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For Maria Adelaide

Preface

The purpose of this book is to present a self-contained and coordinated description of several design methods for nonlinear control systems, with special emphasis on the problem of achieving stability, globally or on arbitrarily large domains, in the presence of model uncertainties. The book is intended to be a continuation of my earlier book *Nonlinear Control Systems*, dealing with the fundamentals of the theory of nonlinear control systems, whose third edition was published in 1995. In this respect, it is written in the form of a "second volume" of a single work, and uses a numbering system that continues the one adopted in the earlier book, with which the overlap is essentially insignificant. The book is intended as a graduate text as well as a reference to scientists and engineers interested in the design of feedback laws for nonlinear control systems.

In the last decade, methods for global stabilization of nonlinear systems have experienced a vigorous growth. The emergence of the concept of ***input-to-state stability***, with its multi-faced consequences in terms of various versions of a ***small-gain criterion*** for the stability analysis of interconnected systems, the idea of making a system ***passive via feedback***, with its far-reaching implications in terms of robustness, and the identification of special structures – such as those of the so-called systems in ***feedback*** or ***feedforward form*** – in which the design of feedback laws can be systematically addressed in a recursive way by dealing each time with a problem in "dimension one" (as in the so-called method of backstepping), have enormously increased our ability of designing feedback laws to the purpose of achieving stability, for various classes of systems modeled by nonlinear differential equations, in the presence of model uncertainties. The objective of this text is to render the reader familiar with major methods and results, and enable him to follow the recent literature.

The book is organized as follows. The first Chapter, namely Chapter 10, is devoted to the exposition of concepts and methods for the stability analysis of systems that can be viewed as a (cascade or feedback) interconnection of lower-dimensional subsystems. In particular, after a quick review of some basic tools for the analysis of the asymptotic behavior of nonlinear systems, the fundamental notion of input-to-state stability is described, along with various different characterizations and features, and it is shown how the so-called input-to-state "gain function" of a system can be evaluated by means of a method which extends the classical method of Lyapunov for stability analysis. Then, this notion is used in the development of a small-gain theorem for input-to-state stable systems, extensively applied later in Chapters 11 and 12 to the purpose of achieving robust stability. The second part of Chapter 10 is dedicated to the exposition of the notion of dissipativity, with particular emphasis on the special cases of systems having a "finite L_2 gain" and of "passive" systems. Again, it is shown how these properties can be tested and how they can be used in the stability analysis of interconnected systems. The last section of the Chapter discusses interesting (and classical) implications of these properties on the transfer function matrix of a linear system.

Chapter 11 describes methods for robust global asymptotic stabilization. For systems modeled by equations in "lower-triangular" form containing unknown parameters, it is shown how to design robustly stabilizing feedback laws, for a series of cases of increasing complexity. The first case describes how the method of backstepping can be used in order to recursively design a feedback law to the purpose of imposing that a fixed (quadratic-like) positive definite function becomes a Lyapunov function for the closed-loop system. The feedback law so obtained is a function of a subset of the set of state variables, the equations modeling the systems are assumed to be linear in the unmeasured state variables, and the internal dynamics associated with the unmeasured state variables are assumed to be globally robustly asymptotically stable. Under appropriate additional hypotheses, this control scheme leads to the design of a dynamic feedback which uses only the output of the system as measured variable. Then, an extension of this method is described, that exploits the small-gain theorem for input-to-state stable systems, in which the hypotheses of linearity in the unmeasured variables and stability of the associated dynamics are no longer required. Finally, the Chapter is complemented by some results regarding the extensions of these methods to the case of systems having many inputs.

Chapter 12 discusses the case in which, instead of looking at the problem of achieving global asymptotic stability, one rather seeks a feedback law able to impose the property that any trajectory corresponding to initial conditions in an arbitrarily large set (semiglobal stabiliz-*ability*) enters, in finite time, an arbitrarily small set (practical stabiliz-*ability*). This setup is particularly convenient in case only the output of the system is available for feedback. In fact, a system in "lower-triangular" form (containing unknown parameters) is globally diffeomorphic to a system in which the states which need to be measured for feedback coincide with the output and a number of its derivatives with respect to time. Such a diffeomorphism may depend on unknown parameters and unmeasured states, but this uncertainty is not a problem if, as explained in the course of the Chapter, output and its derivatives are replaced by estimates which, past a small initial time interval, become arbitrarily accurate. This idea is used to stabilize systems using dynamic output feedback, first under the hypothesis that the zero dynamics are globally robustly asymptotically stable and later also in the more challenging case in which this hypothesis does not hold.

In Chapter 13, the issue of robust stability in the presence of unmodeled dynamics is dealt with via the small-gain theorem for systems having finite L_2 gain, by seeking feedback laws which render the gain of the system, between a fixed disturbance input and a fixed output, sufficiently small. This is the nonlinear version of the so-called problem of disturbance attenuation, and the methods by which the problem is solved are the nonlinear versions of the so-called methods of H_∞ control for robust stabilization of linear systems. The Chapter presents a variety of situations, including the solution of the so-called problem of almost disturbance decoupling, and a comparison with the corresponding results which hold in the case of linear systems is systematically carried out.

Finally, Chapter 14 describes when and how it is possible to achieve global asymptotic stability by means of (state feedback) laws whose amplitude cannot exceed a fixed bound. In the first part of the Chapter, it is shown that a solution of this design problem, which of course exists only under special hypotheses, can be systematically achieved, for systems in "upper-triangular" form, via the recursive synthesis of appropriate (non-quadratic) Lyapunov functions. The second part of the Chapter describes an alternative, and appealing, recursive design procedure in which, at each stage, the desired asymptotic properties of the current subsystem are imposed by means of a feedback law which simply consists in the "saturation" of a suitable linear law. This yields a general design scheme with great potential, known as the scheme of the "nested-saturations", of which some applications are presented at the end of the Chapter.

Although an attempt has been made to present, in a organized way, some of the leading ideas and results appeared in this area in the last decade or so, the exposition is far from being a complete overview of the intended subject, namely robust and global/semiglobal stabilization of nonlinear systems. In particular, the bibliography includes only the titles which have been actually used. Sincere apologies are owed for a number of omissions, unavoidable in any attempt like this one.

I wish to express my deep gratitude to a number of people/institutions which have made this work possible. In particular, I am indebted to Washington University in St. Louis, its School of Engineering and Applied Science, its Department of Systems Science and Mathematics, and the funding agencies NSF and AFOSR, for the generous support, encouragement and advice received. I also wish to thank the NASA Research Center at Langley, VA, where in 1997 I had the opportunity to lecture for the first time, in a systematic manner, on the topics presented in this book. The financial support of the MURST is also gratefully acknowledged. I am indebted to Professors T. Basar, L. Praly, A. Teel and E. Sontag, for discussions, advice and ideas that have helped me enormously in the understanding of the subject. I also wish to thank a number of collaborators and doctoral students with whom I had intensive exchanges of ideas: A. Astolfi, L. Marconi, A. Serrani, B. Schwartz, C. De Persis, R. De Santis, also for their invaluable help in looking at various preliminary drafts of this work.

Rome and St. Louis, June 1999 Alberto Isidori

Table of Contents

10. Stability of Interconnected Nonlinear Systems

10.1 Preliminaries

For convenience of the reader, this section provides a quick review of the notion of *comparison functions* and their role in the well-known criterion of Lyapunov for determining stability and asymptotic stability.

Definition 10.1.1. *A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathcal{K}_\infty$.*

Definition 10.1.2. *A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the function*

$$\begin{array}{rccc} \alpha: & [0,a) & \to & [0,\infty) \\ & r & \mapsto & \beta(r,s) \end{array}$$

belongs to class $\mathcal{K}$ and, for each fixed r, the function

$$\begin{array}{rccc} \varphi: & [0,\infty) & \to & [0,\infty) \\ & s & \mapsto & \beta(r,s) \end{array}$$

is decreasing and $\lim_{s\to\infty} \varphi(s) = 0$.

Class $\mathcal{K}$ and class $\mathcal{KL}$ functions have some interesting features, that can be summarized as follows. The composition of two class $\mathcal{K}$ (respectively, class $\mathcal{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathcal{K}$ (respectively, class $\mathcal{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathcal{K}$ function, defined on $[0,a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique function, $\alpha^{-1} : [0,b) \to [0,a)$, such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \quad \text{for all } r \in [0,a) \\ \alpha(\alpha^{-1}(r)) &= r, \quad \text{for all } r \in [0,b). \end{aligned}$$

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathcal{K}$ function. If $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$. If $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function and $\alpha_1(\cdot), \alpha_2(\cdot)$ are class $\mathcal{K}$ functions, the function thus defined

$$\begin{array}{rccc} \gamma: & [0,a) \times [0,\infty) & \to & [0,\infty) \\ & (r,s) & \mapsto & \alpha_1(\beta(\alpha_2(r),s)) \end{array}$$

is a class $\mathcal{KL}$ function. It is also useful to know that any class $\mathcal{KL}$ function can always be estimated in terms of two other class $\mathcal{K}_\infty$ functions and of the exponential function, as indicated in the following result [1].

Lemma 10.1.1. *Assume $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function. Then, there exist two class $\mathcal{K}_\infty$ functions $\gamma(\cdot)$ and $\theta(\cdot)$ such that*

$$\beta(r,s) \le \gamma(e^{-s}\theta(r))$$

for all $(r,s) \in [0,a) \times [0,\infty)$.

Finally, another important feature of the comparison functions is the following property, which is very useful in establishing the asymptotic convergence to zero of the trajectories of a nonlinear system [2].

Lemma 10.1.2. *Consider the differential equation*

$$\dot{y} = -\alpha(y)$$

where $y \in \mathbb{R}$ and $\alpha(\cdot)$ is a locally Lipschitz class $\mathcal{K}$ function defined on $[0,a)$. For all $0 \le y^\circ < a$, this equation has a unique solution $y(t)$ satisfying $y(0) = y^\circ$, defined for all $t \ge 0$, and

$$y(t) = \varphi(y^\circ, t)$$

where $\varphi(\cdot,\cdot)$ is a class $\mathcal{KL}$ function defined on $[0,a) \times [0,\infty)$.

In what follows we denote, as usual, by $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$ and by B_ε (respectively, by $\bar{B}_\varepsilon$) the open (respectively, closed) ball of radius ε, namely

$$B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}, \qquad \bar{B}_\varepsilon = \{x \in \mathbb{R}^n : \|x\| \le \varepsilon\}.$$

Now, consider a nonlinear system

$$\dot{x} = f(x) \tag{10.1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well known criterion of Lyapunov, which, in terms of comparison function, can be expressed as follows.

[1] For a proof of this result, see Sontag (1998).
[2] For a proof of this result, see Khalil (1996), page 656.

Theorem 10.1.3. *Let $V : B_d \to \mathbb{R}$ be a C^1 function such that, for some class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, defined on $[0,d)$,*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \text{for all } \|x\| < d. \tag{10.2}$$

If

$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \text{for all } \|x\| < d, \tag{10.3}$$

the equilibrium $x = 0$ of (10.1) is stable.

If, for some class $\mathcal{K}$ function $\alpha(\cdot)$, defined on $[0,d)$,

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|) \qquad \text{for all } \|x\| < d, \tag{10.4}$$

the equilibrium $x = 0$ of (10.1) is locally asymptotically stable.

If $d = \infty$ and, in the above inequalities, $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathcal{K}_\infty$ functions, the equilibrium $x = 0$ of (10.1) is globally asymptotically stable.

Proof. From (10.3) it is deduced that, so long as $x(t)$ is defined, $V(x(t))$ is non increasing, i.e.

$$V(x(t)) \le V(x(0)).$$

Suppose $\varepsilon < d$ and define

$$\delta = \overline{\alpha}^{-1}(\underline{\alpha}(\varepsilon)).$$

Then, using (10.2), observe that $\|x(0)\| \le \delta$ implies

$$\underline{\alpha}(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \overline{\alpha}(\|x(0)\|) \le \overline{\alpha}(\delta) = \underline{\alpha}(\varepsilon),$$

i.e.

$$\|x(t)\| \le \varepsilon.$$

This, since $f(x)$ is locally Lipschitz, shows that $x(t)$ is defined for all $t \ge 0$ and proves the property of stability in the sense of Lyapunov.

Set $V(t) = V(x(t))$, $\theta(\cdot) = \alpha(\overline{\alpha}^{-1}(\cdot))$ and observe that, if (10.4) holds, one has

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -\theta(V(t)).$$

Without loss of generality, suppose $\theta(\cdot)$ is locally Lipschitz (if this is not the case, it can be replaced in what follows by any locally Lipschitz class $\mathcal{K}$ function $\bar{\theta}(\cdot)$ such that $\theta(r) \ge \bar{\theta}(r)$). Then, by Lemma 10.1.2, the differential equation

$$\dot{y} = -\theta(y)$$

has a unique solution $y(t)$ satisfying $y(0) = V(0)$, defined for all $t \ge 0$, and

$$y(t) = \varphi(V(0), t)$$

for some class $\mathcal{KL}$ function $\varphi(\cdot,\cdot)$. By the comparison Lemma,

$$V(t) \le \varphi(V(0), t)$$

and this yields

$$\|x(t)\| \le \underline{\alpha}^{-1}(\varphi(\overline{\alpha}(\|x(0)\|), t))$$

Since the right-hand side is a class $\mathcal{KL}$ function in the arguments $\|x(0)\|$ and t, this proves local asymptotic stability. Global asymptotic stability is proven in the same way. ◁

Remark 10.1.1. Note that the inequality on the right-hand side of (10.2) is redundant, since the existence of a function $\overline{\alpha}(x)$ for which this inequality holds is an immediate consequence of the continuity of $V(x)$ (and, of course, if $\underline{\alpha}(\cdot)$ is of class $\mathcal{K}_\infty$, so is necessarily $\overline{\alpha}(\cdot)$). However, the inequality in question turns out to be useful in establishing bounds for the trajectories, as done for instance in the proof of Theorem 10.1.3.

On the other hand, the inequality on the left-hand side of (10.2) is instrumental, together with (10.3), in establishing existence and boundedness of $x(t)$. In particular, suppose the various inequalities considered in Theorem 10.1.3 hold for $d = \infty$. The hypothesis (10.3) that $V(x(t))$ is non-increasing guarantees that $x(t)$ is defined for all $t \ge 0$ and bounded, so long as $x(0)$ is such that $\overline{\alpha}(\|x(0)\|)$ belongs to the domain of the *inverse* of the function $\underline{\alpha}(\cdot)$. If the function $\underline{\alpha}(\cdot)$ is a class $\mathcal{K}_\infty$ function, its inverse is defined on $[0, \infty)$ and therefore existence and boundedness of $x(t)$ are guaranteed for any $x(0)$. ◁

Remark 10.1.2. Arguments similar to those used in the proof of this theorem are very useful in order to establish the *invariance*, in positive time, of certain bounded subsets of $\mathbb{R}^n$. Specifically, suppose the various inequalities considered in Theorem 10.1.3 hold for $d = \infty$ and let Ω_c denote the set of all $x \in \mathbb{R}^n$ for which $V(x) \le c$, namely

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}.$$

Note that, for any c in the image of the function $\underline{\alpha}(\cdot)$ (i.e. for any c such that $c = \underline{\alpha}(r)$ for some $0 \le r < \infty$), the inequality on the left-hand side of (10.2) yields

$$x \in \Omega_c \quad \Rightarrow \quad \|x\| \le \underline{\alpha}^{-1}(c)$$

i.e. the set Ω_c is *bounded*. This property holds for all $c > 0$ if $\underline{\alpha}(\cdot)$ is a class $\mathcal{K}_\infty$ function.

If

$$\frac{\partial V(x)}{\partial x} f(x) < 0$$

at each point x of the boundary of Ω_c, it can be concluded that, for any initial condition in the interior of Ω_c, the solution $x(t)$ of (10.1) is defined for all $t \ge 0$ and such that $x(t) \in \Omega_c$ for all $t \ge 0$. Indeed, existence and uniqueness are guaranteed by the local Lipschitz property so long as $x(t) \in \Omega_c$, because Ω_c is a compact set. The fact that $x(t)$ remains in Ω_c for all $t \ge 0$ is proved by contradiction. For, suppose that, for some trajectory $x(t)$, there is a time t_1 such that $x(t)$ is in the interior of Ω_c at all $t < t_1$ and $x(t_1)$ is on the boundary of Ω_c. Then,

$$\begin{aligned} V(x(t)) &< c \quad \text{for all } t < t_1 \\ V(x(t_1)) &= c \end{aligned}$$

and this contradicts the previous inequality, which shows that the derivative of $V(x(t))$ is strictly negative at $t = t_1$. ◁

Remark 10.1.3. The existence of a class $\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$ satisfying the inequality on the left-hand side of (10.2) is equivalent to the properties that the function $V(x)$ is *positive definite* (i.e. vanishes at $x = 0$ and is positive for all nonzero x) and *proper* (i.e. *for each $c > 0$, the set Ω_c is a compact set*). The necessity of the latter property is an immediate consequence of (10.2), because $x \in \Omega_c$ implies

$$\|x\| \le \underline{\alpha}^{-1}(V(x)) \le \underline{\alpha}^{-1}(c)$$

which shows that Ω_c is bounded. Moreover, Ω_c is closed by definition, so it is compact. Conversely, suppose Ω_c is compact and define, for all $c \ge 0$

$$\rho(c) = \max_{x \in \Omega_c} \|x\| .$$

This function vanishes at $c = 0$, is positive for any nonzero c, is strictly increasing and $\lim_{c \to \infty} \rho(c) = \infty$, because $V(x)$ is continuous. Thus, $\rho(\cdot)$ is a class $\mathcal{K}_\infty$ function. Set

$$\underline{\alpha}(r) = \rho^{-1}(r) .$$

Now, take any x, let $c = V(x)$, and observe that indeed

$$\|x\| \le \max_{x \in \Omega_c} \|x\| .$$

Then,

$$\underline{\alpha}(\|x\|) \le \underline{\alpha}(\max_{x \in \Omega_c} \|x\|) = \underline{\alpha}(\rho(c)) = V(x)$$

i.e. this function satisfies the inequality on the left-hand side of (10.2). ◁

Remark 10.1.4. In the inequality (10.4), the function $\alpha(\cdot)$, was supposed to be of class $\mathcal{K}$. However, one can prove that, if for some $V(x)$ defined for all $x \in \mathbb{R}^n$ the inequalities (10.2) and (10.4) hold, with $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ of class $\mathcal{K}_\infty$ and $\alpha(\cdot)$ of class $\mathcal{K}$, there is another function $\widetilde{V}(x)$ which satisfies similar inequalities but with $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and $\alpha(\cdot)$ all of class $\mathcal{K}_\infty$. In other words, there is no loss of generality in assuming that $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

To this end, let $\rho(\cdot)$ be the function defined by an integral of the form

$$\rho(r) = \int_0^r q(s) ds$$

in which $q(\cdot)$ is a smooth class $\mathcal{K}_\infty$ function. Indeed, also the function $\rho(\cdot)$ is a class $\mathcal{K}_\infty$ function. Define $\widetilde{V}(x)$ as

$$\widetilde{V}(x) = \rho(V(x))$$

and define

$$\tilde{\alpha}(r) = q(\underline{\alpha}(r))\alpha(r) .$$

By construction, the function $\widetilde{V}(x)$ is automatically C^1 and, as it easy to check, satisfies estimates of the form (10.2). Moreover, the function $\tilde{\alpha}(r)$ is a class $\mathcal{K}_\infty$ function, because so are the functions $\underline{\alpha}(\cdot)$ and $q(\cdot)$. Now, observe that

$$\frac{\partial \widetilde{V}}{\partial x}f(x) = q[V(x)]\frac{\partial V}{\partial x}f(x) \leq -q[V(x)]\alpha(\|x\|) .$$

But

$$-q[V(x)]\alpha(\|x\|) \leq -q(\underline{\alpha}(\|x\|))\alpha(\|x\|) = -\tilde{\alpha}(\|x\|)$$

and this completes the proof. ◁

It is well-known that the criterion for asymptotic stability provided by the previous Theorem has a *converse*, namely, the existence of a function $V(x)$ having the properties indicated in Theorem 10.1.3 is *implied* by the property of asymptotic stability of the equilibrium $x = 0$ of (10.1). In particular, the following result holds [3].

Theorem 10.1.4. *Suppose the equilibrium $x = 0$ of (10.1) is locally asymptotically stable. Then, there exist $d > 0$, a C^1 function $V : B_d \to \mathbb{R}$, and class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (10.2) and (10.4) hold. If the equilibrium $x = 0$ of (10.1) is globally asymptotically stable, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (10.2) and (10.4) hold with $d = \infty$.*

Remark 10.1.5. Note that, combining the result of Theorem 10.1.4 with an argument used in the proof of Theorem 10.1.3, it can be deduced that, if the equilibrium $x = 0$ is globally asymptotically stable, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that, for any x°, the solution $x(t)$ of (10.1) with initial condition $x(0) = x^\circ$ satisfies an estimate of the form

$$\|x(t)\| \leq \beta(\|x^\circ\|, t)$$

for all $t \geq 0$. Note also that, using Lemma 10.1.1, this estimate can be replaced by an estimate of the form

$$\|x(t)\| \leq \gamma(e^{-t}\theta(\|x^\circ\|))$$

in which $\gamma(\cdot)$ and $\theta(\cdot)$ are class $\mathcal{K}_\infty$ functions. Since the inverse of $\gamma(\cdot)$ is defined on $[0, \infty)$ and is a class $\mathcal{K}_\infty$ function, this shows that, if the equilibrium $x = 0$ of (10.1) is globally asymptotically stable, any trajectory $x(t)$ satisfies an estimate of the form

$$\tilde{\gamma}(\|x(t)\|) \leq e^{-t}\theta(\|x(0)\|)$$

in which $\tilde{\gamma}(\cdot)$ and $\theta(\cdot)$ are class $\mathcal{K}_\infty$ functions. ◁

[3] For a proof of this result, see Kurzweil (1956).

It is well-known that, for a nonlinear system, the property of asymptotic stability of the equilibrium $x = 0$ does not necessarily imply *exponential* decay to zero of $\|x(t)\|$. If the equilibrium $x = 0$ of system (10.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$ and $\lambda > 0$ such that

$$x(0) \in B_d \quad \Rightarrow \quad \|x(t)\| \leq Me^{-\lambda t}\|x(0)\| \quad \text{for all } t \geq 0$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. In what follows, a characterization of those systems possessing a globally asymptotically and locally exponentially stable equilibrium is given. This characterization, and another interesting property that is presented immediately afterwards, are of great help in addressing certain problems of asymptotic stabilization, discussed in the next Chapters.

Lemma 10.1.5. *The equilibrium $x = 0$ of nonlinear system (10.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function $V(x) : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and real numbers $\delta > 0, a > 0, b > 0$, such that*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$

$$\frac{\partial V}{\partial x}f(x) \leq -\alpha(\|x\|)$$

for all $x \in \mathbb{R}^n$ and

$$\underline{\alpha}(s) = as^2, \qquad \alpha(s) = bs^2$$

for all $s \in [0, \delta]$.

Proof. The equilibrium of the nonlinear system (10.1) is locally exponentially stable [4] if and only if there exists real numbers $r > 0, \underline{a} > 0, \overline{a} > 0, \underline{b} > 0$ and a smooth function $U(x) : B_r \to \mathbb{R}$ such that

$$\begin{aligned} \underline{a}\|x\|^2 \leq U(x) \leq \overline{a}\|x\|^2 \\ \frac{\partial U}{\partial x}f(x) \leq -\underline{b}\|x\|^2 \end{aligned} \tag{10.5}$$

for all $x \in B_r$.

Thus, to prove that the condition stated in the Theorem is sufficient, it is only needed to check that, for all $x \in B_\delta$ and some $\overline{a}$, $V(x) \leq \overline{a}\|x\|^2$, which is indeed the case since $V(x)$ is a smooth function vanishing at $x = 0$ together with its first derivatives.

To prove the necessity, let $U(x)$ be a function satisfying the conditions (10.5) and $V(x)$ be a function satisfying (10.2) and (10.4) with $d = \infty$. We claim that there exist real numbers $k > 0, \delta > 0, \rho > 0, c_1 > 0, c_2 > 0$ such that the set

[4] See Khalil (1996), pages 140 and 149.

$$S = \{x \in \mathbb{R}^n : c_1 \le kV(x) \le c_2\}$$

satisfies

$$\bar{B}_\delta \subset S \subset \bar{B}_\rho \subset B_r$$

and

$$kV(x) \ge U(x) \qquad \text{for all } x \in S\,. \tag{10.6}$$

In fact, choose any $\rho < r$, and $c_1 = \bar{a}\rho^2$, $c_2 = 2\bar{a}\rho^2$. Then, choose k so that

$$\{x \in \mathbb{R}^n : kV(x) \le c_2\} \subset \bar{B}_\rho\,.$$

Such a k indeed exists because, if $k \ge c_2/\underline{\alpha}(\rho)$

$$kV(x) \le c_2 \quad \Rightarrow \quad \|x\| \le \underline{\alpha}^{-1}(\frac{c_2}{k}) \le \rho\,.$$

In this way, $S \subset \bar{B}_\rho \subset B_r$ and

$$x \in S \quad \Rightarrow \quad kV(x) \ge c_1 = \bar{a}\rho^2 \ge U(x)$$

which proves (10.6). Finally, choose $\delta < \min_{\{kV(x)=c_1\}} \|x\|$.

Now, let $\sigma(\cdot)$ be a smooth non-decreasing function, defined on $[0,\infty)$, and such that

$$\sigma(s) = \begin{cases} 0 & \text{if } s \le c_1 \\ 1 & \text{if } c_2 \le s\,. \end{cases}$$

Its derivative, denoted $\sigma'(\cdot)$, satisfies

$$\sigma'(s) \begin{cases} = 0 & \text{if } s \le c_1 \\ \ge 0 & \text{if } c_1 < s < c_2 \\ = 0 & \text{if } c_2 \le s\,. \end{cases}$$

Set

$$\beta(x) = \sigma(kV(x))$$

and consider the function

$$W(x) = \beta(x)kV(x) + (1-\beta(x))U(x)\,,$$

which is well-defined because, by construction, $\|x\| \ge r$ implies $1-\beta(x) = 0$. Then, for all $x \in \mathbb{R}^n$,

$$W(x) \ge \beta(x)k\underline{\alpha}(\|x\|) + (1-\beta(x))\underline{a}\|x\|^2\,.$$

Let $0 < a \le \underline{a}$ be such that

$$as^2 < k\underline{\alpha}(s) \qquad \text{for all } s \in [\delta, \rho]\,.$$

Then, from the previous inequality it can be concluded that there exists a class $\mathcal{K}_\infty$ function $\underline{\tilde{\alpha}}(\cdot)$, satisfying

$$\underline{\tilde{\alpha}}(s) = \begin{cases} as^2 & \text{if } s \le \delta \\ k\underline{\alpha}(s) & \text{if } s \ge \rho\,, \end{cases}$$

such that

$$W(x) \ge \underline{\tilde{\alpha}}(\|x\|)$$

for all $x \in \mathbb{R}^n$.

Moreover

$$\frac{\partial W}{\partial x}f(x) = \beta(x)k\frac{\partial V}{\partial x}f(x) + (1-\beta(x))\frac{\partial U}{\partial x}f(x) + (kV(x)-U(x))\frac{\partial \beta}{\partial x}f(x).$$

Observe that

$$(kV(x)-U(x))\frac{\partial \beta}{\partial x}f(x) = (kV(x)-U(x))\sigma'(kV(x))k\frac{\partial V}{\partial x}f(x) \tag{10.7}$$

and that, by construction,

$$\sigma'(kV(x))(kV(x)-U(x)) \ge 0$$

for all $x \in \mathbb{R}^n$. Thus, the quantity (10.7) is always non-positive, and

$$\begin{aligned} \frac{\partial W}{\partial x}f(x) &\le \beta(x)k\frac{\partial V}{\partial x}f(x) + (1-\beta(x))\frac{\partial U}{\partial x}f(x) \\ &\le -[\beta(x)k\alpha(\|x\|) + (1-\beta(x))\underline{b}\|x\|^2]\,. \end{aligned}$$

Form this one can conclude, as before, that there exists a number $0 < b < \underline{b}$ and a class $\mathcal{K}_\infty$ function $\tilde{\alpha}(\cdot)$, satisfying $\tilde{\alpha}(s) = bs^2$ for all $s \in [0,\delta]$, such that

$$\frac{\partial W}{\partial x}f(x) \le -\tilde{\alpha}(\|x\|)$$

for all $x \in \mathbb{R}^n$. This completes the proof of the necessity. ◁

The following lemma, which concludes the section, provides a useful estimate for the function $\sigma(\|x(t)\|)$ resulting from the composition of a class $\mathcal{K}$ function $\sigma(\cdot)$ with the norm of an integral curve $x(t)$ of system (10.1), under the hypothesis that the latter has a globally asymptotically and locally exponentially stable at $x = 0$.

Lemma 10.1.6. *Consider the system (10.1). Suppose the equilibrium $x = 0$ is globally asymptotically stable and locally exponentially stable. Let $\sigma(\cdot)$ be a class $\mathcal{K}$ function which is differentiable at the origin. Then, there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ and a number $\lambda > 0$ such that, for any $x^\circ \in \mathbb{R}^n$, the integral curve $x(t)$ passing through x° at time $t = 0$ is such that*

$$\sigma(\|x(t)\|) \le \alpha(\|x^\circ\|)e^{-\lambda t}$$

for all $t \ge 0$.

Proof. Set
$$F(t) = \sigma(\|x(t)\|) .$$
By hypothesis, $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ function $\beta(\|x^\circ\|, t)$ and this, in view of Lemma 10.1.1, implies that, for some pair of class $\mathcal{K}_\infty$ functions $\gamma(\cdot)$, $\theta(\cdot)$,
$$\|x(t)\| \le \gamma(e^{-t}\theta(\|x^\circ\|)) .$$
Note that $\gamma(\theta(s)) \ge s$. Consider the class $\mathcal{K}_\infty$ functions $\tilde{\sigma}(\cdot)$ and $\tilde{\gamma}(\cdot)$ defined as
$$\begin{aligned} \tilde{\sigma}(s) &= 2\max\{s, \sigma(s)\} \\ \tilde{\gamma}(s) &= \tilde{\sigma} \circ \gamma(s) . \end{aligned}$$
Then,
$$F(t) \le \tilde{\sigma}(\|x(t)\|) \le \tilde{\gamma}(e^{-t}\theta(\|x^\circ\|)) .$$
Observe also that $\tilde{\gamma}(\theta(s)) > s$ for all $s > 0$.

By hypothesis, the equilibrium $x = 0$ of the system is also locally exponentially stable. This means that there are numbers $M > 0$, $\lambda > 0$ and $d > 0$ such that, if $\|x^\circ\| \le d$, the integral curve passing through x° at time $t = 0$ satisfies
$$\|x(t)\| \le M\|x^\circ\|e^{-\lambda t}$$
for all $t \ge 0$. Moreover, the function $\sigma(\cdot)$ is by hypothesis differentiable at the origin. This implies the existence of numbers $N > 0$ and $\bar{s} > 0$ such that, if $s \in [0, \bar{s}]$, $\sigma(s) \le Ns$. Let
$$R = \min\{d, \frac{\bar{s}}{M}\} .$$
Then, if $\|x^\circ\| \le R$,
$$F(t) \le N\|x(t)\| \le NM\|x^\circ\|e^{-\lambda t} \tag{10.8}$$
for all $t \ge 0$.

Suppose, without loss of generality, $NM > 1$, define
$$A(s) = NM\tilde{\gamma}(\theta(s))\Big(\frac{\theta(s)}{\tilde{\gamma}^{-1}(R)}\Big)^\lambda$$
and consider the case $\|x^\circ\| > R$. Since $\tilde{\gamma}(\theta(s)) > s$ for all $s > 0$,
$$T_1 := \ln \frac{\theta(\|x^\circ\|)}{\tilde{\gamma}^{-1}(R)} > 0$$
and
$$A(\|x^\circ\|)e^{-\lambda T_1} = NM\tilde{\gamma}(\theta(\|x^\circ\|)) .$$
As a consequence, we have, for all $t \in [0, T_1]$,
$$\begin{aligned} F(t) &\le \tilde{\gamma}(\theta(\|x^\circ\|)) \\ &\le NM\tilde{\gamma}(\theta(\|x^\circ\|)) = A(\|x^\circ\|)e^{-\lambda T_1} \le A(\|x^\circ\|)e^{-\lambda t} . \end{aligned}$$

Moreover, by definition of T_1,
$$\|x(T_1)\| \le \gamma(e^{-T_1}\theta(\|x^\circ\|)) \le \tilde{\gamma}(e^{-T_1}\theta(\|x^\circ\|)) = R ,$$
and therefore, for all $t \ge T_1$,
$$\begin{aligned} F(t) &\le NM\|x^\circ(T_1)\|e^{-\lambda(t-T_1)} \\ &\le NMRe^{-\lambda(t-T_1)} = NM\tilde{\gamma}(e^{-T_1}\theta(\|x^\circ\|))e^{-\lambda(t-T_1)} \\ &\le NM\tilde{\gamma}(\theta(\|x^\circ\|))e^{\lambda T_1}e^{-\lambda t} \le A(\|x^\circ\|)e^{-\lambda t} \end{aligned}$$
This proves that, for all $\|x^\circ\| > R$,
$$F(t) \le A(\|x^\circ\|)e^{-\lambda t}$$
for all $t \ge 0$. On the other hand, for all $\|x^\circ\| \le R$, (10.8) holds.

Thus, let $\alpha(\cdot)$ be a class $\mathcal{K}$ function satisfying
$$\begin{aligned} \alpha(s) &\ge NMs \quad \text{for } s \in [0, R] \\ \alpha(s) &\ge A(s) \quad \text{for } s > R . \end{aligned}$$
This proves the Lemma. ◁

10.2 Asymptotic Stability and Small Perturbations

This section contains a result which analyzes the effect of a "small perturbation" affecting a system of the form (10.1), whose equilibrium $x = 0$ is assumed to be locally asymptotically stable. The theorem below, often referred to as the theorem of "total stability", shows that the trajectories of the perturbed system remain arbitrarily close to the equilibrium $x = 0$, although they may not necessarily asymptotically converge to it, provided that the perturbation as well as the initial state are sufficiently small.

Theorem 10.2.1. *Suppose the equilibrium $x = 0$ of (10.1) is locally asymptotically stable. Suppose $g(x, t)$ is piecewise continuous in t and satisfies a Lipschitz condition*
$$\|g(x', t) - g(x'', t)\| \le L\|x' - x''\|$$
for all $t \ge 0$ and all x', x'' in some neighborhood U of $x = 0$. Then, given any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ (both possibly dependent on $\varepsilon > 0$) such that, if
$$\|x^\circ\| \le \delta_1$$
$$\|g(x, t)\| \le \delta_2 \;\; \textit{for all} \;\; \|x\| \le \varepsilon \;\; \textit{and all} \;\; t \ge 0 ,$$
the solution $x(t)$ of the perturbed system
$$\dot{x} = f(x) + g(x, t) \tag{10.9}$$

satisfying $x(0) = x^\circ$ *is such that*

$$\|x(t)\| \le \varepsilon$$

for all $t \ge 0$.

Proof. Let $V(x)$ be a C^1 function satisfying (10.2), (10.4). Since $\dfrac{\partial V}{\partial x}$ is C^0, there exists a number $M > 0$ such that

$$\|\frac{\partial V}{\partial x}\| \le M$$

for all $x \in U$. Suppose, without loss of generality, that ε is such that $B_\varepsilon \subset U$. Fix $\varepsilon > 0$ and let $c > 0$ be such that $c \le \underline{\alpha}(\varepsilon)$. Choose δ_2 such that

$$-\alpha(\overline{\alpha}^{-1}(c)) + M\delta_2 < 0\,.$$

By construction, $x \in \Omega_c$ implies

$$\|x\| \le \varepsilon\,.$$

In fact,

$$\underline{\alpha}(\|x\|) \le V(x) \le c$$

implies

$$\|x\| \le \underline{\alpha}^{-1}(c) \le \underline{\alpha}^{-1}(\underline{\alpha}(\varepsilon)) = \varepsilon.$$

Also, at each point x of the boundary of Ω_c,

$$\alpha(\|x\|) \ge \alpha(\overline{\alpha}^{-1}(V(x))) = \alpha(\overline{\alpha}^{-1}(c))\,.$$

As a consequence, at each point x of the boundary of Ω_c,

$$\frac{\partial V}{\partial x}[f(x) + g(x,t)] \le -\alpha(\|x\|) + \|\frac{\partial V}{\partial x}\|\delta_2 \le -\alpha(\overline{\alpha}^{-1}(c)) + M\delta_2 < 0\,.$$

From this, it can be concluded that, for any initial condition in the interior of Ω_c, the solution $x(t)$ of (10.9) is defined for all $t \ge 0$ and is such that $x(t) \in \Omega_c$ for all $t \ge 0$ (see Remark 10.1.2). To complete the proof it suffices to choose $\delta_1 < \overline{\alpha}^{-1}(c)$, for this guarantees that x° is in the interior of Ω_c. In fact,

$$\overline{\alpha}^{-1}(V(x^\circ)) \le \|x^\circ\| \le \delta_1 < \overline{\alpha}^{-1}(c)$$

implies $V(x^\circ) < c$. ◁

Note that $g(x,t)$ is not assumed to be vanishing at $x = 0$. In other words, the perturbed system may not have an equilibrium at $x = 0$. Or the perturbed system may have an unstable equilibrium at $x = 0$ (as shown in the following example). Nevertheless, in both cases the theorem of total stability guarantees that, if the perturbation is small enough, the trajectories of the perturbed system remain arbitrarily close to the equilibrium $x = 0$ of the unperturbed system.

Example 10.2.1. Consider the system

$$\dot{x} = -x^3 + \gamma x \tag{10.10}$$

where $\gamma > 0$ is a small number. Indeed the "unperturbed" system $\dot{x} = -x^3$ has a globally asymptotically stable equilibrium at $x = 0$. On the other hand, the "perturbed system" has three equilibria, at $x = 0$, $x = +\sqrt{\gamma}$, $x = -\sqrt{\gamma}$. Elementary arguments, based on the principle of stability in the first approximation, show that, since γ is positive, the first one of these equilibria is unstable, while the other two are asymptotically stable. A glance at the graph of $-x^3 + \gamma x$ suggests that

$$\begin{array}{ccccccl}
0 & < & x(0) & < & +\sqrt{\gamma} & \Rightarrow & x(t) \to +\sqrt{\gamma} \\
+\sqrt{\gamma} & < & x(0) & < & +\infty & \Rightarrow & x(t) \to +\sqrt{\gamma} \\
-\sqrt{\gamma} & < & x(0) & < & 0 & \Rightarrow & x(t) \to -\sqrt{\gamma} \\
-\infty & < & x(0) & < & -\sqrt{\gamma} & \Rightarrow & x(t) \to -\sqrt{\gamma}\,.
\end{array}$$

Note that, even though the equilibrium $x = 0$ of the perturbed system is unstable, the boundedness properties indicated in the Theorem of total stability hold. In particular, given any $\varepsilon > 0$, choose $\delta_1 = \varepsilon$ and $\delta_2 = \varepsilon^3$. Then, the perturbation term γx satisfies

$$|\gamma x| \le \delta_2 \quad \text{for all } |x| \le \varepsilon$$

if $\sqrt{\gamma} \le \varepsilon$. If this is the case, the two nonzero equilibria of the system lie in the interval $[-\varepsilon, +\varepsilon]$, and we see form the previous analysis that any trajectory with initial condition $|x(0)| \le \delta_1 = \varepsilon$ remains confined to the set $|x| \le \varepsilon$. ◁

This theorem lends itself to an easy application to the study of the stability of the equilibrium of a pair of *cascade-connected* systems. More precisely, consider the composite system

$$\begin{aligned}
\dot{x} &= f(x,z) \\
\dot{z} &= g(z)
\end{aligned} \tag{10.11}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0) = 0$, and $f(x,z)$, $g(z)$ are locally Lipschitz on a neighborhood U of $(x,z) = (0,0)$. In view of the fact that the state z of the lower subsystem acts as an input to the upper subsystem, we shall sometimes refer to the latter as to the "driven subsystem" and to the former as to the "driving subsystem". As an immediate corollary of the theorem of total stability, it can be deduced that if the equilibrium $x = 0$ of the upper subsystem, driven by $z = 0$, is locally asymptotically stable and the equilibrium $z = 0$ of the lower subsystem is stable, then the equilibrium $(x,z) = (0,0)$ of the cascade is stable.

Corollary 10.2.2. *Consider system (10.11). Suppose the equilibrium* $x = 0$ *of* $\dot{x} = f(x,0)$ *is locally asymptotically stable and the equilibrium* $z = 0$ *of* $\dot{z} = g(z)$ *is stable. Then, the equilibrium* $(x,z) = (0,0)$ *of (10.11) is stable.*

Proof. By hypothesis, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|z^\circ\| \leq \delta$, the integral curve $z^\circ(t)$ of $\dot{z} = g(z)$ satisfying $z^\circ(0) = z^\circ$ is such that $\|z^\circ(t)\| \leq \varepsilon$ for all $t \geq 0$. Express $f(x, z^\circ(t))$ as

$$f(x, z^\circ(t)) = f(x, 0) + g(x, t) \tag{10.12}$$

where

$$g(x, t) = f(x, z^\circ(t)) - f(x, 0)\,.$$

Since $f(x, z)$ is locally Lipschitz, there exist $\eta > 0$ and $M > 0$ such that

$$\|g(x, t)\| \leq M\|z^\circ(t)\| \leq M\varepsilon$$

for all $\|x\| \leq \eta$, all $\|z^\circ\| \leq \delta$ and $t \geq 0$. This bound for $\|g(x, t)\|$ can be rendered arbitrarily small by choosing a sufficiently small δ and the result follows from Theorem 10.2.1. $\lhd$

10.3 Asymptotic Stability of Cascade-Connected Systems

In this section we investigate the *asymptotic* stability of the equilibrium $(x, z) = (0, 0)$ of a pair of cascade connected subsystems of the form

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z)\,, \end{aligned} \tag{10.13}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0) = 0$, and $f(x, z)$, $g(z)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ (see Fig. 10.1).

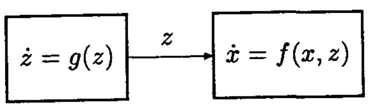

Fig. 10.1. Cascade connection.

The main result of the analysis is that local asymptotic stability of the equilibrium $x = 0$ of the upper subsystem, driven by $z = 0$, and local asymptotic stability of the equilibrium $z = 0$ of the lower subsystem always imply local asymptotic stability of the equilibrium $(x, z) = (0, 0)$ of the cascade. However, *global* asymptotic stability of the equilibrium $x = 0$ of the upper subsystem, driven by $z = 0$, and *global* asymptotic stability of the equilibrium $z = 0$ of the lower subsystem *do not* imply, in general, *global* asymptotic stability of the equilibrium $(x, z) = (0, 0)$ of the cascade. To infer global asymptotic stability of the cascade, a (strong) extra condition is needed, as shown below.

The key of the proof of these facts is the following result.

Theorem 10.3.1. *Consider system (10.13). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable. Let S be a set with the property that, for any $\bar{x}^\circ \in S$, the integral curve $\bar{x}(t)$ of $\dot{\bar{x}} = f(\bar{x}, 0)$ satisfying $\bar{x}(0) = \bar{x}^\circ$ is defined for all $t \geq 0$ and is such that*

$$\lim_{t\to\infty} \bar{x}(t) = 0\,.$$

Pick any z° and let $z^\circ(t)$ denote the integral curve of $\dot{z} = g(z)$ satisfying $z^\circ(0) = z^\circ$. Suppose $z^\circ(t)$ is defined for all $t \geq 0$ and such that

$$\lim_{t\to\infty} z^\circ(t) = 0\,.$$

Pick any $x^\circ \in S$ and let $x^\circ(t)$ denote the integral curve of $\dot{x} = f(x, z^\circ(t))$ satisfying $x^\circ(0) = x^\circ$. Suppose $x^\circ(t)$ is defined for all $t \geq 0$, is bounded and is such that $x^\circ(t) \in S$ for all $t \geq 0$. Then,

$$\lim_{t\to\infty} x^\circ(t) = 0\,.$$

Proof. First of all, observe that, using the theorem of total stability, it is easy to prove that, since the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable, given any ε, there exist δ_1 and δ_2 such that, if $\|\bar{x}^\circ\| \leq \delta_1$ and $\|z(t)\| \leq \delta_2$ for all $t \geq 0$, the solution $\bar{x}(t)$ of

$$\dot{x} = f(x, z(t))$$

satisfying $\bar{x}(0) = \bar{x}^\circ$ is such that $\|\bar{x}(t)\| \leq \varepsilon$ for all $t \geq 0$. To this end, it suffices to split $f(x, z(t))$ as in (10.12) and use arguments similar to those used in the proof of Corollary 10.2.2. As a consequence, the theorem will be proven if we can show that, for any ε, it is possible to find a time T such that $\|x^\circ(T)\| \leq \delta_1$, and $\|z^\circ(t)\| \leq \delta_2$ for all $t \geq T$.

Now, pick $T_1 > 0$, set

$$z^\circ_{T_1}(t) = \begin{cases} z^\circ(t) & \text{if } t \leq T_1 \\ 0 & \text{if } t > T_1. \end{cases}$$

and let $x^\circ_{T_1}(t)$ denote the integral curve of

$$\dot{x} = f(x, z^\circ_{T_1}(t))$$

satisfying $x^\circ_{T_1}(0) = x^\circ$. Indeed, $x^\circ_{T_1}(t) = x^\circ(t)$ for all $0 \leq t \leq T_1$. Moreover, for $t > T_1$, $x^\circ_{T_1}(t)$ is a solution of

$$\dot{x} = f(x, 0)$$

and hence tends to 0 as $t \to \infty$, because $x^\circ_{T_1}(T_1) \in S$. In particular, there exist T_2 such that

$$\|x^\circ_{T_1}(T_1 + T_2)\| \leq \frac{\delta_1}{2}\,. \tag{10.14}$$

The time T_2 thus characterized may depend on $x^\circ_{T_1}(T_1)$ and hence on T_1 but, since by hypothesis $x^\circ(t)$ is bounded, using a compactness argument one can conclude that there exists a T_2 depending only on x° for which the inequality (10.14) holds for any T_1.

Set
$$T = T_1 + T_2 .$$

We will show now that, if T_1 is large enough

(a) $\|z^\circ(t)\| \le \delta_2$ for all $t \ge T$, and

(b) $\|x^\circ(T) - x^\circ_{T_1}(T)\| \le \frac{\delta_1}{2}$,

and this will conclude the proof.

Property (a) is an immediate consequence of the fact that $z^\circ(t)$ converges to 0 as $t \to \infty$. To prove (b), we proceed as follows. Observe that $x^\circ(t)$ and $x^\circ_{T_1}(t)$ are integral curves of
$$\dot{x} = f(x, z^\circ(t))$$
and, respectively,
$$\dot{x} = f(x, 0)$$
satisfying $x^\circ(T_1) = x^\circ_{T_1}(T_1)$. Thus, for $t \ge T_1$,
$$x^\circ(t) = x^\circ(T_1) + \int_{T_1}^{t} f(x^\circ(s), z^\circ(s))ds$$
and
$$x^\circ_{T_1}(t) = x^\circ(T_1) + \int_{T_1}^{t} f(x^\circ_{T_1}(s), 0)ds .$$

Since $x^\circ(s), x^\circ_{T_1}(s), z^\circ(s)$ are defined for all $s \ge T_1$ and bounded and $f(x,z)$ is locally Lipschitz, there exist $L > 0$ and $M > 0$ such that
$$\|f(x^\circ(s), z^\circ(s)) - f(x^\circ_{T_1}(s), 0)\| \le L\|x^\circ(s) - x^\circ_{T_1}(s)\| + M\|z^\circ(s)\|$$
for all $s \ge T_1$. Thus,
$$\|x^\circ(t) - x^\circ_{T_1}(t)\| \le L\int_{T_1}^{t} \|x^\circ(s) - x^\circ_{T_1}(s)\|ds + M\int_{T_1}^{t} \|z^\circ(s)\|ds .$$
Since $z^\circ(s)$ converges to 0 as $t \to \infty$, given any $\delta > 0$, there is T_1 such that $\|z^\circ(s)\| \le \delta$ for all $t \ge T_1$. Thus, for all $t \ge T_1$ we can write
$$\|x^\circ(t) - x^\circ_{T_1}(t)\| \le L\int_{T_1}^{t} \|x^\circ(s) - x^\circ_{T_1}(s)\|ds + M\delta(t - T_1) .$$

Gronwall-Bellman's lemma yields
$$\|x^\circ(t) - x^\circ_{T_1}(t)\| \le \frac{M\delta}{L}(e^{L(t-T_1)} - 1)$$

and, thus,
$$\|x^\circ(T) - x^\circ_{T_1}(T)\| \le \frac{M\delta}{L}(e^{LT_2} - 1) .$$

In the right-hand side, M, L, T_2 are fixed numbers, but δ can be rendered arbitrarily small by choosing an appropriately large T_1. This proves (b) and completes the proof of the theorem. ◃

From this theorem it is easy to deduce the following corollaries, which express the result outlined at the beginning of the section.

Corollary 10.3.2. *Consider the system (10.13). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is locally asymptotically stable and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is locally asymptotically stable. Then, the equilibrium $(x,z) = (0,0)$ of (10.13) is locally asymptotically stable.*

Proof. Indeed, for some sufficiently small $\delta > 0$ one can choose $\mathcal{S} = B_\delta$. By Corollary 10.2.2, the equilibrium $(x,z) = (0,0)$ is stable. Thus, there exist δ_1 and δ_2 such that, if x° and z° satisfy
$$\|x^\circ\| \le \delta_1, \qquad \|z^\circ\| \le \delta_2 ,$$
then $x^\circ(t)$ belongs to B_δ. From this, the result follows. ◃

Corollary 10.3.3. *Consider the system (10.13). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is globally asymptotically stable and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is globally asymptotically stable. Suppose the integral curves of the composite system are defined for all $t \ge 0$ and bounded. Then, the equilibrium $(x,z) = (0,0)$ of (10.13) is globally asymptotically stable.*

Proof. In this case, $\mathcal{S} = \mathbb{R}^n$ and the hypothesis needed to show, using Theorem 10.3.1, that for any initial condition (x°, z°) the trajectory $(x^\circ(t), z^\circ(t))$ converges to zero as $t \to \infty$ is simply the boundedness of $x^\circ(t)$. ◃

10.4 Input-to-State Stability

In the previous section, we have discussed the asymptotic behavior of the response $x(t)$ of a system of the form
$$\dot{x} = f(x, z)$$
to an input $z(t)$ which was, in turn, the response of the autonomous nonlinear system
$$\dot{z} = g(z) .$$
Since the latter was assumed to be locally (respectively, globally) asymptotically stable, the input $z(t)$ driving the first system was a function of time

which, for sufficiently small $z(0)$ (respectively, for any $z(0)$), asymptotically decayed to 0 as $t \to \infty$.

In this section we extend this type of analysis, to the case in which the input driving the first system is simply a *bounded* function of time. Of course, we cannot expect anymore that the state $x(t)$ decays to 0 as $t \to \infty$; rather, we are interested in the case in which $x(t)$ remains bounded, and the bound on the state can be expressed as a (possibly nonlinear) function of the bound on the input. In the special case in which the input tends to 0 (in particular, when the input is identically zero), we still expect that $x(t)$ converges to 0 as time tends to ∞, as it happened in the earlier situation. These requirements altogether lead to the notion of *input-to-state stability*, which was introduced by E. Sontag [5] and can be formally characterized as follows.

Consider a nonlinear system

$$\dot{x} = f(x,u) \tag{10.15}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0,\infty) \to \mathbb{R}^m$ of (10.15) can be any piecewise continuous bounded function. The set of all such functions, endowed with the supremum norm

$$\|u(\cdot)\|_\infty = \sup_{t\geq 0} \|u(t)\|$$

is denoted by L_∞^m.

Definition 10.4.1. *System (10.15) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, called a gain function, such that, for any input $u(\cdot) \in L_\infty^m$ and any $x^\circ \in \mathbb{R}^n$, the response $x(t)$ of (10.15) in the initial state $x(0) = x^\circ$ satisfies*

$$\|x(t)\| \leq \beta(\|x^\circ\|, t) + \gamma(\|u(\cdot)\|_\infty) \tag{10.16}$$

for all $t \geq 0$.

Remark 10.4.1. Since, for any pair $\beta > 0, \gamma > 0$, $\max\{\beta,\gamma\} \leq \beta+\gamma \leq \max\{2\beta, 2\gamma\}$, it is seen that an alternative way to say that a system is input-to-state stable is that there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L_\infty^m$ and any $x^\circ \in \mathbb{R}^n$, the response $x(t)$ of (10.15) in the initial state $x(0) = x^\circ$ satisfies

$$\|x(t)\| \leq \max\{\beta(\|x^\circ\|, t), \gamma(\|u(\cdot)\|_\infty)\} \tag{10.17}$$

for all $t \geq 0$. Both estimates (10.16) and (10.17) will be indifferently used in the sequel. ◁

[5] See Sontag (1989).

In what follows, it will be shown that the property, for a given system, of being input-to-state stable, can be given a characterization which extends the well known criterion of Lyapunov for asymptotic stability. The key tool for this analysis is the notion of *ISS-Lyapunov function*, defined as follows.

Definition 10.4.2. *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (10.15) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\chi(\cdot)$ such that*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \tag{10.18}$$

and

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|) \quad \text{for all } x \in \mathbb{R}^n. \tag{10.19}$$

The reason why a function characterized in this way can play an important role in establishing the property of input-to-state stability can be explained as follows. First of all, observe that if $V(x)$ is an ISS-Lyapunov function for (10.15), evaluation of (10.19) at $u = 0$ yields

$$\frac{\partial V}{\partial x} f(x,0) \leq -\alpha(\|x\|) \quad \text{for all } x \in \mathbb{R}^n ,$$

and therefore $V(x)$ is a Lyapunov function in the usual sense of the term for the autonomous system $\dot{x} = f(x,0)$. Thus, as seen in Section 10.1, the response of (10.15) with $u(t) = 0$ for all $t \geq 0$, and arbitrary initial state $x(0) = x^\circ$, satisfies an estimate of the form

$$\|x(t)\| \leq \beta(\|x^\circ\|, t)$$

for all $t \geq 0$, where $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function, as expected from the definition of input-to-state stability.

Consider now the case of a nonzero input $u(\cdot) \in L_\infty^m$, let

$$M = \|u(\cdot)\|_\infty,$$

and set

$$c = \overline{\alpha}(\chi(M)) .$$

Then, from the inequality on the right-hand side of (10.18), it is deduced that the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$

is such that

$$B_{\chi(M)} \subset \Omega_c .$$

As a consequence, $\|x\| \geq \chi(M)$ at each x on the boundary of Ω_c and therefore, at any $t \geq 0$ such that $x(t)$ is on the boundary of Ω_c, $\|x(t)\| \geq \chi(\|u(t)\|)$. Suppose now that (10.19) holds. Then,

$$\frac{\partial V(x)}{\partial x} f(x(t), u(t)) < 0$$

at any $t \geq 0$ such that $x(t)$ is on the boundary of Ω_c, and (see Section 10.1), it can be concluded that for any initial condition $x'(0)$ in the interior of Ω_c, the solution $x'(t)$ of (10.15) is defined for all $t \geq 0$ and $x'(t) \in \Omega_c$ for all $t \geq 0$.

In particular, for all $t \geq 0$, $x'(t)$ satisfies

$$\|x'(t)\| \leq \underline{\alpha}^{-1}(V(x'(t))) \leq \underline{\alpha}^{-1}(c) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(M))) \,.$$

Setting

$$\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r) \,,$$

we see that, for all $t \geq 0$, $x'(t)$ satisfies

$$\|x'(t)\| \leq \gamma(\|u(\cdot)\|_\infty) \,, \tag{10.20}$$

or, what is the same,

$$\|x'(\cdot)\|_\infty \leq \gamma(\|u(\cdot)\|_\infty) \,.$$

Let now $x(t)$ be the solution of (10.15), satisfying $x(0) = x^\circ$. If $x^\circ \in \Omega_c$, the previous arguments show that $\|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty)$ for all $t \geq 0$ and this proves that the estimate (10.16) holds. If not, i.e. if $V(x^\circ) > c$, observe that, so long as $V(x(t)) > c$, $\|x(t)\| > \chi(\|u(t)\|)$ and this yields

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = \frac{\partial V}{\partial x} f(x(t), u(t)) \leq -\alpha(\|x(t)\|) < 0 \,.$$

Thus, so long as $V(x(t)) > c$, the function $V(x(t))$ is decreasing, and this shows in particular that $x(t)$ is bounded (in fact, $\|x(t)\| \leq \underline{\alpha}^{-1}(V(x(t))) \leq \underline{\alpha}^{-1}(V(x(0)))$). Moreover, it is possible to show that, for some finite time t_0,

$$\begin{aligned} &V(x(t)) > c, \quad \text{for all} \quad 0 \leq t < t_0 \\ &V(t_0) = c \,. \end{aligned}$$

By contradiction, suppose this is not the case. Then $V(x(t)) > c$ for all $t \geq 0$, and therefore,

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leq -\alpha(\overline{\alpha}^{-1}(V(x(t)))) \tag{10.21}$$

for all $t \geq 0$. In other words, the function $V(t) = V(x(t))$ is such that

$$\dot{V}(t) \leq -\alpha(\overline{\alpha}^{-1}(V(t))$$

for all $t \geq 0$ and this shows (see proof of Theorem 10.1.3) that, for some class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$,

$$\|x(t)\| \leq \beta(\|x^\circ\|, t) \tag{10.22}$$

for all $t \geq 0$. In particular, $x(t)$ tends to 0 as $t \to \infty$ and thus $V(x(t))$ tends to 0 as $t \to \infty$, which contradicts the assumption that $V(x(t))$ was bounded from below by $c > 0$ for all $t \geq 0$.

The inequality (10.21) holds for all $t \in [0, t_0)$. Then, an estimate of the form (10.22) holds for all $t \in [0, t_0)$. Since for $t \geq t_0$ we have

$$\|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty)$$

we conclude that

$$\|x(t)\| \leq \max\{\beta(\|x^\circ\|, t), \gamma(\|u(\cdot)\|_\infty)\}$$

for all $t \geq 0$, and this completes the proof of the fact that if $V(x)$ is an ISS-Lyapunov function for (10.15), then this system is input-to-state stable.

Remark 10.4.2. The previous argument shows also how the function $\gamma(\cdot)$ which appears in the estimate (10.17) can be computed from the functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, and $\chi(\cdot)$ which characterize (10.18) and (10.19). As a matter of fact, $\gamma(\cdot)$ can be given the expression

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r) \,. \triangleleft$$

The arguments above show that the existence of an ISS-Lyapunov function is a ***sufficient condition*** for input-to-state stability. As in the case of asymptotic stability, this result has also a ***converse***, namely the existence of a function $V(x)$ having the properties indicated in (10.18) and (10.19) is ***implied*** by the property of input-to-state stability [6]

Theorem 10.4.1. *System (10.15) is input-to-state stable if and only if there exists an ISS-Lyapunov function.*

There is an alternative way to check whether or not a function $V(x)$ is an ISS-Lyapunov function for a given system, which is useful in many circumstances.

Lemma 10.4.2. *Consider system (10.15). A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (10.15) if and only if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that (10.18) holds and*

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \qquad \textit{for all } x \in \mathbb{R}^n \textit{ and all } u \in \mathbb{R}^m \,. \tag{10.23}$$

Proof. Suppose (10.23) holds and define

$$\chi(r) = \alpha^{-1}(k\sigma(r)) \,,$$

with $k > 1$. Then, $\|x\| \geq \chi(\|u\|)$ implies

[6] For a proof of this result, see Sontag and Wang (1995).

$$\frac{1}{k}\alpha(\|x\|) \geq \sigma(\|u\|)$$

which in turn implies

$$\frac{\partial V}{\partial x}f(x,u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \leq -\frac{k-1}{k}\alpha(\|x\|)\,.$$

This shows that a relation of the form (10.19) holds.

Observe now that, if (10.19) holds and $\|x\| \geq \chi(\|u\|)$, then (10.23) holds for any $\sigma(\cdot)$. To complete the proof, define

$$\phi(r) = \max_{\|u\|=r, \|x\|\leq\chi(r)} \{\frac{\partial V}{\partial x}f(x,u) + \alpha(\chi(\|u\|)\}\,.$$

Then, for $\|x\| \leq \chi(\|u\|)$,

$$\frac{\partial V}{\partial x}f(x,u) \leq -\alpha(\|x\|) + \phi(\|u\|)\,.$$

Define

$$\sigma(r) = \max\{0, \phi(r)\}\,,$$

The function $\sigma(\cdot)$ is continuous, nonnegative, and $\sigma(0) = 0$. If $\sigma(\cdot)$ is not a class $\mathcal{K}$ function, majorize it by a class $\mathcal{K}$ function, to have property (10.23) fulfilled. $\triangleleft$

Remark 10.4.3. From the proof of the previous Lemma, it is possible to deduce how the gain function $\gamma(\cdot)$ which appears in the estimate (10.17) can be computed from the functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ and $\sigma(\cdot)$ which characterize (10.18) and (10.23). As a matter of fact, recall that if $V(x)$ is a (positive definite and proper) function satisfying (10.19), an estimate of the form (10.17) holds with a gain function $\gamma(\cdot)$ given by

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r)\,.$$

On the other hand, the proof of the previous Lemma shows that if $V(x)$ is a (positive definite and proper) function satisfying (10.23), then (10.19) holds for $\chi(r) = \alpha^{-1}(k\sigma(r))$ and $k > 1$. Thus, an estimate of the form (10.17) holds with a gain function $\gamma(\cdot)$ given by

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha^{-1} \circ k\sigma(r)$$

where k is any number satisfying $k > 1$. $\triangleleft$

The concepts introduced above are illustrated in the following simple examples.

Example 10.4.1. Consider a linear system

$$\dot{x} = Ax + Bu$$

and suppose that all the eigenvalues of the matrix A have negative real part. Let $P > 0$ denote the unique solution of the Lyapunov equation $PA + A^{\mathrm{T}}P = -I$, observe that the function $V(x) = x^{\mathrm{T}}Px$ satisfies

$$\underline{a}\|x\|^2 \leq V(x) \leq \overline{a}\|x\|^2$$

for suitable $\underline{a} > 0$ and $\overline{a} > 0$, and that

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\|x\|^2 + 2\|x\|\|P\|\|B\|\|u\|\,.$$

Pick any $0 < \varepsilon < 1$ and set

$$c = \frac{2}{1-\varepsilon}\|P\|\|B\|\,, \qquad \chi(r) = cr\,.$$

Then

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}(Ax + Bu) \leq -\varepsilon\|x\|^2\,.$$

Thus, the system is input-to-state stable, with a gain function

$$\gamma(r) = (\overline{a}/\underline{a})cr$$

which is a *linear* function. $\triangleleft$

Example 10.4.2. Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k + bx^p\varphi(u)\,,$$

in which $k \in \mathbb{I}$ is *odd*, $p \in \mathbb{I}$ satisfies

$$p < k\,, \tag{10.24}$$

$a > 0$, and $\varphi(\cdot)$ is a C^1 function satisfying $\varphi(0) = 0$.

Indeed, since $a > 0$ and k is odd, the system is globally asymptotically stable. Choose a candidate ISS-Lyapunov function as $V(x) = \frac{1}{2}x^2$, which yields

$$\dot{V} = \frac{\partial V}{\partial x}f(x,u) = -ax^{k+1} + bx^{p+1}\varphi(u)\,.$$

Let $\theta(\cdot)$ be a class $\mathcal{K}$ function satisfying

$$|\varphi(u)| \leq \theta(|u|)$$

for all $u \in \mathbb{R}$, and observe that, since $k+1$ is even

$$\dot{V} \leq -a|x|^{k+1} + |b||x|^{p+1}\theta(|u|)\,.$$

Set

$$\nu = k - p$$

and obtain

$$\dot{V} \le |x|^{p+1}\Big(-a|x|^{\nu} + |b|\theta(|u|)\Big) .$$

Thus, using the class $\mathcal{K}_\infty$ function

$$\alpha(r) = \varepsilon r^{k+1} ,$$

in which $\varepsilon > 0$, it is deduced that

$$\dot{V} \le -\alpha(|x|)$$

if

$$-a|x|^{\nu} + |b|\theta(|u|) \le -\varepsilon|x|^{\nu} ,$$

i.e. if

$$(a - \varepsilon)|x|^{\nu} \ge |b|\theta(|u|) .$$

Taking, without loss of generality, $\varepsilon < a$, it is concluded that condition (10.19) holds for the class $\mathcal{K}$ function

$$\chi(r) = \left(\frac{|b|\theta(r)}{a-\varepsilon}\right)^{\frac{1}{\nu}} .$$

Thus, the system is input-to-state stable, with a gain function $\gamma(\cdot)$ which is bounded by this function $\chi(\cdot)$. ◁

Example 10.4.3. Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k \mathrm{sgn}(x) + bx^p\varphi(u) ,$$

in which $k \in \mathbb{I}$ is *even*, $p \in \mathbb{I}$ satisfies

$$p < k ,$$

$a > 0$, and $\varphi(\cdot)$ is a C^1 function satisfying $\varphi(0) = 0$. Note that the function $x^k\mathrm{sgn}(x)$ is a C^1 function, for any even integer $k > 0$.

Choose the same candidate ISS-Lyapunov function as in the previous example, $V(x) = \frac{1}{2}x^2$, to obtain

$$\dot{V} = \frac{\partial V}{\partial x}f(x,u) = -ax^{k+1}\mathrm{sgn}(x) + bx^{p+1}\varphi(u) .$$

This, since $k+1$ is odd, yields the same inequality found in the previous example, namely

$$\dot{V} \le -a|x|^{k+1} + |b||x|^{p+1}\theta(|u|) .$$

from which identical conclusions follow. ◁

Example 10.4.4. An important feature of the two previous examples, which made it possible to prove that both systems were input-to-state stable, was the inequality (10.24). In fact, if this inequality does not hold, the system may fail to be input-to-state stable. This can be seen, for instance, in the simple example

$$\dot{x} = -x + xu .$$

In fact, suppose $u(t) = 2$ for all $t \ge 0$. The state response of the system, to this input, from the initial state $x(0) = x^\circ$ coincides with that of the autonomous system

$$\dot{x} = x ,$$

i.e. $x(t) = e^t x^\circ$, which shows that the bound (10.16) cannot hold. ◁

Example 10.4.5. Let $n = 2$, $m = 1$ and consider the system

$$\begin{aligned} \dot{z} &= -z^3 + zy \\ \dot{y} &= az^2 - y + u . \end{aligned}$$

where a is a real parameter. To check whether or not this system might be input-to-state stable, for some value of a, choose the candidate ISS-Lyapunov function

$$V(z,y) = \frac{1}{2}(z^2 + y^2)$$

to obtain

$$\dot{V} = \frac{\partial V}{\partial x}f(x,u) = -z^4 + (1+a)z^2y - y^2 + yu .$$

Recall that

$$z^2y \le \frac{1}{2}z^4 + \frac{1}{2}y^2$$

and that, for any number $\delta > 0$,

$$yu \le \frac{\delta}{2}y^2 + \frac{1}{2\delta}u^2 .$$

Thus, it is seen that

$$\dot{V} \le (-1 + \frac{|1+a|}{2})z^4 + (-1 + \frac{|1+a|}{2} + \frac{\delta}{2})y^2 + \frac{1}{2\delta}u^2 .$$

From this, it is easy to conclude that the system is input-to-state stable if $|a| < 1$. In fact, if this is the case, in the previous inequality the coefficient

$$-1 + \frac{|1+a|}{2}$$

of z^4 is negative, and it is possible to find $\delta > 0$ such that also the coefficient

$$-1 + \frac{|1+a|}{2} + \frac{\delta}{2}$$

of y^2 is negative. In other words, there exist numbers $d_1 > 0$, $d_2 > 0$ such that
$$\dot{V} \leq -(d_1 z^4 + d_2 y^2) + \frac{1}{2\delta}u^2 .$$

Consider now the function
$$W(z,y) = d_1 z^4 + d_2 y^2 .$$

This function is positive definite and also proper because (see Remark 10.1.3), for any $c > 0$, the closed set
$$\Omega_c = \{(z,y) \in \mathbb{R}^2 : W(z,y) \leq c\}$$
is bounded. In fact
$$W(z,y) \leq c \quad \Rightarrow \quad |z| \leq \left(\frac{c}{d_1}\right)^{\frac{1}{4}}, |y| \leq \left(\frac{c}{d_2}\right)^{\frac{1}{2}} .$$

Therefore, there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that
$$\alpha(\|x\|) \leq W(z,y)$$
for all $x \in \mathbb{R}^2$. As a consequence,
$$\dot{V} \leq -\alpha(\|x\|) + \frac{1}{2\delta}|u|^2$$
and it is concluded that an inequality of the form (10.23) holds, for
$$\sigma(r) = \frac{1}{2\delta}r^2 .$$
This shows that, if $|a| < 1$, the system is input-to-state stable. ◁

As shown, for instance, in the third of these examples, in a system of the form (10.15), the global asymptotic stability of the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ *does not* necessarily imply input-to state stability. Nevertheless, it is always possible *to render* such a system input-to-state stable by means of a *feedback transformation* of the form
$$u = \beta(x)v$$
where $\beta(x)$ is an $m \times m$ matrix of smooth functions of x, defined and invertible for all $x \in \mathbb{R}^n$. Note that feedback transformations of this kind, which actually correspond to x-dependent changes of coordinates in the space of input values, have been encountered several times before in the design of nonlinear control systems.

Theorem 10.4.3. *Consider system (10.15). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is globally asymptotically stable. Then, there exists an $m \times m$ matrix $\beta(x)$ of smooth functions of x, which is defined for all $x \in \mathbb{R}^n$ and is nonsingular for all x, such that*
$$\dot{x} = f(x, \beta(x)u) \tag{10.25}$$
is input-to-state stable.

Proof. By the converse Lyapunov theorem, there exists a C^1 positive definite and proper function $V(x)$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that
$$\frac{\partial V}{\partial x} f(x,0) < -\alpha(\|x\|) \quad \text{for all nonzero } x \in \mathbb{R}^n.$$

We will prove the theorem by showing that, for some $\beta(x)$ having the properties indicated above and for some class $\mathcal{K}$ function $\chi(\cdot)$,
$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x,\beta(x)u) \leq -\frac{1}{2}\alpha(\|x\|) \quad \text{for all } x \in \mathbb{R}^n,$$
thus showing that $V(x)$ is an ISS-Lyapunov function for (10.25).

To this end, let $g(\cdot)$ be any smooth function $g : [0,\infty) \to [0,\infty)$, which is positive for all $s \geq 0$ and identically equal to 1 on the interval $[0,1]$, and set
$$\beta(x) = g(\|x\|)I .$$

The matrix $\beta(x)$ thus defined is a matrix of smooth functions of x, invertible for all $x \in \mathbb{R}^n$.

Define
$$\delta(s,r) = \max_{\|x\|=s, \|v\|=r} \{\frac{\partial V}{\partial x} f(x,v) + \frac{1}{2}\alpha(s)\} ,$$
and observe that, for any choice of $g(\cdot)$,
$$\frac{\partial V}{\partial x} f(x,\beta(x)u) + \frac{1}{2}\alpha(\|x\|) \leq \delta(\|x\|, g(\|x\|)\|u\|) , \tag{10.26}$$
for all x and u.

The function $\delta(s,r)$, a continuous function defined for all $(s,r) \in [0,\infty) \times [0,\infty)$, by hypothesis is such that $\delta(s,0) < 0$ for every $s > 0$. Thus, using continuity arguments, it is possible to prove the existence of a continuous function $\rho(s)$, defined for all $s \geq 0$, with $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$, such that
$$r \leq \rho(s) \quad \Rightarrow \quad \delta(s,r) < 0$$
for all $s > 0$. In view of this and of (10.26), the result is proven if one can show the existence of a function $\chi(\cdot)$ and complete the definition of $g(\cdot)$ so that
$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad g(\|x\|)\|u\| \leq \rho(\|x\|) . \tag{10.27}$$

Let $\theta(\cdot)$ be any class $\mathcal{K}_\infty$ function which satisfies

$$\begin{array}{ll} \theta(s) < \rho(s) & \text{for } s \leq 2 \\ \theta(s) < s & \text{for } s \geq 2, \end{array}$$

and set

$$\chi(s) = \theta^{-1}(s) \ .$$

Moreover, let $g(\cdot)$ be such that

$$\begin{array}{ll} g(s) \leq 1 & \text{for all } s \\ g(s) < \dfrac{\rho(s)}{s} & \text{for } s \geq 2 \ . \end{array}$$

By construction, the functions $\theta(\cdot)$ and $g(\cdot)$ are such that

$$g(s)\theta(s) \leq \rho(s) \quad \text{for all } s > 0 \ .$$

Observe now that

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \theta(\|x\|) \geq \|u\| \ .$$

Therefore,

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad g(\|x\|)\|u\| \leq g(\|x\|)\theta(\|x\|) \leq \rho(\|x\|)$$

which shows that (10.27) holds and this completes the proof. ◁

The notion of input-to-state stability lends itself to a number of alternative (equivalent) characterizations, among which the most relevant one can be derived as follows. Recall (see Remark 10.4.1) that a system is input-to-state stable if the response $x(\cdot)$ to an input $u(\cdot) \in L_\infty^m$ satisfies an estimate of the form (10.17). Observe that, for any $t \geq 0$,

$$\beta(\|x^\circ\|, t) \leq \beta(\|x^\circ\|, 0)$$

and $\beta(\cdot, 0)$ is a class $\mathcal{K}$ function. Then, using (10.17), it is seen that, in an input-to-state stable system, the response $x(\cdot)$ to any input $u(\cdot) \in L_\infty^m$ is bounded and, in particular,

$$\|x(\cdot)\|_\infty \leq \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\} \tag{10.28}$$

for some class $\mathcal{K}$ function $\gamma_0(\cdot)$ (while $\gamma(\cdot)$ is the same class $\mathcal{K}$ function for which the estimate (10.17) holds). Moreover, since

$$\lim_{t\to\infty} \beta(\|x^\circ\|, t) = 0 \ ,$$

in an input-to-state stable system the response $x(t)$ to any input $u(\cdot) \in L_\infty^m$ satisfies

$$\limsup_{t\to\infty} \|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty) \tag{10.29}$$

(where $\gamma(\cdot)$ is again the same class $\mathcal{K}$ function for which the estimate (10.17) holds).

The estimate (10.29) can be given an alternative characterization, which only involves the behavior of $\|u(t)\|$ for large t. In fact, the following result holds.

Lemma 10.4.4. *Property (10.29) is equivalent to the property*

$$\limsup_{t\to\infty} \|x(t)\| \leq \gamma(\limsup_{t\to\infty} \|u(t)\|) \ . \tag{10.30}$$

Proof. Since

$$\gamma(\limsup_{t\to\infty} \|u(t)\|) \leq \gamma(\|u(\cdot)\|_\infty) \ ,$$

then (10.30) implies (10.29), actually with the same function $\gamma(\cdot)$. Conversely, suppose (10.29) holds. Pick any $x^\circ \in \mathbb{R}^n$, any $u(\cdot) \in L_\infty^m$ and $\varepsilon > 0$. Let

$$r = \limsup_{t\to\infty} \|u(t)\|$$

and let $h > 0$ be such that

$$\gamma(r + h) - \gamma(r) < \varepsilon \ .$$

By definition of r, there exists $T > 0$ such that $\|u(t)\| \leq r + h$ for all $t \geq T$.

Let $\tilde{x}(t)$ denote the response of system (10.15) from the initial state $\tilde{x}^\circ = x(T)$ and input $\tilde{u}(\cdot)$ defined as

$$\tilde{u}(t) = u(t + T) \ .$$

Clearly, $\tilde{x}(t) = x(t + T)$, where $x(\cdot)$ is the response form the initial state x° and input $u(\cdot)$. By definition of T,

$$\|\tilde{u}(t)\| \leq r + h \qquad \text{for all } t \geq 0$$

i.e. $\|\tilde{u}(\cdot)\|_\infty \leq r + h$. Then (10.29) implies

$$\limsup_{t\to\infty} \|x(t)\| = \limsup_{t\to\infty} \|\tilde{x}(t)\| \leq \gamma(\|\tilde{u}(\cdot)\|_\infty) \leq \gamma(r + h) < \gamma(r) + \varepsilon \ .$$

Letting $\varepsilon \to 0$ yields (10.30), with the same $\gamma(\cdot)$ as in (10.29). ◁

Remark 10.4.4. From the proof of this Lemma it is deduced that, if (10.17) holds, then also (10.28) and (10.30) hold, with the *same* class $\mathcal{K}$ function $\gamma(\cdot)$. Thus, in particular, if it is known that $V(x)$ is an ISS-Lyapunov function for system (10.15), from Remark 10.4.2 it follows that the response $x(\cdot)$ to any input $u(\cdot) \in L_\infty^m$ is such that the estimates (10.28) and (10.30) hold, with $\gamma(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$. ◁

The arguments above show that if the estimate (10.17) holds, then necessarily also (10.28) and (10.30) hold, for the same class $\mathcal{K}$ function $\gamma(\cdot)$ and the class $\mathcal{K}$ function $\gamma_0(\cdot) = \beta(\cdot, 0)$. However, it turns out that the properties expressed by these two inequalities are not just implications, but *equivalent characterizations*, of the property of input-to-state stability [7].

Theorem 10.4.5. *System (10.15) is input-to-state stable if and only if there exists class $\mathcal{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L_\infty^m$ and any $x^\circ \in \mathbb{R}^n$, the response $x(t)$ in the initial state $x(0) = x^\circ$ satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \gamma(\limsup_{t\to\infty} \|u(t)\|) . \end{aligned}$$

Note that, if (10.17) holds, then – as shown before – also (10.28) and (10.30) hold, with the *same* class $\mathcal{K}$ function $\gamma(\cdot)$. Theorem 10.4.5 above says that the fulfillment of estimates of the form (10.28) and (10.30) implies input-to-state stability, i.e the fulfillment of an estimate of the form (10.17). However, is important to stress that this may occur for a possibly *different* function $\gamma(\cdot)$. This can be seen, for instance, in the following example.

Example 10.4.6. Consider the system

$$\dot{x} = -\frac{1}{1+u^2}x ,$$

for which

$$x(t) = x(0)\exp\left(-\int_0^t \frac{1}{1+u^2(\tau)}d\tau\right) .$$

Clearly,

$$|x(t)| \leq |x(0)| \qquad \text{and} \qquad \lim_{t\to\infty} x(t) = 0$$

for any $u(\cdot) \in L_\infty$ and any $x(0)$, so that (10.28) and (10.30) hold for $\gamma_0(s) = s$ and $\gamma(s) = 0$. However, an estimate of the form (10.17) with $\gamma(\cdot) = 0$ cannot hold, because the rate of converge of $|x(t)|$ to zero decreases as $\|u(\cdot)\|_\infty$ increases.

It is easy to see that $V(x) = x^4$ is an ISS-Lyapunov function for this system. In fact, observe that, for any $\varepsilon > 0$,

$$|x| \geq \varepsilon|u| \quad \Rightarrow \quad \frac{4\varepsilon^2 x^4}{\varepsilon^2 + x^2} \leq \frac{4x^4}{1+u^2} .$$

Thus, the two class $\mathcal{K}_\infty$ functions

$$\chi(r) = \varepsilon r \qquad \alpha(r) = \frac{4\varepsilon^2 r^4}{\varepsilon^2 + r^2}$$

[7] For a proof of this result, see Sontag and Wang (1996).

are such that

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}\left(-\frac{1}{1+u^2}x\right) = -\frac{4x^4}{1+u^2} \leq -\alpha(|x|) .$$

Thus, an estimate of the form (10.17) holds, in which $\gamma(\cdot)$ can be given the expression $\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r)$. Since one can pick $\underline{\alpha}(r) = \overline{\alpha}(r) = r^4$, this yields

$$\gamma(r) = \varepsilon r ,$$

where ε is any small (but nonzero) positive number. ◁

10.5 Input-to-State Stability of Cascade-Connected Systems

In this section we discuss a problem similar to that considered in Section 10.3, this time referred to the property of input-to-state stability. More precisely, we shall consider the cascade connection of two subsystems, each one being input-to-state stable, and we will prove that the composed system is still input-to-state stable. As a preliminary step in the analysis, we first discuss a property of input-to-state stable systems, which will be used later in the proof of the main result of this section, but that also has its own independent interest.

Recall that, according to the results discussed in the previous section and, in particular, to Lemma 10.4.2, a system

$$\dot{x} = f(x, u)$$

is input-to-state stable if there exists a continuously differentiable function $V(x)$ satisfying (10.18), a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \tag{10.31}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

The main significance of the latter inequality is essentially the following one. Consider a bounded input $u(\cdot)$, let $\varepsilon > 0$ be any small number. Set

$$d = \alpha^{-1}(\sigma(\|u(\cdot)\|_\infty) + \varepsilon)$$

and

$$c(d) = \max_{\|x\| \leq d} V(x) .$$

By definition,

$$\Omega_{c(d)} \supset \bar{B}_d .$$

If (10.31) holds, the derivative of $V(x(t))$ is strictly negative at each $x(t)$ such that $\|x(t)\| \geq d$ (thus, in particular, on the boundary of $\Omega_{c(d)}$). In fact, at all such points,

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \le -\alpha(\|x(t)\|) + \sigma(\|u(t)\|) \le -\alpha(d) + \sigma(\|u(\cdot)\|_\infty) = -\varepsilon\ .$$

This shows that, under this input, for any trajectory $x(t)$ there is a time t_0 such that $V(x(t)) \le c(d)$ for all $t \ge t_0$. In other words, the composed class $\mathcal{K}$ function $\alpha^{-1}(\sigma(\cdot))$ characterizes how the bound $\|u(\cdot)\|_\infty$ on the input function determines the (finite) number $c(d)$ by which $V(x(t))$ is guaranteed to be bounded for suitably large times.

In view of this, it appears that only the composition of the two functions $\alpha^{-1}(\cdot)$ and $\sigma(\cdot)$ matters in establishing a correspondence between a bound on the input and a bound on the state, and that, of course, infinitely many other such pairs might yield the same result. Motivated by this observation, we will examine below the problem of constructing families of pairs of functions $\alpha(\cdot)$ and $\sigma(\cdot)$ rendering an inequality of the form (10.31) true for a given input-to-state system. For convenience, we will say that a pair $\{\alpha(\cdot), \sigma(\cdot)\}$, in which the former is a class $\mathcal{K}_\infty$ function and the latter a class $\mathcal{K}$ function, is an *ISS-pair* for system (10.15) if, for some C^1 function $V(x)$ satisfying estimates of the form (10.18), the inequality (10.31) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Lemma 10.5.1. *Assume $\{\alpha(\cdot), \sigma(\cdot)\}$ is an ISS-pair for (10.15).*

(i) Let $\tilde\sigma(\cdot)$ be a class $\mathcal{K}$ function such that $\sigma(r) = \mathcal{O}[\tilde\sigma(r)]$ as $r \to \infty$. Then, there exists a class $\mathcal{K}_\infty$ function $\tilde\alpha(\cdot)$ such that $\{\tilde\alpha(\cdot), \tilde\sigma(\cdot)\}$ is an ISS-pair.

(ii) Let $\tilde\alpha(\cdot)$ be a class $\mathcal{K}_\infty$ function such that $\tilde\alpha(r) = \mathcal{O}[\alpha(r)]$ as $r \to 0^+$. Then, there exists a class $\mathcal{K}$ function $\tilde\sigma(\cdot)$ such that $\{\tilde\alpha(\cdot), \tilde\sigma(\cdot)\}$ is an ISS-pair.

Proof. For both parts of the theorem, the proof will be conducted by considering a candidate ISS-Lyapunov function $W(x)$ of the form

$$W(x) = \rho(V(x))$$

where $\rho(\cdot)$ is a class $\mathcal{K}_\infty$ function defined by an integral of the form

$$\rho(s) = \int_0^s q(t)dt$$

in which $q(\cdot)$ is a smooth function $[0,\infty) \to [0,\infty)$, which is non-decreasing and such that $q(s) > 0$ for $s > 0$ (the class of all such functions is often denoted as $\mathcal{SN}$). For this function, we wish to obtain an inequality of the form (10.31). To this end, observe that

$$\frac{\partial W}{\partial x} f(x,u) = q[V(x)]\frac{\partial V}{\partial x} f(x,u) \le q[V(x)][-\alpha(\|x\|) + \sigma(\|u\|)]\ . \tag{10.32}$$

Set $\theta(s) = \overline{\alpha}(\alpha^{-1}(2\sigma(s)))$. Then, it is easy to see that the right-hand side of (10.32) is bounded by

$$-\frac{1}{2}q[V(x)]\alpha(\|x\|) + q[\theta(\|u\|)]\sigma(\|u\|)\ . \tag{10.33}$$

In fact, this is indeed the case (no matter what $\theta(\cdot)$ is) if $\alpha(\|x\|) \ge 2\sigma(\|u\|)$. If, $\alpha(\|x\|) \le 2\sigma(\|u\|)$, observe that $V(x) \le \overline{\alpha}(\|x\|) \le \theta(\|u\|)$, in which case the right-hand side of (10.32) is bounded by the quantity $-q[V(x)]\alpha(\|x\|) + q[\theta(\|u\|)]\sigma(\|u\|)$.

The quantity (10.33) can in turn be bounded by

$$-\frac{1}{2}q[\underline{\alpha}(\|x\|)]\alpha(\|x\|) + q[\theta(\|u\|)]\sigma(\|u\|)\ ,$$

and, therefore, part (i) is proven if one can show that it is possible to find $q(\cdot)$ and $\tilde\alpha(\cdot)$ such that

$$\begin{aligned} q[\underline{\alpha}(s)]\alpha(s) &\ge 2\tilde\alpha(s) \\ q[\theta(r)]\sigma(r) &\le \tilde\sigma(r)\ . \end{aligned} \tag{10.34}$$

Assume, without loss of generality, that $\sigma(\cdot)$ is class $\mathcal{K}_\infty$ (if this is not the case, majorize it by a class $\mathcal{K}_\infty$ function), so that also $\theta(\cdot)$ is class $\mathcal{K}_\infty$ and define

$$\beta(r) = \sigma(\theta^{-1}(r)), \qquad \tilde\beta(r) = \tilde\sigma(\theta^{-1}(r))\ .$$

Both these functions are class $\mathcal{K}_\infty$ and, since $\sigma(r) = \mathcal{O}[\tilde\sigma(r)]$ as $r \to \infty$, also $\beta(r) = \mathcal{O}[\tilde\beta(r)]$ as $r \to \infty$. Using this property, it is easy to see that there exists a class $\mathcal{SN}$ function $q(\cdot)$ such that

$$q(r)\beta(r) \le \tilde\beta(r)$$

for all $r \in [0,\infty)$. Thus,

$$q[\theta(r)]\sigma(r) \le \tilde\sigma(r)\ . \tag{10.35}$$

Define

$$\tilde\alpha(s) = \frac{1}{2}q[\underline{\alpha}(s)]\alpha(s)\ . \tag{10.36}$$

This is a class $\mathcal{K}_\infty$ function, because so is $\alpha(\cdot)$ and $q(\cdot)$ is of class $\mathcal{SN}$. Indeed, (10.35) and (10.36) prove (10.34) and this completes the proof of (i).

To prove part (ii), we need to find $q(\cdot)$ and $\tilde\sigma(\cdot)$ such that (10.34) holds. To this end, define

$$\beta(r) = \frac{1}{2}\alpha(\theta^{-1}(s)), \qquad \tilde\beta(r) = \tilde\alpha(\theta^{-1}(s))\ .$$

These functions are such that $\tilde\beta(r) = \mathcal{O}[\beta(r)]$ as $r \to 0^+$. Using this property, it is easy to see that there exists a class $\mathcal{SN}$ function $q(\cdot)$ such that

$$\tilde\beta(s) \le q(s)\beta(s)$$

for all $s \in [0,\infty)$. Thus

$$-\frac{1}{2}q[\theta(s)]\alpha(s) \le -\tilde\alpha(s)\ . \tag{10.37}$$

Define

$$\tilde\sigma(r) = q[\theta(r)]\sigma(s)\ , \tag{10.38}$$

which is a class $\mathcal{K}_\infty$ function. Indeed, (10.37) and (10.38) prove (10.34) and this completes the proof of (ii). ◁

As an application of this result, we find the proof of the property that the cascade connection of two input-to-state stable systems is input-to-state stable. More precisely, consider a system of the form

$$\begin{aligned} \dot{x} &= f(x,z) \\ \dot{z} &= g(z,u)\,, \end{aligned} \tag{10.39}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0,0) = 0$, and $f(x,z)$, $g(z,u)$ are locally Lipschitz (see Fig. 10.2).

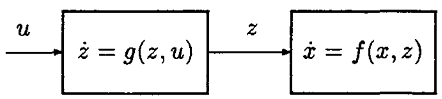

Fig. 10.2. Cascade connection with input.

Theorem 10.5.2. *Suppose that system*

$$\dot{x} = f(x,z)\,, \tag{10.40}$$

viewed as a system with input z and state x is input-to-state stable and that system

$$\dot{z} = g(z,u)\,, \tag{10.41}$$

viewed as a system with input u and state z is input-to-state stable as well. Then, system (10.39) is input-to-state stable.

Proof. By hypothesis, there exist an ISS-pair $\{\alpha(\cdot),\sigma(\cdot)\}$ for system (10.40) and an ISS-pair $\{\beta(\cdot),\zeta(\cdot)\}$ for system (10.41). Define a function $\tilde{\beta}(\cdot)$ in the following way

$$\tilde{\beta}(s) = \begin{cases} \beta(s) & \text{for small } s \\ \sigma(s) & \text{for large } s. \end{cases}$$

Then, by Lemma 10.5.1, part (ii), there exists $\tilde{\zeta}(\cdot)$ such that $\{\tilde{\beta}(\cdot),\tilde{\zeta}(\cdot)\}$ is an ISS-pair for system (10.41). Also, define a function $\tilde{\sigma}(\cdot)$ as

$$\tilde{\sigma}(s) = \frac{1}{2}\tilde{\beta}(s)\,.$$

Then, by Lemma 10.5.1, part (i), there exists $\tilde{\alpha}(\cdot)$ such that $\{\tilde{\alpha}(\cdot),\frac{1}{2}\tilde{\beta}(\cdot)\}$ is an ISS-pair for system (10.40).

As a consequence, there exist positive definite and proper functions $V(x)$ and $U(z)$, such that

$$\begin{aligned} \frac{\partial V}{\partial x} f(x,z) &\leq -\tilde{\alpha}(\|x\|) + \tfrac{1}{2}\tilde{\beta}(\|z\|) \\ \frac{\partial U}{\partial z} g(z,u) &\leq -\tilde{\beta}(\|z\|) + \tilde{\zeta}(\|u\|)\,. \end{aligned}$$

The composite function $W(x,z) = V(x) + U(z)$ satisfies

$$\frac{\partial W}{\partial x} f(x,z) + \frac{\partial W}{\partial z} g(z,u) \leq -\tilde{\alpha}(\|x\|) - \frac{1}{2}\tilde{\beta}(\|z\|) + \tilde{\zeta}(\|u\|)\,.$$

and therefore $W(x,z)$ is an ISS-Lyapunov function for the composite system (10.39). ◁

As an immediate corollary of this theorem, it is possible to derive a result analogous to that established with Corollary 10.3.3.

Corollary 10.5.3. *Consider system (10.13). Suppose that system*

$$\dot{x} = f(x,z)\,,$$

viewed as a system with input z and state x is input-to-state stable and that the equilibrium $z = 0$ of

$$\dot{z} = g(z)\,,$$

is globally asymptotically stable. Then, the equilibrium $(x,z) = (0,0)$ of system (10.13) is globally asymptotically stable.

