(s)}{D(s)}\,.$$

Since the limit (2.3) is finite and (by definition of r) *nonzero*, it is concluded that r is necessarily the difference between the degree of $D(s)$ and the degree of $N(s)$. This motivates the terminology “relative degree”. Finally, note that, if $T(s)$ is expressed (as it is always possible) in the form

$$T(s) = K'\frac{\Pi_{i=1}^{n-r}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)}\,,$$

it necessarily follows that [1]

$$K' = CA^{r-1}B\,.$$

We use now the concept of relative degree to derive a change of variables in the state space, yielding a form of special interest. The following facts are easy consequences of the definition of relative degree.[2]

Proposition 2.1. *The r rows of the $r \times n$ matrix*

$$T_1 = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} \tag{2.4}$$

are linearly independent. As a consequence, $r \le n$.

We see from this that, if r is strictly less than n, it is possible to find – in many ways – a matrix $T_0 \in \mathbb{R}^{(n-r)\times n}$ such that the resulting $n \times n$ matrix

$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = \begin{pmatrix} T_0 \\ C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} \tag{2.5}$$

is *nonsingular*. Moreover, the following result also holds.

Proposition 2.2. *It is always possible to pick T_0 in such a way that the matrix (2.5) is nonsingular and $T_0B = 0$.*

We use now the matrix T introduced above to define a change of variables. To this end, in view of the natural partition of the rows of T in two blocks (the upper block consisting of the $n-r$ rows of T_0 and the lower block consisting of the r rows of the matrix (2.4)) it is natural to choose different notations for the first $n-r$ new state variables and for the last r new state variables, setting

$$z = T_0 x, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} x\,.$$

To determine the equations describing the system in the new coordinates, we take the derivatives of z and ξ with respect to time. For the former, no special structure is found and we simply obtain

$$\dot{z} = T_0\dot{x} = T_0(Ax + Bu) = T_0Ax + T_0Bu\,. \tag{2.6}$$

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[1] The parameter K' is sometimes referred to as the *high-frequency gain* of the system.

[2] For a proof see, e.g., [2, pp. 142-144]

On the contrary, for the latter, a special structure can be displayed. In fact, observing that $\xi_i = CA^{i-1}x$, for $i = 1, \ldots, r$, and using the defining properties (2.2), we obtain

$$\begin{aligned}
\dot{\xi}_1 &= C\dot{x} = C(Ax+Bu) = CAx = \xi_2 \\
\dot{\xi}_2 &= CA\dot{x} = CA(Ax+Bu) = CA^2x = \xi_3 \\
&\cdots \\
\dot{\xi}_{r-1} &= CA^{r-2}\dot{x} = CA^{r-2}(Ax+Bu) = CA^{r-1}x = \xi_r
\end{aligned}$$

and

$$\dot{\xi}_r = CA^{r-1}\dot{x} = CA^{r-1}(Ax+Bu) = CA^r x + CA^{r-1}Bu\,.$$

The equations thus found can be cast in a compact form. To this end, it is convenient to introduce a special triplet of matrices, $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times 1}$, $\hat{C} \in \mathbb{R}^{1\times r}$, which are defined as [3]

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.7}$$

With the help of such matrices, it is easy to obtain

$$\begin{aligned}
\dot{\xi} &= \hat{A}\xi + \hat{B}(CA^r x + CA^{r-1}Bu) \\
y &= Cx = \xi_1 = \hat{C}\xi\,.
\end{aligned} \tag{2.8}$$

Note that the right-hand sides of (2.6) and (2.8) are still expressed in terms of the "original" set of state variables x. To complete the change of coordinates, x should be expressed as a (linear) function of the new state variables z and ξ, that is as a function of the form

$$x = M_0 z + M_1 \xi$$

in which M_0 and M_1 are partitions of the inverse of T, implicitly defined by

$$\begin{pmatrix} M_0 & M_1 \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = I\,.$$

Setting

$$\begin{aligned}
A_{00} &= T_0 A M_0\,, \qquad & A_{01} &= T_0 A M_1\,, \qquad & B_0 &= T_0 B\,, \\
A_{10} &= CA^r M_0\,, \qquad & A_{11} &= CA^r M_1, \qquad & b &= CA^{r-1}B\,,
\end{aligned}$$

the equations in question can be cast in the form

[3] A triplet of matrices of this kind is often referred to as a triplet in *prime form*.

$$\begin{aligned}\dot{z} &= A_{00}z + A_{01}\xi + B_0 u \\ \dot{\xi} &= \hat{A}\xi + \hat{B}(A_{10}z + A_{11}\xi + bu) \\ y &= \hat{C}\xi\,.\end{aligned} \tag{2.9}$$

These equations characterize the so-called *normal form* of the equations (2.1) describing the system. Note that the matrix B_0 can be made equal to 0 if the option described in Proposition 2.2 is used. If this is the case, the corresponding normal form is said to be *strict*. On the contrary, the coefficient b, which is equal to the so-called high-frequency gain of the system, is always nonzero.

In summary, by means of the change of variables indicated above, the original system is transformed into a system described by matrices having the following structure

$$TAT^{-1} = \begin{pmatrix} A_{00} & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} \end{pmatrix}, \qquad TB = \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 0 & \hat{C} \end{pmatrix}. \tag{2.10}$$

One of the most relevant features of the normal form of the equations describing the system is the possibility of establishing a relation between the zeros of the transfer function of the system and certain submatrices appearing in (2.9).