Proof. The driving system is trivially input-to-state stable and so is the cascade (10.13), by Theorem 10.5.2. The latter, which has no input, is therefore globally asymptotically stable. ◁

Another application of this theorem is the following one. Suppose a system

$$\dot{x} = f(x,u)$$

has an ISS-pair $\{\alpha(\cdot),\sigma(\cdot)\}$ in which, for some integer $q > 0$,

$$\sigma(r) = \mathcal{O}[r^q] \qquad \text{as } r \to \infty.$$

Then, for any $K > 0$, the system in question has an ISS-pair $\{\tilde{\alpha}(\cdot),\tilde{\sigma}(\cdot)\}$ in which $\tilde{\sigma}(r) = Kr^q$. Accordingly, for some positive definite and proper C^1 function $V(x)$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -\tilde{\alpha}(\|x\|) + K\|u\|^q\,. \tag{10.42}$$

Suppose now the input function $u : [0,\infty) \to \mathbb{R}^m$ is a piecewise continuous function satisfying

$$\lim_{T\to\infty} \int_0^T \|u(t)\|^q dt < \infty\,.$$

The set of all such functions, endowed with the norm

$$\|u(\cdot)\|_q = \left(\int_0^\infty \|u(t)\|^q dt\right)^{\frac{1}{q}},$$

is denoted by L_q^m.

Integration of (10.42) on the interval $[0,t]$ yields

$$V(x(t)) - V(x^\circ) \le K\int_0^t \|u(s)\|^q ds \le K\int_0^\infty \|u(s)\|^q ds$$

and therefore

$$V(x(t)) \le V(x^\circ) + K\Big[\|u(\cdot)\|_q\Big]^q,$$

which in turn, using the estimates (10.18) yields

$$\|x(t)\| \le \underline{\alpha}^{-1}\Big(\overline{\alpha}(\|x^\circ\|) + K\Big[\|u(\cdot)\|_q\Big]^q\Big).$$

This shows that the response $x(t)$ to any input $u(\cdot) \in L_q^m$, in any initial state x°, exists for all $t \ge 0$ and is bounded, by a quantity which depends on the norm of the initial state and on the norm of the input function.

10.6 The "Small-Gain" Theorem for Input-to-State Stable Systems

In this section we investigate the stability property of *feedback interconnected* nonlinear systems, and we will see that the property of input-to-state stability lends itself to a simple characterization of an important *sufficient condition* under which the feedback interconnection of two globally asymptotically stable systems remains globally asymptotically stable.

Consider the following interconnected system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2, u), \end{aligned} \tag{10.43}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$ and $f_1(0,0) = 0$, $f_2(0,0,0) = 0$ (see Fig. 10.3). We suppose that the first subsystem, viewed as a system with internal state x_1 and input x_2 is input-to-state stable. Likewise, we suppose the second subsystem, viewed as a system with internal state x_2 and inputs x_1 and u is input-to-state stable.

In view of the results discussed at the end of Section 10.4, the hypothesis of input-to-state stability of the first subsystem is equivalent to the existence of two class $\mathcal{K}$ functions $\gamma_{01}(\cdot)$, $\gamma_1(\cdot)$ such that the response $x_1(\cdot)$ to any input $x_2(\cdot) \in L_\infty^{n_2}$ satisfies

$$\|x_1(t)\| \le \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2(\cdot)\|_\infty)\} \tag{10.44}$$

for all $t \ge 0$, and

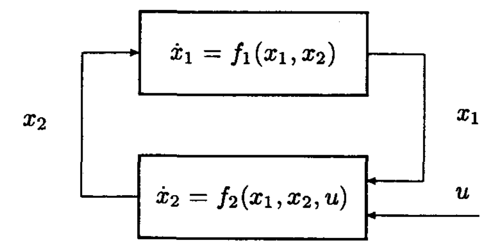

Fig. 10.3. Feedback connection with input.

$$\limsup_{t\to\infty} \|x_1(t)\| \le \gamma_1(\limsup_{t\to\infty} \|x_2(t)\|). \tag{10.45}$$

Likewise the hypothesis of input-to-state stability of the second subsystem is equivalent to the existence of three class $\mathcal{K}$ functions $\gamma_{02}(\cdot)$, $\gamma_2(\cdot)$, $\gamma_u(\cdot)$ such that the response $x_2(\cdot)$ to any input $x_1(\cdot) \in L_\infty^{n_1}$, $u(\cdot) \in L_\infty^m$ satisfies

$$\|x_2(t)\| \le \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\} \tag{10.46}$$

for all $t \ge 0$, and

$$\limsup_{t\to\infty} \|x_2(t)\| \le \max\{\gamma_2(\limsup_{t\to\infty} \|x_1(t)\|), \gamma_u(\limsup_{t\to\infty} \|u(t)\|)\}. \tag{10.47}$$

In what follows we shall prove that if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a *simple contraction*, i.e. if

$$\gamma_1(\gamma_2(r)) < r \quad \text{for all } r > 0, \tag{10.48}$$

the composite system is input-to-state stable. This result is usually referred to as the *small-gain* theorem.

Theorem 10.6.1. *If the condition (10.48) holds, system (10.43), viewed as a system with state $x = (x_1, x_2)$ and input u, is input-to-state stable. In particular, the class $\mathcal{K}$ functions*

$$\begin{aligned} \gamma_0(r) &= \max\{2\gamma_{01}(r), 2\gamma_{02}(r), 2\gamma_1 \circ \gamma_{02}(r), 2\gamma_2 \circ \gamma_{01}(r)\} \\ \gamma(r) &= \max\{2\gamma_1 \circ \gamma_u(r), 2\gamma_u(r)\} \end{aligned}$$

are such that response $x(t)$ to any input $u(\cdot) \in L_\infty^m$ is bounded and

$$\begin{aligned} \|x(\cdot)\|_\infty &\le \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\le \gamma(\limsup_{t\to\infty} \|u(t)\|). \end{aligned}$$

Proof. Pick $x_1^\circ \in \mathbb{R}^{n_1}$, $x_2^\circ \in \mathbb{R}^{n_2}$, $u(\cdot) \in L_\infty^m$. We show first that the corresponding trajectories $x_1(t)$ and $x_2(t)$ exist for all $t \geq 0$ and are bounded. For, suppose this is not the case. Then, for every number $R > 0$, there exists a time $T > 0$, such that the trajectories are defined on $[0,T]$, and

$$\text{either} \quad \|x_1(T)\| > R \quad \text{or} \quad \|x_2(T)\| > R\,. \tag{10.49}$$

Choose R such that

$$\begin{aligned} R &> \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \\ R &> \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\}\,, \end{aligned} \tag{10.50}$$

and let T be such that (10.49) hold. Define, for $i = 1,2$,

$$\begin{aligned} x_i^T(t) &= x_i(t) \quad \text{if} \quad t \in [0,T] \\ &= 0 \qquad\ \ \text{if} \quad t > T\,. \end{aligned}$$

and let $\tilde{x}_1(t)$ denote the response of the top subsystem of (10.43), in the initial state x_1°, to the input $x_2^T(\cdot)$. Since the latter is bounded on $[0,\infty)$, we have

$$\|\tilde{x}_1(t)\| \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2^T(\cdot)\|_\infty)\}$$

for all $t \geq 0$.

Since, by causality,

$$\tilde{x}_1(t) = x_1(t) \qquad \text{for all} \quad t \in [0,T]$$

we deduce that

$$\|x_1^T(\cdot)\|_\infty = \max_{t\in[0,T]} \|x_1(t)\| \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2^T(\cdot)\|_\infty)\}\,. \tag{10.51}$$

Similarly, let $\tilde{x}_2(t)$ denote the response of the bottom subsystem of (10.43), in the initial state x_2°, to the input $x_1^T(\cdot)$, $u(\cdot)$. Since the latter are bounded on $[0,\infty)$, we have

$$\|\tilde{x}_2(t)\| \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1^T(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}$$

for all $t \geq 0$. This, in turn, since

$$\tilde{x}_2(t) = x_2(t) \qquad \text{for all} \quad t \in [0,T]$$

yields

$$\|x_2^T(\cdot)\|_\infty = \max_{t\in[0,T]} \|x_2(t)\| \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1^T(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}\,. \tag{10.52}$$

Observe now that, if $a \leq \max\{b, c, \theta(a)\}$ and $\theta(a) < a$, then necessarily $\max\{b, c, \theta(a)\} = \max\{b, c\}$. Thus, replacing the estimate (10.52) into (10.51) and using the hypothesis that $\gamma_1 \circ \gamma_2(r) < r$ yields

$$\|x_1^T(\cdot)\|_\infty \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\}\,.$$

Observe also that if $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then also $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction. In fact, let $\gamma_1^{-1}(\cdot)$ denote the inverse of the function $\gamma_1(\cdot)$, which is defined on an interval of the form $[0, r_1^*)$ where

$$r_1^* = \lim_{r\to\infty} \gamma_1(r)\,.$$

If $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then

$$\gamma_2(r) < \gamma_1^{-1}(r) \quad \text{for all} \quad 0 < r < r_1^*\,,$$

and this shows that

$$\gamma_2(\gamma_1(r)) < r \quad \text{for all} \quad r > 0\,,$$

i.e. $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction.

Therefore, an argument identical to the one used before shows that

$$\|x_2^T(\cdot)\|_\infty \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\}\,.$$

In particular, using (10.50), we have

$$\begin{aligned} \|x_1(T)\| &\leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} &< R \\ \|x_2(T)\| &\leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\} &< R\,, \end{aligned}$$

which contradicts (10.49).

Having shown that the trajectories are defined for all $t \geq 0$ and bounded, (10.44) and (10.46) yield

$$\begin{aligned} \|x_1(\cdot)\|_\infty &\leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_\infty &\leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}\,, \end{aligned}$$

combining which, and using the property that $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, one obtains

$$\begin{aligned} \|x_1(\cdot)\|_\infty &\leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_\infty &\leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\}\,. \end{aligned}$$

In a similar way, combining now (10.45) and (10.47) (in which all the limits are finite since $x_1(\cdot)$ and $x_2(\cdot)$ are bounded) and using the property that $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, one obtains

$$\begin{aligned} \limsup_{t\to\infty} \|x_1(t)\| &\leq \gamma_1 \circ \gamma_u(\limsup_{t\to\infty} \|u(t)\|) \\ \limsup_{t\to\infty} \|x_2(t)\| &\leq \gamma_u(\limsup_{t\to\infty} \|u(t)\|)\,. \end{aligned} \tag{10.53}$$

From this, observing that

$$\|x(\cdot)\|_\infty \leq \max\{2\|x_1(\cdot)\|_\infty, 2\|x_2(\cdot)\|_\infty\}$$

and

$$\limsup_{t\to\infty} \|x(t)\| \leq \max\{2\limsup_{t\to\infty} \|x_1(t)\|, 2\limsup_{t\to\infty} \|x_2(t)\|\}\,,$$

the result follows. ◁

The condition (10.48), i.e. the condition that the composed function $\gamma_1 \circ \gamma_2(\cdot)$ is a contraction, is usually referred to as the *small gain condition.* Of course, it can be written in different alternative ways depending on how the functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are estimated. For instance, if it is known that $V_1(x_1)$ is an ISS-Lyapunov functions for the upper subsystem of (10.43), i.e. a function such

$$\underline{\alpha}_1(\|x_1\|) \le V_1(x_1) \le \overline{\alpha}_1(\|x_1\|)$$

$$\|x_1\| \ge \chi_1(\|x_2\|) \quad \Rightarrow \quad \frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) \le -\alpha(\|x_1\|)$$

then (see Remark 10.4.4) estimates of the form (10.44) and (10.45) hold with

$$\gamma_1(r) = \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1(r) .$$

Likewise, if $V_2(x_2)$ is a function such that

$$\underline{\alpha}_2(\|x_2\|) \le V_2(x_2) \le \overline{\alpha}_2(\|x_2\|)$$

$$\|x_2\| \ge \max\{\chi_2(\|x_1\|), \chi_u(\|u\|)\} \quad \Rightarrow \quad \frac{\partial V_2}{\partial x_2} f_2(x_1, x_2, u) \le -\alpha(\|x_2\|) ,$$

then, by means of arguments similar to those used in Section 10.4, it is easy to deduce that estimates of the form (10.46) and (10.47) hold with

$$\begin{aligned} \gamma_2(r) &= \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r) \\ \gamma_u(r) &= \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_u(r) . \end{aligned}$$

In this case the small-gain condition can be written in the form

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r) < r .$$

We conclude the section with a simple example.

Example 10.6.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 \\ \dot{x}_2 &= a x_1^2 - x_2 + u . \end{aligned} \qquad (10.54)$$

in which a is a real parameter, viewed as a system with state $(x_1, x_2) \in \mathbb{R}^2$ and input u. We regard this as interconnection of two one-dimensional systems, described by the upper and, respectively, lower equation, and we use the result of Theorem 10.6.1 to determine whether or not there are values of a such that the system is input-to-state stable.

For the upper subsystem

$$\dot{x}_1 = -x_1^3 + x_1 x_2$$

viewed as a system with state x_1 and input x_2, we consider the ISS-Lyapunov function $V(x_1) = \frac{1}{2}x_1^2$ and obtain

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1(x_1, x_2) \le -|x_1|^4 + |x_1|^2 |x_2| .$$

Choose any $0 < \varepsilon < 1$ and observe that

$$(1-\varepsilon)|x_1|^2 \ge |x_2|$$

implies

$$\dot{V} \le -\alpha(|x_1|)$$

for the class $\mathcal{K}_\infty$ function $\alpha(r) = \varepsilon r^4$. Thus, the inequality (10.19) holds with

$$\chi(r) = \frac{\sqrt{r}}{\sqrt{1-\varepsilon}} .$$

Since we can take

$$\underline{\alpha}(r) = \overline{\alpha}(r) = \frac{1}{2} r^2$$

it is deduced that the upper subsystem of (10.54) is input-to-state stable and estimates of the form (10.44) and (10.45) hold, with

$$\gamma_1(r) = \frac{\sqrt{r}}{\sqrt{1-\varepsilon}} .$$

For the lower subsystem of (10.54)

$$\dot{x}_2 = a x_1^2 - x_2 + u$$

viewed as a system with state x_2 and input (x_1, u), consider again a quadratic ISS-Lyapunov function $V(x_2) = \frac{1}{2}x_2^2$ and obtain

$$\dot{V} = \frac{\partial V}{\partial x_2} f_2(x_1, x_2, u) \le |x_2|(|a||x_1|^2 - |x_2| + |u|) .$$

Choose again any $0 < \varepsilon < 1$ and observe that

$$(1-\varepsilon)|x_2| \ge |a||x_1|^2 + |u|$$

yields

$$\dot{V} \le -\alpha(|x_2|)$$

for the class $\mathcal{K}_\infty$ function $\alpha(r) = \varepsilon r^2$. Thus, the class $\mathcal{K}$ functions

$$\chi_2(r) = \frac{2|a|r^2}{1-\varepsilon}, \qquad \chi_u(r) = \frac{2r}{1-\varepsilon}$$

are such that

$$|x_2| \ge \max\{\chi_2(|x_1|), \chi_u(|u|)\} \quad \Rightarrow \quad \dot{V} \le -\alpha(|x_2|) .$$

As a consequence, estimates of the form (10.46) and (10.47) hold, with

$$\gamma_2(r) = \frac{2|a|r^2}{1-\varepsilon}, \qquad \chi_u(r) = \frac{2r}{1-\varepsilon} .$$

Checking the small gain condition on these estimates yields

$$\gamma_2(\gamma_1(r)) = \frac{2|a|r}{(1-\varepsilon)^2} < r ,$$

which can be fulfilled, for all $r > 0$, if $|a| < 1/2$. ◁

10.7 Dissipative Systems

In the previous section, we have characterized how, in a given input-to-state stable system, bounds on the input function determine bounds on the corresponding state response and we have described how such a characterization can be exploited in order to study the asymptotic stability of the interconnection of several subsystems. In the present and in the following section we conduct a somewhat similar analysis, but addressing the case in which also the outputs of the systems enter into the picture. To this end, consider a system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \qquad (10.55)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, $f(0,0) = 0$, $h(0,0) = 0$, and where $f(x,u)$, $h(x,u)$ are locally Lipschitz. A notion which, in the case of systems of this form, plays a role having a number of similarities with the notion of input-to-state stability is the notion of *dissipativity*, whose definition is essentially based on replacing, in the inequality (10.23), the class $\mathcal{K}$ function $\sigma(\|u\|)$, with an arbitrary (thus not necessarily positive) continuous function $q(u,y)$ of the input u and output y, with $q(0,0) = 0$. This function is usually called a *supply rate.*

Definition 10.7.1. *System (10.55) is said to be dissipative, with respect to the supply rate $q(u,y)$, if there exists a C^1 function $V(x)$ satisfying*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad (10.56)$$

for all $x \in \mathbb{R}^n$, where $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are class $\mathcal{K}_\infty$ functions, such that

$$\frac{\partial V}{\partial x} f(x,u) \le q(u,y) \qquad (10.57)$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and all $y = h(x,u)$.

The system is said to be strictly dissipative if, instead of (10.57), the inequality

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) + q(u,y) \qquad (10.58)$$

holds for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$.

A function $V(x)$, satisfying (10.56), for which either (10.57) or (10.58) holds, is called a storage function for (10.55). The inequalities (10.57) and (10.58) are called dissipation inequalities.

For convenience, it is useful to define the notion of dissipativity also in the case of a *memoryless* system, that is a system defined by an input-output map of the form

$$y = \varphi(u) \,, \qquad (10.59)$$

where $\varphi : \mathbb{R}^m \to \mathbb{R}^p$ is continuous and $\varphi(0) = 0$.

Definition 10.7.2. *System (10.59) is dissipative, with respect to the supply rate $q(u,y)$, if*

$$q(u,y) \ge 0$$

for all $u \in \mathbb{R}^m$ and all $y = \varphi(u)$.

Of course, an input-to-state stable system, with ISS-pair $\{\alpha(\cdot), \sigma(\cdot)\}$ is strictly dissipative with respect to the supply rate

$$q(u,y) = \sigma(\|u\|) \,.$$

Conversely, if a system is strictly dissipative with respect to a supply rate $q(u,y)$ satisfying

$$q(u,y) \le \sigma(\|u\|) \qquad \text{for all } u \in \mathbb{R}^m,\ y \in \mathbb{R}^p,$$

for some class $\mathcal{K}$ function $\sigma(\cdot)$, then this system is input-to-state stable.

If a system is strictly dissipative with respect to a supply rate $q(u,y)$ satisfying

$$q(0,y) \le 0 \qquad \text{for all } y \in \mathbb{R}^p, \qquad (10.60)$$

then this system is globally asymptotically stable, in fact with Lyapunov function $V(x)$.

If a system is just dissipative (but possibly not strictly dissipative) with respect to a supply rate satisfying (10.60), $V(x)$ is only guaranteed to satisfy

$$\frac{\partial V}{\partial x} f(x,0) \le 0$$

and thus the system in question is only guaranteed to be stable in the sense of Lyapunov, maybe not globally asymptotically stable. The latter property holds if (10.60) is satisfied in the stronger sense

$$q(0,y) < 0 \qquad \text{for all } y \ne 0, \qquad (10.61)$$

and the system has the following property of *zero-state detectability.*

Definition 10.7.3. *System (10.55) is zero-state detectable if, for any x° such that the solution $x^\circ(t)$ of $\dot{x} = f(x,0)$ satisfying $x^\circ(0) = x^\circ$ is defined for all $t \ge 0$, the condition that $h(x^\circ(t),0) = 0$ for all $t \ge 0$ implies* $\lim_{t\to\infty} x^\circ(t) = 0$.

Remark 10.7.1. In other words, a system is zero-state detectable if, whenever the input is identically zero on $[0,\infty)$, any state trajectory which is defined for all $t \in [0,\infty)$, and such that the associated output is identically zero on $[0,\infty)$, converges to zero as $t \to \infty$. ◁

To see why this property, in a system which is dissipative with respect to a supply rate satisfying (10.61), implies global asymptotic stability, one can proceed as follows. First of all, observe that

$$\frac{\partial V}{\partial x} f(x,0) \le q(0,y) \le 0\,. \tag{10.62}$$

Hence, $V(x(t))$ is non-increasing and this, since $V(x)$ is positive definite and proper, implies that all trajectories are bounded. Set

$$V_\infty = \lim_{t\to\infty} V(x(t))\,.$$

Using standard arguments, pick any trajectory $x(t)$ and let Γ denote its ω-limit set, which – since the trajectory $x(t)$ is bounded – is nonempty, compact and invariant. By definition of ω-limit set, $V(x) = V_\infty$ at each $x \in \Gamma$. Since Γ is invariant under $\dot{x} = f(x,0)$, any trajectory $x^\circ(t)$ of $\dot{x} = f(x,0)$ with initial condition x° in Γ is such that $V(x^\circ(t))$ is constantly equal to V_∞ and (10.62) yields

$$0 \le q(0, h(x^\circ(t), 0)) \le 0\,.$$

Because of (10.61), it follows that necessarily $h(x^\circ(t),0) = 0$ for all $t \ge 0$. Using the property of zero-state detectability, this yields $\lim_{t\to\infty} x^\circ(t) = 0$ and, in turn, $V_\infty = 0$. Since $V(x)$ is continuous and vanishing only at $x = 0$, this proves that also the trajectory $x(t)$ tends to zero as $t \to \infty$.

Besides input-to-state stable systems, there are other relevant classes of dissipative systems. One of these is that of the so-called *finite L_q gain* systems, which are defined as follows.

Definition 10.7.4. *A system is said to have finite L_q gain if it is dissipative with respect a supply rate of the form*

$$q(u,y) = K\|u\|^q - L\|y\|^q\,, \tag{10.63}$$

for some $K > 0, L > 0$.

Note that, in a finite L_q gain system, the supply rate satisfies (10.61). Thus, finite L_q gain systems are globally asymptotically stable, if zero-state detectable.

The terminology "finite L_q gain" is related to the following properties. First of all, observe that if a system is dissipative with respect to a supply rate of the form (10.63), it is also dissipative with respect to a supply rate of the form

$$\tilde{q}(u,y) = \gamma^q\|u\|^q - \|y\|^q\,. \tag{10.64}$$

In fact, it is immediate to see that if $V(x)$ renders the dissipation inequality (10.57) fulfilled for a supply rate $q(x,y)$ of the form (10.63), the inequality

$$\frac{\partial \tilde{V}}{\partial x} f(x,u) \le \gamma^q\|u\|^q - \|y\|^q \tag{10.65}$$

holds, with

$$\tilde{V}(x) = \frac{1}{L}V(x)\,, \qquad \gamma^q = \frac{K}{L}\,,$$

and this shows that the system is dissipative with respect to a supply rate of the form (10.64).

Integration of the inequality (10.65) on the interval $[0,T]$ yields (see also the end of Section 10.5) that for any $u(\cdot) \in L_q^m$, and any initial state x°,

$$\begin{aligned} V(x(T)) &\le V(x^\circ) + \gamma^q \int_0^T \|u(t)\|^q dt - \int_0^T \|y(t)\|^q dt \\ &\le V(x^\circ) + \gamma^q \int_0^\infty \|u(t)\|^q dt\,, \end{aligned}$$

from which it is deduced that the response $x(t)$ of the system is defined for all $t \in [0,\infty)$ and bounded. Now, suppose $x^\circ = 0$ and observe that the previous inequality yields

$$V(x(T)) \le \gamma^q \int_0^T \|u(t)\|^q dt - \int_0^T \|y(t)\|^q dt$$

for any $T > 0$. Since $V(x(T)) \ge 0$, we deduce that

$$\int_0^T \|y(t)\|^q dt \le \gamma^q \int_0^T \|u(t)\|^q dt \le \gamma^q \Big[\|u(\cdot)\|_q\Big]^q$$

for any $T > 0$ and therefore

$$\Big[\|y(\cdot)\|_q\Big]^q \le \gamma^q \Big[\|u(\cdot)\|_q\Big]^q\,,$$

i.e.

$$\|y(\cdot)\|_q \le \gamma\|u(\cdot)\|_q\,.$$

In other words, for any $u(\cdot) \in L_q^m$, the response of the system from the initial state $x(0) = 0$ is defined for all $t \ge 0$, produces an output $y(\cdot)$ which is a function in L_q^p, and the ratio between the L_q norm of the output and the L_q norm of the input is bounded by γ. For this reason, the system is said to have a finite L_q gain, bounded from above by the number γ.

The case $q = 2$ has a special interest, because functions in L_2^m and L_2^p represent signals having *finite energy* over the infinite time interval $[0,\infty)$, and therefore the number γ in the dissipation inequality (10.65) can be given the interpretation of ratio between the energies of output and input. Moreover, another interpretation, which does not necessarily require the consideration of inputs having finite energy, is possible. Suppose the input is a *periodic* function of time, with period T, i.e. that

$$u(t+kT) = u^\circ(t), \quad \text{for all } t \in [0,T),\ k \ge 0$$

for some piecewise continuous function $u^\circ(t)$, defined on $[0,T)$. Also, suppose that, for some suitable initial state $x(0) = x^\circ$, the state response $x^\circ(t)$ of the system is defined for all $t \in [0,T]$ and satisfies

$$x^\circ(T) = x^\circ .$$

Then, it is obvious that $x^\circ(t)$ exists for all $t \geq 0$, and is a *periodic* function, having the same period T of the input, namely

$$x^\circ(t+kT) = x^\circ(t), \quad \text{for all } t \in [0,T),\ k \geq 0$$

and so is the corresponding output response $y(t) = h(x^\circ(t), u(t))$.

For the triplet $\{u(t), x^\circ(t), y(t)\}$ thus defined, integration of the inequality (10.65) (with $q = 2$) over an interval $[t_0, t_0+T]$, with arbitrary $t_0 \geq 0$, yields

$$V(x^\circ(t_0+T)) - V(x^\circ(t_0)) \leq \gamma^2 \int_{t_0}^{t_0+T} \|u(s)\|^2 ds - \int_{t_0}^{t_0+T} \|y(s)\|^2 ds ,$$

i.e., since $V(x^\circ(t_0+T)) = V(x^\circ(t_0))$,

$$\int_{t_0}^{t_0+T} \|y(s)\|^2 ds \leq \gamma^2 \int_{t_0}^{t_0+T} \|u(s)\|^2 ds . \tag{10.66}$$

Observe that the integrals on both sides of this inequality are independent of t_0, because the integrands are periodic functions having period T, and recall that the *root mean square* value of any (possibly vector-valued) periodic function $f(t)$ (which is usually abbreviated as r.m.s. and characterizes the *average power* of the signal represented by $f(t)$) is defined as

$$\|f(\cdot)\|_{\text{r.m.s.}} = \left(\frac{1}{T}\int_{t_0}^{t_0+T} \|f(s)\|^2 ds\right)^{\frac{1}{2}} .$$

With this in mind, (10.66) yields

$$\|y(\cdot)\|_{\text{r.m.s.}} \leq \gamma \|u(\cdot)\|_{\text{r.m.s.}} . \tag{10.67}$$

In other words, in a finite L_2 gain system, the number γ (which appears in the dissipation inequality (10.65)) happens to be also an upper bound for the ratio between the r.m.s. value of the output and the r.m.s value of the input, whenever a periodic input is producing (from an appropriate initial state) a periodic (state and output) response.

We can therefore conclude that, in a system which satisfies a dissipation inequality of the form

$$\frac{\partial V}{\partial x} f(x,u) \leq \gamma^2 \|u\|^2 - \|y\|^2 ,$$

the number γ can be given these two interpretations. If the input represents a signal whose energy over the infinite interval $[0,\infty)$ is finite, then the corresponding output from the initial state $x(0) = 0$ is a function having finite energy over the interval $[0,\infty)$ and the ratio between the energies of output and input is bounded from above by the number γ. On the other hand, if the input is a periodic function which produces, from some appropriate initial state $x(0) = x^\circ$, a periodic state and output response, the number γ provides a bound for the ratio between the average powers of the output and input.

Example 10.7.1. As an illustration of the last concept, suppose a system of the form (10.55) is strictly dissipative, with respect to a supply rate of the form

$$q(u,y) = \gamma^2 \|u\|^2 - \|y\|^2 . \tag{10.68}$$

By definition, this system is globally asymptotically stable and has finite L_2 gain.

Suppose, in addition, that the functions $\underline{\alpha}(\cdot)$ and $\alpha(\cdot)$ which characterize the estimates (10.56) and the dissipation inequality (10.58), for some $\delta > 0, a > 0, b > 0$, are such that

$$\underline{\alpha}(s) = as^2, \quad \alpha(s) = bs^2, \tag{10.69}$$

for all $s \in [0,\delta]$. Then, (see Lemma 10.1.5) the equilibrium $x = 0$ of the system is also locally exponentially stable.

Let the input be a sinusoidal function of time, of period $T = 2\pi/\omega_0$, e.g.

$$\tilde{u}(t) = U u_0 \cos(\omega_0 t) \tag{10.70}$$

where $U > 0$ and $u_0 \in \mathbb{R}^m$ has unitary norm. Bearing in mind the results discussed in Section 8.1 it is easy to realize that, if U is sufficiently small, and the initial state is appropriately set, the system exhibits a response which is a periodic function of period T. In fact, observe that the input thus defined can be viewed as generated by an autonomous system of the form (8.2), with $w \in \mathbb{R}^2$,

$$\begin{aligned} s(w) &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} w \\ p(w) &= u_0 \begin{pmatrix} 1 & 0 \end{pmatrix} w \end{aligned}$$

in the initial state

$$w(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} U .$$

Since this system is neutrally stable and the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is locally exponentially stable, the hypotheses of Proposition 8.1.1 are fulfilled and there exists a mapping $x = \pi(w)$, defined in a neighborhood W° of the origin in $\mathbb{R}^2$, with $\pi(0) = 0$, which satisfies

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), p(w)) . \tag{10.71}$$

If U is sufficiently small, $w(t) \in W^\circ$ for all $t \geq 0$, and the response of system (10.55) in the initial state

$$x^\circ(0) = \pi(\begin{pmatrix} 1 \\ 0 \end{pmatrix} U) \tag{10.72}$$

is given by

$$x^\circ(t) = \pi(w(t)) , \tag{10.73}$$

which, indeed, is a periodic function (of period T). As a matter of fact, the latter is precisely the expression of the steady state response of the system to the input (10.70).

Thus, in view of the interpretation presented above, one can conclude that, for any sufficiently small U, the (output) response $\tilde{y}(t)$ of system (10.55) to the input (10.70) in the initial state (10.72) is such that

$$\|\tilde{y}(\cdot)\|_{\text{r.m.s.}} \le \gamma \|\tilde{u}(\cdot)\|_{\text{r.m.s.}} . \triangleleft$$

Another relevant class of dissipative systems is that of the so-called *passive systems*, which are defined as follows.

Definition 10.7.5. *A system in which $m = p$ is said to be passive (respectively, strictly passive) if it is dissipative (respectively, strictly dissipative) with respect to the supply rate*

$$q(u, y) = y^{\mathrm{T}} u . \tag{10.74}$$

A system which is dissipative with respect to a supply rate of the form

$$q(u, y) = y^{\mathrm{T}} u - \varepsilon \|y\|^2 \tag{10.75}$$

for some $\varepsilon > 0$, it said to be output strictly passive.

Note that the supply rate of a passive system satisfies (10.60), and the supply rate of an output strictly passive system satisfies (10.61). Thus, strictly passive systems are globally asymptotically stable, and output strictly passive systems are globally asymptotically stable if zero-state detectable.

Also, it is easy to see that output strictly passive systems are necessarily finite L_2 gain systems. In fact, observe that, for any $\delta > 0$

$$y^{\mathrm{T}} u \le \|y\| \cdot \|u\| \le \frac{1}{2\delta}\|u\|^2 + \frac{\delta}{2}\|y\|^2 .$$

Suppose a system is output strictly passive, with supply rate (10.75) and choose $\delta = \varepsilon$, to obtain

$$q(u, y) \le \frac{1}{2\varepsilon}\|u\|^2 - \frac{\varepsilon}{2}\|y\|^2$$

Thus, the system is dissipative with respect to the supply rate

$$\tilde{q}(u, y) = \frac{1}{2\varepsilon}\|u\|^2 - \frac{\varepsilon}{2}\|y\|^2$$

and, therefore, also dissipative with respect to a supply rate of the form (10.68) with $\gamma = \varepsilon^{-1}$.

Observe that a *strictly passive* system is a system for which there exists a positive definite and proper C^1 function $V(x)$ satisfying

$$\frac{\partial V}{\partial x} f(x, u) - y^{\mathrm{T}} u \le -\alpha(\|x\|)$$

for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, i.e. such that the difference between the derivative of $V(x)$ along trajectories and the supply rate $y^{\mathrm{T}} u$ is bounded, from above, by a function which is negative for any nonzero x. A weaker version of this notion is that of *output strict passivity*, in which the bound in question is replaced by function of x and u, namely $-\varepsilon\|h(x, u)\|^2$, which is only non-positive. For the sake of completeness, we review below another variant of this notion, in which the bound in question is still a non-positive function x and u, but not necessarily proportional to the square of the norm of the output.

Definition 10.7.6. *A system of the form (10.55), in which $m = p$, is said to be weakly strictly passive if there exists a positive definite and proper function C^1 $V(x)$, and a continuous function $d(x, u)$, defined on $\mathbb{R}^n \times \mathbb{R}^m$ and satisfying*

$$d(0, 0) = 0 , \qquad d(x, u) \ge 0 \ \textit{for all}\ (x, u) \in \mathbb{R}^n \times \mathbb{R}^m ,$$

such that

$$\frac{\partial V}{\partial x} f(x, u) \le -d(x, u) + y^{\mathrm{T}} u , \tag{10.76}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and, moreover, for any x° and any $u^\circ(\cdot)$ such that the solution $x^\circ(t)$ of $\dot{x} = f(x, u^\circ)$ satisfying $x^\circ(0) = x^\circ$ is defined for all $t \ge 0$,

$$d(x^\circ(t), u^\circ(t)) = 0 \ \textit{for all}\ t \ge 0 \qquad \Rightarrow \qquad \lim_{t\to\infty} x^\circ(t) = 0 .$$

Remark 10.7.2. Clearly, any strictly passive system is also weakly strictly passive, in the sense of the Definition above. An output strictly passive system is weakly strictly passive if, for any x° and any $u^\circ(\cdot)$ such that the solution $x^\circ(t)$ of $\dot{x} = f(x, u^\circ)$ satisfying $x^\circ(0) = x^\circ$ is defined for all $t \ge 0$,

$$y(t) = 0 \ \text{for all}\ t \ge 0 \qquad \Rightarrow \qquad \lim_{t\to\infty} x^\circ(t) = 0 .$$

It is also easy to see that a weakly strictly passive system has a globally asymptotically stable equilibrium at $x = 0$. In fact, by definition, its storage function $V(x)$ satisfies

$$\frac{\partial V}{\partial x} f(x, 0) \le -d(x, 0) \le 0 ,$$

and this, as shown before, implies boundedness of all the trajectories. Let Γ denote the ω-limit set of any trajectory $x(t)$, and recall that, for each $x \in \Gamma$, $V(x) = V_\infty = \lim_{t\to\infty} V(x(t))$. Since Γ is invariant under $\dot{x} = f(x,0)$, any trajectory $x^\circ(t)$ of $\dot{x} = f(x,0)$ with initial condition x° in Γ is such that $V(x^\circ(t))$ is constantly equal to V_∞ and the previous inequality yields

$$d(x^\circ(t), 0) = 0 .$$

Therefore, $x^\circ(t)$ must converge to zero as $t \to \infty$. This shows that $V_\infty = 0$ and proves that $x(t)$ tends to zero as $t \to \infty$. ◁

A third relevant class of dissipative systems is that of the so-called *sector bounded* systems.

Definition 10.7.7. *A system in which $m = p$ is said to be sector bounded if it is dissipative with respect to a supply rate of the form*

$$q(u,y) = (y - au)^{\mathrm{T}}(bu - y) , \tag{10.77}$$

for some pair of real numbers $a \le b$.

Remark 10.7.3. In the case of a single-input single-output memoryless system, modeled by a map $y = \varphi(u)$ as in (10.59), to say that the system is dissipative with respect to a supply rate of the form (10.77) is to say that

$$(\varphi(u) - au)(bu - \varphi(u)) \ge 0$$

for all $u \in \mathbb{R}$. The latter is equivalent to

$$a \le \frac{\varphi(u)}{u} \le b$$

and this is why systems dissipative with respect to a supply rate of the form (10.77) are called "sector bounded". ◁

Note that all supply rates (10.68), (10.74), (10.75), (10.77) are *quadratic forms* in u, y, i.e. special cases of a function of the form

$$q(u,y) = \begin{pmatrix} u^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} R & S^{\mathrm{T}} \\ S & Q \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = u^{\mathrm{T}}Ru + 2y^{\mathrm{T}}Su + y^{\mathrm{T}}Qy \tag{10.78}$$

in which R, Q are symmetric matrices. In fact, finite L_2 gain systems correspond to the case in which

$$R = \gamma^2 I, \quad S = 0, \quad Q = -I ,$$

passive systems to the case in which

$$R = 0, \quad S = \frac{1}{2}I, \quad Q = 0 ,$$

output strictly passive systems to the case in which

$$R = 0, \quad S = \frac{1}{2}I, \quad Q = -\varepsilon I ,$$

sector-bounded systems to the case in which

$$R = -abI, \quad S = \frac{1}{2}(a+b)I, \quad Q = -I .$$

If a system is dissipative with respect to a quadratic supply rate and the functions $f(x,u)$ and $h(x,u)$ which characterize the right-hand sides of (10.55) are *affine forms* in u, the dissipation inequality (10.57) can be given a more explicit characterization. More precisely, consider a nonlinear system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) + k(x)u \end{aligned} \tag{10.79}$$

in which $f(0) = 0$ and $h(0) = 0$.

This system is dissipative with respect to a given supply rate $q(u,y)$ if there exists a positive definite and proper C^1 function $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)u] - q(u, h(x) + k(x)u) \le 0 \tag{10.80}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. If $q(u,y)$ is a supply rate of the form (10.78), the left-hand side of this inequality can be viewed as a (complete) form of degree two in u, with coefficients depending on x, in fact equal to

$$\begin{aligned} &\frac{\partial V}{\partial x}f(x) - h^{\mathrm{T}}(x)Qh(x) + [\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S)]u \\ &\quad + u^{\mathrm{T}}[-R - k^{\mathrm{T}}(x)S - S^{\mathrm{T}}k(x) - k^{\mathrm{T}}(x)Qk(x)]u . \end{aligned} \tag{10.81}$$

Thus, the input-affine system (10.79) is dissipative with respect to the quadratic supply rate (10.78) if and only if there exists a positive definite and proper C^1 function $V(x)$ such that, for each x, (10.81) is a negative semi-definite form in u. This observation yields the following criterion for dissipativity.

Proposition 10.7.1. *System (10.79) is dissipative with respect to the supply rate (10.78) if and only if:*

(i) *the symmetric matrix*

$$W(x) = R + k^{\mathrm{T}}(x)S + S^{\mathrm{T}}k(x) + k^{\mathrm{T}}(x)Qk(x)$$

is positive semi-definite for all $x \in \mathbb{R}^n$,

(ii) *there exists a positive definite and proper C^1 function $V(x)$ such that, for all $x \in \mathbb{R}^n$, the set*

$$\mathcal{U}(x) = \{u \in \mathbb{R}^m : W(x)u = \frac{1}{2}[\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S)]^{\mathrm{T}}\}$$

is not empty and, for all $u \in \mathcal{U}(x)$,

$$\frac{\partial V}{\partial x}f(x) - h^{\mathrm{T}}(x)Qh(x) + u^{\mathrm{T}}W(x)u \leq 0 \tag{10.82}$$

Proof. Set

$$\begin{aligned} A(x) &= \frac{\partial V}{\partial x}f(x) - h^{\mathrm{T}}(x)Qh(x) \\ B(x) &= \frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S)\,. \end{aligned}$$

In view of the above observations, system (10.79) is dissipative with respect to the supply rate (10.78) if and only if, for some $V(x)$, at each fixed $x \in \mathbb{R}^n$,

$$A(x) + B(x)u - u^{\mathrm{T}}W(x)u \tag{10.83}$$

is a negative semi-definite function of u. To prove the necessity observe that, if (10.83) is negative semi-definite, necessarily $W(x)$ is positive semi-definite, i.e. condition (i) holds. Moreover, any u such that $W(x)u = 0$ is necessarily such that $B(x)u = 0$ and this shows that the vector $B^{\mathrm{T}}(x)$ is in the image of the matrix $W(x)$. As a consequence, the set $\mathcal{U}(x)$ is not empty. The value of (10.83) at any $u \in \mathcal{U}(x)$ is equal to

$$A(x) + u^{\mathrm{T}}W(x)u$$

and this, since the latter must be a non-positive number, completes the proof of the necessity of (ii).

As far as the sufficiency is concerned, let x be fixed, say $x = \bar{x}$. Pick any $\bar{u} \in \mathcal{U}(\bar{x})$, which is such that $B(\bar{x}) = 2\bar{u}^{\mathrm{T}}W(\bar{x})$. Then, by standard completion of the squares, one obtains, for all $u \in \mathbb{R}^m$,

$$\begin{aligned} A(\bar{x}) + B(\bar{x})u - u^{\mathrm{T}}W(\bar{x})u &= A(\bar{x}) + 2\bar{u}^{\mathrm{T}}W(\bar{x})u - u^{\mathrm{T}}W(\bar{x})u \\ &= A(\bar{x}) + \bar{u}^{\mathrm{T}}W(\bar{x})\bar{u} - (u - \bar{u})^{\mathrm{T}}W(\bar{x})(u - \bar{u}) \end{aligned}$$

which is non positive because, by assumption (i), $W(\bar{x})$ is positive semi-definite and, by assumption (ii),

$$A(\bar{x}) + \bar{u}^{\mathrm{T}}W(\bar{x})\bar{u} \leq 0.$$

Since $\bar{x}$ was any arbitrary point in $\mathbb{R}^n$, this completes the proof of the sufficiency. ◁

Remark 10.7.4. Note that the property of being strictly dissipative can be given a very similar characterization. The reader can easily check that system (10.79) is strictly dissipative with respect to the supply rate (10.78) if and only if conditions (i) and (ii) of the above Proposition hold, with (10.82) replaced by the inequality

$$\frac{\partial V}{\partial x}f(x) - h^{\mathrm{T}}(x)Qh(x) + u^{\mathrm{T}}W(x)u \leq -\alpha(\|x\|)$$

in which $\alpha(\cdot)$ is some class $\mathcal{K}_\infty$ function. ◁

The criterion described in the previous Proposition lends itself to somewhat simpler versions in two extreme cases, the one in which $W(x)$ is nonsingular for all x and the one in which $W(x)$ is identically zero for all x.

Corollary 10.7.2. *Consider system (10.79) and let the supply rate (10.78) be given. Suppose $W(x)$ is nonsingular for all x. Then, the system is dissipative if and only if $W(x)$ is positive definite and there exists a positive definite and proper C^1 function $V(x)$, such that, for all $x \in \mathbb{R}^n$,*

$$\begin{aligned} &\frac{\partial V}{\partial x}f(x) - h^{\mathrm{T}}(x)Qh(x) + \frac{1}{4}[\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S)]\cdot \\ &\cdot W^{-1}(x)[\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S)]^{\mathrm{T}} \leq 0\,. \end{aligned} \tag{10.84}$$

Suppose $W(x) = 0$ for all x. Then, the system is dissipative if and only if there exists a positive definite and proper C^1 function $V(x)$, such that, for all $x \in \mathbb{R}^n$,

$$\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S) = 0 \tag{10.85}$$

and

$$\frac{\partial V}{\partial x}f(x) - h^{\mathrm{T}}(x)Qh(x) \leq 0\,. \tag{10.86}$$

Proof. If $W(x)$ is nonsingular, then the set $\mathcal{U}(x)$ is nonempty for all $x \in \mathbb{R}^n$, and coincides with the single element

$$u = \frac{1}{2}W^{-1}(x)[\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)(Qk(x) + S)]^{\mathrm{T}}\,.$$

Replacing this in (10.82) yields (10.84), and proves the first part of the corollary.

On the other hand, if $W(x) = 0$, the set $\mathcal{U}(x)$ is nonempty if and only if (10.85) holds, in which case the set itself coincides with $\mathbb{R}^m$. Since (10.82) reduces to (10.86), this proves the second part of the corollary. ◁

The conditions of Proposition 10.7.1 and Corollary 10.7.2 are further simplified in the case of a system in which there is no feed-through between input and output, i.e. $k(x) = 0$. In this case, in fact

$$W(x) = R$$

and

$$\mathcal{U}(x) = \{u \in \mathbb{R}^m : Ru = \frac{1}{2}[\frac{\partial V}{\partial x}g(x) - 2h^{\mathrm{T}}(x)S]^{\mathrm{T}}\}\,.$$

For instance, consider the supply rate

$$q(u,y) = \gamma^2 \|u\|^2 - \|y\|^2 .$$

In this case $W(x) = R = \gamma^2 I$ is nonsingular and positive definite. Thus, the system is dissipative (i.e. is a *finite L_2 gain system*) if and only if there exists a positive definite and proper C^1 function $V(x)$ such that, for all $x \in \mathbb{R}^n$,

$$\frac{\partial V}{\partial x} f(x) + h^{\mathrm{T}}(x)h(x) + \frac{1}{4\gamma^2}[\frac{\partial V}{\partial x} g(x)][\frac{\partial V}{\partial x} g(x)]^{\mathrm{T}} \leq 0 .$$

Or, consider the supply rate

$$q(u,y) = y^{\mathrm{T}} u .$$

In this case $W(x) = 0$, and the system is dissipative (i.e. is a ***passive system***) if and only if there exists a positive definite and proper C^1 function $V(x)$ such that, for all $x \in \mathbb{R}^n$,

$$\frac{\partial V}{\partial x} g(x) = h^{\mathrm{T}}(x), \qquad \frac{\partial V}{\partial x} f(x) \leq 0 .$$

10.8 Stability of Interconnected Dissipative Systems

We discuss in this section the stability properties of feedback interconnected dissipative systems. In particular, we consider the case in which the interconnection takes place at the level of inputs and outputs (as opposite to an interconnection occurring at the level of states, such as in (10.43)) and, since this is the most frequently considered case, we will study the case of *negative* feedback interconnection. More precisely, consider two systems Σ_1 and Σ_2, described by equations of the form

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i) \\ y_i &= h_i(x_i, u_i) \end{aligned}$$

with $i = 1, 2$, in which, in addition to the standard hypotheses considered so far for this class of systems (in particular $f_i(0,0) = 0$ and $h_i(0,0) = 0$) we assume also that

$$\begin{aligned} \dim(u_2) &= \dim(y_1) \\ \dim(u_1) &= \dim(y_2) . \end{aligned}$$

Suppose that the functions $h_1(x_1, u_1)$ and $h_2(x_2, u_2)$ are such that the constraint

$$\begin{aligned} u_2 &= y_1 \\ u_1 &= -y_2 \end{aligned}$$

makes sense. In other words, suppose that, for each x_1, x_2 there is a unique pair u_1, u_2 satisfying

$$\begin{aligned} u_2 &= h_1(x_1, u_1) \\ u_1 &= -h_2(x_2, u_2) . \end{aligned}$$

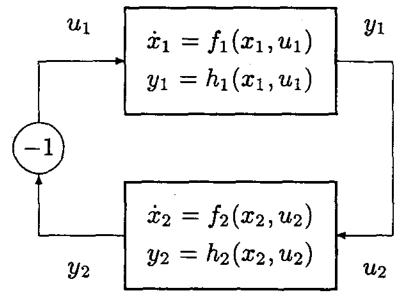

Fig. 10.4. Negative feedback connection.

If this is the case, the interconnected system obtained by imposing the constraint $u_2 = y_1$ and $u_1 = -y_2$ is the *(negative) feedback interconnection* of Σ_1 and Σ_2 (see Fig. 10.4).

Remark 10.8.1. Of course, the feedback interconnection makes always sense if either one of the $h_i(x_i, u_i)$ is independent of u_i. In the case of input affine systems, namely when

$$h_i(x_i, u_i) = h_i(x_i) + k_i(x_i) u_i ,$$

the negative feedback interconnection makes sense if the linear system

$$\begin{aligned} u_2 &= h_1(x_1) + k_1(x_1) u_1 \\ u_1 &= -h_2(x_2) - k_2(x_2) u_2 \end{aligned}$$

as a unique solution u_1, u_2, and this occurs if and only if the matrix

$$\begin{pmatrix} -k_1(x_1) & I \\ I & k_2(x_2) \end{pmatrix}$$

is invertible, i.e. the matrix

$$I + k_2(x_2) k_1(x_1)$$

is nonsingular for all x_1, x_2. ◁

Suppose now that both Σ_1 and Σ_2 are strictly dissipative, with respect to quadratic supply rates $q_1(u_1, y_1)$ and $q_2(u_2, y_2)$. Then, if certain relations hold between the parameters which characterize these supply rates, the negative feedback interconnection of Σ_1 and Σ_2 is globally asymptotically stable, as shown below. In case the two systems are just dissipative, additional conditions of zero-state detectability are requested, for each one of the systems or for their *open loop* interconnection $\Sigma_2 \circ \Sigma_1$, which is defined as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1) \\ \dot{x}_2 &= f_2(x_2, h_1(x_1, u_1)) \\ y_2 &= h_2(x_2, h_1(x_1, u_1)) \end{aligned}$$

Theorem 10.8.1. *Let two supply rates*

$$q_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i \,, \qquad i = 1, 2$$

be given and suppose that, for some $a > 0$, the symmetric matrix

$$M = \begin{pmatrix} Q_1 + aR_2 & -S_1 + aS_2^{\mathrm{T}} \\ -S_1^{\mathrm{T}} + aS_2 & R_1 + aQ_2 \end{pmatrix}$$

is either:

(i) *negative semi-definite,*

(ii) *negative semi-definite, $S_1 = aS_2^{\mathrm{T}}$ and $R_1 + aQ_2$ is nonsingular,*

(iii) *negative definite.*

Then, if both Σ_1 and Σ_2 are strictly dissipative and (i) holds, the negative feedback interconnection is globally asymptotically stable. If both Σ_1 and Σ_2 are dissipative, (ii) holds and $\Sigma_2 \circ \Sigma_1$ is zero-state detectable, the negative feedback interconnection is globally asymptotically stable. If both Σ_1 and Σ_2 are dissipative, (iii) holds and both Σ_1 and Σ_2 are zero-state detectable, the negative feedback interconnection is globally asymptotically stable.

Proof. If Σ_1 and Σ_2 are strictly dissipative, with respect to the given supply rates, there exist positive definite and proper C^1 functions $V_1(x_1)$ and $V_2(x_2)$, such that

$$\frac{\partial V_i}{\partial x_i} f_i(x_i, u_i) \le -\alpha_i(\|x_i\|) + q_i(u_i, y_i)$$

where $\alpha_i(\cdot)$ is a class $\mathcal{K}_\infty$ function. If $a > 0$, the function

$$W(x_1, x_2) = V_1(x_1) + aV_2(x_2)$$

is still positive definite and proper, and such that

$$\begin{aligned} \frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 &\le -\alpha_1(\|x_1\|) - a\alpha_2(\|x_2\|) + q_1(u_1, y_1) + aq_2(u_2, y_2) \\ &= -\alpha_1(\|x_1\|) - a\alpha_2(\|x_2\|) + q_1(-y_2, y_1) + aq_2(y_1, y_2) \\ &= -\alpha_1(\|x_1\|) - a\alpha_2(\|x_2\|) + (\, y_1^{\mathrm{T}} \quad y_2^{\mathrm{T}} \,)\, M \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} . \end{aligned}$$

If $M \le 0$,

$$\frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 \le -\alpha_1(\|x_1\|) - a\alpha_2(\|x_2\|)$$

and this proves global asymptotic stability.

If Σ_1 and Σ_2 are just dissipative (i.e. not strictly dissipative), we have instead

$$\frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 \le (\, y_1^{\mathrm{T}} \quad y_2^{\mathrm{T}} \,)\, M \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} . \tag{10.87}$$

Thus, if $M \le 0$ we conclude that all trajectories (of the negative feedback interconnection of Σ_1 and Σ_2) are bounded and that the equilibrium $(x_1, x_2) = (0, 0)$ is stable. Asymptotic stability is proved using the standard arguments of La Salle's invariance theorem. It is known that on each trajectory $\lim_{t\to\infty} W(x_1(t), x_2(t)) = W_\infty \ge 0$ and that on the ω-limit set Γ of this trajectory $W(x_1, x_2) = W_\infty$. Thus, using (10.87) and the fact that $M \le 0$, it is concluded that

$$(\, y_1^{\mathrm{T}}(t) \quad y_2^{\mathrm{T}}(t) \,)\, M \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = 0$$

along any trajectory with initial condition $(x_1^\circ, x_2^\circ) \in \Gamma$.

If (ii) holds, necessarily $y_2(t) = 0$. Thus, each initial condition in Γ produces a bounded trajectory $(x_1^\circ(t), x_2^\circ(t))$ for which $y_2(t) = u_1(t) = 0$. Since $\Sigma_2 \circ \Sigma_1$ is zero-state detectable, this trajectory necessarily converges to $(0, 0)$ and therefore $W_\infty = 0$. This proves global asymptotic stability.

If (iii) holds, necessarily $y_1(t) = 0$ and $y_2(t) = 0$ and, this time using zero-state detectability of both Σ_1 and Σ_2, global asymptotic stability follows again. ◁

Remark 10.8.2. Note that the conclusions of this Theorem remain valid, with appropriate adaptations, if one of the two systems, e.g. Σ_2, is a memoryless system, i.e. if $y_2 = \varphi(u_2)$, for some smooth function $\varphi(\cdot)$ satisfying $\varphi(0) = 0$. Recall that, in this case, Σ_2 is dissipative with respect to the supply rate $q_2(u_2, y_2)$ if

$$q_2(u_2, \varphi_2(u_2)) \ge 0$$

for all u_2. Now, it is easy to check that the function $V_1(x_1)$ is such that

$$\begin{aligned} \frac{\partial V_1}{\partial x_1}\dot{x}_1 &\le -\alpha_1(\|x_1\|) + q_1(u_1, y_1) + aq_2(u_2, y_2) - aq_2(u_2, y_2) \\ &= -\alpha_1(\|x_1\|) - aq_2(u_2, \varphi(u_2)) + (\, y_1^{\mathrm{T}} \quad y_2^{\mathrm{T}} \,)\, M \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} . \end{aligned}$$

Thus, the negative feedback interconnection is globally asymptotically stable if either:

(i) Σ_1 is strictly dissipative and M is negative semi-definite,

(ii) Σ_1 is dissipative, zero-state detectable, M is negative semi-definite, and

$$q_2(u_2, \varphi_2(u_2)) = 0 \qquad \Rightarrow \qquad u_2 = 0 \,,$$

(iii) Σ_1 is dissipative, zero-state detectable, and M is negative definite. ◁

There are numerous interesting consequences of this result. We discuss here just few of them. The first important special case occurs when both systems have finite L_2 gain, and leads to an alternative version of the property, already seen in Section 10.6, that a "small gain" in a feedback interconnection does not destroy stability.

Corollary 10.8.2. *Suppose Σ_1 and Σ_2 are dissipative with respect supply rates of the form*

$$q_i(u_i, y_i) = \gamma_i^2 u_i^{\mathrm{T}} u_i - y_i^{\mathrm{T}} y_i \qquad i = 1, 2 .$$

If both Σ_1 and Σ_2 are zero-state detectable, and

$$\gamma_1 \gamma_2 < 1 \tag{10.88}$$

the negative feedback interconnection is globally asymptotically stable.

Proof. To prove this result, observe that in this case

$$M = \begin{pmatrix} (-1 + a\gamma_2^2)I & 0 \\ 0 & (\gamma_1^2 - a)I \end{pmatrix} .$$

The matrix in question is negative definite if

$$-1 + a\gamma_2^2 < 0, \qquad \gamma_1^2 - a < 0$$

i.e. if

$$\gamma_1^2 < a < \frac{1}{\gamma_2^2}$$

If the condition (10.88) holds, there exists an $a > 0$ which renders the latter conditions satisfied. Thus, the Corollary follows from case (iii) of Theorem 10.8.1. ◁

Another important special case occurs when both systems are passive. In this case, for example, one can immediately obtain the following results.

Corollary 10.8.3. *Suppose Σ_1 and Σ_2 are dissipative with respect supply rates of the form*

$$q_i(u_i, y_i) = u_i^{\mathrm{T}} y_i - \varepsilon_i y_i^{\mathrm{T}} y_i \qquad i = 1, 2$$

where $\varepsilon_i \geq 0$. If $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and both Σ_1 and Σ_2 are zero-state detectable, the negative feedback interconnection is globally asymptotically stable. If $\varepsilon_1 = 0$, $\varepsilon_2 > 0$ and $\Sigma_2 \circ \Sigma_1$ is zero-state detectable, the negative feedback interconnection is globally asymptotically stable.

Proof. Set $a = 1$ in Theorem 10.8.1, to obtain

$$M = \begin{pmatrix} -\varepsilon_1 I & 0 \\ 0 & -\varepsilon_2 I \end{pmatrix} .$$

If both $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$, case (iii) applies. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, case (i) applies. ◁

Corollary 10.8.4. *Consider the interconnection (see Fig. 10.5) of*

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{10.89}$$

and $u = -\varphi(y)$. Suppose that system (10.89) is passive and zero-state detectable, and that the map $\varphi(\cdot)$ satisfies $\varphi(0) = 0$ and

$$y^{\mathrm{T}} \varphi(y) > 0 \qquad \text{for all } y \neq 0.$$

Then, the interconnection is globally asymptotically stable.

Proof. By hypothesis, the memoryless system $y_2 = \varphi(u_2)$ is such that $y_2^{\mathrm{T}} u_2 = \varphi^{\mathrm{T}}(u_2) u_2 > 0$ for all $u_2 \neq 0$. Thus, this system is dissipative with respect to the supply rate $q_2(u_2, y_2) = y_2^{\mathrm{T}} u_2$ and the result follows from the property indicated in the Remark 10.8.2, case (ii). ◁

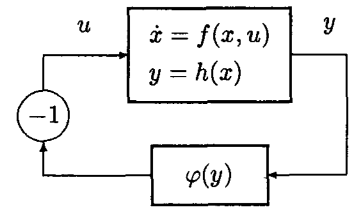

Fig. 10.5. Memoryless negative feedback.

The third case considers slightly more elaborate supply rates.

Corollary 10.8.5. *Suppose Σ_1 is dissipative with respect to a supply rate of the form*

$$q_1(u_1, y_1) = u_1^{\mathrm{T}} u_1 + k u_1^{\mathrm{T}} y_1 \tag{10.90}$$

and Σ_2 is dissipative with respect to a supply rate of the form

$$q_2(u_2, y_2) = (y_2 - \varepsilon u_2)^{\mathrm{T}}((k - \varepsilon)u_2 - y_2) , \tag{10.91}$$

with $k > 2\varepsilon > 0$. If Σ_1 and Σ_2 are zero-state detectable, the negative feedback interconnection is globally asymptotically stable.

Proof. To prove this result, observe that in this case

$$M = \begin{pmatrix} (-a\varepsilon k + a\varepsilon^2)I & \frac{1}{2}(a-1)kI \\ \frac{1}{2}(a-1)kI & (1-a)I \end{pmatrix} .$$

This matrix is negative definite if

$$\begin{aligned} & a > 1 \\ & \varepsilon(k-\varepsilon) > \frac{(a-1)k^2}{4a} . \end{aligned}$$

and these can be satisfied if $a = 1 + \delta$ and $\delta > 0$ is sufficiently small. Thus, the Corollary follows from case (iii) of Theorem 10.8.1. ◁

Note that the hypothesis on system Σ_1 in the last corollary can be given a slightly different form. Set

$$\tilde{y}_1 = y_1 + \frac{1}{k} u_1$$

and observe that

$$q_1(u_1, y_1) = u_1^{\mathrm{T}} u_1 + k u_1^{\mathrm{T}} y_1 = k u_1^{\mathrm{T}} (\frac{1}{k} u_1 + y_1) = k u_1^{\mathrm{T}} \tilde{y}_1 .$$

From this, it is easily deduced that system Σ_1 is dissipative with respect to the supply rate (10.90) if and only if the system $\widetilde{\Sigma}_1$ thus defined

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1) \\ \tilde{y}_1 &= h_1(x_1, u_1) + \frac{1}{k} u_1 \end{aligned}$$

is dissipative with respect to the supply rate

$$\tilde{q}_1(u_1, \tilde{y}_1) = u_1^{\mathrm{T}} \tilde{y}_1 . \tag{10.92}$$

In fact, the function $\widetilde{V}_1(x_1) = \frac{1}{k} V(x_1)$ satisfies by construction the dissipation inequality

$$\frac{\partial \widetilde{V}_1}{\partial x_1} f_1(x_1, u_1) \le \tilde{q}_1(u_1, \tilde{y}_1) .$$

Thus, the hypothesis on Σ_1 in Corollary 10.8.5 can be rephrased by assuming that the auxiliary system $\widetilde{\Sigma}_1$ satisfies the property of being dissipative with respect to the supply rate (10.92). In other words, system $\widetilde{\Sigma}_1$ is assumed to be *passive*.

In case both systems are single-input single-output and system Σ_2 is memoryless, the last corollary specializes in the following well-known criterion for global asymptotic stability under memoryless, sector-bounded, negative feedback.

Corollary 10.8.6. *Consider the interconnection of*

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{10.93}$$

and $u = -\varphi(y)$. Suppose (10.93) is zero-state detectable and that, for some $k > 2\varepsilon > 0$, the system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) + \frac{1}{k} u \end{aligned} \tag{10.94}$$

is passive. Suppose the map $\varphi(\cdot)$ satisfies $\varphi(0) = 0$ and

$$\varepsilon < \frac{\varphi(y)}{y} < k - \varepsilon \qquad \textit{for all } y \neq 0 .$$

Then, the interconnection is globally asymptotically stable.

Remark 10.8.3. Suppose that, in the above Corollary, $f(x, u)$ is affine in u, i.e. $f(x, u) = f(x) + g(x)u$. Then, the criterion given by Corollary 10.7.2 can be used to test the property of passivity of (10.94). In fact, in this case

$$W(x) = \frac{1}{k}$$

and the system is passive if and only if there exists a positive definite and proper C^1 function $V(x)$ such that

$$\frac{\partial V}{\partial x} f(x) + \frac{k}{4} [\frac{\partial V}{\partial x} g(x) - h^{\mathrm{T}}(x)][\frac{\partial V}{\partial x} g(x) - h^{\mathrm{T}}(x)]^{\mathrm{T}} \le 0 .$$

for all $x \in \mathbb{R}^n$.

If, in particular, system (10.93) is a passive system, then (see end of Section 10.7) the previous inequality holds for any k. In this case, the upper bound for $\varphi(y)/y$ can be any arbitrarily large number. ◁

10.9 Dissipative Linear Systems

In the case of linear systems, the conditions for dissipativity presented in the previous section can be given alternative characterizations, in particular involving the frequency response matrix of the system. These characterizations are summarized in this section, beginning with the property, for a system, of having an L_2 gain which does not exceed a fixed number γ.

As a continuation of the Example 10.7.1, consider an asymptotically stable linear system, modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du . \end{aligned} \tag{10.95}$$

In this case, the function $\pi(w)$ which solves the equation (10.71) is a linear function of w, i.e. $\pi(w) = \Pi w$ for some matrix

$$\Pi = (\Pi_1 \quad \Pi_2)$$

and the equation in question reduces to a Sylvester equation of the form

$$\Pi \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} = A\Pi + Bu_0 \,(1 \quad 0) \,.$$

This equation, since all eigenvalues of A have negative real part (because the system is by hypothesis asymptotically stable), has a unique solution Π. An elementary calculation (multiply first both sides on the right by the vector $(1 \quad j)^{\mathrm{T}}$) yields

$$\Pi_1 + j\Pi_2 = (j\omega_0 I - A)^{-1} Bu_0 \,,$$

i.e.

$$\Pi = (\,\mathrm{Re}[(j\omega_0 I - A)^{-1}B]u_0 \quad \mathrm{Im}[(j\omega_0 I - A)^{-1}B]u_0\,) \,.$$

As shown in the Example 10.7.1, if the system is in the initial state

$$x^\circ(0) = \Pi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{Re}[(j\omega_0 I - A)^{-1}B]u_0 \,,$$

the periodic input

$$\tilde{u}(t) = u_0 \cos(\omega_0 t)$$

produces the periodic state response

$$x^\circ(t) = \Pi w(t) = \Pi_1 \cos(\omega_0 t) - \Pi_2 \sin(\omega_0 t) \,, \tag{10.96}$$

and the periodic output response

$$\begin{aligned} \tilde{y}(t) &= Cx^\circ(t) + Du_0 \cos(\omega_0 t) \\ &= \mathrm{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \mathrm{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t) \,, \end{aligned} \tag{10.97}$$

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D \,.$$

Observe now that

$$\int_0^{\frac{2\pi}{\omega_0}} \|\tilde{u}(s)\|^2 ds = \frac{\pi}{\omega_0} \|u_0\|^2$$

and that, in view of the specific form of Π,

$$\int_0^{\frac{2\pi}{\omega_0}} \|\tilde{y}(s)\|^2 ds = \frac{\pi}{\omega_0} \|T(j\omega_0)u_0\|^2 \,.$$

In other words

$$\begin{aligned} \|\tilde{u}(\cdot)\|^2_{\mathrm{r.m.s.}} &= \tfrac{1}{2}\|u_0\|^2 \\ \|\tilde{y}(\cdot)\|^2_{\mathrm{r.m.s.}} &= \tfrac{1}{2}\|T(j\omega_0)u_0\|^2 \end{aligned}$$

Thus, from the interpretation illustrated in the Example 10.7.1 one can conclude that, if the system is strictly dissipative with respect to a supply rate of the form (10.68),

$$\|T(j\omega_0)u_0\|^2 = 2\|\tilde{y}(\cdot)\|^2_{\mathrm{r.m.s.}} \leq \gamma^2 2\|\tilde{u}(\cdot)\|^2_{\mathrm{r.m.s.}} = \gamma^2\|u_0\|^2$$

i.e.

$$\|T(j\omega_0)u_0\| \leq \gamma\|u_0\| \,.$$

Observing that both u_0 and ω_0 are arbitrary, it is seen from this that

$$\sup_{\omega\in\mathbb{R}} \max_{\|u\|=1} \|T(j\omega)u\| \leq \gamma \,. \tag{10.98}$$

The quantity on the left-hand side is, by definition, the so-called H_∞ norm of the matrix $T(j\omega)$. Therefore, one can conclude that a linear system which is strictly dissipative with respect to a supply rate of the form (10.68), is *asymptotically stable* and *the H_∞ norm of its frequency response matrix is bounded from above by the number γ*.

It will be shown now that also the converse of this property holds. To this end, using Proposition 10.7.1, condition (i), observe that the linear system (10.95) is dissipative with respect to a supply rate of the form (10.78) with $R = \gamma^2 I$ and $Q = -I$, that is a supply rate of the form

$$q(u,y) = \gamma^2\|u\|^2 - \|y\|^2 \,, \tag{10.99}$$

only if the matrix

$$W = \gamma^2 I - D^{\mathrm{T}} D$$

is positive semi-definite. Suppose that γ is large enough so that this matrix is positive definite, and thus nonsingular. Then, using this time Corollary 10.7.2, it is seen that the linear system (10.95) is strictly dissipative with respect to a supply rate of the form (10.99) if and only if there exists a positive definite and proper C^1 function $V(x)$ such that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{\partial V}{\partial x}Ax + x^{\mathrm{T}}C^{\mathrm{T}}Cx + \frac{1}{4}[\frac{\partial V}{\partial x}B + 2x^{\mathrm{T}}C^{\mathrm{T}}D]W^{-1}[\frac{\partial V}{\partial x}B + 2x^{\mathrm{T}}C^{\mathrm{T}}D]^{\mathrm{T}} \\ &\qquad \leq -\alpha(\|x\|) \,, \end{aligned} \tag{10.100}$$

where $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

Now, suppose that $V(x)$ is a quadratic form in x, i.e. a function of the form

$$V(x) = x^{\mathrm{T}}Px$$

where P is a symmetric matrix (which is necessarily positive definite if (10.56) holds). Suppose also that $\alpha(r) = \delta r^2$ for some $\delta > 0$. In this case, an easy calculation shows that the inequality (10.100) reduces to matrix inequality

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + [PB + C^{\mathrm{T}}D]W^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0 \,. \tag{10.101}$$

In this way, we have shown that if γ is large enough so that the matrix W is positive definite and *if* there exists a symmetric positive definite solution P of the matrix inequality (10.101), the positive definite and proper function $V(x) = x^{\mathrm{T}}Px$ satisfies the condition (10.100) and, consequently, the system is strictly dissipative, with respect to the supply rate (10.99). Inequalities of this kind are usually referred to as *algebraic Riccati inequalities.* In other words, we can conclude that the existence of a positive definite solution of the algebraic Riccati inequality (10.101) implies strict dissipativity with respect to the supply rate (10.99) and this, bearing in mind the conclusions reached at the beginning of the section, in turn implies (10.98), i.e. the fact that γ is an upper bound for the H_∞ norm of the frequency response matrix of the system.

It will be proved below that if γ is an upper bound for the H_∞ norm of the frequency response matrix of the system, then there exists a positive definite solution of the algebraic Riccati inequality (10.101), thus showing that the properties in question are, actually, equivalent. This will be done by means of a circular proof involving another equivalent version of the property that γ is an upper bound for the H_∞ norm of the frequency response matrix of the system, which is very useful for practical purposes, since it can be rather easily verified.

More precisely, the fact that γ is an upper bound for the H_∞ norm of the frequency response matrix of the system can be checked by looking at the spectrum of a matrix of the form

$$H = \begin{pmatrix} A_0 & R_0 \\ -Q_0 & -A_0^{\mathrm{T}} \end{pmatrix}, \tag{10.102}$$

in which R_0 and Q_0 are *symmetric* matrices which, together with A_0, depend on the matrices A, B, C, D which characterize the system and on the number γ. A matrix with this structure is called an *Hamiltonian matrix* and has the property that its spectrum is symmetric with respect to the imaginary axis. In fact, since

$$\begin{pmatrix} 0 & -I \\ I & 0^{\mathrm{T}} \end{pmatrix}^{-1} H \begin{pmatrix} 0 & -I \\ I & 0^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} -A_0^{\mathrm{T}} & Q_0 \\ -R_0 & A_0 \end{pmatrix} = -H^{\mathrm{T}}$$

the matrices H and $-H^{\mathrm{T}}$ are similar. Thus, if λ is an eigenvalue of H, so is also $-\lambda$. Since the entries of H are real numbers, this shows that if $a + jb$ is an eigenvalue of H, so is also $-a + jb$.

Theorem 10.9.1. *Consider the linear system (10.95) and let $\gamma > 0$ be a fixed number. The following are equivalent:*

(i) *there exists a number $\tilde{\gamma} < \gamma$, such that the system is strictly dissipative with respect to the supply rate*

$$q(u, y) = \tilde{\gamma}^2 \|u\|^2 - \|y\|^2 ,$$

(ii) *all the eigenvalues of A have negative real part and the frequency response matrix of the system $T(j\omega) = C(j\omega I - A)^{-1}B + D$ satisfies*

$$\sup_{\omega \in \mathbb{R}} \max_{\|u\|=1} \|T(j\omega)u\| < \gamma , \tag{10.103}$$

(iii) *all the eigenvalues of A have negative real part, the matrix $W = \gamma^2 I - D^{\mathrm{T}} D$ is positive definite, and the Hamiltonian matrix*

$$H = \begin{pmatrix} A + BW^{-1}D^{\mathrm{T}}C & BW^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}C - C^{\mathrm{T}}DW^{-1}D^{\mathrm{T}}C & -A^{\mathrm{T}} - C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}} \end{pmatrix} \tag{10.104}$$

has no eigenvalues on the imaginary axis,

(iv) *the matrix $W = \gamma^2 I - D^{\mathrm{T}} D$ is positive definite, and there exists a symmetric matrix $P > 0$ such that*

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + [PB + C^{\mathrm{T}}D]W^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0 . \tag{10.105}$$

Proof. We have already proven, at the beginning of the section, that (i) implies that (10.95) is an asymptotically stable system, with a frequency response matrix satisfying

$$\sup_{\omega \in \mathbb{R}} \max_{\|u\|=1} \|T(j\omega)u\| \le \tilde{\gamma} .$$

Thus, (i) $\Rightarrow$ (ii).

To show that (ii) $\Rightarrow$ (iii), first of all that note, since

$$\lim_{\omega \to \infty} T(j\omega) = D$$

it necessarily follows that $\|Du\| < \gamma$ for all u with $\|u\| = 1$ and this implies $\gamma^2 I > D^{\mathrm{T}} D$, i.e. the matrix W is positive definite.

Now observe that the Hamiltonian matrix (10.104) can be expressed in the form

$$H = L + MN$$

for

$$L = \begin{pmatrix} A & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{pmatrix}, \quad M = \begin{pmatrix} B \\ -C^{\mathrm{T}}D \end{pmatrix},$$

$$N = \begin{pmatrix} W^{-1}D^{\mathrm{T}}C & W^{-1}B^{\mathrm{T}} \end{pmatrix} .$$

Suppose, by contradiction, that the matrix H has eigenvalues on the imaginary axis. By definition, there exist a $2n$-dimensional vector x_0 and a number $\omega_0 \in \mathbb{R}$ such that

$$(j\omega_0 I - L)x_0 = MNx_0 .$$

Observe now that the matrix L has no eigenvalues on the imaginary axis, because its eigenvalues coincide with those of A and $-A^{\mathrm{T}}$, and A is by hypothesis stable. Thus $(j\omega_0 I - L)$ is nonsingular. Observe also that the vector $u_0 = Nx_0$ is nonzero because otherwise x_0 would be an eigenvector of L associated with an eigenvalue at $j\omega_0$, which is a contradiction. A simple manipulation yields

$$u_0 = N(j\omega_0 I - L)^{-1}Mu_0 . \tag{10.106}$$

It is easy to check that

$$N(j\omega_0 I - L)^{-1}M = W^{-1}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D] \tag{10.107}$$

where $T(s) = C(sI - A)^{-1}B + D$. In fact, it suffices to compute the transfer function of

$$\begin{aligned}\dot{x} &= Lx + Mu\\ y &= Nx\end{aligned}$$

and observe that $N(sI - L)^{-1}M = W^{-1}[T^{\mathrm{T}}(-s)T(s) - D^{\mathrm{T}}D]$.

Multiply (10.107) on the left by $u_0^{\mathrm{T}}W$ and on the right by u_0, and use (10.106), to obtain

$$u_0^{\mathrm{T}}Wu_0 = u_0^{\mathrm{T}}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D]u_0\,,$$

which in turn, in view of the definition of W, yields

$$\gamma^2\|u_0\|^2 = \|T(j\omega_0)u_0\|^2$$

which contradicts (ii), thus completing the proof.

To show that (iii) $\Rightarrow$ (iv), set

$$\begin{aligned}F &= (A + BW^{-1}D^{\mathrm{T}}C)^{\mathrm{T}}\\ Q &= -BW^{-1}B^{\mathrm{T}}\\ GG^{\mathrm{T}} &= C^{\mathrm{T}}(I + DW^{-1}D^{\mathrm{T}})C\end{aligned}$$

(the latter is indeed possible because $I + DW^{-1}D^{\mathrm{T}}$ is a positive definite matrix, i.e. $I + DW^{-1}D^{\mathrm{T}} = M^{\mathrm{T}}M$ for some nonsingular M).

The pair (F, G) thus defined is stabilizable. In fact, suppose that this is not the case. Then, there is a vector $x \neq 0$ such that

$$x^{\mathrm{T}}\,(\,F - \lambda I \quad G\,) = 0$$

for some λ with non-negative real part. Then,

$$0 = \begin{pmatrix} A + BW^{-1}D^{\mathrm{T}}C - \lambda I\\ MC\end{pmatrix} x$$

and this implies $Cx = 0$, which in turn implies $Ax = \lambda x$, and this is a contradiction because all the eigenvalues of A have negative real part.

Moreover, it is easy to check that

$$H^{\mathrm{T}} = \begin{pmatrix} F & -GG^{\mathrm{T}}\\ -Q & -F^{\mathrm{T}}\end{pmatrix}$$

and this matrix by hypothesis has no eigenvalues on the imaginary axis.

Thus, by Lemma 13.6.2, there is a unique solution Y^- of

$$Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q = 0\,, \quad \sigma(F - GG^{\mathrm{T}}Y^-) \subset \mathbb{C}^-\,, \tag{10.108}$$

the set of solutions Y of the inequality

$$YF + F^{\mathrm{T}}Y - YGG^{\mathrm{T}}Y + Q > 0 \tag{10.109}$$

is nonempty and any Y in this set is such that $Y < Y^-$.

Observe now that

$$\begin{aligned}0 &= Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q\\ &= Y^-(A^{\mathrm{T}} + C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}}) + (A + BW^{-1}D^{\mathrm{T}}C)Y^-\\ &\qquad - Y^-C^{\mathrm{T}}(I + DW^{-1}D^{\mathrm{T}})CY^- - BW^{-1}B^{\mathrm{T}}\\ &= Y^-A^{\mathrm{T}} + AY^- - [Y^-C^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY^- - B^{\mathrm{T}}] - Y^-C^{\mathrm{T}}CY^-,\end{aligned}$$

which yields

$$Y^-A^{\mathrm{T}} + AY^- \geq 0\,.$$

Setting $V(x) = x^{\mathrm{T}}Y^-x$, this inequality shows that, along the trajectories of

$$\dot{x} = A^{\mathrm{T}}x\,, \tag{10.110}$$

the function $V(x(t))$ is non-decreasing, i.e. $V(x(t)) \geq V(x(0))$ for any $x(0)$ and any $t \geq 0$. On the other hand, system (10.110) is by hypothesis asymptotically stable, i.e. $\lim_{t\to\infty} x(t) = 0$. Therefore, necessarily, $V(x(0)) \leq 0$, i.e the matrix Y^- is negative semi-definite. From this, it is concluded that any solution Y of (10.109), that is of the inequality

$$YA^{\mathrm{T}} + AY - [YC^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY - B^{\mathrm{T}}] - YC^{\mathrm{T}}CY > 0\,, \tag{10.111}$$

which necessarily satisfies $Y < Y^- \leq 0$, is a negative definite matrix.

Take any of the solutions Y of (10.111) and consider $P = -Y^{-1}$. By construction, this matrix is a positive definite solution of the inequality in (iv).

To show that (iv) $\Rightarrow$ (i) observe that the left-hand side of (10.105), which is negative definite, by continuity remains negative definite if γ is replaced by $\tilde{\gamma} = \gamma - \varepsilon$ and ε is sufficiently small. Thus, for some $\delta > 0$, the matrix P satisfies

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + [PB + C^{\mathrm{T}}D][\tilde{\gamma}^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D] < -\delta I.$$

As a consequence, the positive definite and proper function $V(x) = x^{\mathrm{T}}Px$ satisfies the inequality (10.100). ◁

Also the property of strict passivity has important implications on the frequency response matrix of a linear system. In particular, we refer, in what follows, to the property of *weak strict passivity.* Observe (see Remark 10.7.2) that in a weakly strictly passive linear system (10.95) the matrix A has all the eigenvalues with negative real part and therefore, as shown at the beginning of the section, the periodic input

$$\tilde{u}(t) = u_0\cos(\omega_0 t)$$

produces, from the initial state

$$x^\circ(0) = \mathrm{Re}[(j\omega_0 I - A)^{-1}B]u_0 \,,$$

the periodic state response (10.96) and the periodic output response (10.97). Assume that

$$\mathrm{rank}(B) = m$$

so that, for any nonzero u_0, $Bu_0 \neq 0$ and $x^\circ(t)$ cannot be identically zero.

Since $x^\circ(t)$ is defined for all $t \geq 0$, by definition of weak strict passivity the constraint

$$d(x^\circ(t), \tilde{u}(t)) = 0$$

implies $\lim_{t\to\infty} x^\circ(t) = 0$, which is a contradiction because $x^\circ(t)$ is a nonzero periodic function. Thus, $d(x^\circ(t), \tilde{u}(t))$ cannot be identically zero. As a consequence, since $d(x,u) \geq 0$ and $d(x^\circ(t), \tilde{u}(t))$ is a periodic function of period $\frac{2\pi}{\omega_0}$, it follows that

$$\int_0^{\frac{2\pi}{\omega_0}} d(x^\circ(t), \tilde{u}(t))dt > 0 \,.$$

Integration of the dissipation inequality

$$\frac{\partial V}{\partial x} f(x, \tilde{u}) \leq \tilde{y}^{\mathrm{T}}\tilde{u} - d(x,u)$$

over the interval $[0, \frac{2\pi}{\omega_0}]$ yields, bearing in mind the fact that

$$x^\circ(0) = x^\circ(\frac{2\pi}{\omega_0}) \,,$$

the inequality

$$\int_0^{\frac{2\pi}{\omega_0}} \tilde{y}^{\mathrm{T}}(t)\tilde{u}(t)dt \geq \int_0^{\frac{2\pi}{\omega_0}} d(x^\circ(t), \tilde{u}(t))dt > 0 \,.$$

In view of the specific form of $\tilde{u}(t)$ and $\tilde{y}(t)$, this inequality in turn yields

$$u_0^{\mathrm{T}}\mathrm{Re}[T(j\omega_0)]u_0 > 0 \,,$$

which indeed holds for every nonzero u_0 and every ω_0. From this, bearing in mind the fact that $T(s)$ is a matrix of rational functions with real coefficients, standard manipulations show that the frequency response matrix $T(j\omega)$ of a weakly strictly passive linear system is such that

$$T(j\omega) + T^{\mathrm{T}}(-j\omega) > 0$$

for all $\omega \in \mathbb{R}$.

The result below shows that also the converse property is true.

Theorem 10.9.2. *Consider the linear system (10.95). Suppose that B has rank m. Suppose also the pair (A,B) is controllable and the pair (C,A) is observable. The following are equivalent:*

(i) *the system is weakly strictly passive,*

(ii) *all the eigenvalues of A have negative real part and the frequency response matrix of the system $T(j\omega) = C(j\omega I - A)^{-1}B + D$ is such that*

$$T(j\omega) + T^{\mathrm{T}}(-j\omega) > 0$$

for all $\omega \in \mathbb{R}$,

(iii) *there exist an $n \times n$ symmetric positive definite matrix P, an $m \times m$ matrix K and an $m \times n$ matrix L such that*

$$\begin{aligned} A^{\mathrm{T}}P + PA &= -L^{\mathrm{T}}L \\ C &= B^{\mathrm{T}}P + K^{\mathrm{T}}L \\ D + D^{\mathrm{T}} &= K^{\mathrm{T}}K \,, \end{aligned} \tag{10.112}$$

and such that

$$\det\begin{pmatrix} A - sI & B \\ L & K \end{pmatrix} = n + m$$

for all s such that $\mathrm{Re}[s] \geq 0$.

Proof. The implication (i) $\Rightarrow$ (ii) has already been proven. To show that (ii) $\Rightarrow$ (iii), we begin by recalling [8] that, if in the linear system (10.95) the matrix A has all eigenvalues with negative real part and the transfer matrix $T(s)$ is such that

$$T(j\omega) + T^{\mathrm{T}}(-j\omega) \geq 0$$

for all $\omega \in \mathbb{R}$, then there exists an $r \times m$ matrix of proper rational functions $V(s)$, in which r is the rank of $T(s) + T^{\mathrm{T}}(-s)$ over the field of rational functions of s, such that: $\mathrm{rank}[V(s)] = r$ for all $s \in \mathbb{C}^+$, all poles of $V(s)$ are in $\mathbb{C}^-$, and

$$T(s) + T^{\mathrm{T}}(-s) = V^{\mathrm{T}}(-s)V(s) \tag{10.113}$$

for all $s \in \mathbb{C}$.

In the present case, condition (ii) implies that the matrix $T(s)+T^{\mathrm{T}}(-s)$ is nonsingular for all $s \in \mathbb{C}^0$, and therefore necessarily $r = m$ and $\mathrm{rank}[V(s)] = m$ for all s with $\mathrm{Re}[s] \geq 0$.

Let F, G, H, K be a minimal realization of $V(s)$. Since $V(s)$ has all poles in $\mathbb{C}^-$, the matrix F has all eigenvalues in $\mathbb{C}^-$ and, since the pair (F,H) is observable, the Lyapunov equation

$$\overline{P}F + F^{\mathrm{T}}\overline{P} = -H^{\mathrm{T}}H \tag{10.114}$$

has a unique solution $\overline{P} > 0$.

[8] See e.g. Youla (1961)

Observe that the transfer function matrix $V^{\mathrm{T}}(-s)V(s)$ has a state-space realization of the form

$$\begin{aligned}\begin{pmatrix}\dot z_1\\ \dot z_2\end{pmatrix} &= \begin{pmatrix}F & 0\\ -H^{\mathrm{T}}H & -F^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \begin{pmatrix}G\\ -H^{\mathrm{T}}K\end{pmatrix}v\\ w &= \begin{pmatrix}K^{\mathrm{T}}H & G^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + K^{\mathrm{T}}Kv\,,\end{aligned}$$

and that the latter, via the change of coordinates

$$\tilde z_2 = z_2 - \bar P z_1\,,$$

because of (10.114) can be transformed in

$$\begin{aligned}\begin{pmatrix}\dot z_1\\ \dot{\tilde z}_2\end{pmatrix} &= \begin{pmatrix}F & 0\\ 0 & -F^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}z_1\\ \tilde z_2\end{pmatrix} + \begin{pmatrix}G\\ -H^{\mathrm{T}}K - \bar P G\end{pmatrix}v\\ w &= \begin{pmatrix}K^{\mathrm{T}}H + G^{\mathrm{T}}\bar P & G^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}z_1\\ \tilde z_2\end{pmatrix} + K^{\mathrm{T}}Kv\,.\end{aligned}$$

This shows that

$$\begin{aligned}&V^{\mathrm{T}}(-s)V(s) =\\ &K^{\mathrm{T}}K + (K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P})(sI - F)^{-1}G + [(K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P})(-sI - F)^{-1}G]^{\mathrm{T}}\end{aligned}$$

and (10.113) yields

$$\begin{aligned}&D + D^{\mathrm{T}} + C(sI - A)^{-1}B + [C(-sI - A)^{-1}B]^{\mathrm{T}} =\\ &K^{\mathrm{T}}K + (K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P})(sI - F)^{-1}G + [(K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P})(-sI - F)^{-1}G]^{\mathrm{T}}\,.\end{aligned} \tag{10.115}$$

This identity, equating the constant term and the term with poles in $\mathbb{C}^-$, yields

$$D + D^{\mathrm{T}} = K^{\mathrm{T}}K\,,$$

which is the last one of the (10.112) and

$$C(sI - A)^{-1}B = (K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P})(sI - F)^{-1}G\,. \tag{10.116}$$

It is possible to show that the pair $F, K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P}$ is observable. To this end, observe that, since the matrix

$$V(s) = K + H(sI - F)^{-1}G$$

has rank m for all s such that $\mathrm{Re}[s] \geq 0$ and

$$\begin{pmatrix}F - sI & G\\ H & K\end{pmatrix} = \begin{pmatrix}I & 0\\ H(F - sI)^{-1} & I\end{pmatrix}\begin{pmatrix}F - sI & G\\ 0 & V(s)\end{pmatrix}\,,$$

it follows that

$$\mathrm{rank}\begin{pmatrix}F - sI & G\\ H & K\end{pmatrix} = n + m$$

for all s such that $\mathrm{Re}[s] \geq 0$.

Now, suppose the pair $F, K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P}$ is not observable. Then, there exists a nonzero vector v such that

$$\begin{pmatrix}F + \lambda I\\ K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P}\end{pmatrix}v = 0$$

for some λ, i.e.

$$\begin{aligned}&Fv = -\lambda v\\ &K^{\mathrm{T}}Hv + G^{\mathrm{T}}\overline{P}v = 0.\end{aligned}$$

Necessarily, $\mathrm{Re}[\lambda] > 0$, because F has all eigenvalues with negative real part. Observe that

$$\begin{pmatrix}F^{\mathrm{T}} - \lambda I & H^{\mathrm{T}}\\ G^{\mathrm{T}} & K^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\overline{P}v\\ Hv\end{pmatrix} = \begin{pmatrix}F^{\mathrm{T}}\overline{P}v - \lambda\overline{P}v + H^{\mathrm{T}}Hv\\ K^{\mathrm{T}}Hv + G^{\mathrm{T}}\overline{P}v\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}\,.$$

This shows (note that $\overline{P}v \neq 0$) that the matrix

$$\begin{pmatrix}F^{\mathrm{T}} - \lambda I & H^{\mathrm{T}}\\ G^{\mathrm{T}} & K^{\mathrm{T}}\end{pmatrix}$$

is singular at some λ with $\mathrm{Re}[\lambda] > 0$, and this contradicts an earlier conclusion.

Identity (10.116) shows that $\{A, B, C\}$ and $\{F, G, K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P}\}$ are both minimal realizations of the same transfer function matrix, and therefore there must exist a nonsingular matrix T such that

$$TAT^{-1} = F, \quad TB = G, \quad CT^{-1} = K^{\mathrm{T}}H + G^{\mathrm{T}}\overline{P}\,.$$

Defining $P = T^{\mathrm{T}}\overline{P}T$ and $L = HT$, this shows that

$$\begin{aligned}&PA + A^{\mathrm{T}}P = -L^{\mathrm{T}}L\\ &C = K^{\mathrm{T}}L + B^{\mathrm{T}}P\,,\end{aligned}$$

which completes the proof of (10.112).

Since the matrix

$$V(s) = K + H(sI - F)^{-1}G = K + L(sI - A)^{-1}B$$

has rank m for all s such that $\mathrm{Re}[s] \geq 0$, an argument identical to one used before shows that

$$\mathrm{rank}\begin{pmatrix}A - sI & B\\ L & K\end{pmatrix} = n + m$$

for all s such that $\mathrm{Re}[s] \geq 0$ and this completes the proof of (iii).

To show that (iii) $\Rightarrow$ (i), define $V(x) = \frac{1}{2}x^{\mathrm{T}}Px$ and, using (10.112), observe that

$$\begin{aligned} 2\dot{V} &= 2x^{\mathrm{T}}PAx + 2x^{\mathrm{T}}PBu \\ &= -x^{\mathrm{T}}L^{\mathrm{T}}Lx + 2u^{\mathrm{T}}(C - K^{\mathrm{T}}L)x \\ &= -x^{\mathrm{T}}L^{\mathrm{T}}Lx + 2u^{\mathrm{T}}(C - K^{\mathrm{T}}L)x + 2u^{\mathrm{T}}Du - 2u^{\mathrm{T}}Du \\ &= -x^{\mathrm{T}}L^{\mathrm{T}}Lx + 2u^{\mathrm{T}}y - 2u^{\mathrm{T}}K^{\mathrm{T}}Lx - u^{\mathrm{T}}K^{\mathrm{T}}Ku \,. \end{aligned}$$

Thus,

$$\dot{V} = y^{\mathrm{T}}u - d(x,u)$$

where

$$d(x,u) = \frac{1}{2}\|Lx + Ku\|^2 \,.$$

Now, $d(x,u) = 0$ implies $Lx + Ku = 0$. Thus, the system would be weakly strictly passive if so happens that any pair $x(\cdot), u(\cdot)$ satisfying

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ 0 &= Lx(t) + Ku(t) \end{aligned}$$

is such that $\lim_{t\to\infty} x(t) = 0$. The Laplace transform of such pair, denoted $\mathcal{X}(s), \mathcal{U}(s)$, is a solution of the equation

$$\begin{pmatrix} A - sI & B \\ L & K \end{pmatrix} \begin{pmatrix} \mathcal{X}(s) \\ \mathcal{U}(s) \end{pmatrix} = \begin{pmatrix} -x(0) \\ 0 \end{pmatrix} ,$$

and the matrix

$$\begin{pmatrix} A - sI & B \\ L & K \end{pmatrix}$$

can lose rank only at values of s having negative real part. Thus, $\mathcal{X}(s)$ and $\mathcal{U}(s)$ are vectors of rational functions having all poles in $\mathbb{C}^-$ and $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the fact that (10.95) is weakly strictly passive. $\triangleleft$

We conclude the section with another interesting property of passive linear systems.

Lemma 10.9.3. *Consider a linear system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \,. \end{aligned} \tag{10.117}$$

Suppose there exists a symmetric matrix $P > 0$ such that

$$\begin{aligned} &PA + A^{\mathrm{T}}P \le 0 \\ &C = B^{\mathrm{T}}P \,. \end{aligned}$$

Then, the system is passive. Moreover, the pair (C, A) is detectable if and only if the pair (A, B) is stabilizable.

Proof. Set $V(x) = \frac{1}{2}x^{\mathrm{T}}Px$, $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$ and observe that $V(x)$ satisfies

$$\frac{\partial V}{\partial x} f(x) \le 0, \qquad \frac{\partial V}{\partial x} g(x) = h^{\mathrm{T}}(x) \,.$$

Therefore, as shown at the end of section 10.7, the system is passive.

Observe now that if the pair (C, A) is detectable, the system is indeed zero-state detectable and therefore, using the result of Corollary 10.8.4, in which we take $\varphi(y) = y$, it can be concluded that the system is globally asymptotically stabilized by the (linear) feedback law $u = -Cx$. This trivially shows that the pair (A, B) is stabilizable.

To show that the converse is also true, observe that, as shown in the previous section, a feedback law of the form

$$\bar{u}(x) = -Cx = -B^{\mathrm{T}}Px$$

is such that

$$\begin{aligned} \frac{\partial V}{\partial x}[Ax + B\bar{u}(x)] &= \frac{1}{2}x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x - x^{\mathrm{T}}PBB^{\mathrm{T}}Px \\ &\le -x^{\mathrm{T}}PBB^{\mathrm{T}}Px = -\|\bar{u}(x)\|^2 \,. \end{aligned} \tag{10.118}$$

Therefore, the system

$$\dot{x} = Ax + B\bar{u}(x) = (A - BC)x \tag{10.119}$$

is stable in the sense of Lyapunov, and the ω-limit set Γ of any trajectory satisfies $\Gamma \subset \{x \in \mathbb{R}^n : B^{\mathrm{T}}Px = 0\}$.

Any initial condition $\tilde{x} \in \Gamma$ produces a trajectory $x(t)$ such that $V(x(t))$ is constant, and therefore

$$\begin{aligned} &x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x(t) = 0 \\ &\bar{u}(x(t)) = 0 \,. \end{aligned}$$

The first one of these, since $(PA + A^{\mathrm{T}}P)$ is symmetric and negative semi-definite, yields $(PA + A^{\mathrm{T}}P)x(t) = 0$, i.e.

$$PAx(t) = -A^{\mathrm{T}}Px(t) \,. \tag{10.120}$$

The second one, observing that $x(t)$ is necessarily a trajectory of $\dot{x} = Ax$ (because $\bar{u}(x(t)) = 0$), yields, for any $k \ge 0$,

$$0 = \frac{\mathrm{d}^k\bar{u}(x(t))}{\mathrm{d}t^k} = \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\left[\frac{\partial \bar{u}}{\partial x}\right]Ax(t) = -\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}B^{\mathrm{T}}PAx(t) = -B^{\mathrm{T}}PA^kx(t)$$

and hence, using repeatedly (10.120),

$$B^{\mathrm{T}}(A^{\mathrm{T}})^kPx(t) = 0 \,.$$

Putting all these constraints together yields

$$x^{\mathrm{T}}(t)P\,(B \quad AB \quad \cdots \quad A^kB\,) = 0. \tag{10.121}$$

Now, suppose the pair (A, B) is stabilizable, decompose the system into controllable/uncontrollable parts

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 + B_1u \\ A_{22}x_2 \end{pmatrix}$$

and the matrix P accordingly. By hypothesis of stabilizability, all the eigenvalues of A_{22} have negative real part, and (A_{11}, B_1) is a controllable pair.

Since

$$A^kB = \begin{pmatrix} A_{11}^kB_1 \\ 0 \end{pmatrix}$$

and (A_{11}, B_1) is a controllable pair, the constraint (10.121) implies

$$x_1^{\mathrm{T}}(t)P_{11} + x_2^{\mathrm{T}}(t)P_{12} = 0\,,$$

i.e. (note that P_{11} is positive definite, and therefore nonsingular)

$$x_1^{\mathrm{T}}(t) = -x_2^{\mathrm{T}}(t)P_{12}P_{11}^{-1}\,.$$

This, since $x_2(t) \to 0$ as $t \to \infty$, shows that also $x_1(t) \to 0$ as $t \to \infty$. As a consequence, $V(x) = 0$ on the ω-limit set Γ of any trajectory of (10.119) and the equilibrium $x = 0$ of the latter is globally asymptotically stable.

Suppose now that there exists a trajectory of system $\dot{x} = Ax$ for which $y = Cx$ is identically zero. Since Cx coincides (modulo the sign) with the feedback law yielding the closed-loop system (10.119), the trajectory in question is necessarily also a trajectory of (10.119) and hence converges to zero as $t \to \infty$. This proves that system (10.117) is zero-state detectable. ◁

11. Feedback Design for Robust Global Stability

11.1 Preliminaries

The purpose of this Chapter is to describe some important tools for the design of feedback laws which globally asymptotically stabilize a nonlinear system in the presence of parameter uncertainties. We consider the case in which the mathematical model of the system to be controlled depends on a vector $\mu \in \mathbb{R}^p$ of parameters, which are assumed to be constant, but whose actual values are unknown to the designer. The vector μ of unknown parameters could be any vector in some ***a priori given*** set $\mathcal{P}$, and the goal of the design is to find a feedback law (obviously *independent* of μ) which globally asymptotically stabilizes the system for each value of $\mu \in \mathcal{P}$. A problem of this type is usually referred to as a problem of *robust* stabilization.

In section 9.2, we have seen that a powerful method to address a problem of global stabilization, for certain classes of nonlinear systems, is a recursive procedure which – based on a repeated use of the result of Lemma 9.2.2 – consists in solving the problem in question for a sequence of subsystems of increasing dimension. This procedure is commonly referred to as the method of *backstepping*. In view of this, a natural point of departure for the analysis of a problem of robust stabilization seems to be the extension of the results of Lemma 9.2.1 and Lemma 9.2.2 to the case of systems affected by parameter uncertainties. In this section we outline the basic ideas which make such an extension possible, and in the subsequent sections we will address a number of specific situations, in order of increasing complexity.

Consider a system modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z, \xi, \mu) \\ \dot{\xi} &= q(z, \xi, \mu) + u \end{aligned} \tag{11.1}$$

with state $(z, \xi) \in \mathbb{R}^n \times \mathbb{R}$, input $u \in \mathbb{R}$, in which $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters and $\mathcal{P}$ is a *compact* set. Assume, as usual, that $f(z, \xi, \mu)$ and $q(z, \xi, \mu)$ are smooth functions of their arguments, and also that $f(0, 0, \mu) = 0$ and $q(0, 0, \mu) = 0$ for every $\mu \in \mathcal{P}$, so that the point $(z, \xi) = (0, 0)$ is an *equilibrium* (when $u = 0$) for every value of μ.

If the parameter μ were known exactly, the problem of stabilizing such a system could be addressed using the result of Lemma 9.2.2, which says that from the knowledge of a smooth function $v^\star(z)$, with $v^\star(0) = 0$, which globally asymptotically stabilizes the equilibrium $z = 0$ of

$$\dot{z} = f(z, v^\star(z), \mu) \tag{11.2}$$

and the knowledge of a Lyapunov function $V(z)$ for (11.2), it is possible to construct a feedback law $u = u(z, \xi)$, with $u(0,0) = 0$, which globally asymptotically stabilizes the equilibrium $(z, \xi) = (0,0)$ of (11.1).

The result in question essentially reposes on the following two observations:

(i) the (globally defined) change of variable

$$y = \xi - v^\star(z) \tag{11.3}$$

transforms system (11.1) into a system having the same structure, namely a system of the form

$$\begin{aligned} \dot{z} &= \tilde{f}(z, y, \mu) \\ \dot{y} &= \tilde{q}(z, y, \mu) + u , \end{aligned} \tag{11.4}$$

where

$$\begin{aligned} \tilde{f}(z, y, \mu) &= f(z, v^\star(z) + y, \mu) \\ \tilde{q}(z, y, \mu) &= q(z, v^\star(z) + y, \mu) - \frac{\partial v^\star}{\partial z} f(z, v^\star(z) + y, \mu) , \end{aligned}$$

but in which, as a consequence of the hypothesis on (11.2), the equilibrium $z = 0$ of

$$\dot{z} = \tilde{f}(z, 0, \mu) \tag{11.5}$$

is globally asymptotically stable, with Lyapunov function $V(z)$,

(ii) once it is known that the equilibrium $z = 0$ of (11.5) is globally asymptotically stable, with Lyapunov function $V(z)$, it is very easy to determine a feedback law $u = \tilde{u}(z, y)$ which globally asymptotically stabilizes the equilibrium $(z, y) = (0, 0)$ of (11.4).

In fact, knowing that $V(z)$ is a Lyapunov function for (11.5), it is easy to determine the expression of a feedback law $u = \tilde{u}(z, y)$ which renders

$$W(z, y) = V(z) + \frac{1}{2} y^2$$

a Lyapunov function for

$$\begin{aligned} \dot{z} &= \tilde{f}(z, y, \mu) \\ \dot{y} &= \tilde{q}(z, y, \mu) + \tilde{u}(z, y) . \end{aligned} \tag{11.6}$$

To this end, it suffices to expand $\tilde{f}(z, y, \mu)$ as

$$\tilde{f}(z, y, \mu) = \tilde{f}(z, 0, \mu) + p(z, y, \mu) y ,$$

and to impose that (recall the proof of Lemma 9.2.1)

$$\begin{aligned} \dot{W} &= \frac{\partial V}{\partial z} \tilde{f}(z, y, \mu) + y\Big(\tilde{q}(z, y, \mu) + \tilde{u}(z, y)\Big) \\ &= \frac{\partial V}{\partial z} \tilde{f}(z, 0, \mu) + \frac{\partial V}{\partial z} y p(z, y, \mu) + y \tilde{q}(z, y, \mu) + y \tilde{u}(z, y) \\ &= \frac{\partial V}{\partial z} \tilde{f}(z, 0, \mu) - y^2 , \end{aligned}$$

which is achieved choosing $\tilde{u}(z, y)$ as

$$\tilde{u}(z, y) = -\frac{\partial V}{\partial z} p(z, y, \mu) - \tilde{q}(z, y, \mu) - y .$$

This feedback law renders $\dot{W}$ negative definite and hence globally asymptotically stabilizes the equilibrium $(z, y) = (0, 0)$ of (11.4).

Reversing the change of variables (11.3) yields a feedback law

$$u(z, \xi) = \tilde{u}(z, \xi - v^\star(z))$$

which globally asymptotically stabilizes the equilibrium $(z, \xi) = (0, 0)$ of (11.1).

An immediate obstruction to the extension of this procedure to the case in which the parameter μ is unknown is the fact that the method in question relies upon the *exact cancellation* of certain terms in the derivative of the candidate Lyapunov function $W(z, y)$. If the actual value of μ is not known, such a cancellation is no longer possible, and the procedure is not applicable. Therefore, in case μ is unknown, the stabilization problem of a system such as (11.4) must be addressed on different grounds. The issue is to determine a feedback law $\tilde{u}(z, y)$ which globally asymptotically stabilizes the equilibrium $(z, y) = (0, 0)$ of (11.6). Since in no way the term $\tilde{u}(z, y)$ can be used to "cancel" the term $\tilde{q}(z, y, \mu)$ and the contribution of $p(z, y, \mu)$ to the derivative of the candidate Lyapunov function $W(z, y)$, the best way to proceed seems to be the following one. Recall that system (11.5), which is obtained by setting $y = 0$ in the upper subsystem of (11.6), is globally asymptotically stable by hypothesis, and observe that the system

$$\dot{y} = \tilde{q}(0, y, \mu) + \tilde{u}(0, y) , \tag{11.7}$$

which is obtained by setting $z = 0$ in the lower subsystem of (11.6), can be rendered globally asymptotically stable, by appropriate choice of $\tilde{u}(0, y)$, if the set $\mathcal{P}$ on which the uncertain parameter μ ranges is a *compact* set. To this end, note that the function $\tilde{q}(0, y, \mu)$, which by hypothesis vanishes at $y = 0$ and is a smooth function of y and μ, can be expressed in the form

$$\tilde{q}(0, y, \mu) = k(y, \mu) y$$

for some $k(y, \mu)$, and, if $\mathcal{P}$ is a compact set, there exists a continuous function $\rho(y)$, positive for all y, such that

$$|k(y,\mu)| \leq \rho(y)$$

for all y and μ. This being the case, it is immediate to check that the input

$$\tilde{u}(0,y) = -2y\rho(y)$$

globally asymptotically stabilizes the equilibrium $y = 0$ of (11.7), actually with Lyapunov function y^2. In fact, this choice yields, along the trajectories of (11.7),

$$\frac{\mathrm{d}y^2}{\mathrm{d}t} = 2y^2 k(y,\mu) - 4y^2\rho(y) \leq -2y^2\rho(y)\ .$$

This being the case, one may look at system (11.6) as to the interconnection of two stable systems, and try to appeal to the *small gain theorem*. Of course, global asymptotic stability of (11.5) and (11.7) is only a prerequisite for the application of the small gain theorem. What one should do is to check that the upper subsystem in (11.6), viewed as a system with input y and state z, is actually input-to-state stable and to determine an upper estimate of a "gain" function, namely a class $\mathcal{K}$ function $\gamma(\cdot)$ which renders the estimate

$$\|z(t)\| \leq \max\{\beta(\|z^\circ\|,t), \gamma(\|y(\cdot)\|_\infty)\}$$

fulfilled (for some class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$). Then, one should design a feedback law $\tilde{u}(z,y)$ so as to obtain an input-to-state stable lower subsystem and to impose, on it, a "gain" function which would render the "small-gain" condition fulfilled. As difficult as it may seem at a first glance, this way of proceeding turns out to be pretty effective and helps in successfully addressing the robust stabilization problem.

This procedure sketches how the robust stabilization problem can be addressed in case the upper system of (11.1) is already input-to-state stable. In some sense, if successful, it would produce a result which is the robust version of Lemma 9.2.1. If that system is not input-to-state stable, then a preliminary step is needed, which fortunately can always be accomplished if the subsystem in question is globally asymptotically *stabilizable* (i.e. if hypotheses similar to those used in Lemma 9.2.2 hold). In fact, suppose there exists a smooth function $v^\star(z)$, with $v^\star(0) = 0$, which globally asymptotically stabilizes the equilibrium $z = 0$ of (11.2). Then, using Theorem 10.4.3, it follows that there exists also a smooth function $\beta^\star(z)$, nowhere zero, such that the system

$$\dot{z} = f(z, v^\star(z) + \beta^\star(z)y, \mu)\ ,$$

viewed as a system with state z and input y, is input-to-state stable. Since $\beta^\star(z)$ is nowhere zero, the procedure of step (i) can be repeated, considering now the change of variable

$$y = [\beta^\star(z)]^{-1}[\xi - v^\star(z)]\ ,$$

which transform system (11.1) into a system having a similar structure, namely

$$\begin{aligned} \dot{z} &= \bar{f}(z,y,\mu) \\ \dot{y} &= \bar{q}(z,y,\mu) + [\beta^\star(z)]^{-1}u\ , \end{aligned}$$

but in which the upper subsystem is by construction input-to-state stable. In this way, the problem can be reduced to the problem considered above, namely the problem of designing a control $u = \tilde{u}(z,y)$ able of making the lower subsystem input-to-state stable, with a "gain" function which would render the "small-gain" condition fulfilled.

In this Chapter, we describe some important situations in which this design procedure can be successfully applied. We begin with the analysis of simple cases in which, due to the particular structure of the system and to some hypotheses, the upper subsystem is already input-to-state stable, and, moreover, the actions needed to secure that the lower subsystem has the appropriate "gain" function are automatically build-in in the procedure used to check that a certain candidate Lyapunov function becomes an actual Lyapunov function, if a suitable feedback law is imposed. After this, we address substantially more general situations.

11.2 Stabilization via Partial State Feedback: a Special Case

We begin this section by addressing the case of a system modeled by equations of the form

$$\begin{aligned} \dot{z} &= F(\mu)z + G(y,\mu)y \\ \dot{y} &= H(y,\mu)z + K(y,\mu)y + b(y,\mu)u\ , \end{aligned} \tag{11.8}$$

in which $z \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters, with $\mathcal{P}$ a *compact* set. The functions $F(\mu)$, $G(y,\mu)$, $H(y,\mu)$, $K(y,\mu)$ and $b(y,\mu)$ are smooth functions of their arguments.

A system of this form is perhaps one of the simplest examples of a nonlinear system which – under appropriate hypotheses – can be globally robustly stabilized, actually by means of a feedback law of the form

$$u = \alpha(y)\ .$$

Note that, if the coefficient $b(y,\mu)$ is nowhere zero, this system can be regarded as a system having uniform relative degree *one* between input u and output y, and a zero dynamics characterized by

$$\dot{z} = F(\mu)z\ . \tag{11.9}$$

The hypotheses on which the robust stabilization method reposes are the asymptotic stability of the dynamics (11.9), and the existence of a positive lower bound for $b(y,\mu)$. To understand, before proceeding with a formal proof, why these hypotheses are successful in the light of the general framework presented in the previous section, observe that if the dynamics (11.9) are asymptotically stable, then automatically the upper subsystem is input-to-state stable. In fact, this system can be viewed as a linear system

$$\dot{z} = F(\mu)z + v$$

driven by the input $v = G(y,\mu)y$ and therefore, if (11.9) is asymptotically stable, its state response $z(t)$ satisfies a bound of the form

$$\|z(t)\| \le Me^{-at}\|z(0)\| + N \max_{0\le s\le t} \|G(y(s),\mu)y(s)\|$$

which indeed guarantees the property of input-to-state stability. On the other hand, the property that $b(y,\mu)$ is bounded away from zero makes it possible to impose a suitably strong control action on the lower subsystem, so as to achieve input-to-state stability and a prescribed gain.

Lemma 11.2.1. *Consider system (11.8). Suppose that:*

(i) *for each μ, the eigenvalues of $F(\mu)$ have negative real part, or, what is the same, there exists a symmetric matrix $P(\mu) > 0$ (continuously depending on μ) such that*

$$P(\mu)F(\mu) + F(\mu)P(\mu) = -I\,,$$

(ii) *there exists a real number $b_0 > 0$ such that*

$$b(y,\mu) \ge b_0$$

for all $y \in \mathbb{R}$ and all $\mu \in \mathcal{P}$.

Then, there exists a smooth function $\gamma(y)$ such that, if the system (11.8) is controlled by the input

$$u = -y\gamma(y)\,, \tag{11.10}$$

the positive definite function

$$V(z,y) = z^{\mathrm{T}}P(\mu)z + y^2$$

satisfies

$$\frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial y}\dot{y} < -\varepsilon(\|z\|^2 + y^2)$$

for some $\varepsilon > 0$. As a consequence, the feedback law (11.10) globally asymptotically stabilizes the system for any value of $\mu \in \mathcal{P}$.

Proof. Observe that

$$\frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial y}\dot{y} =$$

$$-z^{\mathrm{T}}z + 2z^{\mathrm{T}}P(\mu)G(y,\mu)y + 2yH(y,\mu)z + 2[K(y,\mu) - b(y,\mu)\gamma(y)]y^2 =$$

$$-\begin{pmatrix} z \\ y \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I & -P(\mu)G(y,\mu) - H^{\mathrm{T}}(y,\mu) \\ -G^{\mathrm{T}}(y,\mu)P(\mu) - H(y,\mu) & 2[b(y,\mu)\gamma(y) - K(y,\mu)] \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}.$$

It is easy to see that, if $\gamma(y)$ is appropriately chosen,

$$\begin{pmatrix} I & -P(\mu)G(y,\mu) - H^{\mathrm{T}}(y,\mu) \\ -G^{\mathrm{T}}(y,\mu)P(\mu) - H(y,\mu) & 2[b(y,\mu)\gamma(y) - K(y,\mu)] \end{pmatrix} - \varepsilon I > 0 \tag{11.11}$$

for some $\varepsilon > 0$. To this end, recall that if Q_0 and R_0 are symmetric matrices and Q_0 is nonsingular

$$\begin{pmatrix} Q_0 & S_0 \\ S_0^{\mathrm{T}} & R_0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ S_0^{\mathrm{T}}Q_0^{-1} & I \end{pmatrix} \begin{pmatrix} Q_0 & 0 \\ 0 & R_0 - S_0^{\mathrm{T}}Q_0^{-1}S_0 \end{pmatrix} \begin{pmatrix} I & Q_0^{-1}S_0 \\ 0 & I \end{pmatrix},$$

so that the matrix on the left-hand side is positive definite if and only if $Q_0 > 0$ and $R_0 - S_0^{\mathrm{T}}Q_0^{-1}S_0 > 0$.

Now, let ε be any number such that $0 < \varepsilon < 1$. Then, the matrix in (11.11) is positive definite if and only if

$$\begin{aligned} &2[b(y,\mu)\gamma(y) - K(y,\mu)] - \varepsilon \\ &\quad > [P(\mu)G(y,\mu) + H^{\mathrm{T}}(y,\mu)]^{\mathrm{T}} \frac{1}{1-\varepsilon}[P(\mu)G(y,\mu) + H^{\mathrm{T}}(y,\mu)]\,. \end{aligned}$$

Clearly, this inequality holds if $\gamma(y)$ is such that

$$2b(y,\mu)\gamma(y) > \frac{1}{1-\varepsilon}(\|P(\mu)G(y,\mu) + H^{\mathrm{T}}(y,\mu)\|)^2 + 2K(y,\mu) + \varepsilon\,. \tag{11.12}$$

Observe now that, since by hypothesis μ ranges over a compact set, there exist continuous functions $g(y), h(y), k(y)$, and a number $P_M > 0$, such that

$$\begin{aligned} \|G(y,\mu)\| &\le g(y) \\ \|H(y,\mu)\| &\le h(y) \\ |K(y,\mu)| &\le k(y) \\ \|P(\mu)\| &\le P_M \end{aligned} \tag{11.13}$$

for all y and all μ. Then, (11.12) holds if

$$\gamma(y) > \frac{1}{2b_0(1-\varepsilon)}[(P_M g(y) + h(y))^2 + 2k(y) + \varepsilon]\,.$$

If $\gamma(y)$ is chosen in this way

$$\frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial y}\dot{y} < -\varepsilon(\|z\|^2 + y^2)\,,$$

and this concludes the proof. ◁

Remark 11.2.1. Of course, a similar result holds if, instead of condition (ii), a condition of the form

$$b(y,\mu) \le -b_0 < 0\,,$$

with b_0 is a fixed number, is assumed. In this case, the stabilizing law would be $u = y\gamma(y)$.

If the set $\mathcal{P}$ is not compact, then the hypothesis that estimates of the form (11.13) hold must be explicitly invoked. ◁

Remark 11.2.2. Note that the result remains valid in the more general case of a system of the form

$$\begin{aligned}\dot{z} &= F(y,\mu)z + G(y,\mu)y\\ \dot{y} &= H(y,\mu)z + K(y,\mu)y + b(y,\mu)u\,,\end{aligned}$$

so long as there exists a symmetric matrix $P(\mu) > 0$, depending only on μ and not on y, such that

$$P(\mu)F(y,\mu) + F^{\mathrm{T}}(y,\mu)P(\mu) = -I\,,$$

for all $y \in \mathbb{R}$ and all $\mu \in \mathcal{P}$. ◁

Remark 11.2.3. The simple basic mechanism behind the proof of the Lemma is that the choice of a large $\gamma(y)$ helps to (robustly) offset the effect of the off-diagonal terms in the matrix (11.11), so as to render it positive definite. This is indeed a "small-gain" argument, because such an action weakens the coupling between the upper and lower subsystem of (11.8).

To see an alternative proof of this Lemma, which directly uses the small gain Theorem for input-to-state stable systems, one should proceed as follows. Set

$$V_1(z) = z^{\mathrm{T}}P(\mu)z\,, \qquad V_2(y) = y^2\,.$$

Observe that

$$\frac{\partial V_1}{\partial z}\dot{z} = -z^{\mathrm{T}}z + 2z^{\mathrm{T}}P(\mu)G(y,\mu)y \le -z^{\mathrm{T}}z + \delta z^{\mathrm{T}}P(\mu)P(\mu)z + \frac{1}{\delta}\|G(y,\mu)\|^2y^2$$

for any $\delta > 0$.

Recall also that, if $\underline{\lambda} > 0$ and $\overline{\lambda} > 0$ are the smallest and, respectively, the largest eigenvalue of a symmetric matrix $M > 0$,

$$\underline{\lambda}\|x\|^2 \le x^{\mathrm{T}}Mx \le \overline{\lambda}\|x\|^2$$

for all x. If M depends continuously on a parameter μ and μ ranges over a compact set $\mathcal{P}$, then an estimate of the form

$$\underline{c}\|x\|^2 \le x^{\mathrm{T}}Mx \le \overline{c}\|x\|^2\,,$$

in which $0 < \underline{c} < \overline{c}$ are independent of μ, holds for all x and all $\mu \in \mathcal{P}$.

Bearing this fact and the estimates (11.13) in mind, choose δ such that the matrix $I - \delta P(\mu)P(\mu)$ is positive definite for all μ and observe that the previous inequality yields

$$\frac{\partial V_1}{\partial z}\dot{z} \le -a\|z\|^2 + bg^2(y)y^2$$

in which $a > 0$ and $b > 0$ are fixed numbers, and

$$\underline{a}\|z\|^2 \le V_1(z) \le \overline{a}\|z\|^2$$

for some $0 < \underline{a} < \overline{a}$.

Without loss of generality, assume that $g(y) = g(-y)$ and that $g(|y|)$ is non decreasing, so that the latter inequality can be written in the form

$$\frac{\partial V_1}{\partial z}\dot{z} \le -a\|z\|^2 + \sigma(|y|)$$

where $\sigma(\cdot)$ is the class $\mathcal{K}_\infty$ function

$$\sigma(r) = b[g(r)r]^2\,.$$

As a consequence (see Remark 10.4.3), we see that the response $z(\cdot)$ to any bounded $y(\cdot)$ is such that

$$\|z(t)\| \le \max\{\beta_1(\|z^\circ\|,t), \gamma_1(\|y(\cdot)\|_\infty)\}$$

for some class $\mathcal{KL}$ function $\beta_1(\cdot,\cdot)$, where $\gamma_1(\cdot)$ is a class $\mathcal{K}_\infty$ function of the form

$$\gamma_1(r) = \sqrt{\frac{\overline{a}}{\underline{a}}}\sqrt{\frac{k\sigma(r)}{a}}\,,$$

in which $k > 1$. In particular, for some $d > 0$,

$$\gamma_1(r) = dg(r)r\,.$$

Let $\varphi(y)$ be a continuous function satisfying $\varphi(y) = \varphi(-y)$, such that $\varphi(|y|)$ is positive and non-decreasing, and choose $\gamma(y)$ so that

$$2b_0\gamma(y) > h^2(y) + 2k(y) + \varphi^2(y)\,.$$

Then, observe that

$$\begin{aligned}\frac{\partial V_2}{\partial y}\dot{y} &= 2yH(y,\mu)z + 2K(y,\mu)y^2 - 2b(y,\mu)\gamma(y)y^2\\ &\le z^{\mathrm{T}}z + h^2(y)y^2 + 2k(y)y^2 - 2b_0\gamma(y)y^2\\ &\le -\alpha(|y|) + \|z\|^2\,,\end{aligned}$$

where $\alpha(\cdot)$ is the class $\mathcal{K}_\infty$ function

$$\alpha(r) = [\varphi(r)r]^2\,.$$

As a consequence, we see that the response $y(\cdot)$ to any bounded $z(\cdot)$ is such that

$$|y(t)| \le \max\{\beta_2(|y^\circ|,t), \gamma_2(\|z(\cdot)\|_\infty)\}$$

for some class $\mathcal{KL}$ function $\beta_2(\cdot,\cdot)$, where $\gamma_2(\cdot)$ is a class $\mathcal{K}_\infty$ function of the form

$$\gamma_2(r) = \alpha^{-1}(kr^2)\,,$$

in which $k > 1$.

Thus, system (11.8) with input (11.10) can be viewed as interconnection of two input-to-state stable systems. This system is globally asymptotically stable if the "small gain condition"

$$\gamma_1(r) < \gamma_2^{-1}(r)$$

holds for all $r > 0$. This condition reduces to

$$dg(r)r < \frac{1}{\sqrt{k}}\varphi(r)r$$

and can indeed be satisfied by appropriate choice of $\varphi(r)$. ◁

Having shown how to stabilize a system of the form (11.8), we can use the methodology described in the previous section to prove a basic Lemma, which makes it possible to iteratively stabilize classes of systems having a "chained" structure.

Lemma 11.2.2. *Consider a system of the form*

$$\begin{aligned} \dot z &= F(\mu)z + G(x_1,\mu)x_1 \\ \dot x_1 &= H_1(x_1,\mu)z + K_1(x_1,\mu)x_1 + b_1(x_1,\mu)y \\ \dot y &= H_2(x_1,y,\mu)z + K_{21}(x_1,y,\mu)x_1 + K_{22}(x_1,y,\mu)y \\ &\quad + b_2(x_1,y,\mu)u \end{aligned} \tag{11.14}$$

in which $z \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^i$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters, with $\mathcal{P}$ a compact set. The functions $F(\mu)$, $G(x_1,\mu)$, $H_1(x_1,\mu)$, $K_1(x_1,\mu)$, $b_1(x_1,\mu)$, $H_2(x_1,y,\mu)$, $K_{21}(x_1,y,\mu)$, $K_{22}(x_1,y,\mu)$ and $b_2(x_1,y,\mu)$ are smooth functions of their arguments. Suppose that:

(i) *there exist a symmetric matrix $P(\mu) > 0$, continuously depending on μ, an $i \times i$ matrix $M_1(x_1)$ of smooth functions, nonsingular for all x_1, and a $1 \times i$ vector $\gamma_1(x_1)$ of smooth functions, such that*

$$\|M_1(x_1)x_1\|^2 \ge \underline{\alpha}(\|x_1\|) \qquad \text{for all } x_1 \in \mathbb{R}^i ,$$

for some class $\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$, and such that the positive definite function

$$V_1(z,x_1) = z^{\mathrm{T}}P(\mu)z + \|M_1(x_1)x_1\|^2$$

satisfies

$$\begin{aligned} &\begin{pmatrix} \frac{\partial V_1}{\partial z} & \frac{\partial V_1}{\partial x_1} \end{pmatrix} \begin{pmatrix} F(\mu)z + G(x_1,\mu)x_1 \\ H_1(x_1,\mu)z + K_1(x_1,\mu)x_1 + b_1(x_1,\mu)\gamma_1(x_1)x_1 \end{pmatrix} \\ &\qquad \le -\varepsilon(\|z\|^2 + \|M_1(x_1)x_1\|^2) , \end{aligned} \tag{11.15}$$

for some $\varepsilon > 0$,

(ii) *there exists a real number $b_{20} > 0$ such that*

$$b_2(x_1,y,\mu) \ge b_{20}$$

for all $(x_1,y) \in \mathbb{R}^{i+1}$ and all $\mu \in \mathcal{P}$.

Then, there exists a smooth function $\gamma_2(x_1,y)$ such that, if the system (11.14) is controlled by the input

$$u = -\gamma_2(x_1,y)[y - \gamma_1(x_1)x_1] , \tag{11.16}$$

the positive definite function

$$V_2(z,x_1,y) = z^{\mathrm{T}}P(\mu)z + \|M_1(x_1)x_1\|^2 + [y - \gamma_1(x_1)x_1]^2$$

satisfies

$$\frac{\partial V_2}{\partial z}\dot z + \frac{\partial V_2}{\partial x_1}\dot x_1 + \frac{\partial V_2}{\partial y}\dot y \le -\frac{\varepsilon}{2}(\|z\|^2 + \|M_1(x_1)x_1\|^2 + [y - \gamma_1(x_1)x_1]^2) .$$

Proof. For notational convenience, set

$$U_1(x_1) = \|M_1(x_1)x_1\|^2$$

and

$$\dot V_2 = \frac{\partial V_2}{\partial z}\dot z + \frac{\partial V_2}{\partial x_1}\dot x_1 + \frac{\partial V_2}{\partial y}\dot y .$$

Observe also that there exists a matrix $W(x_1)$ of smooth functions such that

$$\frac{\partial U_1}{\partial x_1} = 2x_1^{\mathrm{T}}W(x_1) .$$

Using the hypothesis (i), it is possible to obtain

$$\begin{aligned} \dot V_2 &\le -\varepsilon(\|z\|^2 + \|M_1(x_1)x_1\|^2) + [y - \gamma_1(x_1)x_1]\frac{\partial U_1}{\partial x_1}b_1(x_1,\mu) \\ &\quad + 2[y - \gamma_1(x_1)x_1]\Big(\dot y - \gamma_1(x_1)\dot x_1 - x_1^{\mathrm{T}}\frac{\partial \gamma_1^{\mathrm{T}}}{\partial x_1}\dot x_1\Big) \\ &= -\varepsilon(\|z\|^2 + \|M_1(x_1)x_1\|^2) \\ &\quad + 2[y - \gamma_1(x_1)x_1](x_1^{\mathrm{T}}W(x_1)b_1(x_1,\mu) + \dot y - \tilde\gamma_1(x_1)\dot x_1) , \end{aligned}$$

where we have set

$$\tilde\gamma_1(x_1) = \gamma_1(x_1) + x_1^{\mathrm{T}}\frac{\partial \gamma_1^{\mathrm{T}}}{\partial x_1} .$$

Bearing in mind the expressions of $\dot x_1$ and $\dot y$, write

$$\begin{aligned} &x_1^{\mathrm{T}}W(x_1)b_1(x_1,\mu) + \dot y - \tilde\gamma_1(x_1)\dot x_1 = \\ &A(x_1,y,\mu)z + B(x_1,y,\mu)x_1 + C(x_1,y,\mu)[y - \gamma_1(x_1)x_1] + b_2(x_1,y,\mu)u \end{aligned}$$

where

$$
\begin{aligned}
A(x_1,y,\mu) &= H_2(x_1,y,\mu) - \tilde\gamma_1(x_1)H_1(x_1,\mu)\\
B(x_1,y,\mu) &= b_1^{\mathrm{T}}(x_1,\mu)W^{\mathrm{T}}(x_1) + K_{21}(x_1,y,\mu)\\
&\quad - \tilde\gamma_1(x_1)\Big(K_1(x_1,\mu) + b_1(x_1,\mu)\gamma_1(x_1)\Big)\\
&\quad + K_{22}(x_1,y,\mu)\gamma_1(x_1)\\
C(x_1,y,\mu) &= K_{22}(x_1,y,\mu) - \tilde\gamma_1(x_1)b_1(x_1,\mu)\,.
\end{aligned}
$$

Choosing the control u as in (11.16) yields

$$
\dot V_2 \le - \begin{pmatrix} z \\ x_1 \\ [y-\gamma_1(x_1)x_1] \end{pmatrix}^{\mathrm{T}} Q(x_1,y,\mu) \begin{pmatrix} z \\ x_1 \\ [y-\gamma_1(x_1)x_1] \end{pmatrix}
$$

where

$$
Q(x_1,y,\mu) =
\begin{pmatrix}
\varepsilon I & 0 & -A^{\mathrm{T}}(x_1,y,\mu)\\
0 & \varepsilon M_1^{\mathrm{T}}(x_1)M_1(x_1) & -B^{\mathrm{T}}(x_1,y,\mu)\\
-A(x_1,y,\mu) & -B(x_1,y,\mu) & 2[b_2(x_1,y,\mu)\gamma_2(x_1,y) - C(x_1,y,\mu)]
\end{pmatrix}.
$$

Since by hypothesis μ ranges over a compact set, there exist continuous functions $a(x_1,y)$, $b(x_1,y)$, and $c(x_1,y)$ such that the matrices $A(x_1,y,\mu)$, $B(x_1,y,\mu)$ and $C(x_1,y,\mu)$ satisfy

$$
\begin{aligned}
\|A(x_1,y,\mu)\| &\le a(x_1,y)\\
\|B(x_1,y,\mu)\| &\le b(x_1,y)\\
\|C(x_1,y,\mu)\| &\le c(x_1,y)
\end{aligned}
$$

for all $(x_1,y) \in \mathbb{R}^{i+1}$ and all $\mu \in \mathcal{P}$. Therefore, by means of arguments identical to the ones used in the proof of the previous Lemma, it can be deduced that there exists a smooth function $\gamma_2(x_1,y)$ such that

$$
Q(x_1,y,\mu) > \frac{\varepsilon}{2}\begin{pmatrix} I & 0 & 0\\ 0 & M_1^{\mathrm{T}}(x_1)M_1(x_1) & 0\\ 0 & 0 & 1 \end{pmatrix}
$$

and this concludes the proof. ◁

Remark 11.2.4. Note that, setting

$$
x_2 = \mathrm{col}(x_1,y)
$$

and

$$
M_2(x_2) = \begin{pmatrix} M_1(x_1) & 0\\ -\gamma_1(x_1) & 1 \end{pmatrix},
$$

the function $V_2(z,x_1,y)$ can be written in the form

$$
\tilde V_2(z,x_2) = z^{\mathrm{T}}P(\mu)z + \|M_2(x_2)x_2\|^2\,,
$$

which shows that this function has a structure identical to that of the function $V_1(z,x_1)$. Moreover, since by hypothesis $M_1(x_1)$ is such that

$$
\underline{\alpha}(\|x_1\|) \le \|M_1(x_1)x_1\|^2
$$

for some class $\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$, it is possible to show that also $M_2(x_2)$ is nonsingular for all x_2 and such that

$$
\underline{\tilde\alpha}(\|x_2\|) \le \|M_2(x_2)x_2\|^2
$$

for some class $\mathcal{K}_\infty$ function $\underline{\tilde\alpha}(\cdot)$.

To this end, it suffices to use the property (see Remark 10.1.3) that a positive definite function $V(x)$ satisfies an estimate of the form $\underline{\alpha}(\|x\|) \le V(x)$, with $\underline{\alpha}(\cdot)$ a class $\mathcal{K}_\infty$ function, if and only if, for any $c > 0$, the set of all x such that $V(x) \le c$ is a compact set. In particular, suppose

$$
\|M_2(x_2)x_2\|^2 \le c\,. \tag{11.17}
$$

Then, necessarily,

$$
\begin{aligned}
\|M_1(x_1)x_1\|^2 &\le c\\
|y| &\le |\gamma_1(x_1)x_1| + \sqrt{c}\,.
\end{aligned}
$$

The first one of these, since by hypothesis $\|M_1(x_1)x_1\|^2$ is bounded from below by a class $\mathcal{K}_\infty$ function of $\|x_1\|$, implies that x_1 belongs to a compact set, say Ω_c^1. Then, the second one yields

$$
|y| \le \max_{x_1\in\Omega_c^1}[|\gamma_1(x_1)x_1| + \sqrt{c}]\,.
$$

Since both x_1 and $|y|$ are bounded, the set of all x_2 which render (11.17) true is compact and therefore there exists a class $\mathcal{K}_\infty$ function of $\|x_2\|$ which bounds $\|M_2(x_2)x_2\|^2$ from below. ◁

The arguments used in the previous remark show that also the function

$$
z^{\mathrm{T}}P(\mu)z + \|M_1(x_1)x_1\|^2 + [y-\gamma_1(x_1)x_1]^2\,,
$$

for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, is bounded from below by $\alpha(\|\mathrm{col}(z,x_1,y)\|)$. Thus, we can conclude, from the Lemma, that if the system (11.14) is controlled by the input (11.16), the function $V_2(z,x_1,y)$ is such that

$$
\frac{\partial V_2}{\partial z}\dot z + \frac{\partial V_2}{\partial x_1}\dot x_1 + \frac{\partial V_2}{\partial y}\dot y \le -\frac{\varepsilon}{2}\alpha(\|\mathrm{col}(z,x_1,y)\|)
$$

which enables us to assert that the feedback (11.16) is globally asymptotically stabilizing. Moreover, the function

$$
V(t) = V_2(z(t),x_1(t),y(t))
$$

is such that

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le -aV$$

for some $a > 0$ and this, because of standard comparison arguments, proves that

$$V(t) \le e^{-at}V(0)\,,$$

i.e. that $V(t)$ decays to zero exponentially.

Using the previous two Lemma, it is immediate to deduce the existence of a globally stabilizing feedback law for a system of the form

$$\begin{aligned}
\dot z &= F(\mu)z + G(\xi_1,\mu)\xi_1\\
\dot\xi_1 &= H_1(\xi_1,\mu)z + K_1(\xi_1,\mu)\xi_1 + b_1(\xi_1,\mu)\xi_2\\
\dot\xi_2 &= H_2(\xi_1,\xi_2,\mu)z + \sum_{i=1}^{2} K_{2i}(\xi_1,\xi_2,\mu)\xi_i + b_2(\xi_1,\xi_2,\mu)\xi_3\\
&\cdots\\
\dot\xi_r &= H_r(\xi_1,\ldots,\xi_r,\mu)z + \sum_{i=1}^{r} K_{ri}(\xi_1,\ldots,\xi_r,\mu)\xi_i + b_r(\xi_1,\ldots,\xi_r,\mu)u
\end{aligned}\tag{11.18}$$

in which $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, for $i = 1,\ldots,r$, $u \in \mathbb{R}$ and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters.

Theorem 11.2.3. *Consider system (11.18), with μ ranging over a compact set $\mathcal{P}$. Suppose that:*

(i) *for each μ, the eigenvalues of $F(\mu)$ have negative real part,*

(ii) *there exist real numbers $b_{i0} > 0$ such that*

$$b_i(\xi_1,\ldots,\xi_i,\mu) \ge b_{i0}$$

for all $1 \le i \le r$, $(\xi_1,\ldots,\xi_i) \in \mathbb{R}^i$, and all $\mu \in \mathcal{P}$.

Then, there exists a smooth feedback law

$$u = u(\xi_1,\ldots,\xi_r)\,,$$

with $u(0,\ldots,0)$, which globally asymptotically stabilizes the system for any value of $\mu \in \mathcal{P}$.

Proof. If $r = 1$, the hypotheses of the Theorem coincide with those of Lemma 11.2.1, and therefore one can assert the existence of a smooth function $\gamma_1(\xi_1)$ such that the positive definite function

$$V_1(z,\xi_1) = z^{\mathrm{T}}P(\mu)z + \xi_1^2$$

satisfies

$$\begin{pmatrix}\dfrac{\partial V_1}{\partial z} & \dfrac{\partial V_1}{\partial \xi_1}\end{pmatrix}\begin{pmatrix}F(\mu)z + G(\xi_1,\mu)\xi_1\\ H_1(\xi_1,\mu)z + K_1(\xi_1,\mu)\xi_1 + b_1(\xi_1,\mu)\gamma_1(\xi_1)\xi_1\end{pmatrix}$$

$$< -\varepsilon(\|z\|^2 + \|\xi_1\|^2)\,,$$

for some $\varepsilon > 0$.

Proceeding by induction, suppose $r > 1$, set

$$x_1 = \mathrm{col}(\xi_1,\ldots,\xi_i)\,,\quad y = \xi_{i+1}\,,$$

rewrite the first $n+i+1$ equations of (11.18), with ξ_{i+2} replaced by u, in the form (11.14) and suppose hypothesis (i) of Lemma 11.2.2 holds. In other words, suppose there exist a $1\times i$ vector $\gamma_1(x_1)$ of smooth functions and an $i \times i$ matrix $M_1(x_1)$ of smooth functions, such that the positive definite function

$$V_1(z,x_1) = z^{\mathrm{T}}P(\mu)z + \|M_1(x_1)x_1\|^2$$

satisfies the inequality (11.15) for some $\varepsilon > 0$ (as shown above, this is the case if $i = 1$). Then, by Lemma 11.2.2, there exists a smooth function $\gamma_2(x_1,y)$ such that the positive definite function

$$V_2(z,x_1,y) = z^{\mathrm{T}}P(\mu)z + \|M_1(x_1)x_1\|^2 + [y - \gamma_1(x_1)x_1]^2$$

satisfies, when $u = -\gamma_2(x_1,y)[y - \gamma_1(x_1)x_1]$,

$$\dot V_2 < -\frac{\varepsilon}{2}(\|z\|^2 + \|M_1(x_1)x_1\|^2 + [y - \gamma_1(x_1)x_1]^2)\,.$$

Writing

$$\|M_1(x_1)x_1\|^2 + [\xi_{i+1} - \gamma_1(x_1)x_1]^2 = \left\|\begin{pmatrix}M_1(x_1) & 0\\ -\gamma_1(x_1) & 1\end{pmatrix}\begin{pmatrix}x_1\\ \xi_{i+1}\end{pmatrix}\right\|^2,$$

and

$$\begin{aligned}
&-\gamma_2(x_1,\xi_{i+1})[\xi_{i+1} - \gamma_1(x_1)x_1]\\
&\quad= \begin{pmatrix}\gamma_2(x_1,\xi_{i+1})\gamma_1(x_1) & -\gamma_2(x_1,\xi_{i+1})\end{pmatrix}\begin{pmatrix}x_1\\ \xi_{i+1}\end{pmatrix},
\end{aligned}$$

and bearing in mind Remark 11.2.4, it is concluded that the hypothesis (i) of Lemma 11.2.2 holds for the system consisting of the first $n+i+2$ equations of (11.18). Thus, by induction, it is concluded that there exist a feedback law

$$u = u(\xi_1,\ldots,\xi_r)\,,$$

with $u(0,\ldots,0) = 0$, and a positive definite function $V(z,\xi_1,\ldots,\xi_r)$, satisfying

$$\underline{\alpha}(\|\mathrm{col}(z,\xi_1,\ldots,\xi_r)\|) \le V(z,\xi_1,\ldots,\xi_r)$$

for some class $\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$, with the property that

$$\dot V \le -\alpha(\|\mathrm{col}(z,\xi_1,\ldots,\xi_r)\|)$$

for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$. This proves the Theorem. ◁

We conclude this section with some remarks about the structure of system (11.18). As it can be seen, the state of this system is split into two subsets, namely z and $\xi_1, \ldots, \xi_r$, and the fact that the nonlinear terms of the model depend in a triangular manner on the individual ξ_i's makes it possible to implement the recursive design procedure based on Lemma 11.2.2. Of course the class of systems modeled by equations of the form (11.18) includes, as a special case, the case of systems which – for a suitable choice of coordinates – exhibit a triangular dependence *for all* the state variables. In this respect, the set of top n equations in (11.18), namely

$$\dot{z} = F(\mu)z + g(\xi_1, \mu)\xi_1 \ ,$$

should be viewed as representing some subsystem in which a triangular dependence on the individual components of the state variables *cannot* be found. In order to cope with the fact that for this subsystem a triangular internal structure may not exist, the expressions on the right-hand side of (11.18) have been assumed to be *affine* functions of z and this, as it is clear from the previous arguments, has simplified the synthesis of the stabilizing feedback.

Another crucial hypothesis on (11.18) was the stability of the internal dynamics of the upper subsystem. In view of the general principles illustrated at the beginning of the Chapter, it may seem that this hypothesis could be removed, if the upper subsystem is stabilizable by some feedback law $\xi_1 = v^*(z)$. However, it must be observed that the change of variables

$$y = \xi_1 - v^*(z) \ ,$$

needed to transform the system into one in which the top part is stable, in general destroys the particular structure of (11.18), as the various expressions on the right-hand side may no longer be affine in z after the transformation. It is for this reason that the assumption in question was somewhat necessary in the present setup. We will see however in section 11.4 how, by appealing to the small-gain Theorem for input-to-state stable systems, such an assumption can eventually be removed.

11.3 Stabilization via Output Feedback: a Special Case

In this section, we address the problem of robustly globally stabilizing a nonlinear system using output feedback only. If a system has uniform relative degree one and the special structure of system (11.8), we know from Lemma 11.2.1 that global stabilization is possible using a feedback which depends only on y, namely a *memoryless* output feedback . In this section, we consider a special class of systems having uniform relative degree larger than one, and we will show how, using the result of Theorem 11.2.3, global stabilization can be achieved using *dynamic* output feedback.

The basic idea on which this stabilization method reposes is the following one. Suppose, without loss of generality, that the system (by hypothesis assumed to have relative degree $r \geq 2$) is modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x,y,\mu) + g(x,y,\mu)u \\ \dot{y} &= h(x,y,\mu) \ , \end{aligned} \tag{11.19}$$

in which $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$ and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters. Suppose to augment its dynamics by adding the following set of $r-1$ equations

$$\begin{aligned} \dot{\xi}_2 &= -\lambda_1\xi_2 + \xi_3 \\ \dot{\xi}_3 &= -\lambda_2\xi_3 + \xi_4 \\ &\cdots \\ \dot{\xi}_r &= -\lambda_{r-1}\xi_r + \gamma(y)u \ , \end{aligned}$$

in which the λ_i's, for $i = 1, \ldots, r-1$, are positive numbers, u and y are input and output of system (11.19) and $\gamma(y)$ is a smooth function, bounded away from zero. In other words, consider the "augmented" system

$$\begin{aligned} \dot{x} &= f(x,y,\mu) + g(x,y,\mu)u \\ \dot{y} &= h(x,y,\mu) \\ \dot{\xi} &= A\xi + B\gamma(y)u \end{aligned} \tag{11.20}$$

in which

$$\xi = \mathrm{col}(\xi_2, \ldots, \xi_r)$$

and

$$A = \begin{pmatrix} -\lambda_1 & 1 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -\lambda_{r-2} & 1 \\ 0 & 0 & \cdots & 0 & -\lambda_{r-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}.$$

Set also

$$C = (1 \quad 0 \quad \cdots \quad 0 \quad 0) \ ,$$

so that $\xi_2 = C\xi$.

Suppose, now, that there exists a change of coordinates

$$\tilde{x} = x - D(\mu)\xi \tag{11.21}$$

which transforms the augmented system (11.20) into a system of the form

$$\begin{aligned} \dot{\tilde{x}} &= F(\mu)\tilde{x} + G(y,\mu)y + d(\mu)C\xi \\ \dot{y} &= H(\mu)\tilde{x} + K(y,\mu)y + b(\mu)C\xi \\ \dot{\xi} &= A\xi + B\gamma(y)u \ , \end{aligned} \tag{11.22}$$

with $b(\mu)$ satisfying

$$b(\mu) \geq b_0 > 0$$

for all μ. If this is the case, the additional change of coordinates

$$z = \tilde{x} - \frac{1}{b(\mu)} d(\mu) y$$

transforms the latter system into a system of the form

$$\begin{aligned}
\dot{z} &= \tilde{F}(\mu)z + \tilde{G}(y,\mu)y \\
\dot{y} &= \tilde{H}(\mu)z + \tilde{K}(y,\mu)y + b(\mu)\xi_2 \\
\dot{\xi}_2 &= -\lambda_1\xi_2 + \xi_3 \\
\dot{\xi}_3 &= -\lambda_2\xi_3 + \xi_4 \\
&\cdots \\
\dot{\xi}_r &= -\lambda_{r-1}\xi_r + \gamma(y)u ,
\end{aligned} \tag{11.23}$$

in which

$$\begin{aligned}
\tilde{F}(\mu) &= F(\mu) - \frac{1}{b(\mu)} d(\mu)H(\mu) \\
\tilde{G}(y,\mu) &= G(y,\mu) + \frac{1}{b(\mu)}\Big[F(\mu)d(\mu) - \frac{1}{b(\mu)} d(\mu)H(\mu)d(\mu) - d(\mu)K(y,\mu)\Big] \\
\tilde{H}(\mu) &= H(\mu) \\
\tilde{K}(y,\mu) &= K(y,\mu) + \frac{1}{b(\mu)} H(\mu)d(\mu) .
\end{aligned}$$

System (11.23) turns out to be a system having the same structure as system (11.18). If it satisfies the hypotheses of Theorem 11.2.3, global robust stabilization is possible, by means of a feedback law of the form

$$u = u(y,\xi) .$$

Of course, this feedback stabilizes also system (11.20), which differs from (11.23) by a simple change of coordinates. But to say that this (memoryless) feedback law stabilizes (11.20) is the same as to say that the system

$$\begin{aligned}
\dot{\xi} &= A\xi + B\gamma(y)u(y,\xi) \\
u &= u(y,\xi) ,
\end{aligned} \tag{11.24}$$

viewed as a system with input y, output u and internal state $\xi \in \mathbb{R}^{r-1}$, is a *dynamical* feedback law which stabilizes system (11.19), and this is precisely the design goal that was desired to accomplish.

In summary, bearing in mind the conditions of Theorem 11.2.3, it can be concluded that system (11.19) can be globally stabilized via dynamic output feedback if:

(i) it is possible to transform system (11.20), via a change of coordinates of the form (11.21), into a system of the form (11.22),

(ii) for each μ, the eigenvalues of the matrix $\tilde{F}(\mu)$ have negative real part,

(iii) there exists a real number $b_0 > 0$ such that

$$\begin{aligned}
b(\mu) &\geq b_0 \\
\gamma(y) &\geq b_0
\end{aligned}$$

for all $y \in \mathbb{R}$ and all $\mu \in \mathcal{P}$.

We now address these points separately. In order to have condition (i) fulfilled, observe that this implies

$$\begin{aligned}
f(\tilde{x} + D(\mu)\xi, y, \mu) + g(\tilde{x} + D(\mu)\xi, y, \mu)u - D(\mu)A\xi - D(\mu)B\gamma(y)u \\
= F(\mu)\tilde{x} + G(y,\mu)y + d(\mu)C\xi .
\end{aligned}$$

Thus, necessarily,

$$f(\tilde{x} + D(\mu)\xi, y, \mu) - D(\mu)A\xi = F(\mu)\tilde{x} + G(y,\mu)y + d(\mu)C\xi \tag{11.25}$$

and

$$g(\tilde{x} + D(\mu)\xi, y, \mu) = D(\mu)B\gamma(y) . \tag{11.26}$$

From the first one of these, we see that, necessarily,

$$\begin{aligned}
f(x,0,\mu) &= F(\mu)x \\
f(0,y,\mu) &= G(y,\mu)y
\end{aligned}$$

and

$$f(x,y,\mu) = F(\mu)x + G(y,\mu)y .$$

If this holds, then (11.25) holds if and only if

$$F(\mu)D(\mu) - D(\mu)A = d(\mu)C .$$

On the other hand, (11.26) implies

$$g(x,y,\mu) = \bar{g}(\mu)\gamma(y)$$

where

$$\bar{g}(\mu) = D(\mu)B .$$

Identical arguments prove, starting from

$$h(\tilde{x} + D(\mu)\xi, y, \mu) = H(\mu)\tilde{x} + K(y,\mu)y + b(\mu)C\xi ,$$

that necessarily

$$h(x,y,\mu) = H(\mu)x + K(y,\mu)y$$

and

$$H(\mu)D(\mu) = b(\mu)C .$$

In summary, condition (i) can be fulfilled if and only if system (11.19) has the special form

$$\begin{aligned} \dot{x} &= F(\mu)x + G(y,\mu)y + \bar{g}(\mu)\gamma(y)u \\ \dot{y} &= H(\mu)x + K(y,\mu)y \end{aligned} \tag{11.27}$$

and there exist $D(\mu)$, $d(\mu)$, $b(\mu)$ such that

$$\begin{aligned} F(\mu)D(\mu) - D(\mu)A &= d(\mu)C \\ D(\mu)B &= \bar{g}(\mu) \\ H(\mu)D(\mu) &= b(\mu)C\,. \end{aligned} \tag{11.28}$$

It turns out that the latter equations can always be solved, and therefore it is possible to conclude that the necessary and sufficient condition which the system has to fulfill so that (i) holds is simply the condition that the system has the special form (11.27). For the sake of convenience, we state and prove separately the claim about the solvability of (11.28).

Lemma 11.3.1. *Suppose system (11.27) has uniform relative degree r. Then the equations (11.28) have a unique solution $D(\mu)$, $d(\mu)$, $b(\mu)$. In particular*

$$d(\mu) = p(F(\mu))\bar{g}(\mu)\,, \tag{11.29}$$

where $p(\lambda)$ is the polynomial

$$p(\lambda) = (\lambda + \lambda_1)\cdots(\lambda + \lambda_{r-1})\,, \tag{11.30}$$

and

$$b(\mu) = H(\mu)F^{r-2}(\mu)\bar{g}(\mu)\,. \tag{11.31}$$

Proof. Partition $D(\mu)$ into its $r-1$ columns as

$$D(\mu) = (\, d_1(\mu) \quad d_2(\mu) \quad \cdots \quad d_{r-1}(\mu)\,)$$

and observe that the second of (11.28) reduces to

$$d_{r-1}(\mu) = \bar{g}(\mu)\,.$$

The first condition, read column by column, yields, for $i = r-1, r-2, \ldots, 2$

$$d_{i-1}(\mu) = [F(\mu) + \lambda_i I]d_i(\mu)$$

and

$$d(\mu) = [F(\mu) + \lambda_1 I]d_1(\mu)\,.$$

Thus, recursively, we get

$$\begin{aligned} d_{r-1}(\mu) &= \bar{g}(\mu) \\ d_{r-2}(\mu) &= [F(\mu) + \lambda_{r-1}I]\bar{g}(\mu) \\ &\cdots \\ d(\mu) &= [F(\mu) + \lambda_1 I]\cdots[F(\mu) + \lambda_{r-1}I]\bar{g}(\mu) \end{aligned}$$

The last one of these proves (11.29).

So far, we have seen that the first two equations in (11.28) have a unique solution $D(\mu)$, $d(\mu)$. It remains to show that this solution is able to render the last equation fulfilled, for some $b(\mu)$. To this purpose we use the hypothesis that system (11.27) has relative degree r. Computing the first r derivatives of y, we deduce that, necessarily,

$$H(\mu)\bar{g}(\mu) = H(\mu)F(\mu)\bar{g}(\mu) = \cdots = H(\mu)F^{r-3}(\mu)\bar{g}(\mu) = 0$$

and

$$H(\mu)F^{r-2}(\mu)\bar{g}(\mu) \neq 0\,.$$

Using these, and bearing in mind the expressions previously found for the $d_i(\mu)$'s, we see that

$$H(\mu)\,(\, d_2(\mu) \quad \cdots \quad d_{r-1}(\mu)\,) = (\, 0 \quad \cdots \quad 0\,)$$

and

$$H(\mu)d_1(\mu) = H(\mu)F^{r-2}(\mu)\bar{g}(\mu)$$

which prove the last of (11.28). ◁

We address now condition (ii) above. To this end, recall that by hypothesis system (11.27) has uniform relative degree r. Therefore, this system has a well-defined *zero dynamics* which can be identified in the following manner. Simple calculations show that the constraint $y = 0$ implies

$$\begin{aligned} 0 &= H(\mu)x \\ 0 &= H(\mu)F(\mu)x \\ &\cdots \\ 0 &= H(\mu)F^{r-2}(\mu)x \\ 0 &= H(\mu)F^{r-1}(\mu)x + H(\mu)F^{r-2}(\mu)\bar{g}(\mu)\gamma(0)u\,. \end{aligned}$$

From this it is seen that the zero dynamics manifold Z^* of (11.27) is the subspace

$$Z^* = \{x \in \mathbb{R}^n : H(\mu)F^i(\mu)x = 0, i = 0, \ldots, r-2\} \tag{11.32}$$

and, also, that the latter is rendered invariant by the control

$$u = -\frac{1}{H(\mu)F^{r-2}(\mu)\bar{g}(\mu)\gamma(0)}H(\mu)F^{r-1}(\mu)x\,.$$

Assuming, without loss of generality, that $\gamma(0) = 1$ (if this is not the case, simply redefine $\bar{g}(\mu)$ and $\gamma(y)$ so as to have this condition fulfilled), and bearing in mind the formula (11.31), this control can be rewritten as

$$u = -\frac{1}{b(\mu)}H(\mu)F^{r-1}(\mu)x\,.$$

We can conclude, therefore, that the zero dynamics of (11.27) are those of the *restriction* of the linear mapping

$$F(\mu) - \frac{1}{b(\mu)}\bar{g}(\mu)H(\mu)F^{r-1}(\mu) \tag{11.33}$$

to its invariant subspace Z^*.

It is possible to show that condition (ii) is actually equivalent to the property that the eigenvalues of the linear mapping

$$[F(\mu) - \frac{1}{b(\mu)}\bar{g}(\mu)H(\mu)F^{r-1}(\mu)]\Big|_{Z^*} \tag{11.34}$$

have negative real part.

Lemma 11.3.2. *Assume $\lambda_1 > 0, \ldots, \lambda_{r-1} > 0$. The eigenvalues of the matrix $\tilde{F}(\mu)$ have negative real part if and only if the eigenvalues of (11.34) have negative real part.*

Proof. The proof of this result is essentially based on the following property. Let A be a linear mapping $\mathbb{R}^n \to \mathbb{R}^n$ and let $\mathcal{V}$ be a k-dimensional invariant subspace. Then, it is well known that, choosing coordinates in such a way that vectors in $\mathcal{V}$ are vectors whose the last $n-k$ entries are zero, the mapping A is represented by a matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \tag{11.35}$$

and A_{11} is a matrix representation of the restriction of A to $\mathcal{V}$. Note also that, since $\mathcal{V}$ is invariant for A, the mapping

$$\begin{array}{rccc} A^* : & \mathbb{R}^n/\mathcal{V} & \to & \mathbb{R}^n/\mathcal{V} \\ & \{x\} & \mapsto & \{Ax\} \end{array}$$

(in which $\{x\}$ denotes the equivalence class of x, namely the set of all x' such that $(x'-x) \in \mathcal{V}$) is well defined, and A_{22} is a matrix representation of A^*. Thus, the spectrum of A is the union of the spectra of $A\Big|_{\mathcal{V}}$ and of A^*.

Consider now our case and set

$$\mathcal{V} = \text{span}\{\bar{g}(\mu), F(\mu)\bar{g}(\mu), \ldots, F(\mu)^{r-2}\bar{g}(\mu)\}\ .$$

This subspace is an invariant subspace of $\tilde{F}(\mu)$. To see this, express the polynomial (11.30) in the form

$$p(\lambda) = \lambda^{r-1} + a_{r-2}\lambda^{r-2} + \ldots + a_1\lambda + a_0\ ,$$

and observe that a simple calculation yields

$$\tilde{F}(\mu)\begin{pmatrix} \bar{g}(\mu) & F(\mu)\bar{g}(\mu) & \cdots & F(\mu)^{r-2}\bar{g}(\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \bar{g}(\mu) & F(\mu)\bar{g}(\mu) & \cdots & F(\mu)^{r-2}\bar{g}(\mu) \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ 0 & 1 & \cdots & -a_2 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & -a_{r-2} \end{pmatrix}.$$

This proves that $\mathcal{V}$ is invariant under $\tilde{F}(\mu)$ and that

$$A_{11} = \begin{pmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ 0 & 1 & \cdots & -a_2 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & -a_{r-2} \end{pmatrix} \tag{11.36}$$

is a matrix representation of $\tilde{F}(\mu)\Big|_{\mathcal{V}}$.

Consider now the subspace

$$Z^* = \ker \begin{pmatrix} H(\mu) \\ H(\mu)F(\mu) \\ \cdots \\ H(\mu)F^{r-2}(\mu) \end{pmatrix}.$$

This subspace is not necessarily invariant for $\tilde{F}(\mu)$. However, it is complementary to $\mathcal{V}$. In fact, no non-trivial linear combination of $\bar{g}(\mu)$, $F(\mu)\bar{g}(\mu), \ldots,$ $F(\mu)^{r-2}\bar{g}(\mu)$ can be annihilated by all $H(\mu), H(\mu)F(\mu), \ldots, H(\mu)F^{r-2}(\mu)$'s because, otherwise, this would contradict the fact that $H(\mu)F^{r-2}(\mu)\bar{g}(\mu)$ is nonzero.

As a consequence of this, it is seen that, for each $x \in \mathbb{R}^n$, there exists a unique $z \in Z^*$ such that

$$\{x\} = z + \mathcal{V}\ ,$$

and, in particular, the mapping

$$\tilde{F}^* : \{x\} \mapsto \{\tilde{F}(\mu)x\}$$

is such that

$$\tilde{F}^*\{x\} = \{\tilde{F}(\mu)z\}\ .$$

Now, it turns out that, by construction, for all $z \in Z^*$

$$\tilde{F}(\mu)z = [F(\mu) - \frac{1}{b(\mu)}\bar{g}(\mu)H(\mu)F^{r-1}(\mu)]z \qquad \text{mod } \mathcal{V}\ .$$

Thus,

$$\{\tilde{F}(\mu)z\} = \{[F(\mu) - \frac{1}{b(\mu)}\bar{g}(\mu)H(\mu)F^{r-1}(\mu)]z\}$$

and this shows that the mappings $\tilde{F}^*$ and

$$[F(\mu) - \frac{1}{b(\mu)}\bar{g}(\mu)H(\mu)F^{r-1}(\mu)]\Big|_{Z^*} \tag{11.37}$$

are isomorphic. As a consequence, it is found that the matrix A_{22} in the bottom-right corner of the representation (11.35) of $\tilde{F}(\mu)$ is in fact also a matrix representation of (11.37).

We can thus conclude the spectrum of $\tilde{F}(\mu)$ is the union of the spectra of (11.36) and of (11.37) and this proves the claim. ◁

In summary, the arguments above show that the fulfillment of conditions (i), (ii) is possible if and only if the system (which was assumed to have uniform relative degree r) has the special form (11.27) and its zero dynamics are asymptotically stable. Therefore, it can be concluded the following result holds.

Proposition 11.3.3. *Consider system*

$$\begin{aligned} \dot{x} &= F(\mu)x + G(y,\mu)y + \bar{g}(\mu)\gamma(y)u \\ \dot{y} &= H(\mu)x + K(y,\mu)y \,. \end{aligned}$$

in which $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters and $\mathcal{P}$ is a compact set. Assume the following

(i) *$H(\mu)F^i(\mu)\bar{g}(\mu) = 0$ for $i = 1, \ldots, r-3$ and, for some $b_0 > 0$,*

$$\begin{aligned} &H(\mu)F^{r-2}(\mu)\bar{g}(\mu) \geq b_0 \\ &\gamma(y) \geq b_0 \end{aligned}$$

for all y and $\mu \in \mathcal{P}$,

(ii) *for each $\mu \in \mathcal{P}$, the eigenvalues of*

$$[F(\mu) - \frac{1}{b(\mu)}\bar{g}(\mu)H(\mu)F^{r-1}(\mu)]\Big|_{Z^*}$$

have negative real part.

Then, this system is globally asymptotically stabilizable by dynamic output feedback.

11.4 Stabilization of Systems in Lower Triangular Form

We present in this section a method for robust global stabilization of systems described by equations having a *lower-triangular* structure, such as

$$\begin{aligned} \dot{z} &= f(z,\xi_1,\mu) \\ \dot{\xi}_1 &= q_1(z,\xi_1,\mu) + b_1(z,\xi_1,\mu)\xi_2 \\ \dot{\xi}_2 &= q_2(z,\xi_1,\xi_2,\mu) + b_2(z,\xi_1,\xi_2,\mu)\xi_3 \\ &\cdots \\ \dot{\xi}_r &= q_r(z,\xi_1,\ldots,\xi_r,\mu) + b_r(z,\xi_1,\ldots,\xi_r,\mu)u \end{aligned} \tag{11.38}$$

in which $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, for $i = 1, \ldots, r$, $u \in \mathbb{R}$ and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters.

These systems are often referred to as systems in *feedback form*, in consideration of the fact that they correspond to a cascade interconnection of $r+1$ subsystems, starting with the lower subsystem of (11.38) and ending with the upper subsystem of (11.38), in which the i-th subsystem is fed by the "outputs" $\xi_{i-1}, \ldots, \xi_1, z$ of all subsequent subsystems in the cascade (see Fig. 11.1). This class of systems can be seen as an extension of the class of systems modeled by the equations (11.18), specifically because the expressions on the right-hand sides are no longer assumed to be affine functions in z. Moreover, also the hypothesis that the upper subsystem, with $\xi_1 = 0$, has an asymptotically stable equilibrium at $z = 0$ can be dropped.

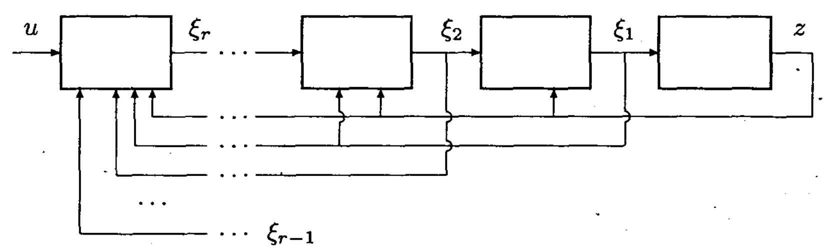

Fig. 11.1. System in feedback form.

The fundamental result on which the possibility of robustly stabilizing a system of the form (11.38) is based is a Lemma which can be considered as the robust version of Lemma 9.2.1.

Lemma 11.4.1. *Consider the system*

$$\begin{aligned} \dot{z} &= f(z,\xi,\mu) \\ \dot{\xi} &= \phi(z,\xi,\mu) + b(z,\xi,\mu)u \end{aligned} \tag{11.39}$$

in which $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$, $f(0,0,\mu) = 0$ and $\phi(0,0,\mu) = 0$. Suppose the following:

(i) *for each μ the upper subsystem in (11.39) is input-to-state stable and, in particular, a class $\mathcal{K}_\infty$ function $\gamma(\cdot)$, independent of μ, is known such that the response $z(\cdot)$ to any bounded $\xi(\cdot)$ satisfies*

$$\begin{aligned} \|z(\cdot)\|_\infty &\leq \max\{\beta(\|z^\circ\|,t), \gamma(\|\xi(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|z(t)\| &\leq \gamma(\limsup_{t\to\infty} \|\xi(t)\|) \end{aligned} \tag{11.40}$$

for some class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$,

(ii) *there exists a number* $b_0 > 0$ *such that* $b(z,\xi,\mu) \geq b_0$ *for all* (z,ξ) *and all* μ,

(iii) *there exist class* $\mathcal{K}$ *functions* $\rho_0(\cdot)$ *and* $\rho_1(\cdot)$, *which are locally Lipschitz at the origin, such that*

$$\max\{|\phi(z,\xi,\mu)|, |\xi||b(z,\xi,\mu)|^2\} \leq \max\{\rho_0(|\xi|), \rho_1(\|z\|)\}$$

for all (z,ξ) *and all* μ,

(iv) *the function* $\rho_1(\gamma(\cdot))$ *is locally Lipschitz at the origin.*

Then, there exists a smooth function $k(\xi)$, *with* $k(0)=0$, *such that, under the control law*

$$u = k(\xi) + v\,, \tag{11.41}$$

the closed-loop system (11.39) – (11.41), viewed as a system with input v *and state* (z,ξ), *is input-to-state stable and, in particular, a class* $\mathcal{K}_\infty$ *function* $\tilde\gamma(\cdot)$, *independent of* μ, *can be found such that the response* $x(\cdot) = (z(\cdot),\xi(\cdot))$ *to any bounded* $v(\cdot)$ *satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\beta(\|x^\circ\|,t), \tilde\gamma(\|v(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \tilde\gamma(\limsup_{t\to\infty} \|v(t)\|) \end{aligned} \tag{11.42}$$

for some class $\mathcal{KL}$ *function* $\beta(\cdot,\cdot)$.

Proof. The first step in proving this Lemma consists in showing how, given any *arbitrary* class $\mathcal{K}_\infty$ function $g(\cdot)$, it is possible to find a smooth function $k(\xi)$ such that the system

$$\dot\xi = \phi(z,\xi,\mu) + b(z,\xi,\mu)[k(\xi)+v] \tag{11.43}$$

viewed as a system with input (z,v) and state ξ, is input-to-state stable, and – in particular – such that the response to any bounded $z(\cdot)$ and $v(\cdot)$ satisfies

$$|\xi(t)| \leq \max\{\beta(|\xi^\circ|,t), g(\|z(\cdot)\|_\infty), \gamma_v(\|v(\cdot)\|_\infty)\} \tag{11.44}$$

for some class $\mathcal{K}$ function $\gamma_v(\cdot)$, independent of μ, and some class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$. For convenience, this fact is stated and proved separately.

Lemma 11.4.2. *Consider the system*

$$\dot x = \phi(z,x,\mu) + b(z,x,\mu)[u+v]$$

in which $x \in \mathbb{R}$, $z \in \mathbb{R}^n$, $v \in \mathbb{R}$, $u \in \mathbb{R}$. *Suppose there exists a number* $b_0 > 0$ *such that* $b(z,x,\mu) \geq b_0$ *for all* (z,x) *and all* μ. *Suppose there exist class* $\mathcal{K}$ *functions* $\rho_0(\cdot)$ *and* $\rho_1(\cdot)$, *which are locally Lipschitz at the origin, such that*

$$\max\{|\phi(z,x,\mu)|, |x||b(z,x,\mu)|^2\} \leq \max\{\rho_0(|x|), \rho_1(\|z\|)\}$$

for all (z,x) *and all* μ. *Let* $g(\cdot)$ *be some fixed class* $\mathcal{K}_\infty$ *function and suppose that* $\rho_1(g^{-1}(\cdot))$ *is locally Lipschitz at the origin. Then there exists a smooth feedback law* $u = k(x)$ *such that the positive definite function* $V(x) = x^2$ *satisfies*

$$|x| \geq \max\{d|v|, g(\|z\|)\} \;\Rightarrow\; \frac{\partial V}{\partial x}\Big[\phi(z,x,\mu) + b(z,x,\mu)[k(x)+v]\Big] \leq -\varepsilon|x|^2\,, \tag{11.45}$$

for some $d > 0$ *and* $\varepsilon > 0$.

Proof of Lemma 11.4.2. Choose

$$k(x) = -x - \alpha(x)$$

in which $\alpha(x)$ is a smooth strictly increasing function, with $\alpha(0)=0$, and $\alpha(-x) = -\alpha(x)$. Thus, $x\alpha(x) = |x|\alpha(|x|)$. Observe that

$$\begin{aligned} &\frac{\partial V}{\partial x}[\phi(z,x,\mu) + b(z,x,\mu)k(x) + b(z,x,\mu)v] \\ &\quad = +\,2x\phi(z,x,\mu) - 2x^2 b(z,x,\mu) - 2x\alpha(x)b(z,x,\mu) + 2xvb(z,x,\mu) \\ &\quad \leq +\,2|x||\phi(z,x,\mu)| - 2b_0|x|^2 - 2b_0|x|\alpha(|x|) + |x|^2|b(z,x,\mu)|^2 + |v|^2 \\ &\quad = +\,|x|[2|\phi(z,x,\mu)| - 2b_0\alpha(|x|) + |x||b(z,x,\mu)|^2] - 2b_0|x|^2 + |v|^2\,. \end{aligned}$$

The inequality in (11.45) can be obtained if, for some $\varepsilon > 0$,

$$-2b_0|x|^2 + |v|^2 \leq -\varepsilon|x|^2$$

and if

$$2b_0\alpha(|x|) \geq 2|\phi(z,x,\mu)| + |x||b(z,x,\mu)|^2\,.$$

Choosing $\varepsilon < 2b_0$, first one of these conditions is indeed fulfilled if $|x| \geq d|v|$, with $d = 1/\sqrt{2b_0 - \varepsilon}$). The second one is fulfilled if

$$2b_0\alpha(|x|) \geq 3\max\{\rho_0(|x|), \rho_1(\|z\|)\}\,.$$

This condition suggests to choose

$$\alpha(|x|) \geq \frac{3}{2b_0}\max\{\rho_0(|x|), \rho_1(g^{-1}(|x|))\}\,, \tag{11.46}$$

so that, $2b_0\alpha(|x|) \geq 3\rho_0(|x|)$ and

$$|x| \geq g(\|z\|) \quad\Rightarrow\quad 2b_0\alpha(|x|) \geq 3\rho_1(\|z\|)\,.$$

This choice of $\alpha(|x|)$ renders (11.45) satisfied. Finally, the hypotheses that $\rho_0(\cdot)$ and $\rho_1(g^{-1}(\cdot))$ are locally Lipschitz at the origin guarantee that it is possible to choose a smooth $\alpha(x)$ such that (11.46) holds. This concludes the proof of Lemma 11.4.2. ◁

Proof of Lemma 11.4.1 (continued). Property (11.45) shows that $V(\xi) = \xi^2$ is an ISS-Lyapunov function for system

$$\dot\xi = \phi(z,\xi,\mu) + b(z,\xi,\mu)[k(\xi)+v] \tag{11.47}$$

viewed as a system with input (z,v) and state ξ. As a consequence (see Remark 10.4.2), its response to any bounded $z(\cdot)$ and $v(\cdot)$ satisfies an estimate of the form

$$|\xi(t)| \le \max\{\beta(|\xi^\circ|,t), \gamma_z(\|z(\cdot)\|_\infty), \gamma_v(\|v(\cdot)\|_\infty)\}$$

for some class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, where

$$\gamma_z(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ g(r)$$

and $\underline{\alpha}, \overline{\alpha}$ are such that

$$\underline{\alpha}(|\xi|) \le V(\xi) \le \overline{\alpha}(|\xi|) .$$

In the present case one can choose, trivially, $\underline{\alpha}(r) = \overline{\alpha}(r) = r^2$, and therefore $\gamma_z(r) = g(r)$, and this proves that an estimate of the form (11.44) holds. The same argument also shows that $\gamma_v(r) = dr$, from which it is seen that this function is independent of μ.

From (11.44), as shown in section 10.4, it follows that the response of (11.47) to bounded $z(\cdot)$ and $v(\cdot)$ is such that

$$\begin{aligned} \|\xi(\cdot)\|_\infty &\le \max\{\beta(\|\xi^\circ\|,t), g(\|z(\cdot)\|_\infty), \gamma_v(\|v(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|\xi(t)\| &\le \max\{g(\limsup_{t\to\infty} \|z(t)\|), \gamma_v(\limsup_{t\to\infty} \|v(t)\|)\} . \end{aligned} \tag{11.48}$$

Consider now the system

$$\begin{aligned} \dot{z} &= f(z,\xi,\mu) \\ \dot{\xi} &= \phi(z,\xi,\mu) + b(z,\xi,\mu)[k(\xi) + v] . \end{aligned} \tag{11.49}$$

Suppose the various hypotheses of Lemma 11.4.1 hold. In particular, the hypothesis (i) is that, in the upper subsystem, the response $z(\cdot)$ to bounded $\xi(\cdot)$ satisfies the estimates (11.40). Define a class $\mathcal{K}_\infty$ function $g(\cdot)$ as

$$g(r) = \frac{1}{2}\gamma^{-1}(r) .$$

In view of the assumption (iv), the function

$$g^{-1}(r) = \gamma(2r)$$

satisfies the hypotheses of Lemma 11.4.2 and therefore it is possible to choose $k(\xi)$ so that, in the lower subsystem, the response $\xi(\cdot)$ to bounded $z(\cdot)$ and $v(\cdot)$ satisfies the estimates (11.48).

Note also that $g^{-1}(r) > \gamma(r)$, i.e.

$$\gamma \circ g(r) < r \qquad \text{for all } r > 0 .$$

Thus, system (11.49) satisfies the hypothesis of the small-gain Theorem and, consequently, the system – viewed as system with state $x = (z,\xi)$ and

input v – becomes input-to-state stable. In particular (see Theorem 10.6.1), the response $x(\cdot) = (z(\cdot),\xi(\cdot))$ to any bounded $v(\cdot)$ satisfies estimates of the form (11.42), with

$$\tilde{\gamma}(r) = \max\{2\gamma \circ \gamma_v(r), 2\gamma_v(r)\} .$$

This completes the proof of Lemma 11.4.1. ◁

Remark 11.4.1. It is worth to stress that the main argument on which the proof of this Lemma is based is the fact that the lower subsystem of (11.49), which is a one-dimensional system, can be rendered – under the appropriate technical hypotheses (ii), (iii) and (iv) – input-to-state stable, with a *prescribed gain function* $g(\cdot)$, between input z and state ξ. As a consequence, no matter what the gain function $\gamma(\cdot)$ of the upper subsystem is, it is possible to choose $g(\cdot)$ so as to have the small gain condition fulfilled, and the result of the Lemma immediately follows by the small gain Theorem. An equivalent argument was also, as observed, the main ingredient of the proof of Lemma 11.2.1, which is a special case of the present result. ◁

Lemma 11.4.1 contains all the ingredients needed to set up a recursive design procedure for robust stabilization of a system having the form (11.38). In fact, it shows – under the technical hypotheses (ii), (iii) and (iv) – that, if the upper system of (11.39) is input-to-state stable and, in particular, estimates of the form (11.40) hold with a function $\gamma(\cdot)$ which is independent of μ, then a feedback law $k(\xi)$ can be found such that the entire system, with control

$$u = k(\xi) + v ,$$

i.e. system (11.49), is input-to-state stable and, in particular, estimates of the form (11.42) hold with a function $\tilde{\gamma}(\cdot)$ which is independent of μ.

To see how the recursion proceeds consider, for instance, a system modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z,\xi,\mu) \\ \dot{\xi} &= \phi(z,\xi,\mu) + b(z,\xi,\mu)\zeta \\ \dot{\zeta} &= \psi(z,\xi,\zeta,\mu) + d(z,\xi,\zeta,\mu)u . \end{aligned} \tag{11.50}$$

The change of variable

$$v = \zeta - k(\xi)$$

transforms (11.50) into a system, modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z,\xi,\mu) \\ \dot{\xi} &= \phi(z,\xi,\mu) + b(z,\xi,\mu)k(\xi) + b(z,\xi,\mu)v \\ \dot{v} &= \bar{\psi}(z,\xi,v,\mu) + \bar{d}(z,\xi,v,\mu)u , \end{aligned} \tag{11.51}$$

in which the subsystem consisting of the upper two equations, viewed a system with state (z,ξ) and input v, satisfies an hypothesis which exactly corresponds to the hypothesis (i) of Lemma 11.4.1. Thus, if the gain function $\tilde{\gamma}(\cdot)$ and the functions $\bar{\psi}(z,\xi,v,\mu)$ and $\bar{d}(z,\xi,v,\mu)$ which appear in the lower equation are such that the conditions corresponding to the technical hypotheses (ii), (iii) and (iv) of Lemma 11.4.1 hold, a control law

$$u = \bar{k}(v) + w$$

can be found, such that the corresponding closed loop system, viewed as a system with input w and state $x = (z,\xi,v)$, is input-to-state stable and, in particular, its response $x(\cdot)$ to any bounded $w(\cdot)$ will satisfy estimates of the form

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\beta(\|x^\circ\|,t), \bar{\tilde{\gamma}}(\|w(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \bar{\tilde{\gamma}}(\limsup_{t\to\infty} \|w(t)\|) \end{aligned}$$

where $\bar{\tilde{\gamma}}(\cdot)$ is a class $\mathcal{K}$ function which is independent of μ.

We conclude the section by showing how the hypothesis (i) of Lemma 11.4.1 can be weakened, assuming instead that there is a feedback law $\xi = v^\star(z)$ such that system

$$\dot{z} = f(z, v^\star(z), \mu)$$

is robustly globally asymptotically stable, with a Lyapunov function which is independent of μ, and an extra technical condition is fulfilled.

More precisely, suppose there exist a smooth function $v^\star(z)$, with $v^\star(0) = 0$, and a smooth function $V(z)$, satisfying estimates of the form

$$\underline{\alpha}(\|z\|) \leq V(z) \leq \overline{\alpha}(\|z\|)\,, \tag{11.52}$$

such that

$$\frac{\partial V(z)}{\partial z} f(z, v^\star(z), \mu) \leq -\alpha(\|z\|)\,,$$

where $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$ are class $\mathcal{K}_\infty$ functions (independent of μ).

Then, consider the function $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\delta(s,r) = \max_{\|z\|=s} \Big[\frac{\partial V(z)}{\partial z} f(z, v^\star(z) + r, \mu) + \frac{1}{2}\alpha(s)\Big]\,.$$

By hypothesis, the value of this function at $r = 0$ is negative for any $s > 0$. What we need to assume in the following is that this function continues to be negative in a region of the form

$$\mathcal{R} = \{(s,r) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : s > 0, |r| \leq \rho(s)\}$$

for some continuous function $\rho(s)$, independent of μ, defined for all $s \geq 0$, with $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$.

Lemma 11.4.3. *Consider the system*

$$\dot{z} = f(z, \xi, \mu)\,. \tag{11.53}$$

Suppose there exist a smooth function $v^\star(z) = 0$, with $v^\star(0) = 0$, a smooth function $V(z)$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$, and a continuous function $\rho(s)$, defined for all $s \geq 0$, with $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$, all independent of μ, such that (11.52) hold, and

$$\max_{\|z\|=s} \Big[\frac{\partial V(z)}{\partial z} f(z, v^\star(z) + r, \mu) + \frac{1}{2}\alpha(s)\Big] < 0 \tag{11.54}$$

for all $s > 0$ and all $|r| \leq \rho(s)$. Then, there exist a smooth (positive-valued) function $\beta^\star(z)$, such that

$$0 < \beta^\star(z) \leq 1$$

for all z, and a class $\mathcal{K}_\infty$ function $\chi(\cdot)$, both independent of μ, such that

$$\|z\| \geq \chi(|v|) \quad \Rightarrow \quad \frac{\partial V(z)}{\partial z} f(z, v^\star(z) + \beta^\star(z)v, \mu) \leq -\frac{1}{2}\alpha(\|z\|)$$

for all z.

Proof. From the proof of Theorem 10.4.3, it is known that, starting from the function $\rho(\cdot)$, it is possible to construct a smooth function $\beta^\star(z)$, with $0 < \beta^\star(z) \leq 1$ and a class $\mathcal{K}_\infty$ function $\chi(\cdot)$ such that

$$\|z\| \geq \chi(|v|) \quad \Rightarrow \quad |\beta^\star(z)v| \leq \rho(\|z\|)\,.$$

The functions $\beta^\star(z)$ and $\chi(\cdot)$ thus constructed are independent of μ because so is the function $\rho(\cdot)$. Then, the result of the Lemma is a direct consequence of the hypothesis (11.54). ◁

The use of this result in the problem of robustly stabilizing a system of the form (11.39) (or, more in general, a system of the form (11.38)), is straightforward. Suppose the top subsystem of (11.39) satisfies the hypothesis of the Lemma, and consider the change of variable

$$v = [\beta^\star(z)]^{-1}[\xi - v^\star(z)]\,.$$

This yields a system modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z, v^\star(z) + \beta^\star(z)v, \mu) \\ \dot{v} &= \bar{\phi}(z, v, \mu) + \bar{b}(z, v, \mu)u \end{aligned} \tag{11.55}$$

in which the upper subsystem satisfies an hypothesis which exactly corresponds to the hypothesis (i) of Lemma 11.4.1. In fact, in view of the result of Lemma 11.4.3, it can be asserted that this subsystem – viewed as a system with state z and input v – is input-to-state stable, and in particular satisfies estimates of the form (11.40), with $\gamma(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$ independent of μ.

If $b(z,\xi,\mu) > b_0$, then also the hypothesis corresponding to hypothesis (ii) of Lemma 11.4.1 holds, because

$$\bar{b}(z,v,\mu) = [\beta^\star(z)]^{-1} b(z, v^\star(z) + \beta^\star(z)v, \mu)$$

and $\beta^\star(z) \leq 1$. Thus, if also the remaining two technical hypotheses hold, a result corresponding to that of Lemma 11.4.1 follows.

The results presented in this section are illustrated in the following simple examples.

Example 11.4.1. Consider a system modeled by equations of the form

$$\begin{aligned} \dot{z} &= -az^2\mathrm{sgn}(z) + bz(\xi + \xi^2) \\ \dot{\xi} &= z + u\,, \end{aligned} \tag{11.56}$$

in which a and b are parameters satisfying

$$0 < a_0 \leq a \leq a_1\,, \qquad |b| \leq b_0\,.$$

The upper subsystem, which is a special case of the system considered in the Example 10.4.3, is input-to-state stable. In particular, from the conclusions of this Example, it can be deduced that the subsystem in question satisfies condition (11.40) of Lemma 11.4.1 for a gain function

$$\gamma(r) = \frac{|b|}{a-\varepsilon}(r + r^2)\,,$$

in which $0 < \varepsilon < a$. Since the uncertain parameters a and b are supposed to vary within fixed bounds (and the former, in particular, is always positive) it is possible to choose ε and to find a number $k > 0$ such that

$$\frac{|b|}{a-\varepsilon} < k$$

for all admissible values of a and b, and therefore condition (11.40) of Lemma 11.4.1 holds for a gain function

$$\gamma(r) = k(r + r^2)\,,$$

which is independent of (a, b). In other words, the hypothesis (i) of Lemma 11.4.1 is fulfilled.

Condition (ii) of this Lemma trivially holds, and so does condition (iii), which reads as

$$\max\{|z|, |\xi|\} \leq \max\{\rho_0(|\xi|), \rho_1(|z|)\}\,,$$

for $\rho_0(r) = \rho_1(r) = r$. Finally, also condition (iv) holds, since the function $\rho_1(\gamma(r)) = \gamma(r)$ is locally Lipschitz at $r = 0$.

Following the proof of Lemma 11.4.1, choose a class $\mathcal{K}_\infty$ function $g(\cdot)$ such that the (small gain) condition

$$\gamma \circ g(r) < r \qquad \text{for all } r > 0$$

or, equivalently, the condition

$$\gamma(r) < g^{-1}(r) \qquad \text{for all } r > 0$$

hold. For example, choose

$$g^{-1}(r) = 2k(r + r^2)\,.$$

Suppose, without loss of generality, that $2k > 1$, so that

$$2k(r + r^2) \geq r$$

and define

$$\alpha(\xi) = \begin{cases} 3k(\xi + \xi^2) & \text{if } \xi \geq 0 \\ -\alpha(-\xi) & \text{if } \xi \leq 0\,. \end{cases}$$

By construction, this function satisfies the condition (11.46). Therefore, in view of the result of the Lemma, it can be concluded that the control law

$$u = -\xi - \alpha(\xi) + v$$

renders system (11.56) input-to-state stable. ◁

Example 11.4.2. Consider a system modeled by equations of the form

$$\begin{aligned} \dot{z} &= az + b(1 + z^2)\xi \\ \dot{\xi} &= \xi z + u\,, \end{aligned} \tag{11.57}$$

in which a and b are parameters satisfying

$$0 < a_0 \leq a \leq a_1\,, \qquad 0 < b_0 \leq b \leq b_1\,.$$

The upper subsystem is unstable, and therefore the first step in the design of a stabilizing feedback for (11.57) consists in finding a control law $\xi = v^\star(z)$ rendering this subsystem robustly globally asymptotically stable. Clearly, a law of the form $v^\star(z) = -kz$, if $k > 0$ is large enough, achieves this goal. In fact, this yields a system of the form

$$\dot{z} = -c_1 z - c_3 z^3$$

in which, if $k > 0$ is large enough,

$$c_1 = bk - a\,, \qquad c_3 = bk$$

are both positive, for any value of a and b.