We begin by observing that

$$T(s) = \frac{\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix}}{\det(A - sI)}\,. \tag{2.11}$$

This is a simple consequence of a well-know formula for the determinant of a partitioned matrix. [4] Using the latter, we obtain

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det(A - sI)\det(-C(A - sI)^{-1}B)$$

from which, bearing in mind the fact that $C(A - sI)^{-1}B$ is a scalar quantity, the identity (2.11) immediately follows. Note that in this way we have identified simple and appealing expressions for the numerator and denominator polynomial of $T(s)$.

Next, we determine an expansion of the numerator polynomial.

Proposition 2.3. *The following expansion holds*

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = (-1)^r CA^{r-1}B\det([A_{00} - \frac{1}{b}B_0A_{10}] - sI)\,. \tag{2.12}$$

[4] The formula in question is

$$\det\begin{pmatrix} S & P \\ Q & R \end{pmatrix} = \det(S)\det(R - QS^{-1}P)\,.$$

Proof. Observe that the left-hand side of (2.12) remains unchanged if A,B,C is replaced by TAT^{-1},TB,CT^{-1}. In fact

$$\begin{pmatrix} TAT^{-1}-sI & TB \\ CT^{-1} & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A-sI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

from which, using the fact that the determinant of a product is the product of the determinants and that

$$\det \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \det \begin{pmatrix} T^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} TT^{-1} & 0 \\ 0 & 1 \end{pmatrix} = 1$$

we obtain

$$\det \begin{pmatrix} A-sI & B \\ C & 0 \end{pmatrix} = \det \begin{pmatrix} TAT^{-1}-sI & TB \\ CT^{-1} & 0 \end{pmatrix}.$$

Furthermore, observe also that the left-hand side of (2.12) remains unchanged if A is replaced by $A+BF$, regardless of how the matrix $F \in \mathbb{R}^{1\times n}$ is chosen. This derives from the expansion

$$\begin{pmatrix} A+BF-sI & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A-sI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ F & 1 \end{pmatrix}$$

and from the fact that the determinant of the right-hand factor in the product above is simply equal to 1.

With these observations in mind, use for T the transformation that generates the normal form (2.10), to arrive at

$$\begin{aligned} \det \begin{pmatrix} A-sI & B \\ C & 0 \end{pmatrix} &= \det \begin{pmatrix} \begin{pmatrix} A_{00}-sI & A_{01} \\ \hat{B}A_{10} & \hat{A}+\hat{B}A_{11}-sI \end{pmatrix} & \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{C} \end{pmatrix} & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} \begin{pmatrix} A_{00}-sI & A_{01} \\ \hat{B}A_{10} & \hat{A}+\hat{B}A_{11}-sI \end{pmatrix} + \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} F & \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{C} \end{pmatrix} & 0 \end{pmatrix}. \end{aligned}$$

The choice

$$F = \frac{1}{b} \begin{pmatrix} -A_{10} & -A_{11} \end{pmatrix}$$

yields

$$\begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} F = \begin{pmatrix} -\frac{1}{b}B_0A_{10} & -\frac{1}{b}B_0A_{11} \\ -\hat{B}A_{10} & -\hat{B}A_{11} \end{pmatrix}.$$

and hence it follows that

$$\det \begin{pmatrix} A-sI & B \\ C & 0 \end{pmatrix} = \det \begin{pmatrix} [A_{00}-\frac{1}{b}B_0A_{10}]-sI & A_{01}-\frac{1}{b}B_0A_{11} & 0 \\ 0 & \hat{A}-sI & \hat{B}b \\ 0 & \hat{C} & 0 \end{pmatrix}.$$

The matrix on the right-hand side is block-triangular, hence

$$\det\begin{pmatrix} A-sI & B \\ C & 0 \end{pmatrix} = \det([A_{00} - \frac{1}{b}B_0A_{10}] - sI)\,\det\begin{pmatrix} \hat{A}-sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix}.$$

Finally, observe that

$$\begin{pmatrix} \hat{A}-sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix} = \begin{pmatrix} -s & 1 & 0 & \cdots & 0 & 0 \\ 0 & -s & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -s & b \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Expanding the determinant according to the entries of last row we obtain

$$\det\begin{pmatrix} \hat{A}-sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix} = (-1)^{r+2}\det\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -s & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -s & b \end{pmatrix} = (-1)^r b$$

which, bearing in mind the definition of b, yields (2.12). ◁

With this expansion in mind, we return to the formula (2.11), from which we deduce that

$$T(s) = \frac{(-1)^r CA^{r-1}B\,\det([A_{00} - \frac{1}{b}B_0A_{10}] - sI)}{\det(A - sI)}.$$

Changing the signs of both matrices in the numerator and denominator yields the final expression

$$T(s) = CA^{r-1}B\,\frac{\det(sI - [A_{00} - \frac{1}{b}B_0A_{10}])}{\det(sI - A)}. \tag{2.13}$$

If the triplet A,B,C is a minimal realization of its transfer function, i.e. if the pair (A,B) is reachable and the pair (A,C) is observable , the numerator and denominator polynomial of this fraction cannot have common factors.[5] Thus, we can conclude that if the pair (A,B) is reachable and the pair (A,C) is observable, the $n-r$ eigenvalues of the matrix

$$[A_{00} - \frac{1}{b}B_0A_{10}].$$

can be identified with *zeros of the transfer function* $T(s)$.

[5] Otherwise, $T(s)$ could be written as strictly proper rational function in which the denominator is a polynomial of degree *strictly less* than n. This would imply the existence of a realization of dimension strictly less than n, contradicting the minimality of A,B,C.

Remark 2.1. Note that if, in the transformation T used to obtain the normal form, the matrix T_0 has been chosen so as to satisfy $TB = 0$, the structure of the normal form (2.10) is simplified, and $B_0 = 0$. In this case, the zeros of the transfer function coincide with the eigenvalues of the matrix A_{00}. $\triangleleft$

2.2 The hypothesis of minimum-phase

We consider in this chapter the case of a linear single-input single-output system of fixed dimension n, whose coefficient matrices may depend on a *vector* μ of (possibly) *uncertain parameters*. The value of μ is not known, nor is available for measurement (neither directly nor indirectly through an estimation filter) but it is assumed to be *constant* and to range over a fixed, and *know*, *compact set* $\mathbb{M}$.

Accordingly, the equations (2.1) will be written in the form

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u \\ y &= C(\mu)x\,. \end{aligned} \tag{2.14}$$

The theory described in what follows is based on the following basic hypothesis.

Assumption 2.1 *$A(\mu), B(\mu), C(\mu)$ are matrices of continuous functions of μ. For every $\mu \in \mathbb{M}$, the pair $(A(\mu), B(\mu))$ is reachable and the pair $(A(\mu), C(\mu))$ is observable. Moreover:*

(i) The relative degree of the system is the same for all $\mu \in \mathbb{M}$.
(ii) The zeros of the transfer function $C(\mu)(sI - A(\mu))^{-1}B(\mu)$ have negative real part for all $\mu \in \mathbb{M}$.

In the classical theory of servomechanisms, systems whose transfer function has zeros only in the (closed) left-half complex plane have been often referred to as *minimum-phase systems*. This a terminology that dates back more or less to the works of H.W. Bode.[6] For convenience, we keep this terminology to express the property that a system satisfies condition (ii) of Assumption 2.1, even though the latter (which, as it will be seen, is instrumental in the proposed robust stabilization strategy) excludes the occurrence of zeros on the imaginary axis. Thus, in what follows, a (linear) system satisfying condition (ii) of Assumption 2.1 will be referred to as a minimum-phase system.

Letting r denote the relative degree, system (2.14) can be put in *strict* normal form, by means of a change of variables

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0(\mu) \\ T_1(\mu) \end{pmatrix} x$$

in which

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[6] See [1] and also [3, p. 283].

$$T_1(\mu) = \begin{pmatrix} C(\mu) \\ C(\mu)A(\mu) \\ \cdots \\ C(\mu)A^{r-1}(\mu) \end{pmatrix}$$

and $T_0(\mu)$ satisfies $T_0(\mu)B(\mu) = 0$. Note that $T_1(\mu)$ is by construction a continuous function of μ and $T_0(\mu)$ can always be chosen as a continuous function of μ.

The normal form in question is written as

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u \\ y &= \xi_1\,, \end{aligned}$$

with $z \in \mathbb{R}^{n-r}$ or, alternatively, as

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u] \\ y &= \hat{C}\xi \end{aligned}$$

in which $\hat{A}, \hat{B}, \hat{C}$ are the matrices introduced in (2.7). Moreover,

$$b(\mu) = C(\mu)A^{r-1}(\mu)B(\mu)\,.$$

As a consequence of Assumption 2.1:

(i) $b(\mu) \neq 0$ for all $\mu \in \mathbb{M}$. By continuity, $b(\mu)$ it is either positive or negative for all $\mu \in \mathbb{M}$. In what follows, without loss of generality, it will be assumed that

$$b(\mu) > 0 \qquad \text{for all } \mu \in \mathbb{M}. \tag{2.15}$$

(ii) The eigenvalues of $A_{00}(\mu)$ have negative real part for all $\mu \in \mathbb{M}$. This being the case, it is known from the converse Lyapunov Theorem [7] that there exists a unique, symmetric and *positive definite*, matrix $P(\mu)$, of dimension $(n-r) \times (n-r)$, of *continuous* functions of μ such that

$$P(\mu)A_{00}(\mu) + A_{00}^{\mathrm{T}}(\mu)P(\mu) = -I \qquad \text{for all } \mu \in \mathbb{M}. \tag{2.16}$$

2.3 The case of relative degree 1

Perturbed systems that belong to the class of systems characterized by Assumption 2.1 can be *robustly stabilized* by means of a very simple-minded feedback strategy,

[7] See Theorem A.3 in Appendix A.

as it will be described in what follows. The equations (2.14) define a *set* of systems, one for each value of μ. A *robust stabilizer* is a feedback law that stabilizes each member of this set. The feedback law in question is not allowed to know which one is the individual member of the set that is being controlled, i.e. it is has to be the same for each member of the set. In the present context of systems modeled as in (2.14), the robustly stabilizing feedback must be a fixed dynamical system not depending on the value of μ.

For simplicity, we address first in this section the case of a system having relative degree is 1. In this case ξ is a vector of dimension 1 (i.e. a scalar quantity) and the normal form reduces to

$$\begin{aligned}\dot z &= A_{00}(\mu)z + A_{10}(\mu)\xi\\ \dot\xi &= A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u\\ y &= \xi\,.\end{aligned} \tag{2.17}$$

Consider the control law [8]

$$u = -ky\,. \tag{2.18}$$

This yields a closed loop system of the form

$$\begin{aligned}\dot z &= A_{00}(\mu)z + A_{01}(\mu)\xi\\ \dot\xi &= A_{10}(\mu)z + [A_{11}(\mu) - b(\mu)k]\xi\,,\end{aligned}$$

or, what is the same

$$\begin{pmatrix}\dot z\\ \dot\xi\end{pmatrix} = \begin{pmatrix}A_{00}(\mu) & A_{01}(\mu)\\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix}\begin{pmatrix}z\\ \xi\end{pmatrix}. \tag{2.19}$$