The second step in the design consists in changing the state variable ξ in

$$v = \xi - v^\star(z)$$

which transforms the first subsystem in a system of the form

$$\dot{z} = -c_1 z - c_3 z^3 + b(1+z^2)v \,.$$

In order to be able to proceed with the design, this system must be input-to-state stable, which however is not obvious, because of the presence of the term $(1+z^2)$ multiplying the input v. Nevertheless, it is easy to render this system input-to-state stable, changing v into $\beta^\star(z)v$, with

$$\beta^\star(z) = \frac{1}{(1+z^2)} \,.$$

In other words, consider for (11.57) the change of variable

$$v = [\beta^\star(z)]^{-1}[\xi - v^\star(z)] = (1+z^2)(\xi + kz) \,, \tag{11.58}$$

which yields a system of the form

$$\begin{aligned} \dot{z} &= -c_1 z - c_3 z^3 + bv \\ \dot{v} &= \phi(z,v,a,b,k) + (1+z^2)u \,, \end{aligned} \tag{11.59}$$

in which c_1, c_2 have the form indicated above, and

$$\phi(z,v,a,b,k) = zv + (1+z^2)k\dot{z} - kz^2 + \frac{v}{1+z^2}2z\dot{z} \,.$$

In this system, the upper subsystem is now input-to-state stable, with a quadratic ISS-Lyapunov function $V(z) = z^2$ and a gain function which is bounded by a linear function independent of a and b. In fact, routine calculations show that

$$\dot{V} \leq -2c_1 z^2 + 2bzv$$

from which it is deduced that

$$\dot{V} \leq -2\varepsilon z^2$$

(with $0 < \varepsilon < c_1$) if

$$|z| \geq \frac{|b|}{c_1 - \varepsilon}|v| \,.$$

Hence, the gain of the subsystem is bounded by a function of the form

$$\gamma(r) = dr$$

where d is any number satisfying

$$d > \frac{|b|}{c_1 - \varepsilon} \,.$$

From this point, the design proceeds as in the previous example. Observe that, in system (11.59), hypothesis (i) of Lemma 11.4.1 holds and so does hypothesis (ii). To check hypothesis (iii), observe that $|\phi(z,v,a,b,k)|$ can be bounded (for all a and b) by a polynomial in $|z|$ and $|v|$. A similar property holds for the term $|v||(1+z^2)|^2$. Hence, it is possible to find polynomial class $\mathcal{K}_\infty$ functions $\rho_0(\cdot)$ and $\rho_1(\cdot)$ for which hypothesis (iii) holds. Finally, also condition (iv) holds, since the function $\rho_1(\gamma(r)) = \rho_1(dr)$ is smooth at $r = 0$.

Following the proof of the Lemma, choose a class $\mathcal{K}_\infty$ function $g(\cdot)$ such that the (small gain) condition

$$\gamma(r) < g^{-1}(r) \qquad \text{for all } r > 0$$

holds. For example, choose

$$g^{-1}(r) = 2dr \,.$$

Then, choose a function $\alpha(\cdot)$ so as to satisfy the condition (11.46).

In view of the result of the Lemma, it can be concluded that the control law

$$u = -v - \alpha(v)$$

robustly globally asymptotically stabilizes system (11.59). Reversing the change of variable (11.58), we obtain a control law

$$u = -(1+z^2)(\xi + kz) - \alpha((1+z^2)(\xi + kz))$$

which robustly globally asymptotically stabilizes system (11.57). ◁

11.5 Design for Multi-Input Systems

The purpose of this section is to study how the stabilization procedures described in section 9.2 and in this Chapter can be implemented in the case of systems having $m > 1$ inputs. Of course, the point of departure is the study of conditions under which global normal forms similar to those discussed in section 9.1 exist for the system under consideration. For convenience, we skip the intermediate stage of multivariable systems having *vector relative degree*, i.e. the case of systems which, by means of suitable globally defined diffeomorphisms, can be changed into systems described by equations having the normal form studied in section 5.1, and we address directly the more general problem of the existence of a globally defined diffeomorphism yielding normal forms of the type introduced in section 6.1. The case of systems having vector relative degree will be briefly discussed later, as a special case.

The existence of the normal forms in question relies upon the algorithm, described in section 6.1 in coordinate free terms as well as in the actual system coordinates, for the characterization of the zero dynamics of a multivariable nonlinear system. We begin by summarizing the basic steps of this algorithm, and we take this opportunity to strengthen the various "regularity" hypotheses already considered in section 6.1, so as to prepare ourselves to the subsequent derivation of globally defined normal forms.

Consider a nonlinear system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{11.60}$$

with the *same* number m of input and output components and state x defined in $\mathbb{R}^n$. Assume that $f(x)$ and the m columns of $g(x)$ are smooth vector fields, that the m entries of $h(x)$ are smooth functions, and that $f(0) = 0$ and $h(0) = 0$.

Step 0. Suppose that dh has constant rank, say s_0, for all x and that – possibly after reorder – the first s_0 rows of dh are linearly independent at each x. Define $H_0(x)$ to be the vector consisting of the first s_0 rows of $h(x)$.

Step 1. Suppose that $L_gH_0(x)$ has constant rank, say r_0, for all x and that – possibly after reorder – the first r_0 rows of $L_gH_0(x)$ are linearly independent at each x. Then, there exists a unique $(s_0 - r_0) \times r_0$ matrix $\bar{R}_0(x)$ of smooth functions of x, such that

$$(\bar{R}_0(x) \quad I)\, L_gH_0(x) = 0$$

for all x. Set

$$R_0(x) = (\bar{R}_0(x) \quad I)$$

and define $\Phi_0(x)$ to be the vector

$$\Phi_0(x) = R_0(x) L_f H_0(x) .$$

Suppose that $\mathrm{col}(dH_0\,,\, d\Phi_0)$ has constant rank, say $s_0 + s_1$, for all x and that – possibly after reorder in the rows of $d\Phi_0$ – the first $s_0 + s_1$ rows of $\mathrm{col}(dH_0\,,\, d\Phi_0)$ are linearly independent at each x. Define $H_1(x)$ to be the vector consisting of the first $s_0 + s_1$ rows of $\mathrm{col}(H_0\,,\, \Phi_0)$.

Step 2. Suppose that $L_gH_1(x)$ has constant rank, say r_1, for all x and that – possibly after reorder in the last s_1 rows of $L_gH_1(x)$ – the $r_1 \times m$ matrix consisting of the first r_0 rows of $L_gH_0(x)$ and of the first $r_1 - r_0$ rows of $L_g\Phi_0(x)$ has rank r_1 at each x. Then, there exists a unique $(s_1 - r_1 + r_0) \times (r_1 - r_0)$ matrix $\bar{Q}_0(x)$ and a $(s_1 - r_1 + r_0) \times s_0$ matrix $P_0(x)$ of smooth functions of x, such that

$$\begin{pmatrix} R_0(x) & 0 \\ P_0(x) & (\bar{Q}_0(x) \quad I) \end{pmatrix} L_gH_1(x) = 0$$

for all x. Set

$$R_1(x) = \begin{pmatrix} R_0(x) & 0 \\ P_0(x) & (\bar{Q}_0(x) \quad I) \end{pmatrix},$$

and define $\Phi_1(x)$ to be the vector

$$\Phi_1(x) = (P_0(x) \quad (\bar{Q}_0(x) \quad I))\, L_fH_1(x) .$$

Suppose that $\mathrm{col}(dH_1\,,\, d\Phi_1)$ has constant rank, say $s_0+s_1+s_2$, for all x and that – possibly after reorder in the rows of $d\Phi_1$ – the first $s_0 + s_1 + s_2$ rows of $\mathrm{col}(dH_1\,,\, d\Phi_1)$ are linearly independent at each x. Define $H_2(x)$ to be the vector consisting of the first $s_0 + s_1 + s_2$ rows of $\mathrm{col}(H_1\,,\, \Phi_1)$.

Step $k + 1 \geq 3$. Suppose that $L_gH_k(x)$ has constant rank, say r_k, for all x and that – possibly after reorder in the last s_k rows of $L_gH_k(x)$ – the $r_k \times m$ matrix consisting of the first r_0 rows of $L_gH_0(x)$, of the first $r_1 - r_0$ rows of $L_g\Phi_0(x), \ldots$, and of the first $r_k - r_{k-1}$ rows of $L_g\Phi_{k-1}(x)$ has rank r_k at each x. Then, there exists a unique $(s_k - r_k + r_{k-1}) \times (r_k - r_{k-1})$ matrix $\bar{Q}_{k-1}(x)$ and a $(s_k - r_k + r_{k-1}) \times (s_0 + \cdots + s_{k-1})$ matrix $P_{k-1}(x)$ of smooth functions of x, such that

$$\begin{pmatrix} R_{k-1}(x) & 0 \\ P_{k-1}(x) & (\bar{Q}_{k-1}(x) \quad I) \end{pmatrix} L_gH_k(x) = 0$$

for all x. Set

$$R_k(x) = \begin{pmatrix} R_{k-1}(x) & 0 \\ P_{k-1}(x) & (\bar{Q}_{k-1}(x) \quad I) \end{pmatrix},$$

and define $\Phi_k(x)$ to be the vector

$$\Phi_k(x) = (P_{k-1}(x) \quad (\bar{Q}_{k-1}(x) \quad I))\, L_fH_k(x) .$$

Suppose that $\mathrm{col}(dH_k\,,\, d\Phi_k)$ has constant rank, say $s_0 + s_1 + \cdots + s_{k+1}$, for all x and that – possibly after reorder in the rows of $d\Phi_k$ – the first $s_0 + s_1 + \cdots + s_{k+1}$ rows of $\mathrm{col}(dH_k\,,\, d\Phi_k)$ are linearly independent at each x. Define $H_{k+1}(x)$ to be the vector consisting of the first $s_0 + s_1 + \cdots + s_{k+1}$ rows of $\mathrm{col}(H_k\,,\, \Phi_k)$.

As shown in section 6.1, the construction ends at some step $k^\star < n$. Assuming that

$$r_{k^\star} = \mathrm{rank}[L_gH_{k^\star}(x)] = m ,$$

it can be deduced (see again section 6.1) that $s_0 = m$, $s_1 = s_0 - r_0$, and for all $k \geq 1$, $s_{k+1} = s_k - r_k + r_{k-1}$, and this, in turn, implies that

$$H_0(x) = h(x)$$

and

$$H_{k+1}(x) = \mathrm{col}(H_k(x), \Phi_k(x)) .$$

Moreover, the set

$$Z^\star = \{x \in \mathbb{R}^n : H_{k^\star}(x) = 0\}$$

is a smooth embedded submanifold of $\mathbb{R}^n$, of codimension

$$r = s_0 + s_1 + \cdots + s_{k^\star} .$$

The unique solution $u^\star(x)$ of

$$L_fH_{k^\star}(x) + L_gH_{k^\star}(x)u^\star(x) = 0$$

renders $Z^\star$ invariant with respect to the vector field $f^\star(x) = f(x)+g(x)u^\star(x)$.

As in section 6.1, define $T_0(x) = h(x)$ and $T_k(x)$ to be the $m \times 1$ vector in which the first $m - s_k$ elements are zero, and the last s_k elements are exactly those of $\Phi_{k-1}(x)$, for $1 \le k \le k^\star$. Let $T(x)$ be the $m \times (k^\star+1)$ matrix defined by

$$T(x) = \begin{pmatrix} T_0(x) & T_1(x) & \cdots & T_{k^\star}(x) \end{pmatrix} .$$

Let n_i denote the number of nonzero entries in the i-th row of matrix $T(x)$, and observe (see again section 6.1) that $n_1 \le n_2 \le \cdots \le n_m$. Moreover

$$n_1 + n_2 + \cdots + n_m = s_0 + s_1 + \cdots + s_{k^\star} = r .$$

Let $\xi^i_k(x)$ be the entry found at the i-th row and k-th column of the matrix $T(x)$, for $1 \le k \le n_i$, and $1 \le i \le m$. Set

$$\begin{aligned} a^i(x) &= L_g\xi^i_{n_i}(x) \\ b^i(x) &= L_f\xi^i_{n_i}(x) , \end{aligned}$$

for $i = 1, \ldots, m$. Lengthy, but simple, manipulations show that

$$\begin{aligned} L_g\xi^1_k(x) &= 0 \\ L_f\xi^1_k(x) &= \xi^1_{k+1}(x) \end{aligned} \tag{11.61}$$

for $1 \le k \le n_1 - 1$, and that there are functions $\delta^i_{k,j}(x)$ such that

$$L_g\xi^i_k(x) = \sum_{j=1}^{i-1} \delta^i_{k,j}(x)a^j(x) \tag{11.62}$$

and

$$L_f\xi^i_k(x) = \sum_{j=1}^{i-1} \delta^i_{k,j}(x)b^j(x) + \xi^i_{k+1}(x) \tag{11.63}$$

for $1 \le k \le n_i - 1$, $2 \le i \le m$.

Finally, define

$$A(x) = \begin{pmatrix} a^1(x) \\ \vdots \\ a^m(x) \end{pmatrix}, \quad b(x) = \begin{pmatrix} b^1(x) \\ \vdots \\ b^m(x) \end{pmatrix},$$

and observe that, by construction, the matrix $A(x)$ is invertible for all x. Using the matrix $A(x)$ and the vector $b(x)$ thus defined, set

$$\begin{aligned} \tilde{f}(x) &= f(x) - g(x)A^{-1}(x)b(x) \\ \tilde{g}(x) &= g(x)A^{-1}(x) , \end{aligned}$$

and

$$Y^k_m(x) = (-1)^{k-1}\mathrm{ad}^{k-1}_{\tilde{f}}\tilde{g}_m(x)$$

for $1 \le k \le n_m$, and

$$\begin{aligned} Y^1_j(x) &= \tilde{g}_j(x) - \sum_{l=j+1}^{m}\sum_{i=2}^{n_l}\delta^l_{n_l-i+1,j}(x)Y^i_l(x) , \\ Y^k_j(x) &= (-1)^{k-1}\mathrm{ad}^{k-1}_{\tilde{f}}Y^1_j(x) \end{aligned}$$

for $1 \le j \le m-1$, $1 < k \le n_j$.

Then, the following result holds, which is the global version of Proposition 6.1.5.

Proposition 11.5.1. *Suppose that the vector fields*

$$Y^k_j(x) , \quad 1 \le j \le m , \quad 1 \le k \le n_j$$

are complete. Then $Z^\star$ is connected and there is a globally defined diffeomorphism

$$\Psi : \mathbb{R}^n \to Z^\star \times \mathbb{R}^r ,$$

which changes system (11.60) into a system described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z,(\xi^1,\ldots,\xi^m)) + g_0(z,(\xi^1,\ldots,\xi^m))u \\ \dot{\xi}^1_1 &= \xi^1_2 \\ &\cdots \\ \dot{\xi}^1_{n_1-1} &= \xi^1_{n_1} \\ \dot{\xi}^1_{n_1} &= b^1(x) + a^1(x)u \\ \dot{\xi}^2_1 &= \xi^2_2 + \delta^2_{11}(x)(b^1(x) + a^1(x)u) \\ &\cdots \\ \dot{\xi}^2_{n_2-1} &= \xi^2_{n_2} + \delta^2_{n_2-1,1}(x)(b^1(x) + a^1(x)u) \\ \dot{\xi}^2_{n_2} &= b^2(x) + a^2(x)u \\ &\cdots \\ \dot{\xi}^i_1 &= \xi^i_2 + \textstyle\sum_{j=1}^{i-1}\delta^i_{1j}(x)(b^j(x) + a^j(x)u) \\ &\cdots \\ \dot{\xi}^i_{n_i-1} &= \xi^i_{n_i} + \textstyle\sum_{j=1}^{i-1}\delta^i_{n_i-1,j}(x)(b^j(x) + a^j(x)u) \\ \dot{\xi}^i_{n_i} &= b^i(x) + a^i(x)u \\ &\cdots \end{aligned} \tag{11.64}$$

with

$$y_i = \xi^i_1$$

for $i = 1,\ldots,m$, where $\xi^i = (\xi^i_1,\ldots,\xi^i_{n_i})$, and $x = \Psi^{-1}(z,(\xi^1,\ldots,\xi^m))$.

Remark 11.5.1. Note, comparing with the local normal form exhibited in Proposition 6.1.5, that the terms $\sigma^i_k(x)u$ are missing. This is a consequence of the fact that, in the present situation, the ranks of the matrices $L_gH_k(x)$ were assumed to be constant for all $x \in \mathbb{R}^n$ and not just on the sets M^c_k, as in section 6.1. ◁

Proof. By definition of the vector fields $\tilde{f}$ and $\tilde{g}$ and from equations (11.61) and (11.62) we have that

$$\begin{array}{rcll} L_{\tilde{f}}\xi^i_k(x) &=& \xi^i_{k+1}(x) & 1 \le i \le m, \quad 1 \le k \le n_i - 1, \\ L_{\tilde{f}}\xi^i_{n_i}(x) &=& 0 & 1 \le i \le m\,, \end{array}$$

and

$$\begin{array}{rcll} L_{\tilde{g}}\xi^i_k(x) &=& [\delta^i_{k,1}(x)\ \delta^i_{k,2}(x)\ \cdots\ \delta^i_{k,i-1}(x)\ \ 0_{1\times(m-i+1)}] & 2 \le i \le m \\ &&& 1 \le k \le n_i - 1\,, \\ L_{\tilde{g}}\xi^i_{n_i}(x) &=& [0_{1\times(i-1)}\ \ 1\ \ 0_{1\times(m-i)}] & 1 \le i \le m\,, \\ L_{\tilde{g}}\xi^1_k(x) &=& 0_{1\times m} & 1 \le k \le n_i - 1\,. \end{array}$$

Set $\phi^k_i(x) = L^{k-1}_{\tilde{f}}h_i(x)$, for $1 \le i \le m$, $1 \le k \le n_i$. Then the identity

$$\langle dL^s_f\phi(x)\,,\,[f\,,g](x)\rangle = L_f\,\langle dL^s_f\phi(x)\,,\,g(x)\rangle - \left\langle dL^{s+1}_f\phi(x)\,,\,g(x)\right\rangle\,,$$

the above equations and the definition of the vector fields Y^k_j , $1 \le j \le m$, $1 \le k \le n_j$, yield

$$L_{Y^k_j}\phi^s_i(x) = \begin{cases} 1 & \text{for } i = j \text{ and } s + k = n_j + 1 \\ 0 & \text{otherwise,} \end{cases} \tag{11.65}$$

for $1 \le i, j \le m$, and $1 \le k \le n_j$, $1 \le s \le n_i$.

Let $\Phi^Y_t(x)$ denote the *flow* of the vector field Y, and define the map

$$\Phi : Z^\star \times \mathbb{R}^r \to \mathbb{R}^n$$

by

$$(z, (\xi^1, \xi^2, \ldots, \xi^m)) \mapsto \Phi^{Y^1_1}_{\xi^1_{n_1}} \circ \cdots \circ \Phi^{Y^1_m}_{\xi^m_{n_m}} \circ \Phi^{Y^2_1}_{\xi^1_{n_1-1}} \circ \Phi^{Y^2_2}_{\xi^2_{n_2-1}} \circ \cdots \circ \Phi^{Y^{n_m}_m}_{\xi^m_1}(z)$$

where $\xi^i = (\xi^i_1, \ldots, \xi^i_{n_i})$. Φ is clearly smooth as it is the composition of smooth maps.

Let $x \in \mathbb{R}^n$ and let $q = \Phi^{Y^k_j}_{\bar{t}}(x)$. Thus,

$$\begin{array}{rcl} \phi^s_i(q) - \phi^s_i(x) &=& \displaystyle\int_0^{\bar{t}} \frac{\partial}{\partial t}\phi^s_i(\Phi^{Y^k_j}_t(x))dt \\ &=& \displaystyle\int_0^{\bar{t}} L_{Y^k_j}\phi^s_i(\Phi^{Y^k_j}_t(x))dt \end{array}$$

Therefore, from equation (11.65),

$$\phi^s_i(q) - \phi^s_i(x) = \begin{cases} \bar{t} & \text{for } i = j \text{ and } s + k = n_j + 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi^s_i(q) = \begin{cases} \bar{t} + \phi^s_i(x) & \text{for } i = j \text{ and } s + k = n_j + 1 \\ \phi^s_i(x) & \text{otherwise.} \end{cases}$$

Now, let

$$\varphi(x) = \Phi^{Y^{n_m}_m}_{-\phi^1_m(x)} \circ \cdots \circ \Phi^{Y^2_2}_{-\phi^{n_2-1}_2(x)} \circ \Phi^{Y^2_1}_{-\phi^{n_1-1}_1(x)} \circ \Phi^{Y^1_m}_{-\phi^{n_m}_m(x)} \circ \cdots \circ \Phi^{Y^1_1}_{-\phi^{n_1}_1(x)}(x)$$

and observe that

$$\phi^s_i(\varphi(x)) = -\phi^s_i(x) + \phi^s_i(x) = 0\,,$$

for $1 \le i \le m$, $1 \le s \le n_i$. But this implies that $\varphi(x) \in Z^\star$, as $\phi^k_i(x) = \xi^k_i(x)$ and by the definition of $Z^\star$. Thus the map

$$\Psi : x \mapsto (\varphi(x), (\phi^1_1(x), \ldots, \phi^{n_1}_1(x), \phi^1_2(x), \ldots, \phi^{n_2}_2(x), \ldots, \phi^1_m(x), \ldots, \phi^{n_m}_m(x))\,,$$

maps $\mathbb{R}^n$ into $Z^\star \times \mathbb{R}^r$, and is clearly smooth. Further, for $(z, (\xi^1, \xi^2, \ldots, \xi^m)) \in Z^\star \times \mathbb{R}^r$,

$$\begin{array}{l} \phi^s_i(\Phi^{Y^1_1}_{\xi^1_{n_1}} \circ \cdots \circ \Phi^{Y^1_m}_{\xi^m_{n_m}} \circ \Phi^{Y^2_1}_{\xi^1_{n_1-1}} \circ \Phi^{Y^2_2}_{\xi^2_{n_2-1}} \circ \cdots \circ \Phi^{Y^{n_m}_m}_{\xi^m_1}(z)) \\ \qquad = \xi^i_{(n_i+1-(n_i+1-s))} - \phi^s_i(z) = \xi^i_s \end{array}$$

as $z \in Z^\star$, and

$$\varphi(\Phi^{Y^1_1}_{\xi^1_{n_1}} \circ \cdots \circ \Phi^{Y^1_m}_{\xi^m_{n_m}} \circ \Phi^{Y^2_1}_{\xi^1_{n_1-1}} \circ \Phi^{Y^2_2}_{\xi^2_{n_2-1}} \circ \cdots \circ \Phi^{Y^{n_m}_m}_{\xi^m_1}(z)) = z\,,$$

which implies, $\Psi \circ \Phi(z, (\xi^1, \xi^2, \ldots, \xi^m)) = (z, (\xi^1, \xi^2, \ldots, \xi^m))$. Also, $\Phi \circ \Psi(x) = x$ by construction.

Therefore, $\Psi = \Phi^{-1}$ and since both Φ and Ψ are smooth mappings and as $\mathbb{R}^n$ is connected, it can be concluded that $Z^\star$ is connected and Ψ is a (globally defined) diffeomorphism. The special structure of the normal form (11.64) derives, as in section 6.1, from the special choice of the coordinates $\xi^i_k(x)$. ◁

Proposition 11.5.2. *Suppose, in addition to the hypotheses of Proposition 11.5.1, that the vector fields*

$$Y^k_j(x)\,, \quad 1 \le j \le m\,, \quad 1 \le k \le n_j$$

commute, i.e.

$$[Y^s_i, Y^k_j] = 0$$

for $1 \le i, j \le m$, $1 \le s \le n_i$, $1 \le k \le n_j$. *Then (11.60) is globally diffeomorphic to a system having the normal form (11.64) but with special structure*

$$\dot{z} = f_0(z, (\xi_1^1, \xi_1^2, \dots, \xi_1^m)).$$

Moreover, the zero dynamics of the system, in the new coordinates, are described by

$$\dot{z} = f^\star(z) = f_0(z, (0, 0, \dots, 0)).$$

Proof. First note that,

$$\frac{\partial}{\partial \xi_{n_1}^1} \Phi(z, (\xi^1, \xi^2, \dots, \xi^m)) = Y_1^1 \circ \Phi(z, (\xi^1, \xi^2, \dots, \xi^m)).$$

Hence,

$$\Phi_\star(\frac{\partial}{\partial \xi_{n_1}^1}) = Y_1^1 \circ \Phi,$$

by the properties of flows. Now write,

$$\Phi_\star\left(\frac{\partial}{\partial \xi_{n_i}^i}\right) = \left(\Phi_{\xi_{n_1}^1}^{Y_1^1}\right)_\star \circ \cdots \circ \left(\Phi_{\xi_{n_{i-1}}^{i-1}}^{Y_{i-1}^1}\right)_\star Y_i^1 \left(\Phi_{\xi_{n_i}^i}^{Y_i^1} \circ \cdots \circ \Phi_{\xi_1^m}^{Y_m^{n_m}}(z)\right) =$$

$$\left(\Phi_{\xi_{n_1}^1}^{Y_1^1}\right)_\star \circ \cdots \circ \left(\Phi_{\xi_{n_{i-1}}^{i-1}}^{Y_{i-1}^1}\right)_\star Y_i^1 \left(\Phi_{-\xi_{n_{i-1}}^{i-1}}^{Y_{i-1}^1} \circ \cdots \circ \Phi_{-\xi_{n_1}^1}^{Y_1^1} \left(\Phi\left(z, (\xi^1, \dots, \xi^m)\right)\right)\right)$$

for $1 \le i \le m$. Recall that, if the vector fields τ and X commute, function $v(t) = (\Phi_{-t}^\tau)_\star X \circ \Phi_t^\tau(x)$ is constant and equal to $X(x)$. Thus,

$$\Phi_\star(\frac{\partial}{\partial \xi_{n_i}^i}) = Y_i^1 \circ \Phi,$$

for $1 \le i \le m$, or in general,

$$\Phi_\star(\frac{\partial}{\partial \xi_k^i}) = Y_i^{n_i-k+1} \circ \Phi,$$

for $1 \le i \le m$ and $1 \le k \le n_i$.

Since $D\Psi(\Phi(z, (\xi^1, \xi^2, \dots, \xi^m)))D\Phi(z, (\xi^1, \xi^2, \dots, \xi^m)) = I$, it is clear from the previous equation that

$$\begin{aligned} D\Psi(\Phi(z, (\xi^1, \xi^2, \dots, \xi^m)))Y_i^{n_i-k+1} \circ \Phi(z, (\xi^1, \xi^2, \dots, \xi^m)) \\ = (0 \ \cdots \ 0 \ 0 \ 1 \ \cdots \ 0)^{\mathrm{T}}, \end{aligned}$$

where the element 1 appears in the $(n - r + (n_1 + n_2 + \cdots + n_{i-1}) + k)$*-th* position. But by construction,

$$\tilde{g}_m(x) = Y_m^1(x),$$

and

$$\tilde{g}_j(x) = Y_j^1(x) + \sum_{l=j+1}^{m} \sum_{i=2}^{n_l} \delta_{n_l-i+1,j}^l(x) Y_l^i(x),$$

for $1 \le j \le m - 1$.

Therefore, $(D\Psi)\tilde{g} \circ \Phi$ is a matrix in which the first $n-r$ rows are identically zero and so

$$(D\Psi)g \circ \Phi = (D\Psi)\tilde{g}A \circ \Phi$$

is a matrix in which the first $n - r$ rows are identically zero, i.e. in the new coordinates the first $n - r$ equations are independent of the input u.

Finally, by construction,

$$\Phi_\star(\frac{\partial}{\partial \xi_k^i}) = Y_i^{n_i-k+1} \circ \Phi = (-1)^{n_i-k} \mathrm{ad}_{\bar{f}} Y_i^1$$

for $1 \le i \le m$, and $1 \le k \le n_i$. Let

$$\bar{f} = \Phi_\star^{-1} \tilde{f} \circ \Phi, \text{ and let } \bar{Y}_i^k = \Phi_\star^{-1} Y_i^k \circ \Phi.$$

Thus,

$$\bar{f} = f_0(z, (\xi^1, \xi^2, \dots, \xi^m))\frac{\partial}{\partial z} + \sum_{i=1}^{m} \sum_{k=2}^{n_i} \xi_k^i \frac{\partial}{\partial \xi_{k-1}^i},$$

and

$$\bar{Y}_i^{n_i-k+1} = \frac{\partial}{\partial \xi_k^i}.$$

Therefore,

$$\mathrm{ad}_{\bar{f}} \bar{Y}_m^1 = -\frac{\partial f_0(z, (\xi^1, \xi^2, \dots, \xi^m))}{\partial \xi_{n_m}^m} \frac{\partial}{\partial z} - \frac{\partial}{\partial \xi_{n_m-1}^m}.$$

But, as Φ is a morphism of Lie brackets,

$$\mathrm{ad}_{\bar{f}} \bar{Y}_m^1 = \Phi_\star^{-1}(-Y_m^2 \circ \Phi) = -\frac{\partial}{\partial \xi_{n_m-1}^m},$$

which implies

$$\frac{\partial f_0(z, (\xi^1, \xi^2, \dots, \xi^m))}{\partial \xi_{n_m}^m} = 0,$$

i.e. f_0 is independent of $\xi_{n_m}^m$. So now,

$$\mathrm{ad}_{\bar{f}}^2 \bar{Y}_m^1 = \frac{\partial f_0(z, (\xi^1, \xi^2, \dots, \xi^m))}{\partial \xi_{n_m-1}^m} \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi_{n_m-2}^m}.$$

But,

$$\mathrm{ad}_{\bar{f}}^2 \bar{Y}_m^1 = \Phi_\star^{-1}(Y_m^3 \circ \Phi) = \frac{\partial}{\partial \xi_{n_m-2}^m},$$

implying

$$\frac{\partial f_0(z,(\xi^1,\xi^2,\ldots,\xi^m))}{\partial \xi^m_{n_m-1}} = 0\,,$$

i.e. f_0 is independent of $\xi^m_{n_m-1}$. Proceeding inductively in this way, it can be shown that f_0 is independent of all $\xi^1_{n_1},\ldots,\xi^1_2,\xi^2_{n_2},\ldots,\xi^2_2,\ldots,\xi^m_{n_m},\ldots,\xi^m_2$. i.e. f_0 is properly denoted $f_0(z,(\xi^1_1,\xi^2_1,\ldots,\xi^m_1))$. ◁

Bearing in mind Remark 6.1.8, it is easily seen that the results described in the previous two Propositions include, as expected, the special case in which the original system (11.60) has *uniform vector relative degree*, i.e. has vector relative degree $\{r_1,\ldots,r_m\}$ at each $x^\circ \in \mathbb{R}^n$. If this is the case, the regularity hypotheses assumed in the description of the zero dynamics algorithm are automatically satisfied and, in particular, $n_i = r_i$ for all $1 \le i \le m$, and $\delta^i_{k,j}(x) = 0$ for all $1 \le k \le n_i - 1, 1 \le j \le i-1, 2 \le i \le m$. Thus, if the vector fields Y^k_j are complete, the system is globally diffemorphic to a system described by equations of the form (5.7) – (5.8). If, in addition, these vector fields commute, the system is globally diffeomorphic to a system described by equations of the form

$$\begin{array}{rcl}
\dot z &=& f_0(z,(\xi^1_1,\xi^2_1,\ldots,\xi^m_1))\\
\dot\xi^1_1 &=& \xi^1_2\\
&\cdots&\\
\dot\xi^1_{n_1-1} &=& \xi^1_{n_1}\\
\dot\xi^1_{n_1} &=& b^1(x)+a^1(x)u\\
\dot\xi^2_1 &=& \xi^2_2\\
&\cdots&\\
\dot\xi^2_{n_2-1} &=& \xi^2_{n_2}\\
\dot\xi^2_{n_2} &=& b^2(x)+a^2(x)u\\
&\cdots&\\
\dot\xi^i_1 &=& \xi^i_2\\
&\cdots&\\
\dot\xi^i_{n_i-1} &=& \xi^i_{n_i}\\
\dot\xi^i_{n_i} &=& b^i(x)+a^i(x)u\\
&\cdots&
\end{array} \qquad (11.66)$$

with

$$y_i = \xi^i_1$$

for $i = 1,\ldots,m$, in which the upper subsystem is driven only by $\xi^1_1,\ldots,\xi^m_1$.

Indeed, the feedback law

$$u = A^{-1}(x)(-b(x)+v)$$

changes system (11.66) into a new system decomposed into a set of m "chains of integrators" (where m is the number of input/output channels) that drive the $(n-r)$-dimensional subsystem

$$\dot z = f_0(z,(y_1,\ldots,y_m))\,,$$

whose autonomous dynamics are precisely the zero dynamics of the original system. An immediate byproduct of such a decomposition is that the various stabilization techniques described in section 9.1 and in this Chapter can be implemented. Thus, for instance, if there exist smooth functions $v^\star_1(z),\ldots,v^\star_m(z)$, with $v^\star_1(0)=\cdots=v^\star_m(0)=0$, which globally asymptotically stabilize the equilibrium $z=0$ of

$$\dot z = f_0(z,(v^\star_1(z),\ldots,v^\star_m(z)))\,,$$

it is possible, "stepping-back" through the m chains of integrators, to find a feedback law $u^\star(x)$ which globally asymptotically stabilizes the equilibrium $x=0$ of the full system.

Example 11.5.1. As a simple illustration, consider (see Fig. 11.2) a system with two inputs, u_1 and u_2, and two outputs, $y_1=\xi^1_1$ and $y_2=\xi^2_1$, described by equations of the form

$$\begin{array}{rcl}
\dot z &=& f(z,y_1,y_2)\\
\dot y_1 &=& v_1\\
\dot y_2 &=& \xi^2_2\\
\dot\xi^2_2 &=& v_2\,.
\end{array} \qquad (11.67)$$

Suppose there exist functions $v^\star_1(z), v^\star_2(z)$, with $v^\star_1(0)=v^\star_2(0)=0$, which globally asymptotically stabilize the equilibrium $z=0$ of

$$\dot z = f(z,v^\star_1(z),v^\star_2(z))\,.$$

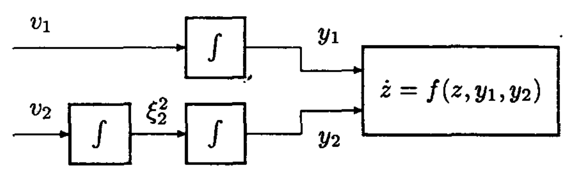

Fig. 11.2. Two-input system having vector relative degree.

In order to find a globally stabilizing feedback for (11.67), construct first, using Lemma 9.2.2, a feedback law $u^\star_1(z,y_1)$ which globally asymptotically stabilizes the equilibrium $(z,y_1)=(0,0)$ of

$$\begin{aligned}\dot z &= f(z, y_1, v_2^\star(z))\\ \dot y_1 &= u_1^\star(z, y_1)\,.\end{aligned}$$

Then, look at the system

$$\begin{aligned}\dot z &= f(z, y_1, y_2)\\ \dot y_1 &= u_1^\star(z, y_1)\\ \dot y_2 &= \xi_2^2\\ \dot\xi_2^2 &= v_2\,,\end{aligned} \tag{11.68}$$

as to a system in which a chain of two integrators drives, through y_2, the subsystem

$$\begin{aligned}\dot z &= f(z, y_1, y_2)\\ \dot y_1 &= u_1^\star(z, y_1)\,,\end{aligned}$$

which, by hypothesis, is globally asymptotically stabilized by $y_2 = v_2^\star(z)$. Using Theorem 9.2.3, it is possible to find, for (11.68), a globally asymptotically stabilizing feedback $v_2 = u_2^\star(z, y_1, y_2, \xi_2^2)$, and this completes the design procedure. ◁

The technique of backstepping, however, cannot as easily be implemented if the system *does not* have a vector relative degree, i.e. is not globally diffeomorphic to a system of the form (11.66), but – rather – is diffeomorphic to a system in the more general normal form considered by Proposition 11.5.2, i.e. form (11.64) with the additional constraint that

$$\dot z = f_0(z, (\xi_1^1, \xi_1^2, \ldots, \xi_1^m))\,.$$

As a matter of fact, extra hypotheses are needed, as illustrated first in the following simple example.

Example 11.5.2. Consider a system (see Fig. 11.3) with two inputs, u_1 and u_2, and two outputs, $y_1 = \xi_1^1$ and $y_2 = \xi_1^2$, described by equations of the form

$$\begin{aligned}\dot z &= f(z, y_1, y_2)\\ \dot y_1 &= v_1\\ \dot y_2 &= \xi_2^2 + \delta(z, y_1, y_2, \xi_2^2)v_1\\ \dot\xi_2^2 &= v_2\,.\end{aligned} \tag{11.69}$$

Clearly, if $\delta(z, y_1, y_2, \xi_2^2) \neq 0$, the system in question does not have vector relative degree as the input u_1 appears "too soon" in the derivative of the output y_2 (see Example 5.4.1). Hence, it is not immediately evident how to "step-back" a feedback law designed for $\dot z = f(z, y_1, y_2)$ all the way to the actual inputs u_1, u_2. As a matter of fact, a normal form like (11.69) does not, in general, lend itself to the possibility of implementing a backstepping design procedure. However, there is a special situation in which this is possible.

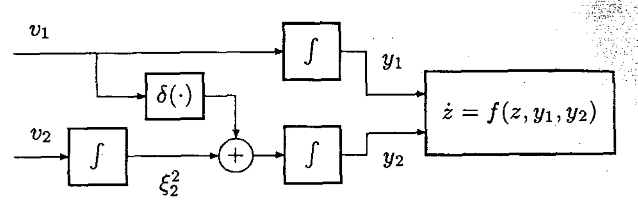

Fig. 11.3. Two-input system not having vector relative degree.

Suppose

$$\delta(z, y_1, y_2, \xi_2^2) = \delta(z, y_1, y_2)\,.$$

As in the previous example, suppose there exists functions $v_1^\star(z), v_2^\star(z)$, with $v_1^\star(0) = v_2^\star(0) = 0$, which globally asymptotically stabilize the equilibrium $z = 0$ of

$$\dot z = f(z, v_1^\star(z), v_2^\star(z))\,.$$

Backstepping once from a feedback $y_1 = v_1^\star(z)$ stabilizing the system

$$\dot z = f(z, y_1, v_2^\star(z)),$$

one can determine, as in the previous example, a feedback law $u_1^\star(z, y_1)$ stabilizing the system

$$\begin{aligned}\dot z &= f(z, y_1, v_2^\star(z))\\ \dot y_1 &= u_1^\star(z, y_1)\,.\end{aligned}$$

Setting $v_1 = u_1^\star(z, y_1)$ yields a system

$$\begin{aligned}\dot z &= f(z, y_1, y_2)\\ \dot y_1 &= u_1^\star(z, y_1)\\ \dot y_2 &= \xi_2^2 + \delta(z, y_1, y_2)u_1^\star(z, y_1)\\ \dot\xi_2^2 &= v_2\,,\end{aligned}$$

which can be regarded as a subsystem

$$\begin{aligned}\dot z &= f(z, y_1, y_2)\\ \dot y_1 &= u_1^\star(z, y_1)\,,\end{aligned}$$

by hypothesis globally asymptotically stabilized by $y_2 = v_2^\star(z)$, driven by

$$\begin{aligned}\dot y_2 &= \xi_2^2 + \delta(z, y_1, y_2)u_1^\star(z, y_1)\\ \dot\xi_2^2 &= v_2\,.\end{aligned}$$

The latter is no longer a chain of two integrators (as in the previous example), but has anyway the lower-triangular structure that renders backstepping possible. Thus, in two iterations, from $y_2 = v_2^\star(z)$, it is possible to find a feedback law $v_2 = u_2^\star(z, y_1, y_2, \xi_2^2)$ which stabilizes the system

$$\begin{aligned}
\dot{z} &= f(z, y_1, y_2) \\
\dot{y}_1 &= u_1^\star(z, y_1) \\
\dot{y}_2 &= \xi_2^2 + \delta(z, y_1, y_2) u_1^\star(z, y_1) \\
\dot{\xi}_2^2 &= u_2^\star(z, y_1, y_2, \xi_2^2)\,,
\end{aligned}$$

and this completes the design. ◃

In general, if a system satisfies the hypotheses of Proposition 11.5.2, and hence admits a normal form of the type (11.64) with the additional property that the upper subsystem is driven only by $\xi_1^1, \ldots, \xi_1^m$, i.e. by $y_1, \ldots, y_m$, a preliminary control law

$$u = A^{-1}(x)(-b(x) + v)$$

yields a normal form with the following structure

$$\begin{aligned}
\dot{z} &= f_0(z, y_1, \ldots, y_m) \\
&\cdots \\
\dot{\xi}_{n_1-1}^1 &= \xi_{n_1}^1 \\
\dot{\xi}_{n_1}^1 &= v_1 \\
\dot{\xi}_1^2 &= \xi_2^2 + \delta_{11}^2(x) v_1 \\
&\cdots \\
\dot{\xi}_{n_2-1}^2 &= \xi_{n_2}^2 + \delta_{n_2-1,1}^2(x) v_1 \\
\dot{\xi}_{n_2}^2 &= v_2 \\
&\cdots \\
\dot{\xi}_1^i &= \xi_2^i + \textstyle\sum_{j=1}^{i-1} \delta_{1j}^i(x) v_j \\
&\cdots \\
\dot{\xi}_{n_i-1}^i &= \xi_{n_i}^i + \textstyle\sum_{j=1}^{i-1} \delta_{n_i-1,j}^i(x) v_j \\
\dot{\xi}_{n_i}^i &= v_i \\
&\cdots
\end{aligned}$$

with

$$y_i = \xi_1^i$$

for $i = 1, \ldots, m$.

In this system, it is possible to "step-back" feedback laws

$$y_1 = v_1^\star(z),\ \ldots,\ y_m = v_m^\star(z)$$

to feedback laws

$$\begin{aligned}
v_1 &= u_1^\star(z, \xi_1^1, \ldots, \xi_1^m, \xi^1) \\
v_2 &= u_2^\star(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \xi^2) \\
&\cdots \\
v_m &= u_m^\star(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \xi^2, \ldots, \xi^m)
\end{aligned}$$

if the various coefficients $\delta_{k,j}^i$ exhibit a "triangular" dependence on state variables. That is, if the function $\delta_{k,j}^i$ depends only on variables $\xi_{\ell_b}^{\ell_p}$ belonging to the group with superscript $i-1$ and smaller ($\ell_p \le i-1$), or on variables having superscript i but with subscript that is less than or equal to k ($\ell_p = i$ & $\ell_b \le k$), or on any of the "leading" variables in a group, namely any of the variables $\xi_1^{\ell_p}$ ($1 \le \ell_p \le m$), i.e.

$$\begin{aligned}
&\delta_{1,1}^2(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \xi_1^2) \\
&\delta_{2,1}^2(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \xi_1^2, \xi_2^2) \\
&\cdots \\
&\delta_{n_2-1,1}^2(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \xi_1^2, \xi_2^2, \ldots, \xi_{n_2-1}^2) \\
&\cdots \\
&\delta_{1,j}^i(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \ldots, \xi^{i-1}, \xi_1^i) \\
&\delta_{2,j}^i(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \ldots, \xi^{i-1}, \xi_1^i, \xi_2^i) \\
&\cdots \\
&\delta_{n_i-1,j}^i(z, \xi_1^1, \ldots, \xi_1^m, \xi^1, \ldots, \xi^{i-1}, \xi_1^i, \xi_2^i, \ldots, \xi_{n_i-1}^i)\,.
\end{aligned}$$

It is easy to observe that if this type of "triangularity" property holds then the backstepping design methods can be directly applied to the system in normal form. The backstepping commences with the subsystem

$$\dot{z} = f_0(z, (v_1^\star(z), v_2^\star(z), \ldots, v_m^\star(z)))\,,$$

first backstepping n_1 times through the first group of variables (variables with the superscript 1) to obtain the feedback law

$$u_1^\star(z, \xi_1^1, v_2^\star(z), \ldots, v_m^\star(z), \xi^1)\,,$$

then backstepping n_2 times through the second group of variables (variables with the superscript 2) to obtain the feedback law

$$u_2^\star(z, \xi_1^1, \xi_1^2, v_3^\star(z), \ldots, v_m^\star(z), \xi^1, \xi^2)\,,$$

continuing group by group in the numbered order 1 through m, backstepping n_i times through group i (variables with the superscript i) to discover the feedback law

$$u_i^\star(z, \xi_1^1, \xi_1^2, \ldots, \xi_1^i, v_{i+1}^\star(z), \ldots, v_m^\star(z), \xi^1, \xi^2, \ldots, \xi^i)\,.$$

While considering group i for backstepping the leading variables $\xi_1^{i+1}, \ldots, \xi_1^m$ are of course fixed to the functions $v_{i+1}^\star(z), \ldots, v_m^\star(z)$ respectively.

It is simple to check whether the special dependence of the functions $\delta_{k,j}^i$ illustrated above holds. This dependence can be expressed by the geometric conditions

$$L_{Y_l^{n_l-s+1}} \delta_{k,j}^i(x) = 0$$

for $l > i, s \neq 1$ and for $l = i$ and $k < s$. These conditions mandate that

$$\delta^i_{k,j}(\Phi(z,(\xi^1,\ldots,\xi^m))) = \delta^i_{k,j}(z,\xi^1_1,\ldots,\xi^m_1,\xi^1,\ldots,\xi^{i-1},\xi^i_1,\xi^i_2,\ldots,\xi^i_k)$$

for $j = 1,\ldots,i-1$, where the $Y_l^{n_l-s+1}$ are the vector fields introduced earlier in this section, and $\Phi(z,(\xi^1,\ldots,\xi^m))$ is the global diffeomorphism considered in Proposition 11.5.2. Note that the condition thus found is coordinate free, for the functions $\delta^i_{k,j}$ were found from the zero dynamics algorithm.

12. Feedback Design for Robust Semiglobal Stability

12.1 Achieving Semiglobal and Practical Stability

In section 9.3 we have introduced the concept of semiglobal stabilizability, and we have shown (Theorem 9.3.1) how, using a linear feedback, it is possible to stabilize in a semiglobal sense (i.e. imposing that the domain of attraction of the equilibrium contains a prescribed compact set) a system of the form (9.23), under the hypothesis that the equilibrium $z = 0$ of its zero dynamics is globally asymptotically stable. In this section, in preparation to the subsequent study of the problem of robust semiglobal stabilization using output feedback, we extend the result of Theorem 9.3.1 to the case of a system modeled by equations of the form

$$\begin{aligned}
\dot z &= f_0(z,\xi_1)\\
\dot\xi_1 &= \xi_2\\
\dot\xi_2 &= \xi_3\\
&\cdots\\
\dot\xi_r &= q(z,\xi_1,\ldots,\xi_r,\mu) + b(z,\xi_1,\ldots,\xi_r,\mu)u\,,
\end{aligned}\tag{12.1}$$

in which $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, for $i = 1,\ldots,r$, $u \in \mathbb{R}$ and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters, ranging over a compact set $\mathcal{P}$.

As in Theorem 9.3.1, we assume that

(i) $f_0(0,0) = 0$ and the equilibrium $z = 0$ of the subsystem

$$\dot z = f_0(z,0)$$

is globally asymptotically stable.

Moreover, we also assume that

(ii) for some $b_0 > 0$,

$$b(z,\xi_1,\ldots,\xi_r,\mu) \geq b_0\,.$$

Note that we do not make any specific assumption on $q(z,\xi_1,\ldots,\xi_r,\mu)$, other than the standard hypothesis of smoothness. Thus, $q(0,0,\ldots,0,\mu)$ is not required to be 0 for all μ, which means that $(z,\xi_1,\ldots,\xi_r) = (0,0,\ldots,0)$ is not necessarily an equilibrium point of (12.1) for $u = 0$.

The notion of semiglobal stabilizability essentially requires that, for any given compact set, there exists a feedback law such that, in the corresponding closed-loop system, each trajectory with initial condition in this set asymptotically converges to the prescribed equilibrium. But, in the case of an uncertain system such as system (12.1), in which the equilibrium point may depend on the value of some unknown parameter, this notion has to be weakened, since it may no longer be possible that all trajectories converge to the same point regardless of the value of the unknown parameter. Thus, the next best alternative is to ask that all trajectories converge to a prescribed set, which hopefully will be some (small) neighborhood of a prescribed point, for instance the point $(z, \xi_1, \ldots, \xi_r) = (0, 0, \ldots, 0)$. To formalize this concept, the following terminology is often used: a trajectory is said to be *captured by the set* Q if it is defined for all $t \in [0, \infty)$, enters the set Q at some finite time T and remains in this set for all $t \geq T$. A system is said to be *semiglobally practically stabilizable* if, for any (arbitrarily large) set K and any *arbitrarily small* set Q, there is a feedback law, which in general depends on both K and Q, such that any trajectory with initial condition in K is captured by the set Q.

In what follows we will show that system (12.1) is semiglobally practically stabilizable, by means of a linear feedback law. For convenience, let $\bar{Q}^n_c$ denote the closed cube in $\mathbb{R}^n$, centered at the origin, of side $2c$, namely

$$\bar{Q}^n_c = \{x \in \mathbb{R}^n : |x_i| \leq c, i = 1, \ldots, n\}.$$

Theorem 12.1.1. *Consider system (12.1) and assume hypotheses* (i) *and* (ii) *hold. Given any arbitrarily large number $R > 0$ and any arbitrarily small number $\varepsilon > 0$, there exists a feedback law of the form*

$$u = k_1\xi_1 + \ldots + k_r\xi_r \tag{12.2}$$

(in which the numbers $k_1, \ldots, k_r$ depend on the choice of R and ε) such that, in the closed loop system (12.1) - (12.2), any initial condition in $\bar{Q}^{n+r}_R$ produces a trajectory which is captured by the set $\bar{Q}^{n+r}_\varepsilon$.

Proof. Let $a_0, a_1, \ldots, a_{r-2}$ be such that the polynomial

$$\lambda^{r-1} + a_{r-2}\lambda^{r-2} + \cdots + a_1\lambda + a_0$$

has all roots with negative real part, and set

$$x = \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}),$$

$$\zeta = \xi_r + k^{r-1}a_0\xi_1 + k^{r-2}a_1\xi_2 + \cdots + ka_{r-2}\xi_{r-1}.$$

where $k > 0$ is a number to be determined later.

Rewrite system (12.1) in the form

$$\begin{aligned} \dot{x} &= F(x) + G\zeta \\ \dot{\zeta} &= \bar{q}(x, \zeta, \mu) + \bar{b}(x, \zeta, \mu)u \end{aligned} \tag{12.3}$$

where

$$F(x) = \begin{pmatrix} f_0(z, \xi_1) \\ \xi_2 \\ \vdots \\ \xi_{r-1} \\ -k^{r-1}a_0\xi_1 - k^{r-2}a_1\xi_2 - \cdots - ka_{r-2}\xi_{r-1} \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

and

$$\begin{aligned} \bar{q}(x, \zeta, \mu) &= q(z, \xi_1, \ldots, \xi_{r-1}, \zeta - k^{r-1}a_0\xi_1 - \ldots - ka_{r-2}\xi_{r-1}, \mu) \\ &\quad + k^{r-1}a_0\xi_2 + \cdots + k^2a_{r-3}\xi_{r-1} \\ &\quad + ka_{r-2}(\zeta - k^{r-1}a_0\xi_1 - k^{r-2}a_1\xi_2 - \cdots - ka_{r-2}\xi_{r-1}) \\ \bar{b}(x, \zeta, \mu) &= b(z, \xi_1, \ldots, \xi_{r-1}, \zeta - k^{r-1}a_0\xi_1 - \ldots - ka_{r-2}\xi_{r-1}, \mu). \end{aligned}$$

It is known from Theorem 9.3.1 that, given any $R > 0$, there is a number $k^* > 0$ such that, if $k \geq k^*$, the equilibrium $x = 0$ of the system

$$\dot{x} = F(x) \tag{12.4}$$

is locally asymptotically stable and, moreover, any initial condition $x^\circ \in \bar{Q}^{n+r-1}_R$ produces a trajectory which converges to this equilibrium as $t \to \infty$. In other words, $\bar{Q}^{n+r-1}_R$ is a subset of the domain of attraction $\mathcal{A}$ of the equilibrium $x = 0$ of this system. Pick, and *fix*, any k larger than or equal to k^*. Recall that $\mathcal{A}$ is an open set and that there is a diffeomorphism,

$$\begin{aligned} \Phi : \mathcal{A} &\to \mathbb{R}^{n+r-1} \\ x &\mapsto \tilde{x} \end{aligned}$$

with $\Phi(0) = 0$. This diffeomorphism transform the restriction of system (12.4) to $\mathcal{A}$ into a system

$$\dot{\tilde{x}} = \tilde{F}(\tilde{x}) = \Phi_*F(\Phi^{-1}(\tilde{x})),$$

defined on $\mathbb{R}^{n+r-1}$, whose equilibrium $\tilde{x} = 0$ is globally asymptotically stable. Hence, by the converse Lyapunov Theorem, there exists a smooth function $\tilde{V}(\tilde{x})$, which is positive definite and proper, such that

$$\frac{\partial \tilde{V}}{\partial \tilde{x}}\tilde{F}(\tilde{x}) < 0$$

for all nonzero $\tilde{x}$. Now observe that the image $\tilde{Q}$ of the set $\bar{Q}^{n+r-1}_R \subset \mathcal{A}$ under the mapping Φ is a compact set. Therefore, there exists a number $c > 1$ such that

$$\tilde{x} \in \tilde{Q} \quad \Rightarrow \quad \tilde{V}(\tilde{x}) \leq c.$$

Consider now the smooth function, defined on $\mathcal{A}$,

$$V(x) = \tilde{V}(\Phi(x)),$$

and let Δ_c denote the set

$$\Delta_c = \{x \in \mathbb{R}^{n+r-1} : V(x) \le c\} .$$

Clearly, by construction,

$$\bar{Q}_R^{n+r-1} \subset \Delta_c \subset \Delta_{c+1} \subset \mathcal{A} ,$$

and

$$\frac{\partial V}{\partial x} F(x) < 0, \qquad \text{for all } x \in \Delta_{c+1}, x \neq 0 .$$

Set

$$\ell = (1 + k^{r-1}a_0 + \cdots + ka_{r-2})$$

(recall that k and all a_i's are positive numbers) and

$$d = \ell^2 R^2 .$$

Assume, without loss of generality that $R > 1$ so that $d > 1$, and note that

$$\begin{array}{rcl} |\xi_i| \le R \text{ for all } i = 1, \ldots, r & \Rightarrow & |\zeta|^2 \le d \\ |\zeta| \le \delta, |\xi_i| \le \delta \text{ for all } i = 1, \ldots, r-1 & \Rightarrow & |\xi_r| \le \ell\delta \le \sqrt{d}\delta . \end{array} \tag{12.5}$$

Now, consider the function

$$W(x, \zeta) = \frac{cV(x)}{c + 1 - V(x)} + \frac{d\zeta^2}{d + 1 - \zeta^2} \tag{12.6}$$

which is defined in the interior of the set $\Delta_{c+1} \times \{\zeta \in \mathbb{R} : \zeta^2 \le d + 1\}$ and positive definite. Set, as usual

$$\Omega_a = \{(x, \zeta) \in \mathbb{R}^{n+r} : W(x, \zeta) \le a\} .$$

Observe that, if $W(x, \zeta) \le c^2 + d^2 + 1$,

$$V(x) \le (c+1)\frac{c^2 + d^2 + 1}{c^2 + d^2 + 1 + c}, \qquad \zeta^2 \le (d+1)\frac{c^2 + d^2 + 1}{c^2 + d^2 + 1 + d}, \tag{12.7}$$

which shows that

$$(x, \zeta) \in \Omega_{c^2+d^2+1} \quad \Rightarrow \quad V(x) < c + 1, \text{ and } \zeta^2 < d + 1 . \tag{12.8}$$

Note also that

$$V(x) \le c \text{ and } \zeta^2 \le d \quad \Rightarrow \quad W(x, \zeta) \le c^2 + d^2 .$$

Choose now $u = -\bar{k}\zeta$ in (12.3) and compute the derivative of the function $W(x, \zeta)$ along the trajectories of the vector field thus obtained, namely of

$$\begin{array}{rcl} \dot{x} & = & F(x) + G\zeta \\ \dot{\zeta} & = & \bar{q}(x, \zeta, \mu) - \bar{b}(x, \zeta, \mu)\bar{k}\zeta . \end{array} \tag{12.9}$$

This yields

$$\begin{array}{rcl} \dot{W}(x, \zeta) & = & \dfrac{c(c+1)}{(c + 1 - V(x))^2} \dfrac{\partial V}{\partial x}[F(x) + G\zeta] \\ & & + \dfrac{d(d+1)}{(d + 1 - \zeta^2)^2} 2\zeta[\bar{q}(x, \zeta, \mu) - \bar{b}(x, \zeta, \mu)\bar{k}\zeta] . \end{array}$$

From (12.7) we obtain

$$\begin{array}{ccccc} \dfrac{c}{c+1} & \le & \dfrac{c(c+1)}{(c + 1 - V(x))^2} & \le & \dfrac{(c^2 + d^2 + 1 + c)^2}{c(c+1)} \\ \dfrac{d}{d+1} & \le & \dfrac{d(d+1)}{(d + 1 - \zeta^2)^2} & \le & \dfrac{(c^2 + d^2 + 1 + d)^2}{d(d+1)} . \end{array}$$

Thus, if $(x, \zeta) \in \Omega_{c^2+d^2+1}$,

$$\begin{array}{rcl} \dot{W}(x, \zeta) & \le & \dfrac{c(c+1)}{(c + 1 - V(x))^2} \dfrac{\partial V}{\partial x} F(x) - 2b_0 \dfrac{d}{d+1}\bar{k}\zeta^2 + \\ & & \left[\dfrac{(c^2 + d^2 + 1 + c)^2}{c(c+1)} \left|\dfrac{\partial V}{\partial x}\right| + 2\dfrac{(c^2 + d^2 + 1 + d)^2}{d(d+1)} |\bar{q}(x, \zeta, \mu)|\right] |\zeta| . \end{array}$$

Now, choose any arbitrarily small number $\rho > 0$ and consider the compact sets

$$\begin{array}{rcl} S & = & \{(x, \zeta) : \rho \le W(x, \zeta) \le c^2 + d^2 + 1\} \\ S_0 & = & S \cap \{(x, \zeta) : \zeta = 0\} . \end{array}$$

By construction, $(x, \zeta) = (0, 0)$ is not in S_0, because $(x, \zeta) = (0, 0) \notin S$, and (see (12.8))

$$S_0 \subset \{(x, \zeta) : V(x) < c + 1, \zeta = 0\} \subset \Delta_{c+1} \times \{0\} .$$

Observe that, by hypothesis, if $x \in \Delta_{c+1}$ and $x \neq 0$,

$$\dot{W}(x, 0) = \frac{c(c+1)}{(c + 1 - V(x))^2} \frac{\partial V}{\partial x} F(x) < 0 .$$

Thus, the function $\dot{W}(x, \zeta)$ is negative at each point of the compact set S_0. By continuity, $\dot{W}(x, \zeta)$ is negative at each point of some open set $U \supset S_0$.

Consider now the compact set $\tilde{S} = S \setminus U$. Since $\zeta \neq 0$ at each point of $\tilde{S}$ there exists $m > 0$ such that

$$\zeta^2 > m ,$$

for all $(x, \zeta) \in \tilde{S}$. Also, there exists $M > 0$ such that

$$\left[\frac{(c^2 + d^2 + 1 + c)^2}{c(c+1)} \left|\frac{\partial V}{\partial x}\right| + 2\frac{(c^2 + d^2 + 1 + d)^2}{d(d+1)} |\bar{q}(x, \zeta, \mu)|\right] |\zeta| \le M$$

for all $(x, \zeta) \in \tilde{S}$ and all $\mu \in \mathcal{P}$. Thus, at each $(x, \zeta) \in \tilde{S}$

$$\dot{W}(x,\zeta) \leq -2b_0 \frac{d}{d+1}\bar{k}m + M .$$

This, as in the Proof of Theorem 9.3.1, shows that there is a number $\bar{k}^* > 0$ such that, if $\bar{k} > \bar{k}^*$, the function $\dot{W}(x,\zeta)$ is negative at each point of S. Note that $\bar{k}^*$ depends on the choice of ρ.

Since

$$\begin{aligned}(z,\xi_1,\ldots,\xi_r) \in \bar{Q}_R^{n+r} \quad &\Rightarrow \quad x \in \bar{Q}_R^{n+r-1} \quad \text{and } |\xi_i| \leq R \text{ for all } 1 \leq i \leq r \\ &\Rightarrow \quad V(x) \leq c, \quad \text{and } |\zeta|^2 \leq d\end{aligned}$$

we have that

$$(z,\xi_1,\ldots,\xi_r) \in \bar{Q}_R^{n+r} \quad \Rightarrow \quad W(x,\zeta) \leq c^2 + d^2 ,$$

i.e.

$$\bar{Q}_R^{n+r} \subset \Omega_{c^2+d^2} .$$

Thus, if the initial condition of (12.9) is in $\bar{Q}_R^{n+r}$, the corresponding trajectory remains in $\Omega_{c^2+d^2+1}$ for all $t \geq 0$, because $\dot{W}(x(t),\zeta(t))$ is negative at each point of the boundary of $\Omega_{c^2+d^2+1}$. Moreover, it is possible to see that at some finite time T the trajectory enters the set Ω_ρ and remains in Ω_ρ for all $t \geq T$. For, suppose this is not true. Then, the previous argument shows that $W(x(t),\zeta(t))$ is always decreasing and converges to a nonnegative limit $W_0 \geq \rho$. Let Γ denote the ω-limit set of the trajectory in question, and note that $\Gamma \subset S$. It is known that $W(x,\zeta) = W_0$ at each point of this set. Pick any initial condition in Γ and observe that the function $W(x,\zeta)$ is constant along the corresponding trajectory. Thus, $\dot{W}(x,\zeta) = 0$ along this trajectory, which is a contradiction, because it was supposed that Γ was in S and $\dot{W}(x,\zeta) < 0$ on S. Thus, $(x(t),\zeta(t))$ must enter the set Ω_ρ in finite time and it can never leave this set afterwards, because $\dot{W}(x,\zeta)$ is negative at each point of its boundary.

In summary, we have shown that, given any $\rho > 0$, there is a number $\bar{k}^*$ such that, if $\bar{k} \geq \bar{k}^*$, any initial condition in $\bar{Q}_R^{n+r}$ produces a trajectory which is captured by the set

$$\Omega_\rho = \{(x,\zeta) : W(x,\zeta) \leq \rho\} .$$

To complete the proof, it remains to show that, if ρ is small enough,

$$\Omega_\rho \subset \bar{Q}_\varepsilon^{n+r} .$$

Now, observe that

$$W(x,\zeta) \leq \rho \quad \Rightarrow \quad V(x) \leq \frac{(c+1)\rho}{c+\rho}, \text{ and } |\zeta|^2 \leq \left(\frac{(d+1)\rho}{d+\rho}\right) .$$

Since $V(x)$ is positive definite, there exists a class $\mathcal{K}$ function $\underline{\alpha}(\cdot)$ such that

$$\underline{\alpha}(\|x\|) \leq V(x) .$$

Thus, if ρ is small enough, so that $(c+1)\rho/(c+\rho)$ is in the image of $\underline{\alpha}(\cdot)$,

$$W(x,\zeta) \leq \rho \quad \Rightarrow \quad \|x\| \leq \underline{\alpha}^{-1}(\frac{(c+1)\rho}{c+\rho}) \text{ and } |\zeta|^2 \leq \left(\frac{(d+1)\rho}{d+\rho}\right) .$$

Both estimates for $\|x\|$ and $|\zeta|$ on the right-hand side converge to 0 as $\rho \to 0$. Thus, there exists ρ such that $W(x,\zeta) \leq \rho$ implies

$$\|x\| \leq \underline{\alpha}^{-1}(\frac{(c+1)\rho}{c+\rho}) \leq \frac{\varepsilon}{\sqrt{d}}, \quad \text{and} \quad |\zeta| \leq \left(\frac{(d+1)\rho}{d+\rho}\right)^{\frac{1}{2}} \leq \frac{\varepsilon}{\sqrt{d}}$$

This, using the second of (12.5) and the fact that $d > 1$, shows that $|\xi_r| \leq \varepsilon$, i.e.

$$W(x,\zeta) \leq \rho \quad \Rightarrow \quad (x,\xi_r) \in \bar{Q}_\varepsilon^{n+r}$$

and completes the proof. Note that the feedback law which solves the problem has the form

$$u = -\bar{k}[\xi_r + k^{r-1}a_0\xi_1 + k^{r-2}a_1\xi_2 + \cdots + ka_{r-2}\xi_{r-1}] \tag{12:10}$$

which is precisely a feedback law of the form (12.2). ◁

Remark 12.1.1. The result of this Theorem can be extended to cover the case in which the upper subsystem of (12.1) contains an uncertain parameter μ, namely has the form

$$\dot{z} = f_0(z,\xi_1,\mu)$$

so long as it is known that there exists a function $V(z)$, independent of μ, satisfying

$$\begin{aligned}\underline{\alpha}(\|z\|) \leq V(z) \leq \overline{\alpha}(\|z\|) \\ \frac{\partial V}{\partial z} f(z,0,\mu) \leq -\alpha(\|z\|)\end{aligned}$$

for all z and μ, where $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$ are class $\mathcal{K}_\infty$ functions. ◁

As observed in the introduction, there is no guarantee that the closed-loop system (12.1) – (12.2) possesses an equilibrium at $(z,\xi_1,\ldots,\xi_r) = (0,0,\ldots,0)$, simply because it was not assumed that $q(z,\xi_1,\ldots,\xi_r,\mu)$ is vanishing at $(z,\xi_1,\ldots,\xi_r) = (0,0,\ldots,0)$, for all μ. On the contrary, if

$$q(0,0,\ldots,0,\mu) = 0 , \tag{12.11}$$

the point $(z,\xi_1,\ldots,\xi_r) = (0,0,\ldots,0)$ is indeed an equilibrium of the closed-loop system. If this is the case, it might well happen that the trajectories of the closed-loop system, which are guaranteed to enter in finite time an arbitrary small neighborhood of the origin (which is invariant in positive time), actually converge, as $t \to \infty$, to the equilibrium in question. In other words, it may happen that the feedback law (12.2) renders the equilibrium $(z,\xi_1,\ldots,\xi_r) = (0,0,\ldots,0)$ locally asymptotically stable, with a region of attraction which contains the set $\bar{Q}_R^{n+r}$. In what follows, it will be shown that, in order to have this occurring, it suffices to assume that the equilibrium $z = 0$ of

$$\dot{z} = f_0(z, 0)$$

is not simply globally asymptotically, but also locally exponentially stable, i.e. that the eigenvalues of the matrix

$$F_0 = \left[\frac{\partial f_0}{\partial z}\right]_{(0,0)}$$

are all in the left-half plane.

Under this extra hypothesis, in fact, the feedback law (12.10), for large k and $\bar{k}$ is a feedback law which asymptotically stabilizes the linear approximation of (12.1) at the equilibrium $(z, \xi_1, \ldots, \xi_r, u) = (0, 0, \ldots, 0, 0)$, as the reader may easily verify on the basis of simple calculations. As a consequence, the arguments of the previous proof can be adapted to show that the law in question actually solves a problem of semiglobal stabilization (and not just a problem of semiglobal and practical stabilization).

Corollary 12.1.2. *Consider system (12.1) and assume hypotheses* (i) *and* (ii) *hold. Assume also that all the eigenvalues of the matrix F_0 above have negative real part and that (12.11) holds. Given any arbitrarily large number $R > 0$, there exists a feedback law of the form (12.2), in which the numbers $k_1, \ldots, k_r$ depend on the choice of R, such that, in the closed loop system (12.1) – (12.2), the equilibrium $(z, \xi) = (0, 0)$ is locally exponentially stable and, moreover, any initial condition in $\bar{Q}_R^{n+r}$ produces a trajectory which asymptotically converges to $(0, 0)$ as $t \to \infty$.*

Proof. Consider the closed-loop system (12.1) – (12.2), and observe that the Jacobian matrix of the vector field $F(x)$ at $x = 0$ has a block triangular structure

$$\left[\frac{\partial F}{\partial x}\right]_{x=0} = \begin{pmatrix} \left[\frac{\partial f_0}{\partial z}\right]_{(0,0)} & * \\ 0 & A \end{pmatrix} \tag{12.12}$$

in which A is a matrix whose eigenvalues have by hypothesis negative real part. As a consequence, in the hypothesis of the Corollary, all the eigenvalues of (12.12) have negative real part and the equilibrium $x = 0$ of $\dot{x} = F(x)$ is locally exponentially stable.

The same property holds for the equilibrium $\tilde{x} = 0$ of the system

$$\dot{\tilde{x}} = \tilde{F}(\tilde{x}) \tag{12.13}$$

obtained by means of the diffeomorphism $\tilde{x} = \Phi(x)$ considered in the proof of Theorem 12.1.1. In fact, since

$$\left[\frac{\partial \tilde{F}}{\partial \tilde{x}}\right]_{\tilde{x}=0} = \Phi_*(0)\left[\frac{\partial F}{\partial x}\right]_{x=0} \Phi_*^{-1}(0) ,$$

and $\Phi_*(0)$ is nonsingular, all the eigenvalues of this matrix have negative real part. As a consequence, in view of Lemma 10.1.5, it can be asserted that there exists a function $\tilde{V}(\tilde{x})$ such that

$$\underline{\tilde{\alpha}}(\|\tilde{x}\|) \leq \tilde{V}(\tilde{x}), \qquad \frac{\partial \tilde{V}}{\partial \tilde{x}}\tilde{F}(\tilde{x}) \leq -\tilde{\alpha}(\|\tilde{x}\|)$$

for all $\tilde{x} \in \mathbb{R}^n$, where $\underline{\tilde{\alpha}}(\cdot)$ and $\tilde{\alpha}(\cdot)$ are class $\mathcal{K}_\infty$ functions which, for some $\tilde{s}_0 > 0$, $\tilde{a} > 0$ and $\tilde{b} > 0$ satisfy

$$\underline{\tilde{\alpha}}(s) = \tilde{a}s^2, \qquad \tilde{\alpha}(s) = \tilde{b}s^2$$

for all $s \in [0, \tilde{s}_0]$.

It is not difficult to check that similar properties hold also for the function $V(x) = \tilde{V}(\Phi(x))$. For instance, consider the inequality

$$\tilde{a}\|\Phi(x)\|^2 = \tilde{a}\|\tilde{x}\|^2 \leq \tilde{V}(\Phi(x)) = V(x)$$

which holds for sufficiently small $\|x\|$ and observe that $\Phi(x)$ can be expanded in the form

$$\Phi(x) = Tx + R(x)$$

where T is a nonsingular matrix and $R(x)$ a residual whose entries vanish at $x = 0$ together with their first derivatives. Then,

$$\|\Phi(x)\|^2 = x^{\mathrm{T}}Qx + S(x)$$

where $Q > 0$ and $S(x)$ is a function satisfying

$$\lim_{\|x\| \to 0} \frac{S(x)}{\|x\|^2} = 0 .$$

As a consequence, it is readily seen that, for some $a > 0$ and some sufficiently small s_0,

$$a\|x\|^2 \leq V(x)$$

for all x such that $\|x\| \leq s_0$. An identical argument shows that, for some $b > 0$,

$$\frac{\partial V}{\partial x}F(x) \leq -b\|x\|^2$$

for all x such that $\|x\| \leq s_0$.

Consider now again the Lyapunov function $W(x, \zeta)$ introduced in the proof of Theorem 12.1.1 and observe that, for all $\|x\| \leq s_0$

$$\frac{cV(x)}{c + 1 - V(x)} \geq \frac{c}{c+1}V(x) \geq \frac{c}{c+1}a\|x\|^2 .$$

Likewise, for all $|\zeta|^2 < d + 1$

$$\frac{d\zeta^2}{d+1-\zeta^2} \geq \frac{d}{d+1}\zeta^2 .$$

Thus, there is a number $\hat{a}$ such that, for all (x,ζ) whose norm does not exceed s_0, the function $W(x,\zeta)$ is bounded from below as

$$\hat{a}\|(x,\zeta)\|^2 \leq W(x,\zeta) . \tag{12.14}$$

The derivative of $W(x,\zeta)$ along the trajectories of the closed loop system satisfies, for all (x,ζ) whose norm does not exceed s_0,

$$\dot{W}(x,\zeta) \leq -cb\|x\|^2 - 2b_0\frac{d}{d+1}\bar{k}\zeta^2 + M(x,\zeta,\mu)\zeta$$

where

$$M(x,\zeta,\mu) = \left[\frac{c(c+1)}{(c+1-V(x))^2}\frac{\partial V}{\partial x}G + 2\frac{d(d+1)}{d+1-\zeta^2}\bar{q}(x,\zeta,\mu)\right] .$$

Since

$$\left[\frac{\partial V}{\partial x}G\right](0) = 0, \qquad \bar{q}(0,0,\mu) = 0$$

it follows that the function $M(x,\zeta,\mu)$, a smooth function of (x,ζ,μ), can be expressed in the form

$$2L_1(x,\zeta,\mu)x + L_2(x,\zeta,\mu)\zeta$$

where $L_1(x,\zeta,\mu)$ and $L_2(x,\zeta,\mu)$ are smooth functions. As a consequence, for all (x,ζ) whose norm does not exceed s_0, $\dot{W}(x,\zeta)$ can be bounded from above as

$$\dot{W}(x,\zeta) \leq -\begin{pmatrix} x \\ \zeta \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} cbI & -L_1^{\mathrm{T}}(x,\zeta,\mu) \\ -L_1(x,\zeta,\mu) & \frac{2b_0 d}{d+1}\bar{k} - L_2(x,\zeta,\mu) \end{pmatrix} \begin{pmatrix} x \\ \zeta \end{pmatrix} .$$

Since μ ranges over a compact set, given any positive number $\hat{b} < cb$ there is a number $\bar{k}^*$ such that, for all $\bar{k} \geq \bar{k}^*$ and all (x,ζ) whose norm does not exceed s_0

$$\begin{pmatrix} cbI & -L_1^{\mathrm{T}}(x,\zeta,\mu) \\ -L_1(x,\zeta,\mu) & \frac{2b_0 d}{d+1}\bar{k} - L_2(x,\zeta,\mu) \end{pmatrix} \geq \hat{b}I$$

i.e.

$$\dot{W}(x,\zeta) \leq -\hat{b}\|(x,\zeta)\|^2 . \tag{12.15}$$

Inequalities (12.14) and (12.15) prove, by Lemma 10.1.5, that, if $\bar{k}$ is large enough, the equilibrium $(x,\zeta) = (0,0)$ of the closed loop system (12.1) - (12.2) is locally exponentially stable. In particular, observe that these inequalities hold for all (x,ζ) whose norm does not exceed a number s_0 which *is independent* of $\bar{k}$. Let $c_0 > 0$ be such that

$$\Omega_{c_0} \subset B_{s_0} .$$

Since $\dot{W}(x,\zeta)$ is negative at each point of the boundary of the set Ω_{c_0}, the latter (which only depends on the function $W(x,\zeta)$ and c_0, and *does not* depend on the actual value of $\bar{k}$) is invariant for positive time. Moreover, since $W(x,\zeta) \leq \hat{c}\|(x,\zeta)\|^2$ for some $\hat{c}$ and all $(x,\zeta) \in B_{s_0}$, any initial condition in Ω_{c_0} produces a trajectory along which

$$\dot{W}(x(t),\zeta(t)) \leq -\frac{\hat{b}}{\hat{c}}W(x(t),\zeta(t)) := -a_0 W(x(t),\zeta(t)) .$$

As a consequence, any of such trajectories satisfies

$$W(x(t),z(t)) \leq e^{-a_0 t}W(x(0),\zeta(0)) ,$$

i.e. satisfies

$$\|(x(t),\zeta(t))\| \leq \frac{1}{\hat{a}}e^{-a_0 t}W(x(0),\zeta(0))$$

for all $t \geq 0$. In particular, any of such trajectories converges to $(0,0)$ as $t \to \infty$. Now, let ε be such that

$$\bar{Q}_\varepsilon^{n+r} \subset \Omega_{c_0} .$$

Theorem 12.1.1 shows that, if $\bar{k}$ is large enough, any trajectory with initial condition in $\bar{Q}_R^{n+r}$ enters in finite time the set $\bar{Q}_\varepsilon^{n+r}$, hence in particular the set Ω_{c_0}. Therefore, it can be concluded that any trajectory with initial condition in $\bar{Q}_R^{n+r}$ converges to $(0,0)$ as $t \to \infty$ and this completes the proof of the Corollary. ◃

12.2 Semiglobal Stabilization via Partial State Feedback

The key argument of the proof of Theorem 12.1.1 in the previous section was essentially the fact that, in a system of the form

$$\begin{aligned} \dot{x} &= F(x) + G\zeta \\ \dot{\zeta} &= q(x,\zeta) - b(x,\zeta)k\zeta \end{aligned} \tag{12.16}$$

(with $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$), under the hypotheses that:

(i) the equilibrium $x = 0$ of $\dot{x} = F(x)$ is locally asymptotically stable and its domain of attraction $\mathcal{A}$ contains the set $\bar{Q}_R^n$,

(ii) $b(x,\zeta) \geq b_0$ for some $b_0 > 0$,

given any arbitrarily small $\varepsilon > 0$, there is a number k^* such that, if $k \geq k^*$, all trajectories with initial condition in $\bar{Q}_R^{n+1}$ are captured by the set $\bar{Q}_\varepsilon^{n+1}$.

As observed after the Theorem, this does not necessarily guarantee asymptotic convergence of all such trajectories to the point $(x,\zeta) = (0,0)$. Even assuming that the point in question is an equilibrium point for the system (hence assuming that $q(0,0) = 0$), additional hypotheses are needed to be able to conclude that all trajectories with initial condition in $\bar{Q}_R^{n+1}$ eventually converge to the equilibrium $(x, \zeta) = (0,0)$. For instance, as shown in Corollary 12.1.2, this is the case if the equilibrium $x = 0$ of $\dot{x} = F(x)$ satisfies the extra hypothesis of being locally exponentially stable. However, the proof of the Corollary only uses this extra hypothesis in order to guarantee that the equilibrium $(x, \zeta) = (0,0)$ of (12.16) is locally *asymptotically* stable, with a region of attraction which contains a neighborhood of the equilibrium which *does not* depend on k (as a matter of fact, the property that this equilibrium is locally *exponentially* stable, which actually holds under the said hypothesis if k is large enough, was not needed as such in the conclusion of the proof of the Corollary). Thus, one may argue that a result of asymptotic stability with prescribed domain of attraction such as that of Corollary 12.1.2 might hold under hypotheses weaker than those considered in this Corollary. In this section we wish to elaborate further on this subject, and discuss when, in a system with an interconnected structure such as that of system (12.16), the equilibrium $(x, \zeta) = (0,0)$ is locally asymptotically stable with a region of attraction that contains a prescribed set.

The need of extra hypotheses and the fact that the extra hypotheses of Corollary 12.1.2 might be weakened can be better understood with the aid of two simple examples.

Example 12.2.1. Consider the system

$$\begin{aligned} \dot{x} &= -x^3 + \zeta \\ \dot{\zeta} &= x - k\zeta\,. \end{aligned} \tag{12.17}$$

In this system, the equilibrium $(x, \zeta) = (0,0)$ cannot be locally asymptotically stable, no matter how k is chosen. In fact, its linear approximation

$$\begin{pmatrix} \dot{x} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix} \begin{pmatrix} x \\ \zeta \end{pmatrix}$$

has a characteristic polynomial $\lambda^2 + k\lambda - 1$ with one root having positive real part for any k.

Note that this is not in contrast with the result of Theorem 12.1.1. In fact, system (12.17), for $k > 0$, has three equilibria

$$\begin{aligned} (x, \zeta) &= (0,0) \\ (x, \zeta) &= (1/\sqrt{k}, 1/\sqrt{k^3}) \\ (x, \zeta) &= (-1/\sqrt{k}, -1/\sqrt{k^3})\,. \end{aligned}$$

The former of these is unstable, but the other two are locally asymptotically stable and, for any $\varepsilon > 0$, there is a number k^* such that, if $k > k^*$, both the stable equilibria are in $\bar{Q}_\varepsilon^2$ (see Fig. 12.1).

This example is perhaps the simplest explanation of why, in system (12.16), the asymptotic stability of the equilibrium of $\dot{x} = F(x)$ is not sufficient to guarantee, even for large $k > 0$, local asymptotic stability of the

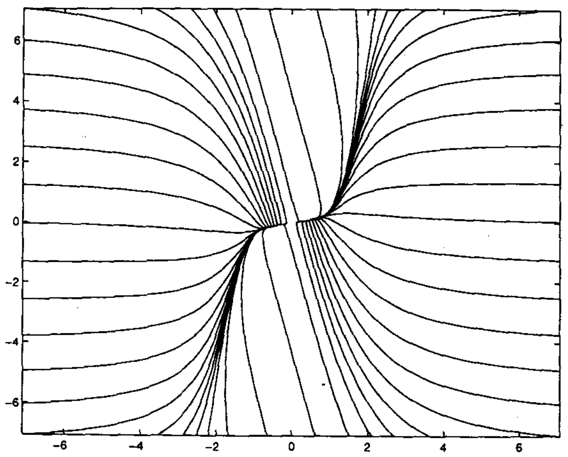

Fig. 12.1. Phase portrait of the system.

equilibrium $(x, \zeta) = (0,0)$. To find an example that the extra condition of Corollary 12.1.2 is sufficient but not necessary, it suffices to look at the system

$$\begin{aligned} \dot{x} &= -x^3 + \zeta \\ \dot{\zeta} &= x^3 - k\zeta\,. \end{aligned} \tag{12.18}$$

Consider the positive definite function

$$V(x, \zeta) = \frac{1}{4}x^4 + \frac{1}{2}\zeta^2\,.$$

Its derivative along the trajectories of (12.18) reads as

$$\dot{V} = -x^6 + x^3\zeta + \zeta x^3 - k\zeta^2 = -\begin{pmatrix} x^3 & \zeta \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & k \end{pmatrix} \begin{pmatrix} x^3 \\ \zeta \end{pmatrix}$$

and is negative definite for $k > 1$. Thus, for $k > 1$, the equilibrium $(x, \zeta) = (0,0)$ of (12.18) is globally asymptotically stable.

Another alternative way to check this conclusion is to use the small gain theorem. The upper subsystem, viewed as a system with state x and input ζ, is input to state stable, with gain function

$$\gamma_1(r) = r^{\frac{1}{3}}\,.$$

If $k > 0$, the lower subsystem, viewed as a system with state ζ and input x, is input to state stable, with gain function

$$\gamma_2(r) = \frac{1}{k}r^3\,.$$

Thus, if $k > 1$, the small gain condition is fulfilled. ◁

The second part of this example suggests to use, in order to check the property of asymptotic stability of system (12.16) in situations other than that of local exponential stability of the equilibrium $x = 0$ of $\dot{x} = F(x)$, the *small gain theorem*. In the present context, however, only *local* asymptotic stability is of interest, because the convergence to an arbitrarily small neighborhood of the equilibrium is guaranteed by other arguments. Thus, it is expected that a sort of "local" version of the property of input-to-state stability, on which the small gain theorem is based, would suffice. This notion, together with an application to the present problem, is described hereafter.

Consider a nonlinear system

$$\dot{x} = f(x,u) \tag{12.19}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. Let X be an open subset of $\mathbb{R}^n$ containing the origin and let U be a positive number.

Definition 12.2.1. *System (12.19) is said to be input-to-state stable with restriction X on x° and restriction U on $u(\cdot)$ if there exist class $\mathcal{K}$ functions $\gamma_0(\cdot)$ and $\gamma_u(\cdot)$ such that, for any $x^\circ \in X$ and any input $u(\cdot) \in L_\infty^m$ satisfying $\|u(\cdot)\|_\infty < U$, the response $x(t)$ in the initial state $x(0) = x^\circ$ satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(\|x^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \gamma_u(\limsup_{t\to\infty} \|u(t)\|)\,. \end{aligned}$$

The function $\gamma_u(\cdot)$ for which the estimates above hold is sometimes referred to as a *local* gain function, in recognition of the fact that the estimates in question hold only for those $u(\cdot)$ whose L_∞ norm does not exceed the number U.

In close analogy to the study presented in section 10.6, we show hereafter that, if the small gain condition is fulfilled, the feedback interconnection of two nonlinear systems that are both input-to-state stable, with restrictions on initial state and input, is – in turn – input-to-state stable, with restrictions on initial state and input.

More precisely, consider the following interconnected system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2, u)\,, \end{aligned} \tag{12.20}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$ and $f_1(0,0) = 0$, $f_2(0,0,0) = 0$. Suppose that the first subsystem, viewed as a system with internal state x_1 and input x_2 is input-to-state stable with restrictions X_1 on x_1° and Δ_2 on $x_2(\cdot)$. That is, suppose there exist class $\mathcal{K}$ functions $\gamma_{01}(\cdot)$ and $\gamma_1(\cdot)$ such that for any $x_1^\circ \in X_1$ and any $\|x_2(\cdot)\|_\infty < \Delta_2$, the response $x_1(t)$ satisfies

$$\begin{aligned} \|x_1(\cdot)\|_\infty &\leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x_1(t)\| &\leq \gamma_1(\limsup_{t\to\infty} \|x_2(t)\|)\,. \end{aligned}$$

Likewise, suppose that the second subsystem, viewed as a system with internal state x_2 and inputs x_1 and u is input-to-state stable with restrictions X_2 on x_2°, Δ_1 on $x_1(\cdot)$ and U on $u(\cdot)$. That is, suppose there exist class $\mathcal{K}$ functions $\gamma_{02}(\cdot)$, $\gamma_2(\cdot)$ and $\gamma_u(\cdot)$ such that for any $x_2^\circ \in X_2$, any $\|x_1(\cdot)\|_\infty < \Delta_1$ and any $\|u(\cdot)\|_\infty < U$, the response $x_2(t)$ satisfies

$$\begin{aligned} \|x_2(\cdot)\|_\infty &\leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x_2(t)\| &\leq \max\{\gamma_2(\limsup_{t\to\infty} \|x_1(t)\|), \gamma_u(\limsup_{t\to\infty} \|u(t)\|)\}\,. \end{aligned}$$

The same arguments used in the proof of Theorem 10.6.1 show that, if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, i.e. if

$$\gamma_1(\gamma_2(r)) < r \quad \text{for all } r > 0, \tag{12.21}$$

the composite system (12.20) is input-to-stable, with appropriate restrictions on the initial states x_1°, x_2° and the input $u(\cdot)$. The only extra precaution needed here is to carefully keep track of the restrictions that have to be imposed in order to take advantage of the inequalities above, which characterize the property of input-to-state stability. To this end, recall that the key of the proof in question, used first to proof boundedness of all trajectories and then the property of input-to-state stability, was the substitution of the estimates provided by the second set of inequalities into the first set (and vice-versa). Now, suppose that $x_2^\circ \in X_2$, $\|x_1(\cdot)\|_\infty < \Delta_1$ and $\|u(\cdot)\|_\infty < U$, so that

$$\|x_2(\cdot)\|_\infty \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}\,. \tag{12.22}$$

Suppose also that $x_1^\circ \in X_1$ and $\|x_2(\cdot)\|_\infty < \Delta_2$, so that

$$\|x_1(\cdot)\|_\infty \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2(\cdot)\|_\infty)\}\,. \tag{12.23}$$

Replacing (12.22) into (12.23) yields

$$\begin{aligned} \|x_1(\cdot)\|_\infty \leq \max\{&\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \\ &\gamma_1 \circ \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\}\,, \end{aligned}$$

which, using the small gain condition (12.21), reduces to

$$\|x_1(\cdot)\|_\infty \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\}\,.$$

Since $x_1(\cdot)$ was supposed to satisfy $\|x_1(\cdot)\|_\infty < \Delta_1$, we need to assume that

$$\begin{aligned} \gamma_{01}(\|x_1^\circ\|) &< \Delta_1 \\ \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|) &< \Delta_1 \\ \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty) &< \Delta_1\,. \end{aligned}$$

Likewise, after having replaced (12.23) into (12.22), we see that the condition $\|x_2(\cdot)\|_\infty < \Delta_2$ requires

$$\begin{aligned}\gamma_{02}(\|x_2^\circ\|) &< \Delta_2\\ \gamma_2\circ\gamma_{01}(\|x_1^\circ\|) &< \Delta_2\\ \gamma_u(\|u(\cdot)\|_\infty) &< \Delta_2 .\end{aligned}$$

If these constraints have to hold, x_1° must be restricted to the set

$$\tilde{X}_1 = \{x_1 \in X_1 : \gamma_{01}(\|x_1\|) < \Delta_1, \gamma_2\circ\gamma_{01}(\|x_1\|) < \Delta_2\} ,$$

and x_2° must be restricted to the set

$$\tilde{X}_2 = \{x_2 \in X_2 : \gamma_{02}(\|x_2\|) < \Delta_2, \gamma_1\circ\gamma_{02}(\|x_2\|) < \Delta_1\} .$$

Moreover, $u(\cdot)$ must be such that

$$\|u(\cdot)\|_\infty < \tilde{U}$$

for some $\tilde{U}$ satisfying

$$\tilde{U} \le U, \qquad \gamma_u(\tilde{U}) \le \Delta_2, \qquad \gamma_1\circ\gamma_u(\tilde{U}) \le \Delta_1 .$$

The relations thus found identify appropriate candidates for the restrictions on the initial states x_1°, x_2° and the input $u(\cdot)$ of system (12.20). Formally, following exactly the same procedure used to prove Theorem 10.6.1, it is possible to show that the following holds.

Theorem 12.2.1. *If the condition (12.21) holds, system (12.20), viewed as a system with state $x = (x_1, x_2)$ and input u is input-to-state stable, with restrictions $\tilde{X}_1$ on x_1°, $\tilde{X}_2$ on x_2° and $\tilde{U}$ on $u(\cdot)$. In particular, the class $\mathcal{K}$ functions*

$$\begin{aligned}\gamma_0(r) &= \max\{2\gamma_{01}(r), 2\gamma_{02}(r), 2\gamma_1\circ\gamma_{02}(r), 2\gamma_2\circ\gamma_{01}(r)\}\\ \gamma(r) &= \max\{2\gamma_1\circ\gamma_u(r), 2\gamma_u(r)\}\end{aligned}$$

are such that response $x(t)$ satisfies

$$\begin{aligned}\|x(\cdot)\|_\infty &\le \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\}\\ \limsup_{t\to\infty}\|x(t)\| &\le \gamma(\limsup_{t\to\infty}\|u(t)\|) .\end{aligned}$$

Remark 12.2.1. Also the property of input-to-state stability with restrictions can be checked by seeking the existence of appropriate Lyapunov functions. In fact, let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function satisfying

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) ,$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$. Suppose there exists numbers δ_x and δ_u and a class $\mathcal{K}$ function $\chi(\cdot)$ such that

$$\|x\| \ge \chi(\|u\|) \quad\Rightarrow\quad \frac{\partial V}{\partial x}f(x,u) \le -\alpha(\|x\|) \quad \text{for all } \|x\|\le\delta_x, \|u\|\le\delta_u , \tag{12.24}$$

where $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

Define

$$c^* = \underline{\alpha}(\delta_x) ,$$

set

$$X = \Omega_{c^*}$$

and let U be such that

$$\overline{\alpha}\circ\chi(U) < c^*, \qquad \text{and} \qquad U < \delta_u .$$

Then, it is readily seen that,

$$\Omega_{c^*} \in B_{\delta_x} .$$

Moreover, if u is such that $\|u\| < U$ and x is on the boundary of Ω_{c^*},

$$\overline{\alpha}\circ\chi(\|u\|) < c^* \le \overline{\alpha}(\|x\|) ,$$

i.e. $\|x\| \ge \chi(\|u\|)$, so that $\dot{V}$ is negative at each point of the boundary of Ω_c. Thus, from arguments identical to those used at the beginning of section 10.4, it follows that for any $x^\circ \in X$ and any $u(\cdot)$ satisfying $\|u(\cdot)\|_\infty < U$

$$\|x(t)\| \le \max\{\beta(\|x^\circ\|, t), \gamma(\|u(\cdot)\|_\infty)\}$$

for some class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, with

$$\gamma(r) = \underline{\alpha}^{-1}\circ\overline{\alpha}\circ\chi(r) .$$

It follows that the system is input-to-state stable with restriction X on x° and restriction U on $u(\cdot)$. ◁

The notion of input-to-state stability with restrictions is helpful in weakening the hypotheses of Corollary 12.1.2. In fact, let the system (12.1) – (12.2) be rewritten in the form (12.9), namely

$$\begin{aligned}\dot{x} &= F(x) + G\zeta\\ \dot{\zeta} &= \bar{q}(x,\zeta,\mu) - \bar{b}(x,\zeta,\mu)\bar{k}\zeta .\end{aligned} \tag{12.25}$$

Suppose the upper subsystem, viewed as a system with state x and input ζ, is input-to-state stable with restrictions X for x° and Δ_ζ for $\zeta(\cdot)$, for some neighborhood X of $x = 0$ and some $\Delta_\zeta > 0$. Moreover, suppose the lower subsystem, viewed as a system with state ζ and input x, is input-to-state stable with restrictions Z for ζ° and Δ_x for $x(\cdot)$, for some neighborhood Z of $\zeta = 0$ and some $\Delta_x > 0$. Suppose also that these restrictions as well as the estimates which characterize the property of input-to-state stability are independent of $\bar{k}$ and μ. If the (local) gain functions of the two subsystems satisfy the small gain condition (12.21), the equilibrium point $(x,\zeta) = (0,0)$ is locally asymptotically stable, with a region of attraction that is guaranteed to contain a neighborhood $\mathcal{A}_0$ of the origin which does not depend on $\bar{k}$. Thus, as shown at the end of the proof of Corollary 12.1.2, if $\bar{k}$ is large enough, all trajectories with initial conditions in $\bar{Q}_R^{n+r}$ asymptotic converge to the equilibrium $(x,\zeta)=(0,0)$ as $t\to\infty$.