We want to prove that, under the standing assumptions, if k is large enough this system is stable for all $\mu \in \mathbb{M}$. To this end, consider the positive definite $n \times n$ matrix

$$\begin{pmatrix}P(\mu) & 0\\ 0 & 1\end{pmatrix}$$

in which $P(\mu)$ is the matrix defined in (2.16). If we are able to show that the matrix

$$\begin{aligned}Q(\mu) = &\begin{pmatrix}P(\mu) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}A_{00}(\mu) & A_{01}(\mu)\\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix}\\ &+\begin{pmatrix}A_{00}(\mu) & A_{01}(\mu)\\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}P(\mu) & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

is *negative definite*, then by the direct Lyapunov Theorem,[9] we can assert that system (2.19) has all eigenvalues with negative real part. A simple calculation shows that, because of (2.16),

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[8] The negative sign is a consequence of the standing hypothesis (2.15). If $b(\mu) < 0$, the sign must be reversed.

[9] See Theorem A.2 in Appendix A.

$$Q(\mu) = \begin{pmatrix} -I & [P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)] \\ [P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]^{\mathrm{T}} & 2[A_{11}(\mu) - b(\mu)k] \end{pmatrix} \tag{2.20}$$

To check positive definiteness, we change sign to $Q(\mu)$ and appeal to Sylvester's criterion for positive definiteness, i.e. we check the sign of all leading principal minors. Because of the special form of $-Q(\mu)$, all its leading principal minors of order $1, 2, \ldots, n-1$ are equal to 1 (and hence positive). Thus the matrix in question is positive definite if (and only if) its determinant is positive. To compute the determinant, we observe that the matrix in question has the form

$$-Q(\mu) = \begin{pmatrix} I & d(\mu) \\ d^{\mathrm{T}}(\mu) & q(\mu) \end{pmatrix}.$$

Hence, thanks to the formula for the determinant of a partitioned matrix

$$\det[-Q(\mu)] = \det[I]\det[q(\mu) - d^{\mathrm{T}}(\mu)I^{-1}d(\mu)] = q(\mu) - d^{\mathrm{T}}(\mu)d(\mu) = q(\mu) - \|d(\mu)\|^2.$$

Thus, the conclusion is that $Q(\mu)$ is negative definite for all $\mu \in \mathbb{M}$ if (and only if)

$$q(\mu) - \|d(\mu)\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Reverting to the notations of (2.20) we can say that $Q(\mu)$ is negative definite, or – what is the same – system (2.19) has all eigenvalues with negative real part, for all $\mu \in \mathbb{M}$ if

$$-2[A_{11}(\mu) - b(\mu)k] - \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Since $b(\mu) > 0$, the inequality is equivalent to

$$k > \frac{1}{2b(\mu)}\Big(2A_{11}(\mu) + \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2\Big) \qquad \text{for all } \mu \in \mathbb{M}.$$

Now, set

$$k^* := \max_{\mu \in \mathbb{M}} \Big(\frac{2A_{11}(\mu) + \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2}{2b(\mu)}\Big).$$

This maximum exists because the functions are continuous functions of μ and $\mathbb{M}$ is a compact set. Then we are able to conclude that if

$$k > k^*$$

the control law

$$u = -ky$$

stabilizes the closed loop system, regardless of what the particular value of $\mu \in \mathbb{M}$ is. In other words, this control *robustly* stabilizes the given set of systems.

Remark 2.2. The control law $u = -ky$ is usually referred to as a *high-gain* output feedback. As it is seen from the previous analysis, a large value of k makes the matrix $Q(\mu)$ negative definite. The matrix in question has the form

$$Q(\mu) = \begin{pmatrix} -I & -d(\mu) \\ -d^{\mathrm{T}}(\mu) & -2b(\mu)k + q_0(\mu) \end{pmatrix}.$$

The role of a large k is to render the term $-2b(\mu)k$ *sufficiently negative*, so as: (i) to overcome the uncertain term $q_0(\mu)$, and (ii) to overcome the effect of the uncertain off-diagonal terms. Reverting to the equations (2.19) that describe the closed-loop system, one may observe that the upper equation can be seen as a stable subsystem with state z and input ξ, while the lower equation can be seen as a subsystem with state ξ and input z. The role of a *large* k is: (i) to render the lower subsystem stable, and (ii) to *lower* the effect of the *coupling* between the two subsystems. This second role, which is usually referred to as a *small-gain property*, will be described and interpreted in full generality in the next Chapter. ◁

2.4 The case of higher relative degree: partial state feedback

Consider now the case of a system having higher relative degree $r > 1$. This system can be "artificially" reduced to a system to which the stabilization procedure described in the previous section is applicable, by means of a simple strategy. Let the variable ξ_r of the normal form be replaced by a new state variable defined as

$$\theta = \xi_r + a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} \tag{2.21}$$

in which $a_0, a_1, \ldots, a_{r-2}$ are design parameters. With this change of variable (it's only a change of variables, no control has been chosen yet !), a system is obtained which has the form

$$\begin{aligned}
\dot{z} &= A_{00}(\mu)z + \tilde{a}_{01}(\mu)\xi_1 + \cdots + \tilde{a}_{0,r-1}(\mu)\xi_{r-1} + \tilde{a}_{0r}(\mu)\theta \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= -(a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1}) + \theta \\
\dot{\theta} &= A_{10}(\mu)z + \tilde{a}_{11}(\mu)\xi_1 + \cdots + \tilde{a}_{1,r-1}(\mu)\xi_{r-1} + \tilde{a}_{1r}(\mu)\theta + b(\mu)u \\
y &= \xi_1\,,
\end{aligned}$$

in which the $\tilde{a}_{0i}(\mu)$'s and $\tilde{a}_{1i}(\mu)$'s are appropriate coefficients.[10]

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[10] These coefficients can be easily derived as follows. Let

$$A_{01}(\mu) = \begin{pmatrix} a_{01,1} & a_{01,2} & \cdots & a_{01,r} \end{pmatrix}.$$

Hence

$$A_{01}(\mu)\xi = a_{01,1}\xi_1 + a_{01,2}\xi_2 + \cdots + a_{01,r}\xi_r\,.$$

This system can be formally viewed as a system having relative degree 1, with input u and output θ. To this end, in fact, it suffices to set

$$\zeta = \begin{pmatrix} z \\ \xi_1 \\ \cdots \\ \xi_{r-1} \end{pmatrix}$$

and rewrite the system as

$$\dot{\zeta} = \begin{pmatrix} A_{00}(\mu) & \tilde{a}_{01}(\mu) & \tilde{a}_{02}(\mu) & \cdots & \tilde{a}_{0,r-1}(\mu) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix} \zeta + \begin{pmatrix} \tilde{a}_{0r}(\mu) \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \theta$$

$$\dot{\theta} = \begin{pmatrix} A_{10}(\mu) & \tilde{a}_{11}(\mu) & \tilde{a}_{12}(\mu) & \cdots & \tilde{a}_{1,r-1}(\mu) \end{pmatrix} \zeta + \tilde{a}_{1r}(\mu)\theta + b(\mu)u\,.$$

The latter has the structure of a system in normal form

$$\begin{aligned} \dot{\zeta} &= F_{00}(\mu)\zeta + F_{01}(\mu)\theta \\ \dot{\theta} &= F_{10}(\mu)\zeta + F_{11}(\mu)\theta + b(\mu)u \end{aligned} \tag{2.22}$$

in which $F_{00}(\mu)$ is a $(n-1)\times(n-1)$ block-triangular matrix

$$F_{00}(\mu) = \begin{pmatrix} A_{00}(\mu) & \tilde{a}_{01}(\mu) & \tilde{a}_{02}(\mu) & \cdots & \tilde{a}_{0,r-1}(\mu) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix} := \begin{pmatrix} A_{00}(\mu) & * \\ 0 & A_0 \end{pmatrix}$$

with

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix}.$$

Since

$$\xi_r = \theta - (a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1})$$

we see that

$$A_{01}(\mu)\xi = [a_{01,1} - a_{01,r}a_0]\xi_1 + [a_{01,2} - a_{01,r}a_1]\xi_2 + \cdots + [a_{01,r-1} - a_{01,r}a_{r-2}]\xi_{r-1} + a_{01,r}\theta$$

The latter can be rewritten as

$$\tilde{a}_{01}(\mu)\xi_1 + \cdots + \tilde{a}_{0,r-1}(\mu)\xi_{r-1} + \tilde{a}_{0r}(\mu)\theta\,.$$

A similar procedure is followed to transform $A_{11}(\mu)\xi + a_0\dot{\xi}_1 + \cdots + a_{r-1}\dot{\xi}_{r-1}$.

By Assumption 2.1, all eigenvalues of the submatrix $A_{00}(\mu)$ have negative real part for all μ. On the other hand, the characteristic polynomial of the submatrix A_0, which is a matrix in companion form, coincides with the polynomial

$$p_a(\lambda) = a_0 + a_1\lambda + \cdots + a_{r-2}\lambda^{r-2} + \lambda^{r-1}\,. \tag{2.23}$$

The design parameters $a_0, a_1, \ldots, a_{r-2}$ can be chosen in such a way that all eigenvalues of A_0 have negative real part. If this is the case, we can conclude that *all* the $n-1$ eigenvalues of $F_{00}(\mu)$ have negative real part, for all μ.

Thus, system (2.22) can be seen as a system having relative degree 1 which satisfies all the assumptions used in the previous section to obtain robust stability. In view of this, it immediately follows that there exists a number k^* such that, if $k > k^*$, the control law

$$u = -k\theta$$

robustly stabilizes such system.

Note that the control thus found, expressed in the original coordinates, reads as

$$u = -k[a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} + \xi_r]$$

that is as a linear combination of the components of the vector ξ. This is a *partial state* feedback, which can be written, in compact form, as

$$u = H\xi\,. \tag{2.24}$$

Remark 2.3. It it worth to observe that, by definition,

$$\xi_1(t) = y(t), \quad \xi_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \quad \ldots \quad , \quad \xi_r(t) = \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}\,.$$

Thus, the variable θ is seen to satisfy

$$\theta(t) = a_0 y(t) + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \cdots + a_{r-2}\frac{\mathrm{d}^{r-2}y(t)}{\mathrm{d}t^{r-2}} + \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}\,.$$

In other words, θ can be seen as output of a system with input y and transfer function

$$D(s) = a_0 + a_1 s + \cdots + a_{r-2}s^{r-2} + s^{r-1}\,.$$

It readily follows that, if $T(s,\mu) = C(\mu)(sI - A(\mu))^{-1}B(\mu)$ is the transfer function of (2.14), the transfer function of system (2.22), seen as a system with input u and output θ, is equal to $D(s)T(s,\mu)$. As confirmed by the state space analysis and in particular from the structure of $F_{00}(\mu)$, this new system has $n-1$ zeros, $n-r$ of which coincide with the original zeros of (2.14), while the additional $r-1$ zeros coincide with the roots of the polynomial (2.23). In other words, the indicated design method can be interpreted as an addition of $r-1$ zeros having negative real part (so as to lower the relative degree to the value 1 while keeping the property that all zeros have negative real part) followed by high-gain output feedback on the resulting output.