The reader can easily check that the hypotheses thus indicated are weaker than those of Corollary 12.1.2. In fact, if the equilibrium $x=0$ of $\dot{x}=F(x)$ is locally exponentially stable, the upper subsystem of (12.25), for some (small) neighborhood X of $x=0$ and $\Delta_\zeta>0$, is indeed input-to-state stable with restrictions X and Δ_ζ, and has a (local) gain function $\gamma_1(\cdot)$ of the form

$$\gamma_1(r)=cr\,.$$

Also the lower subsystem, for some (small) neighborhood Z of $\zeta=0$ and $\Delta_x>0$, both independent of $\bar{k}$ and μ, is input-to-state stable with restrictions Z and Δ_x, and has a (local) gain function that, for large $\bar{k}$, can be bounded by a function $\gamma_2(\cdot)$ of the form

$$\gamma_2(r)\le\frac{1}{2c}r$$

Therefore, for large $\bar{k}$ the small gain condition (12.21) is fulfilled.

12.3 A Proof of Theorem 9.6.2

Before proceeding with the description of how, for selected classes of nonlinear uncertain systems, the problem of semiglobal practical stabilization can be solved *using output feedback*, it is convenient to examine in detail how the problem in question can be solved in the absence of parameter uncertainties. We recall that this problem was addressed in section 9.6, where it was shown that, if a system

$$\begin{aligned}\dot{x}&=f(x)+g(x)u\\ y&=h(x)\,,\end{aligned}\qquad(12.26)$$

is globally asymptotically *stabilizable using* a memoryless smooth *state feedback* and it is also *uniformly observable*, then semiglobal stabilization using dynamic output feedback is possible (Theorem 9.6.2). In this section we provide a detailed proof of this result, which is interesting per se and also because it introduces some arguments that will be used in the later sections in dealing with similar stabilization problems for uncertain systems.

Consider a system of the form (12.26), with state $x\in\mathbb{R}^n$, input $u\in\mathbb{R}$, output $y\in\mathbb{R}$, in which it is assumed that $f(x)$, $g(x)$, $h(x)$ are smooth functions of their arguments, and $f(0)=0, h(0)=0$. Suppose there exists a smooth feedback law $u=\alpha(x)$, with $\alpha(0)=0$ which renders the equilibrium $x=0$ of the system

$$\dot{x}=f(x)+g(x)\alpha(x)$$

globally asymptotically stable. As a consequence (Theorem 9.2.3), there exists a smooth feedback law

$$\bar{u}=\theta(x,v_0,\ldots,v_{n-1})\qquad(12.27)$$

which globally asymptotically stabilizes the extended $2n$-dimensional system

$$\begin{aligned}\dot{x}&=f(x)+g(x)v_0\\ \dot{v}_0&=v_1\\ &\cdots\\ \dot{v}_{n-2}&=v_{n-1}\\ \dot{v}_{n-1}&=\bar{u}\,.\end{aligned}\qquad(12.28)$$

Moreover, suppose system (12.26) is uniformly observable and recall that this property, as shown in section 9.6, guarantees that the mapping $w=\Phi(x,v)$ defined as in (9.58) has global inverse $x=\Psi(w,v)$.

The controller which in section 9.6 was claimed to be able to semiglobally asymptotically stabilize system (12.26) was a dynamical system modeled by equations of the form

$$\begin{pmatrix}\dot{v}_0\\ \dot{v}_1\\ \cdot\\ \dot{v}_{n-2}\\ \dot{v}_{n-1}\end{pmatrix}=\begin{pmatrix}v_1\\ v_2\\ \cdots\\ v_{n-1}\\ \theta(\Psi^*(\eta,v),v_0,\ldots,v_{n-1})\end{pmatrix},\qquad(12.29)$$

$$\begin{pmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \cdot\\ \dot{\eta}_{n-1}\\ \dot{\eta}_n\end{pmatrix}=\begin{pmatrix}\eta_2\\ \eta_3\\ \cdots\\ \eta_n\\ \varphi_n(\Psi^*(\eta,v),v_0,\ldots,v_{n-1})\end{pmatrix}+\begin{pmatrix}gc_{n-1}\\ g^2c_{n-2}\\ \cdots\\ g^{n-1}c_1\\ g^nc_0\end{pmatrix}(y-\eta_1)\,,\qquad(12.30)$$

and

$$u=v_0\,,\qquad(12.31)$$

in which:

(i) $v=\mathrm{col}(v_0,v_1,\ldots,v_{n-2})$, $\eta=\mathrm{col}(\eta_1,\eta_2,\ldots,\eta_n)$,

(ii) $\theta(x,v_0,v_1,\ldots,v_{n-1})$ is the function (12.27),

(iii) $\varphi_n(x,v_0,v_1,\ldots,v_{n-1})$ is the n-th function in the sequence (9.57),

(iv) $g>0$ is a design parameter which depends on the size of the set of initial conditions that must be asymptotically steered to the equilibrium, and $c_0,c_1,\ldots,c_{n-1}$ are coefficients of a Hurwitz polynomial

$$p(\lambda)=\lambda^n+c_{n-1}\lambda^{n-1}+\cdots+c_1\lambda+c_0\,,$$

(v) $\Psi^*(\eta,v)$ is a function defined as follows

$$\Psi^*(\eta, v) = \begin{cases} \Psi(\eta, v) & \text{if } \|\Psi(\eta, v)\| < M \\ \dfrac{\Psi(\eta, v)}{\|\Psi(\eta, v)\|} M & \text{if } \|\Psi(\eta, v)\| \geq M , \end{cases} \tag{12.32}$$

where $M > 0$ is another design parameter which depends on the size of the set of initial conditions that must be asymptotically steered to the equilibrium. Note that $\Psi^*(\eta, v)$ is a function which coincides with $\Psi(\eta, v)$ for all (η, v) such that the norm of $\Psi(\eta, v)$ is less than a fixed number M, and bounded (in norm) by M elsewhere.

To prove Theorem 9.6.2, consider the closed-loop system (12.26) - (12.29) - (12.30) - (12.31), define

$$e = \mathrm{col}(e_1, e_2, \ldots, e_n)$$

by

$$\begin{aligned} e_1 &= g^{n-1}(\varphi_0(x) - \eta_1) \\ e_i &= g^{n-i}(\varphi_{i-1}(x, v_0, \ldots, v_{i-2}) - \eta_i), \qquad 2 \leq i \leq n , \end{aligned}$$

and observe that

$$e = D_g(\Phi(x, v) - \eta)$$

where

$$D_g = \mathrm{diag}\{g^{n-1}, \ldots, g, 1\} .$$

Finally, set

$$z = \mathrm{col}(x, v_0, v_1, \ldots, v_{n-1}) .$$

This yields equations of the form

$$\begin{aligned} \dot{z} &= F(z) + p_1(z, e) \\ \dot{e} &= gAe + p_2(z, e) \end{aligned} \tag{12.33}$$

in which

$$A = \begin{pmatrix} -c_{n-1} & 1 & 0 & \cdots & 0 \\ -c_{n-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -c_1 & 0 & 0 & \cdots & 1 \\ -c_0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$p_1(z, e) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_1(z, e) \end{pmatrix}, \quad p_2(z, e) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_2(z, e) \end{pmatrix},$$

and

$$\begin{aligned} \phi_1(z, e) &= \theta(\Psi^*(\Phi(x, v) - D_g^{-1}e, v), v_0, \ldots, v_{n-1}) - \theta(x, v_0, \ldots, v_{n-1}) \\ \phi_2(z, e) &= \varphi_n(x, v_0, \ldots, v_{n-1}) - \varphi_n(\Psi^*(\Phi(x, v) - D_g^{-1}e, v), v_0, \ldots, v_{n-1}). \end{aligned}$$

By construction, system

$$\dot{z} = F(z)$$

has a globally asymptotically stable equilibrium at $z = 0$. Thus, there exists a smooth real-valued function $V(z)$ satisfying

$$\begin{aligned} \underline{\alpha}(\|z\|) \leq V(z) \leq \overline{\alpha}(\|z\|) \\ \frac{\partial V}{\partial z} F(z) \leq -\alpha(V(z)) \end{aligned}$$

for all z, where $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$ are class $\mathcal{K}_\infty$ functions.

By hypothesis,

$$\{z(0), \eta(0)\} \in \mathcal{S}_z \times \mathcal{S}_\eta$$

where $\mathcal{S}_z$ and $\mathcal{S}_\eta$ are fixed compact sets. Choose c such that

$$\Omega_c = \{z : V(z) \leq c\} \supset \mathcal{S}_z$$

and then choose the parameter M in the definition of the function $\Psi^*(\eta, v)$ as

$$M = \max_{z \in \Omega_{c+1}} \|x\| + 1 .$$

With this choice, it is easy to check that the two vectors $p_1(z, e)$ and $p_2(z, e)$ defined before vanish at $e = 0$, so long as $z \in \Omega_{c+1}$. In fact, recall that $x = \Psi(\Phi(x, v), v)$. Thus, if $z \in \Omega_{c+1}$, we have $\|\Psi(\Phi(x, v), v)\| < M$. As a consequence, if $z \in \Omega_{c+1}$

$$[\Psi^*(\eta, v)]_{e=0} = \Psi^*(\Phi(x, v), v) = \Psi(\Phi(x, v), v) = x,$$

and this shows that if $z \in \Omega_{c+1}$,

$$p_1(z, 0) = 0, \quad p_2(z, 0) = 0.$$

Note that, for all $(z, e) \in \Omega_{c+1} \times \mathbb{R}^n$ and for all $g > 0$,

$$\|\Psi^*(\Phi(x, v) - D_g^{-1}e, v)\| \leq M .$$

Thus, there exist positive numbers β_1, β_2, independent of g, such that

$$\|p_i(z, e)\| \leq \beta_i, \text{ for all } (z, e) \in \Omega_{c+1} \times \mathbb{R}^n . \tag{12.34}$$

Let k_1 be such that

$$\left\| \frac{\partial V(z)}{\partial z} \right\| \leq k_1 \text{ for all } z \in \Omega_{c+1} .$$

Then

$$\frac{\partial V(z)}{\partial z}(F(z) + p_1(z, e)) \leq -\alpha(V(z)) + k_1\beta_1 \tag{12.35}$$

for all $(z, e) \in \Omega_{c+1} \times \mathbb{R}^n$.

Let P be the positive definite solution of $PA + A^T P = -I$, and let $k_2 = \|P\|$. Then, using standard inequalities, observe that for all $(z,e) \in \Omega_{c+1} \times \mathbb{R}^n$, the function $Q(e) = e^T Pe$ satisfies

$$\frac{\partial Q}{\partial e}(gAe + p_2(z,e)) \leq -g\|e\|^2 + 2\|e^T P\|\beta_2 \leq -\left(g - \frac{k_2}{\nu}\right)\|e\|^2 + \nu k_2 \beta_2^2$$

where $\nu > 0$ is any number. Let $k_3 > 0$ and $k_4 > 0$ be such that

$$k_3\|e\|^2 \leq Q(e) \leq k_4\|e\|^2$$

and set

$$a(g) = \left(g - \frac{k_2}{\nu}\right)\frac{1}{k_4}.$$

Assuming that $g > 0$ is large enough (so that $a(g) > 0$), this yields

$$\frac{\partial Q}{\partial e}(gAe + p_2(z,e)) \leq -a(g)Q(e) + \nu k_2 \beta_2^2 \tag{12.36}$$

for all $(z,e) \in \Omega_{c+1} \times \mathbb{R}^n$.

Inequalities (12.35) and (12.36) show that, if $z(t) \in \Omega_{c+1}$ and g is large enough,

$$\frac{dV(z(t))}{dt} \leq k_1\beta_1, \quad \frac{dQ(e(t))}{dt} \leq \nu k_2 \beta_2^2, \tag{12.37}$$

with $k_1, \beta_1, \nu, k_2, \beta_2$ independent of g.

From these it can be seen that there is a fixed time $T > 0$ (independent of g) such that, for every initial state $z(0), \eta(0) \in \mathcal{S}_z \times \mathcal{S}_\eta$, the solution $z(t), \eta(t)$ is defined for all $t \in [0,T]$ and, in particular, $z(t) \in \Omega_{c+1}$ for all $t \in [0,T]$. In fact, integration of the first one of (12.37) on the interval $[0,t]$ yields

$$V(z(t)) - V(z(0)) \leq k_1\beta_1 t.$$

Choosing $T = (2k_1\beta_1)^{-1}$, it is concluded that necessarily $V(t) \leq V(0) + 1/2$ for all $t \in [0,T]$, i.e. $z(t) \in \Omega_{c+1/2}$ for all $t \in [0,T]$, because otherwise this inequality would be contradicted. Then, the second one of (12.37) shows that also $e(t)$, and thus $\eta(t)$, is defined for all $t \in [0,T]$.

We prove now the following Lemma.

Lemma 12.3.1. *For any ϵ there exists a number $g^* > 0$ (independent of $z(0)$, $\eta(0)$) such that, if $g > g^*$,*

$$\|e(T)\| \leq \epsilon,$$

and, moreover, for all $T' > T$

$$z(t) \in \Omega_{c+1} \textit{ for all } t \in [T,T'] \Rightarrow \|e(t)\| \leq \epsilon \textit{ for all } t \in [T,T'].$$

Proof. Consider again (12.36). Using the comparison Lemma, it is deduced that

$$Q(e(t)) \leq e^{-a(g)t}Q(e(0)) + \frac{1 - e^{-a(g)t}}{a(g)}\nu k_2 \beta_2^2$$

and this yields

$$\|e(t)\|^2 \leq \frac{1}{k_3}\left[k_4 e^{-a(g)t}\|e(0)\|^2 + \frac{1 - e^{-a(g)t}}{a(g)}\nu k_2 \beta_2^2\right].$$

Fix ϵ and choose ν to satisfy $2k_2\beta_2^2\nu \leq k_3\epsilon^2$, so that, if $a(g) > 1$,

$$\|e(t)\|^2 \leq \frac{k_4}{k_3}e^{-a(g)t}\|e(0)\|^2 + \frac{\epsilon^2}{2}.$$

Observe that, for any fixed choice of initial conditions $x(0), v(0), \eta(0)$,

$$\lim_{g \to \infty} e^{-a(g)T}\|e(0)\|^2 = 0,$$

because $\|e(0)\| = \|D_g(\Phi(x(0), v(0)) - \eta(0))\|$ is bounded by a polynomial of degree $n-1$ in g. In particular, since $x(0)$, $v(0)$ and $\eta(0)$ range over a compact set, there is a number $g^* > 0$ such that

$$\frac{k_4}{k_3}e^{-a(g)T}\|e(0)\|^2 \leq \frac{\epsilon^2}{2}$$

for every $g > g^*$ and every $x(0)$, $v(0)$, $\eta(0)$ and the result follows. ◁

Using this Lemma, it can be proven that the trajectories of the system are bounded and, actually, converge to an arbitrarily small neighborhood of the origin. To this end, we observe first that there exists a positive non-decreasing function $\gamma : [0,\infty) \to [0,\infty)$, with $\gamma(0) = 0$, which is independent of g (if $g > 1$), such that

$$\|p_1(z,e)\| \leq \gamma(\|e\|), \text{ for all } (z,e) \in \Omega_{c+1} \times \mathbb{R}^n. \tag{12.38}$$

In fact, set

$$\tilde{\phi}_1(z,e) = \theta(\Psi(\Phi(x,v) - D_g^{-1}e, v), v_0, \ldots, v_{n-1}) - \theta(x, v_0, \ldots, v_{n-1}).$$

and observe that, since this is a smooth function of z, e which vanishes at $e = 0$, there is a positive non-decreasing function $\tilde{\gamma} : [0,\infty) \to [0,\infty)$, with $\tilde{\gamma}(0) = 0$, independent of g (if $g > 1$), such that

$$|\tilde{\phi}_1(z,e)| \leq \tilde{\gamma}(\|e\|),$$

for all $z \in \Omega_{c+1}$ and all $e \in \mathbb{R}^n$.

For any $z \in \Omega_{c+1}$, let $\mathcal{E}_z$ denote the set of all $e \in \mathbb{R}^n$ such that

$$\phi_1(z,e) = \tilde{\phi}_1(z,e),$$

and observe that, in view of the specific choice of M, the point $e = 0$ is in the interior of $\mathcal{E}_z$. Thus, a bound of the form

$$\|p_1(z,e)\| \le \tilde{\gamma}(\|e\|)$$

holds for all $z \in \Omega_{c+1}$ and all $e \in \mathcal{E}_z$. On the other hand, we know that there exists a number β_1 such that $\|p_1(z,e)\| \le \beta_1$ for all $z \in \Omega_{c+1}$ and all $e \in \mathbb{R}^n$. Thus, it is possible to find a function $\gamma(\cdot)$, with the properties indicated above, for which the bound (12.38) holds.

Now, let $\delta > 0$ and $\rho > 0$ be such that the set

$$\bar{B}_\delta = \{z : \|z\| \le \delta\}$$

satisfies

$$\Omega_\rho \subset \bar{B}_\delta \subset \Omega_c\,.$$

Then, using the function $\gamma(\cdot)$ for which (12.38) holds, choose ϵ so that

$$k_1\gamma(\epsilon) < \alpha(\rho)$$

and consider the set

$$S = \{z \in \Omega_{c+1} : V(z) \ge \rho\}.$$

By construction, so long as

$$z(t) \in S, \quad \text{and} \quad \|e(t)\| \le \epsilon$$

the function $V(z(t))$ satisfies

$$\frac{dV(z(t))}{dt} \le -\alpha(V(z(t))) + k_1\gamma(\epsilon) < 0 \tag{12.39}$$

i.e. is decreasing.

We have already shown that, for all $t \in [0,T]$, the solution $z(t)$, $\eta(t)$ is defined and $z(t) \in \text{int}(\Omega_{c+1})$. If $g > g^*$, we know from the previous Lemma that $\|e(T)\| \le \epsilon$. Then, using again this Lemma, we see that $z(t)$ cannot leave the set Ω_{c+1} and $\|e(t)\|$ is bounded by ϵ for all $t \ge T$. In fact, $z(t)$ cannot reach the boundary of the set Ω_{c+1}, where $V(z) > V(z(T))$ without contradicting (12.39).

It can also be shown that, in finite time, $z(t)$ enters the set Ω_ρ and remains there for all subsequent times. For, suppose this is not true. Then, $V(z(t))$ is always decreasing and converges to a nonnegative limit $V_0 \ge \rho$. Let Γ denote the ω-limit set of the trajectory in question. It is well-known that $V(z) = V_0$ at each point of Γ. Pick any initial condition in Γ and observe that the function $V(z)$ is constant along the corresponding trajectory. Thus we have

$$0 \le -\alpha(V_0) + k_1\gamma(\epsilon),$$

i.e.

$$V_0 < \rho$$

which is a contradiction. Once $z(t)$ has entered Ω_ρ, it can never leave this set, because $\dot{V}(z(t))$ is negative at each point of its boundary.

So far, we have proven that, in finite time, $(z(t), e(t))$ enters an arbitrarily small neighborhood of the equilibrium $(z,e) = (0,0)$. In order to complete the proof, observe that if $g > 1$, then $\|D_g^{-1}e\| \le \|e\|$. Therefore, if ϵ and δ are sufficiently small,

$$\Psi^*(\Phi(x,v) - D_g^{-1}e, v) = \Psi(\Phi(x,v) - D_g^{-1}e, v).$$

As a consequence, for all $\|z\| \le \delta$, $\|e\| \le \epsilon$ and $g > 1$, the function $p_2(z,e)$, which vanishes at $e = 0$, can be bounded as

$$\|p_2(z,e)\| = k_5\|e\|\,.$$

Once $z(t)$ has entered the set B_δ, the function $Q(e)$ satisfies

$$\frac{\partial Q}{\partial e}\dot{e} \le -g\|e\|^2 + 2k_2k_5\|e\|^2\,.$$

and therefore, if g is large enough,

$$\lim_{t\to\infty} e(t) = 0.$$

At this point, a straightforward application of Theorem 10.3.1 shows that also $z(t) \to 0$ as $t \to \infty$ and this completes the proof.

12.4 Stabilization of Minimum-Phase Systems in Lower-Triangular Form

We consider in this section the problem of robust stabilization of systems described by equations having a *lower-triangular* structure, such as equations (11.38), which are re-written here for convenience

$$\begin{aligned}
\dot{z} &= f_0(z,\xi_1)\\
\dot{\xi}_1 &= q_1(z,\xi_1,\mu) + b_1(z,\xi_1,\mu)\xi_2\\
\dot{\xi}_2 &= q_2(z,\xi_1,\xi_2,\mu) + b_2(z,\xi_1,\xi_2,\mu)\xi_3\\
&\cdots\\
\dot{\xi}_r &= q_r(z,\xi_1,\ldots,\xi_r,\mu) + b_r(z,\xi_1,\ldots,\xi_r,\mu)u\,.
\end{aligned} \tag{12.40}$$

It is assumed that $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, for $i = 1,\ldots,r$, $u \in \mathbb{R}$ and $\mu \in \mathcal{P} \subset \mathbb{R}^p$ is a vector of unknown parameters. In particular, we address the problem of determining conditions under which such a system is semiglobally practically stabilizable by means of a (possibly dynamic) feedback law which uses as information *only the output*

$$y = \xi_1$$

of the system.

The first step in the analysis consists in showing that there exists a globally defined partial change of coordinates, transforming system (12.40) into a system modeled by a simpler set of equations, which have the form

$$\begin{aligned} \dot z &= f_0(z,\zeta_1) \\ \dot\zeta_1 &= \zeta_2 \\ \dot\zeta_2 &= \zeta_3 \\ &\cdots \\ \dot\zeta_r &= q(z,\zeta_1,\ldots,\zeta_r,\mu) + b(z,\zeta_1,\ldots,\zeta_r,\mu)u\,. \end{aligned} \tag{12.41}$$

and in which

$$\zeta_1 = y\,.$$

The reason why this change of coordinates is useful is that, by construction, for each $i = 1,\ldots,r$, the (new) state variable ζ_i coincides with the derivative of order $i-1$ of the output y with respect to time. Moreover, these equations are precisely equations of the form (12.1), for which we have shown in section 12.1 that it is possible solve a problem of semiglobal and practical stabilization using a linear feedback in the variables $\zeta_1,\ldots\zeta_r$.

In order to obtain the form (12.41), set

$$\begin{aligned} \chi_0 &= 0 \\ \psi_0 &= 1\,, \end{aligned}$$

define

$$\begin{aligned} \chi_1(z,\xi_1,\mu) &= q_1(z,\xi_1,\mu) \\ \psi_1(z,\xi_1,\mu) &= b_1(z,\xi_1,\mu)\,, \end{aligned}$$

and recursively, for $i = 2,\ldots,r-1$,

$$\begin{aligned} \chi_i(z,\xi_1,\ldots,\xi_i,\mu) &= \frac{\partial[\chi_{i-1}+\psi_{i-1}\xi_i]}{\partial z} f_0(z,\xi_1) \\ &+ \frac{\partial[\chi_{i-1}+\psi_{i-1}\xi_i]}{\partial \xi_1}[q_1(z,\xi_1,\mu)+b_1(z,\xi_1,\mu)\xi_2] + \ldots \\ &+ \frac{\partial[\chi_{i-1}+\psi_{i-1}\xi_i]}{\partial \xi_{i-1}}[q_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu)+b_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu)\xi_i] \\ &+ \psi_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu)q_i(z,\xi_1,\ldots,\xi_i,\mu) \end{aligned}$$

and

$$\psi_i(z,\xi_1,\ldots,\xi_i,\mu) = \psi_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu)b_i(z,\xi_1,\ldots,\xi_i,\mu)\,.$$

The way these formulas are constructed is such that, along the trajectories of system (12.40)

$$\frac{\mathrm{d}[\chi_{i-1}+\psi_{i-1}\xi_i]}{\mathrm{d}t} = \chi_i + \psi_i\xi_{i+1}\,.$$

Therefore, for all $i = 1,\ldots,r-1$, the variable

$$\zeta_i = \chi_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu) + \psi_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu)\xi_i$$

is such that

$$\dot\zeta_i = \zeta_{i+1}$$

which is precisely what is sought in the equations (12.41).

Because of the special structure of the functions thus defined, the substitution of the state variables $\xi_1,\ldots,\xi_r$ with the variables

$$\begin{aligned} \zeta_1 &= \xi_1 \\ \zeta_2 &= \chi_1(z,\xi_1,\mu) + \psi_1(z,\xi_1,\mu)\xi_2 \\ \zeta_3 &= \chi_2(z,\xi_1,\xi_2,\mu) + \psi_2(z,\xi_1,\xi_2,\mu)\xi_3 \\ &\cdots \\ \zeta_r &= \chi_{r-1}(z,\xi_1,\ldots,\xi_{r-1},\mu) + \psi_{r-1}(z,\xi_1,\ldots,\xi_{r-1},\mu)\xi_r \end{aligned} \tag{12.42}$$

is globally defined if the $\psi_i(z,\xi_1,\ldots,\xi_i,\mu)$'s are nowhere zero. This is the case if – as assumed in Chapter 11 – there exist $b_{i0} > 0$ such that

$$b_i(z,\xi_1,\ldots,\xi_i,\mu) > b_{i0} \tag{12.43}$$

for all $i = 1,\ldots,r$.

Assuming this is the case, the variables $\xi_1,\ldots,\xi_r$ can be uniquely expressed in the form

$$\xi_i = \theta_i(z,\zeta_1,\ldots,\zeta_i,\mu)$$

where the $\theta_i(z,\zeta_1,\ldots,\zeta_i,\mu)$'s is are smooth functions of their arguments. Therefore, it can be easily observed that

$$\dot\zeta_r = q(z,\zeta_1,\ldots,\zeta_r,\mu) + b(z,\zeta_1,\ldots,\zeta_r,\mu)u$$

which is precisely the last equation of (12.41). Moreover, for some $b_0 > 0$,

$$b(z,\zeta_1,\ldots,\zeta_r,\mu) > b_0\,. \tag{12.44}$$

In summary, we have shown that, if there exists numbers $b_{i0} > 0$ such that (12.43) hold for all $i = 1,\ldots,r$, there is a smooth mapping

$$\begin{aligned} \mathbb{R}^r &: \to \mathbb{R}^r \\ \xi &: \mapsto \Phi(z,\xi,\mu)\,, \end{aligned}$$

which has a globally defined smooth inverse, such that equations (12.40) are transformed into equations of the form (12.41), in which the property (12.44) holds.

As shown in section 12.1, the form (12.41), if the subsystem

$$\dot{z} = f_0(z,0)\,,$$

possesses a globally asymptotically stable equilibrium at $z = 0$, lends itself to the design of a very simple state feedback law, actually a *linear* feedback law of the form $u = K\zeta$, able to solve a problem of semiglobal and practical robust stabilization. It is important, however, to stress that the change of coordinates (12.42) yielding the special form (12.41) explicitly *depends* on the vector z of state variables of (12.40) and on the vector μ of unknown parameters. In other words, the feedback law $u = K\zeta$, in terms of the original state variables (z,ξ), is a function of the form

$$u = K\Phi(z,\xi,\mu)$$

which *cannot be* implemented in practice, because the vector μ is unknown (and perhaps the state z is not accessible for feedback). We shall see however, in the second half of this section, that this apparent inconvenience can be overcome, because the various components of the vector ζ needed for the implementation of this feedback law coincide with a certain number of derivatives with respect to time of the output y of the system, namely

$$\begin{pmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \cdots \\ \zeta_r(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y^{(1)}(t) \\ \cdots \\ y^{(r-1)}(t) \end{pmatrix},$$

and, as shown in the previous section, the latter can be reasonably estimated with the aid of a suitable auxiliary dynamical system which uses as information only the actual output y.

In what follows we will show that, for systems of the form (12.40), semiglobal practical stabilization can be achieved by means of a dynamic, output feedback, controller of the form

$$\begin{pmatrix} \dot\eta_1 \\ \dot\eta_2 \\ \cdot \\ \dot\eta_{r-1} \\ \dot\eta_r \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_3 \\ \cdots \\ \eta_r \\ 0 \end{pmatrix} + \begin{pmatrix} gc_{r-1} \\ g^2c_{r-2} \\ \cdots \\ g^{r-1}c_1 \\ g^rc_0 \end{pmatrix}(y-\eta_1) \tag{12.45}$$

$$u = -\sigma_\ell\Big(\bar{k}[\eta_r + k^{r-1}a_0\eta_1 + k^{r-2}a_1\eta_2 + \cdots + ka_{r-2}\eta_{r-1}]\Big) \tag{12.46}$$

in which the function $\sigma_\ell(\cdot)$ is a saturation function, defined as

$$\sigma_\ell(r) = \begin{cases} r & \text{if } |r| < \ell \\ \mathrm{sgn}(r)\ell & \text{if } |r| \geq \ell\,, \end{cases}$$

the c_i's and a_i's are (fixed) coefficients of two polynomials

$$\lambda^r + c_{r-1}\lambda^{r-1} + \cdots + c_1\lambda + c_0$$
$$\lambda^{r-1} + a_{r-2}\lambda^{r-2} + \cdots + a_1\lambda + a_0$$

having all roots with negative real part, and $\bar{k}, k, g, \ell$ are design parameters, to be tuned in accordance with the data of the design problem.

The standing hypotheses, on system (12.40), under which the proposed controller succeeds in securing semiglobal and practical stability are the following ones:

(i) $f_0(0,0) = 0$ and the equilibrium $z = 0$ of the subsystem

$$\dot{z} = f_0(z,0)$$

is globally asymptotically stable,

(ii) for $i = 1,\ldots,r$, there exists $b_{i0} > 0$ such that

$$b_i(z,\xi_1,\ldots,\xi_i,\mu) > b_{i0}\,,$$

(iii) for $i = 1,\ldots,r$

$$q_i(0,0,\ldots,0,\mu) = 0\,,$$

for all $(z,\xi_1,\ldots,\xi_i)$ and all $\mu \in \mathcal{P}$.

Theorem 12.4.1. *Consider system (12.40) and assume hypotheses* (i), (ii) *and* (iii) *hold. Given any arbitrarily large number $R > 0$ and any arbitrarily small number $\varepsilon > 0$, there are numbers $\bar{k} > 0, k > 0, g > 0, \ell > 0$ such that, in the closed loop system (12.40) – (12.45) – (12.46), any initial condition in $\bar{Q}_R^{n+2r}$ produces a trajectory which is captured by the set $\bar{Q}_\varepsilon^{n+2r}$.*

Proof. Define new state variables

$$\zeta_i = \chi_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu) + \psi_{i-1}(z,\xi_1,\ldots,\xi_{i-1},\mu)\xi_i$$

and

$$e_i = g^{r-i}(\zeta_i - \eta_i),$$

for $i = 1,\ldots,r$. In a more convenient notation, these relations can be rewritten as

$$\begin{aligned} \zeta &= \Phi(z,\xi,\mu) \\ e &= D_g(\zeta - \eta) \end{aligned}$$

where $\Phi(z,\xi,\mu)$ is the mapping defined by (12.42) and

$$D_g = \mathrm{diag}\{g^{r-1},\ldots,g,1\}\,.$$

Set

$$K = -\begin{pmatrix} \bar{k}k^{r-1}a_0 & \bar{k}k^{r-2}a_1 & \cdots & \bar{k}ka_{r-2} & \bar{k} \end{pmatrix}$$

so that (12.46) becomes

$$u = \sigma_\ell(K\eta) = \sigma_\ell(K(\zeta - D_g^{-1}e)) .$$

Finally, set $x = \mathrm{col}(z, \zeta)$. This yields equations of the form

$$\begin{aligned} \dot{x} &= F(x,\mu) + p_1(x,e,\mu) \\ \dot{e} &= gAe + p_2(x,e,\mu) \end{aligned} \tag{12.47}$$

in which

$$F(x,\mu) = \begin{pmatrix} f_0(z,\zeta_1) \\ \zeta_2 \\ \vdots \\ \zeta_r \\ q(z,\zeta_1,\ldots,\zeta_r,\mu) + b(z,\zeta_1,\ldots,\zeta_r,\mu)K\zeta \end{pmatrix},$$

$$A = \begin{pmatrix} -c_{r-1} & 1 & 0 & \cdots & 0 \\ -c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -c_1 & 0 & 0 & \cdots & 1 \\ -c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{12.48}$$

$$p_1(x,e,\mu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_1(x,e,\mu) \end{pmatrix}, \quad p_2(x,e,\mu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_2(x,e,\mu) \end{pmatrix},$$

and

$$\begin{aligned} \phi_1(x,e,\mu) &= b(z,\zeta_1,\ldots,\zeta_r,\mu)[\sigma_\ell(K(\zeta - D_g^{-1}e)) - K\zeta] \\ \phi_2(x,e,\mu) &= q(z,\zeta_1,\ldots,\zeta_r,\mu) + b(z,\zeta_1,\ldots,\zeta_r,\mu)\sigma_\ell(K(\zeta - D_g^{-1}e)) . \end{aligned}$$

Now, observe that, as a consequence of the hypothesis (iii), the function $\chi_i(z,\xi_1,\ldots,\xi_i,\mu)$ is such that

$$\chi_i(0,0,\ldots,0,\mu) .$$

Therefore, the mapping $\Phi(z,\xi,\mu)$ satisfies

$$\Phi(0,0,\mu) = 0 .$$

Since the mapping $\Phi(z,\xi,\mu)$ is a smooth mapping and μ ranges over a compact set, given any $R > 0$ there exists $R' > 0$ such that

$$(z,\xi) \in \bar{Q}_R^{n+r} \quad \Rightarrow \quad x \in \bar{Q}_{R'}^{n+r}$$

and, also, given any $\varepsilon > 0$, there exists $\varepsilon' > 0$

$$x \in \bar{Q}_{\varepsilon'}^{n+r} \quad \Rightarrow \quad (z,\xi) \in \bar{Q}_\varepsilon^{n+r} \text{ and } \|\zeta\| \le \frac{\varepsilon}{2} .$$

In view of this, the proof of the Theorem amounts to show that, for every initial condition $x^\circ \in \bar{Q}_{R'}^{n+r}$ and $\eta^\circ \in \bar{Q}_R^r$, the trajectory of the system (12.47) is such that $x(t)$ is captured by the set $\bar{Q}_{\varepsilon'}^{n+r}$ and $e(t)$ is captured by the set $\bar{Q}_{\varepsilon/2}^r$. Note, in fact, that this requirement on $e(t)$ guarantees, if $g > 1$, that $\eta(t)$ is captured by the set $\bar{Q}_\varepsilon^r$, since

$$\eta = \zeta - D_g^{-1}e .$$

Consider the system

$$\dot{x} = F(x,\mu) ,$$

and observe that this is precisely the system whose asymptotic properties were studied in the proof of Theorem 12.1.1. From that proof, it can be claimed that, given $R' > 0$, there is a choice of $k > 0$ and there exists a positive definite function (the function (12.6), which in the present context will be rewritten, for consistency, as $W(x)$), defined on a bounded set which contains $\bar{Q}_{R'}^{n+r}$ in its interior, such that, for some $a > 0$ satisfying

$$\bar{Q}_{R'}^{n+r} \subset \Omega_a ,$$

the property

$$\frac{\partial W}{\partial x} F(x,\mu) < 0 \tag{12.49}$$

holds at each point of a set of the form

$$S = \{x : \rho \le W(x) \le a + 1\} , \tag{12.50}$$

where ρ is a number which can be rendered arbitrarily small by increasing the value of $\bar{k}$. Bearing in mind the fact that $W(x)$ is positive definite, choose $\rho > 0$ such that, for some $\delta > 0$ and $\rho' > 0$

$$\Omega_\rho \subset \bar{B}_\delta \subset \Omega_{\rho'} \subset \bar{Q}_{\varepsilon'}^{n+r} ,$$

and fix the value of $\bar{k}$ accordingly (i.e. such that (12.49) holds on (12.50)). The values of k and $\bar{k}$ thus chosen remain fixed throughout the rest of proof.

Note also that there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that

$$\frac{\partial W}{\partial x} F(x,\mu) \le -\alpha(\|x\|), \qquad \text{for all } x \in \Omega_{a+1} \setminus B_\delta . \tag{12.51}$$

Now, choose ℓ as

$$\ell = \max_{x \in \Omega_{a+1}} |K\zeta| + 1 .$$

With this choice, it is easy to check there exist positive numbers β_1, β_2, independent of g, such that

$$\|p_i(x,e,\mu)\| \le \beta_i, \text{ for all } (x,e) \in \Omega_{a+1} \times \mathbb{R}^r .$$

Moreover, the vector $p_1(x,e,\mu)$ defined before vanishes at $e=0$, so long as $x \in \Omega_{a+1}$, and there exists a positive non-decreasing function $\gamma(\cdot)$, with $\gamma(0)=0$, which is independent of g (if $g>1$), such that

$$\|p_1(x,e,\mu)\| \le \gamma(\|e\|), \text{ for all } (x,e) \in \Omega_{a+1} \times \mathbb{R}^r.$$

Let k_1 be such that

$$\left\|\frac{\partial W}{\partial x}\right\| \le k_1 \text{ for all } x \in \Omega_{a+1}.$$

Then

$$\frac{\partial W}{\partial x}(F(x,\mu) + p_1(x,e,\mu)) \le -\alpha(\|x\|) + k_1\beta_1 \tag{12.52}$$

for all $(x,e) \in (\Omega_{a+1} \setminus B_\delta) \times \mathbb{R}^r$.

Let P be the positive definite solution of $PA + A^TP = -I$, and let $k_2 = \|P\|$. Then, as in the proof of Theorem 9.6.2, we obtain, for any $\nu > 0$, that

$$\frac{\partial Q}{\partial e}(gAe + p_2(x,e,\mu)) \le -a(g)Q(e) + \nu k_2 \beta_2^2 \tag{12.53}$$

for all $(x,e) \in \Omega_{a+1} \times \mathbb{R}^r$, where

$$a(g) = \left(g - \frac{k_2}{\nu}\right)\frac{1}{k_4}.$$

Inequalities (12.52) and (12.53) show (see proof of Theorem 9.6.2) that there is a fixed time $T>0$ (independent of g) such that the solution $x(t)$, $e(t)$ of (12.47) is defined for all $t \in [0,T]$ and, in particular, $x(t) \in \Omega_{a+1}$ for all $t \in [0,T]$.

Indeed, a Lemma identical to Lemma 12.3.1 holds, and thus one can claim that for any ϵ there exists a number $g^* > 0$ (independent of $x(0)$, $\eta(0)$) such that, if $g > g^*$,

$$\|e(T)\| \le \epsilon,$$

and, moreover, for all $T' > T$

$$x(t) \in \Omega_{a+1} \text{ for all } t \in [T,T'] \Rightarrow \|e(t)\| \le \epsilon \text{ for all } t \in [T,T'].$$

This, as in the proof of Theorem 9.6.2, shows that trajectories are bounded. In fact, so long as

$$x(t) \in \{x : \rho' \le W(x) \le a+1\}, \quad \text{and} \quad \|e(t)\| \le \epsilon\,,$$

the function $W(x(t))$ satisfies

$$\frac{\mathrm{d}W(x(t))}{\mathrm{d}t} \le -\alpha(\|x\|) + k_1\gamma(\epsilon)\,.$$

Pick ϵ such that $\alpha(\delta) > k_1\gamma(\epsilon)$. Then, so long as

$$x(t) \in \{x : \rho' \le W(x) \le a+1\}, \quad \text{and} \quad \|e(t)\| \le \epsilon\,,$$

we have

$$\frac{\mathrm{d}W(x(t))}{\mathrm{d}t} < 0\,,$$

i.e. $W(x(t))$ is decreasing. This, as in the proof of Theorem 9.6.2, shows that in finite time the $x(t)$ is captured by the set $\Omega_{\rho'}$, and therefore captured by the set $\bar{Q}^{n+r}_{\varepsilon'}$. If $\epsilon < \varepsilon/2$, then also $e(t)$ is captured by the set $\bar{Q}^r_{\varepsilon/2}$ and this, as remarked before, completes the proof. ◁

As in section 12.1, it can be proven that if the eigenvalues of the matrix

$$F_0 = \left[\frac{\partial f_0}{\partial z}\right]_{(0,0)}$$

are all in the left-half plane, the proposed feedback law is able to locally exponentially stabilize the system.

Corollary 12.4.2. *Consider system (12.40) and assume hypotheses* (i), (ii) *and* (iii) *hold. Assume also that all the eigenvalues of the matrix F_0 above have negative real part. Given any arbitrarily large number $R>0$ there are numbers $\bar{k}>0, k>0, g>0, \ell>0$ such that, in the closed loop system (12.40) – (12.45) – (12.46), the equilibrium $(z,\xi,\eta)=(0,0,0)$ is locally exponentially stable and, moreover, any initial condition in $\bar{Q}^{n+2r}_R$ produces a trajectory which asymptotically converges to $(0,0,0)$ as $t\to\infty$.*

12.5 Stabilization via Output Feedback Without a Separation Principle

In this section we illustrate a simple recursive design method for stabilizing a linear system by output feedback, which does not use any technique for pole assignment nor the separation principle , but rather uses concepts reminiscent of some classical design methods based on root-locus properties. Of course, this does not lead to any specific breakthrough in the case of linear systems, where pole assignment or the solution of a diophantine equation suffice to stabilize using output feedback, but its nonlinear version, described in the next section, provides a design tool which may prove useful in the problem of robustly stabilizing, via output feedback, a possibly unstable and non-minimum phase nonlinear system.

Consider a single-input single-output linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,. \end{aligned} \tag{12.54}$$

Let n denote the dimension of its state space and let r denote its relative degree. It is well-known, and it can be deduced – in particular – as a corollary of Proposition 9.1.1 (because in the case of a linear system the completeness and commutativity hypotheses on the vector fields (9.2) are always fulfilled), that such a system can always be represented in the form

$$\begin{aligned} \dot z &= F_0 z + G_0 \xi_1 \\ \dot\xi_1 &= \xi_2 \\ &\cdots \\ \dot\xi_{r-1} &= \xi_r \\ \dot\xi_r &= H_0 z + a_1\xi_1 + \cdots + a_r\xi_r + bu \\ y &= \xi_1 \end{aligned} \tag{12.55}$$

in which $z \in \mathbb{R}^{n-r}$ and $b \neq 0$. Set

$$\zeta = \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}), \qquad v = \xi_r \,,$$

and

$$F = \begin{pmatrix} F_0 & G_0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

$$H = (\, H_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{r-1}\,), \qquad J = a_r$$

$$K = (\, 0 \quad 1 \quad 0 \quad 0 \quad \cdots \quad 0\,) \,,$$

and rewrite the system in the form

$$\begin{aligned} \dot\zeta &= F\zeta + Gv \\ \dot v &= H\zeta + Jv + bu \\ y &= K\zeta \,, \end{aligned} \tag{12.56}$$

in which $\zeta \in \mathbb{R}^{n-1}$ and v coincides with $y^{(r-1)}$, the $(r-1)$-th derivative of the output with respect to time.

With this system, we associate an *auxiliary subsystem*, with input u_a, output y_a, and state z, defined as follows

$$\begin{aligned} \dot\zeta &= F\zeta + Gu_a \\ y_a &= H\zeta + Ju_a \,. \end{aligned} \tag{12.57}$$

The main purpose of this section is to show that, if the auxiliary subsystem thus defined is stabilizable by (dynamic) output feedback, so is the full system (12.56). This result is the direct consequence of two Lemmas. The first Lemma shows how to stabilize (12.56) using a dynamic output feedback which uses v as input.

Lemma 12.5.1. *Suppose system*

$$\begin{aligned} \dot\varphi &= L\varphi + My_a \\ u_a &= N\varphi \end{aligned}$$

stabilizes system (12.57). Then there is a number k^* such that, for all $k > k^*$ the feedback law

$$\begin{aligned} \dot\varphi &= L\varphi + Mk(v - N\varphi) \\ u &= \frac{1}{b}\Big[N[L\varphi + Mk(v - N\varphi)] - k(v - N\varphi)\Big] \end{aligned} \tag{12.58}$$

stabilizes system (12.56).

Proof. Consider the interconnection of (12.56) with (12.58) and change the state variable v into a state variable θ defined as

$$\theta = v - N\varphi \,,$$

which yields

$$\begin{aligned} \dot\zeta &= F\zeta + G(N\varphi + \theta) \\ \dot\varphi &= L\varphi + Mk\theta \\ \dot\theta &= H\zeta + J(N\varphi + \theta) - k\theta \,. \end{aligned}$$

Change now the state variables φ into new state variables σ defined as

$$\sigma = \varphi + M\theta \,,$$

to obtain a system of the form

$$\begin{aligned} \begin{pmatrix} \dot\zeta \\ \dot\sigma \end{pmatrix} &= \begin{pmatrix} F & GN \\ MH & L + MJN \end{pmatrix} \begin{pmatrix} \zeta \\ \sigma \end{pmatrix} + \begin{pmatrix} G - GNM \\ MJ - (L + MJN)M \end{pmatrix} \theta \\ \dot\theta &= -(k - J + JNM)\theta + (\, H \quad JN\,) \begin{pmatrix} \zeta \\ \sigma \end{pmatrix} . \end{aligned} \tag{12.59}$$

The latter can be viewed as the feedback interconnection of two subsystems. The upper subsystem, viewed as a system with state (ζ, σ) and input θ, is stable by hypothesis and has some finite L_2-gain γ, which is independent of k. The lower subsystem, viewed as a system with state θ and input (ζ, σ), is stable for large k, and its L_2-gain decreases to 0 as $k \to \infty$. As a consequence, by the small gain theorem, if k is large enough, the system in question is stable. ◁

The second Lemma shows how, in the previous stabilizer, v can be replaced by an estimate generated by a filter which receives y as input. Define

$$P = \begin{pmatrix} -gc_{r-1} & 1 & 0 & \cdots & 0 \\ -g^2c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -g^{r-1}c_1 & 0 & 0 & \cdots & 1 \\ -g^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} gc_{r-1} \\ g^2c_{r-2} \\ \cdot \\ g^{r-1}c_1 \\ g^r c_0 \end{pmatrix} \tag{12.60}$$

$$R = (0 \;\; 0 \;\; 0 \;\; \cdots \;\; 1)\,,$$

where $c_0, c_1, \ldots, c_{r-1}$ are the coefficients of a fixed polynomial having all roots with negative real part, and $g > 0$ is a parameter to be determined.

Lemma 12.5.2. *Suppose system*

$$\begin{aligned} \dot\varphi &= A\varphi + Bv \\ u &= C\varphi + Dv \end{aligned}$$

stabilizes system (12.56). Then, there is a number g^* such that, for all $g > g^*$ the feedback law

$$\begin{aligned} \dot\eta &= P\eta + Qy \\ \dot\varphi &= A\varphi + BR\eta \\ u &= C\varphi + DR\eta \end{aligned} \tag{12.61}$$

stabilizes system (12.55).

Proof. Define $e_i = g^{r-i}(\xi_i - \eta_i)$, for $i = 1, 2, \ldots, r$ and observe that, since $R\eta = v - e_r$, the interconnection of (12.55) and (12.61) becomes

$$\begin{aligned} \dot\zeta &= F\zeta + Gv \\ \dot v &= H\zeta + Jv + b(C\varphi + Dv) - bDe_r \\ \dot\varphi &= A\varphi + Bv - Be_r \\ \dot e &= gAe + B(H\zeta + Jv + b(C\varphi + Dv) - bDe_r)\,, \end{aligned} \tag{12.62}$$

in which A is the matrix (12.48), having all eigenvalues in $\mathbb{C}^-$, and $B = \mathrm{col}(0, 0, \ldots, 0, 1)$. This can be viewed as the feedback interconnection of two subsystems. One subsystem, modeled by the first three equations, with input e_r and internal state (ζ, v, φ), is stable by construction and has an L_2-gain bounded by a fixed number which is independent of g. The other subsystem, modeled by the last equation, with input (ζ, v, φ) and internal state e, is stable for large g and, as an elementary calculation shows, its L_2-gain decreases to 0 as $g \to \infty$. Thus, again by the small gain theorem, the interconnection is stable for large g. ◃

Indeed, the combination of these two Lemmas shows that if system

$$\begin{aligned} \dot\varphi &= L\varphi + My_a \\ u_a &= N\varphi \end{aligned}$$

is a dynamic, output feedback, stabilizer for the auxiliary system (12.57), then the system

$$\begin{aligned} \dot\eta &= P\eta + Qy \\ \dot\varphi &= L\varphi + Mk(R\eta - N\varphi) \\ u &= \frac{1}{b}\Big[N[L\varphi + Mk(R\eta - N\varphi)] - k(R\eta - N\varphi)\Big] \end{aligned}$$

is a dynamic, output feedback, stabilizer for system (12.54).

The hypothesis on which this stabilization scheme works is that the auxiliary system (12.57) is stabilizable by output feedback. Thus, it seems natural to pose the question of how much such an hypothesis is restrictive, in the context of the problem of stabilizing a linear system using output feedback. Well, it turns out that this hypothesis is not restrictive since it can be shown that, possibly up to a memoryless output feedback transformation, it must necessarily hold if the original system (12.56) is stabilizable by output feedback.

Proposition 12.5.3. *Suppose system (12.56) is stabilizable by output feedback and consider the memoryless feedback transformation*

$$u \mapsto u + ky$$

which changes system (12.56) into system

$$\begin{aligned} \dot\zeta &= F\zeta + Gv \\ \dot v &= (H + bkK)\zeta + Jv + bu \\ y &= K\zeta\,. \end{aligned} \tag{12.63}$$

Then, for every $k \in \mathbb{R}$ except possibly one single value k_0, the auxiliary subsystem associated to (12.63), i.e. the system

$$\begin{aligned} \dot\zeta &= F\zeta + Gu_a \\ y_a &= (H + bkK)\zeta + Ju_a\,. \end{aligned}$$

is stabilizable by output feedback.

Proof. Using the normal form recalled at the beginning of the section, observe that

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} F & G & 0 \\ H & J & b \\ K & 0 & 0 \end{pmatrix} = \begin{pmatrix} F_0 & G_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ H_0 & a_1 & a_2 & \cdots & a_r & b \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

A straightforward application of standard stabilizability/detectability tests, shows that if the pair (A,B) is stabilizable, so is the pair (F_0,G_0) and, likewise, if the pair (A,C) is detectable, so is the pair (F_0,H_0).

Consider now the auxiliary system associated with (12.63) and define

$$\begin{pmatrix} F & G \\ \tilde{H} & J \end{pmatrix} = \begin{pmatrix} F & G \\ H+bkK & J \end{pmatrix} = \begin{pmatrix} F_0 & G_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ H_0 & a_1+kb & a_2 & \cdots & a_{r-1} & a_r \end{pmatrix}.$$

Using the property that the pair (F_0,G_0) is stabilizable, it is not difficult to show that, for any λ with nonnegative real part,

$$\mathrm{rank}\,(\,F-\lambda I \quad G\,) = n-1 \qquad (12.64)$$

and therefore the pair (F,G) is stabilizable. Using the property that the pair (F_0,H_0) is detectable, it is not difficult to show that, if λ has nonnegative real part and $\lambda \neq 0$,

$$\mathrm{rank}\begin{pmatrix} F-\lambda I \\ \tilde{H} \end{pmatrix} = n-1 \qquad (12.65)$$

for any k. The matrix in (12.65) has rank $n-1$ at $\lambda=0$ (and hence the pair $(F,\tilde{H})$ is detectable) if and only if

$$\det\begin{pmatrix} F_0 & G_0 \\ H_0 & a_1+kb \end{pmatrix} \neq 0\,.$$

Observe that

$$\det\begin{pmatrix} F_0 & G_0 \\ H_0 & a_1+kb \end{pmatrix} = (a_1+kb)\det(F_0) + \det\begin{pmatrix} F_0 & G_0 \\ H_0 & 0 \end{pmatrix}.$$

The right-hand side of this identity, a polynomial of first degree in k, cannot be identically vanishing because otherwise

$$\det(F_0)=0, \qquad \text{and} \qquad \det\begin{pmatrix} F_0 & G_0 \\ H_0 & 0 \end{pmatrix} = 0$$

would contradict the fact that the triplet (F_0,G_0,H_0) is stabilizable and detectable. Thus, there is at most one single value of k for which $(F,\tilde{H})$ loses detectability. $\triangleleft$

Remark 12.5.1. System (12.63) is obtained from system (12.56) via a memoryless output feedback of the form $u \mapsto u+ky$ and this indeed leaves the property of stabilizability by output feedback unchanged. The previous Proposition shows that, in case the auxiliary system associated with (12.56) *is not* stabilizable by output feedback, the auxiliary system associated with (12.63) *is* stabilizable by output feedback *for every nonzero k*. $\triangleleft$

This Proposition shows that there is no loss of generality in approaching the problem of stabilization via output feedback in the manner suggested above, i.e. by "reducing" the problem to that of stabilizing an auxiliary lower-dimensional subsystem. Indeed, the process can be iterated all the way down to the case in which the auxiliary subsystem has dimension one (where a simple gain would suffice to stabilize). This yields a systematic method for stabilization via output feedback, which does not use any technique for pole assignment nor the idea of solving separately a problem of state-feedback stabilization and a problem of asymptotic state reconstruction. Rather, it is based on the recursive update of a sequence of "dynamic" output feedback stabilizers for a sequence of subsystems of increasing dimension.

12.6 Stabilization via Output Feedback of Non-Minimum-Phase Systems

Many of the methods, described so far, for robust stabilization of a nonlinear system using partial state feedback rely upon the hypothesis that the system has a stable zero dynamics . The main reason why this hypothesis is assumed is that the methods in question use "high-gain" feedback in order to offset the presence of certain unwanted terms in the dynamics of the closed-loop system. The consequence of this high-gain control is to enforce a behavior whose asymptotic properties are directly influenced by the asymptotic properties of the zero dynamics. In particular, asymptotic stabilization may occur only if the latter is asymptotically stable, i.e. if the system is minimum-phase . For example, in the stabilization of a system of the form

$$\begin{aligned} \dot{z} &= f_0(z,y,\mu) \\ \dot{y} &= f_1(z,y,\mu)+u \end{aligned}$$

robust stability (with respect to the uncertain parameter μ) can be achieved by choosing the control u in such a way as to offset the (possibly unpleasant) presence on the term $f_1(z,y,\mu)$ in the derivative of a candidate Lyapunov function of the form

$$U(z,y) = V(z) + (y-y^*(z))^2\,,$$

where $V(z)$ is a positive definite and proper function such that

$$\frac{\partial V}{\partial z} f_0(z,y^*(z),\mu) < 0 \qquad (12.66)$$

for all $z \neq 0$. If the input u is only allowed to depend on y (and not on z), this goal, i.e. the fact that

$$\frac{\partial V}{\partial z} f_0(z,y,\mu) + 2(y-y^*(z))[f_1(z,y,\mu)+u-\frac{\partial y^*}{\partial z} f_0(z,y,\mu)] < 0\,, \qquad (12.67)$$

can only be achieved if $y^*(z) = 0$ and $V(z)$ is a Lyapunov function for the zero dynamics

$$\dot{z} = f_0(z, 0, \mu) .$$

If this is the case (i.e. if (12.66) holds for $y^*(z) = 0$), the linear law $u = -ky$, where k is a sufficiently large number, solves the problem of semiglobal and practical stabilization, essentially because the term $-ky^2$ in (12.67) is able to dominate all other non-negative terms. If, on the contrary, the zero dynamics are unstable, some (output-based) estimate of a nontrivial $y^*(z)$ is needed, and this might not be compatible with a control action which, by dominating the term $f_1(z, y, \mu)$, would render z poorly observable from y.

The method described in the previous section does not rely at all upon a similar procedure, but rather aims at taking *explicit advantage* of the term $f_1(z, y, \mu)$, to the purpose of determining a stabilizer for what was called the "auxiliary subsystem", in this case system

$$\begin{aligned} \dot{z} &= f_0(z, u_a, \mu) \\ y_a &= f_1(z, u_a, \mu) . \end{aligned}$$

We describe in this section a nonlinear version of that method.

Consider a nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1, \mu) \\ \dot{\xi}_1 &= q_1(z, \xi_1, \mu) + b_1(\xi_1)\xi_2 \\ \dot{\xi}_2 &= q_2(z, \xi_1, \xi_2, \mu) + b_2(\xi_1)\xi_3 \\ &\cdots \\ \dot{\xi}_r &= q_r(z, \xi_1, \ldots, \xi_r, \mu) + b_r(\xi_1)u \\ y &= \xi_1 , \end{aligned} \tag{12.68}$$

in which $z \in \mathbb{R}^{n-r}$ and μ is a (possibly vector-valued) unknown parameter, ranging over a compact set $\mathcal{P}$.

Assume that:

(i) $f_0(0, 0, \mu) = 0$,

(ii) for $i = 1, \ldots, r$, there exists $b_{i0} > 0$ such that

$$b_i(\xi_1) > b_{i0} ,$$

(iii) for $i = 1, \ldots, r$

$$q_i(0, 0, \ldots, 0, \mu) = 0 ,$$

for all $(z, \xi_1, \ldots, \xi_i)$ and all $\mu \in \mathcal{P}$.

Then, as shown in section 12.4, there exists a globally defined diffeomorphism

$$\xi \mapsto \zeta = \Phi(z, \xi, \mu) ,$$

such that $\Phi(0, 0, \mu) = 0$, transforming system (12.68) into a system described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \zeta_1, \mu) \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\cdots \\ \dot{\zeta}_r &= q(z, \zeta_1, \ldots, \zeta_r, \mu) + b(\zeta_1)u \\ y &= \zeta_1 . \end{aligned} \tag{12.69}$$

Moreover,

$$\begin{aligned} &q(0, 0, \ldots, 0, \mu) = 0 \\ &b(\zeta_1) > b_0 \end{aligned}$$

for some $b_0 > 0$.

As remarked before, the global diffeomorphism changing (12.68) into (12.69) is μ-dependent and z-dependent, but this is not an obstacle to semiglobal practical stabilization via output feedback.

With system (12.69), we associate an ***auxiliary system***

$$\begin{aligned} \dot{x}_a &= f_a(x_a, u_a, \mu) \\ y_a &= h_a(x_a, u_a, \mu) \end{aligned} \tag{12.70}$$

in which

$$x_a = \begin{pmatrix} z \\ \zeta_1 \\ \cdot \\ \zeta_{r-2} \\ \zeta_{r-1} \end{pmatrix}, \qquad f_a(x_a, u_a, \mu) = \begin{pmatrix} f_0(z, \zeta_1, \mu) \\ \zeta_2 \\ \cdot \\ \zeta_{r-1} \\ u_a \end{pmatrix},$$

and

$$h_a(x_a, \mu) = q(z, \zeta_1, \ldots, \zeta_{r-1}, u_a, \mu) .$$

Motivated by the arguments presented in the previous section, we assume the following on the auxiliary system (12.70):

(iv) there exists a dynamical system of the form

$$\begin{aligned} \dot{\varphi} &= L(\varphi) + M y_a \\ u_a &= N(\varphi) , \end{aligned} \tag{12.71}$$

in which $\varphi \in \mathbb{R}^\nu$ and a positive definite and proper smooth function $V(x_a, \varphi)$ whose derivative along the trajectories of the interconnected system (12.70) – (12.71), namely system

$$\begin{aligned} \dot{x}_a &= f_a(x_a, N(\varphi), \mu) \\ \dot{\varphi} &= L(\varphi) + M h_a(x_a, N(\varphi), \mu) , \end{aligned} \tag{12.72}$$

is negative definite, i.e.

$$\frac{\partial V}{\partial x_a} f_a(x_a, N(\varphi), \mu) + \frac{\partial V}{\partial \varphi}[L(\varphi) + M h_a(x_a, N(\varphi), \mu)] < 0$$

for all $(x_a, \varphi) \neq (0,0)$.

In other words, the system (12.71) is supposed to globally asymptotically stabilize the auxiliary system (12.70), and the interconnection (12.70) – (12.71) possesses a Lyapunov function $V(x_a, \varphi)$ which is independent of μ.

Consider now, for the original plant (12.69), the dynamic feedback law

$$\begin{aligned} \dot\varphi &= L(\varphi) + Mk[\zeta_r - N(\varphi)] \\ u &= \frac{1}{b(\zeta_1)}\Big[\frac{\partial N}{\partial \varphi}\Big(L(\varphi) + Mk[\zeta_r - N(\varphi)]\Big) - k[\zeta_r - N(\varphi)]\Big] \end{aligned} \qquad (12.73)$$

where k is a positive number.

We claim that a controller with this structure is able to robustly semiglobally practically stabilize the plant (12.69). To this end, observe that the feedback interconnection of (12.69) and (12.73), namely system

$$\begin{aligned} \dot x_a &= f_a(x_a, \zeta_r, \mu) \\ \dot\zeta_r &= h_a(x_a, \zeta_r, \mu) + \frac{\partial N}{\partial \varphi}\Big(L(\varphi) + Mk[\zeta_r - N(\varphi)]\Big) - k[\zeta_r - N(\varphi)] \\ \dot\varphi &= L(\varphi) + Mk[\zeta_r - N(\varphi)]\,, \end{aligned} \qquad (12.74)$$

changing the state variable ζ_r into the new variable

$$\theta = \zeta_r - N(\varphi)\,,$$

becomes

$$\begin{aligned} \dot x_a &= f_a(x_a, \theta + N(\varphi), \mu) \\ \dot\theta &= h_a(x_a, \theta + N(\varphi), \mu) - k\theta \\ \dot\varphi &= L(\varphi) + Mk\theta\,. \end{aligned} \qquad (12.75)$$

This system can be viewed as the system resulting from the interconnection of a system with output θ and input v, modeled by equations of the form

$$\begin{aligned} \dot x_a &= f_a(x_a, \theta + N(\varphi), \mu) \\ \dot\theta &= h_a(x_a, \theta + N(\varphi), \mu) + v \\ \dot\varphi &= L(\varphi) - Mv \end{aligned} \qquad (12.76)$$

and a memoryless system (with input θ and output v)

$$v = -k\theta\,. \qquad (12.77)$$

System (12.76) has uniform relative degree one between input v and output θ, and its "high-frequency gain" coefficient is equal to one. Thus, the system has a (globally defined) zero dynamics manifold, the set

$$Z^* = \{(x_a, \theta, \varphi) : \theta = 0\}\,,$$

which is rendered invariant by

$$v = v^*(x_a, \varphi) = -h_a(x_a, N(\varphi), \mu)\,.$$

Consequently, the zero dynamics of (12.76) are those of

$$\begin{aligned} \dot x_a &= f_a(x_a, N(\varphi), \mu) \\ \dot\varphi &= L(\varphi) + M h_a(x_a, N(\varphi), \mu)\,. \end{aligned} \qquad (12.78)$$

These dynamics have by hypothesis a globally asymptotically stable equilibrium at $(x_a, \varphi) = (0,0)$. Therefore, in view of the results discussed in section 12.1, a feedback law of the form (12.77) is able to solve a problem of semiglobal and practical stability.

As a matter of fact, consider the positive definite and proper function

$$W(x_a, \varphi, \theta) = V(x_a, \varphi + M\theta) + \theta^2$$

and let Ω_a to denote the set

$$\Omega_a = \{(x_a, \varphi, \theta) : W(x_a, \varphi, \theta) \leq a\}\,.$$

The following result holds.

Lemma 12.6.1. *For any $R > 0$ and $\varepsilon > 0$, and for any $\rho > 0$ and $c > 0$ such that*

$$\Omega_\rho \subset \bar Q_\varepsilon^{n+\nu} \subset \bar Q_R^{n+\nu} \subset \Omega_c$$

there is a number k^* such that, if $k > k^*$, the derivative of the function $W(x_a, \varphi, \theta)$ along the trajectories of (12.75) is negative at each point of the set

$$S = \{(x_a, \varphi, \theta) : \rho \leq W(x_a, \varphi, \theta) \leq c\}\,.$$

Proof. Let $\dot W(x_a, \varphi, \theta)$ denote the derivative of the function $W(x_a, \varphi, \theta)$ along the trajectories of (12.75). Then,

$$\begin{aligned} \dot W(x_a, \varphi, \theta) = {} & \frac{\partial V}{\partial x_a} f_a(x_a, \theta + N(\varphi), \mu) + 2\theta[h_a(x_a, \theta + N(\varphi), \mu) - k\theta] \\ & + \Big[\frac{\partial V}{\partial \varphi}\Big]_{\varphi + M\theta}\Big[L(\varphi) + Mk\theta + M h_a(x_a, \theta + N(\varphi), \mu) - Mk\theta\Big]\,. \end{aligned}$$

The expression on the right-hand side can be put in the form

$$\frac{\partial V}{\partial x_a} f_a(x_a, N(\varphi), \mu) + \frac{\partial V}{\partial \varphi}[L(\varphi) + M h_a(x_a, N(\varphi), \mu)] - 2k\theta^2 + R(x_a, \varphi, \theta)\theta\,,$$

where $R(x_a, \varphi, \theta)$ is an appropriate smooth function. In this expression, the sum of the first two terms is a negative definite function of (x_a, φ), and therefore $\dot W(x_a, \varphi, \theta)$ is negative on the set

$$S_0 = S \cap \{(x_a, \varphi, \theta) : \theta = 0\} .$$

Thus, as observed in the proof of Theorem 12.1.1, it follows that the function $\dot{W}(x_a, \varphi, \theta)$ is negative at each point of some open subset $U \supset S_0$. On the other hand, θ^2 is nowhere zero on the set $\tilde{S} = S \setminus U$. Thus, for some large k, $\dot{W}(x_a, \varphi, \theta)$ is negative on the entire set S. ◁

The dynamic controller (12.73) uses the state variable ζ_r, i.e. the derivative of order $r-1$ of the output y of system (12.69), as input. Thus, in order to find an output feedback controller, this variable must be replaced by an appropriate estimate which, as seen in section 12.5, can be provided by a dynamical system of the form

$$\dot{\eta} = P\eta + Qy$$

in which the matrices Q and P have the form (12.60). In this case, however, as shown in section 12.4, it is appropriate to saturate the resulting control law, so as to avoid the occurrence of finite escape times for large values of g. Thus, in the controller (12.73), the term

$$k[\zeta_r - N(\varphi)]$$

is replaced by the term

$$\sigma_\ell(k[\eta_r - N(\varphi)]) ,$$

in which $\sigma_\ell(\cdot)$ is a saturation function

$$\sigma_\ell(r) = \begin{cases} r & \text{if } |r| < \ell \\ \mathrm{sgn}(r)\bar{M} & \text{if } |r| \geq \ell . \end{cases}$$

This yields a controller modeled by equations of the form

$$\begin{aligned} \dot{\eta} &= P\eta + Qy \\ \dot{\varphi} &= L(\varphi) + M\sigma_\ell(k[\eta_r - N(\varphi)]) \\ u &= \frac{1}{b(y)}\Big[\frac{\partial N}{\partial \varphi}\Big(L(\varphi) + M\sigma_\ell(k[\eta_r - N(\varphi)])\Big) - \sigma_\ell(k[\eta_r - N(\varphi)])\Big] . \end{aligned} \tag{12.79}$$

A controller of this type is able to robustly semiglobally practically stabilize the plant (12.69). To see that this is the case, consider the corresponding closed-loop system,

$$\begin{aligned} \dot{x}_a &= f_a(x_a, \zeta_r, \mu) \\ \dot{\zeta}_r &= h_a(x_a, \zeta_r, \mu) + \frac{\partial N}{\partial \varphi}\Big(L(\varphi) + M\sigma_\ell(k[\eta_r - N(\varphi)])\Big) - \sigma_\ell(k[\eta_r - N(\varphi)]) \\ \dot{\varphi} &= L(\varphi) + M\sigma_\ell(k[\eta_r - N(\varphi)]) \\ \dot{\eta} &= P\eta + Q\zeta_1 , \end{aligned} \tag{12.80}$$

and again the change of variables $\theta = \zeta_r - N(\varphi)$, to obtain

$$\begin{aligned} \dot{x}_a &= f_a(x_a, \theta + N(\varphi), \mu) \\ \dot{\theta} &= h_a(x_a, \theta + N(\varphi), \mu) - \sigma_\ell(k[\eta_r - N(\varphi)]) \\ \dot{\varphi} &= L(\varphi) + M\sigma_\ell(k[\eta_r - N(\varphi)]) \\ \dot{\eta} &= P\eta + Q\zeta_1 . \end{aligned} \tag{12.81}$$

Define

$$e_i = g^{r-i}(\zeta_i - \eta_i)$$

for $i = 1, \ldots, r$ (recall that $\zeta_1, \ldots, \zeta_{r-1}$ are components of the vector x_a), i.e

$$e = D_g(\zeta - \eta) ,$$

in which D_g is the matrix, already used before,

$$D_g = \mathrm{diag}\{g^{r-1}, \ldots, g, 1\} .$$

Then, system (12.81) can be rewritten in the form

$$\begin{aligned} \dot{x}_a &= f_a(x_a, \theta + N(\varphi), \mu) \\ \dot{\theta} &= h_a(x_a, \theta + N(\varphi), \mu) - k\theta + \phi_1(\theta, e) \\ \dot{\varphi} &= L(\varphi) + Mk\theta - M\phi_1(\theta, e) \\ \dot{e} &= gAe + B\phi_2(x_a, \theta, \varphi, e) , \end{aligned} \tag{12.82}$$

in which A is the matrix (12.48), having all eigenvalues in $\mathbb{C}^-$, $B = \mathrm{col}(0, 0, \ldots, 0, 1)$, and

$$\begin{aligned} \phi_1(\theta, e) &= k\theta - \sigma_\ell(k\theta - ke_r) \\ \phi_2(x_a, \theta, \varphi, e) &= h_a(x_a, \theta + N(\varphi), \mu) + \frac{\partial N}{\partial \varphi}[L(\varphi) + M\sigma_\ell(k\theta - ke_r)] \\ &\quad - \sigma_\ell(k\theta - ke_r) . \end{aligned}$$

From this, using the result of Lemma 12.6.1, it is possible conclude the following.

Theorem 12.6.2. *Consider system (12.68) and assume hypotheses* (i), (ii), (iii) *and* (iv) *hold. Given any arbitrarily large number $R > 0$ and any arbitrarily small number $\varepsilon > 0$, there are numbers $k > 0, g > 0, \ell > 0$ such that, in the closed loop system (12.68) – (12.79), any initial condition in $\bar{Q}_R^{n+\nu+r}$ produces a trajectory which is captured by the set $\bar{Q}_\varepsilon^{n+\nu+r}$.*

Proof. Since the changes of coordinates transforming system (12.68) – (12.79) into system (12.81) are globally defined diffeomorphisms which preserve the origin, it suffices to prove the Theorem for system (12.81). The proof uses the same arguments used in the proof of Theorem 12.4.1, and will be only summarily sketched. Let $W(x_a, \varphi, \theta)$ and Ω_a be as in Lemma 12.6.1 and choose ρ, δ, ρ', c such that

$$\Omega_\rho \subset \bar{Q}_\delta^{n+\nu} \subset \Omega_{\rho'} \subset \bar{Q}_\varepsilon^{n+\nu} \subset \bar{Q}_R^{n+\nu} \subset \Omega_c \subset \Omega_{c+1}$$

and, using the result of Lemma 12.6.1, *fix* a number k such that the derivative of the function $W(x_a, \varphi, \theta)$ along the trajectories of (12.75) is negative at each point of the set

$$S = \{(x_a, \varphi, \theta) : \rho \le W(x_a, \varphi, \theta) \le c+1\} .$$

Then, choose the saturation level as

$$\ell = \max_{(x_a,\varphi,\theta)\in\Omega_{c+1}} |k\theta| + 1 .$$

Observe that, for all $((x_a, \varphi, \theta), e) \in \Omega_{c+1} \times \mathbb{R}^r$

$$\begin{aligned} |\phi_1(\theta, e)| &\le \beta_1 \\ |\phi_2(x_a, \theta, \varphi, e)| &\le \beta_2 \\ |\phi_1(\theta, e)| &\le \gamma(\|e\|) \end{aligned}$$

in which β_1, β_2 are fixed numbers, and $\gamma(\cdot)$ is a positive non-decreasing function such that $\gamma(0) = 0$, all independent of g.

From this, the proof uses the same arguments as the proof of Theorem 12.4.1. ◁

We conclude the section with some remarks about the structure of the controller used to stabilize the "auxiliary subsystem" (12.70). The previous Theorem considers the case in which the stabilizer for (12.70) is a strictly proper system, namely a system with no feed-through between y_a and u_a, and has the special structure (12.71), where the right-hand side is affine in the input y_a and the vector multiplying y_a is a constant vector. It is possible, however, to show that the structure thus assumed is not restrictive, from the point of view here considered.

As a matter of fact, it is possible to prove that if a nonlinear system, such as system (12.70), is globally asymptotically stabilized by an output feedback controller of a nonlinear general structure, then it is also semiglobally stabilizable by means of a system having the special structure (12.71).

Consider two subsystems

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) , \end{aligned} \tag{12.83}$$

$$\begin{aligned} \dot{\varphi} &= \theta(\varphi, v) \\ u &= \kappa(\varphi, v) , \end{aligned} \tag{12.84}$$

in which $x \in \mathbb{R}^n$, $\varphi \in \mathbb{R}^\nu$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $v \in \mathbb{R}$, $f(0,0) = 0$, $h(0,0) = 0$, $\theta(0,0) = 0$ and $\kappa(0,0) = 0$. Suppose that, for each x, φ, the equation

$$h(x, \kappa(\varphi, v)) = v$$

has a unique solution

$$v = v(x, \varphi) ,$$

which is a smooth function of (x, φ) (necessarily with $v(0,0) = 0$). Suppose also that

$$u = \kappa(\varphi, v(x, \varphi))$$

is the unique solution of the equation

$$\kappa(\varphi, h(x, u)) = u .$$

In this case, the feedback interconnection of (12.83) and (12.84) via $v = y$ is well-defined, and can be written as

$$\begin{aligned} \dot{x} &= f(x, \kappa(\varphi, v(x, \varphi))) \\ \dot{\varphi} &= \theta(\varphi, v(x, \varphi)) . \end{aligned} \tag{12.85}$$

Proposition 12.6.3. *Suppose the equilibrium $(x, \varphi) = (0,0)$ of the interconnected system (12.85) is globally asymptotically and locally exponentially stable. Suppose also that, for all $(x, \varphi) \in \mathbb{R}^n \times \mathbb{R}^\nu$ and all $\xi \in \mathbb{R}$,*

$$\xi[\xi + v(x, \varphi) - h(x, \kappa(\varphi, \xi + v(x, \varphi)))] \ge \alpha(\xi) ,$$

where $\alpha(\cdot)$ is a class $\mathcal{K}$ function satisfying, for some $a > 0$ and $\delta > 0$,

$$\alpha(s) = as^2 \qquad \text{for all } s \in [0, \delta].$$

Consider the (strictly proper) system

$$\begin{aligned} \dot{\varphi} &= \theta(\varphi, v) \\ \dot{v} &= -Mv + My \\ u &= \kappa(\varphi, v) . \end{aligned} \tag{12.86}$$

Then, given any arbitrarily large number $R > 0$, there exists a number M such that, in the closed loop system (12.83) - (12.86), the equilibrium $(x, \varphi, v) = (0,0,0)$ is locally exponentially stable and, moreover, any initial condition in $\bar{Q}_R^{n+\nu+1}$ produces a trajectory which asymptotically converges to $(0,0,0)$ as $t \to \infty$.

Proof. To prove the result, change v into

$$\xi = v - v(x, \varphi)$$

to obtain

$$\begin{aligned} \dot{x} &= f(x, \kappa(\varphi, \xi + v(x, \varphi))) \\ \dot{\varphi} &= \theta(\varphi, \xi + v(x, \varphi)) \\ \dot{\xi} &= -M[\xi + v(x, \varphi) - h(x, \kappa(\varphi, \xi + v(x, \varphi)))] \\ &\quad - \frac{\partial v}{\partial x} f(x, \kappa(\varphi, \xi + v(x, \varphi))) - \frac{\partial v}{\partial \varphi} \theta(\varphi, \xi + v(x, \varphi)) . \end{aligned} \tag{12.87}$$

For $\xi = 0$ the upper two equations reduce to the feedback interconnection of (12.83) and (12.84), globally asymptotically and locally exponentially stable by hypothesis. On the other hand, it is easy to see that

$$\frac{\mathrm{d}\xi^2}{\mathrm{d}t} \le -2M\alpha(\xi) + (\xi^2 + |\xi||x| + |\xi||\varphi|)q(x,\varphi,\xi)$$

for some smooth function $q(x,\varphi,\xi)$. Then, the result follows from arguments identical to those used before in the proof of Theorem 12.1.1 and Corollary 12.1.2. ◁

In other words, this Proposition determines (sufficient) conditions under which a nonlinear system, such as system (12.70), which is globally asymptotically stabilizable by an output feedback controller of a nonlinear general structure, it is also semiglobally stabilizable by means of a system having the special structure (12.71).

Note that the feedback interconnection of (12.83) and (12.86) differs from the feedback interconnection of (12.83) and (12.84) by the "insertion", between y and v, of a system characterized by the transfer function

$$T(s) = \frac{1}{1+\tau s}$$

in which the time constant $\tau = 1/M$, as shown in the proof of the Proposition, is appropriately small (see Fig. 12.2).

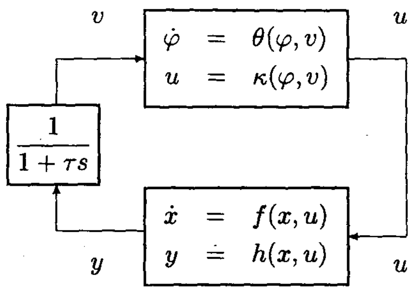

Fig. 12.2. Block-diagram of the system.

12.7 Examples

The purpose of the following example is to show that there are systems which are globally asymptotically stabilizable by static state feedback and uniformly observable , but nevertheless not globally asymptotically stabilizable by any

continuous, static or dynamic, output feedback. For these systems, the result of Theorem 9.6.2 is, in this respect, the best possible stabilization result available: global asymptotic stability via output feedback can never be achieved, but local asymptotic stability with a domain of attraction which contains any arbitrary large bounded set can.

Example 12.7.1. Consider the system

$$\begin{aligned} \dot y &= x \\ \dot x &= 2x^3 + u \end{aligned} \qquad (12.88)$$

with input $u \in \mathbb{R}$, output $y \in \mathbb{R}$ and state $(y, x) \in \mathbb{R}^2$.

This system is globally asymptotically stabilizable by static state feedback (the law $u = -2x^3 - x - y$ does the job) and is also uniformly observable (in fact, the mapping (9.58) is the identity map). We will show that, no matter how a locally Lipschitz, static or dynamic, output feedback

$$\begin{aligned} \dot\xi &= \eta(\xi, y) \\ u &= \theta(\xi, y) \end{aligned} \qquad (12.89)$$

is chosen, there are always initial conditions for which finite escape time occurs.

To this end, consider the differential equation

$$\dot x = cx^3 \,,$$

assume $c > 0$ and $x(0) > 0$, and observe that the corresponding integral curve, given by

$$x(t) = \frac{x(0)}{(1 - 2c[x(0)]^2 t)^{\frac{1}{2}}} \,,$$

is monotonically increasing, and has finite escape time at

$$T = \frac{1}{2c[x(0)]^2} \,.$$

Observe also that, if $x \ge 1$ and $|u| \le 1$,

$$x^3 \le 2x^3 + u \le 3x^3 \,. \qquad (12.90)$$

Thus, if the input $u(t)$ to system (12.88) is such that

$$|u(t)| \le 1 \qquad \text{for all } t \ge 0$$

and $x(0) \ge 1$, the response $x(t)$ of (12.88), for all t for which it is defined, is always increasing and satisfies

$$\dot x(t) \ge x^3(t) \,.$$

Hence, in view of the comparison Lemma,

$$x(t) \geq \frac{x(0)}{(1-2[x(0)]^2 t)^{\frac{1}{2}}} ,$$

from which it is seen that $x(t)$ escapes to infinity at some time

$$T^* \leq \frac{1}{2[x(0)]^2} . \tag{12.91}$$

On the other hand, using the right-hand side of (12.90), it is seen that, on the interval $[0, T^*)$, $x(t)$ satisfies

$$\dot{x}(t) \leq 3x^3(t) ,$$

and therefore, for any $0 < \tau < T^*$,

$$x(t) \leq \frac{x(\tau)}{[1-6[x(\tau)]^2(t-\tau)]^{\frac{1}{2}}} ,$$

for all $t \geq \tau$ satisfying

$$t - \tau < \frac{1}{6[x(\tau)]^2} .$$

This inequality can be used to show that the integral of $x(t)$ over the interval $[0, T^*)$ is finite. To this end, define a sequence of times $t_0 = 0 < t_1 < \ldots < t_k < \ldots$ in the following recursive way. Let x_k denote the value of $x(t)$ at time $t = t_k$, i.e.

$$x_k = x(t_k) ,$$

and define t_{k+1} as

$$t_{k+1} = t_k + T_k ,$$

where $t_k + T_k$ is the finite escape time of the function

$$U_k(t) = \frac{x_k}{[1-6x_k^2(t-t_k)]^{\frac{1}{2}}} ,$$

i.e.

$$T_k = \frac{1}{6x_k^2} .$$

Note that, since $U_k(t)$ is an upper bound for the function $x(t)$ on the time interval $[t_k, t_{k+1})$, the value x_{k+1} is well defined and the recursion makes sense. Clearly,

$$\lim_{k\to\infty} t_k = T^* .$$

Since the function $x(t)$, on the time interval $[t_k, t_{k+1})$, has a lower bound

$$L_k(t) = \frac{x_k}{[1-2x_k^2(t-t_k)]^{\frac{1}{2}}} ,$$

it follows that

$$x_{k+1}^2 \geq \frac{x_k^2}{[1-2x_k^2 T_k]} = \frac{3}{2}x_k^2 =: \frac{1}{a^2}x_k^2$$

with $a = \sqrt{2/3} < 1$. As a consequence,

$$x_k \geq \frac{1}{a^k}x(0) . \tag{12.92}$$

Now, observe that

$$A_k = \int_{t_k}^{t_{k+1}} x(t)dt \leq \int_0^{T_k} \frac{x_k}{[1-6x_k^2 t]^{\frac{1}{2}}}dt = \frac{1}{3x_k} .$$

Therefore, using (12.92),

$$A_k \leq \frac{a^k}{3x(0)} .$$

This, in turn, yields

$$\int_0^{T^*} x(t)dt = \sum_{k=0}^{\infty} A_k \leq \sum_{k=0}^{\infty} \frac{a^k}{3x(0)} = \frac{1}{3x(0)}\frac{a}{1-a} .$$

which proves that the integral of $x(t)$ over the interval $[0, T^*)$ is finite.

As a consequence, if $y(0) = 0$ we obtain, for all $t \in [0, T^*)$,

$$y(t) = \int_0^t x(s)ds \leq \frac{1}{3x(0)}\frac{a}{1-a} \leq d$$

with $d := \sqrt{2/3}/3(1-\sqrt{2/3})$ (recall that it was assumed $x(0) \geq 1$). In summary, there is a number $d > 0$, independent of $x(0)$, such that, if $y(0) = 0$, if $x(0) \geq 1$ and $|u(t)| \leq 1$,

$$|y(t)| \leq d$$

so long as $x(t)$ is defined.

Suppose system (12.88) is controlled by an output feedback of the form (12.89) and suppose, without loss of generality, that $\eta(0,0) = 0$ and $\theta(0,0) = 0$. Let $x(0) \geq 1$, $y(0) = 0$, $\xi(0) = 0$. Since $u(0) = 0$, using continuity arguments and the fact that $|y(t)|$ is bounded by a quantity which is independent of $x(0)$ so long as $|u(t)| \leq 1$, it is deduced that there is a time $T_0 > 0$, independent of $x(0)$, such that $|u(t)| \leq 1$ for all $t \in [0, T_0)$. If $x(0)$ is such that

$$\frac{1}{2x^2(0)} < T_0 ,$$

the previous analysis (see (12.91)) shows that $x(t)$ escapes to infinity in finite time. Thus, the feedback (12.89) cannot globally asymptotically stabilize the equilibrium $(y, x, \xi) = (0, 0, 0)$.

The reason why system (12.88) cannot be globally asymptotically stabilized via output feedback is that there are internal states for which the trajectory escapes to infinity at some finite time $t = T$, while the corresponding output remains bounded on the interval $[0, T)$. Thus, the phenomenon of finite escape time cannot be observed and the controller cannot react to it. ◁

The second example is an application of the stabilization method described in section 12.6 to a system which is not minimum phase.

Example 12.7.2. Consider the three state approximation of the nonlinear Moore-Greitzer model of a compressor [1]

$$\begin{aligned} \dot{R} &= \sigma R(1-\Phi^2-R) \qquad R(0)>0 \\ \dot{\Phi} &= -\Psi+\Psi_C(\Phi)-3\Phi R \\ \dot{\Psi} &= \frac{1}{\beta^2}(\Phi+1-\gamma\sqrt{\Psi})\,. \end{aligned} \tag{12.93}$$

Here $R=A^2/4$, where A is the rotating stall amplitude, Φ is the scaled annuls-averaged flow, Ψ is the plenum pressure rise, $\sigma>0$ and β are fixed parameters, γ is the throttle opening used as a control input, and

$$\Psi_C(\Phi)=\Psi_{C0}+1+\frac{3}{2}\Phi-\frac{1}{2}\Phi^3$$

is the compressor characteristic.

For any Φ_0, this system has an equilibrium at

$$(0,\Phi_0,\Psi_C(\Phi_0))$$

while, if $|\Phi_0|<1$, the system has one additional equilibrium at

$$(1-\Phi_0^2,\Phi_0,\Psi_C(\Phi_0)-3\Phi_0(1-\Phi_0^2))\,.$$

The equilibria in question are enforced by the constant input

$$\gamma=\frac{\Phi_0+1}{\sqrt{\Psi}}\,.$$

Consider, for instance, the case in which $\Phi_0=1$, set

$$\gamma=\frac{2+u}{\sqrt{\Psi}}$$

and translate the coordinate so as to bring the equilibrium $(0,1,\Psi_C(1))$ to the origin, namely set

$$\begin{aligned} \phi &= \Phi-1 \\ \psi &= \Psi-\Psi_C(1)\,. \end{aligned}$$

This yields

$$\begin{aligned} \dot{R} &= -\sigma R^2-\sigma R(2\phi+\phi^2) \\ \dot{\phi} &= -(3/2)\phi^2-(1/2)\phi^3-3R\phi-3R-\psi \\ \dot{\psi} &= \frac{1}{\beta^2}(\phi-u)\,, \end{aligned} \tag{12.94}$$

[1] See Krstic *et al.* (1998) for details.

Note that the half-space $R\geq 0$ is invariant, for every choice of u.

Suppose Ψ, i.e. ψ in the model (12.94), is taken as ***output*** of the system. With such choice of output (and input), the system has relative degree 1 and has a zero dynamics

$$\begin{aligned} \dot{R} &= -\sigma R^2-\sigma R(2\phi+\phi^2) \\ \dot{\phi} &= -(3/2)\phi^2-(1/2)\phi^3-3R\phi-3R \end{aligned}$$

which is not globally asymptotically stable, even in the invariant half-space $R\geq 0$ (for $\sigma=1$, there are two equilibria at $(R,\phi)=(0,0)$ and $(R,\phi)=(0,-3)$, with the phase portrait depicted in Fig. 12.3).

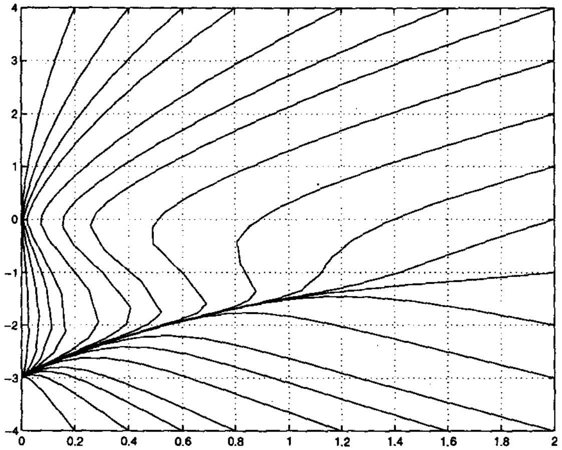

Fig. 12.3. Phase portrait of the (unstable) zero dynamics of the compressor.

In order to apply the design methodology described above, consider the auxiliary system

$$\begin{aligned} \dot{R} &= -\sigma R^2-\sigma R(2\phi+\phi^2) \\ \dot{\phi} &= -(3/2)\phi^2-(1/2)\phi^3-3R\phi-3R-u_a \\ y_a &= \frac{1}{\beta^2}(\phi)\,. \end{aligned} \tag{12.95}$$

Using the inequality

$$R(2\phi+\phi^2)\leq\frac{1}{2}R^2+\frac{1}{2}(2\phi+\phi^2)^2$$

observe that

$$\dot{R}\leq-\frac{\sigma}{2}R^2+\frac{\sigma}{2}(2\phi+\phi^2)^2\,.$$

This shows that $R(t)$ is decreasing whenever

$$R > 2\phi + \phi^2 .$$

Thus, the upper subsystem of (12.95), viewed as a system with input ϕ and state R, is input-to-state stable, with gain function

$$\gamma(r) = 2r + r^2 .$$

As a consequence, one can think of using the result of Lemma 11.4.1 to globally asymptotically stabilize system (12.95). Condition (i) of this Lemma has been just checked, condition (ii) trivially holds, condition (iii) holds because

$$|-(3/2)\phi^2 - (1/2)\phi^3 - 3R\phi - 3R| + |\phi|$$

is locally Lipschitz at the origin, and also condition (iv) holds, because $\gamma(\cdot)$ is locally Lipschitz at the origin. Therefore, the system in question can be globally asymptotically stabilized, by means of a feedback law of the form $u_a = k(\phi)$, which would be a nonlinear function of ϕ. If just semiglobal stabilization is sought, a linear law suffices.

To see why this is the case, observe that, given any number $M > 0$, it is possible to find a number $a > 0$ such that

$$(2\phi + \phi^2)^2 \le a^2\phi^2 \qquad \text{for all } |\phi| \le M .$$

Therefore, for all such ϕ we have

$$\dot{R} \le -\frac{\sigma}{2}R^2 + \frac{\sigma}{2}a^2\phi^2 ,$$

and, consequently, $R(t)$ is decreasing whenever $R > a|\phi|$. From this, it is possible to deduce (see section 12.2) that the upper subsystem of (12.95) is input-to-state stable, with restriction $X_R = \{R \in \mathbb{R} : R \ge 0\}$ on $R(0)$, restriction M on $\phi(\cdot)$ and linear gain function $\gamma_\phi(r) = ar$. Note that the number a depends on M.