Example 2.1. Consider the problem of robustly stabilizing a rocket's upright orientation in the initial phase of the launch. The equation describing the motion on a vertical plane is similar to those that describes the motion of an inverted pendulum (see Figure 2.1) and has the form [11]

$$J_t \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = mg\ell\sin(\varphi) - \gamma\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \ell\cos(\varphi)u\,.$$

in which ℓ is the length of the pendulum, m is the mass concentrated at the tip of the pendulum, $J_t = J + m\ell^2$ is the total moment of inertia, γ is a coefficient of rotational viscous friction and u is a force applied at the base.

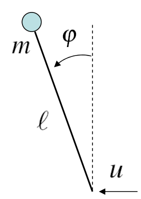

Fig. 2.1 An inverted pendulum

If the angle φ is sufficiently small, one can use the approximations $\sin(\varphi) \approx \varphi$ and $\cos(\varphi) \approx 1$ and obtain a linear model. Setting $\xi_1 = \varphi$ and $\xi_2 = \dot{\varphi}$, the equation can be put in state space form as

$$\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= q_1\xi_1 + q_2\xi_2 + bu\end{aligned}$$

in which

$$q_1 = \frac{mg\ell}{J_t}, \qquad q_2 = -\frac{\gamma}{J_t}, \qquad b = \frac{\ell}{J_t}.$$

Note that the system is unstable (because q_1 is positive) and that, if φ is considered as output, the system has relative degree 2, and hence is trivially minimum phase.

According to the procedure described above, we pick (compare with (2.21))

$$\theta = \xi_2 + a_0\xi_1$$

with $a_0 > 0$, and obtain (compare with (2.22))

[11] See [3, pp. 36-37].

$$\begin{aligned}\dot{\xi}_1 &= -a_0\xi_1 + \theta \\ \dot{\theta} &= (q_1 - a_0q_2 - a_0^2)\xi_1 + (q_2 + a_0)\theta + bu\,.\end{aligned}$$

This system is going to be controlled by

$$u = -k\theta\,,$$

which results in (compare with (2.19))

$$\begin{pmatrix}\dot{\xi}_1 \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}-a_0 & 1 \\ (q_1 - a_0q_2 - a_0^2) & (q_2 + a_0) - bk\end{pmatrix}\begin{pmatrix}\xi_1 \\ \theta\end{pmatrix}.$$

It is known from the theory described above that, if k is sufficiently large, the system is stable. To determine the minimal value of k as a function of the parameters, we impose that the matrix

$$\begin{pmatrix}-a_0 & 1 \\ (q_1 - a_0q_2 - a_0^2) & (q_2 + a_0) - bk\end{pmatrix}$$

has eigenvalues with negative real part. This is the case if

$$\begin{aligned}&(q_2 + a_0) - bk < 0 \\ &(-a_0)(q_2 + a_0 - bk) - (q_1 - a_0q_2 - a_0^2) > 0\,,\end{aligned}$$

that is

$$k > \max\{\frac{q_2 + a_0}{b}, \frac{q_1}{a_0b}\}\,.$$

Reverting to the original parameters, this yields

$$k > \max\{\frac{a_0J_t - \gamma}{\ell}, \frac{mg}{a_0}\}\,.$$

A conservative estimate is obtained if the term $-\gamma$ is neglected (which is reasonable, since the value of γ, the coefficient of viscous friction, may be subject to large variations), obtaining

$$k > \max\{\frac{a_0(J + m\ell^2)}{\ell}, \frac{mg}{a_0}\}\,.$$

Once the ranges of the parameters J, m, ℓ are specified, this expression can be used to determine the design parameters a_0 and k. Expressed in the original variable the stabilizing control is

$$u = -k(\xi_2 + a_0\xi_1) = -k\dot{\varphi} - ka_0\varphi\,, \tag{2.25}$$

that is, the classical "proportional-derivative" feedback. ◁

2.5 The case of higher relative degree: output feedback

We have seen in the previous section that a system satisfying Assumption 2.1 can be robustly stabilized by means of a feedback law which is a linear form in the states $\xi_1, \ldots, \xi_r$ that characterize its normal form. In general, the components of the state ξ are not directly available for feedback, nor they can be retrieved from the original state x, since the transformation that defines ξ in terms of x depends on the uncertain parameter μ. We see now how this problem can be overcome, by designing a dynamic controller that provides appropriate "replacements" for the components of ξ in the control law (2.24). Observing that these variables coincide, by definition, with the measured output y and with its first $r-1$ derivatives with respect to time, it seems reasonable to try to generate the latter by means of a dynamical system of the form

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \kappa c_{r-1}(y - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \kappa^2 c_{r-2}(y - \hat{\xi}_1) \\
&\cdots \\
\dot{\hat{\xi}}_{r-1} &= \hat{\xi}_r + \kappa^{r-1} c_1(y - \hat{\xi}_1) \\
\dot{\hat{\xi}}_r &= \kappa^r c_0(y - \hat{\xi}_1)\,.
\end{aligned} \tag{2.26}$$

In fact, if $\hat{\xi}_1(t)$ where identical to $y(t)$, it would follow that $\hat{\xi}_i(t)$ coincides with $y^{(i-1)}(t)$, that is with $\xi_i(t)$, for all $i = 1, 2, \ldots, r$. In compact form, the system thus defined can be rewritten as

$$\dot{\hat{\xi}} = \hat{A}\hat{\xi} + D_\kappa G_0(y - \hat{C}\hat{\xi})\,,$$

in which

$$G_0 = \begin{pmatrix} c_{r-1} \\ c_{r-2} \\ \cdots \\ c_1 \\ c_0 \end{pmatrix}, \qquad D_\kappa = \begin{pmatrix} \kappa & 0 & \cdots & 0 \\ 0 & \kappa^2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \kappa^r \end{pmatrix},$$

and $\hat{A}, \hat{C}$ are the matrices defined in (2.7).