Consider now the lower subsystem of (12.95), and set $u_a = K\phi + v$. The candidate ISS-Lyapunov function $V(\phi) = \frac{1}{2}\phi^2$ satisfies

$$\begin{array}{rcl} \dot{V} & = & -\frac{3}{2}\phi^3 - \frac{1}{2}\phi^4 - 3R\phi^2 - 3R\phi - K\phi^2 - \phi v \\ & \le & -\frac{3}{2}\phi^3 + \frac{9}{b^2}[\phi^4 + \phi^2] + (\frac{1}{2} - K)\phi^2 + \frac{b^2}{2}R^2 + \frac{1}{2}v^2 , \end{array}$$

where b is any number. Again, it is possible to find a number $L > 0$ such that

$$-\frac{3}{2}\phi^3 + \frac{9}{b^2}[\phi^4 + \phi^2] \le L\phi^2 \qquad \text{for all } |\phi| \le M ,$$

and therefore, choosing K such that $L + \frac{1}{2} - K \le -1$ we have, for all such ϕ,

$$\dot{V} \le -\phi^2 + \frac{b^2}{2}R^2 + \frac{1}{2}v^2 .$$

Note that b is arbitrary, and K depends on b and M.

From this property, is possible to deduce (see again section 12.2) that the lower subsystem of (12.95) with control $u_a = K\phi + v$, viewed as a system with input (R, v) and state ϕ, is input-to-state stable, with restriction $X_\phi = \{\phi \in \mathbb{R} : |\phi| \le M\}$ on $\phi(0)$, restriction M/b on $R(\cdot)$, restriction M on $v(\cdot)$, and linear gain functions $\gamma_R(r) = br$ and $\gamma_v(r) = r$.

Choose b so that

$$b < 1, \qquad ab < 1 ,$$

and fix K accordingly. Then, using Theorem 12.2.1, it is possible to conclude that system (12.95) with control $u_a = K\phi + v$, viewed as a system with input v and state (R, ϕ) is input-to-state stable, with restriction

$$X = \{(R, \phi) \in \mathbb{R}^2 : \|(R, \phi)\| \le M\}$$

on $(R(0), \phi(0))$, restriction M on $v(\cdot)$, and linear gain function $\gamma_v(r) = cr$ (for some $c > 0$).

This proves, in particular, that for any M there exists K such that the equilibrium of (12.95), with control $u_a = K\phi$ is locally asymptotically stable, with a domain of attraction which contains the set $\{(R, \phi) \in \mathbb{R}^2 : \|(R, \phi)\| \le M\}$.

We have found in this way a "semiglobal" stabilizer for system (12.95). However, this is not yet of the required form. To find a controller of the form (12.71), an additional step is needed. Consider again system (12.95) and add an integrator on the input, i.e. set

$$\begin{array}{rcl} u_a & = & \varphi \\ \dot{\varphi} & = & u . \end{array} \qquad (12.96)$$

Changing, as usual, the variable φ into $v = \varphi - K\phi$, we obtain

$$\dot{\phi} = -(3/2)\phi^2 - (1/2)\phi^3 - 3R\phi - 3R - K\phi - v$$

which is exactly the same equation found before, and

$$\dot{v} = -K\dot{\phi} + u . \qquad (12.97)$$

As it is clear from the previous analysis, choosing $u = -Hv$, it should be possible to enforce the desired properties of input-to-state stability also on the latter subsystem, viewed as a system with state v and input (R, ϕ), with linear gains characterized by arbitrarily small gain factors. More precisely, set $U(v) = v^2$ and observe that

$$\dot{U} = 3K\phi^2 v + K\phi^3 v + 6KR\phi v + 6KRv + 2K^2\phi v + (2K - 2H)v^2 .$$

As before, it is seen that given any number $N > 0$ and any (arbitrarily small) number ε, there is a choice of H (the number K has already been fixed) such that, if $\|(R(t), \phi(t))\| \leq N$, for all $t \geq 0$,

$$\dot{U} \leq -v^2 + \frac{\varepsilon^2}{2} R^2 + \frac{\varepsilon^2}{2} \phi^2 .$$

Thus system (12.97), with control $u = -Hv$, is input-to-state stable, with no restriction on $v(0)$, restriction N on $(R(\cdot), \phi(\cdot))$, and linear gain function $\gamma_{(R,\phi)}(r) = \varepsilon r$. Note that H depends on N and ε.

Choose ε so that

$$\varepsilon < 1, \qquad \varepsilon c < 1 .$$

Then, using again Theorem 12.2.1, it can be proven that the system (12.95) - (12.96), with control $u = -Hv$, is locally asymptotically stable, with a domain of attraction which contains the set $\{(R, \phi, v) : \|(R, \phi, v)\| < M\}$.

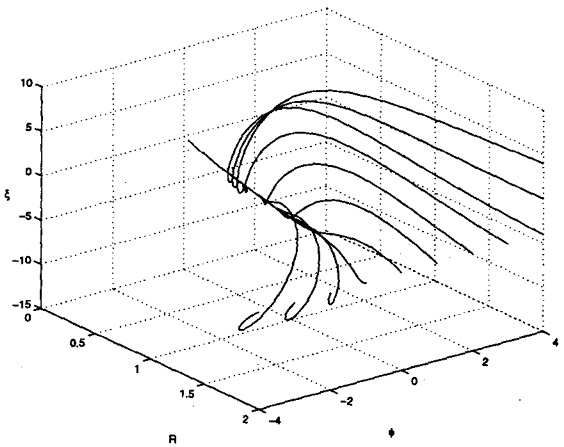

Fig. 12.4. Phase portrait of the stabilized auxiliary subsystem

Reversing the change of coordinates in (12.96), this shows that semiglobal stability, for system (12.95), can be obtained by means of a dynamic controller of the form

$$\begin{aligned} \dot{\varphi} &= -H\varphi + HK\beta^2 y_{\mathrm{a}} \\ u_{\mathrm{a}} &= \varphi , \end{aligned} \qquad (12.98)$$

which has precisely the required structure. The phase portrait of the corresponding interconnection (12.95)-(12.98) is shown in Fig. 12.4.

Having determined a stabilizer for the auxiliary system (12.95), the theory presented in the previous section shows that one can construct a dynamic output feedback stabilizer for the original system (12.94). In this case, the

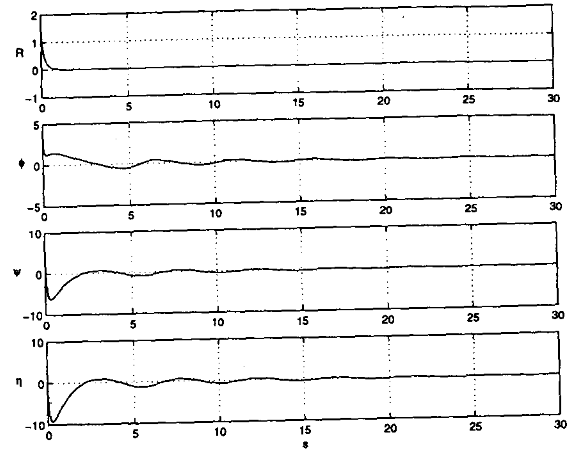

Fig. 12.5. Time history of the stabilized compressor.

result of Lemma 12.6.1 suffices, since system (12.94) has relative degree $r = 1$ and there is no need to estimate derivatives of y. The resulting stabilizer has the form

$$\begin{aligned} \dot{\varphi} &= -H\varphi + HK\beta^2 k(y - \varphi) \\ u &= \beta^2 H\varphi - \beta^2 (HK\beta^2 - +1) k(y - \varphi) , \end{aligned} \qquad (12.99)$$

with

$$y = \psi .$$

Fig. 12.5 shows a trajectory of system (12.94) controlled by (12.99) (for $\sigma = 1$, $\beta = 1$, $K = 6$, $H = 10$, $G = 0.5$).

13. Disturbance Attenuation

13.1 Robust Stability via Disturbance Attenuation

In this Chapter we will study problems of global stabilization of systems that can be modeled as feedback interconnection of two subsystems, one of which is accurately known while the other one is uncertain but has a finite L_2 gain, for which an upper bound is available. More precisely, we consider systems modeled by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, h_2(x_2), u) \\ \dot{x}_2 &= f_2(x_2, h_1(x_1)) , \end{aligned} \tag{13.1}$$

which describe the feedback interconnection of a system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, w, u) \\ y &= h_1(x_1) \end{aligned} \tag{13.2}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $w \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, and $f_1(0,0,0) = 0$, $h_1(0) = 0$, and a system

$$\begin{aligned} \dot{x}_2 &= f_2(x_2, y) \\ w &= h_2(x_2) \end{aligned} \tag{13.3}$$

in which $x_2 \in \mathbb{R}^{n_2}$ and $f_2(0,0) = 0$, $h_2(0) = 0$.

The model of system (13.2) is supposed to be accurately known, while the model of system (13.3) is possibly unknown, but it is known that, for some γ_2, this system is *strictly dissipative*, with respect to the supply rate

$$q_2(w, y) \doteq \gamma_2^2 y^2 - w^2 .$$

Global asymptotic stability of the uncertain system (13.1) can be achieved in the following way. Suppose $u = u(x_1)$ is a feedback law (assume, as usual, $u(0) = 0$) such that the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, w, u(x_1)) \\ y &= h_1(x_1) \end{aligned} \tag{13.4}$$

is strictly dissipative with respect to the supply rate

$$q_1(y,w) = \gamma_1^2 w^2 - y^2 \,,$$

and that

$$\gamma_1 \gamma_2 < 1 \,.$$

Then, by the Theorem 10.8.1, the system (see Fig. 13.1)

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, h_2(x_2), u(x_1)) \\ \dot{x}_2 &= f_2(x_2, h_1(x_1)) \end{aligned} \qquad (13.5)$$

is globally asymptotically stable. In other words, the feedback law $u = u(x_1)$ has globally robustly stabilized the uncertain system (13.1).

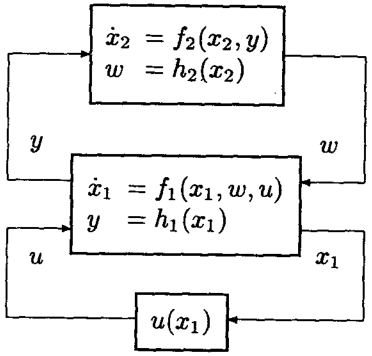

Fig. 13.1. Robust stabilization via disturbance attenuation.

This observation motivates the following problem, known as *Problem of Disturbance Attenuation* (in the sense of the L_2-gain), *with Stability*. Given a system modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x,w,u) \\ y &= h(x) \end{aligned} \qquad (13.6)$$

in which $x \in \mathbb{R}^n$, $w \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, and $f(0,0,0) = 0$, $h(0) = 0$, and a number $\gamma > 0$, find, if possible, a feedback law $u = u(x)$ such that the resulting closed-loop system is strictly dissipative with respect to the supply rate

$$q(w,y) = \gamma^2 w^2 - y^2 \,. \qquad (13.7)$$

In this setup, the input w of (13.6) is called the ***disturbance input***, while the input u is called the *control input.*

We describe hereafter a series of results, which are useful to address the problem of Disturbance Attenuation with Stability for an important class of nonlinear systems. Recall that, in view of the definition of dissipativity, with respect to a supply rate of the form (13.7), the issue is to find a law $u = u(x)$ such that, for some smooth function $V(x)$ which is positive definite and proper, i.e. satisfies estimates of the form

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \,,$$

with $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot)$ class $\mathcal{K}_\infty$ functions, the dissipation inequality

$$\frac{\partial V}{\partial x} f(x,w,u(x)) \le -\alpha(\|x\|) + \gamma^2 w^2 - h^2(x) \qquad (13.8)$$

holds for all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}$, with $\alpha(\cdot)$ a class $\mathcal{K}_\infty$ function.

In what follows, it will be shown that, for nonlinear systems that can be described by equations having a special lower-triangular structure, the problem of Disturbance Attenuation with Stability can be addressed by means of a technique of backstepping, which closely follows the procedure for global stabilization described in section 9.2, i.e. by recursively solving the problem in question for a sequence of systems of increasing dimension. The points of departure are the appropriate versions of Lemma 9.2.1 and 9.2.2.

Lemma 13.1.1. *Consider a system described by equations of the form*

$$\begin{aligned} \dot{z} &= f(z,\xi,w) \\ \dot{\xi} &= q(z,\xi,w) + u \\ y &= h(z,\xi) \end{aligned} \qquad (13.9)$$

in which $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$, $f(0,0,0) = 0$, $q(0,0,0) = 0$ *and* $h(0,0) = 0$. *Assume that*

$$f(z,\xi,w) - f(z,\xi,0)$$

is independent of ξ. *Assume also that, for some smooth real-valued function* $R_1(z,\xi)$,

$$|q(z,\xi,w) - q(z,\xi,0)| \le R_1(z,\xi)|w|$$

for all (z,ξ) *and all* w.

Suppose there exist a number $\gamma > 0$, *a smooth real-valued function* $V(z)$, *which is positive definite and proper, and a class* $\mathcal{K}_\infty$ *function* $\alpha_0(\cdot)$ *such that*

$$\frac{\partial V}{\partial z} f(z,0,w) \le -\alpha_0(\|z\|) + \gamma^2 w^2 - h^2(z,0)$$

for all z *and all* w. *Then, for every* $\varepsilon > 0$, *there exist a smooth feedback law* $u = u(z,\xi)$, *a smooth real-valued function* $W(z,\xi)$, *which is positive definite and proper, and a class* $\mathcal{K}_\infty$ *function* $\alpha(\cdot)$ *such that*

$$\begin{aligned} &\frac{\partial W}{\partial z} f(z,\xi,w) + \frac{\partial W}{\partial \xi}[q(z,\xi,w) + u(z,\xi)] \\ &\qquad \le -\alpha(\|x\|) + (\gamma+\varepsilon)^2 w^2 - h^2(z,\xi) \end{aligned} \qquad (13.10)$$

for all (z,ξ) *and all* w, *where* $x = \mathrm{col}(z,\xi)$.

Proof. Set $f_0(z,\xi) = f(z,\xi,0)$, and $p_0(z,w) = f(z,\xi,w) - f(z,\xi,0)$. Set $f_1(z,\xi) = q(z,\xi,0)$ and express $q(z,\xi,w)$ as

$$q(z,\xi,w) = f_1(z,\xi) + p_1(z,\xi,w)$$

where, by hypothesis, $|p_1(z,\xi,w)| \le R_1(z,\xi)|w|$. System (13.9) becomes

$$\begin{aligned} \dot z &= f_0(z,\xi) + p_0(z,w) \\ \dot\xi &= f_1(z,\xi) + p_1(z,\xi,w) + u \\ y &= h(z,\xi)\,. \end{aligned} \tag{13.11}$$

Set

$$W(z,\xi) = V(z) + \frac{1}{2}\xi^2$$

and observe that

$$\begin{aligned} \frac{\partial W}{\partial z} f(z,\xi,w) &= \frac{\partial W}{\partial z} f_0(z,\xi) + \frac{\partial W}{\partial z} p_0(z,w) \\ &= \frac{\partial W}{\partial z} f_0(z,0) + \frac{\partial W}{\partial z} p_0(z,w) + A(z,\xi)\xi \\ &= \frac{\partial W}{\partial z} f(z,0,w) + A(z,\xi)\xi \end{aligned}$$

where $A(z,\xi)$ is a suitable smooth function. Moreover,

$$\frac{\partial W}{\partial \xi}[q(z,\xi,w) + u] = f_1(z,\xi)\xi + p_1(z,\xi,w)\xi + u\xi\,.$$

Finally, note that

$$h^2(z,\xi) = h^2(z,0) + B(z,\xi)\xi$$

where $B(z,\xi)$ is a suitable smooth function.

Thus, the left-hand side of (13.10), henceforth referred to as $\dot W$, satisfies

$$\begin{aligned} \dot W &= \frac{\partial W}{\partial z} f(z,0,w) + A(z,\xi)\xi + f_1(z,\xi)\xi + p_1(z,\xi,w)\xi + u\xi \\ &\le -\alpha_0(\|z\|) + \gamma^2 w^2 - h^2(z,0) - B(z,\xi)\xi + B(z,\xi)\xi \\ &\quad + A(z,\xi)\xi + f_1(z,\xi)\xi + p_1(z,\xi,w)\xi + u\xi\,. \end{aligned}$$

Choose

$$u = -d\xi - B(z,\xi) - A(z,\xi) - f_1(z,\xi) + v$$

with $d > 0$, to obtain

$$\dot W \le -\alpha_0(\|z\|) + \gamma^2 w^2 - h^2(z,\xi) - d\xi^2 + p_1(z,\xi,w)\xi + v\xi\,.$$

To prove the Lemma, we choose now v so as to render

$$v\xi + p_1(z,\xi,w)\xi \le \varepsilon^2 w^2\,. \tag{13.12}$$

To this end, note that, for any $\varepsilon > 0$

$$v\xi + p_1(z,\xi,w)\xi \le v\xi + R_1(z,\xi)|w||\xi| \le v\xi + \frac{R_1^2(z,\xi)\xi^2}{4\varepsilon^2} + \varepsilon^2 w^2\,,$$

so that, to obtain (13.12), we choose

$$v = -\frac{R_1^2(z,\xi)\xi}{4\varepsilon^2}\,.$$

At this point, we have

$$\begin{aligned} \dot W &\le -\alpha_0(\|z\|) - d\xi^2 + \gamma^2 w^2 + \varepsilon^2 w^2 - h^2(z,\xi) \\ &\le -\alpha_0(\|z\|) - d\xi^2 + (\gamma+\varepsilon)^2 w^2 - h^2(z,\xi)\,. \end{aligned} \tag{13.13}$$

Note that, since $\alpha_0(\cdot)$ is a class $\mathcal{K}_\infty$ function, the function $\alpha_0(\|z\|) + d\xi^2$ is a positive definite and proper function of $x = \mathrm{col}(z,\xi)$. Thus (see Remark 10.1.3), there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$\alpha(\|x\|) \le \alpha_0(\|z\|) + d\xi^2\,.$$

This, in view of (13.13), completes the proof. ◃

Lemma 13.1.2. *Consider a system described by equations of the form (13.9). Assume that*

$$f(z,\xi,w) - f(z,\xi,0)$$

is independent of ξ. Assume also that, for some smooth real-valued functions $R_0(z), R_1(z,\xi)$,

$$\begin{aligned} \|f(z,\xi,w) - f(z,\xi,0)\| &\le R_0(z)|w| \\ |q(z,\xi,w) - q(z,\xi,0)| &\le R_1(z,\xi)|w| \end{aligned} \tag{13.14}$$

for all (z,ξ) and all w.

Suppose there exist a number $\gamma > 0$, a smooth real-valued function $v(z)$, with $v(0) = 0$, a smooth real-valued function $V(z)$, which is positive definite and proper, and a class $\mathcal{K}_\infty$ function $\alpha_0(\cdot)$ such that

$$\frac{\partial V}{\partial z} f(z,v(z),w) \le -\alpha_0(\|z\|) + \gamma^2 w^2 - h^2(z,v(z)) \tag{13.15}$$

for all z and all w. Then, for every $\varepsilon > 0$, there exist a smooth feedback law $u = u(z,\xi)$, a smooth real-valued function $W(z,\xi)$, which is positive definite and proper, and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$\begin{aligned} &\frac{\partial W}{\partial z} f(z,\xi,w) + \frac{\partial W}{\partial \xi}[q(z,\xi,w) + u(z,\xi)] \\ &\qquad \le -\alpha(\|x\|) + (\gamma+\varepsilon)^2 w^2 - h^2(z,\xi) \end{aligned}$$

for all (z,ξ) and all w, where $x = \mathrm{col}(z,\xi)$.

Proof. Consider the globally defined change of coordinates $\eta = \xi - v(z)$ and note that, in the new coordinates, system (13.9) becomes

$$\begin{aligned}
\dot{z} &= f(z,\eta + v(z), w) := \tilde{f}(z,\eta,w)\\
\dot{\eta} &= q(z,\eta+v(z),w) - \frac{\partial v}{\partial z} f(z,\eta+v(z),w) + u := \tilde{q}(z,\eta,w) + u\\
y &= h(z,\eta+v(z)) .
\end{aligned}$$

Indeed,

$$|\tilde{q}(z,\eta,w) - \tilde{q}(z,\eta,0)| \le R_1(z,\eta+v(z))|w| + \left\|\frac{\partial v}{\partial z}\right\| R_0(z)|w| .$$

Thus, this system satisfies the hypotheses of Lemma 13.1.1, and the result follows. ◁

This Lemma can be used repeatedly, to prove that the fulfillment of an inequality of the form (13.15) suffices to determine a solution of the problem of Disturbance Attenuation with Stability for systems having the following "lower-triangular" structure

$$\begin{aligned}
\dot{z} &= f(z,\xi_1,w)\\
\dot{\xi}_1 &= \xi_2 + q_1(z,\xi_1,w)\\
\dot{\xi}_2 &= \xi_3 + q_2(z,\xi_1,\xi_2,w)\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r + q_{r-1}(z,\xi_1,\ldots,\xi_{r-1},w)\\
\dot{\xi}_r &= u + q_r(z,\xi_1,\ldots,\xi_r,w)\\
y &= h(z,\xi_1) ,
\end{aligned} \tag{13.16}$$

so long as certain assumptions hold. More precisely, assume, as before, that $f(z,\xi_1,w) - f(z,\xi_1,0)$ is independent of ξ_1, and rewrite system (13.16) in the form

$$\begin{aligned}
\dot{z} &= f_0(z,\xi_1) + p_0(z,w)\\
\dot{\xi}_1 &= \xi_2 + f_1(z,\xi_1) + p_1(z,\xi_1,w)\\
\dot{\xi}_2 &= \xi_3 + f_2(z,\xi_1,\xi_2) + p_2(z,\xi_1,\xi_2,w)\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r + f_{r-1}(z,\xi_1,\ldots,\xi_{r-1}) + p_{r-1}(z,\xi_1,\ldots,\xi_{r-1},w)\\
\dot{\xi}_r &= u + f_r(z,\xi_1,\ldots,\xi_r) + p_r(z,\xi_1,\ldots,\xi_r,w)\\
y &= h(z,\xi_1) ,
\end{aligned} \tag{13.17}$$

where

$$\begin{aligned}
f_0(z,\xi) &:= f(z,\xi_1,0)\\
p_0(z,w) &:= f(z,\xi_1,w) - f(z,\xi_1,0)
\end{aligned}$$

and

$$\begin{aligned}
f_i(z,\xi_1,\ldots,\xi_i) &:= q_i(z,\xi_1,\ldots,\xi_i,0)\\
p_i(z,\xi_1,\ldots,\xi_i,w) &:= q_i(z,\xi_1,\ldots,\xi_i,w) - q_i(z,\xi_1,\ldots,\xi_i,0) ,
\end{aligned}$$

for $i = 1,\ldots,r$.

Then, we have the following result.

Theorem 13.1.3. *Consider system (13.17), in which $f_0(0,0) = 0$, $f_1(0,0) = 0, \ldots, f_r(0,0,\ldots,0) = 0$. Assume that, for some smooth functions $R_0(z)$, $R_1(z,\xi_1)$, ..., $R_r(z,\xi_1,\ldots,\xi_r)$,*

$$\|p_0(z,w)\| \le R_0(z)|w| \tag{13.18}$$

and

$$|p_i(z,\xi_1,\ldots,\xi_i,w)| \le R_i(z,\xi_1,\ldots,\xi_i)|w| \tag{13.19}$$

for $i = 1,\ldots,r$.

Suppose there exist a number $\gamma > 0$, a smooth real-valued function $\xi_1 = v(z)$, with $v(0) = 0$, and a smooth real-valued function $V(z)$, which is positive definite and proper, and a class $\mathcal{K}_\infty$ function $\alpha_0(\cdot)$ such that

$$\frac{\partial V}{\partial z}[f_0(z,v(z)) + p_0(z,w)] \le -\alpha_0(\|z\|) + \gamma^2 w^2 - h^2(z,v(z)) \tag{13.20}$$

for all z and all w. Then, for every $\varepsilon > 0$, there exists a smooth feedback law $u = u(z,\xi_1,\ldots,\xi_r)$ such that the resulting closed-loop system is strictly dissipative with respect to the supply rate

$$q(w,y) = (\gamma+\varepsilon)^2 w^2 - y^2 .$$

Proof. Pick any $1 \le i < r$ and define

$$\tilde{z} = \begin{pmatrix} z\\ \xi_1\\ \cdots\\ \xi_i \end{pmatrix}, \quad \tilde{f}_0(\tilde{z},\xi_{i+1}) = \begin{pmatrix} f_0(z,\xi_1)\\ \xi_2 + f_1(z,\xi_1)\\ \cdots\\ \xi_{i+1} + f_i(z,\xi_1,\ldots,\xi_i) \end{pmatrix},$$

$$\tilde{p}_0(\tilde{z},w) = \begin{pmatrix} p_0(z,w)\\ p_1(z,\xi_1,w)\\ \cdots\\ p_i(z,\xi_1,\ldots,\xi_i,w) \end{pmatrix},$$

and

$$\begin{aligned}
\tilde{f}_1(\tilde{z},\xi_{i+1}) &= f_{i+1}(z,\xi_1,\ldots,\xi_{i+1})\\
\tilde{p}_1(\tilde{z},\xi_{i+1},w) &= p_{i+1}(z,\xi_1,\ldots,\xi_{i+1},w)\\
\tilde{h}(\tilde{z},\xi_{i+1}) &= h(z,\xi_1) .
\end{aligned}$$

Observe that system

$$\begin{aligned}
\dot{\tilde{z}} &= \tilde{f}_0(\tilde{z},\xi_{i+1}) + \tilde{p}_0(\tilde{z},w)\\
\dot{\xi}_{i+1} &= u + \tilde{f}_1(\tilde{z},\xi_{i+1}) + \tilde{p}_1(\tilde{z},\xi_{i+1},w)\\
y &= h(\tilde{z},\xi_{i+1})
\end{aligned}$$

is such tl

$$\begin{aligned} \|\tilde{p}_0(\tilde{z},w)\| &\leq \tilde{R}_0(\tilde{z})|w| \\ |\tilde{p}_1(\tilde{z},\xi_{r+1},w)| &\leq \tilde{R}_1(\tilde{z},\xi_{i+1})|w| \end{aligned}$$

for some$(\tilde{z},\xi_{i+1})$. Thus, a simple induction, based on Lemma 13.1.2, pBult. ◁

Remark ɛ that the hypotheses (13.18) and (13.19) are trivially satisfied $\xi_1,\ldots,\xi_i,w)$'s, which by definition vanish at $w=0$, depend lthe disturbance input w. Another situation in which these hyp automatically satisfied in the one in which it is known that, for. number $K>0$, $|w(t)|\leq K$ for all $t\geq 0$. In fact, since $p_i(:w)$ is smooth and vanishing at $w=0$, there exists $S_i(z,\xi_1,$..ch that

$$z,\xi_1,\ldots,\xi_i,w)=S_i(z,\xi_1,\ldots,\xi_i,w)w$$

and $R_i(z$such that

$$S_i(z,\xi_1,\ldots,\xi_i,w)|\leq R_i(z,\xi_1,\ldots,\xi_i)$$

for all $|w$

In coɛ see from this Theorem that, in a system of the form (13.17), tıe to be addressed in order to be able to solve the problem of Disturnuation with Stability, is to find a smooth real-valued function $v(0)=0$, a smooth real-valued function $V(z)$, which is positive cproper, rendering the inequality (13.20) satisfied, for all z and all

It is to compare the dissipation inequality (13.20) with the original dnequality (13.8), whose fulfillment is the basic goal of the problrbance Attenuation with Stability. To this end, rewrite $f_0(z,\xi_1)$ as $f(z,\xi_1,w)$ and observe that the inequality (13.20), rewritten

$$z),w)\leq -\alpha(\|z\|)+\gamma^2w^2-h^2(z,v(z)) \tag{13.21}$$

is preciseıality which expresses the property that, under the feedback lawhe "subsystem"

$$\begin{aligned} \dot{z} &= f(z,v,w) \\ y &= h(z,v) \end{aligned} \tag{13.22}$$

is strictly̧, with respect to the supply rate (13.7). The result of Theoremvs that, for a system of the form (13.17), the problem of finding aw $u=u(x)$ rendering the dissipation inequality (13.8) fulfilled ced to the problem of finding a feedback law $v=v(z)$

rendering the dissipation inequality (13.21) fulfilled, i.e. rendering the subsystem (13.22) strictly dissipative with respect to the supply rate (13.7). This subsystem has a more general structure than subsystem (13.6), because its output is allowed to depend on the control input v, which was not the case for system (13.6), but has a lower dimension.

In the next sections of the Chapter, we will often consider the special case of systems modeled by equations, such as (13.22), whose right-hand sides are *affine* functions of the disturbance input and of the control input. In these cases, the inequality (13.21) can be given a simpler expression, which does not call for the explicit existence of feedback laws, such as $v(z)$, but just the existence of a function $V(z)$. In fact, consider a system modeled by equations of the form

$$\dot{x}=f(x)+g(x)u+p(x)w$$

with $f(0)=0$, and set

$$y=\begin{pmatrix} h(x) \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ r(x) \end{pmatrix}u$$

with $h(0)=0$ and $r(x)\neq 0$ for all x. A feedback law $u=u(x)$ renders this system strictly dissipative, with respect to the supply rate (13.7), if and only if there exists a positive definite and proper function $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x)+g(x)u(x)+p(x)w]\leq -\alpha(\|x\|)+\gamma^2w^2-h^2(x)-r^2(x)u^2(x)\,, \tag{13.23}$$

and the following holds.

Lemma 13.1.4. *A function $V(x)$ satisfies (13.23) for some $u(x)$ if and only if*

$$\frac{\partial V}{\partial x}f(x)+h^2(x)-\frac{1}{4r^2(x)}\left[\frac{\partial V}{\partial x}g(x)\right]^2+\frac{1}{4\gamma^2}\left[\frac{\partial V}{\partial x}p(x)\right]^2\leq -\alpha(\|x\|)\,. \tag{13.24}$$

Proof. Observe that, for any $(v,w)\in\mathbb{R}^2$,

$$\begin{aligned} &\frac{\partial V}{\partial x}[f(x)+g(x)u+p(x)w]+h^2(x)+r^2(x)u^2-\gamma^2w^2 \\ &\quad=\frac{\partial V}{\partial x}f(x)+h^2(x)-\frac{1}{4r^2(x)}\left[\frac{\partial V}{\partial x}g(x)\right]^2+\frac{1}{4\gamma^2}\left[\frac{\partial V}{\partial x}p(x)\right]^2 \\ &\qquad+\left[r(x)u+\frac{1}{2r(x)}\frac{\partial V}{\partial x}g(x)\right]^2-\left[\gamma w-\frac{1}{2\gamma}\frac{\partial V}{\partial x}p(x)\right]^2. \end{aligned}$$

Suppose that, for some $u(x)$, $V(x)$ satisfies (13.23) for all x and w. Then, choosing

$$w=\frac{1}{2\gamma^2}\frac{\partial V}{\partial x}p(x)$$

it is seen that (13.24) necessarily holds. Conversely, if the latter holds, then

$$u(x)=-\frac{1}{2r^2(x)}\frac{\partial V}{\partial x}g(x)$$

renders condition (13.23) satisfied for all x and w. ◁

13.2 The Case of Linear Systems

The purpose of this section is to show that the *sufficient* conditions of Theorem 13.1.3 for the solution of a problem of Disturbance Attenuation with Stability become, in the case of linear systems, also *necessary*, and that this fact can be exploited to the purpose of determining the *optimal* value of the achievable disturbance attenuation, namely a number γ^* with the property that the problem in question is solvable for each $\gamma > \gamma^*$ and is not solvable for $\gamma < \gamma^*$.

Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Pw + Bu \\ y &= Cx \end{aligned} \tag{13.25}$$

in which $x \in \mathbb{R}^n$, $w \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$. It follows from Theorem 10.9.1 that there exists a *linear* feedback law $u = Kx$ which renders the resulting closed-loop system strictly dissipative with respect to the supply rate

$$q(w,y) = \tilde{\gamma}^2 w^2 - y^2$$

for some $\tilde{\gamma} < \gamma$, if and only if the inequality

$$(A+BK)^{\mathrm{T}}S + S(A+BK) + \frac{1}{\gamma^2}SPP^{\mathrm{T}}S + C^{\mathrm{T}}C < 0 \tag{13.26}$$

holds for some symmetric matrix $S > 0$. Thus, the ability of solving this matrix inequality for some K and some symmetric $S > 0$ determines the ability of solving the problem of Disturbance Attenuation with Stability for system (13.25), by means of a linear feedback law.

In what follows, we will show that the inequality (13.26) can be solved if and only if γ is larger than a fixed number γ^*, which can be rather easily determined once system (13.25) has been expressed in special new coordinates. To this end, recall (see 12.55) that *any* linear system of the form (13.25) can always be expressed, by means of suitable changes of coordinates, in the following form,

$$\begin{aligned} \dot{z} &= Fz + G\xi_1 + Qw \\ \dot{\xi}_1 &= \xi_2 + p_1 w \\ \dot{\xi}_2 &= \xi_3 + p_2 w \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r + p_{r-1} w \\ \dot{\xi}_r &= Hz + a_1\xi_1 + \ldots a_r\xi_r + bu + p_r w \\ y &= \xi_1 , \end{aligned} \tag{13.27}$$

(with $z \in \mathbb{R}^{n-r}$) which is the linear counterpart of the form (13.17). The first step in the analysis consists in showing that the optimal level of achievable disturbance attenuation γ^* only depends on the parameters which characterize the z-subsystem.

Lemma 13.2.1. *Consider the linear system (13.27) and let γ be a fixed number. The following are equivalent:*

(i) *there exists a linear feedback law which renders the resulting closed-loop system strictly dissipative with respect to the supply rate*

$$q(w,y) = \tilde{\gamma}^2 w^2 - y^2$$

for some $\tilde{\gamma} < \gamma$,

(ii) *there exist a matrix K and a symmetric matrix $S > 0$ such that (13.26) holds,*

(iii) *there exist a matrix L and a symmetric matrix $Y > 0$ such that*

$$Y(F+GL) + (F+GL)^{\mathrm{T}}Y + \frac{1}{\gamma^2}YQQ^{\mathrm{T}}Y + L^{\mathrm{T}}L < 0 , \tag{13.28}$$

(iv) *there exists a symmetric matrix $Z > 0$ such that*

$$FZ + ZF^{\mathrm{T}} + \frac{1}{\gamma^2}QQ^{\mathrm{T}} - GG^{\mathrm{T}} < 0 . \tag{13.29}$$

Proof. As observed before, (i) $\Rightarrow$ (ii) by Theorem 10.9.1.

(ii) $\Rightarrow$ (iii). For $i = 1, \ldots, r$, set

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{i\times i}, \quad B_i = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}_{i\times 1}$$

$$C_i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{1\times i}, \quad P_i = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_i \end{pmatrix} .$$

Set also

$$F_i = \begin{pmatrix} F & GC_i \\ 0 & A_i \end{pmatrix}, \quad G_i = \begin{pmatrix} 0 \\ B_i \end{pmatrix}, \quad Q_i = \begin{pmatrix} Q \\ P_i \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & C_i \end{pmatrix} ,$$

and

$$\bar{K} = \begin{pmatrix} H & a_1 & \cdots & a_r \end{pmatrix} .$$

With this choice, we see that system (13.27) can be rewritten in the form

$$\begin{aligned} \dot{x} &= F_r x + Q_r w + G_r(bu + \bar{K}x) \\ y &= H_r x \end{aligned}$$

where $x = \mathrm{col}(z, \xi_1, \ldots, \xi_r)$.

Suppose (ii) holds. Then, it is reaat there exist a matrix K_r and a matrix $X_r = X_r^{\mathrm{T}} > 0$ which sat

$$X_r(F_r + G_rK_r) + (F_r + G_rK_r)^{\mathrm{T}} X_r Q_r^{\mathrm{T}} X_r + H_r^{\mathrm{T}} H_r = -M_r \,,$$

where $M_r = M_r^{\mathrm{T}} > 0$. From this, by)ne can prove that, for $1 \leq i < r$, there exist a matrix K_i and ma$X_i^{\mathrm{T}} > 0$ and $M_i = M_i^{\mathrm{T}} > 0$ such that

$$X_i(F_i + G_iK_i) + (F_i + G_iK_i)^{\mathrm{T}} X_i + \frac{1}{\gamma}{}_i + H_i^{\mathrm{T}} H_i = -M_i \,. \quad (13.30)$$

To this end, suppose (13.30) hold$n - r + i$ and consider the linear equation

$$(0_{1\times(d_i-1)} \quad 1)\, X_i \left(I_{(d_i\cdot} \right. = 0_{1\times(d_i-1)} \,,$$

in the unknown K_{i-1}. Since $X_i > 0$, ithe bottom-right position is nonzero, and this equation has a uniq Define

$$\begin{aligned} X_{i-1} &= (I_{(d_i-1)\times(d_i-1)} \quad \cdot \left. \begin{matrix} I_{(d_i-1)\times(d_i-1)} \\ K_{i-1} \end{matrix} \right) \\ M_{i-1} &= (I_{(d_i-1)\times(d_i-1)} \quad \cdot \left. \begin{matrix} I_{(d_i-1)\times(d_i-1)} \\ K_{i-1} \end{matrix} \right) \,, \end{aligned}$$

multiply (13.30) on the left by

$$T_i = \begin{pmatrix} I_{(d_i-1)\times \cdot 1} \\ 0_{1\times(d_i} \end{pmatrix}$$

and on the right by T_i^{T}, and take the lock. This, since

$$T_i X_i T_i^{\mathrm{T}} = \begin{pmatrix} X_{i-)\times 1} \\ 0_{1\times(d_i} \end{pmatrix} \,,$$

and

$$(I_{(d_i-1)\times(d_i-1)} \quad {}_{\imath}H_{i-1}$$

for $i > 1$, yields an identity of the formith i replaced by $i - 1$.

One last iteration of this process, ;) written for $i = 1$, shows that an inequality of the form (13.28)

(iii) $\Rightarrow$ (i). As observed, system (13.:tructure of system (13.17), and

$$f_0(z, \xi_1) + p_0(z, w) := + Qw \,.$$

Define

$$V(z) =: z^{\mathrm{T}} Y z, \ ;z \,.$$

Suppose (13.28) holds and note that, since its left-hand side is sign definite, the same inequality continues to hold if γ is replaced by $\tilde{\gamma} = \gamma - \varepsilon$ and $\varepsilon > 0$ is sufficiently small. Therefore, the functions $V(z)$ and $v(z)$ are such that

$$\frac{\partial V}{\partial z}[Fz + Gv(z)] + \frac{1}{4\tilde{\gamma}^2}\frac{\partial V}{\partial z} QQ^{\mathrm{T}} \frac{\partial V}{\partial z}^{\mathrm{T}} + [v(z)]^2 \leq -c\|z\|^2$$

for some $c > 0$, i.e. such that (recall Corollary 10.7.2 and subsequent observations)

$$\frac{\partial V}{\partial z}[Fz + Gv(z) + Qw] \leq -c\|z\|^2 + \tilde{\gamma}^2 w^2 - [v(z)]^2$$

for all z and all w. In other words, an inequality of the form (13.20) holds. Since all other hypotheses of Theorem 13.1.3 are satisfied, it is possible to conclude the existence of a feedback law $u = u(z, \xi_1, \ldots, \xi_r)$ such that the resulting closed-loop system is strictly dissipative with respect to the supply rate

$$q(w, y) = \tilde{\gamma}^2 w^2 - y^2 \,.$$

In particular, the construction underlying the various Lemmas used in the proof of Theorem 13.1.3 shows that $u = u(z, \xi_1, \ldots, \xi_r)$ is a linear feedback law. Thus, it is concluded that (i) must hold.

(iii) $\Leftrightarrow$ (iv). Add and subtract $YGG^{\mathrm{T}}Y$ to the left-hand side of (13.28) to obtain

$$YF + F^{\mathrm{T}}Y + \frac{1}{\gamma^2} YQQ^{\mathrm{T}}Y - YGG^{\mathrm{T}}Y + (YG + L^{\mathrm{T}})(YG + L^{\mathrm{T}})^{\mathrm{T}} < 0 \,.$$

Then, $Z = Y^{-1} > 0$ satisfies (13.29). Conversely, suppose Z satisfies (13.29). Set $Y = Z^{-1}$ and $L = -G^{\mathrm{T}}Y$, to obtain (13.28). ◁

The implication (ii) $\Rightarrow$ (iii) of this Lemma shows that, in the case of linear systems, the basic condition of Theorem 13.1.3, namely the existence of a function $V(z)$ (in this case a quadratic form $z^{\mathrm{T}}Yz$) and a function $v(z)$ (in this case a linear function Lz) which satisfy the inequality (13.20) (equivalent, in this case, to an inequality of the form (13.28)) is also a *necessary* condition for the solution of the problem of Disturbance Attenuation with Stability. As a consequence, the optimal value of the achievable disturbance attenuation is precisely the infimum of all possible values of γ for which the inequality (13.28) can be fulfilled. This Lemma also shows that, in this specific setup, the quadratic matrix inequality (13.28) can actually be replaced by a simpler linear matrix inequality. As a matter of fact, the Lemma shows that the existence of a symmetric solution $Z > 0$ of the Sylvester inequality (13.29) is a necessary and sufficient condition for the existence of a linear feedback law yielding internal stability and a level $\tilde{\gamma} < \gamma$ of attenuation between the disturbance input w and the output y in system (13.27). Thus, again, the optimal value of the achievable disturbance attenuation coincides with the infimum of all possible values of γ for which the inequality (13.29) can be fulfilled.

A further simplification is possible. Without loss of generality suppose that the z-subsystem of (13.27), after a suitable change of coordinates, is put in the form

$$\begin{aligned} \dot z_1 &= F_1 z_1 + G_1\xi_1 + Q_1 w \\ \dot z_2 &= F_2 z_2 + G_2\xi_1 + Q_2 w \\ \dot z_3 &= F_3 z_3 + G_3\xi_1 + Q_3 w \end{aligned} \tag{13.31}$$

where $\sigma(F_1) \subset \mathbb{C}^-$, $\sigma(F_2) \subset \mathbb{C}^0$ and $\sigma(F_3) \subset \mathbb{C}^+$. Then, it is easy to realize that the following holds.

Lemma 13.2.2. *There exists a symmetric matrix $Z > 0$ such that (13.29) holds if and only if there exists a symmetric matrix $\tilde Z > 0$ rendering*

$$\tilde F\tilde Z + \tilde Z\tilde F^{\mathrm T} + \frac{1}{\gamma^2}\tilde Q\tilde Q^{\mathrm T} - \tilde G\tilde G^{\mathrm T} < 0\,, \tag{13.32}$$

where

$$\tilde F = \begin{pmatrix} F_2 & 0 \\ 0 & F_3\end{pmatrix}, \quad \tilde Q = \begin{pmatrix} Q_2 \\ Q_3\end{pmatrix}, \quad \tilde G = \begin{pmatrix} G_2 \\ G_3\end{pmatrix}.$$

Proof. Suppose (13.29) holds and set

$$Z = \begin{pmatrix} Z_1 & S \\ S^{\mathrm T} & \tilde Z\end{pmatrix},$$

where $\dim(Z_1) = \dim(F_1)$. Then $\tilde Z > 0$ necessarily satisfies (13.32). Conversely, suppose the latter holds. Set

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & \tilde Z\end{pmatrix},$$

where the matrix Z_1 has to be determined. Rewrite the left-hand side of (13.29) as

$$\begin{pmatrix} M_1 & N \\ N^{\mathrm T} & \tilde M\end{pmatrix}$$

where

$$\tilde M = \tilde F\tilde Z + \tilde Z\tilde F^{\mathrm T} + \frac{1}{\gamma^2}\tilde Q\tilde Q^{\mathrm T} - \tilde G\tilde G^{\mathrm T} < 0$$

and

$$M_1 = F_1 Z_1 + Z_1 F_1^{\mathrm T} + \frac{1}{\gamma^2}Q_1 Q_1^{\mathrm T} - G_1 G_1^{\mathrm T}\,,$$

and N does not contain Z_1. From the identity

$$\begin{aligned} &\begin{pmatrix} I & -N\tilde M^{-1} \\ 0 & I\end{pmatrix}\begin{pmatrix} M_1 & N \\ N^{\mathrm T} & \tilde M\end{pmatrix}\begin{pmatrix} I & 0 \\ -\tilde M^{-1}N^{\mathrm T} & I\end{pmatrix} \\ &= \begin{pmatrix} M_1 - N\tilde M^{-1}N^{\mathrm T} & 0 \\ 0 & \tilde M\end{pmatrix} \end{aligned} \tag{13.33}$$

it is seen that (13.29) holds if and only if $M_1 - N\tilde M^{-1}N^{\mathrm T}$ is negative definite, i.e.

$$F_1 Z_1 + Z_1 F_1^{\mathrm T} < -\frac{1}{\gamma^2}Q_1 Q_1^{\mathrm T} + G_1 G_1^{\mathrm T} + N\tilde M^{-1}N^{\mathrm T}\,.$$

Let $R > 0$ be any symmetric matrix such that

$$R > \frac{1}{\gamma^2}Q_1 Q_1^{\mathrm T} - G_1 G_1^{\mathrm T} - N\tilde M^{-1}N^{\mathrm T}\,.$$

Since $\sigma(F_1) \subset \mathbb{C}^-$, the unique solution Z_1 of the Lyapunov equation

$$F_1 Z_1 + Z_1 F_1^{\mathrm T} = -R$$

is positive definite. Such a Z_1 renders the upper-left block of (13.33) negative definite and this completes the proof. ◁

We see from this and from the previous Lemma that a level γ of disturbance attenuation can be achieved if and only if for this number γ the inequality (13.32) has a positive definite solution $\tilde Z$. This result can be used in order to compute an explicit expression of the infimum of all possible values of γ for which the problem of Disturbance Attenuation with Stability can be solved. To this end, define

$$\begin{aligned} U_3 &= \int_0^\infty e^{-F_3 s}(G_3 G_3{}^{\mathrm T})e^{-(F_3 s)^{\mathrm T}}ds \\ V_3 &= \int_0^\infty e^{-F_3 s}(Q_3 Q_3{}^{\mathrm T})e^{-(F_3 s)^{\mathrm T}}ds\,, \end{aligned} \tag{13.34}$$

where the integrals are well-defined because by hypothesis all eigenvalues of $-F_3$ are in $\mathbb{C}^-$. Then, we have the following result.

Proposition 13.2.3. *There exists a symmetric matrix $\tilde Z > 0$ which solves the inequality (13.32) if and only if*

$$\frac{1}{\gamma^2}V_3 < U_3 \tag{13.35}$$

and

$$\frac{1}{\gamma^2}x^\star Q_2 Q_2{}^{\mathrm T}x < x^\star G_2 G_2{}^{\mathrm T}x \tag{13.36}$$

for any (possibly complex) eigenvector x of $-F_2^{\mathrm T}$.

Proof. Rewrite the inequality in question as

$$\bar Y\begin{pmatrix} \bar F_1 & 0 \\ 0 & \bar F_2\end{pmatrix} + \begin{pmatrix} \bar F_1^{\mathrm T} & 0 \\ 0 & \bar F_2^{\mathrm T}\end{pmatrix}\bar Y + \begin{pmatrix} \bar Q_1 & \bar Q_{12} \\ \bar Q_{12}^{\mathrm T} & \bar Q_2\end{pmatrix} > 0\,, \tag{13.37}$$

with

$$\bar F_1 = -F_3^{\mathrm T}, \qquad \bar F_2 = -F_2^{\mathrm T}\,,$$

and

$$\bar{Q}_1 = -\frac{}{\gamma}_3 G_3^{\mathrm{T}}, \qquad \bar{Q}_2 = -\frac{1}{\gamma^2}Q_2Q_2^{\mathrm{T}} + G_2G_2^{\mathrm{T}}.$$

Specializing tohe conditions of Theorem 13.6.3, it is easy to see that a solutionts if and only if the unique solution Y_1 of

$$_1 + \bar{F}_1^{\mathrm{T}}Y_1 + \bar{Q}_1 = 0\,, \tag{13.38}$$

is positive definite.

$$x^*\bar{Q}_2x > 0$$

for any (possibly genvector x of $\bar{F}_2$. Now, the equation (13.38) has a unique soluti

$$= \int_0^\infty e^{\bar{F}_1^{\mathrm{T}}s}\bar{Q}_1e^{\bar{F}_1s}ds$$

which, in view of tlefinitions given, is positive definite if and only if (13.35) holds. Thndition is precisely condition (13.36). ◁

The first one olitions can be further simplified. To this end, observe that if (13 be fulfilled for some γ, the matrix U_3 needs to be invertible. In2) is equivalent to (13.29), which in turn is equivalent to (13.28er holds, for some $Y > 0$, L and γ, the matrix $F + GL$ must have nvalues in $\mathbb{C}^-$, because Y is a solution of the Lyapunov inequalit$L) + (F + GL)^{\mathrm{T}}Y < 0$. Thus, the pair F, G must be stabilizablview of the decomposition (13.31), where by hypothesis F_3 has aes in $\mathbb{C}^+$, shows that (F_3, G_3) needs to be a controllable pair. Aence of well-known properties, the matrix U_3 needs to be invertib

In view of this, (3.35) can be rewritten as

$$\gamma > \rho(U_3^{-1}V_3) \tag{13.39}$$

where the notation es the spectral radius of a matrix.

We can summarie discussion as follows. Set

$$\gamma^* := \max\left\{\rho(U_3^- \quad \max_{\text{: eigenvector of } -\bar{F}_2^{\mathrm{T}}}\left\{\frac{x^*Q_2Q_2{}^{\mathrm{T}}x}{x^*G_2G_2{}^{\mathrm{T}}x}\right\}\right\}. \tag{13.40}$$

The problem of Disttenuation with Stability is solvable for each $\gamma > \gamma^*$ and is not so $< \gamma^*$.

13.3 Disturbance Attenuation

We return now to the problem of Disturbance Attenuation with Stability for a nonlinear system of the special form (13.16), and we elaborate further on the possibility of fulfilling the main condition of Theorem 13.1.3, i.e. the existence of a smooth real-valued function $v(z)$, with $v(0) = 0$, and a smooth, real-valued, positive definite and proper function $V(z)$, such that

$$\frac{\partial V}{\partial z}f(z, v(z), w) \leq -\alpha(\|z\|) + \gamma^2w^2 - h^2(z, v(z)) \tag{13.41}$$

for all z and all w, where $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

Motivated by the analysis presented in the previous section for linear systems, we consider the case in which

$$h(z, \xi_1) = \xi_1$$

and we assume that the z-subsystem

$$\dot{z} = f(z, \xi_1, w),$$

can be split as

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, \xi_1, w) \\ \dot{z}_2 &= f_2(z_2, \xi_1, w)\,, \end{aligned} \tag{13.42}$$

where the z_1-subsystem is viewed as the "stable component" in the sense that $z_1 = 0$ is a globally asymptotically stable equilibrium for $z_2 = 0$, $\xi_1 = 0$, and $w = 0$, while the z_2-subsystem represents a possibly "unstable" but "stabilizable component" in the sense that for some smooth function $v(z_2)$, the subsystem $\dot{z}_2 = f_2(z_2, v(z_2), 0)$ has a globally asymptotically stable equilibrium at $z_2 = 0$. The decomposition in (13.42) can be viewed as a nonlinear version of the decomposition in (13.31). Our aim is to identify conditions under which the lower bound for the achievable level of disturbance attenuation is determined by properties of the *unstable* component of (13.42), namely the z_2-subsystem.

Lemma 13.3.1. *Consider the system (13.42). Suppose that:*

(i) *there exists a smooth real-valued function $V_1(z_1)$, that is positive definite and proper, such that*

$$\frac{\partial V_1}{\partial z_1}f_1(z_1, z_2, \xi_1, w) \leq -\alpha_1(\|z_1\|) + \gamma_0^2w^2 + \gamma_0^2\|z_2\|^2 + \gamma_0^2\xi_1^2 \tag{13.43}$$

for some class $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ and some positive real number γ_0,

(ii) *there exist a smooth real-valued function $v_2(z_2)$, with $v_2(0) = 0$, and a smooth real-valued function $V_2(z_2)$, that is positive definite and proper, such that*

$$\frac{\partial V_2}{\partial z_2}f_2(z_2, v_2(z_2), w) \leq -\alpha_2(\|z_2\|) + \gamma^2w^2 - v_2^2(z_2) \tag{13.44}$$

for some class $\mathcal{K}_\infty$ function $\alpha_2(\cdot)$ and some positive real number γ,

(iii) *for some $r_1 > 0$ and some constant a*

$$\frac{r^2}{\alpha_2(r)} \le a \quad \forall r \in [r_1, \infty).$$

Then for every $\varepsilon > 0$ there exist a smooth function $v(z)$, with $v(0) = 0$, and a real-valued, positive definite, proper function $V(z)$ such that

$$\frac{\partial V}{\partial z} f(z, v(z), w) \le -\alpha(\|z\|) + (\gamma + \varepsilon)^2 w^2 - v^2(z)$$

for all z and all w, where $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

Proof. Suppose condition (i) is fulfilled for some γ_0 and multiply both sides of (13.43) by ε^2/γ_0^2, where ε is any arbitrary positive number, to obtain a similar inequality, in which γ_0 is replaced by ε. This shows that, without loss of generality, we can assume that (13.43) holds with γ_0 replaced by an arbitrary number $\varepsilon > 0$.

Define

$$\bar{\alpha}_2(r) = \begin{cases} \alpha_2(r) & \text{if } \alpha_2(r) \le \varepsilon^2 r^2 \\ \varepsilon^2 r^2 & \text{if } \alpha_2(r) > \varepsilon^2 r^2 . \end{cases} \tag{13.45}$$

Since $\bar{\alpha}_2(\cdot)$ is a class $\mathcal{K}_\infty$ function that satisfies

$$\bar{\alpha}_2(r) \le \varepsilon^2 r^2, \quad -\alpha_2(r) \le -\bar{\alpha}_2(r) \tag{13.46}$$

this $\bar{\alpha}_2(\cdot)$ can replace $\alpha_2(\cdot)$ in (13.44).

Also,

$$\frac{r^2}{\bar{\alpha}_2(r)} = \begin{cases} \dfrac{r^2}{\alpha_2(r)} & \text{if } \alpha_2(r) \le \varepsilon^2 r^2 \\ 1/\varepsilon^2 & \text{if } \alpha_2(r) > \varepsilon^2 r^2 \end{cases} \tag{13.47}$$

which in turn implies, utilizing technical assumption (iii), that

$$\frac{r^2}{\bar{\alpha}_2(r)} \le \max\{a, 1/\varepsilon^2\} \quad \forall r \in [r_1, \infty). \tag{13.48}$$

So Lemma 10.5.1 can be applied to deduce the existence of a class $\mathcal{K}_\infty$ function $\tilde{\alpha}_1(\cdot)$ and a smooth, positive definite, proper function $\tilde{V}_1(z_1)$, satisfying

$$\frac{\partial \tilde{V}_1}{\partial z_1} f_1(z_1, z_2, \xi_1, w) \le -\tilde{\alpha}_1(\|z_1\|) + \frac{1}{2}\bar{\alpha}_2(|w|) + \frac{1}{2}\bar{\alpha}_2(\|z_2\|) + \frac{1}{2}\bar{\alpha}_2(|\xi_1|). \tag{13.49}$$

Let $\tilde{V}_2(z_2) = \left(1 + \frac{\varepsilon^2}{2}\right) V_2(z_2)$. Hence, (13.44) yields

$$\begin{aligned} &\frac{\partial \tilde{V}_2}{\partial z_2} f_2(z_2, v_2(z_2), w) \\ &\quad \le -\left(1 + \tfrac{\varepsilon^2}{2}\right)\bar{\alpha}_2(\|z_2\|) + \gamma^2\left(1 + \tfrac{\varepsilon^2}{2}\right)|w|^2 - \left(1 + \tfrac{\varepsilon^2}{2}\right)|v_2(z_2)|^2 . \end{aligned} \tag{13.50}$$

Let $V(z) = \tilde{V}_1(z_1) + \tilde{V}_2(z_2)$. Then,

$$\begin{aligned} &\frac{\partial V}{\partial z} f(z, v_2(z), w) \\ &\le -\tilde{\alpha}_1(\|z_1\|) + \tfrac{1}{2}\bar{\alpha}_2(|w|) + \tfrac{1}{2}\bar{\alpha}_2(\|z_2\|) + \tfrac{1}{2}\bar{\alpha}_2(|v_2(z_2)|) \\ &\quad -\left(1 + \tfrac{\varepsilon^2}{2}\right)\bar{\alpha}_2(\|z_2\|) + \gamma^2\left(1 + \tfrac{\varepsilon^2}{2}\right)|w|^2 - \left(1 + \tfrac{\varepsilon^2}{2}\right)|v_2(z_2)|^2 \\ &\le -\tilde{\alpha}_1(\|z_1\|) + \tfrac{1}{2}\bar{\alpha}_2(|w|) - \tfrac{1}{2}\bar{\alpha}_2(\|z_2\|) + \tfrac{1}{2}\bar{\alpha}_2(|v_2(z_2)|) \\ &\quad -\tfrac{\varepsilon^2}{2}\bar{\alpha}_2(\|z_2\|) + \gamma^2\left(1 + \tfrac{\varepsilon^2}{2}\right)|w|^2 - \left(1 + \tfrac{\varepsilon^2}{2}\right)|v_2(z_2)|^2 \\ &\le -\tilde{\alpha}_1(\|z_1\|) + \tfrac{\varepsilon^2}{2}|w|^2 - \tfrac{1}{2}\bar{\alpha}_2(\|z_2\|) + \tfrac{\varepsilon^2}{2}|v_2(z_2)|^2 \\ &\quad +\gamma^2\left(1 + \tfrac{\varepsilon^2}{2}\right)|w|^2 - \left(1 + \tfrac{\varepsilon^2}{2}\right)|v_2(z_2)|^2 \\ &\le -\tilde{\alpha}_1(\|z_1\|) - \tfrac{1}{2}\bar{\alpha}_2(\|z_2\|) + \left[\gamma^2 + \tfrac{\varepsilon^2}{2}(1 + \gamma^2)\right]|w|^2 - |v_2(z_2)|^2 . \end{aligned}$$

This concludes the proof. ◁

Note that condition (i) expresses a special property of input-to-state stability for the first subsystem of (13.42), viewed as a system with state z_1 and inputs w, z_2 and ξ_1. The lemma just proven shows that this subsystem has no influence in determining the lower bound for the achievable level of disturbance attenuation. Note also that value of γ_0 which renders (13.43) satisfied is, in this context, immaterial.

The result described in this Lemma is, to some extent, a counterpart, for nonlinear systems modeled by equations of the form (13.16) and in which the z-subsystem admits a decomposition as in (13.42), of the result expressed by Lemma 13.2.2, which relates the achievable level of disturbance attenuation to a property of the "unstable" component of the z-subsystem. We will pursue later, in Section 13.5, the issue of seeking an explicit bound for γ (which would provide a sort of counterpart of the bound determined, for linear systems, at the end of the previous section). In the meanwhile, we discuss the special case in which the problem of Disturbance Attenuation, with Stability, can be solved for any arbitrarily small value of γ.

13.4 Almost Disturbance Decoupling

Consider the following design problem. When is it possible, for a given system of the form (13.6), to find, for *any* (hence arbitrarily small) value of γ, a feedback law (which indeed will depend on the chosen γ) which solves the problem of Disturbance Attenuation with Stability? This situation is often referred to as the *Problem of Almost Disturbance Decoupling* (with Stability).

The analysis presented in section 13.2 immediately yields necessary and sufficient conditions for the solvability of such a problem in the case of a *linear* system. In fact, from Proposition 13.2.3, we see that it is possible to solve the problem of Disturbance Attenuation, with Stability, for any arbitrarily small value of γ, if and only if the number γ^* given by (13.40) is 0, and this occurs if and only if $V_3 = 0$, which in turn implies $Q_3 = 0$, and ${Q_2}^{\mathrm{T}}x = 0$ for any eigenvector x of $-{F_2}^{\mathrm{T}}$. In other words, Almost Disturbance Decoupling with Stability is possible if and only if the decomposition (13.31) has the special form

$$\begin{aligned} \dot{z}_1 &= F_1 z_1 + G_1\xi_1 + Q_1 w \\ \dot{z}_2 &= F_2 z_2 + G_2\xi_1 + Q_2 w \\ \dot{z}_3 &= F_3 z_3 + G_3\xi_1 \end{aligned} \tag{13.51}$$

in which $\sigma(F_1) \subset \mathbb{C}^-$, $\sigma(F_2) \subset \mathbb{C}^0$, $\sigma(F_3) \subset \mathbb{C}^+$, and every eigenvector x of $-{F_2}^{\mathrm{T}}$ is annihilated by ${Q_2}^{\mathrm{T}}$.

In the special case in which the matrix F has no eigenvalues in $\mathbb{C}^0$, the necessary and sufficient condition for Almost Disturbance Decoupling with Stability is simply that the unstable component of the z-subsystem is not affected at all by the disturbance w.

A similar situation for the class of nonlinear systems considered in this Chapter is the one in which the z_2-subsystem of (13.42) is independent of the disturbance w, namely system (13.42) has the following special form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, \xi_1, w) \\ \dot{z}_2 &= f_2(z_2, \xi_1)\,. \end{aligned} \tag{13.52}$$

In this case, we can derive from Lemma 13.3.1 the following result, which provides a sufficient condition for Almost Disturbance Decoupling with Stability.

Corollary 13.4.1. *Consider the system (13.52). Suppose that condition* (i) *of Lemma 13.3.1 holds and that*

(ii) *there exist a smooth real-valued function $v_2(z_2)$, with $v_2(0) = 0$, and a smooth real-valued function $V_2(z_2)$, that is positive definite and proper, such that*

$$\frac{\partial V_2}{\partial z_2} f_2(z_2, v_2(z_2)) + v_2^2(z_2) \le -\alpha_2(\|z_2\|) \tag{13.53}$$

for some class $\mathcal{K}_\infty$ function α_2.

Then for every $\gamma > 0$ there exist a smooth function $v(z)$, with $v(0) = 0$, and a real-valued, positive definite, proper function $V(z)$ such that

$$\frac{\partial V}{\partial z} f(z, v(z), w) \le -\alpha(\|z\|) + \gamma^2 w^2 - v^2(z)$$

for all z and all w, where $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

Proof. Let $\tilde{\alpha}_2(\cdot)$ be any class $\mathcal{K}_\infty$ function satisfying

$$\tilde{\alpha}_2(r) = \begin{cases} \alpha_2(r) & \text{for small } r \\ r^2/a^2 & \text{for large } r\,. \end{cases}$$

Note that $\tilde{\alpha}_2(r) = \mathcal{O}(\alpha_2(r))$ as $r \to 0^+$. Thus, as shown in the proof of Lemma 10.5.1, there exists a smooth function $q : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, positive and non decreasing, satisfying

$$q(V_2(z_2))\alpha_2(\|z_2\|) \ge \tilde{\alpha}_2(\|z_2\|)\,.$$

Without loss of generality, suppose $q(r) \ge 1$.

Define

$$\rho(r) = \int_0^r q(s)ds$$

and set $\widetilde{V}_2(z_2) = \rho(V_2(z_2))$. In view of condition (ii), this yields

$$\begin{aligned} \frac{\partial \widetilde{V}_2}{\partial z_2} f_2(z_2, v_2(z_2)) &= q(V_2(z_2))\frac{\partial V_2}{\partial z_2} f_2(z_2, v_2(z_2)) \\ &\le -\tilde{\alpha}_2(\|z_2\|) - q(V_2(z_2))|v_2(z_2)|^2 \\ &\le -\tilde{\alpha}_2(\|z_2\|) - |v_2(z_2)|^2\,. \end{aligned}$$

This shows that condition (ii) of Lemma 13.3.1 holds with $\gamma = 0$, for a function $\tilde{\alpha}_2(r)$ which by construction satisfies condition (iii) of Lemma 13.3.1. Thus, the result follows. ◃

To conclude the section, we consider now the particular case in which the function $f_2(z_2, \xi_1)$ is affine in ξ_1, i.e. the case in which system (13.52) assumes the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, \xi_1, w) \\ \dot{z}_2 &= f_2(z_2) + g_2(z_2)\xi_1\,. \end{aligned}$$

In this case, in view of Lemma 13.1.4, the inequality appearing in condition (ii) of Corollary 13.4.1 can be given the expression

$$\frac{\partial V_2}{\partial z_2} f_2(z_2) - \frac{1}{4}\left[\frac{\partial V_2}{\partial z_2} g_2(z_2)\right]^2 \le -\alpha_2(\|z_2\|)\,.$$

To simplify the notation, let the latter be rewritten as

$$\frac{\partial V}{\partial x} f(x) - \frac{1}{4}\left[\frac{\partial V}{\partial x} g(x)\right]^2 \le -\alpha(\|x\|)\,, \tag{13.54}$$

(note that this is the particular form assumed by the inequality (13.24) when $h(x) = 0$, $p(x) = 0$ and $r(x) = 1$) and observe that, if an inequality of this form holds, necessarily $V(x)$ must be such that

$$\frac{\partial V}{\partial x} g(x) = 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|)\,. \tag{13.55}$$

In other words, a function $V(x)$ which satisfies (13.54) is necessarily a *control Lyapunov function* (see section 9.4) for the system

$$\dot{x} = f(x) + g(x)u\,. \tag{13.56}$$

In view of this, the problem naturally arises of when, i.e. under what hypotheses, from the knowledge of a control Lyapunov function $V(x)$, it is possible to determine a function (perhaps different from $V(x)$ itself) which satisfies the inequality (13.54). This problem is partially answered in the following result.

Lemma 13.4.2. *Suppose $V(x)$ is a smooth control Lyapunov function for (13.56), i.e. a smooth real-valued positive definite and proper function satisfying (13.55) for some positive definite function $\alpha(\cdot)$. Suppose there exist a neighborhood S of $x = 0$ and a real number $K > 0$ such that*

$$\frac{\partial V}{\partial x}f(x) - K\Big[\frac{\partial V}{\partial x}g(x)\Big]^2 < 0 \tag{13.57}$$

for all $x \in S \setminus \{0\}$. Then there exists a smooth real-valued positive definite and proper function $U(x)$ satisfying, for all x,

$$\frac{\partial U}{\partial x}f(x) - \frac{1}{4}\Big[\frac{\partial U}{\partial x}g(x)\Big]^2 \le -\tilde{\alpha}(\|x\|) \tag{13.58}$$

for some positive definite function $\tilde{\alpha}(\cdot)$. If $\alpha(\cdot)$ in (13.55) is a class $\mathcal{K}_\infty$ function, then there exists a smooth real-valued positive definite and proper function $U(x)$ satisfying (13.58) for some class $\mathcal{K}_\infty$ function $\tilde{\alpha}(\cdot)$.

Proof. By hypothesis, the left-hand side of (13.57) is negative definite in a neighborhood of $x = 0$. Thus, after possibly having redefined the function $\alpha(\cdot)$ for which (13.55) holds, it can be observed that, for some $\bar{a} > 0$

$$\|x\| < \bar{a} \quad \Rightarrow \quad \frac{\partial V}{\partial x}f(x) - K\Big[\frac{\partial V}{\partial x}g(x)\Big]^2 \le -\alpha(\|x\|)\,. \tag{13.59}$$

Set

$$M = \{x \in \mathbb{R}^n : \frac{\partial V}{\partial x}f(x) + \alpha(\|x\|) \ge 0\}\,,$$

and observe that $\dfrac{\partial V}{\partial x}g(x) \ne 0$ on $M \setminus \{0\}$. Define, for any $a \in (0,\infty)$,

$$\sigma(a) = \max_{\{x:\|x\|=a\}\cap M} \frac{\dfrac{\partial V}{\partial x}f(x) + \alpha(\|x\|)}{\Big[\dfrac{\partial V}{\partial x}g(x)\Big]^2}\,.$$

By (13.59), $\sigma(a) \le K$ for small a. Let $\underline{\alpha}(\cdot)$ be any class $\mathcal{K}_\infty$ function satisfying $\underline{\alpha}(|x|) \le V(x)$. Let $q : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be any smooth, positive and non decreasing, function such that $q(r) > 4\sigma\circ\underline{\alpha}^{-1}(r)$. Then, by construction, at each $x \in M \setminus \{0\}$,

$$q(V(x)) > 4\sigma(\|x\|) \ge 4\frac{\dfrac{\partial V}{\partial x}f(x) + \alpha(\|x\|)}{\Big[\dfrac{\partial V}{\partial x}g(x)\Big]^2}\,. \tag{13.60}$$

Consider the class $\mathcal{K}_\infty$ function $\rho(\cdot)$ defined in the proof of Corollary 13.4.1 and set $U(x) = \rho(V(x))$. Then

$$\frac{\partial U}{\partial x}f(x) - \frac{1}{4}\Big[\frac{\partial U}{\partial x}g(x)\Big]^2 = q(V(x))\frac{\partial V}{\partial x}f(x) - \Big[\frac{q(V(x))}{2}\frac{\partial V}{\partial x}g(x)\Big]^2\,. \tag{13.61}$$

From this, it is easy to check that

$$\frac{\partial U}{\partial x}f(x) - \frac{1}{4}\Big[\frac{\partial U}{\partial x}g(x)\Big]^2 \le -q(V(x))\alpha(\|x\|)\,. \tag{13.62}$$

In fact, if $x \notin M$, then

$$q(V(x))\frac{\partial V}{\partial x}f(x) < -q(V(x))\alpha(\|x\|)\,,$$

while, if $x \in M \setminus \{0\}$, then (13.60) yields

$$-\Big[\frac{q(V(x))}{2}\frac{\partial V}{\partial x}g(x)\Big]^2 < -q(V(x))\frac{\partial V}{\partial x}f(x) - q(V(x))\alpha(\|x\|)\,.$$

Since $-q(V(x))\alpha(\|x\|) \le -[q\circ\underline{\alpha}](\|x\|)\alpha(\|x\|)$ and $[q\circ\underline{\alpha}](\cdot)\alpha(\cdot)$ is a positive definite function, (13.62) completes the proof. ◃

Addressing the problem of Almost Disturbance Decoupling with Stability, this result can be used in order to check the fulfillment of condition (ii) of Corollary 13.4.1. In fact, if the second equation of (13.52) is affine in ξ_1 and a control Lyapunov function is known which satisfies a condition of the form (13.57), condition (ii) of the Corollary holds.

From a different point of view, it may be worth observing that, if $V(x)$ is a control Lyapunov function for (13.56) which satisfies the special condition (13.57), then this system can be globally asymptotically stabilized by *smooth* feedback (recall – from section 9.4 – that, if no special assumptions are made, the knowledge of a control Lyapunov function implies stabilizability by means of a feedback law which is guaranteed to be smooth only on $\mathbb{R}^n \setminus \{0\}$).

Corollary 13.4.3. *If there exists a control Lyapunov function $V(x)$ satisfying the condition (13.57) of Lemma 13.4.2, then there exists a smooth real-valued positive definite and proper function $U(x)$ such that, for any $k \ge 1$, the smooth feedback law*

$$u(x) = -\frac{k}{4}\frac{\partial U}{\partial x}g(x) \tag{13.63}$$

globally asymptotically stabilizes the equilibrium $x = 0$ of (13.56).

Proof. For any $k \geq 1$

$$\frac{\partial U}{\partial x}[f(x) + g(x)u(x)] = \frac{\partial U}{\partial x}f(x) - \frac{k}{4}\Big[\frac{\partial U}{\partial x}g(x)\Big]^2 < 0$$

for all nonzero x, and this completes the proof. ◁

Remark 13.4.1. Note that a smooth stabilizing feedback for (13.56) may exist under hypotheses weaker than (13.57). For, suppose there exist a neighborhood S of $x = 0$ and a smooth function $k : S \to \mathbb{R}$, with $k(0) = 0$, such that

$$\frac{\partial V}{\partial x}f(x) + k(x)\frac{\partial V}{\partial x}g(x) < 0 \qquad (13.64)$$

for all $x \in S \backslash \{0\}$. Then, it is easy to show that there exists a smooth feedback law $u = u^*(x)$ which globally asymptotically stabilizes the equilibrium $x = 0$ of (13.56). In fact, in section 9.4 it was shown that, if $V(x)$ is a control Lyapunov function for (13.56), then there exists a smooth function $\bar{k} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\bar{k}(x)] < 0$$

for all nonzero x. Let $c : \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying

$$\begin{array}{ll} c(x) = 1 & \text{for } x \in S' \subset S \\ 0 \leq c(x) \leq 1 & \\ c(x) = 0 & \text{for } x \notin S \end{array}$$

where S' is some neighborhood of $x = 0$. Then

$$u^*(x) = c(x)k(x) + (1 - c(x))\bar{k}(x) .$$

defines a smooth function on $\mathbb{R}^n$. The feedback law $u = u^*(x)$ is such that

$$\begin{aligned} &\frac{\partial V}{\partial x}[f(x) + g(x)u^*(x)] \\ &\quad = [1 - c(x)]\frac{\partial V}{\partial x}[f(x) + g(x)\bar{k}(x)] + c(x)\frac{\partial V}{\partial x}[f(x) + g(x)k(x)] < 0 \end{aligned}$$

for all nonzero x. Thus, if $V(x)$ is a control Lyapunov function for (13.56) which satisfies (13.64), there exists a smooth feedback which globally asymptotically stabilizes the equilibrium $x = 0$ of (13.56). Note also that (13.57) trivially implies (13.64). However, (13.57) implies – as shown in the previous corollary – the existence of a smooth stabilizing feedback of the special form (13.63). ◁

13.5 An Estimate of the Minimal Level of Disturbance Attenuation

In this section, we address in part the issue of determining an estimate of the "minimal" value of γ for which an inequality of the form (13.44) can be solved. As explained in Section 13.3, such an estimate will provide also an estimate of the minimal level of disturbance attenuation achievable in a system of the form (13.17). For simplicity, we consider the case in which the underlying system

$$\dot{z}_2 = f_2(z_2, \xi_1, w)$$

is affine in both ξ_1 and w, i.e. in which system (13.42) assumes the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, \xi_1, w) \\ \dot{z}_2 &= f_2(z_2) + g_2(z_2)\xi_1 + p_2(z_2)w . \end{aligned}$$

In view of Lemma 13.1.4, the inequality (13.44) is equivalent to the inequality

$$\frac{\partial V_2}{\partial z_2}f_2(z_2) - \frac{1}{4}\Big[\frac{\partial V_2}{\partial z_2}g_2(z_2)\Big]^2 + \frac{1}{4\gamma^2}\Big[\frac{\partial V_2}{\partial z_2}p_2(z_2)\Big]^2 \leq -\alpha_2(\|z_2\|) .$$

which, in order to simplify the notation, will be rewritten as

$$\frac{\partial V}{\partial x}f(x) - \frac{1}{4}\Big[\frac{\partial V}{\partial x}g(x)\Big]^2 + \frac{1}{4\gamma^2}\Big[\frac{\partial V}{\partial x}p(x)\Big]^2 \leq -\alpha(\|x\|) , \qquad (13.65)$$

(note that this is the particular form assumed by the inequality (13.24) when $h(x) = 0$, and $r(x) = 1$).

In what follows, we address the problem of constructing a solution $V(x)$ of the inequality (13.65) from the knowledge of the solution of a similar inequality written for a system of lesser dimension. The price that we pay for this "reduction" of dimensionality , however, is that we cannot obtain a solution valid for all $x \in \mathbb{R}^n$, but valid only on some compact subset of $\mathbb{R}^n$. As we will see in a moment, this limitation is acceptable, from a certain point of view, if the set in question can be arbitrarily large. The approach is to some extent similar to the one that replaces the requirement of global stability by that of asymptotic stability with arbitrarily large basin of attraction. In the problem of asymptotic stabilization, the requirement was to obtain, via feedback, convergence to zero for all initial conditions in any prescribed (but arbitrarily large) compact set. In the problem of Disturbance Attenuation, the corresponding idea is to obtain, via feedback, a prescribed level γ of disturbance attenuation, ***for any bounded set*** of disturbances in $L_2[0, \infty)$.

In order to see that achieving disturbance attenuation for all disturbances in a bounded subset of $L_2[0, \infty)$ actually corresponds to solving a dissipation inequality for all x in a suitable compact subset of $\mathbb{R}^n$, consider again a system of the (general) form

$$\begin{aligned}\dot{x} &= f(x,w,u)\\ y &= h(x,u)\end{aligned}$$

and suppose that, for some smooth feedback law $u = u(x)$, the basic dissipation inequality for the estimation of the L_2 gain, namely

$$\frac{\partial V}{\partial x} f(x,w,u(x)) \le -\alpha(\|x\|) + \gamma^2 w^2 - h^2(x,u(x)) \,, \tag{13.66}$$

holds for all $w \in \mathbb{R}$, but only for all $x \in \Omega_{\gamma^2 K}$, where

$$\Omega_{\gamma^2 K} = \{x \in \mathbb{R}^n : V(x) \le \gamma^2 K\}$$

and $K > 0$ is a fixed number. Suppose also that the disturbance input satisfies

$$\int_0^\infty |w(\tau)|^2 d\tau \le K \,.$$

Then, it is easy to realize that any trajectory of

$$\begin{aligned}\dot{x} &= f(x,w,u(x))\\ y &= h(x,u(x))\end{aligned}$$

starting at $x(0) = 0$ remains in $\Omega_{\gamma^2 K}$ for all $t \ge 0$ and, consequently,

$$\int_0^\infty |y(\tau)|^2 d\tau \le \gamma^2 \int_0^\infty |w(\tau)|^2 d\tau \,. \tag{13.67}$$

In fact, integration of (13.66) along any of such trajectories yields

$$V(x(t)) \le \gamma^2 \int_0^t |w(\tau)|^2 d\tau - \int_0^t |y(\tau)|^2 d\tau \le \gamma^2 \int_0^t |w(\tau)|^2 d\tau \le \gamma^2 K \,,$$

which shows that $x(t)$ cannot leave $\Omega_{\gamma^2 K}$ and therefore (13.66) again yields (13.67). Moreover, inequality (13.66) also shows that, if $w = 0$, any trajectory with $x(0) \in \Omega_{\gamma^2 K}$ remains in $\Omega_{\gamma^2 K}$ for all $t \ge 0$ and converges to 0 as $t \to \infty$.

Therefore, having an input $u(x)$ which renders the inequality (13.66) fulfilled for all w and for all $x \in \Omega_{\gamma^2 K}$ guarantees that in the resulting closed-loop system:

(i) the equilibrium $x = 0$ is asymptotically stable, with basin of attraction which contains $\Omega_{\gamma^2 K}$,

(ii) for any disturbance input whose $L_2[0,\infty)$ norm does not exceed K, the L_2 gain between $w(\cdot)$ and $y(\cdot)$ does not exceed γ.

With this and all prior results developed in this Chapter in mind, we consider now the following problem. Let $\gamma > 0$ be a fixed number. Given any $K > 0$, find a smooth function $V(x)$, that is positive definite and proper, such that the inequality (13.65) holds for all x satisfying

$$V(x) \le \gamma^2 K \,.$$

Theorem 13.5.1. *Consider the system*

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,x_2) + p_1(x_1,x_2)w\\ \dot{x}_2 &= f_2(x_1,x_2) + p_2(x_1,x_2)w + u \,,\end{aligned} \tag{13.68}$$

with $x_1 \in \mathbb{R}^{n-1}$ and $x_2 \in \mathbb{R}$, in which $f_1(x_1,x_2)$, $f_2(x_1,x_2)$, $p_1(x_1,x_2)$, $p_2(x_1,x_2)$ are smooth functions and $f_1(0,0) = 0$, $f_2(0,0) = 0$.

Set

$$\delta(x_1,x_2) = 1 - \frac{p_2^2(x_1,x_2)}{\gamma^2}$$

and

$$\begin{aligned}f_1(x_1,x_2) &= f_{11}(x_1) + f_{12}(x_1,x_2)x_2\\ f_2(x_1,x_2) &= f_{21}(x_1) + f_{22}(x_1,x_2)x_2 \,,\end{aligned}$$

and

$$\begin{aligned}p_1^\star(x_1) &= p_1(x_1,0)\\ p_2^\star(x_1) &= p_2(x_1,0)\\ \delta^\star(x_1) &= \delta(x_1,0)\end{aligned}$$

and

$$f_{11}^\star(x_1) = \delta^\star(x_1)f_{11}(x_1) + \frac{1}{\gamma^2} f_{21}(x_1)p_2^\star(x_1)p_1^\star(x_1) \,.$$

Suppose that, for some $\bar{\epsilon} > 0$,

$$\delta(x_1,x_2) \ge \bar{\epsilon} \tag{13.69}$$

for all x_1, x_2, and that there exist a smooth positive definite and proper function $V_1(x_1)$, and a class $\mathcal{K}$ function $\alpha_1(\cdot)$, satisfying the inequality

$$\frac{\partial V_1}{\partial x_1} f_{11}^\star(x_1) + \frac{1}{4\gamma^2}\left(\frac{\partial V_1}{\partial x_1} p_1^\star(x_1)\right)^2 + (f_{21}(x_1))^2 \le -\alpha_1(\|x_1\|) \tag{13.70}$$

and such that, for some $c > 0$,

$$\max\left\{|f_{21}|^2, \left|f_{21}\frac{\partial V_1}{\partial x_1}\right|, \left|\frac{\partial V_1}{\partial x_1} f_{11}\right|, \left|\frac{\partial V_1}{\partial x_1}\right|^2\right\} \le c\alpha_1(\|x_1\|) \tag{13.71}$$

for all x_1.

Then, given any $K > 0$, there exist a smooth positive definite and proper function $V(x)$ and a class $\mathcal{K}$ function $\alpha(\cdot)$ such that

$$\frac{\partial V}{\partial x} f(x) - \frac{1}{4}\left[\frac{\partial V}{\partial x} g(x)\right]^2 + \frac{1}{4\gamma^2}\left[\frac{\partial V}{\partial x} p(x)\right]^2 \le -\alpha(\|x\|) \tag{13.72}$$

where $x = (x_1, x_2)$ and

$$f(x) = \begin{pmatrix} f_1(x_1,x_2)\\ f_2(x_1,x_2)\end{pmatrix}, \quad p(x) = \begin{pmatrix} p_1(x_1,x_2)\\ p_2(x_1,x_2)\end{pmatrix}, \quad g(x) = \begin{pmatrix} 0\\ 1\end{pmatrix},$$

for all $x \in \{x \in \mathbb{R}^n : V(x) \le \gamma^2 K\}$.

Proof. Fix $K > 0$. Let $V(x) = V_1(x_1) + \beta x_2^2$, $\beta \geq 1$. Let $\mathcal{B}_1$ be any compact set such that

$$\{x_1 \in \mathbb{R}^{n-1} : V_1(x_1) \leq \gamma^2 K\} \subset \mathcal{B}_1 ,$$

and let

$$\mathcal{B}_2 = \{x_2 \in \mathbb{R} : |x_2| \leq \gamma\sqrt{K}\} .$$

Then, it is easily seen that $V(x) \leq \gamma^2 K$ implies $x_1 \in \mathcal{B}_1$ and $x_2 \in \mathcal{B}_2$, for all $\beta \geq 1$. Thus,

$$\{x \in \mathbb{R}^n : V(x) \leq \gamma^2 K\} \subset \mathcal{B}_1 \times \mathcal{B}_2 .$$

Simple, but lengthy, manipulations show [1] that there exists a number β^* such that, for all $\beta > \beta^*$, the function $V(x)$ thus defined satisfies (13.72) for all $x \in \mathcal{B}_1 \times \mathcal{B}_2$. ◁

As anticipated, this Theorem reduces the problem of seeking a solution of the inequality (13.72) to the problem of seeking a solution of the auxiliary inequality (13.70). The latter can be interpreted as an inequality expressing the property that the $(n-1)$-dimensional auxiliary system

$$\begin{aligned} \dot{x}_1 &= f_{11}^\star(x_1) + p_1^\star(x_1)u \\ y &= f_{21}(x_1) , \end{aligned}$$

has an L_2-gain which is less than or equal to γ.

Remark 13.5.1. Note that the requirements that $\delta^\star(x_1) > 0$ and that function $V_1(x_1)$ satisfies the inequality

$$\frac{\partial V_1}{\partial x_1} f_{11}^\star(x_1) + \frac{1}{4\gamma^2}\Big(\frac{\partial V_1}{\partial x_1} p_1^\star(x_1)\Big)^2 + (f_{21}(x_1))^2 < 0 \qquad (13.73)$$

for $x_1 \neq 0$, imply that

$$\delta^\star(x_1)\frac{\partial V_1}{\partial x_1} f_{11}(x_1)$$

$$+ \left(f_{21}(x_1) + \frac{p_2^\star(x_1)}{2\gamma^2}\frac{\partial V_1}{\partial x_1} p_1^\star(x_1)\right)^2 + \frac{\delta^\star(x_1)}{4\gamma^2}\left(\frac{\partial V_1}{\partial x_1} p_1^\star(x_1)\right)^2 < 0$$

for $x_1 \neq 0$, i.e. in particular

$$\frac{\partial V_1}{\partial x_1} f_{11}(x_1) < 0 , \text{ for } x_1 \neq 0 . \qquad (13.74)$$

This shows that the asymptotic stability of the subsystem

$$\dot{x}_1 = f_{11}(x_1)$$

is a prerequisite for (13.70) to hold. ◁

[1] For more details, see Isidori *et al.* (1999).

Theorem 13.5.1 also contains the technical condition (13.71). However, it is easy to see that such a condition is immaterial if the class $\mathcal{K}$ function $\alpha_1(\cdot)$ which appears in (13.70) is quadratic. As a matter of fact, since all functions on the left-hand-side of (13.71) vanish at $x_1 = 0$ together with their first derivatives, it is readily seen that, for any compact subset $\mathcal{S} \subset \mathbb{R}^{n-1}$, there exists $c > 0$ such that

$$\max\left\{|f_{21}|^2, \left|f_{21}\frac{\partial V_1}{\partial x_1}\right|, \left|\frac{\partial V_1}{\partial x_1} f_{11}\right|, \left|\frac{\partial V_1}{\partial x_1}\right|^2\right\} \leq c\|x_1\|^2$$

for all $x_1 \in \mathcal{S}$.

This yields the following simplified version of Theorem 13.5.1.

Corollary 13.5.2. *Consider the system (13.68). Suppose that, for some $\bar{\epsilon} > 0$, $\delta(x_1, x_2) \geq \bar{\epsilon}$ for all x. Suppose that there exists a smooth positive definite and proper function $V_1(x_1)$ satisfying the inequality (13.70), with $\alpha_1(\|x_1\|) = \epsilon_1\|x_1\|^2$.*

Then, given any $K > 0$, there exist a smooth positive definite and proper function $V(x)$ and $a > 0$ such that

$$\frac{\partial V}{\partial x} f(x) - \frac{1}{4}\Big[\frac{\partial V}{\partial x} g(x)\Big]^2 + \frac{1}{4\gamma^2}\Big[\frac{\partial V}{\partial x} p(x)\Big]^2 \leq -a\|x\|^2$$

for all $x \in \{x \in \mathbb{R}^n : V(x) \leq \gamma^2 K\}$.

As an example of how the result of Theorem 13.5.1 can be successfully utilized in order to determine a lower estimate of the minimal value of achievable disturbance attenuation, consider the simple case in which $n = 2$.

Write

$$\begin{aligned} f_{11}(x_1) &= a_{11}(x_1)x_1 \\ f_{21}(x_1) &= a_{21}(x_1)x_1 . \end{aligned}$$

Let $V_1(x_1) = 2\int_0^{x_1} A(s)s ds$, where $A(s)$ is continuous, positive and bounded away from zero. This yields

$$\frac{\partial V_1}{\partial x_1} f_{11}(x_1) = 2A(x_1)a_{11}(x_1)x_1^2 .$$

In order to satisfy the necessary condition (13.74), assume that $a_{11}(x_1) < 0$. Requirement (13.70), with $\alpha_1(r) = \epsilon_1^2 r^2$, can be fulfilled when

$$\begin{aligned} &2A(x_1)\left(-\delta^\star(x_1)a_{11}(x_1) - a_{21}(x_1)p_1^\star(x_1)p_2^\star(x_1)\gamma^{-2}\right) \\ &\quad -\gamma^{-2}p_1^{\star 2}(x_1)A^2(x_1) - a_{21}^2(x_1) > \epsilon_1^2 . \end{aligned} \qquad (13.75)$$

A positive solution $A(x_1)$ to (13.75) can only be found if there are real, distinct, positive roots. This combined with the condition that $\delta^\star(x_1)$ must be positive yields the following *explicit* estimate for γ

$$\gamma > \frac{|p_1^\star(x_1)a_{21}(x_1)|}{2|a_{11}(x_1)|} + \frac{1}{2}\left|\frac{p_1^\star(x_1)a_{21}(x_1)}{|a_{11}(x_1)|} + 2p_2^\star(x_1)\right| + \frac{|p_1^\star(x_1)|}{|a_{11}(x_1)|}\epsilon_1 . \qquad (13.76)$$

13.6 L_2-gain Design for Linear Systems

In the previous sections, we have considered a number of special cases of the dissipation inequality (13.23), or of the Hamilton-Jacobi-type inequality (13.24), which is equivalent to (13.23) if $r(x) \neq 0$ for all x: the inequality (13.8), which corresponds to the special case in which $r(x) = 0$, the inequality (13.54), which corresponds to the special case in which $h(x) = 0$, $p(x) = 0$ and $r(x) = 1$, and the inequality (13.65), which corresponds to the special case in which $h(x) = 0$ and $r(x) = 1$. In this and in the next section we discuss the possibility of determining solutions of the dissipation inequality (13.23) in the more general case, not examined so far, in which both $h(x)$ and $r(x)$ are nonzero. In other words, we consider a system modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w \\ y &= \begin{pmatrix} h(x) \\ r(x)u \end{pmatrix} \end{aligned} \tag{13.77}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$, disturbance input $w \in \mathbb{R}^r$ and output $y \in \mathbb{R}^2$, with $f(0) = 0$, $h(0) = 0$ and $r(x) \neq 0$ for all x, and we seek a feedback law $u = u(x)$ rendering the system strictly dissipative with respect to the supply rate

$$q(w, y) = \gamma^2 w^2 - \|y\|^2 , \tag{13.78}$$

where γ is a fixed number, i.e. rendering the inequality (13.23) fulfilled, for some positive definite and proper $V(x)$, for all w and x.

For convenience, and also in order to present some material that will be used in the sequel, we examine first the case in which system (13.77) is a *linear* system and a *linear feedback* law is sought which renders the system dissipative with respect to the supply rate (13.78). Without any extra complication we can handle the case of a system in which $u \in \mathbb{R}^m$, $w \in \mathbb{R}^r$ and $y \in \mathbb{R}^{p+m}$, i.e. the case of a system modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ y &= \begin{pmatrix} Cx \\ Ru \end{pmatrix} \end{aligned} \tag{13.79}$$

where we assume that the $m \times m$ matrix R is *nonsingular.*

First of all, it is important to observe that, from Theorem 10.9.1 and Lemma 13.1.4 it is easy to deduce the following result.

Lemma 13.6.1. *Consider the linear system (13.79) and let γ be a fixed number. The following are equivalent:*

(i) *there exists a linear feedback law which renders the resulting closed-loop system strictly dissipative with respect to the supply rate*

$$q(w, y) = \tilde{\gamma}^2 \|w\|^2 - \|y\|^2$$

for some $\tilde{\gamma} < \gamma$,

(ii) *there exist a matrix K and a symmetric matrix $X > 0$ such that*

$$(A+BK)^{\mathrm{T}}X + X(A+BK) + C^{\mathrm{T}}C + K^{\mathrm{T}}R^{\mathrm{T}}RK + \frac{1}{\gamma^2}XPP^{\mathrm{T}}X < 0 , \tag{13.80}$$

(iii) *there exists a symmetric matrix $X > 0$ such that*

$$A^{\mathrm{T}}X + XA + C^{\mathrm{T}}C - XB(R^{\mathrm{T}}R)^{-1}B^{\mathrm{T}}X + \frac{1}{\gamma^2}XPP^{\mathrm{T}}X < 0 . \tag{13.81}$$

Proof. The implication (i)$\Leftrightarrow$(ii) is a direct consequence of Theorem 10.9.1. Moreover, the proof of Lemma 13.1.4 shows that if (13.80) holds for some K and some symmetric matrix $X > 0$, and R is a nonsingular matrix, then necessarily X satisfies (13.81). Conversely, if a symmetric matrix $X > 0$ satisfies (13.81), then this X satisfies (13.80) with

$$K = -(R^{\mathrm{T}}R)^{-1}B^{\mathrm{T}}X .$$

This completes the proof. $\triangleleft$

Thus, the problem we are considering can be solved if and only if there exists a symmetric matrix $X > 0$ which satisfies the matrix Riccati inequality (13.81). In what follows, we describe a set of necessary and sufficient conditions, all of which are easily verifiable, for the existence of the solution of a matrix inequality of this type. For convenience, in order to have inequality in question rewritten in a standard form, set

$$F = -A^{\mathrm{T}}, \qquad G = C^{\mathrm{T}}, \qquad Q = B(R^{\mathrm{T}}R)^{-1}B^{\mathrm{T}} - \frac{1}{\gamma^2}PP^{\mathrm{T}}$$

and

$$Y = X^{-1}$$

(recall that the required solution X of (13.81) is nonsingular). Then, to find a solution $X > 0$ of the inequality (13.81) reduces to find a solution $Y > 0$ of an inequality of the form

$$YF + F^{\mathrm{T}}Y - YGG^{\mathrm{T}}Y + Q > 0 . \tag{13.82}$$

Now, if the pair (F, G) is *stabilizable*, the existence of a solution of this inequality can easily be determined by means of the following Lemma [2], which was already used in the proof of Theorem 10.9.1, to prove the equivalence of different conditions ensuring that the L_2 gain does not exceed a fixed number $\gamma > 0$.

[2] For a proof, see Knobloch *et al.* (1993), Appendix A.

Lemma 13.6.2. *Suppose the pair (F, G) is stabilizable. The following three properties are equivalent:*

(i) *there exists a symmetric matrix Y solving the inequality (13.82),*

(ii) *there exists a symmetric matrix Y^- such that*

$$Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q = 0\,, \quad \sigma(F - GG^{\mathrm{T}}Y^-) \subset \mathbb{C}^- \tag{13.83}$$

(iii) *the Hamiltonian matrix*

$$H = \begin{pmatrix} F & -GG^{\mathrm{T}} \\ -Q & -F^{\mathrm{T}} \end{pmatrix} \tag{13.84}$$

has no eigenvalues on the imaginary axis.

Suppose one of these conditions holds. Then, the solution Y^- in (ii) *is unique and is such that the stable invariant subspace of H can be expressed as*

$$\mathcal{V}_s = \operatorname{span}\begin{pmatrix} I \\ Y^- \end{pmatrix}. \tag{13.85}$$

Moreover, any solution Y of (13.82) is such that $Y < Y^-$. Finally, there exist a number $\varepsilon_0 > 0$ and a family of symmetric matrices Y_ε, defined for $\varepsilon \in (0, \varepsilon_0)$ and continuously depending on ε, such that $\lim_{\varepsilon \to 0} Y_\varepsilon = Y^-$, which satisfy

$$Y_\varepsilon F + F^{\mathrm{T}}Y_\varepsilon - Y_\varepsilon GG^{\mathrm{T}}Y_\varepsilon + Q > 0\,.$$

Thus, if the pair (F, G) is stabilizable, a positive definite symmetric solution Y of (13.82) exists if and only if the Hamiltonian matrix H has no eigenvalues and its stable invariant subspace can be expressed in the form (13.85) with Y^- a positive definite matrix.

In the more general case in which the pair (F, G) is *not stabilizable*, a more elaborate test is needed, which can be described as follows. First of all, observe that the inequality remains unchanged under a transformation of the form

$$F \mapsto SFS^{-1}, \qquad G \mapsto SG, \qquad Q \mapsto (S^{-1})^{\mathrm{T}}QS^{-1}, \qquad Y \mapsto (S^{-1})^{\mathrm{T}}YS^{-1}\,,$$

with S a nonsingular matrix.

Then, observe also that S can be chosen in such a way that

$$SFS^{-1} = \begin{pmatrix} F_1 & G_1K_2 & G_1K_3 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{pmatrix}, \qquad SG = \begin{pmatrix} G_1 \\ 0 \\ 0 \end{pmatrix}, \tag{13.86}$$

in which the pair (F_1, G_1) is stabilizable, the eigenvalues of F_2 are in $\mathbb{C}^0$, the eigenvalues of F_3 are in $\mathbb{C}^+$ and K_2, K_3 are suitable matrices. To this end, take any transformation highlighting the decomposition into controllable/uncontrollable parts

$$TFT^{-1} = \begin{pmatrix} F_c & F_{cu} \\ 0 & F_u \end{pmatrix}, \qquad TG = \begin{pmatrix} G_c \\ 0 \end{pmatrix}.$$

Then, using the fact that (F_c, G_c) is controllable, choose a matrix L such that the none of the eigenvalues of $F_c + G_cL$ is an eigenvalue of F_u and solve for M the Sylvester equation

$$F_{cu} + MF_u = (F_c + G_cL)M\,.$$

Then, it easily seen that

$$\begin{pmatrix} I & M \\ 0 & I \end{pmatrix}\begin{pmatrix} F_c & F_{cu} \\ 0 & F_u \end{pmatrix}\begin{pmatrix} I & -M \\ 0 & I \end{pmatrix} = \begin{pmatrix} F_c & G_cLM \\ 0 & F_u \end{pmatrix}.$$

In this way, we have found a transformation $\overline{T}$ such that

$$\overline{T}F\overline{T}^{-1} = \begin{pmatrix} F_c & G_cK \\ 0 & F_u \end{pmatrix}, \qquad \overline{T}G = \begin{pmatrix} G_c \\ 0 \end{pmatrix}, \tag{13.87}$$

for some K. Now, split the Jordan form of F_u into three blocks, corresponding to eigenvalues in $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$. Grouping together the first one of these blocks with the upper block of the decomposition (13.87) yields exactly a decomposition of the form (13.86) with the required properties.

Let Q_1, Q_{12} and Q_2 be the (1,1), (1,2) and (2,2) blocks of the corresponding partition of $(S^{-1})^{\mathrm{T}}QS^{-1}$. Then, the following Theorem [3] provides a general test for the solvability of the inequality (13.82).

Theorem 13.6.3. *There exists a symmetric matrix Y solving the inequality (13.82) if and only if there exist a symmetric matrix Y_1 and a matrix Y_{12} such that*

$$\begin{aligned} Y_1F_1 + F_1^{\mathrm{T}}Y_1 - Y_1G_1G_1^{\mathrm{T}}Y_1 + Q_1 = 0\,, \\ \sigma(F_1 - G_1G_1^{\mathrm{T}}Y_1) \subset \mathbb{C}^- \end{aligned} \tag{13.88}$$

$$(F_1 - G_1G_1^{\mathrm{T}}Y_1)Y_{12} + Y_{12}F_2 + Y_1G_1K_2 + Q_{12} = 0 \tag{13.89}$$

$$x^*[Q_2 + K_2^{\mathrm{T}}K_2 - (K_2 - G_1^{\mathrm{T}}Y_{12})^{\mathrm{T}}(K_2 - G_1^{\mathrm{T}}Y_{12})]x > 0 \tag{13.90}$$

hold for any (possibly complex) eigenvector x of F_2.

There exists a symmetric matrix $Y > 0$ solving the inequality (13.82) if and only if there exist a symmetric matrix $Y_1 > 0$ and a matrix Y_{12} such that the conditions above hold.

As anticipated, the verification of the conditions in this Theorem is not difficult. First of all, observe that, since by hypothesis the pair (F_1, G_1) is stabilizable, one can use Lemma 13.6.2 to check the existence of Y_1 satisfying (13.88). This occurs if and only if the Hamiltonian matrix

[3] For a proof, see Scherer (1992).

$$\begin{pmatrix} F_1 & -G_1G_1^{\mathrm{T}} \\ -Q_1 & -F_1^{\mathrm{T}} \end{pmatrix}$$

has no eigenvalues on the imaginary axis. If this is the case, then Y_1 is unique and can be computed by expressing the stable invariant subspace of this Hamiltonian matrix in the form

$$\mathcal{V}_s = \text{span}\begin{pmatrix} I \\ Y_1 \end{pmatrix}.$$

Then, since $(F_1 - G_1G_1^{\mathrm{T}}Y_1)$ has all eigenvalues in $\mathbb{C}^-$ and F_2 has all eigenvalues in $\mathbb{C}^0$, the Sylvester equation (13.89) has a unique solution Y_{12}. Using this matrix Y_{12} the third condition (13.90) must be checked, which involves only a finite number of tests.

13.7 Global L_2-gain Design for a Class of Nonlinear Systems

We return now to the case of a nonlinear system modeled by equations of the form (13.77), with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$, disturbance input $w \in \mathbb{R}$ and output $y \in \mathbb{R}^2$, with $f(0) = 0$, $h(0) = 0$ and $r(x) \neq 0$ for all x, and we seek a feedback law $u = u(x)$ rendering the system strictly dissipative with respect to the supply rate (13.78), where γ is a fixed number, i.e. rendering the inequality (13.23) fulfilled, for some positive definite and proper $V(x)$, for all w and x. As shown before, this is the same as finding a pair $\{u(x), V(x)\}$ which renders the inequality

$$\frac{\partial V(x)}{\partial x}[f(x)+g(x)u(x)]+\frac{1}{4\gamma^2}\left[\frac{\partial V(x)}{\partial x}p(x)\right]^2+h^2(x)+r^2(x)u^2(x) \leq -\alpha(\|x\|) \tag{13.91}$$

fulfilled for all x.

To find a pair $\{u(x), V(x)\}$ which solves this inequality for all x in a (possibly small) neighborhood of the point $x = 0$ is not terribly difficult, and can be accomplished by determining the solution of a suitable Riccati inequality constructed from the parameters which characterize the linear approximation of the system (13.77) at this point.

In fact, let

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ y &= \begin{pmatrix} Cx \\ Ru \end{pmatrix} \end{aligned} \tag{13.92}$$

denote the linear approximation, at $x = 0$, of the system in question, where

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0}, \quad B = g(0), \quad P = p(0)$$
$$C = \left[\frac{\partial h}{\partial x}\right]_{x=0}, \quad R = r(0).$$

Suppose there exist a $1 \times n$ matrix K and an $n \times n$ symmetric and positive definite matrix X which render the strict Riccati inequality

$$X(A+BK) + (A+BK)^{\mathrm{T}}X + \frac{1}{\gamma^2}XPP^{\mathrm{T}}X + C^{\mathrm{T}}C + R^2K^{\mathrm{T}}K < 0 \tag{13.93}$$

satisfied. Then, an elementary calculation shows that the feedback law $u(x) = Kx$ and the positive definite and proper function $V(x) = x^{\mathrm{T}}Xx$ satisfy (13.91) for all x in a neighborhood of $x = 0$, with $\alpha(r) = cr^2$ and $c > 0$.

Obviously, the *linear* feedback law $u(x) = Kx$ and the *quadratic* function $V(x) = x^{\mathrm{T}}Xx$ thus determined fail, in general, to provide a *global* solution of (13.91). One may conjecture, however, that in some special cases the information obtained from the solution of the Riccati inequality (13.93) suffices to determine a feedback law (which we indeed expect to be a *nonlinear* function of x) providing a global solution of the inequality (13.91) for some suitable $V(x)$. The purpose of this section is to show that this conjecture is true for a special class of nonlinear systems, if the coefficient $r(x)$ which weights the control u in the output y of (13.77) satisfies certain bounds. More specifically, we will prove that, for that class of nonlinear systems, starting from a solution pair $\{K, X\}$ of the strict Riccati inequality (13.93), it is always possible to construct a pair $\{u(x), V(x)\}$ which solves the inequality (13.91) if $|r(x)|$ does not exceed a suitable bound $r^*(x)$, a continuous function which is equal to R in a neighborhood of $x = 0$ (as it should be) but possibly decays to zero as x grows.

More precisely, suppose the system in question can be described by equations of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + p_1(x_1)w \\ \dot{x}_2 &= x_3 + p_2(x_1, x_2)w \\ &\cdots \\ \dot{x}_{n-1} &= x_n + p_{n-1}(x_1, x_2, \ldots, x_{n-1})w \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + u + p_n(x_1, x_2, \ldots, x_n)w \end{aligned} \tag{13.94}$$

in which $p_1(x_1), \ldots, p_n(x_1, x_2, \ldots, x_n)$, $f_n(x_1, x_2, \ldots, x_n)$ are smooth functions, $f_n(0, 0, \ldots, 0) = 0$, and

$$h(x) = x_1.$$

For convenience this system will be represented in the form

$$\begin{aligned} \dot{x} &= Ex + Bf_n(x) + Bu + p(x)w \\ y &= Cx \end{aligned} \tag{13.95}$$

where

$$E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad p(x) = \begin{pmatrix} p_1(x_1) \\ p_2(x_1,x_2) \\ \cdots \\ p_n(x_1,x_2,\ldots,x_n) \end{pmatrix},$$

and its linear approximation of (13.95) at $x = 0$ in the form

$$\begin{aligned} \dot{x} &= (E+BF)x + Bu + Pw \\ y &= Cx, \end{aligned} \tag{13.96}$$

where

$$F = \left[\frac{\partial f_n}{\partial x}\right]_{x=0}, \qquad P = p(0).$$

We will show that, if the Riccati inequality

$$X(E+BF+BK)+(E+BF+BK)^{\mathrm{T}}X+\frac{1}{\gamma^2}XPP^{\mathrm{T}}X+C^{\mathrm{T}}C+R^2K^{\mathrm{T}}K<0 \tag{13.97}$$

is solved by some pair $\{K, X\}$, it is possible to find a pair $\{u(x), V(x)\}$ which solves the Hamilton-Jacobi inequality

$$\frac{\partial V(x)}{\partial x}[Ex+Bf_n(x)+Bu(x)]+\frac{1}{4\gamma^2}\left[\frac{\partial V(x)}{\partial x}p(x)\right]^2+h^2(x)+r^2(x)u^2(x)\le 0 \tag{13.98}$$

provided $|r(x)|$ is conveniently bounded. To this end, we need a few preliminary results.

Set, for $i = 1, \ldots, n$,

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{i\times i}, \qquad B_i = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}_{i\times 1}$$

$$C_i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{1\times i}, \qquad P_i = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_i \end{pmatrix},$$

and observe that $A_n = E, B_n = B, C_n = C, P_n = P$.

It is easy to prove – by induction – that, if the inequality (13.97) holds, then, for each $i = n-1, n-2 \ldots, 1$, there exist K_i and symmetric matrices $X_i > 0$ and $Q_i > 0$ such that

$$X_i(A_i+B_iK_i)+(A_i+B_iK_i)^{\mathrm{T}}X_i+\frac{1}{\gamma^2}X_iP_iP_i^{\mathrm{T}}X_i+C_i^{\mathrm{T}}C_i=-Q_i\,. \tag{13.99}$$

This depends on the following Lemma, whose proof uses arguments already used to prove the implication (ii) ⇒ (iii) in Lemma 13.2.1 and is not repeated here.

Lemma 13.7.1. *Suppose there exist a $1\times i$ matrix K_i and $i\times i$ matrices $X_i = X_i^{\mathrm{T}} > 0$ and $Q_i = Q_i^{\mathrm{T}} > 0$ such that (13.99) holds. Let K_{i-1} be the unique solution of*

$$\begin{pmatrix} 0_{1\times(i-1)} & 1 \end{pmatrix} X_i \begin{pmatrix} I_{(i-1)\times(i-1)} \\ K_{i-1} \end{pmatrix} = 0_{1\times(i-1)}\,.$$

Define

$$X_{i-1} = \begin{pmatrix} I_{(i-1)\times(i-1)} & K_{i-1}^{\mathrm{T}} \end{pmatrix} X_i \begin{pmatrix} I_{(i-1)\times(i-1)} \\ K_{i-1} \end{pmatrix}$$

$$Q_{i-1} = \begin{pmatrix} I_{(i-1)\times(i-1)} & K_{i-1}^{\mathrm{T}} \end{pmatrix} Q_i \begin{pmatrix} I_{(i-1)\times(i-1)} \\ K_{i-1} \end{pmatrix}.$$

Then $K_{i-1}, X_{i-1} > 0, Q_{i-1} > 0$ are such that

$$\begin{aligned} &X_{i-1}(A_{i-1}+B_{i-1}K_{i-1})+(A_{i-1}+B_{i-1}K_{i-1})^{\mathrm{T}}X_{i-1} \\ &\quad+\frac{1}{\gamma^2}X_{i-1}P_{i-1}P_{i-1}^{\mathrm{T}}X_{i-1}+C_{i-1}^{\mathrm{T}}C_{i-1}=-Q_{i-1}\,. \end{aligned} \tag{13.100}$$

For any C^1 function $v(x)$ which vanishes at $x = 0$, let $v^{[1]}(x)$ denote its first-order approximation at $x = 0$, i.e.

$$v^{[1]}(x) = \left[\frac{\partial v}{\partial x}\right]_{x=0} x\,.$$

Likewise, for any C^2 real-valued function $V(x)$ which vanishes at $x = 0$ together with all its first partial derivatives, let $V^{[2]}(x)$ denote its second-order approximation at $x = 0$, i.e.

$$V^{[2]}(x) = x^{\mathrm{T}}\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}}\right]_{x=0} x\,.$$

Set also

$$\mathbf{x}_i = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_i \end{pmatrix},$$

and

$$\mathbf{p}_i(\mathbf{x}_i) = \begin{pmatrix} p_1(x_1) \\ p_2(x_1,x_2) \\ \cdots \\ p_i(x_1,\ldots,x_i) \end{pmatrix}.$$

Then, the following result holds.

Lemma 13.7.2. *Suppose $K_i, X_i = X_i^{\mathrm{T}} > 0, Q_i = Q_i^{\mathrm{T}} > 0$ are such that (13.99) holds and let $K_{i-1}, X_{i-1}, Q_{i-1}$ be defined as in Lemma 13.7.1. Suppose there exist*

(i) *a positive definite and proper smooth function $V_{i-1}(\mathbf{x}_{i-1})$ such that*

$$V_{i-1}^{[2]}(\mathbf{x}_{i-1}) = \mathbf{x}_{i-1}^{\mathrm{T}} X_{i-1} \mathbf{x}_{i-1} ,$$

(ii) *a set of smooth functions $\varphi_1(x_1), \varphi_2(\mathbf{x}_2), \ldots, \varphi_{i-2}(\mathbf{x}_{i-2})$ vanishing at zero together with their first partial derivatives,*

(iii) *a smooth function $v_{i-1}(\mathbf{x}_{i-1})$ vanishing at zero and such that $v_{i-1}^{[1]}(\mathbf{x}_{i-1}) = K_{i-1}\mathbf{x}_{i-1}$,*

satisfying

$$\begin{aligned} &\frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}(A_{i-1}\mathbf{x}_{i-1} + B_{i-1}v_{i-1}(\mathbf{x}_{i-1})) + (C_{i-1}\mathbf{x}_{i-1})^2 \\ &\quad + \frac{1}{4\gamma^2}\Big(\frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}\mathbf{p}_{i-1}(\mathbf{x}_{i-1})\Big)^2 = -S_{i-1}(\mathbf{x}_{i-1}) \end{aligned} \tag{13.101}$$

for all $\mathbf{x}_{i-1}$, where

$$S_{i-1}(\mathbf{x}_{i-1}) = \begin{pmatrix} x_1 \\ x_2 - \varphi_1(x_1) \\ \cdots \\ x_{i-1} - \varphi_{i-2}(\mathbf{x}_{i-2}) \end{pmatrix}^{\mathrm{T}} Q_{i-1} \begin{pmatrix} x_1 \\ x_2 - \varphi_1(x_1) \\ \cdots \\ x_{i-1} - \varphi_{i-2}(\mathbf{x}_{i-2}) \end{pmatrix}$$

Then, there exist

(iv) *a positive definite and proper smooth function $V_i(\mathbf{x}_i)$ such that*

$$V_i^{[2]}(\mathbf{x}_i) = \mathbf{x}_i^{\mathrm{T}} X_i \mathbf{x}_i ,$$

(v) *a smooth function $\varphi_{i-1}(\mathbf{x}_{i-1})$ vanishing at zero together with its first partial derivatives,*

(vi) *a smooth function $v_i(\mathbf{x}_i)$ vanishing at zero and such that $v_i^{[1]}(\mathbf{x}_i) = K_i\mathbf{x}_i$,*

satisfying

$$\frac{\partial V_i}{\partial \mathbf{x}_i}(A_i\mathbf{x}_i + B_i v_i(\mathbf{x}_i)) + \frac{1}{4\gamma^2}\Big(\frac{\partial V_i}{\partial \mathbf{x}_i}\mathbf{p}_i(\mathbf{x}_i)\Big)^2 + (C_i\mathbf{x}_i)^2 = -S_i(\mathbf{x}_i) \tag{13.102}$$

for all $\mathbf{x}_i$, where

$$S_i(\mathbf{x}_i) = \begin{pmatrix} x_1 \\ x_2 - \varphi_1(x_1) \\ \cdots \\ x_i - \varphi_{i-1}(\mathbf{x}_{i-1}) \end{pmatrix}^{\mathrm{T}} Q_i \begin{pmatrix} x_1 \\ x_2 - \varphi_1(x_1) \\ \cdots \\ x_i - \varphi_{i-1}(\mathbf{x}_{i-1}) \end{pmatrix} \tag{13.103}$$

Proof. Set

$$\begin{aligned} Z &= (\,0_{1\times(i-1)} \quad 1\,) X_i (\,0_{1\times(i-1)} \quad 1\,)^{\mathrm{T}} , \\ T_i &= \begin{pmatrix} I_{(i-1)\times(i-1)} & K_{i-1}^{\mathrm{T}} \\ 0_{1\times(i-1)} & 1 \end{pmatrix} , \end{aligned}$$

and observe that

$$T_i X_i T_i^{\mathrm{T}} = \begin{pmatrix} X_{i-1} & 0_{(i-1)\times 1} \\ 0_{1\times(i-1)} & Z \end{pmatrix} . \tag{13.104}$$

The proof consists in choosing

$$V_i(\mathbf{x}_i) = V_{i-1}(\mathbf{x}_{i-1}) + Z(x_i - v_{i-1}(\mathbf{x}_{i-1}))^2$$

and showing that one can find $v_i(\mathbf{x}_i)$ and $\varphi_{i-1}(\mathbf{x}_{i-1})$ such that (13.102) holds.

To begin with, using (13.104), note that the function $V_i(\mathbf{x}_i)$ thus defined satisfies the property indicated in (iv). To prove that (13.102) can be enforced, observe that, for this choice of $V_i(\mathbf{x}_i)$,

$$\begin{aligned} &\frac{\partial V_i}{\partial \mathbf{x}_i}(A_i\mathbf{x}_i + B_i v_i(\mathbf{x}_i)) = \\ &\frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}(A_{i-1}\mathbf{x}_{i-1} + B_{i-1}v_{i-1}(\mathbf{x}_{i-1})) + \frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}B_{i-1}(x_i - v_{i-1}(\mathbf{x}_{i-1})) \\ &\quad + 2Z(x_i - v_{i-1}(\mathbf{x}_{i-1}))\Big(v_i(\mathbf{x}_i) - \frac{\partial v_{i-1}}{\partial \mathbf{x}_{i-1}}(A_{i-1}\mathbf{x}_{i-1} + B_{i-1}x_i)\Big) , \end{aligned}$$

and

$$\begin{aligned} \frac{\partial V_i}{\partial \mathbf{x}_i}\mathbf{p}_i(\mathbf{x}_i) &= \frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}\mathbf{p}_{i-1}(\mathbf{x}_{i-1}) \\ &\quad + 2Z(x_i - v_{i-1}(\mathbf{x}_{i-1}))\Big(p_i(\mathbf{x}_i) - \frac{\partial v_{i-1}}{\partial \mathbf{x}_{i-1}}\mathbf{p}_{i-1}(\mathbf{x}_{i-1})\Big) . \end{aligned}$$

Therefore the left-hand side of (13.102) reduces to an expression of the form

$$\begin{aligned} &\frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}(A_{i-1}\mathbf{x}_i + B_{i-1}v_{i-1}(\mathbf{x}_{i-1})) + \frac{1}{4\gamma^2}\Big(\frac{\partial V_{i-1}}{\partial \mathbf{x}_{i-1}}\mathbf{p}_{i-1}(\mathbf{x}_{i-1})\Big)^2 \\ &\quad + (C_{i-1}\mathbf{x}_{i-1})^2 + (x_i - v_{i-1}(\mathbf{x}_{i-1}))(2Zv_i(\mathbf{x}_i) - a(\mathbf{x}_i)) \end{aligned}$$

with $a(\mathbf{x}_i)$ a smooth function which vanishes at $\mathbf{x}_i = 0$.

Set

$$S_i(\mathbf{x}_i) = \begin{pmatrix} x_1 \\ x_2 - \varphi_1(x_1) \\ \cdots \\ x_{i-1} - \varphi_{i-2}(\mathbf{x}_{i-2}) \\ x_i - v_{i-1}(\mathbf{x}_{i-1}) \end{pmatrix}^{\mathrm{T}} T_i Q_i T_i^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 - \varphi_1(x_1) \\ \cdots \\ x_{i-1} - \varphi_{i-2}(\mathbf{x}_{i-2}) \\ x_i - v_{i-1}(\mathbf{x}_{i-1}) \end{pmatrix} ,$$

and observe that, because of the special form of T_i, this function has the required structure (13.103), in which

$$\varphi_{i-1}(\mathbf{x}_{i-1}) = v_{i-1}(\mathbf{x}_{i-1}) - K_{i-1}\mathbf{x}_{i-1} + K_{i-1}\begin{pmatrix} 0 \\ \varphi_1(x_1) \\ \cdots \\ \varphi_{i-2}(\mathbf{x}_{i-2}) \end{pmatrix}$$

is a smooth function vanishing at $\mathbf{x}_{i-1} = 0$ together with its first derivatives. Also, note that $S_i(\mathbf{x}_i)$ can be written in the form

$$S_i(\mathbf{x}_i) = S_{i-1}(\mathbf{x}_{i-1}) + (x_i - v_{i-1}(\mathbf{x}_{i-1}))b(\mathbf{x}_i)$$

with $b(\mathbf{x}_i)$ a smooth function which vanishes at $\mathbf{x}_i = 0$. Thus,

$$v_i(\mathbf{x}_i) = \frac{1}{2Z}(a(\mathbf{x}_i) - b(\mathbf{x}_i))$$

renders (13.102) satisfied.

To complete the proof, it remains to show that the property indicated in (v) holds. Set $v_i^{[1]}(\mathbf{x}_i) = H_i\mathbf{x}_i$ and consider the second order approximation of both sides of (13.102) near $\mathbf{x}_i = 0$. This yields

$$X_i(A_i + B_iH_i) + (A_i + B_iH_i)^{\mathrm{T}}X_i + \frac{1}{\gamma^2}X_iP_iP_i^{\mathrm{T}}X_i + C_i^{\mathrm{T}}C_i = -Q_i\,.$$

Subtracting off from (13.99) one obtains

$$X_iB_i(H_i - K_i) + (H_i - K_i)^{\mathrm{T}}B_i^{\mathrm{T}}X_i = 0\,,$$

which is a Sylvester equation with unique solution $H_i = K_i$, because X_i is positive definite, and this completes the proof. ◁

Using these two Lemmas it is possible to prove the following intermediate result.

Lemma 13.7.3. *Consider system (13.95). Suppose that, for some $\gamma > 0$, there exist a $1\times n$ matrix K and $n\times n$ matrices $X = X^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ such that*

$$X(E+BF+BK)+(E+BF+BK)^{\mathrm{T}}X+\frac{1}{\gamma^2}XPP^{\mathrm{T}}X+C^{\mathrm{T}}C = -Q\,. \quad (13.105)$$

Then there exist a smooth state feedback law $u(x)$, a smooth function $V(x)$ which is positive definite and proper, and a smooth function $S(x)$, which is positive definite, such that

$$\begin{aligned} u^{[1]}(x) &= Kx \\ V^{[2]}(x) &= x^{\mathrm{T}}Xx \\ S^{[2]}(x) &= x^{\mathrm{T}}Qx\,, \end{aligned} \quad (13.106)$$

which render the identity

$$\frac{\partial V}{\partial x}[Ex + Bf_n(x) + Bu(x)] + \frac{1}{4\gamma^2}\left[\frac{\partial V}{\partial x}p(x)\right]^2 + [Cx]^2 = -S(x) \quad (13.107)$$

satisfied for all x.

Proof. Starting from (13.105), use Lemma 13.7.1 "backward" n times, arriving at an identity of the form

$$2X_1(A_1 + B_1K_1) + \frac{1}{\gamma^2}X_1^2P_1^2 + 1 = -Q_1\,.$$

Set $V_1(x_1) = X_1x_1^2$ and observe that there exists a function $v_1(x_1)$, vanishing at $x_1 = 0$ and such that $v_1^{[1]}(x_1) = K_1x_1$, which satisfies

$$\frac{\partial V_1}{\partial x_1}[A_1x_1 + B_1v_1(x_1)] + \frac{1}{4\gamma^2}\left[\frac{\partial V_1}{\partial x_1}p_1(x_1)\right]^2 + [C_1x_1]^2 = -Q_1x_1^2\,.$$

From this, using Lemma 13.7.2 "forward" n times, arrive at an identity of the form

$$\frac{\partial V}{\partial x}[Ex + Bv(x)] + \frac{1}{4\gamma^2}\left[\frac{\partial V}{\partial x}p(x)\right]^2 + [Cx]^2 = -S(x)$$

in which $V(x)$ is positive definite and proper, $S(x)$ is positive definite, and

$$\begin{aligned} v^{[1]}(x) &= (F+K)x \\ V^{[2]}(x) &= x^{\mathrm{T}}Xx \\ S^{[2]}(x) &= x^{\mathrm{T}}Qx\,. \end{aligned}$$

Setting

$$u(x) = v(x) - f_n(x)$$

proves the Lemma. ◁

We are now ready to draw the desired conclusion.

Theorem 13.7.4. *Consider system (13.95). Suppose that, for some pair of real numbers $\gamma > 0$ and $R > 0$, there exist a $1\times n$ matrix K and $n\times n$ matrices $X = X^{\mathrm{T}} > 0$ and $Y = Y^{\mathrm{T}} > 0$ such that*

$$\begin{aligned} X(E + BF + BK) + (E + BF + BK)^{\mathrm{T}}X + \frac{1}{\gamma^2}XPP^{\mathrm{T}}X \\ + C^{\mathrm{T}}C + R^2K^{\mathrm{T}}K = -Y\,. \end{aligned} \quad (13.108)$$

Set $Q = Y + R^2K^{\mathrm{T}}K$ and (based on Lemma 13.7.3) find a smooth state feedback $u(x)$, a smooth positive definite and proper function $V(x)$, and a smooth positive definite function $S(x)$ such that (13.106) holds and the identity (13.107) is fulfilled for all x.

Set

$$\begin{aligned} r^*(x) &= R \qquad\qquad \text{if} \quad R^2u^2(x) \le S(x) \\ r^*(x) &= \frac{\sqrt{S(x)}}{|u(x)|} \quad \text{if} \quad R^2u^2(x) > S(x)\,. \end{aligned} \tag{13.109}$$

Then, the function $r^*(x)$ is well-defined and continuous at each $x \in \mathbb{R}^n$, and satisfies

$$r^*(0) = R, \quad 0 \le r^*(x) \le R\,.$$

Moreover, if $|r(x)| \le r^*(x)$,

$$\frac{\partial V}{\partial x}[Ex + Bf_n(x) + Bu(x)] + \frac{1}{4\gamma^2}\Big[\frac{\partial V}{\partial x}p(x)\Big]^2 + [Cx]^2 + r^2(x)u^2(x) \le 0 \tag{13.110}$$

for all x.

Proof. Let Γ denote the set

$$\Gamma = \{x \in \mathbb{R}^n : R^2u^2(x) \le S(x)\}\,.$$

We first show that Γ contains an open neighborhood of $x = 0$. In fact, by construction,

$$\begin{aligned} S^{[2]}(x) &= x^{\mathrm{T}}(Y + R^2K^{\mathrm{T}}K)x, \\ u^{[1]}(x) &= Kx\,. \end{aligned}$$

Thus, the function $S(x) - R^2u^2(x)$ is positive definite in a neighborhood of $x = 0$ (recall that Y is by hypothesis positive definite) and $x = 0$ is in the interior of Γ.

From this, since $S(x) > 0$ for any nonzero x, we can deduce that the entire set

$$\Gamma_0 = \{x \in \mathbb{R}^n : u(x) = 0\}$$

is in the interior of Γ.

Thus, the function $r^*(x)$ is well defined at each $x \in \mathbb{R}^n$. The same argument also shows that the function

$$\frac{\sqrt{S(x)}}{|u(x)|}$$

is continuous on some open set $\Gamma_1 \supset \mathbb{R}^n - \Gamma$ and since its value on $\partial\Gamma$ is equal to R, the function $r^*(x)$ is continuous at each $x \in \mathbb{R}^n$. Indeed, $r^*(0) = R$ and $0 \le r^*(x) \le R$.

Finally, by construction, $u(x), V(x), Q(x)$ are such that, for any function $r^*(x)$,

$$\begin{aligned} &\frac{\partial V}{\partial x}[Ex + Bf_n(x) + Bu(x)] + \frac{1}{4\gamma^2}\Big[\frac{\partial V}{\partial x}p(x)\Big]^2 + [Cx]^2 + r^{*2}(x)u^2(x) \\ &= -S(x) + r^{*2}(x)u^2(x)\,. \end{aligned}$$

The choice of $r^*(x)$ indicated in the theorem yields

$$-S(x) + r^{*2}(x)u^2(x) \le 0$$

for all x and this completes the proof. ◁

Remark 13.7.1. Note that, in the previous Theorem, the inequality (13.91) has been solved for $\alpha(\cdot) = 0$. Thus, the resulting closed loop system, namely system

$$\begin{aligned} \dot{x} &= Ex + Bf_n(x) + Bu(x) + p(x)w \\ y &= \begin{pmatrix} Cx \\ r(x)u(x) \end{pmatrix}, \end{aligned}$$

is dissipative but possible not *strictly* dissipative with respect to the supply rate $q(w, y) = \gamma^2w^2 - \|y\|^2$. Nevertheless, the system is guaranteed to be globally asymptotically stable, since it is zero-state detectable, as a simple check shows (see section 10.7). ◁

14. Stabilization Using Small Inputs

14.1 Achieving Global Stability via Small Inputs

In this Chapter, we describe methods for global (robust) stabilization of nonlinear systems, by means of memoryless feedback, in cases in which the amplitude of the control input cannot exceed a fixed bound. Of course, if such a hard constraint is imposed on the amplitude of the control input, one cannot expect – in general – that *global* asymptotic stability is possible, unless the uncontrolled system already possesses this property to a certain extent. The simplest case in which so happens is when there exists a positive definite and proper function, whose derivative along the trajectories of the uncontrolled system is negative *semi-definite* but, possibly, not negative definite. In this case, in fact, under mild hypotheses, it is possible to find a smooth feedback law, whose amplitude does not exceed any (arbitrarily small) a priori fixed number, yielding global asymptotic stability. We discuss this special case first, as a point of departure for the analysis of more general structures that will be presented in the subsequent sections of the Chapter.

As an introduction to the main result of the section, we begin by describing a simple, but important, stability result which is a special case of Corollary 10.8.4.

Proposition 14.1.1. *Consider the system*

$$\dot{x} = f(x) + g(x)u \tag{14.1}$$

in which $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $f(x)$ and $g(x)$ are smooth vector fields and $f(0) = 0$. Suppose there exists a smooth positive definite and proper function $V(x)$ satisfying

$$L_f V(x) \leq 0 \quad \text{for all } x\,. \tag{14.2}$$

Suppose also that the system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= L_g V(x) \end{aligned} \tag{14.3}$$

is zero-state detectable. Then, for any $\varepsilon > 0$, the feedback law

$$u = -\varepsilon L_g V(x) \tag{14.4}$$

globally asymptotically stabilizes the equilibrium $x = 0$.

Proof. Observe that, by construction, system (14.1), with output

$$y = L_g V(x)$$

is passive (see end of section 10.7) and zero-state detectable. Thus, the result follows from Corollary 10.8.4, taking $\varphi(y) = \varepsilon y$. ◁

The feedback law (14.4) is characterized by an arbitrarily small "gain factor" ε, but does not necessarily respect an amplitude constraint. However, still using Corollary 10.8.4, it is immediate to find a feedback law whose *amplitude* is bounded by an arbitrarily small number.

Proposition 14.1.2. *Consider system (14.1) and suppose the hypotheses of Proposition 14.1.1 hold. Consider the function*

$$\begin{aligned} \sigma : \mathbb{R} &\to \mathbb{R} \\ s &\mapsto \frac{s}{\sqrt{1+s^2}} . \end{aligned} \tag{14.5}$$

Then, for any $\varepsilon > 0$, the feedback law

$$u = -\varepsilon\sigma(L_g V(x)) \tag{14.6}$$

globally asymptotically stabilizes the equilibrium $x = 0$.

Proof. The result follows again from Corollary 10.8.4, taking this time $\varphi(y) = \varepsilon\sigma(y)$. ◁

The feedback law shown in this Proposition is such that

$$|u(x)| < \varepsilon \quad \text{for all } x ,$$

and therefore particularly suited in all practical cases in which the control input u of (14.1) is required to satisfy a constraint of the form

$$|u(t)| < U$$

for all t.

Without any special extra effort, it is possible to extend the result of this Proposition to the case of systems which are modeled by equations that are not necessarily affine in the control input, i.e. systems described by equations of the general form

$$\dot{x} = f(x,u) ,$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, in which $f(x,u)$ is a smooth function and $f(0,0) = 0$.

To this end, note that, without loss of generality, the function $f(x,u)$ can be expressed in the form

$$f(x,u) = f(x) + g_1(x)u + g_2(x,u)u^2$$

where $f(x)$ and $g_1(x)$ are smooth vector fields, $g_2(x,u)$ is a smooth function, and $f(0) = 0$. Then the following holds.

Theorem 14.1.3. *Consider the system*

$$\dot{x} = f(x) + g_1(x)u + g_2(x,u)u^2 , \tag{14.7}$$

in which $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $f(x)$ and $g_1(x)$ are smooth vector fields, $g_2(x,u)$ is a smooth function, and $f(0) = 0$. Suppose there exists a smooth positive definite and proper function $V(x)$ satisfying

$$L_f V(x) \le 0 \quad \text{for all } x . \tag{14.8}$$

Suppose also that the system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= L_{g_1} V(x) \end{aligned} \tag{14.9}$$

is zero-state detectable.

Let $\sigma(\cdot)$ be the function defined in Proposition 14.1.2, let ε be a fixed number satisfying $0 < \varepsilon < 1$ and let $\rho(x)$ be any smooth function such that

$$\rho(x) \ge \max_{|u|\le\varepsilon} \|g_2(x,u)\| .$$

The feedback law

$$u(x) = -\lambda(x)\sigma(L_{g_1} V(x)) ,$$

in which

$$\lambda(x) = \frac{\varepsilon}{\left(1 + \rho^2(x)\left\|\frac{\partial V}{\partial x}\right\|^2\right)} ,$$

globally asymptotically stabilizes the equilibrium $x = 0$.

Proof. Suppose $\lambda(x)$ is a smooth function satisfying

$$0 < \lambda(x) \le \varepsilon$$

for all x, and set

$$u(x) = -\lambda(x)\sigma(L_{g_1} V(x)) = -\lambda(x)\frac{L_{g_1} V(x)}{\sqrt{1 + [L_{g_1} V(x)]^2}} .$$

Since $|\sigma(s)| < 1$ for all s, we have

$$|u(x)| < \varepsilon$$

for all x, and $\|g_2(x,u(x))\| \le \rho(x)$ for all x.

Along the trajectories of the closed loop system, we have

$$\begin{aligned} \dot{V} &= L_f V(x) + L_{g_1} V(x)u(x) + \frac{\partial V}{\partial x} g_2(x,u(x))u^2(x) \\ &\le L_f V(x) - \lambda(x)\frac{[L_{g_1} V(x)]^2}{\sqrt{1 + [L_{g_1} V(x)]^2}} + \left\|\frac{\partial V}{\partial x}\right\|\rho(x)\lambda^2(x)\frac{[L_{g_1} V(x)]^2}{1 + [L_{g_1} V(x)]^2} \\ &= L_f V(x) + \lambda(x)\frac{[L_{g_1} V(x)]^2}{\sqrt{1 + [L_{g_1} V(x)]^2}}\left(-1 + \frac{\lambda(x)}{\sqrt{1 + [L_{g_1} V(x)]^2}}\left\|\frac{\partial V}{\partial x}\right\|\rho(x)\right). \end{aligned}$$

If
$$\lambda(x)\Big\|\frac{\partial V}{\partial x}\Big\|\rho(x) < 1 \tag{14.10}$$
we have
$$\lambda(x)\Big(-1+\frac{\lambda(x)}{\sqrt{1+[L_{g_1}V(x)]^2}}\Big\|\frac{\partial V}{\partial x}\Big\|\rho(x)\Big) := \chi(x) < 0$$
for all x, and therefore
$$\dot V \le L_f V(x) + \chi(x)\frac{[L_{g_1}V(x)]^2}{\sqrt{1+[L_{g_1}V(x)]^2}} \le 0 \tag{14.11}$$
for all x.

Condition (14.10) is indeed satisfied with
$$\lambda(x) = \frac{\varepsilon}{\Big(1+\rho^2(x)\Big\|\frac{\partial V}{\partial x}\Big\|^2\Big)}\,,$$
and this shows that the feedback law indicated in the theorem renders (14.11) fulfilled.

Since $V(x)$ is a proper function, this proves boundedness of all trajectories and stability in the sense of Lyapunov. As a consequence, any trajectory $x(t)$ has a nonempty ω-limit set Γ, on which the function $V(x)$ is constant. We see from (14.11) that, necessarily,
$$L_{g_1}V(x) = 0 \quad \text{for all } x \in \Gamma$$
and this in turn implies that $u(x) = 0$ at each point of Γ. This shows also that any trajectory $\tilde x(t)$ entirely contained in Γ is necessarily a trajectory of
$$\dot x = f(x)$$
for which
$$L_{g_1}V(\tilde x(t)) = 0\,.$$
The assumption that system (14.9) is zero-state detectable implies that $\tilde x(t) \to 0$ as $t \to \infty$ and this proves that $V(x)$ is 0 on Γ. Thus, also $x(t) \to 0$ as $t \to \infty$, because $V(x)$ vanishes only at $x = 0$. ◁

Note that also the feedback law derived in this Theorem is such that
$$|u(x)| < \varepsilon \quad \text{for all } x\,.$$

Remark 14.1.1. The results of the previous two Propositions can be immediately extended to a system
$$\dot x = f(x) + g(x)u \tag{14.12}$$
with $m > 1$ inputs. In this case, in fact, since system (14.12) with output $y = [L_gV(x)]^{\mathrm T}$ is by construction passive, if the hypothesis that system (14.3) is zero-state detectable is replaced by the hypothesis that
$$\begin{aligned} \dot x &= f(x) \\ y &= [L_gV(x)]^{\mathrm T} \end{aligned} \tag{14.13}$$
is zero-state detectable, the result of Corollary 10.8.4 still applies. Thus, system (14.12) is stabilized by the feedback law
$$u = -\varepsilon[L_gV(x)]^{\mathrm T}$$
or by the bounded feedback law
$$u = -\varepsilon\sigma([L_gV(x)]^{\mathrm T})$$
where now the function $\sigma(\cdot)$ is defined as
$$\begin{aligned} \sigma:\quad & \mathbb{R}^m \to \mathbb{R}^m \\ & (s_1,\ldots,s_m) \mapsto \Big(\frac{s_1}{\sqrt{1+s_1^2}},\ldots,\frac{s_m}{\sqrt{1+s_m^2}}\Big)\,. \end{aligned} \tag{14.14}$$

Also a multi-input version of Theorem 14.1.3 holds, whose derivation is left to the reader. ◁

One of the key assumptions in the previous results is the hypothesis of the zero-state detectability, of system (14.3) or of system (14.9). Sometimes, in order to check this hypothesis, the following simple property may turn out to be useful.

Lemma 14.1.4. *Let $f(x)$ and $g(x)$ be smooth vector fields of $\mathbb{R}^n$, with $f(0) = 0$, and suppose there exists a smooth positive definite and proper function $V(x)$ such that $L_fV(x) \le 0$ for all x. Then, system*
$$\begin{aligned} \dot x &= f(x) \\ y &= L_gV(x) \end{aligned} \tag{14.15}$$
is zero-state detectable if and only if so is system
$$\begin{aligned} \dot x &= f(x) \\ y &= \begin{pmatrix} L_gV(x) \\ L_fV(x) \end{pmatrix}. \end{aligned}$$

Proof. The "only if" part is trivially true by definition. To prove the "if" part, observe that, using the same arguments of the proof of the previous Theorem, system

$$\dot{x} = f(x) - g(x)L_gV(x) \tag{14.16}$$

is globally asymptotically stable. In fact

$$\frac{\partial V}{\partial x}[f(x) - g(x)L_gV(x)] = L_fV(x) - [L_gV(x)]^2 \leq 0$$

and on the ω-limit set of each trajectory

$$L_gV(x) = L_fV(x) = 0 .$$

Now, consider any trajectory $\bar{x}(t)$ of (14.15) for which $L_gV(\bar{x}(t)) = 0$. This is indeed a trajectory of (14.16) and hence $\bar{x}(t) \to 0$ as $t \to \infty$. This shows that (14.15) is zero-state detectable. ◃

We conclude the section with some additional results that hold in the special case in which system (14.1) is a linear system. Consider a system, with $m \geq 1$ inputs, modeled by equations of the form

$$\dot{x} = Ax + Bu , \tag{14.17}$$

and suppose that (A, B) is ***stabilizable*** and there exists a symmetric matrix $P > 0$ such that

$$A^{\mathrm{T}}P + PA \leq 0 . \tag{14.18}$$

It is known from Lemma 10.9.3 that this system, with output $y = B^{\mathrm{T}}Px$, is detectable. Hence, by Proposition 14.1.2 and its multi-input version (see Remark 14.1.1), it can be globally asymptotically stabilized by the feedback law

$$u(x) = -\varepsilon\sigma(B^{\mathrm{T}}Px) = \varepsilon\sigma(-B^{\mathrm{T}}Px) \tag{14.19}$$

in which $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is the function

$$\sigma(y) : (y_1, \dots, y_m) \mapsto (\frac{y_1}{\sqrt{1+y_1^2}}, \dots, \frac{y_m}{\sqrt{1+y_m^2}}) ,$$

and $\varepsilon > 0$ is any fixed number.

It is possible to show that the stabilizing property of the feedback law (14.19) with respect to system (14.17) holds not just for the specific function $\sigma(\cdot)$ indicated above, but also *for any* other function in a more general class of functions, called *saturation functions*, and, moreover, that the resulting closed loop system turns out to have a property of input-to-state stability with restrictions (in the sense of Definition 12.2.1), with respect to additive disturbances.

As a matter of fact, observe first that the function $\sigma(\cdot)$ is just a special member of the class of functions characterized by the following definition.

Definition 14.1.1. *A locally Lipschitz function $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is said to be a saturation function if:*

- $\sigma(0) = 0$ and $s\sigma(s) > 0$ for all $s \neq 0$,

- there exist $\underline{k}$ and $\overline{k}$ such that

$$|\sigma(s)| \leq \overline{k} \quad \text{for all } s \in \mathbb{R}$$
$$\liminf_{|s|\to\infty} |\sigma(s)| \geq \underline{k} ,$$

- $\sigma(s)$ is differentiable in a neighborhood of $s = 0$ and

$$\sigma'(0) = 1 .$$

A function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is an R^m-valued saturation function if

$$\sigma : (x_1, \dots, x_m) \mapsto (\sigma_1(x_1), \dots, \sigma_m(x_m))$$

and, for all $i = 1, \dots, m$, $\sigma_i(\cdot)$ is a saturation function.

Remark 14.1.2. It is easily verified that, if $\sigma(\cdot)$ is a saturation function, then there exists a number $K > 0$ such that

$$|\sigma(s) - s| \leq Ks\sigma(s) \tag{14.20}$$

for all $s \in \mathbb{R}$. In fact, observe that, since $\sigma'(0) = 1$ and $s\sigma(s) > 0$ for all $s \neq 0$, there exist $\delta > 0$ and $A > 0$ such that

$$|\sigma(s) - s| \leq As\sigma(s) , \qquad \text{for all } |s| < \delta.$$

Moreover,

$$|\sigma(s) - s| \leq \overline{k} + |s| , \qquad \text{for all } s,$$

and there exists $\varepsilon > 0$ such that

$$|\sigma(s)| \geq \varepsilon , \qquad \text{for all } |s| \geq \delta.$$

This yields, for any $K > 0$,

$$K\sigma(s)s \geq K\varepsilon|s| , \qquad \text{for all } |s| \geq \delta.$$

Choosing K so that $K \geq A$ and $K\varepsilon|\delta| \geq \overline{k} + |\delta|$ completes the proof of (14.20).

Using (14.20) it is also easy to see that there exists a number $H \geq 1$ such that

$$|\sigma(s)| \leq \min\{H|s|, \overline{k}\} \tag{14.21}$$

for all $s \in \mathbb{R}$. ◃

Then, the following result can be proven.

Proposition 14.1.5. *Consider system (14.17). Suppose (A, B) is stabilizable and there exists a symmetric matrix $P > 0$ such that (14.18) holds. Let $\sigma(\cdot)$ be any $\mathbb{R}^m$-valued saturation function, and consider the closed loop system*

$$\dot{x} = Ax + B\sigma(-B^{\mathrm{T}}Px + v) + w . \tag{14.22}$$

Then, there exists a number $\delta > 0$ such that (14.22) is input-to-state stable, with no restriction on x°, and restriction δ on $v(\cdot)$ and $w(\cdot)$. In particular, there exist a class $\mathcal{K}$ function $\gamma_0(\cdot)$ and numbers $g_v > 0$, $g_w > 0$ such that, for any $x^\circ \in \mathbb{R}^n$, for any input $v(\cdot) \in L^m_\infty$ satisfying $\|v(\cdot)\|_\infty < \delta$, and any input $w(\cdot) \in L^n_\infty$ satisfying $\|w(\cdot)\|_\infty < \delta$, the response $x(t)$ in the initial state $x(0) = x^\circ$ satisfies

$$\begin{aligned} \|x(\cdot)\|_\infty &\le \max\{\gamma_0(\|x^\circ\|), \gamma_v(\|v(\cdot)\|_\infty), \gamma_w(\|w(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\le \max\{\gamma_v(\limsup_{t\to\infty} \|v(t)\|), \gamma_w(\limsup_{t\to\infty} \|w(t)\|)\} , \end{aligned}$$

where

$$\gamma_v(r) = g_v r, \quad \gamma_w(r) = g_w r$$

for all r.

Proof. From the previous discussion, it is known that system

$$\dot{x} = (A - BB^{\mathrm{T}}P)x$$

is asymptotically stable. Thus, there exists a symmetric matrix $Q > 0$ such that

$$(A - BB^{\mathrm{T}}P)^{\mathrm{T}}Q + Q(A - BB^{\mathrm{T}}P) = -2I .$$

Set

$$z = -B^{\mathrm{T}}Px + v .$$

Along the trajectories of (14.22), the quadratic form

$$V_0(x) = \frac{1}{2}x^{\mathrm{T}}Qx$$

satisfies

$$\begin{aligned} \dot{V}_0 &= x^{\mathrm{T}}Q[(A - BB^{\mathrm{T}}P)x + BB^{\mathrm{T}}Px + B\sigma(z) + w] \\ &= -\|x\|^2 + x^{\mathrm{T}}Q[B\sigma(z) - Bz + Bv + w] \\ &= -\|x\|^2 + x^{\mathrm{T}}QB[\sigma(z) - z] + x^{\mathrm{T}}QBv + x^{\mathrm{T}}Qw . \end{aligned}$$

Using the fact that $\|\sigma(z) - z\| \le Kz^{\mathrm{T}}\sigma(z)$ for some $K > 0$ (see (14.20)), and standard estimates for quadratic forms, it is deduced that there exist positive numbers a_1, a_2, a_3, only depending on Q and B, such that

$$\dot{V}_0 \le -\|x\|^2 + a_1K\|x\|z^{\mathrm{T}}\sigma(z) + a_2\|x\|\|v\| + a_3\|x\|\|w\| .$$

Consider now the positive definite and proper function

$$V_1(x) = \frac{1}{3}(x^{\mathrm{T}}Px)^{3/2} .$$

Along the trajectories of (14.22),

$$\begin{aligned} \dot{V}_1 &= |x^{\mathrm{T}}Px|^{1/2}x^{\mathrm{T}}P[Ax + B\sigma(z) + w] \\ &= \frac{1}{2}|x^{\mathrm{T}}Px|^{1/2}x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + |x^{\mathrm{T}}Px|^{1/2}[x^{\mathrm{T}}PB\sigma(z) + x^{\mathrm{T}}Pw] \\ &\le |x^{\mathrm{T}}Px|^{1/2}[-z^{\mathrm{T}}\sigma(z) + v^{\mathrm{T}}\sigma(z) + x^{\mathrm{T}}Pw] . \end{aligned}$$

Using the fact that $z^{\mathrm{T}}\sigma(z) \ge 0$, that $\|\sigma(z)\| \le \overline{k}$ for some $\overline{k} > 0$ and standard estimates for quadratic forms, it is deduced that there exist positive numbers b_1, b_2, b_3, only depending on P, such that

$$\dot{V}_1 \le -b_1\|x\|z^{\mathrm{T}}\sigma(z) + b_2\overline{k}\|x\|\|v\| + b_3\|x\|^2\|w\| .$$

Set

$$V(x) = V_0(x) + \lambda V_1(x)$$

with $\lambda > 0$. Then, along the trajectories of (14.22),

$$\begin{aligned} \dot{V} \le\ & (-1 + \lambda b_3\|w\|)\|x\|^2 + (a_1K - \lambda b_1)\|x\|z^{\mathrm{T}}\sigma(z) \\ & + (a_2 + \lambda b_2\overline{k})\|x\|\|v\| + a_3\|x\|\|w\| . \end{aligned}$$

Choose λ so that

$$a_1K - \lambda b_1 = 0$$

and δ such that

$$-1 + \lambda b_3\delta = -\frac{1}{2} .$$

Then, if

$$\|w(t)\| \le \delta \quad \text{for all } t \ge 0,$$

we obtain

$$\dot{V} \le \|x\|\left(-\frac{1}{2}\|x\| + (a_2 + \lambda b_2\overline{k})\|v\| + a_3\|w\|\right) .$$

We have in this way shown that, setting, for instance

$$c_v = 8(a_2 + \lambda b_2\overline{k}), \qquad c_w = 8a_3$$

the function $V(x)$ is such that

$$\|x\| \ge \max\{c_v\|v\|, c_w\|w\|\}$$
$$\Rightarrow \quad \frac{\partial V}{\partial x}[Ax + B\sigma(-B^{\mathrm{T}}Px + v) + w] \le -\frac{\|x\|^2}{4} \tag{14.23}$$

for all $x \in \mathbb{R}^n$, all $v \in \mathbb{R}^m$ and all $w \in \mathbb{R}^n$ satisfying $\|w\| \le \delta$.

Observe that there exist positive numbers $\underline{a}_0, \overline{a}_0, \overline{a}_1$, such that the function $V(x)$ satisfies

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) ,$$

with

$$\underline{\alpha}(r) = \underline{a}_0 r^2, \quad \overline{\alpha}(r) = \overline{a}_0 r^2 + \overline{a}_1 r^3 .$$

Thus, the inequality (14.23) above shows that the system is input-to-state stable, with no restriction on x°, no restriction on $v(\cdot)$, restriction δ on $w(\cdot)$, and "gain" functions

$$\gamma_v(r) = \underline{\alpha}^{-1} \circ \overline{\alpha}(c_v r) , \qquad \gamma_w(r) = \underline{\alpha}^{-1} \circ \overline{\alpha}(c_w r) .$$

Since, for any $d > 0$, there exists $\tilde{a}_0 > 0$ such that

$$\overline{\alpha}(r) \le \tilde{a}_0 r^2$$

for all $r \le d$, it follows that for $r \in [0, d]$ the functions $\gamma_v(\cdot)$ and $\gamma_w(\cdot)$ can be estimated by linear functions. This proves the Proposition, if it is assumed that also $v(\cdot)$ is restricted by δ, i.e. $\|v(\cdot)\|_\infty \le \delta$. ◃

14.2 Stabilization of Systems in Upper Triangular Form

In Chapters 11 and 12, we have discussed the problem of global asymptotic stabilization of certain classes of nonlinear systems described by equations having a *lower-triangular* structure, namely systems which are modeled by equations of the form (11.38). We examine in this section the case of systems described by equations having an *upper-triangular* structure, such as

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u) \\
\dot{x}_2 &= f_2(x_2, x_3, \ldots, x_n, u) \\
&\cdots \\
\dot{x}_{n-1} &= f_{n-1}(x_{n-1}, x_n, u) \\
\dot{x}_n &= f_n(x_n, u) ,
\end{aligned} \tag{14.24}$$

in which the functions $f_i(x_i, x_{i+1}, \ldots, x_n, u)$ are supposed to satisfy appropriate hypotheses, which will be introduced in the course of the exposition. These systems are often referred to as systems in *feedforward form* , in consideration of the fact that they correspond to a cascade interconnection of n subsystems, starting with the lower subsystem of (14.24) and ending with the upper subsystem of (14.24), in which the i-th subsystem is fed by the "outputs" $x_{i+1}, \ldots, x_n$ of all previous subsystems in the cascade (see Fig. 14.1).

Because of this triangular structure, the design of stabilizing feedback can be achieved in a recursive way, based on the following procedure. Suppose that a feedback law $u = \alpha_n(x)$ is known which stabilizes the lower subsystem of the chain

$$\dot{x}_n = f_n(x_n, u) ,$$

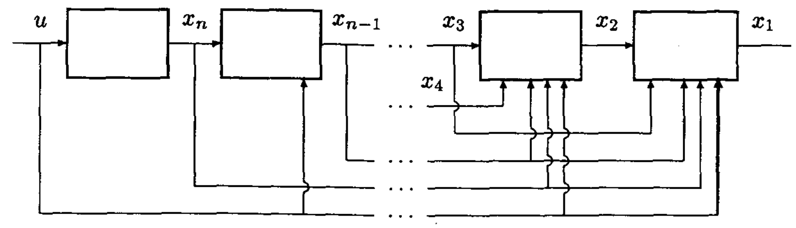

Fig. 14.1. System in feedforward form.

and set

$$u = \alpha_n(x_n) + u_n . \tag{14.25}$$

This yields a system described by equations whose structure is the same as the structure of (14.24), with u replaced by u_n, but – in addition – the subsystem corresponding to the last equation of the chain is stable, by construction, if $u_n = 0$. Consider now the subsystem formed by the last two equations, which have the form

$$\begin{aligned}
\dot{x}_{n-1} &= f_{n-1}(x_{n-1}, x_n, \alpha_n(x_n) + u_n) := \tilde{f}_{n-1}(x_{n-1}, x_n, u_n) \\
\dot{x}_n &= f_n(x_n, \alpha_n(x) + u_n) := \tilde{f}_n(x_n, u_n) ,
\end{aligned}$$

and suppose of being able to design a feedback law $u_n = \alpha_{n-1}(x_n, x_{n-1})$ which renders it stable. Set

$$u_n = \alpha_{n-1}(x_n, x_{n-1}) + u_{n-1} ,$$

which, actually, is the same as setting

$$u = \alpha_n(x_n) + \alpha_{n-1}(x_n, x_{n-1}) + u_{n-1}$$

on the original system (14.24). By doing this, a system is obtained in which the upper $n-2$ equations have the same structure as that of (14.24), with u replaced by u_{n-1}, and in which the subsystem corresponding to the lower two equations of the chain is stable, by construction, if $u_{n-1} = 0$.

At this point, the last two equations of the chain can be considered as a single subsystem, with state $\tilde{x}_{n-1} = (x_n, x_{n-1})$ and input u_{n-1}, which is stable when $u_{n-1} = 0$. The entire system can still be regarded as a system modeled by equations of the form (14.24), with n replaced by $n-1$, x_{n-1} replaced by $\tilde{x}_{n-1}$, and u replaced by u_{n-1}, and – in addition – the last subsystem of the chain is stable by construction when the input is zero. In other words, the entire system is precisely in the same conditions as the original system (14.24) was after the imposition of the control (14.25), and one can proceed in the same manner.

It is seen from these considerations that a system modeled by equations of the form (14.24) can be stabilized if:

- it is known how to stabilize the lower subsystem of the chain,
- it is known how to stabilize a system of the form

$$\begin{aligned} \dot{z} &= \psi(z,\xi,u) \\ \dot{\xi} &= \phi(\xi,u)\,, \end{aligned} \tag{14.26}$$

knowing that the lower subsystem is stable when the input is zero.

In this section we describe conditions under which the second one of these two design problems can be successfully addressed. We assume, as usual, that $\psi(z,\xi,u)$ and $\phi(z,u)$ are smooth functions of their arguments and that $\psi(0,0,0)=0, \phi(0,0)=0$, and – to begin with – also that their are *affine* in the control input u. The more general case of an arbitrary nonlinear dependence on u will be dealt with, using the result of Theorem 14.1.3, at the end of the section. This being the case, system (14.26) is rewritten, for convenience, in the form

$$\begin{aligned} \dot{z} &= a(z)+p(z,\xi)+b(z,\xi)u \\ \dot{\xi} &= \varphi(\xi)+\theta(\xi)u\,, \end{aligned} \tag{14.27}$$

with $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^\nu$, $u \in \mathbb{R}$, and where $a(0)=0$, $p(z,0)=0$, $\varphi(0)=0$. In line with the previous discussion, it is assumed that the equilibrium $\xi=0$ of

$$\dot{\xi}=\varphi(\xi) \tag{14.28}$$

is globally asymptotically stable.

Note that, setting

$$x=\begin{pmatrix} z \\ \xi \end{pmatrix}, \quad f(x)=\begin{pmatrix} a(z)+p(z,\xi) \\ \varphi(\xi) \end{pmatrix}, \quad g(x)=\begin{pmatrix} b(z,\xi) \\ \theta(\xi) \end{pmatrix},$$

system (14.27) can be regarded as a system of the form

$$\dot{x}=f(x)+g(x)u\,. \tag{14.29}$$

Thus, the problem of stabilizing system (14.27) could seem as difficult as any other nonlinear stabilization problem. However, in the present setup, one may take advantage of the upper triangular structure of $f(x)$ and of the fact that (14.28) is by hypothesis globally asymptotically stable. Of course, it would be trivial to assume that the entire system

$$\dot{x}=f(x)$$

is globally asymptotically stable (in which case $u=0$ would solve the stabilization problem), but it is not trivial – and relevant in many important cases – to consider the "next easier" assumption, namely the one in which, as a consequence of the hypothesis of the existence of a suitable Lyapunov function, this system is guaranteed to have bounded trajectories and to be stable in the sense of Lyapunov (but possibly not globally asymptotically stable). If this is the case, in fact, the result of Proposition 14.1.1 can be invoked, which shows that, if a positive definite and proper function $V(x)$ is known such that $L_f V(x) \le 0$ for all x, it is not terribly difficult to globally asymptotically stabilize a system of the form (14.29), provided that appropriate additional conditions hold.

In view of this, we discuss now the problem of identifying situations in which the main hypothesis of Proposition 14.1.1, namely the existence of a smooth function $V(x)$ such that (14.2) holds and (14.3) is zero-state detectable, is fulfilled. As shown below, the existence of a function $V(x)$ satisfying (14.2) is guaranteed when, in system (14.27):

- the function $a(z)$ is *linear* in z, namely $a(z)=Az$, and there exists a symmetric matrix $P>0$ such that

$$PA+A^{\mathrm{T}}P \le 0$$

(i.e. the equilibrium $\tilde{z}=0$ of the associated linear system $\dot{\tilde{z}}=A\tilde{z}$ is stable in the sense of Lyapunov),

- the equilibrium $\xi=0$ of the system $\dot{\xi}=\varphi(\xi)$ is *globally asymptotically* stable and also *locally exponentially* stable,
- the function $p(z,\xi)$ has a *linear* growth in z, namely, is such that

$$\|p(z,\xi)\| \le \gamma(\|\xi\|)(1+\|z\|) \quad \text{for all} \quad z,\xi$$

where $\gamma(\cdot)$ is a class $\mathcal{K}$ function, differentiable at the origin.

Before addressing the issue of the existence (and possibly the construction) of such a function under the said hypotheses, it is useful to start with the analysis of a special case, the one in which, in system (14.27),

$$\begin{aligned} a(z) &= Az && \text{and } A+A^{\mathrm{T}}=0 \text{ (i.e. } A \text{ is a skew}-\text{symmetric matrix)} \\ p(z,\xi) &= p(\xi) && \text{and } p(\xi) \text{ is a polynomial in the components of } \xi \\ \varphi(\xi) &= F\xi && \text{and all eigenvalues of } F \text{ have negative real part}\,. \end{aligned}$$

In other words, we examine the special case in which the system $\dot{x}=f(x)$ has the form

$$\begin{aligned} \dot{z} &= Az+p(\xi) \\ \dot{\xi} &= F\xi \end{aligned} \tag{14.30}$$

with A, $p(\xi)$, F satisfying the hypotheses indicated above. In this case, a proper function $V(z)$ such that $L_f V(z) \le 0$ can be constructed by appealing to the following interesting result [1].

Lemma 14.2.1. *Let $A \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{\nu\times\nu}$ be fixed matrices. Let Q denote the vector space consisting of all polynomials of degree ρ in $\xi_1, \xi_2, \ldots, \xi_\nu$, with coefficients in $\mathbb{R}$, which vanish at $\xi=0$. Let $\mathcal{F}$ denote the linear operator*

[1] See Byrnes *et al.* (1997), page 13, for a proof.

$$\begin{aligned}\mathcal{F} \;:\; \mathcal{Q}^n &\to \mathcal{Q}^n \\ q(\xi) &\mapsto \frac{\partial q}{\partial \xi} F\xi \,.\end{aligned}$$

Suppose that none of the eigenvalues of $\mathcal{F}$ is an eigenvalue of A. Then, given any element $p(\xi)$ of $\mathcal{Q}^n$, there is a unique element $\pi(\xi)$ of $\mathcal{Q}^n$ such that

$$\frac{\partial \pi}{\partial \xi} F\xi = A\pi(\xi) + p(\xi) \,. \tag{14.31}$$

By means of the function $\pi(\xi)$, it is possible to transform system (14.30) into a decoupled system. In fact, set

$$\zeta = z - \pi(\xi)$$

and, using (14.31), observe that system (14.30) is transformed into

$$\begin{aligned}\dot{\zeta} &= A\zeta \\ \dot{\xi} &= F\xi \,.\end{aligned}$$

This system is stable in the sense of Lyapunov. In particular, if $U(\xi)$ is any Lyapunov function for the lower subsystem, i.e. a positive definite and proper function such that

$$\frac{\partial U}{\partial \xi} F\xi < 0 \quad \text{for all} \quad \xi \neq 0 \,,$$

the (positive definite and proper) function thus defined

$$\overline{W}(\zeta, \xi) = \zeta^{\mathrm{T}}\zeta + U(\xi)$$

is such that (recall that A by hypothesis satisfies $A = -A^{\mathrm{T}}$)

$$\frac{\partial \overline{W}}{\partial \zeta}\dot{\zeta} + \frac{\partial \overline{W}}{\partial \xi}\dot{\xi} = \frac{\partial U}{\partial \xi} F\xi \leq 0 \quad \text{for all} \quad (\zeta, \xi) \,. \tag{14.32}$$

Consider now the (positive definite and proper) function obtained from $\overline{W}(z, \xi)$ by reversing the change of coordinates, namely the function

$$W(z, \xi) = \overline{W}(z - \pi(\xi), \xi)$$

which can be written in the form

$$W(z, \xi) = z^{\mathrm{T}} z + U(\xi) + \Psi(z, \xi) \tag{14.33}$$

with a "cross-term" $\Psi(z, \xi)$ defined as

$$\Psi(z, \xi) = -2\pi^{\mathrm{T}}(\xi) z + \pi^{\mathrm{T}}(\xi)\pi(\xi) \,.$$

Indeed, we expect that the derivative of this (positive definite and proper) function along the trajectories of system (14.30) is equal to the expression

obtained by reversing the change of coordinates in the right-hand side of (14.32), i.e. just equal to

$$\frac{\partial U(\xi)}{\partial \xi} F\xi \,.$$

Thus, since

$$\frac{\partial (z^{\mathrm{T}} z)}{\partial z} Az = 0 \,,$$

we expect that the cross-term $\Psi(z, \xi)$ satisfies

$$\frac{\partial (z^{\mathrm{T}} z)}{\partial z} p(\xi) + \frac{\partial \Psi}{\partial z}[Az + p(\xi)] + \frac{\partial \Psi}{\partial \xi} F\xi = 0 \,. \tag{14.34}$$

Of course this is the case, as a straightforward calculation, which uses (14.31) and the property that $A = -A^{\mathrm{T}}$, shows.

Remark 14.2.1. Note that the same conclusions remain valid if the lower subsystem of (14.30), instead of being a linear system, is a nonlinear system, namely

$$\dot{\xi} = \varphi(\xi) \,,$$

so long as there is a (globally defined) solution $\pi(\xi)$ of

$$\frac{\partial \pi(\xi)}{\partial \xi}\varphi(\xi) = A\pi(\xi) + p(\xi) \,. \tag{14.35}$$

The set

$$\mathcal{M} = \{(z, \xi) \in \mathbb{R}^n \times \mathbb{R}^\nu : z = \pi(\xi)\}$$

is the stable manifold of

$$\begin{aligned}\dot{z} &= Az + p(\xi) \\ \dot{\xi} &= \varphi(\xi) \,.\end{aligned} \tag{14.36}$$

Thus, to say that equation (14.35) has a globally defined solution is equivalent to say that the stable manifold of (14.36) can be represented in the form above, namely as the graph of a globally defined function $\pi(\xi)$. ◁

The fact that, in this particular case, it was possible to find a Lyapunov function for (14.30) with a cross-term $\Psi(z, \xi)$ whose derivative along the trajectories cancels out with the sign indefinite term

$$\frac{\partial (z^{\mathrm{T}} z)}{\partial z} p(\xi)$$

suggests seeking a similar result for the more general case in which the system $\dot{x} = f(x)$ has the form

$$\begin{aligned}\dot{z} &= Az + p(z, \xi) \\ \dot{\xi} &= \varphi(\xi) \,.\end{aligned} \tag{14.37}$$

For this system, suppose there is a symmetric matrix $P > 0$ such that

$$PA + A^{\mathrm{T}}P \le 0\,,$$

and suppose there exists a positive definite and proper function $U(\xi)$ such that

$$\frac{\partial U}{\partial \xi}\varphi(\xi) < 0 \quad \text{for all } \xi \neq 0\,.$$

Moreover, suppose there exists a cross-term $\Psi(z,\xi)$ such that the function

$$W(z,\xi) = z^{\mathrm{T}}Pz + U(\xi) + \Psi(z,\xi) \tag{14.38}$$

is positive definite and proper and such that

$$\frac{\partial(z^{\mathrm{T}}Pz)}{\partial z}p(z,\xi) + \frac{\partial \Psi}{\partial z}[Az + p(z,\xi)] + \frac{\partial \Psi}{\partial \xi}\varphi(\xi) = 0\,. \tag{14.39}$$

Then, the derivative of the function $W(z,\xi)$ along the trajectories of (14.37) would satisfy

$$\frac{\partial W}{\partial z}[Az + p(z,\xi)] + \frac{\partial W}{\partial \xi}\varphi(\xi) = z^{\mathrm{T}}(PA + A^{\mathrm{T}}P)z + \frac{\partial U}{\partial \xi}\varphi(\xi)\,,$$

i.e. would be negative semi-definite on $\mathbb{R}^n \times \mathbb{R}^\nu$, thus complying with the hypothesis (14.2) of Proposition 14.1.1. If this function would also be such that system (14.3) is zero-state detectable, then the stabilizing feedback provided by Proposition 14.1.1 could be used.

Of course, existence of such a cross-term implies, as a byproduct, that the trajectories of system (14.37) are bounded for any initial condition. Thus, we check first that this property holds. Instrumental, in this part of the analysis, are the hypotheses of *linear growth* in z for the coupling term in (14.37), and of *local exponential stability* of the equilibrium $\xi = 0$ of the lower subsystem (in addition to the already assumed existence of $P > 0$ and $U(\xi)$, which reflects the stability in the sense of Lyapunov of $\dot{z} = Az$, and the global asymptotic stability of the lower subsystem).

Lemma 14.2.2. *Consider system (14.37). Suppose:*

(i) *there exists a symmetric matrix $P > 0$ such that $PA + A^{\mathrm{T}}P \le 0$,*

(ii) *the equilibrium $\xi = 0$ of the system $\dot{\xi} = \varphi(\xi)$ is globally asymptotically stable and also locally exponentially stable,*

(iii) *the function $p(z,\xi)$ is such that*

$$\|p(z,\xi)\| \le \gamma(\|\xi\|)(1 + \|z\|) \quad \textit{for all } z,\xi\,,$$

where $\gamma(\cdot)$ is a class $\mathcal{K}$ function, differentiable at the origin.

Then, for every $(z^\circ,\xi^\circ) \in \mathbb{R}^n \times \mathbb{R}^\nu$, the integral curve of (14.37) passing through (z°,ξ°) at time $t = 0$ is bounded on $[0,\infty)$.

Proof. Of course, we only need to establish boundedness of $z(t)$. Let

$$V(t) = z^{\mathrm{T}}(t)Pz(t)$$

and observe that, by hypothesis

$$\begin{aligned}\dot{V}(t) &= 2z^{\mathrm{T}}(t)P[Az(t) + p(z(t),\xi(t))]\\ &\le 2\|z(t)\|\|P\|\|p(z(t),\xi(t))\| \le 2\|z(t)\|\|P\|\gamma(\|\xi(t)\|)(1 + \|z(t)\|)\,.\end{aligned}$$

From hypothesis (ii), using the property indicated in Lemma 10.1.6, we know that there exist a class $\mathcal{K}$ function $\theta(\cdot)$ and a number $b > 0$, such that

$$\gamma(\|\xi(t)\|) \le \theta(\|\xi^\circ\|)e^{-bt}$$

for all $t \ge 0$. Assuming – without loss of generality – that $\|z(t)\| \ge 1$, and using the fact that $\|z\|^2 \le cz^{\mathrm{T}}Pz$ for some $c > 0$, we obtain

$$\dot{V}(t) \le 4\|P\|\|z\|^2\theta(\|\xi^\circ\|)e^{-bt} \le V(t)d\theta(\|\xi^\circ\|)e^{-bt}$$

for some $d > 0$. This, by the Gronwall-Bellman inequality, yields

$$V(t) \le V(0)e^{d\theta(\|\xi^\circ\|)\int_0^t e^{-bs}ds} \le V(0)e^{(d/b)\theta(\|\xi^\circ\|)}$$

which shows that $V(t)$ is bounded on $[0,\infty)$. Thus, since $z^{\mathrm{T}}Pz$ is bounded from below by $(1/c)\|z\|^2$, also $z(t)$ is bounded. ◃

Having shown that, under the hypotheses indicated in this Lemma, the trajectories of system (14.37) are bounded, we proceed with the determination of a function $\Psi(z,\xi)$ satisfying (14.39). To explain how this can be achieved, set again, as before,

$$x = \begin{pmatrix} z \\ \xi \end{pmatrix}, \quad f(x) = \begin{pmatrix} Az + p(z,\xi) \\ \varphi(\xi) \end{pmatrix},$$

and also

$$G(x) = \Psi(z,\xi), \quad F(x) = 2z^{\mathrm{T}}Pp(z,\xi)\,.$$

In these notations, (14.39) reduces to the partial differential equation

$$\frac{\partial G}{\partial x}f(x) = -F(x) \tag{14.40}$$

in the unknown $G(x)$.

Let $\overline{x}(x,t)$ denote the value at time $t \ge 0$ of the integral curve of $\dot{x} = f(x)$ passing through x at time $t = 0$. As shown before, the function $\overline{x}(x,t)$ is defined for all $(x,t) \in (\mathbb{R}^n \times \mathbb{R}^\nu) \times [0,\infty)$. It is possible to show that the function $F(\overline{x}(x,t))$ has the following interesting properties.

Lemma 14.2.3. *Suppose the hypotheses* (i), (ii), (iii) *of Lemma 14.2.2 hold. Then, then function* $F(\overline{x}(x,t))$ *is such that*

$$\lim_{t\to\infty} F(\overline{x}(x,t)) = 0\,, \tag{14.41}$$

$$\int_0^\infty F(\overline{x}(x,s))ds < \infty\,, \tag{14.42}$$

for all $x \in (\mathbb{R}^n \times \mathbb{R}^\nu)$. *Moreover, the function*

$$G(x) = \int_0^\infty F(\overline{x}(x,s))ds\,, \tag{14.43}$$

is a smooth function.

Proof. Let $\overline{z}(z,\xi,t)$ and $\overline{\xi}(\xi,t)$ denote the upper and, respectively, lower components of $\overline{x}(x,t)$ (note that the argument z is not present in the lower component of $\overline{x}(x,t)$ because the lower subsystem of (14.37) is independent of z), and observe that

$$|F(\overline{x}(x,t))| \le 2\|P\|\|\overline{z}(z,\xi,t)\|(1+\|\overline{z}(z,\xi,t)\|)\gamma(\|\overline{\xi}(\xi,t)\|)\,.$$

We have shown in Lemma 14.2.2 that, for every (z,ξ), there is a number $M>0$, depending on (z,ξ), such that $\|\overline{z}(z,\xi,t)\| \le M$ for all $t\ge 0$. Also, we know that, for some $b>0$ and class $\mathcal{K}$ function $\theta(\cdot)$, $\gamma(\|\overline{\xi}(\xi,t)\|) \le \theta(\|\xi\|)e^{-bt}$ for all $t \ge 0$ and all ξ. Thus

$$|F(\overline{x}(x,t))| \le 2\|P\|M(1+M)\theta(\|\xi\|)e^{-bt}$$

and both (14.41) and (14.42) immediately follow [2]. ◁

Now, set

$$x^\circ(t) = \overline{x}(x^\circ,t)$$

and, using the well-know semi-group property of $\overline{x}(x^\circ,t)$, note that

$$G(x^\circ(t)) = \int_0^\infty F(\overline{x}(x^\circ(t),s))ds = \int_0^\infty F(\overline{x}(x^\circ,t+s))ds = \int_t^\infty F(\overline{x}(x^\circ,s'))ds'.$$

Taking derivatives with respect to time yields

$$\left[\frac{\partial G}{\partial x}f(x)\right]_{x=x^\circ(t)} = -F(\overline{x}(x^\circ,t)) = -F(x^\circ(t))\,.$$

Since x° was arbitrary, this shows that

$$\frac{\partial G}{\partial x}f(x) = -F(x)$$

i.e. that the function (14.43) solves the p.d.e. (14.40).

[2] The proof that $G(x)$ is a smooth function can be found in Jankovic *et al.* (1996).

We have in this way found a function which renders the condition (14.39) fulfilled. In terms of z,ξ arguments, this function is expressed as

$$\Psi(z,\xi) = \int_0^\infty 2[\overline{z}(z,\xi,s)]^{\mathrm{T}}Pp(\overline{z}(z,\xi,s),\overline{\xi}(\xi,s))ds\,. \tag{14.44}$$

In addition, it is also possible to show that the resulting function (14.38) is positive definite and proper [3].

Lemma 14.2.4. *Suppose the hypotheses* (i), (ii), (iii) *of Lemma 14.2.2 hold. Then, then function*

$$W(z,\xi) = z^{\mathrm{T}}Pz + U(\xi) + \Psi(z,\xi) \tag{14.45}$$

is positive definite and proper.

Remark 14.2.2. It is left to the reader, as an exercise, to see that, going through the same arguments of the previous discussion, similar conclusions can be reached in case the linear term Az in system (14.37) is substituted by a nonlinear term $a(z)$, so long as there exists a positive definite and proper function $V(z)$ such that $L_aV(z) \le 0$ and such that

$$\left\|\frac{\partial V}{\partial z}\right\|\|z\| \le cV(z)\,.$$

for some $c>0$. In this case, the cross-term $\Psi(z,\xi)$ would have the expression

$$\Psi(z,\xi) = \int_0^\infty \left[\frac{\partial V}{\partial z}\right]_{z=\overline{z}(z,\xi,s)} p(\overline{z}(z,\xi,s),\overline{\xi}(\xi,s))ds\,.$$

However, to establish continuous differentiability of $\Psi(z,\xi)$ appropriate additional hypotheses on $a(z)$ are needed [4]. ◁

The next (and last) step needed to render the result of Proposition 14.1.1 applicable in this problem is to check that, choosing $V(x) = W(z,\xi)$, with $W(z,\xi)$ as in (14.45), the zero-state detectability hypothesis of this Proposition is fulfilled. This requires some extra assumptions on system (14.27).

For convenience, we use in this instance the result of Lemma 14.1.4 and, instead of checking the detectability of the system corresponding to (14.3), we check the detectability of the system corresponding to

$$\begin{aligned} \dot{x} &= f(x) \\ y &= \begin{pmatrix} L_gV(x) \\ L_fV(x) \end{pmatrix}\,. \end{aligned}$$

Observe that the system we are dealing with, namely system (14.27), in the present case (i.e. when $a(z) = Az$), can be written as

[3] For the proof, see Jankovic *et al.* (1996).

[4] See Sepulchre *et al.* (1996), for further details.

$$\begin{aligned} \dot{z} &= Az + p(z,\xi) + b(z,\xi)u \\ \dot{\xi} &= \varphi(\xi) + \theta(\xi)u \end{aligned} \tag{14.46}$$

and therefore the issue is to establish whether or not the system

$$\begin{aligned} \dot{z} &= Az + p(z,\xi) \\ \dot{\xi} &= \varphi(\xi) \\ y_1 &= \frac{\partial W}{\partial z} b(z,\xi) + \frac{\partial W}{\partial \xi}\theta(\xi) \\ y_2 &= \frac{\partial W}{\partial z}[Az + p(z,\xi)] + \frac{\partial W}{\partial \xi}\varphi(\xi) \end{aligned} \tag{14.47}$$

is zero-state detectable.

It will be shown that this is the case if:

- the linear approximation of system (14.46) at the equilibrium $(z,\xi,u) = (0,0,0)$ is *stabilizable*,
- the functions $p(z,\xi)$ and $b(z,\xi)$ are such that

$$\left[\frac{\partial p}{\partial \xi}\right]_{(z,0)} = M = \text{constant}, \qquad b(z,0) = B = \text{constant}. \tag{14.48}$$

Note, to this end, that the linear approximation of system (14.46) at the equilibrium $(z,\xi,u) = (0,0,0)$ is a linear system of the form

$$\begin{aligned} \dot{z} &= Az + M\xi + Bu \\ \dot{\xi} &= F\xi + Gu \end{aligned} \tag{14.49}$$

in which

$$M = \left[\frac{\partial p}{\partial \xi}\right]_{(0,0)}, \quad B = b(0,0), \quad F = \left[\frac{\partial \varphi}{\partial \xi}\right]_{(0)}, \quad G = \theta(0).$$

Lemma 14.2.5. *Consider system (14.46). Suppose the hypotheses* (i), (ii), (iii) *of Lemma 14.2.2 hold. Suppose also that the linear approximation (14.49) of system (14.46) at the equilibrium $(z,\xi,u) = (0,0,0)$ is stabilizable and that (14.48) hold. Then, system (14.47) is zero-state detectable.*

Proof. Consider the linear system (14.49) and note that it satisfies all the hypotheses of Lemma 14.2.4. Thus, formula (14.45) provides the expression of a positive definite function whose derivative along the trajectories of (14.49) with $u = 0$ is negative semi-definite. In particular, choose $U(\xi)$ to be a quadratic function (which is indeed possible because the system in question is linear) and observe that the cross-term reduces to the function

$$\Psi_L(z,\xi) = \int_0^\infty 2[\bar{z}_L(z,\xi,s)]^{\mathrm{T}} P M e^{Fs}\xi ds$$

where $\bar{z}_L(z,\xi,t)$ is the solution of

$$\dot{z}(t) = Az(t) + Me^{Ft}\xi, \qquad \text{with } z(0) = z.$$

Since the latter is a linear function in (z,ξ), the cross-term $\Psi(z,\xi)$ is quadratic. Therefore, the entire function (14.45) is a quadratic form, which can be written as

$$W_L(z,\xi) = \begin{pmatrix} z \\ \xi \end{pmatrix}^{\mathrm{T}} P_L \begin{pmatrix} z \\ \xi \end{pmatrix},$$

with $P_L > 0$. By construction, the derivative of this function along the trajectories of (14.49) with $u = 0$ is a negative semi-definite quadratic form. Thus, system (14.49), which is assumed to be stabilizable, satisfies the hypotheses of Lemma 10.9.3. As a consequence, the system

$$\begin{aligned} \dot{z} &= Az + M\xi \\ \dot{\xi} &= F\xi \\ y &= (B^{\mathrm{T}} \quad G^{\mathrm{T}}) P_L \begin{pmatrix} z \\ \xi \end{pmatrix} \end{aligned} \tag{14.50}$$

is detectable. Note that particular trajectories of this system are those of the form $(z(t), 0)$ in which $z(t)$ satisfies $\dot{z} = Az$. Thus, since system (14.50) is detectable, so is also the system

$$\begin{aligned} \dot{z} &= Az \\ w &= (B^{\mathrm{T}} \quad G^{\mathrm{T}}) P_L \begin{pmatrix} z \\ 0 \end{pmatrix}. \end{aligned} \tag{14.51}$$

Now, return to the nonlinear system (14.46), and recall that the issue is to establish that system (14.47) is zero-state detectable. Since by construction

$$y_2 = z^{\mathrm{T}}(PA + A^{\mathrm{T}}P)z + \frac{\partial U}{\partial \xi}\varphi(\xi) \le \frac{\partial U}{\partial \xi}\varphi(\xi)$$

and the term on the right-hand side is a negative definite function of ξ, $y_2 = 0$ implies $\xi = 0$. Thus, to study the zero-state detectability of (14.47) reduces (recall that $p(z,0) = 0$) to study the zero-state detectability of

$$\begin{aligned} \dot{z} &= Az \\ y &= \left[\frac{\partial W}{\partial z}\right]_{(z,0)} b(z,0) + \left[\frac{\partial W}{\partial \xi}\right]_{(z,0)} \theta(0). \end{aligned} \tag{14.52}$$

Now, it is easy to check that the output of (14.52) coincides (modulo a constant factor) with the output of (14.51), which was proven to be detectable. Thus also (14.52) is detectable and this proves the Lemma. To see that this is true, observe that the output w of (14.51) is

$$w = (z^{\mathrm{T}} \quad 0) P_L \begin{pmatrix} B \\ G \end{pmatrix} = \frac{1}{2}\left[\frac{\partial W_L}{\partial z}\right]_{(z,0)} B + \frac{1}{2}\left[\frac{\partial W_L}{\partial \xi}\right]_{(z,0)} G,$$

and therefore, in view of the second one of hypotheses (14.48) and the definition of B and G

$$2w = \left[\frac{\partial W_L}{\partial z}\right]_{(z,0)} b(z,0) + \left[\frac{\partial W_L}{\partial \xi}\right]_{(z,0)} \theta(0) .$$

Comparing with the output of (14.52), we see that it suffices to check that

$$\left[\frac{\partial W}{\partial z}\right]_{(z,0)} = \left[\frac{\partial W_L}{\partial z}\right]_{(z,0)}, \qquad \left[\frac{\partial W}{\partial \xi}\right]_{(z,0)} = \left[\frac{\partial W_L}{\partial \xi}\right]_{(z,0)} .$$

The first one is trivially true, because

$$W_L(z,0) = W(z,0) = z^{\mathrm{T}} P z .$$

As far as the second one is concerned, we need to check that

$$\left[\frac{\partial \Psi_L}{\partial \xi}\right]_{(z,0)} = \left[\frac{\partial \Psi}{\partial \xi}\right]_{(z,0)} . \tag{14.53}$$

Using the first one of the hypotheses (14.48), it is seen that

$$\begin{aligned}\left[\frac{\partial \Psi}{\partial \xi}\right]_{(z,0)} &= \int_0^\infty 2[\bar{z}(z,0,s)]^{\mathrm{T}} P \left[\frac{\partial p}{\partial \xi}\right]_{(z,0)} \left[\frac{\partial \bar{\xi}}{\partial \xi}\right]_{(0)} ds \\ &= \int_0^\infty 2[\bar{z}(z,0,s)]^{\mathrm{T}} P M \left[\frac{\partial \bar{\xi}}{\partial \xi}\right]_{(0)} ds = \int_0^\infty 2[\bar{z}(z,0,s)]^{\mathrm{T}} P M e^{Fs} ds\end{aligned}$$

where, in the last passage, we have used the fact that

$$\left[\frac{\partial \bar{\xi}}{\partial \xi}\right]_{(0)} = e^{Fs} .$$

On the other hand,

$$\left[\frac{\partial \Psi_L}{\partial \xi}\right]_{(z,0)} = \int_0^\infty 2[\bar{z}_L(z,0,s)]^{\mathrm{T}} P M e^{Fs} ds .$$

Since $\bar{z}(z,0,s) = \bar{z}_L(z,0,s)$, this shows that (14.53) holds and completes the proof. ◁

We have thus shown that, under the hypotheses of this Lemma, system (14.46), thanks to the result of Proposition 14.1.1, can be globally asymptotically stabilized, and this accomplishes one of the main design goals stated at the beginning on the section (namely, stabilization of a system of the form (14.26)). One of the hypotheses of the last Lemma is that the equilibrium $\xi = 0$ of the lower subsystem of (14.46), when $u = 0$, is also locally exponentially stable. Thus, in view of seeking a recursive use of the previous results in order to be able to stabilize a system of the form (14.24), we should check that the globally asymptotically stabilizing law proposed for (14.46) also achieves local exponential stability. This is indicated in the following result, which – for convenience – repeats all the hypotheses introduced in the course of the previous analysis.

Theorem 14.2.6. *Consider system*

$$\begin{aligned}\dot{z} &= Az + p(z,\xi) + b(z,\xi)u \\ \dot{\xi} &= \varphi(\xi) + \theta(\xi)u .\end{aligned} \tag{14.54}$$

Assume the following:
(i) *there exists a symmetric matrix $P > 0$ such that $PA + A^{\mathrm{T}}P \le 0$,*
(ii) *the equilibrium $\xi = 0$ of the system $\dot{\xi} = \varphi(\xi)$ is globally asymptotically stable and also locally exponentially stable,*
(iii) *the function $p(z,\xi)$ is such that*

$$\|p(z,\xi)\| \le \gamma(\|\xi\|)(1 + \|z\|) \quad \textit{for all } z, \xi ,$$

where $\gamma(\cdot)$ is a class $\mathcal{K}$ function, differentiable at the origin.
(iv) *the functions $p(z,\xi)$ and $b(z,\xi)$ are such that*

$$\left[\frac{\partial p}{\partial \xi}\right]_{(z,0)} = \textit{constant}, \qquad b(z,0) = \textit{constant} ,$$

(v) *the linear approximation of (14.54) at the equilibrium $(z,\xi,u) = (0,0,0)$ is stabilizable.*

Let $U(\xi)$ be a smooth function such that, for all ξ,

$$\underline{\alpha}(\|\xi\|) \le U(\xi) \le \overline{\alpha}(\|\xi\|)$$
$$L_\varphi U(\xi) < -\alpha(\|\xi\|)$$

where $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$ are class $\mathcal{K}_\infty$ functions with the property that, for some $\delta > 0, a > 0, b > 0$, $\underline{\alpha}(s) = as^2$ and $\alpha(s) = bs^2$ for all $s \in [0,\delta]$.