Let now ξ be replaced by $\hat{\xi}$ in the expression of the control law (2.24). In this way, we obtain a dynamic controller, described by equations of the form

$$\begin{aligned}
\dot{\hat{\xi}} &= \hat{A}\hat{\xi} + D_\kappa G_0(y - \hat{C}\hat{\xi}) \\
u &= H\hat{\xi}\,.
\end{aligned} \tag{2.27}$$

It will be shown in what follows that, if the parameters κ and $c_0, \ldots, c_{r-1}$ which characterize (2.27) are chosen appropriately, this *dynamic – output feedback – control* law does actually robustly stabilize the system.

Controlling the system (assumed to be expressed in strict normal form) by means of the control (2.27) yields a closed loop system

$$\begin{aligned}
\dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\
\dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H\hat{\xi}] \\
\dot{\hat{\xi}} &= \hat{A}\hat{\xi} + D_\kappa G_0(y - \hat{C}\hat{\xi})\,.
\end{aligned}$$

To analyze this closed-loop system, we first perform a change of coordinates, letting the $\hat{\xi}_i$'s be replaced by variables e_i's defined as

$$e_i = \kappa^{r-i}\,(\xi_i - \hat{\xi}_i), \qquad i = 1,\dots,r\,.$$

According to the definition of the matrix D_κ, it is observed that

$$e = \kappa^r D_\kappa^{-1}(\xi - \hat{\xi})\,,$$

that is

$$\hat{\xi} = \xi - \kappa^{-r} D_\kappa e\,.$$

The next step in the analysis is to determine the differential equations for the new variables e_i, for $i = 1,\dots,r$. Setting

$$e = \mathrm{col}(e_1,\dots,e_r)\,.$$

a simple calculation yields

$$\dot{e} = \kappa(\hat{A} - G_0\hat{C})e + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H\hat{\xi}]\,.$$

Replacing also $\hat{\xi}$ with its expression in terms of ξ and e we obtain, at the end, a description of the closed loop system in the form

$$\begin{aligned}
\dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\
\dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H(\xi - \kappa^{-r}D_\kappa e)] \\
\dot{e} &= \kappa(\hat{A} - G_0\hat{C})e + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H(\xi - \kappa^{-r}D_\kappa e)]\,.
\end{aligned}$$

To simplify this system further, we set

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

and define the matrices

$$\begin{aligned}
F_{00}(\mu) &= \begin{pmatrix} A_{00}(\mu) & A_{01}(\mu) \\ \hat{B}A_{10}(\mu) & \hat{A} + \hat{B}[A_{11}(\mu) + b(\mu)H] \end{pmatrix} \\
F_{01}(\mu) &= -\begin{pmatrix} 0 \\ \hat{B}b(\mu)H \end{pmatrix} \\
F_{10}(\mu) &= (\,\hat{B}A_{10}(\mu) \quad \hat{B}[A_{11}(\mu) + b(\mu)H]\,) \\
F_{11}(\mu) &= -\hat{B}b(\mu)H
\end{aligned}$$

in which case the equations of the closed-loop system will be rewritten as

$$\begin{aligned}\dot{\tilde{x}} &= F_{00}(\mu)\tilde{x} + F_{01}(\mu)\kappa^{-r}D_\kappa e\\ \dot{e} &= F_{10}(\mu)\tilde{x} + [\kappa(\hat{A} - G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa]e\,.\end{aligned}$$

The advantage of having the system written in this form is that we know that the matrix $F_{00}(\mu)$, if H has been chosen as described in the earlier section, has eigenvalues with negative real part for all μ. Hence, there is a positive definite symmetric matrix $P(\mu)$ such that

$$P(\mu)F_{00}(\mu) + F_{00}(\mu)^{\mathrm{T}}P(\mu) = -I\,.$$

Moreover, it is readily seen that the characteristic polynomial of the matrix $(\hat{A} - G_0\hat{C})$ coincides with the polynomial

$$p_c(\lambda) = c_0 + c_1\lambda + \cdots + c_{r-1}\lambda^{r-1} + \lambda^r\,. \tag{2.28}$$

Thus, the coefficients $c_0, c_1, \ldots, c_{r-1}$ can be chosen in such a way that all eigenvalues of $(\hat{A} - G_0\hat{C})$ have negative real part. If this is done, there exists a positive definite symmetric matrix $\hat{P}$ such that

$$\hat{P}(\hat{A} - G_0\hat{C}) + (\hat{A} - G_0\hat{C})^{\mathrm{T}}\hat{P} = -I\,.$$