Let $\Psi(z,\xi)$ be the function defined in (14.44) and set

$$u^*(z,\xi) = 2z^{\mathrm{T}} P b(z,\xi) + \frac{\partial \Psi}{\partial z} b(z,\xi) + \frac{\partial \Psi}{\partial \xi}\theta(\xi) + \frac{\partial U}{\partial \xi}\theta(\xi) . \tag{14.55}$$

The feedback law

$$u = -\varepsilon u^*(z,\xi)$$

for any $\varepsilon > 0$ globally asymptotically and locally exponentially stabilizes the equilibrium $(z,\xi) = (0,0)$ of (14.54).

Proof. All has already been proven, except the local exponential stability. This follows from arguments similar to those used in the proof of the previous Lemma. The proof of this Lemma has shown that

$$u^*(z,0) = \left[\frac{\partial W}{\partial z}\right]_{(z,0)} b(z,0) + \left[\frac{\partial W}{\partial \xi}\right]_{(z,0)} \theta(0)$$

and that the latter is a linear function of z, which can be given the form

$$2\,(B^{\mathrm{T}} \quad G^{\mathrm{T}})\,P_L \begin{pmatrix} z \\ 0 \end{pmatrix} .$$

In a similar way, one can show that the *linear approximation* of the function $u^*(0,\xi)$ at $\xi = 0$ can be given the form

$$2\,(B^{\mathrm{T}} \quad G^{\mathrm{T}})\,P_L \begin{pmatrix} 0 \\ \xi \end{pmatrix} .$$

From this, it is easily seen that the linear approximation of $-\varepsilon u^*(z,\xi)$ at $(z,\xi) = 0$ coincides with the law

$$u_L(z,\xi) = -2\varepsilon\,(B^{\mathrm{T}} \quad G^{\mathrm{T}})\,P_L \begin{pmatrix} z \\ \xi \end{pmatrix} ,$$

which is known (using Proposition 14.1.1 and the property that (14.50) is detectable) to asymptotically stabilize the linear approximation of (14.54) at the equilibrium $(z,\xi,u) = (0,0,0)$. This próves the result. ◁

As shown at the beginning of the section, a result of this kind lends itself to the implementation of a recursive design procedure for the synthesis of a stabilizing feedback law for a system of the form

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + p_1(x_1,x_2,\ldots,x_n) + b_1(x_1,x_2,\ldots,x_n)u \\ \dot{x}_2 &= A_2x_2 + p_2(x_2,x_3,\ldots,x_n) + b_2(x_2,x_3,\ldots,x_n)u \\ &\cdots \\ \dot{x}_{n-1} &= A_{n-1}x_{n-1} + p_{n-1}(x_{n-1},x_n) + b_{n-1}(x_{n-1},x_n)u \\ \dot{x}_n &= \varphi(x_n) + \theta(x_n)u , \end{aligned} \qquad (14.56)$$

in which it is assumed that the matrices A_i, for $i = 1,\ldots,n-1$, are such that the associated system $\dot{\tilde{x}}_i = A_i\tilde{x}_i$ is stable in the sense of Lyapunov (but possibly not asymptotically stable), and $p_i(x_i,0,\ldots,0) = 0$.

If the lower system of the chain, when $u = 0$, is globally asymptotically and locally exponentially stable, if the terms $p_{n-1}(x_{n-1},x_n)$ and $b_{n-1}(x_{n-1},x_n)$ satisfy hypotheses (iii) and (iv), and if the linear approximation of

$$\begin{aligned} \dot{x}_{n-1} &= A_{n-1}x_{n-1} + p_{n-1}(x_{n-1},x_n) + b_{n-1}(x_{n-1},x_n)u \\ \dot{x}_n &= \varphi(x_n) + \theta(x_n)u , \end{aligned}$$

is stabilizable, one can find a feedback law $u_{n-1}(x_{n-1},x_n)$ which renders the system

$$\begin{aligned} \dot{x}_{n-1} &= A_{n-1}x_{n-1} + p_{n-1}(x_{n-1},x_n) + b_{n-1}(x_{n-1},x_n)(u_{n-1}(x_{n-1},x_n)+u) \\ \dot{x}_n &= \varphi(x_n) + \theta(x_n)(u_{n-1}(x_{n-1},x_n)+u) , \end{aligned}$$

globally asymptotically and locally exponentially stable, when $u = 0$, and then iterate the process. Note that, by proceeding in this way, i.e. replacing u by $u(x_{n-1},x_n)+u$, the interconnection terms $p_i(x_i,\ldots,x_1)$ and $b_i(x_i,\ldots,x_1)$ are changed and therefore the hypotheses (iii) and (iv) must be re-checked (the fulfillment of hypothesis (v) is guaranteed, at each stage, by the hypothesis that the linear approximation of the entire system (14.56) is stabilizable, thanks to the triangular structure and to the fact that a state feedback does not destroy stabilizability).

One situation which the fulfillment of hypotheses (iii) and (iv) is guaranteed in advance, is when the interconnection terms satisfy

$$\begin{aligned} \|p_i(x_i,x_{i+1},\ldots,x_n)\| &\le \gamma_i(\|(x_{i+1},\ldots,x_n)\|)(1+\|x_i\|) \\ \|b_i(x_i,x_{i+1},\ldots,x_n)\| &\le \chi_i(\|(x_{i+1},\ldots,x_n)\|)(1+\|x_i\|) \\ \left[\frac{\partial p_i}{\partial(x_{i+1},\ldots,x_n)}\right]_{(x_i,0,\ldots,0)} &= \text{constant} \\ b_i(x_i,0,\ldots,0) &= \text{constant} , \end{aligned}$$

for appropriate class $\mathcal{K}$ functions $\gamma_i(\cdot)$ and $\chi_i(\cdot)$.

In fact, consider for instance the case in which $n = 3$. In this case, after having designed the feedback law $u_2(x_2,x_3)$ and replaced u by $u_2(x_2,x_3)+u$ one obtains a system of the form

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + \tilde{p}_1(x_1,x_2,x_3) + \tilde{b}_1(x_1,x_2,x_3)u \\ \dot{x}_2 &= A_2x_2 + \tilde{p}_2(x_2,x_3) + \tilde{b}_2(x_2,x_3)u \\ \dot{x}_3 &= \tilde{f}_3(x_2,x_3,u) , \end{aligned}$$

in which

$$\begin{aligned} \tilde{p}_1(x_1,x_2,x_3) &= p_1(x_1,x_2,x_3) + b_1(x_1,x_2,x_3)u_2(x_2,x_3) \\ \tilde{b}_1(x_1,x_2,x_3) &= b_1(x_1,x_2,x_3) . \end{aligned}$$

Trivially, if $b_1(x_1,0,0)$ is constant so is $\tilde{b}_1(x_1,0,0)$. Moreover, since $u_2(0,0) = 0$, also

$$\left[\frac{\partial \tilde{p}_1}{\partial(x_2,x_3)}\right]_{(x_1,0,0)} = \left[\frac{\partial p_1}{\partial(x_2,x_3)}\right]_{(x_1,0,0)} + b_1(x_1,0,0)\left[\frac{\partial u_2}{\partial(x_2,x_3)}\right]_{(0,0)}$$

is constant. Finally,

$$\begin{aligned} &\|\tilde{p}_1(x_1,x_2,x_3)\| \\ &\quad \le \gamma_1(\|(x_2,x_3)\|)(1+\|x_1\|) + \chi_1(\|(x_2,x_3)\|)(1+\|x_1\|)\|u_1(x_2,x_3)\| \end{aligned}$$

which establishes the desired growth condition for $\tilde{p}_1(x_1,x_2,x_3)$.

To conclude the section, we observe that the results described above can also be used for global asymptotic stabilization of systems described by equations of the form

$$\begin{aligned}\dot{x}_1 &= A_1x_1 + g_1(x_1, x_2, \ldots, x_n, u)\\ \dot{x}_2 &= A_2x_2 + g_2(x_2, x_3, \ldots, x_n, u)\\ &\cdots\\ \dot{x}_{n-1} &= A_{n-1}x_{n-1} + g_{n-1}(x_{n-1}, x_n, u)\\ \dot{x}_n &= g_n(x_n, u)\,,\end{aligned}$$

in which the right-hand side is not an affine function of the control input u. To this end, in fact, it suffices to split the $g_i(x_i, \ldots, x_n, u)$'s as

$$g_i(x_i, \ldots, x_n, u) = p_i(x_i, \ldots, x_n) + b_i(x_i, \ldots, x_n)u + q_i(x_i, \ldots, x_n, u)u^2$$

in which

$$\begin{aligned}p_i(x_i, \ldots, x_n) &= g_i(x_i, \ldots, x_n, 0)\\ b_i(x_i, \ldots, x_n) &= \left[\frac{\partial g_i}{\partial u}\right]_{(x_i,\ldots,x_n,0)}\end{aligned}$$

and use the result of Theorem 14.1.3. At each iteration of the recursive procedure, one has to deal with a system of the form

$$\begin{aligned}\dot{z} &= Az + p(z,\xi) + b(z,\xi)u + q(z,\xi,u)u^2\\ \dot{\xi} &= \varphi(\xi) + \theta(\xi)u + \gamma(\xi,u)u^2\,.\end{aligned} \tag{14.57}$$

If $A, p(z,\xi), b(z,\xi), \varphi(\xi), \theta(\xi)$ are such that the hypotheses of Theorem 14.2.6 hold, it is possible to find a positive definite proper function $W(z,\xi)$ such that

$$\frac{\partial W}{\partial z}[Az + p(z,\xi)] + \frac{\partial W}{\partial \xi}\varphi(\xi) \le 0$$

and such that system

$$\begin{aligned}\dot{z} &= Az + p(z,\xi)\\ \dot{\xi} &= \varphi(\xi)\\ y &= \frac{\partial W}{\partial z}b(z,\xi) + \frac{\partial W}{\partial \xi}\theta(\xi)\,,\end{aligned}$$

is zero-state detectable. Thus, using Theorem 14.1.3, it is possible to find a feedback law which globally asymptotically stabilizes (14.57). As shown before, at each stage of the iteration, the hypotheses of Theorem 14.2.6 have to be re-checked, unless particular hypotheses are set on the various functions which characterize the system, so as to guarantee these properties in advance. The details on how this can be achieved are left as an exercise to the reader.

14.3 Stabilization Using Saturation Functions

The basic structure analyzed in the previous section (see (14.26)) was that of a nonlinear system

$$\dot{\xi} = \phi(\xi, u)\,,$$

which was supposed to possess a globally asymptotically and locally exponentially stable equilibrium at $(\xi, u) = (0, 0)$, cascaded with a system

$$\dot{z} = \psi(z, \xi, u)$$

in which

$$\psi(z, 0, 0) = Az\,.$$

Under the hypothesis that $PA + A^{\mathrm{T}}P \le 0$ for some symmetric matrix $P > 0$, and appropriate additional conditions on the function $\psi(z,\xi,u)$, it was shown that a system of this form can be globally asymptotically stabilized, using a control input with bounded amplitude.

In this section, we describe a different approach to the stabilization of systems having this structure, which is in a certain sense more appealing, because the control law that will result is not based on the explicit availability of a Lyapunov function, as it happens for the feedback law (14.55) determined in the previous section. It must be stressed, in fact, that the explicit calculation of the law (14.55) requires the prior determination of the term $\Psi(z,\xi)$ via the integral (14.44), and this may result, in general, in a difficult computational task.

The first stage in the analysis consists in the study of the case in which the dynamics of a linear system

$$\dot{z} = Az + Bu\,,$$

in which the pair (A, B) stabilizable and $PA + A^{\mathrm{T}}P \le 0$ for some symmetric matrix $P > 0$, are perturbed by the addition of a nonlinear term of the form $g(\xi, u)$, with ξ the state of a nonlinear system

$$\dot{\xi} = f(\xi, u)\,,$$

i.e. the case of a system having the following structure

$$\begin{aligned}\dot{z} &= Az + Bu + g(\xi, u)\\ \dot{\xi} &= f(\xi, u)\,.\end{aligned} \tag{14.58}$$

Indeed, this can be viewed as a system the form (14.57), if it is assumed that $b(z,\xi)$ and $q(z,\xi,u)$ are independent of z.

In what follows, it will be described how system (14.58) can be stabilized by means of a feedback law of the form

$$u = u(z, v)\,, \tag{14.59}$$

in which v represents an additional control input, introduced here for the purpose of setting the stage for a recursive design in the case of systems having a feedforward structure. Note that this feedback depends only on the state z and not on the state ξ, as opposite to the feedback derived in the previous section, which was depending on both z and ξ. We shall see, however, that this structure of the control is not particularly restrictive in case the lower subsystem of (14.58) possesses a globally asymptotically equilibrium at $(\xi, u) = (0, 0)$, as a natural intuition seems to suggest.

The control law (14.59) induces a feedback coupling between the two subsystems, which would be otherwise only cascade coupled. As a matter of fact, system (14.58) with feedback law (14.59) can be regarded as the interconnection of a system with inputs u_1 and v, state x_1 and output y_1, modeled by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1, v) \\ y_1 &= h_1(x_1, v)\,, \end{aligned} \tag{14.60}$$

and a system with input u_2, state x_2 and output y_2, modeled by equations of the form

$$\begin{aligned} \dot{x}_2 &= f_2(x_2, u_2) \\ y_2 &= h_2(x_2, u_2)\,, \end{aligned} \tag{14.61}$$

via (see Fig. 14.2)

$$u_2 = y_1 \qquad u_1 = y_2\,. \tag{14.62}$$

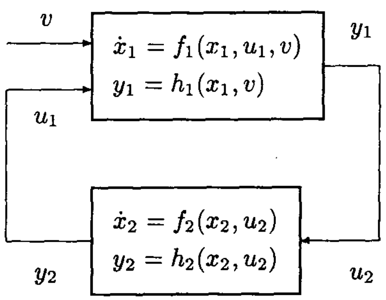

Fig. 14.2. Feedback connection with input v.

To see that this is the case, it suffices to set $x_1 = z$, $x_2 = \xi$, and

$$\begin{aligned} f_1(x_1, u_1, v) &= Ax_1 + Bu(x_1, v) + u_1 \\ h_1(x_1, v) &= u(x_1, v) \\ f_2(x_2, u_2) &= f(x_2, u_2) \\ h_2(x_2, u_2) &= g(x_2, u_2)\,. \end{aligned} \tag{14.63}$$

Stability of such an interconnected system can be studied via the small-gain theorem. In the present case, taking advantage of the fact that the control laws that are to be used have bounded amplitude, it is convenient to appeal to a modified version of this theorem, which is explained below and uses a concept of "gain", introduced by A. Teel [5], which considers *only* bounds on the *asymptotic* behavior of the response, as $t \to \infty$.

For a piecewise-continuous function $u : [0, \infty) \to \mathbb{R}^m$, define

$$\|u(\cdot)\|_a = \limsup_{t\to\infty}\{\max_{1\le i\le m} |u_i(t)|\}\,.$$

The quantity thus introduced is referred to as the *asymptotic* "norm" of $u(\cdot)$.

For a system having inputs and outputs, modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u)\,, \end{aligned} \tag{14.64}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and $f(0,0) = 0$, $h(0,0) = 0$, the definition which follows characterizes the notion of "gain" in terms of asymptotic bounds on the input and on the corresponding output response.

Definition 14.3.1. *System (14.64) is said to satisfy an asymptotic (input-output) bound , with restriction X on x° and restriction U on $u(\cdot)$, if there exists a class $\mathcal{K}$ function $\gamma_u(\cdot)$, called a gain function, such that, for any $x^\circ \in X$ and for any piecewise-continuous input $u(\cdot)$ satisfying $\|u(\cdot)\|_a < U$, the response $x(t)$ in the initial state $x(0) = x^\circ$ exists for all $t \ge 0$ and is such that $y(t) = h(x(t), u(t))$ satisfies*

$$\|y(\cdot)\|_a \le \gamma_u(\|u(\cdot)\|_a)\,.$$

Remark 14.3.1. It is obvious that the above definition, in the way it is expressed, only characterizes how the asymptotic bound on the output is affected by the asymptotic bound on the input and no special care is taken of bounds on the state, if any. In order to compare this with earlier definitions, it is convenient to specialize it to the case in which $h(x, u) = x$, i.e. the output of (14.64) coincides with its state. In this case, the definition says that, for any $x^\circ \in X$,

$$\|x(\cdot)\|_a \le \gamma_u(\|u(\cdot)\|_a)$$

but does not prescribe any special relation between $\|x(t)\|$ and $\|x^\circ\|$ (as the definition of input-to-state stability, with restrictions, does). If, in particular, X is a neighborhood of the origin and $u(t) = 0$, the definition implies that

$$\lim_{t\to\infty} \|x(t)\| = 0\,,$$

but does not necessarily imply local asymptotic stability of the origin, since this point is not required to be stable in the sense of Lyapunov. It is because of this fact that, in the subsequent applications, in order to check that

[5] See Teel (1996a).

certain systems have a (global) asymptotically stable equilibrium, the issue of local asymptotic stability will be addressed separately via other methods, specifically the principle of stability in the first approximation. ◁

Suppose now that system (14.60) satisfies an asymptotic input-output bound, with restriction X_1 on x_1°, restriction U_1 on $u_1(\cdot)$ and restriction V on $v(\cdot)$, that is, suppose there exist gain functions $\gamma_1(\cdot)$ and $\gamma_v(\cdot)$ such that, for any $x_1(0) \in X_1$, the response $x_1(t)$ to piecewise-continuous $u_1(\cdot)$ and $v(\cdot)$ (each one satisfying the indicated restriction) exists for all $t \geq 0$ and

$$\|y_1(\cdot)\|_a \leq \max\{\gamma_1(\|u_1(\cdot)\|_a), \gamma_v(\|v(\cdot)\|_a)\}\,.$$

Likewise, suppose that system (14.61) satisfies an asymptotic input-output bound, with restriction X_2 on x_2° and restriction U_2 on $u_2(\cdot)$, that is, suppose there exists a gain function $\gamma_2(\cdot)$ such that, for any $x_2(0) \in X_2$, the response $x_2(t)$ to a piecewise-continuous $u_2(\cdot)$ (satisfying the indicated restriction) exists for all $t \geq 0$ and

$$\|y_2(\cdot)\|_a \leq \gamma_2(\|u_2(\cdot)\|_a)\,.$$

The following Theorem shows that, if the gain functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ satisfy the small-gain condition, the interconnected system (14.60) – (14.61) – (14.62) satisfies an asymptotic input-output bound, with appropriate restrictions.

Theorem 14.3.1. *Consider the interconnected system of Fig. 14.2 and suppose both subsystems satisfy asymptotic input-output bounds, with restrictions and gain functions as indicated above. Suppose $U_1 = \infty$. Suppose also that*

$$\lim_{r\to\infty} \gamma_1(r) < \infty, \qquad \lim_{r\to\infty} \gamma_1(r) \leq U_2\,.$$

Let $\tilde V$ be any number satisfying

$$\tilde V \leq V, \qquad \gamma_v(\tilde V) \leq U_2\,.$$

Suppose that, for all $(x_1(0), x_2(0)) \in X_1 \times X_2$ and any piecewise-continuous $v(\cdot)$ satisfying $\|v(\cdot)\|_a \leq \tilde V$, the response $(x_1(t), x_2(t))$ exists for all $t \geq 0$.

Then, if

$$\gamma_1 \circ \gamma_2(r) < r$$

for all $r > 0$, the interconnected system satisfies an asymptotic input-output bound, with restriction $X_1 \times X_2$ on (x_1°, x_2°), restriction $\tilde V$ on $v(\cdot)$ and

$$\begin{aligned} \|y_1(\cdot)\|_a &\leq \gamma_v(\|v(\cdot)\|_a) \\ \|y_2(\cdot)\|_a &\leq \gamma_2 \circ \gamma_v(\|v(\cdot)\|_a)\,. \end{aligned} \tag{14.65}$$

Proof. If $\|v(\cdot)\|_a \leq \tilde V \leq V$, since there is no restriction on $\|u_1(\cdot)\|_a$, we have

$$\|y_1(\cdot)\|_a \leq \max\{\gamma_1(\|y_2(\cdot)\|_a), \gamma_v(\|v(\cdot)\|_a)\}\,.$$

Assuming that $\|v(\cdot)\|_a$ is finite (otherwise, there is nothing to prove), this shows – in view of the hypothesis on $\gamma_1(\cdot)$ and the definition of $\tilde V$ – that $\|y_1(\cdot)\|_a$ is finite and

$$\|y_1(\cdot)\|_a \leq U_2\,.$$

Therefore,

$$\|y_2(\cdot)\|_a \leq \gamma_2(\|y_1(\cdot)\|_a)\,,$$

and also $\|y_2(\cdot)\|_a$ is finite.

Combining the inequalities and using the small-gain condition yields (14.65). ◁

This particular version of the small-gain theorem will be used to prove the desired stabilization result for system (14.58). To this end, an auxiliary property is needed, which is an immediate consequence of Proposition 14.1.5.

Corollary 14.3.2. *Consider the linear system (14.17). Suppose (A, B) is stabilizable and there exists a symmetric matrix $P > 0$ such that (14.18) holds. Then, the matrix $A - BB^{\mathrm T}P$ has all eigenvalues in $\mathbb{C}^-$. Let $\sigma(\cdot)$ be any $\mathbb{R}^m$-valued saturation function and consider the system*

$$\begin{aligned} \dot x &= Ax + B\sigma(-B^{\mathrm T}Px + v) + w \\ y &= x\,. \end{aligned} \tag{14.66}$$

Then, there exists a number $\delta' > 0$, such that (14.66) satisfies an asymptotic (input-output) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions $\gamma_v(\cdot)$ and $\gamma_w(\cdot)$.

Proof. If $v(\cdot)$ and $w(\cdot)$ are piecewise-continuous, the response $x(t)$ exists for all $t \geq 0$, for any $x(0)$. Set $c = \max\{\sqrt{m}, \sqrt{n}\}$ and $\delta' = \delta/2c$. Then, for any $v(\cdot)$ and $w(\cdot)$ such that

$$\max\{\|v(\cdot)\|_a, \|w(\cdot)\|_a\} \leq \delta'$$

there is $T > 0$ such that

$$\|v(t)\| \leq \delta, \quad \|w(t)\| \leq \delta, \qquad \text{for all } t \in [T, \infty).$$

The estimates in Proposition 14.1.5 hold for any x°. Thus, no matter what the value of $x(T)$ is, the result follows from the fact that the restrictions of $v(\cdot)$ and $w(\cdot)$ to the interval $[T, \infty)$ satisfy the required bounds. ◁

In other words, this Corollary says that, if (A, B) is stabilizable and there exists a symmetric matrix $P > 0$ such that (14.18) holds, the set of matrices K such that $A + BK$ has all eigenvalues in $\mathbb{C}^-$ and

$$\begin{aligned} \dot{x} &= Ax + B\sigma(Kx + v) + w \\ y &= x \end{aligned} \tag{14.67}$$

satisfies an asymptotic (input-output) bound, with no restriction on x° and restriction $\delta' > 0$ on $v(\cdot)$ and $w(\cdot)$, with *linear* gain functions, is *not empty.* This fact is exploited in the derivation of the next result, which describes how a control law for a system of the form (14.58) can be effectively designed, simply using *linear functions* and *saturation functions.*

Theorem 14.3.3. *Consider the system*

$$\begin{aligned} \dot{z} &= Az + Bu + g(\xi, u) \\ \dot{\xi} &= f(\xi, u)\,, \end{aligned} \tag{14.68}$$

in which $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^\nu$, $u \in \mathbb{R}^m$, $g(\xi, u)$ and $f(\xi, u)$ are locally Lipschitz, and $g(0,0) = 0$, $f(0,0) = 0$. Assume that:

(i) *(A, B) is stabilizable and there exists a symmetric matrix $P > 0$ such that $PA + A^\mathrm{T}P \leq 0$,*

(ii) *the system*

$$\begin{aligned} \dot{\xi} &= f(\xi, u) \\ y &= \xi \end{aligned}$$

satisfies an asymptotic (input-output) bound, with restriction Ξ on ξ° and restriction $U > 0$ on $u(\cdot)$, with linear gain function.

(iii) *the function $g(\xi, u)$ is such that*

$$\lim_{\|(\xi,u)\| \to 0} \frac{\|g(\xi, u)\|}{\|(\xi, u)\|} = 0\,.$$

Let $\sigma(\cdot)$ be any $\mathbb{R}^m$-valued saturation function. Pick an $n \times m$ matrix K such that $A + BK$ has all eigenvalues in $\mathbb{C}^-$ and, for some $\delta' > 0$, system (14.67) satisfies an asymptotic (input-output) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions. Pick two $m \times m$ matrices Γ and Ω.

Then, there exist numbers $\lambda > 0$ and $V > 0$ such that system (14.68), with control

$$u^*(z, v) = \lambda\sigma\Big(\frac{Kz + \Gamma v}{\lambda}\Big) + \Omega v\,, \tag{14.69}$$

and output $y = \mathrm{col}(z, \xi)$ satisfies an asymptotic (input-output) bound, with restriction $\mathbb{R}^n \times \Xi$ on (z°, ξ°) and restriction V on $v(\cdot)$, with linear gain function.

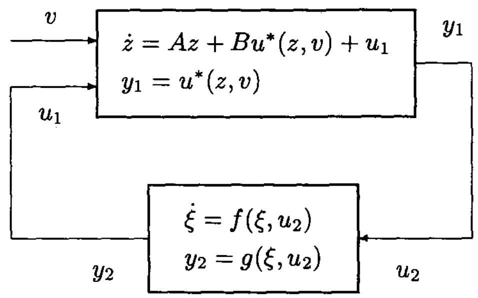

Fig. 14.3. Feedback connection with input v.

Proof. The feedback law (14.69) yields the closed loop system of Fig. 14.3, which is a structure to which it is possible to apply the small-gain Theorem 14.3.1. For convenience, the proof is split into a few steps.

(a) We first show that, if $v(\cdot)$ is piecewise continuous, the trajectories are defined for all t. To this end, suppose the trajectories are defined on $[0, T)$ and observe that, from the properties of a saturation function,

$$\|y_1(t)\| \leq m\lambda\bar{k} + \|\Omega v(t)\|\,.$$

Thus, $y_1(t)$ is bounded on $[0, T)$. Consider now, for the lower subsystem of Fig. 14.3, the input thus defined

$$\begin{aligned} u_2(t) &= y_1(t) \quad \text{if} \quad t \in [0, T) \\ &= 0 \quad \text{if} \quad t \geq T\,, \end{aligned}$$

and let $\tilde{\xi}(t)$ denote the response from the initial state ξ°. By hypothesis (ii), since $\|u_2(\cdot)\|_a = 0$, $\tilde{\xi}_2(t)$ is defined for all $t \geq 0$ and hence bounded on $[0, T)$. By causality, $\tilde{\xi}(t) = \xi(t)$ for all $t \in [0, T)$ and therefore $\xi(t)$ is bounded on $[0, T)$.

Now, consider the upper subsystem of Fig. 14.3. Since both $v(\cdot)$ and $y_2(t)$ are bounded on $[0, T)$, an identical argument shows that $z(t)$ is bounded on $[0, T)$. This shows that no finite escape time is possible, i.e. the trajectories are defined for all $t \geq 0$.

(b) Observe that, if K is chosen as indicated in the Theorem, then, for each $\lambda > 0$, the system

$$\begin{aligned} \dot{z} &= Az + B\lambda\sigma\Big(\frac{Kz + v}{\lambda}\Big) + w \\ y &= z \end{aligned} \tag{14.70}$$

satisfies an asymptotic (input-output) bound, with no restriction on z° and restriction $\lambda\delta'$ on $v(\cdot)$ and $w(\cdot)$, with the same linear gain functions, say

$$\gamma_v(r) = L_v r, \qquad \gamma_w(r) = L_w r\,.$$

To check that this is the case, it suffices to define the new coordinate $x = z/\lambda$, which changes the previous system into

$$\begin{aligned}\dot{x} &= Ax + B\sigma\Big(Kx + \frac{v}{\lambda}\Big) + \frac{w}{\lambda}\\ y &= \lambda x\,.\end{aligned}$$

As a consequence, if

$$\|\Gamma v(\cdot)\|_a \le \lambda\delta', \qquad \|B\Omega v(\cdot) + u_1(\cdot)\|_a \le \lambda\delta'\,,$$

for any z°, the response $z(t)$ of the upper subsystem in Fig. 14.3 satisfies

$$\|z(\cdot)\|_a \le \max\{L_v\|\Gamma v(\cdot)\|_a, L_w\|B\Omega v(\cdot) + u_1(\cdot)\|_a\}\,.$$

Indeed, it is possible to find a number $\delta'' > 0$ and a number $N > 0$ such that, if

$$\|v(\cdot)\|_a \le \lambda\delta'', \qquad \|u_1(\cdot)\|_a \le \lambda\delta''\,,$$

for any z°, the response $z(t)$ of the upper subsystem in Fig. 14.3 satisfies

$$\|z(\cdot)\|_a \le N\max\{\|v(\cdot)\|_a, \|u_1(\cdot)\|_a\}\,. \tag{14.71}$$

Now observe, using the property (14.21), that, for each $i = 1, \ldots, m$

$$\begin{aligned}|[u^*(z,v)]_i| &\le \lambda|\sigma_i\Big(\frac{[Kz+\Gamma v]_i}{\lambda}\Big)| + |[\Omega v]_i|\\ &\le \lambda\min\{H\frac{|[Kz+\Gamma v]_i|}{\lambda}, \bar{k}\} + |[\Omega v]_i|\\ &\le \max\{\min\{2\lambda\bar{k}, L\max_{1\le j\le n}|z_j|\}, L\max_{1\le j\le m}|v_j|\}\}\end{aligned} \tag{14.72}$$

where $L > 0$ is a constant depending on H, Γ, Ω. This, using (14.71), shows that the output response $y_1(\cdot)$ of the upper subsystem of Fig. 14.3, if

$$\|v(\cdot)\|_a \le \lambda\delta'', \qquad \|u_1(\cdot)\|_a \le \lambda\delta''\,, \tag{14.73}$$

for any z° satisfies

$$\|y_1(\cdot)\|_a \le \max\{\min\{2\lambda\bar{k}, LN\max\{\|u_1(\cdot)\|_a, \|v(\cdot)\|_a\}\}, L\|v(\cdot)\|_a\}\,.$$

Since, for any triplet of nonnegative real numbers a, b, c

$$\min\{a, \max\{b,c\}\} \le \max\{c, \min\{a,b\}\}\,,$$

the latter estimate yields

$$\|y_1(\cdot)\|_a \le \max\{\min\{2\lambda\bar{k}, LN\|u_1(\cdot)\|_a\}, L(N+1)\|v(\cdot)\|_a\}\,. \tag{14.74}$$

The estimate in question holds if (14.73) holds, in which $\|u_1(\cdot)\|_a \le \lambda\delta''$. To show that this holds for any $\|u_1(\cdot)\|_a$, it suffices to assume, without loss of generality, that $N \ge 2\bar{k}/L\delta''$, which yields

$$\|u_1(\cdot)\|_a \ge \lambda\delta'' \quad \Rightarrow \quad \min\{2\lambda\bar{k}, LN\|u_1(\cdot)\|_a\} = 2\lambda\bar{k}\,.$$

If this is the case, in fact, (14.74) reduces to

$$\|y_1(\cdot)\|_a \le \max\{2\lambda\bar{k}, L(N+1)\|v(\cdot)\|_a\}\,,$$

which indeed holds because (14.72) implies

$$|[u^*(z,v)]_i| \le \max\{2\lambda\bar{k}, L\max_{1\le j\le m}|v_j|\}\,.$$

It can therefore be concluded that (14.74) holds for any x°, for any $\|u_1(\cdot)\|_a$ and for $\|v(\cdot)\|_a \le \lambda\delta''$.

Now, consider a class $\mathcal{K}$ function $\gamma_1(\cdot)$ satisfying

$$\begin{aligned}\gamma_1(r) &= LNr, && \text{if } r \le 2\lambda\bar{k}/LN\\ \gamma_1(r) &\le LNr, && \text{for all } r\\ \lim_{r\to\infty}\gamma_1(r) &= 3\lambda\bar{k}\,,\end{aligned}$$

and set

$$\bar{\gamma}_v(r) = L(N+1)r\,.$$

Then, it is concluded that the upper system of Fig. 14.3 satisfies an asymptotic input-output bound, with no restriction on z°, restriction $U_1 = \infty$ on $u_1(\cdot)$, restriction $\lambda\delta''$ on $v(\cdot)$, and gain functions $\gamma_1(\cdot)$, $\bar{\gamma}_v(\cdot)$, with

$$\lim_{r\to\infty}\gamma_1(r) = 3\lambda\bar{k}$$

and $\bar{\gamma}_v(\cdot)$ a linear function.

(c) By assumption (ii), the response $\xi(\cdot)$ of the lower subsystem of Fig. 14.3, for any $\xi^\circ \in \Xi$ and for any $u_2(\cdot)$ such that $\|u_2(\cdot)\|_a \le U$, satisfies

$$\|\xi(\cdot)\|_a \le G\|u_2(\cdot)\|_a\,, \tag{14.75}$$

for some $G > 0$.

Moreover, using assumption (iii), given any number $\epsilon > 0$, there exists $\delta > 0$, such that

$$\max_{\substack{1\le i\le \nu\\ 1\le j\le m}}\{|\xi_i|, |u_j|\} \le \delta \quad \Rightarrow \quad \max_{1\le i\le n}|[g(\xi,u)]_i| \le \epsilon\max_{\substack{1\le i\le \nu\\ 1\le j\le m}}\{|\xi_i|, |u_j|\}\,.$$

Thus, using (14.75), it is concluded that, given any number $M > 0$, there exists $U_2 > 0$ such that the response $y_2(\cdot)$ of the lower subsystem of Fig. 14.3, for any $\xi^\circ \in \Xi$ and for any $u_2(\cdot)$ such that $\|u_2(\cdot)\|_a \le U_2$, satisfies

$$\|y_2(\cdot)\|_a \le \gamma_2(\|u_2(\cdot)\|_a)\,,$$

with

$$\gamma_2(r) = Mr\,.$$

Choose M such that

$$MLN < 1 \tag{14.76}$$

and fix U_2 accordingly.

(d) In the previous two steps we have shown that the two subsystems of Fig. 14.3 satisfy the hypotheses assumed for (14.60) and (14.61). In order to be able to use the result of Theorem 14.3.1, we have to choose $\tilde{V}$ such that $\tilde{V} \le \lambda\delta''$ and

$$\max\{\lim_{r\to\infty} \gamma_1(r), \bar{\gamma}_v(\tilde{V})\} \le \max\{3\lambda\bar{k}, L(N+1)\lambda\delta''\} \le U_2\,,$$

which is indeed always possible.

Since, by (14.76), the small gain condition

$$\gamma_1 \circ \gamma_2(r) < r$$

holds for all $r > 0$, Theorem 14.3.1 shows that, for any $(z^\circ, \xi^\circ) \in \mathbb{R}^n \times \Xi$, and all $v(\cdot)$ satisfying $\|v(\cdot)\|_a \le \tilde{V}$, we have

$$\begin{aligned}\|y_1(\cdot)\|_a &\le \bar{\gamma}_v(\|v(\cdot)\|_a)\\ \|y_2(\cdot)\|_a &\le \gamma_2 \circ \bar{\gamma}_v(\|v(\cdot)\|_a)\,,\end{aligned}$$

where the gain functions on the right-hand sides are linear functions. Combining these inequalities with those provided by hypothesis (ii) and by (14.71) it is seen that there exist a number $V \le \tilde{V}$ and linear gain function for which the result of the theorem holds. ◁

If the lower subsystem of Fig. 14.3 has an equilibrium, at $(\xi, u) = (0,0)$, which is asymptotically stable in the first approximation, then the feedback law (14.69), with $v = 0$, is such that the equilibrium $(z,\xi) = (0,0)$ of the corresponding closed loop system is asymptotically stable, and all trajectories with initial conditions in $\mathbb{R}^n \times \Xi$ converge to zero as $t \to \infty$. In particular, if $\Xi = \mathbb{R}^\nu$, the equilibrium in question is globally asymptotically stable.

Corollary 14.3.4. *Consider the system (14.68). Suppose the hypotheses of Theorem 14.3.3 hold with $\Xi = \mathbb{R}^\nu$ and choose K as indicated in the Theorem. Suppose, also, that $f(\xi,u)$ is differentiable at $(\xi,u) = (0,0)$ and that the equilibrium $\xi = 0$ of $\dot{\xi} = f(\xi, 0)$ is asymptotically stable in the first approximation. Then, the interconnected system (14.68) – (14.69) is globally asymptotically stable, when $v = 0$.*

Proof. Note that the linear approximation of the system at the equilibrium $(z,\xi,v) = (0,0,0)$ is characterized by a matrix of the form

$$\begin{pmatrix} A+BK & 0 \\ * & F \end{pmatrix}$$

in which $A + BK$ and F have all eigenvalues in $\mathbb{C}^-$. Thus, this equilibrium is locally asymptotically stable. By Theorem 14.3.3, if $v(t) = 0$,

$$\lim_{t\to\infty} (x(t), \xi(t)) = (0,0)$$

for any $(x(0), \xi(0))$. Thus, the result follows. ◁

The result of Theorem 14.3.3 can also be used to the purpose of recursively stabilizing systems in feedforward form. To this end, some additional hypotheses are needed, as is indicated in the result which follows.

Lemma 14.3.5. *Consider the system*

$$\begin{aligned}\dot{z} &= Az + g_i(\xi_i, u)\\ \dot{\xi}_i &= f_i(\xi_i, u)\end{aligned} \tag{14.77}$$

in which $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}^\nu$, $u \in \mathbb{R}^m$, $g_i(\xi_i,u)$ and $f_i(\xi_i,u)$ are locally Lipschitz, differentiable at $(\xi_i,u) = (0,0)$, and $g_i(0,0) = 0$, $f_i(0,0) = 0$. Assume that:

(i) *there exists a symmetric matrix $P > 0$ such that $PA + A^{\mathrm{T}}P \le 0$,*

(ii) *the linear approximation of (14.77) at the equilibrium $(z, \xi_i, u) = (0,0,0)$ is stabilizable,*

Moreover, assume that there exists a function

$$\begin{aligned}\alpha_i :\ \mathbb{R}^\nu \times \mathbb{R}^m &\to \mathbb{R}^m\\ (\xi_i, v) &\mapsto \alpha_i(\xi_i, v)\,,\end{aligned}$$

with $\alpha_i(0,0) = 0$, which is locally Lipschitz, differentiable at $(\xi_i, v) = (0,0)$, with the following properties:

(iiia) *the matrix*

$$\left[\frac{\partial \alpha_i(\xi_i, v)}{\partial v}\right]_{(0,0)}$$

is nonsingular,

(iiib) *the matrix*

$$\left[\frac{\partial f_i(\xi_i, \alpha_i(\xi_i, v))}{\partial \xi_i}\right]_{(0,0)}$$

has all eigenvalues in $\mathbb{C}^-$,

(iiic) *the system*

$$\begin{aligned}\dot{\xi}_i &= f_i(\xi_i, \alpha_i(\xi_i, v))\\ y &= \xi_i\end{aligned}$$

satisfies an asymptotic (input-output) bound, with restriction X_i on ξ_i°, restriction $V > 0$ on $v(\cdot)$, with linear gain function.

Set $\xi_{i+1} = \mathrm{col}(z, \xi_i)$, $\tilde{\nu} = n + \nu$,

$$f_{i+1}(\xi_{i+1}, u) = \begin{pmatrix} Az + g_i(\xi_i, u) \\ f_i(\xi_i, u) \end{pmatrix} ,$$

and

$$\begin{aligned} F_{i+1} &= \left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_i(\xi_i, v))}{\partial \xi_{i+1}}\right]_{(0,0)} \\ G_{i+1} &= \left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_i(\xi_i, v))}{\partial v}\right]_{(0,0)} . \end{aligned}$$

Then, the pair (F_{i+1}, G_{i+1}) satisfies the hypotheses of Corollary 14.3.2. Let $\sigma(\cdot)$ be any $\mathbb{R}^m$-valued saturation function. Pick a $\tilde{\nu} \times m$ matrix K_{i+1} such that $(F_{i+1} + G_{i+1}K_{i+1})$ has all eigenvalues in $\mathbb{C}^-$ and, for some $\delta' > 0$, system

$$\begin{aligned} \dot{x} &= F_{i+1}x + G_{i+1}\sigma(K_{i+1}x + v) + w \\ y &= x \end{aligned} \tag{14.78}$$

satisfies an asymptotic (input-output) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions. Pick two $m \times m$ matrices Γ and Ω such that $\Gamma + \Omega$ is nonsingular.

Consider the function

$$\begin{aligned} \alpha_{i+1} : \quad \mathbb{R}^{\tilde{\nu}} \times \mathbb{R}^m &\to \mathbb{R}^m \\ (\xi_{i+1}, v) &\mapsto \alpha_i\Big(\xi_i, \lambda\sigma\Big(\frac{K_{i+1}\xi_{i+1} + \Gamma v}{\lambda}\Big) + \Omega v\Big) . \end{aligned} \tag{14.79}$$

Then, there exist numbers $\lambda > 0$ and $\tilde{V} > 0$ such that

(a) *the matrix*

$$\left[\frac{\partial \alpha_{i+1}(\xi_{i+1}, v)}{\partial v}\right]_{(0,0)}$$

is nonsingular,

(b) *the matrix*

$$\left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_{i+1}(\xi_{i+1}, v))}{\partial \xi_{i+1}}\right]_{(0,0)}$$

has all eigenvalues in $\mathbb{C}^-$,

(c) *the system*

$$\begin{aligned} \dot{\xi}_{i+1} &= f_{i+1}(\xi_{i+1}, \alpha_{i+1}(\xi_{i+1}, v)) \\ y &= \xi_{i+1} \end{aligned} \tag{14.80}$$

satisfies an asymptotic (input-output) bound, with restriction $X_{i+1} = \mathbb{R}^n \times X_i$ on ξ_{i+1}°, restriction $\tilde{V} > 0$ on $v(\cdot)$, with linear gain function.

Proof. It is easy to check that the matrix F_{i+1} has the following structure

$$F_{i+1} = \begin{pmatrix} A & R_i \\ 0 & \left[\frac{\partial f_i}{\partial \xi_i}\right]_{(0,0)} \end{pmatrix} + \begin{pmatrix} Q_i \\ \left[\frac{\partial f_i}{\partial u}\right]_{(0,0)} \end{pmatrix} \begin{pmatrix} 0 & \left[\frac{\partial \alpha_i}{\partial \xi_i}\right]_{(0,0)} \end{pmatrix} ,$$

that the matrix G_{i+1} has the following structure

$$G_{i+1} = \begin{pmatrix} Q_i \\ \left[\frac{\partial f_i}{\partial u}\right]_{(0,0)} \end{pmatrix} \left[\frac{\partial \alpha_i}{\partial v}\right]_{(0,0)} ,$$

and that

$$\left[\frac{\partial f_i}{\partial \xi_i}\right]_{(0,0)} + \left[\frac{\partial f_i}{\partial u}\right]_{(0,0)} \left[\frac{\partial \alpha_i}{\partial \xi_i}\right]_{(0,0)} = \left[\frac{\partial f_i(\xi_i, \alpha_i(\xi_i, v))}{\partial \xi_i}\right]_{(0,0)} .$$

Thus, using the hypotheses (i), (ii), (iiia) and (iiib), it is immediate to check that the pair (F_{i+1}, G_{i+1}) is stabilizable and that there exists a symmetric matrix $P_{i+1} > 0$ such that $P_{i+1}F_{i+1} + F_{i+1}^{\mathrm{T}}P_{i+1} \le 0$, i.e. that the pair (F_{i+1}, G_{i+1}) satisfies the hypotheses of Corollary 14.3.2.

Since

$$\left[\frac{\partial \alpha_{i+1}}{\partial v}\right]_{(0,0)} = \left[\frac{\partial \alpha_i}{\partial v}\right]_{(0,0)} (\Gamma + \Omega) ,$$

property (a) indeed holds. Since

$$\left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_{i+1}(\xi_{i+1}, v))}{\partial \xi_{i+1}}\right]_{(0,0)} = F_{i+1} + G_{i+1}K_{i+1} .$$

then also property (b) holds. To prove (c), set

$$g_{i+1}(\xi_i, w) = f_{i+1}(\xi_{i+1}, \alpha_i(\xi_i, w)) - F_{i+1}\xi_{i+1} - G_{i+1}w ,$$

where the notation on the left-hand side reflects the fact that the function on the right-hand side depends only on ξ_i and w and not on the z-component of ξ_{i+1}, and consider the auxiliary system

$$\begin{aligned} \dot{\zeta} &= F_{i+1}\zeta + G_{i+1}w + g_{i+1}(\xi, w) \\ \dot{\xi} &= f_i(\xi, \alpha_i(\xi, w)) , \end{aligned} \tag{14.81}$$

driven by

$$w = \lambda\sigma\Big(\frac{K_{i+1}\zeta + \Gamma v}{\lambda}\Big) + \Omega v ,$$

in the initial condition $(\zeta^\circ, \xi^\circ) = (\xi_{i+1}^\circ, \xi_i^\circ)$.

By construction, the response $\zeta(t)$ coincides with the response $\xi_{i+1}(t)$ of system (14.80). Now, system (14.81) has precisely the structure of the system considered in Theorem 14.3.3. Since

$$\lim_{\|(\xi,w)\| \to 0} \frac{\|g_{i+1}(\xi, w)\|}{\|(\xi, w)\|} = 0 ,$$

all hypotheses of this Theorem hold and thus property (c) follows. ◁

Clearly, this result can be repeatedly used to globally asymptotically stabilize a system in feedforward form, such as a system modeled by the equations

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + g_1(x_2,\dots,x_n,u)\\ \dot{x}_2 &= A_2x_2 + g_2(x_3,\dots,x_n,u)\\ &\cdots\\ \dot{x}_{n-1} &= A_{n-1}x_{n-1} + g_{n-1}(x_n,u)\\ \dot{x}_n &= f_n(x_n,u)\,, \end{aligned} \tag{14.82}$$

under the hypotheses that each of the upper $n-1$ subsystems, when the corresponding input (i.e. $(x_{i+1},\dots,x_n,u)$ for the i-th subsystem) is zero, is stable in the sense of Lyapunov, and the n-th subsystem, by means of some feedback law $u = \alpha_n(x_n,v)$, can be changed into a system which satisfies an asymptotic (input-output) bound, with some nonzero restriction on $v(\cdot)$ and linear gain function.

Theorem 14.3.6. *Consider a system of the form (14.82), in which $x_i \in \mathbb{R}^{n_i}$, $u \in \mathbb{R}^m$, the $g_i(x_{i+1},\dots,x_n,u)$'s are locally Lipschitz, differentiable at $(x_{i+1},\dots,x_n,u) = (0,\dots,0,0)$, and $g_i(0,\dots,0,0) = 0$. Assume that:*

(i) *for each $i = 1,\dots,n-1$, there exists a symmetric matrix $P_i > 0$ such that $P_iA_i + A_i^{\mathrm{T}}P_i \le 0$,*

(ii) *the linear approximation of (14.82) at the equilibrium $(x_1,\dots,x_n,u) = (0,\dots,0,0)$ is stabilizable.*

If there exists a function

$$\begin{aligned} \alpha_n : \ \mathbb{R}^{n_n}\times\mathbb{R}^m &\to \mathbb{R}^m\\ (x_n,v) &\mapsto \alpha_n(x_n,v)\,, \end{aligned}$$

with $\alpha_n(0,0) = 0$, which is locally Lipschitz and differentiable at $(x_n,v) = (0,0)$, with the following properties:

(iiia) *the matrix*

$$\left[\frac{\partial \alpha_n(x_n,v)}{\partial v}\right]_{(0,0)}$$

is nonsingular,

(iiib) *the matrix*

$$\left[\frac{\partial f_n(x_n,\alpha_n(x_n,v))}{\partial x_n}\right]_{(0,0)}$$

has all eigenvalues in $\mathbb{C}^-$,

(iiic) *the system*

$$\begin{aligned} \dot{x}_n &= f_n(x_n,\alpha_n(x_n,v))\\ y &= x_n \end{aligned}$$

satisfies an asymptotic (input-output) bound, with no restriction on x_n°, restriction $V > 0$ on $v(\cdot)$, with linear gain function.

Then, there exists a feedback law

$$u = u^*(x_1,x_2,\dots,x_n)$$

which globally asymptotically stabilizes the equilibrium point $(x_1,\dots,x_n) = (0,\dots,0)$.

14.4 Applications and Extensions

The design methods described in the previous sections lend themselves to a variety of applications and extensions. For instance, the result of Theorem 14.3.6 can be used to show that a linear system

$$\dot{x} = Ax + Bu$$

in which (A,B) is stabilizable, if the matrix A does not have eigenvalues with positive real part, can be globally stabilized using a feedback law whose amplitude does not exceed any (arbitrarily small) fixed bound. The special case in which the eigenvalues of A on the imaginary axis have unitary geometric multiplicity, since there exists a matrix $P > 0$ such that $PA + A^{\mathrm{T}}P \le 0$, can be dealt with using the feedback law described in Proposition 14.1.5. If the matrix A does not have eigenvalues with positive real part and the eigenvalues on the imaginary axis have higher geometric multiplicity, a preliminary change of coordinates in the state space can reduce the system to a form in which Theorem 14.3.6 is immediately applicable.

For instance, consider the case of a stabilizable single-input system in which the matrix A has one eigenvalue at zero, of multiplicity $r-1$, and all remaining eigenvalues with negative real part. Then, in suitable coordinates, the system can be described by equations of the form

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_{12}x_2 + \cdots + A_{1r}x_r + B_1u\\ \dot{x}_2 &= x_3\\ &\cdots\\ \dot{x}_{r-1} &= x_r\\ \dot{x}_r &= u \end{aligned} \tag{14.83}$$

in which A_1 has all the eigenvalues in $\mathbb{C}^-$. This system can be regarded as a system of the form (14.82), in which $A_2 = \cdots = A_{r-1} = 0$. Indeed, the hypotheses (i) and (ii) of Theorem 14.3.6 hold. As far as the hypotheses (iii) are concerned, observe that the last equation characterizes a subsystem to which Corollary 14.3.2 is applicable. In particular, the feedback law

$$\alpha_r(x_r,v) = \sigma(-x_r + v)\,,$$

in which $\sigma(\cdot)$ is any saturation function, is such that

$$\begin{aligned}\dot{x}_r &= \alpha_r(x_r, v)\\ y &= x_r\end{aligned}$$

satisfies an asymptotic (input-output) bound, with no restriction on x_r°, restriction $V > 0$ on $v(\cdot)$, with linear gain function. Thus, this feedback law is such that three hypotheses (iiia), (iiib) and (iiic) of Theorem 14.3.6 hold. As a consequence, the system in question can be globally asymptotically stabilized, by means of a feedback law which has the form

$$u = \sigma(-x_r + v(x_r, \ldots, x_2)) .$$

As shown in the proof of Lemma 14.3.5, the function $v(x_r, \ldots, x_2)$ is a composition of saturation functions, which has the form

$$v(x_r, \ldots, x_2) =$$

$$\lambda_1\sigma\Big(\frac{k_0^1 x_r + k_1^1 x_{r-1}}{\lambda_1} + \frac{\lambda_2}{\lambda_1}\sigma\Big(\cdots + \frac{\lambda_{r-2}}{\lambda_{r-3}}\sigma\Big(\frac{k_0^{r-2}x_r + \cdots + k_{r-2}^{r-2}x_2}{\lambda_{r-2}}\Big)\Big)\Big) .$$

Note that the recursive process does not need to involve the first subsystem, since the latter is a linear stable system driven by an input which, under the indicated feedback law, asymptotically decays to zero as t tends to ∞.

Remark 14.4.1. It is useful to observe that the class of systems considered above is the largest class of linear systems which can be ***globally*** stabilized using bounded feedback. In fact, if the matrix A has an eigenvalue with positive real part and the control is bounded, there are initial states that can never be driven to the origin. This can be easily seen, for instance, if the system has a positive eigenvalue, by changing coordinates in such a way as to isolate a one-dimensional subsystem of the form

$$\dot{x} = ax + bu$$

with $a > 0$. If $|u| \le U$, any initial state x° satisfying $x^\circ \ge (|bU|/a)$ produces a trajectory that diverges as t tends to ∞. $\triangleleft$

As an example of how the results of the previous two sections can be extended to more general classes of systems, we consider here the case of systems modeled by a "perturbed" version of the equations (14.56), such as equations of the form

$$\begin{aligned}\dot{x}_1 &= a_1(x_1, x_2, \ldots, x_n) + b_1(x_1, x_2, \ldots, x_n)u + q_1(x_1, x_2, \ldots, x_n, u)u^2\\ \dot{x}_2 &= a_2(x_2, x_3, \ldots, x_n) + b_2(x_2, x_3, \ldots, x_n)u + q_2(x_1, x_2, \ldots, x_n, u)u^2\\ &\cdots\\ \dot{x}_{n-1} &= a_{n-1}(x_{n-1}, x_n) + b_{n-1}(x_{n-1}, x_n)u + q_{n-1}(x_1, x_2, \ldots, x_n, u)u^2\\ \dot{x}_n &= a_n(x_n) + b(x_n)u + q_n(x_1, x_2, \ldots, x_n, u)u^2 ,\end{aligned} \tag{14.84}$$

in which $u \in \mathbb{R}$, and the functions $a_i(x_i, \ldots, x_n)$, $b_i(x_i, \ldots, x_n)$ satisfy hypotheses similar to those considered for the corresponding functions in the form (14.56). These equations no longer have an upper-triangular structure, but the extra terms which destroy this structure contain u^2 as a factor. Thus, it is reasonable to expect that their presence could be rendered, to some extent, "ineffective" by choosing an input having sufficiently small amplitude (and this can be done by taking explicit advantage of the fact that arbitrarily small inputs are, in this case, allowed by the hypotheses assumed on the various functions which characterize the upper-triangular part of (14.84)).

For simplicity, as an illustration on how to proceed in more general situations, we restrict the analysis to the case in which the decomposition (14.84) consists only of two subsystems, of which the lower one is one-dimensional. More precisely, we consider the case of systems modeled by equations of the form

$$\begin{aligned}\dot{z} &= Az + M\xi + Bu + p(\xi) + g(\xi)u + q(z, \xi, u)u^2\\ \dot{\xi} &= u + \phi(z, \xi, u)u^2\end{aligned} \tag{14.85}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, A is such that $PA + A^{\mathrm{T}}P \le 0$ for some $P > 0$, $p(\xi)$ vanishes as $\xi = 0$ together with its first derivative $p'(\xi)$, and $g(\xi)$ vanishes as $\xi = 0$. Moreover, we assume that, for some $K > 0$ and for some continuous positive-valued function $\gamma(\xi)$,

$$\|q(z, \xi, u)\| \le K\gamma(\xi)(1 + \|z\|) \tag{14.86}$$

and

$$|\phi(z, \xi, u)| \le K(1 + |\xi|) , \tag{14.87}$$

for all $(z, \xi) \in \mathbb{R}^n \times \mathbb{R}$ and all $|u| \le 1$.

Let ε be a number satisfying $0 < \varepsilon < 1$, let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be any continuous function satisfying

$$0 < f(s) \le 1 , \qquad \text{for all } s \in \mathbb{R},$$

let $\mu(\cdot) : \mathbb{R} \to \mathbb{R}$ denote the function

$$\mu(s) = \frac{1}{\sqrt{1 + s^2}} ,$$

set $x = (z, \xi)$ and define

$$\lambda(x) = \varepsilon f(\|z\|)\mu(\xi) .$$

We begin discussing the effect, on system (14.85) of a control law of the form

$$u = \lambda(x)\sigma\Big(\frac{-\xi}{\lambda(x)} + v\Big) \tag{14.88}$$

in which $\lambda(x)$ is a function of the type indicated above and the function $\sigma(\cdot)$ is the saturation function

$$\sigma(s) = \frac{s}{\sqrt{1+s^2}}$$

already considered in section 14.1. As shown below, this input is able to induce certain properties of input-to-state stability, with restriction on v, on the lower subsystem of (14.85).

Lemma 14.4.1. *There is a number $\varepsilon_1 < 1$, depending only on K, such that, for all $\varepsilon \leq \varepsilon_1$, there exists a number $M > 0$ satisfying*

$$2K|\xi|(1+|\xi|) \leq \sqrt{1 + \left(\frac{-\xi}{\lambda(x)} + v\right)^2} \leq \frac{M(1+\xi^2)}{f(\|z\|)} \qquad (14.89)$$

for all $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$ and all $|v| \leq 1$.

Proof. The inequality on the left-hand side is equivalent to an inequality of the form

$$1 + (a-v)^2 \geq b^2$$

with

$$a = \frac{\xi}{\lambda(x)}, \qquad b = 2K|\xi|(1+|\xi|).$$

If $b \leq 1$ there is nothing to prove. Thus, there exists $\xi_0 > 0$, depending only on K, such that the left-hand side of (14.89) holds for all $|\xi| \leq \xi_0$. For $b > 1$, an elementary calculation shows that the inequality in question holds, for all $|v| \leq 1$, if $|a| \geq \sqrt{b^2-1}$ and $|a \pm \sqrt{b^2-1}| \geq 1$. The latter are implied by $|a| \geq |b| + 1$, i.e.

$$\frac{|\xi|}{\lambda(x)} \geq 2K|\xi|(1+|\xi|) + 1 \qquad \text{for all } |\xi| \geq \xi_0.$$

Since $|f(\|z\|)| \leq 1$, the latter holds if

$$|\xi| \geq \varepsilon \frac{2K|\xi|(1+|\xi|)+1}{\sqrt{1+\xi^2}} \qquad \text{for all } |\xi| \geq \xi_0,$$

which indeed holds if ε is small enough. The inequality on the right-hand side is straightforward, if $|v| \leq 1$. ◁

Lemma 14.4.2. *Consider the system*

$$\dot{\xi} = u + \phi(z(t),\xi,u)u^2 \qquad (14.90)$$

in which $\phi(z,\xi,u)$ satisfies the hypothesis (14.87), for all $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$ and all $|u| \leq 1$. Moreover, suppose $z(t)$ is a continuous function defined for all $t \geq 0$. Choose the input (14.88), with $\varepsilon \leq \varepsilon_1$. Then, if $|v| \leq 1$, the function $U(\xi) = \xi^2$ is such that

$$|\xi| \geq \sqrt{2}\lambda(x)|v| \quad \Rightarrow \quad \dot{U} \leq -\frac{f(\|z\|)}{2M}\alpha(\xi) \qquad (14.91)$$

where

$$\alpha(\xi) = \frac{\xi^2}{1+\xi^2}$$

is a class $\mathcal{K}$ function. In particular, since $\lambda(x) \leq \varepsilon$,

$$|\xi(t)| \leq \max\{|\xi(0)|, \sqrt{2}\varepsilon\}. \qquad (14.92)$$

Moreover, if $f(\|z(t)\|)$ is bounded away from zero on $[0,\infty)$, for any $v(\cdot)$ satisfying $\|v(\cdot)\|_\infty \leq 1$,

$$|\xi(t)| \leq \max\{\theta(|\xi(0)|)e^{-bt}, \sqrt{2}\varepsilon\|v(\cdot)\|_\infty\}, \qquad (14.93)$$

in which $\theta(\cdot)$ is a class $\mathcal{K}$ function and $b > 0$.

Proof. Observe that, in view of the specific form of the function $\sigma(\cdot)$,

$$\dot{U} \leq 2\frac{-\xi^2 + \lambda(x)v\xi}{\sqrt{1+\left(\frac{-\xi}{\lambda(x)}+v\right)^2}} + 2\frac{K|\xi|(1+|\xi|)}{\sqrt{1+\left(\frac{-\xi}{\lambda(x)}+v\right)^2}}\frac{(-\xi+\lambda(x)v)^2}{\sqrt{1+\left(\frac{-\xi}{\lambda(x)}+v\right)^2}},$$

which, in view of the left-hand side of (14.89), yields

$$\dot{U} \leq \frac{(-\xi^2/2 - \xi^2/2 + \lambda^2(x)v^2)}{\sqrt{1+\left(\frac{-\xi}{\lambda(x)}+v\right)^2}}.$$

This, using the right-hand side of (14.89), yields (14.91).

From (14.91) it is immediate to deduce (14.92). Finally, if $f(\|z(t)\|)$ is bounded away from zero on $[0,\infty)$, we have that

$$|\xi| \geq \sqrt{2}\varepsilon|v| \quad \Rightarrow \quad \dot{U} \leq -c\alpha(\xi)$$

for some fixed $c > 0$. This, since $\alpha(\xi)$ in a neighborhood of $\xi = 0$ can be bounded from below by $a\xi^2$ for some $a > 0$, shows (in view of Lemma 10.1.5 and 10.1.6) that if $v = 0$

$$|\xi(t)| \leq \theta(|\xi(0)|)e^{-bt}$$

and this, bearing in mind the properties of an input-to-state stable system described in section 10.4, completes the proof. ◁

As a first consequence of this Lemma, it is possible to prove the following stabilization result.

Proposition 14.4.3. *Consider system (14.85). Suppose that* $PA+A^{\mathrm{T}}P \le 0$ *for some* $P>0$, *that* $p(\xi)$, $p'(\xi)$, *and* $g(\xi)$ *vanish at* $\xi=0$ *and that the bounds (14.86) and (14.87) hold for all* $(z,\xi)\in\mathbb{R}^n\times\mathbb{R}$ *and all* $|u|\le 1$. *Suppose also that the pair*

$$\begin{pmatrix} A & M \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} B \\ 1 \end{pmatrix}$$

is stabilizable. Then, there exists a feedback law $u=u^*(z,\xi)$ *which globally asymptotically stabilizes the equilibrium* $(z,\xi)=(0,0)$.

Proof. Set

$$u(\xi)=\varepsilon\mu(\xi)\sigma\Big(\frac{-\xi}{\varepsilon\mu(\xi)}\Big),$$

and consider system (14.85) with feedback law

$$u=u(\xi)+v\,.$$

This yields a system of the form

$$\begin{aligned} \dot z &= Az+p(z,\xi)+b(z,\xi)v+\tilde q(z,\xi,v)v^2 \\ \dot\xi &= \varphi(z,\xi)+\theta(z,\xi)v+\tilde\phi(z,\xi,v)v^2 \end{aligned} \tag{14.94}$$

in which

$$\begin{aligned} p(z,\xi) &= M\xi+Bu(\xi)+p(\xi)+g(\xi)u(\xi)+q(z,\xi,u(\xi))u^2(\xi) \\ b(z,\xi) &= B+g(\xi)+\Big[\frac{\partial q}{\partial u}\Big]_{u=u(\xi)}u^2(\xi)+q(z,\xi,u(\xi))2u(\xi) \\ \varphi(z,\xi) &= u(\xi)+\phi(z,\xi,u(\xi))u^2(\xi) \\ \theta(z,\xi) &= 1+\Big[\frac{\partial\phi}{\partial u}\Big]_{u=u(\xi)}u^2(\xi)+\phi(z,\xi,u(\xi))2u(\xi)\,. \end{aligned}$$

The result of the Proposition can be proven by showing that the system obtained from (14.94) by neglecting the terms of order higher than 1 in v, namely the system

$$\begin{aligned} \dot z &= Az+p(z,\xi)+b(z,\xi)v \\ \dot\xi &= \varphi(z,\xi)+\theta(z,\xi)v\,, \end{aligned} \tag{14.95}$$

satisfies hypotheses identical (or equivalent to) those of Theorem 14.2.6. As a consequence, there exists a positive definite proper function $W(z,\xi)$ such that

$$\frac{\partial W}{\partial z}[Az+p(z,\xi)]+\frac{\partial W}{\partial\xi}\varphi(z,\xi)\le 0$$

and such that system

$$\begin{aligned} \dot z &= Az+p(z,\xi) \\ \dot\xi &= \varphi(z,\xi) \\ y &= \frac{\partial W}{\partial z}b(z,\xi)+\frac{\partial W}{\partial\xi}\theta(z,\xi)\,, \end{aligned}$$

is zero-state detectable. Thus, using Theorem 14.1.3, it is possible to find a feedback law which globally asymptotically stabilizes (14.94).

An immediate check shows that hypotheses (i), (iii), (iv) and (v) of Theorem 14.2.6 hold. As far as hypothesis (ii) is concerned, observe that $u(\xi)$ can be viewed as an input of the form (14.88), with $v=0$ and $f(z)=1$. Thus, in the system

$$\begin{aligned} \dot z &= Az+p(z,\xi) \\ \dot\xi &= \varphi(z,\xi)\,, \end{aligned} \tag{14.96}$$

so long as $z(t)$ is defined, the response $\xi(t)$ satisfies the estimates of Lemma 14.4.2. In particular, $\xi(t)$ is bounded by $\max\{|\xi(0)|,\sqrt{2}\varepsilon\}$. This, using the growth condition

$$\|p(z,\xi)\|\le\gamma(|\xi|)(1+\|z\|)\,,$$

shows that, so long as $z(t)$ is defined,

$$\|Az(t)+p(z(t),\xi(t))\|\le C_0+C_1\|z(t)\|$$

for some $C_0>0$ and $C_1>0$. Thus

$$\|z(t)\|\le\|z(0)\|+\int_0^t[C_0+C_1\|z(s)\|]ds$$

from which, using Gronwall-Bellman's inequality, it can be concluded that $z(t)$ cannot escape to ∞ in finite time. In other words, $z(t)$ is defined for all t and therefore, by Lemma 14.4.2, the function $U(\xi)=\xi^2$ is such that

$$\frac{\partial U}{\partial\xi}\varphi(z,\xi)\le -c\frac{\xi^2}{1+\xi^2}$$

and

$$|\xi(t)|\le\theta(|\xi(0)|)e^{-bt}\,,$$

for some $c>0, b>0$ and class $\mathcal{K}$ function $\theta(\cdot)$. This shows that system (14.95) satisfies an hypothesis equivalent to the hypothesis (ii) of Theorem 14.2.6 and also that the function $U(\xi)$ has the properties indicated in that Theorem.

From this, it is not difficult to prove – adjusting to the present case the arguments presented in section 14.2 – that the function

$$W(z,\xi)=z^{\mathrm{T}}Pz+U(\xi)+\Psi(z,\xi)\,,$$

in which the cross-term $\Psi(z,\xi)$ has the expression

$$\Psi(z,\xi)=\int_0^\infty 2[\overline z(z,\xi,s)]^{\mathrm{T}}Pp(\overline z(z,\xi,s),\overline\xi(z,\xi,s))ds\,,$$

with $\overline z(z,\xi,s)$ and $\overline\xi(z,\xi,s)$ denoting the value at time $t=s$ of the integral curve of system (14.96) passing through (z,ξ) at time $t=0$, is a positive definite and proper smooth function which possesses the properties indicated above. ◁

The actual construction of the stabilizing feedback law whose existence has been proven in the previous Proposition requires the explicit calculation of the Lyapunov function $W(z,\xi)$, which in turn requires the explicit integration of the ordinary differential equation (14.96) for any initial condition (z,ξ). This may prove to be a difficult computational task. A way to avoid this problem is to look at the possibility of recursive use of saturated controls, following the pattern of ideas explained in section 14.3. In what follows, we show how this can be achieved in the (easy) case in which the upper subsystem of (14.85) is one-dimensional. In more general cases, appropriate extensions can be worked-out.

Proposition 14.4.4. *Consider the system*

$$\begin{aligned} \dot z &= \xi + u + p(\xi) + g(\xi)u + q(z,\xi,u)u^2 \\ \dot\xi &= u + \phi(z,\xi,u)u^2 \end{aligned} \tag{14.97}$$

in which $z \in \mathbb{R}$, $\xi \in \mathbb{R}$, $p(\xi)$, $p'(\xi)$ and $g(\xi)$ vanish at $\xi = 0$, and $q(z,\xi,u)$, $\phi(z,\xi,u)$ satisfy the hypotheses (14.86) and (14.87), for all $(z,\xi) \in \mathbb{R}\times\mathbb{R}$ and all $|u| \le 1$. Choose the input

$$u = \lambda(x)\sigma\Big(-\frac{\xi}{\lambda(x)} + \frac{1}{4}\sigma\Big(-\frac{z}{\mu(z)}\Big)\Big),$$

in which

$$\lambda(x) = \varepsilon\mu(z)\mu(\xi) = \varepsilon\frac{1}{\sqrt{1+z^2}}\frac{1}{\sqrt{1+\xi^2}}. \tag{14.98}$$

Then, there is a number $\varepsilon^* > 0$ such that, for all $0 < \varepsilon \le \varepsilon^*$, the equilibrium of the closed loop system (14.97) – (14.98) is globally asymptotically stable.

Proof. Observe, first of all, that the control (14.98) has the form

$$u = \lambda(x)\sigma\Big(-\frac{\xi}{\lambda(x)} + v\Big), \qquad \text{with} \qquad v = \frac{1}{4}\sigma\Big(-\frac{z}{\mu(z)}\Big).$$

For convenience, the proof is split in three steps.

(a) In the first step it is shown that, if ε is small enough, $\xi(t)$ and $z(t)$ are defined for all $t \ge 0$, and there exists a finite time $T^* > 0$ such that

$$|\xi(t)| \le \lambda(x(t)) \qquad \text{for all } t \ge T^*. \tag{14.99}$$

To this end, assume $\varepsilon < \varepsilon_1$ and note that, so long as $z(t)$ is defined, since $|v| \le 1$, from the proof of Lemma 14.4.2, with $f(s) = \mu(s)$, one obtains

$$|\xi(t)| \le L$$

for some number $L > 0$. Thus, using the fact that $|u| < 1$, that $\mu(\xi) \le 1$, that $|\sigma(s)| \le 1$ and (14.86), we have

$$\begin{aligned} \dot z &\le |\xi| + 1 + |p(\xi)| + |g(\xi)| + K\gamma(\xi)(1+|z|)\varepsilon^2\mu^2(\xi)\mu^2(z) \\ &\le C_1 + C_2(1+|z|)\mu^2(z), \end{aligned}$$

where C_1 and C_2 are such that

$$\begin{aligned} &|\xi| + 1 + |p(\xi)| + |g(\xi)| \le C_1 \\ &K\gamma(\xi)\varepsilon^2\mu^2(\xi) \le C_2 \end{aligned}$$

for all $|\xi| \le L$. Thus,

$$|z(t)| \le |z(0)| + C_1 t + \int_0^t \frac{1+|z(s)|}{1+z^2(s)}ds \le \overline{C}_0 + \overline{C}_1 t,$$

for suitable $\overline{C}_0, \overline{C}_1$. This shows that $z(t)$ cannot escape to ∞ in finite time.

Observe that, if $|s| \ge 1$ and $|v| < \frac{1}{4}$,

$$\mathrm{sgn}(s)\sigma(-s+v) < -\frac{1}{2} - \delta$$

for some $\delta > 0$. Observe also that, since $|\sigma(s)| \le 1$, $\mu(z) \le 1$, and $(1+|\xi|)\mu(\xi) \le \sqrt{2}$, then

$$|\phi(z,\xi,u)u^2| \le K(1+|\xi|)\varepsilon^2\lambda^2(x) \le K(1+|\xi|)\varepsilon^2\mu(\xi)\mu(z)\lambda(x) \le \sqrt{2}K\varepsilon^2\lambda(x).$$

Now, suppose that

$$|\xi(t)| \ge \lambda(x(t)) \qquad \text{for all } t \ge 0. \tag{14.100}$$

Then, we have

$$\begin{aligned} \frac{d|\xi|}{dt} = \mathrm{sgn}(\xi)\dot\xi &\le \mathrm{sgn}(\xi)\lambda(x)\sigma\Big(-\frac{\xi}{\lambda(x)} + v\Big) + \sqrt{2}K\varepsilon^2\lambda(x) \\ &\le \lambda(x)\Big(-\frac{1}{2} - \delta + \sqrt{2}K\varepsilon^2\Big). \end{aligned}$$

Let ε_2 be such that $\sqrt{2}K\varepsilon_2^2 = \delta$. Then, for all $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$, we have

$$\frac{d|\xi|}{dt} \le -\frac{1}{2}\lambda(x(t)), \tag{14.101}$$

which implies

$$|\xi(t)| - |\xi(0)| \le -\frac{1}{2}\int_0^t \lambda(x(s))ds. \tag{14.102}$$

Now, recall that $|\xi(t)| \le L$ for all $t \ge 0$ and that

$$\lambda(x(t)) = \varepsilon\frac{1}{\sqrt{1+\xi^2(t)}}\frac{1}{\sqrt{1+z^2(t)}} \ge \frac{1}{\sqrt{1+L^2}}\frac{1}{\sqrt{1+(\overline{C}_0+\overline{C}_1 t)^2}},$$

which yields

$$\lim_{t\to\infty}\int_0^t \lambda(x(s))ds = \infty .$$

Thus, (14.102) cannot hold for all $t \geq 0$, i.e. (14.100) cannot hold for all $t \geq 0$.

This being the case, let T be any time such that $|\xi(T)| = \lambda(x(T))$, observe that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|\xi(t)|}{\lambda(x(t))}\right)\right]_{t=T} = \frac{1}{\lambda(x(T))}\left[\frac{\mathrm{d}|\xi|}{\mathrm{d}t}\right]_{t=T} - \frac{|\xi(T)|}{\lambda(x(T))}\frac{\dot\lambda(x(T))}{\lambda(x(T))}$$

and therefore, using (14.101), that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|\xi(t)|}{\lambda(x(t))}\right)\right]_{t=T} \leq -\frac{1}{2} + \left|\frac{\dot\lambda(x(T))}{\lambda(x(T))}\right| . \tag{14.103}$$

On the other hand

$$\left|\frac{\dot\lambda(x(t))}{\lambda(x(t))}\right| \leq \frac{|\xi\dot\xi|}{1+\xi^2} + \frac{|z\dot z|}{1+z^2} .$$

Using the hypotheses that the functions $p(\xi), g(\xi)$ vanish at $\xi = 0$, and the bounds (14.86) and (14.87), it is easy to check that there is a number ε_3 such that, if $|\xi| \leq \varepsilon_3$ and $|u| \leq \varepsilon_3$

$$\frac{|\xi||u+\phi(z,\xi,u)u|}{1+\xi^2} + \frac{|z||\xi+u+p(\xi)+g(\xi)u+q(z,\xi,u)u^2|}{1+z^2} < \frac{1}{2} .$$

This, using the facts that $|u| \leq \varepsilon$ and, at time $t = T$, $|\xi(T)| = \lambda(x(T)) \leq \varepsilon$, shows that, if $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$,

$$\left[\left|\frac{\dot\lambda(x(t))}{\lambda(x(t))}\right|\right]_{t=T} < \frac{1}{2}$$

and hence, using (14.103), that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|\xi(t)|}{\lambda(x(t))}\right)\right]_{t=T} < 0 .$$

It is concluded from this that, for some $T^* \geq 0$, (14.99) necessarily holds.

(b) Consider now the upper equation of (14.97) on the time interval $[T^*, \infty)$. Add and subtract $-\lambda(x)v$, to obtain

$$\dot z = \lambda(x)v + [\xi - \lambda(x)v + u] + p(\xi) + g(\xi)u + q(z,\xi,u)u^2. \tag{14.104}$$

Observe – using the property (14.20) of the saturation function – that

$$\begin{aligned}
|\xi - \lambda(x)v + u| &= |\xi - \lambda(x)v + \lambda(x)\sigma\Big(\frac{-\xi}{\lambda(x)} + v\Big)| \\
&= \lambda(x)\Big|\sigma\Big(\frac{-\xi}{\lambda(x)} + v\Big) - \Big(\frac{-\xi}{\lambda(x)} + v\Big)\Big| \\
&\leq \lambda(x)\sigma\Big(\frac{-\xi}{\lambda(x)} + v\Big)\Big(\frac{-\xi}{\lambda(x)} + v\Big) \leq \lambda(x)\Big(\frac{-\xi}{\lambda(x)} + v\Big)^2 .
\end{aligned}$$

Since $p(\xi)$, $p'(\xi)$ and $g(\xi)$ vanish at $\xi = 0$, there exists functions $\tilde p(\xi)$ and $\tilde g(\xi)$ such that

$$p(\xi) + g(\xi)u = \tilde p(\xi)\xi^2 + \tilde g(\xi)\xi u$$

and hence, using the fact that $|\xi| \leq \lambda(x)$ and $|u| \leq \lambda(x)$

$$|p(\xi) + g(\xi)u| \leq \lambda(x)(|\tilde p(\xi)| + |\tilde g(\xi)|)|\xi| .$$

Using similar arguments, and the bound (14.86), observe also that

$$\begin{aligned}
|q(z,\xi,u)|u^2 &\leq K\gamma(\xi)(1+|z|)\varepsilon^2\mu(z)\mu(\xi)\lambda(x)\sigma^2\Big(\frac{-\xi}{\lambda(x)} + v\Big) \\
&\leq \sqrt{2}K\gamma(\xi)\varepsilon^2\lambda(x)\Big(\frac{-\xi}{\lambda(x)} + v\Big)^2 .
\end{aligned}$$

Finally, using the facts that $|\xi| \leq \lambda(x)$ and $\lambda(x) \leq \varepsilon$, note that

$$\lambda(x)\Big(\frac{-\xi}{\lambda(x)} + v\Big)^2 = (-\xi + \lambda(x)v)^2 \leq \lambda(x)[|\xi| + 2\varepsilon|v| + \varepsilon v^2] .$$

Putting all these estimates into (14.104) yields

$$\dot z = \lambda(x)v + \varphi(z,\xi,v) , \tag{14.105}$$

in which

$$|\varphi(z,\xi,v)| \leq \lambda(x)[a(\xi,\varepsilon)|v| + b(\xi,\varepsilon)v^2 + c(\xi,\varepsilon)|\xi|]$$

with

$$\begin{aligned}
a(\xi,\varepsilon) &= [1 + \sqrt{2}K\gamma(\xi)\varepsilon^2]2\varepsilon \\
b(\xi,\varepsilon) &= [1 + \sqrt{2}K\gamma(\xi)\varepsilon^2]\varepsilon \\
c(\xi,\varepsilon) &= 1 + \sqrt{2}K\gamma(\xi)\varepsilon^2 + |\tilde p(\xi)| + |\tilde g(\xi)| .
\end{aligned}$$

Clearly, given any $\delta > 0$, and any $k > 1 + |\tilde p(0)| + |\tilde g(0)|$, there exists ε_4 such that, for all $\varepsilon \leq \varepsilon_4$,

$$|\xi| \leq \varepsilon \quad \Rightarrow \quad |\varphi(z,\xi,v)| \leq \lambda(x)[(1/2)|v| + \delta v^2 + k|\xi|]$$

Recall now that

$$v = \frac{1}{4}\sigma\Big(\frac{-z}{\mu(z)}\Big) .$$

Suppose $\varepsilon \leq \min_{1\leq i\leq 4}\{\varepsilon_i\}$, so that, if $|z(t)| > 0$,

$$\frac{d|z|}{dt} \le \frac{\lambda(x)}{8}\Big[-\sigma\Big(\frac{|z|}{\mu(z)}\Big) + \frac{1}{2}(\delta + 8k\varepsilon)\Big] .$$

Observing that the saturation function $\sigma(\cdot)$ has an inverse $\sigma^{-1}(\cdot)$ defined on the interval $(-1, 1)$, choose δ and ε such that

$$\delta + 8k\varepsilon < 1 .$$

Then, we have that

$$\frac{d|z|}{dt} < 0 , \qquad \text{for all} \quad \frac{|z|}{\mu(z)} > \sigma^{-1}(1/2) = \sqrt{1/3},$$

i.e, for all

$$|z|\sqrt{1+z^2} > \sqrt{1/3} .$$

This shows that $|z(t)|$ is decreasing whenever $|z(t)|$ is large and hence $|z(t)|$ is bounded. Moreover, it is easy to see that there exists a finite time $T^\circ \ge T^*$ such that

$$|z(t)| \le 1$$

for all $t \ge T^\circ$. In fact, if $|z(t)| > 1$ for all $t \ge T^*$, then

$$\sigma\Big(\frac{|z(t)|}{\mu(z(t))}\Big) > \sigma(1) ,$$

and

$$\frac{d|z|}{dt} \le \frac{\lambda(x)}{8}[-\sigma(1) + 1/2] .$$

Since the function $\lambda(x(t))$ is bounded away from zero (because both $\xi(t)$ and $z(t)$ are bounded), the right-hand side of this inequality is negative and bounded away from zero on $[T^*, \infty)$. Integrating on $[T^*, \infty)$ yields a contradiction, because the integral of the left-hand side on this interval is finite.

(c) Having shown that $z(t)$ is bounded, the function $\mu(|z(t)|)$ is bounded away from zero. Hence, the conclusion (14.93) of Lemma 14.4.2 holds for $\xi(t)$. As far as $z(t)$ is concerned, observe that the function $V(z) = \frac{1}{2}z^2$, if δ and ε are chosen as indicated in the previous step, since v is an odd function of z, satisfies

$$\begin{aligned}
\dot V &\le \lambda(x)[-\tfrac{1}{2}|z||v| + |z|\delta v^2] + \lambda(x)|z|k|\xi| \\
&\le \frac{\varepsilon\mu(\xi)}{\sqrt{1+\Big(\frac{z}{\mu(z)}\Big)^2}}\Big[-\frac{1}{8}z^2 + \frac{\delta}{16}z^2\sigma\Big(\frac{|z|}{\mu(z)}\Big)\Big] + \lambda(x)|z|k|\xi| \\
&\le \frac{\varepsilon\mu(\xi)}{\sqrt{1+\Big(\frac{z}{\mu(z)}\Big)^2}}\Big[\Big(-\frac{1}{8} + \frac{\delta}{16}\Big)z^2 + |z|k|\xi|\sqrt{\mu^2(z)+z^2}\Big] .
\end{aligned}$$

Since, for some $\bar k > 0$,

$$k\sqrt{\mu^2(z)+z^2} \le k\sqrt{1+z^2} \le \bar k(1+|z|)$$

and $|\xi| \le \varepsilon$, we have

$$|z|k|\xi|\sqrt{\mu^2(z)+z^2} \le \bar k|z||\xi| + \varepsilon\bar k z^2 .$$

This yields

$$\dot V \le \frac{\varepsilon\mu(\xi)}{\sqrt{1+\Big(\frac{z}{\mu(z)}\Big)^2}}[-az^2 + \bar k|z||\xi|] ,$$

with

$$a = \frac{1}{8} - \frac{\delta}{16} - \bar k\varepsilon .$$

If δ and ε are small enough, $a > 0$. Moreover, on the time interval $[T^\circ, \infty)$,

$$1 \le \sqrt{1+\Big(\frac{z}{\mu(z)}\Big)^2} \le \sqrt{3} , \quad \text{and} \quad \frac{1}{\sqrt{1+\varepsilon^2}} \le \mu(\xi) \le 1 .$$

Thus, the previous inequality shows (see e.g. the proof of Lemma 14.4.2) that

$$|z(t)| \le \max\{\theta(|z(0)|)e^{-ct}, L\|\xi(\cdot)\|_\infty\} , \tag{14.106}$$

in which $\theta(\cdot)$ is a class $\mathcal{K}$ function and $c > 0, L > 0$ are fixed numbers.

This, together with (14.93), the fact that $v(\cdot)$ respects the restriction $|v(t)| \le 1$ and the fact that $|v(t)| \le |z(t)|$ on the time interval $[T^\circ, \infty)$, shows, via the small-gain theorem, that if ε is small enough (so that $\sqrt{2}\varepsilon L < 1$), all trajectories of the closed loop system converge to the equilibrium $(z, \xi) = (0, 0)$ as $t \to \infty$ and that the latter is stable in the sense of Lyapunov. This completes the proof. ◁

Bibliographical Notes

Chapter 10. Additional material on the comparison functions and their role in the study of the asymptotic properties of the trajectories of a nonlinear system can be found in the books of Hahn (1967) and Khalil (1996). In particular, the first of these books refers to Theorem 10.2.1 as to the "theorem of total stability". Theorem 10.3.1 is due to Sontag (1990). The notion of "input-to-state" stability and that of ISS-Lyapunov function have been introduced and thoroughly studied by E. Sontag, in a series of fundamental papers beginning with Sontag (1989). An overview of the main features of the notion of input-to-state stability can be found in Sontag (1995). The full proof of Theorem 10.4.1 can be found in Sontag-Wang (1995). The proof of Theorem 10.4.3 is taken from Sontag (1990). The use of the alternative characterization of the notion of input-to-state stability provided by the pair of inequalities (10.28) and (10.29) was introduced by A. Teel (see also, in this respect, Jiang *et al.* (1994) and Coron *et al.* (1995)). The proof of Lemma 10.4.4 is taken from Sontag-Wang (1996). The full proof of the equivalence of the pair of inequalities (10.28) and (10.29) with the property of input-to-state stability, namely Theorem 10.4.5, can be found in Sontag-Wang (1996). The proof of Lemma 10.5.1 and its application in Theorem 10.5.2 are due to Sontag-Teel (1995). The proof of the small-gain theorem for input-to-state stable systems, namely Theorem 10.6.1, is the proof suggested by Teel (1996a) (see also Coron *et al.* (1995)). The notions given here of a "dissipative" nonlinear system, and the related notions of "finite L_2 gain" and of "passive" system, follow – with minor modifications – the seminal work of Willems (1972). The stability of nonlinear interconnected dissipative systems has been the subject of many papers, beginning with the work of Hill-Moylan (1976). In particular, Theorem 10.8.1 is from Hill-Moylan (1977). Section 10.9 summarizes a number of classical results, from the theory of linear systems, concerning the relation (Theorem 10.9.1) between the property of being stable and having a finite L_2 gain not exceeding $\bar{\gamma} < \gamma$ and the property of having a transfer function matrix whose H_∞ norm does not exceed γ, and the relation (Theorem 10.9.2) between the property of being weakly strictly passive and the property of having a transfer function matrix which is positive real. These results are usually referred to as "bounded-real lemma" and "positive-real lemma", or Kalman-Yakubovic-Popov lemma. The notion of "weak strict passivity" used here was suggested by Lozano-Joshi (1990), and helps to obtain the characterization given in Theorem 10.9.2. Also Lemma 10.9.3 is a classical result in linear system theory.

Chapter 11. The recursive design method, outlined in the first section of this Chapter, and known as the method of backstepping, has become a very popular tool for the design of feedback laws in the presence of parameter uncertainties. The potential of the idea on which this method reposes has been successfully demonstrated in the work of Kanellakopoulos *et al.* (1991), where a systematic method for the design of adaptive control laws was introduced. The results presented here in section 11.2 are essentially based on the works of Freeman-Kokotovic (1993) – (1996a). The results presented in section 11.3 are essentially based on the works of Marino-Tomei (1993a) – (1993b), though the derivation described here is slightly different. Abundant sources of additional material on the method of backstepping are the books of Krstic *et al.* (1995) and Marino-Tomei (1995). The results presented here in section 11.4, in which at each stage of the recursion the goal of the design is to impose – via feedback – the "gain function" of an input-to-state stable system, are based on the work of Jiang *et al.* (1994) (see also Coron *et al.* (1995)). The derivation of normal forms for multi-input systems, for the general case in which the system in question does not possess a vector relative degree, is based on our earlier zero-dynamics algorithm presented in Isidori (1995), and is taken from the recent work of Schwartz *et al.* (1999).

Chapter 12. The material described in this Chapter is taken in large part from the fundamental work of Teel-Praly (1995). In particular, from this work is taken Theorem 12.1.1, which is an extension of Theorem 9.3.1, Corollary 12.1.2, the extended version of this Corollary presented at the end of section 13.2, and their respective proofs. One of the key features of Theorem 12.1.1 is the fact that, using high-gain feedback, it is possible to steer the trajectories to an arbitrarily small neighborhood of the origin, despite of the presence of "matched" uncertainties. The proof of this fact uses an argument which extends, to the setup of semiglobal and practical stabilizability, the argument suggested by Bacciotti (1992) for the proof of Theorem 9.3.1. The notion of input-to-state stability with restrictions and the corresponding small-gain Theorem 12.2.1 have been introduced by Teel (1996a). Section 13.3 provides a proof of Theorem 9.6.2, which is different from the proof originally given by Teel-Praly (1994). The arguments introduced in this alternative proof are used also in the proofs of some other results presented later in the Chapter, such as Theorem 12.4.1 and Theorem 12.6.2. Lemma 12.3.1 captures the fundamental idea of Khalil-Esfandiari (1993) that, if the first $n-1$ derivatives of the output of the system are estimated by an "observer" such as the one defined by equations of the form (12.30), past an initial fixed time interval (on which the trajectories of the system are guaranteed to exist because the effect of possibly large estimation errors is neutralized by the "saturation" included in the feedback law), the estimation error can be rendered arbitrarily small. The results presented in sections 13.5 and 13.6 are taken from Isidori (1999). In particular, this work seems to be – to the best of our knowledge – the first one to have called the attention on the importance, in problems of nonlinear stabilization via output feedback, of being able of stabilizing the so-called auxiliary system (12.70), a property that turns out to be necessary in the case of linear systems. The first example in section 13.7 is taken from Mazenc *et al.* (1994). The second example, which is taken from Isidori (1999), seems to suggest that the (semiglobal) stabilization approach, based on the results presented here in section 13.6, compares favorably with other methods presented in the literature, and summarized in Krstic *et al.* (1998).

Chapter 13. The method, described in section 13.1, for the design of a feedback law to the purpose of achieving asymptotic stability and a prescribed upper bound on the L_2 gain, between disturbance and output in a system of the form (13.16), is based on the works of Marino *et al.* (1989), (1994). The role played, in the case of linear systems, by the properties of the "unstable component" of the zero dynamics on establishing the minimal level of disturbance attenuation, was indeed widely acknowledged in the literature, but the explicit characterization given here has been – to the best of our knowledge – only pointed out recently by Isidori *et al.* (1999). The same work introduces the analysis presented here in sections 13.3 and 13.5, which in some way extends the results given in section 13.2 to the case of nonlinear systems. The results on almost disturbance decoupling presented in section 13.4, an extension of the results of Marino *et al.* (1994), are based on the work of Isidori (1996b). Section 13.6 describes classical, necessary and sufficient, conditions for the solution – in the case of a linear system – of the problem of achieving internal stability and a prescribed upper bound on the L_2 gain, or – what is the same – a prescribed bound on the H_∞ norm of the transfer function matrix. Theorem 13.6.3, which is taken from the fundamental work of Scherer (1992), shows how the existence of the solution of the characterizing Riccati inequality (13.81) can be tested in a simple way. The result presented in section 13.7 are due to Isidori-Lin (1998).

Chapter 14. Propositions 14.1.1 and 14.1.2, which show that a system whose uncontrolled dynamics possesses a Lyapunov function whose derivative is only semi-definite can be stabilized – if zero-state detectable – by means of a feedback law given by the derivative of this function along the vector field which weights the input, are corollaries of Corollary 10.8.4 and due to Hill-Moylan (1977). The extension given by Theorem 14.1.3 is due to W. Lin (1996). Proposition 14.1.5, a special case of a more general property which holds for the indicated class of systems, is due to Liu *et al.* (1996). The proof given here of this Proposition is the one suggested in that paper. The possibility of recursively stabilizing systems in upper-triangular, or "feedforward" form, whose antecendents can be found in the work of Teel (1992), was independently investigated by Teel (1996a), by Mazenc-Praly (1996) and by Jankovic *et al.* (1996). The exposition here in section 14.2 is based on the work of Jankovic *et al.* (1996). Additional material on this specific approach, based on the construction of a cross-term for a Lyapunov function via a line-integral along the trajectories of the uncontrolled system, can be found in the book of Sepulchre *et al.* (1997). The results presented here in section 14.3, which lead to the synthesis of globally stabilizing laws consisting in the composition of linear laws and saturation functions, are due to Teel (1996a). The exposition of these results follows very closely that in the original source. The first application presented in section 14.4 is actually a corollary from Teel (1996a), while the extension to systems (14.84), not in upper-triangular form, is based – with slight modifications – on the work of Lin-Li (1999).

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