This being the case, we proceed now to show that the direct criterion of Lyapunov is fulfilled, for the positive definite matrix

$$\begin{pmatrix} P(\mu) & 0\\ 0 & \hat{P}\end{pmatrix}$$

if the number κ is large enough. To this end, we need to check that the matrix

$$\begin{aligned} Q = &\begin{pmatrix} P(\mu) & 0\\ 0 & \hat{P}\end{pmatrix}\begin{pmatrix} F_{00}(\mu) & F_{01}(\mu)\kappa^{-r}D_\kappa\\ F_{10}(\mu) & \kappa(\hat{A} - G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa\end{pmatrix}\\ &+ \begin{pmatrix} F_{00}(\mu) & F_{01}(\mu)\kappa^{-r}D_\kappa\\ F_{10}(\mu) & \kappa(\hat{A} - G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} P(\mu) & 0\\ 0 & \hat{P}\end{pmatrix}\end{aligned}$$

is negative definite. In view of the definitions of $P(\mu)$ and $\hat{P}$, we see that $-Q$ has the form

$$-Q = \begin{pmatrix} I & -d(\mu,\kappa)\\ -d^{\mathrm{T}}(\mu,\kappa) & \kappa I - q(\mu,\kappa)\end{pmatrix}.$$

in which

$$\begin{aligned} d(\mu,\kappa) &= [P(\mu)F_{01}(\mu)\kappa^{-r}D_\kappa + F_{10}^{\mathrm{T}}(\mu)\hat{P}]\\ q(\mu,\kappa) &= [\hat{P}F_{11}(\mu)\kappa^{-r}D_\kappa + \kappa^{-r}D_\kappa F_{11}^{\mathrm{T}}(\mu)\hat{P}]\,.\end{aligned}$$

We want the matrix $-Q$ to be positive definite. According to Schur's Lemma,[12] this is the case if and only if the Schur's complement

$$\kappa I - q(\mu,\kappa) - d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa) \tag{2.29}$$

is positive definite. This is actually the case if κ is large enough. To check this claim, assume, without loss of generality, that $\kappa \geq 1$ and observe that in this case the diagonal matrix

$$\kappa^{-r}D_{\kappa} = \mathrm{diag}(\kappa^{-r+1},\dots,\kappa^{-1},1),$$

has norm 1. If this is the case, the positive number

$$\kappa^* = \sup_{\substack{\mu\in\mathbb{M}\\ \kappa\geq 1}} \|q(\mu,\kappa) + d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa)\|$$

is well-defined. To say that the quadratic form (2.29) is positive definite is to say that, for any nonzero $z \in \mathbb{R}^r$,

$$\kappa z^{\mathrm{T}} z > z^{\mathrm{T}}[q(\mu,\kappa) + d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa)]z.$$

Clearly, if $\kappa > \max\{1,\kappa^*\}$, this inequality holds and the matrix (2.29) is positive definite.

It is therefore concluded that if system (2.14) is controlled by (2.27), with H chosen as indicated in the previous section, and $\kappa > \max\{1,\kappa^*\}$, the resulting closed loop system has all eigenvalues with negative real part, for any $\mu \in \mathbb{M}$.

In summary, we have shown that the uncertain system (2.1), under Assumption 2.1, can be *robustly stabilized* by means of a dynamic output-feedback control law of the form

$$\dot{\hat{\xi}} = \begin{pmatrix} -\kappa c_{r-1} & 1 & 0 & \cdots & 0\\ -\kappa^2 c_{r-2} & 0 & 1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ -\kappa^{r-1}c_1 & 0 & 0 & \cdots & 1\\ -\kappa^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}\hat{\xi} + \begin{pmatrix} \kappa c_{r-1}\\ \kappa^2 c_{r-2}\\ \cdots\\ \kappa^{r-1}c_1\\ \kappa^r c_0 \end{pmatrix} y$$

$$u = -k\,(a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{r-2} \quad 1)\,\hat{\xi},$$

in which $c_0,c_1,\dots,c_{r-1}$ and, respectively, $a_0,a_1,\dots,a_{r-2}$ are such that the polynomials (2.28) and (2.23) have negative real part, and κ and k are large (positive) parameters.

Remark 2.4. Note the striking similarity of the arguments used to show the negative definiteness of Q with those used in section 2.3. The large value of κ is instrumental in overcoming the effects of an additive term in the bottom-right block and of the off-diagonal terms. Both these terms depend now on κ (this was not the case in

[12] See (A.1) in Appendix A.

section 2.3) but fortunately, if $\kappa \geq 1$, such terms have bounds that are independent of κ. Also in this case, we can interpret the resulting system as interconnection of a stable subsystem with state x and input e, connected to a subsystem with state e and input z. The role of a large κ is: (i) to render the lower subsystem stable, and (ii) to lower the effect of the coupling between the two subsystems. ◁

Example 2.2. Consider again the system of Example 2.1 and suppose that only the variable φ is available for feedback. In this case, we use a control

$$u = -k(a_0\hat{\xi}_1 + \hat{\xi}_2)\,,$$

in which $\hat{\xi}_1, \hat{\xi}_2$ are provided by the dynamical system

$$\dot{\hat{\xi}} = \begin{pmatrix} -\kappa c_1 & 1 \\ -\kappa^2 c_0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} \kappa c_1 \\ \kappa^2 c_0 \end{pmatrix} \varphi\,.$$

For convenience, we can take $c_0 = 1$ and $c_1 = 2$, in which case the characteristic polynomial of this system becomes $(s+\kappa)^2$. Setting $\varepsilon = 1/\kappa$ and computing the transfer function $T_{\rm c}(s)$ of the controller, between φ and u, it is seen that

$$T_{\rm c}(s) = -k\,\frac{(1+2a_0\varepsilon)s + a_0}{(\varepsilon s + 1)^2}\,.$$

Or course, as $\varepsilon \to 0$, the control approaches the proportional-derivative control (2.25) . ◁

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