Optimal Control

DEPARTMENT OF COMPUTER, CONTROL, AND
MANAGEMENT ENGINEERING ANTONIO RUBERTI

SAPIENZA
UNIVERSITÀ DI ROMA

Lecture 4

Prof. Daniela Iacoviello

THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

Course outline

- Introduction to optimal control
- Nonlinear optimization
- Dynamic programming
- Calculus of variations
- Calculus of variations and optimal control
- LQ problem
- Minimum time problem

R.F.Hartl, S.P.Sethi. R.G.Vickson, *A Survey of the Maximum Principles for Optimal Control Problems with State Constraints*, SIAM Review, Vol.37, No.2,

Pontryagin

Lev Semenovich **Pontryagin** (3 September 1908 – 3 May 1988) was a Soviet Russian mathematician. He was born in Moscow and
lost his eyesight in a stove explosion when he was 14.
Despite his blindness he was able to
become a mathematician due
to the help of his mother who read
mathematical books and papers to him.

He made major discoveries in a number of fields of mathematics, including the geometric parts of topology.

The Pontryagin principle

Problem 1: Consider the dynamical system: $\boxed{\dot{x} = f(x,u)}$

with:

$$\boxed{x(t) \in R^n, \quad u(t) \in U \subset R^p \quad f, \frac{\partial f}{\partial x_i} \in C^0(R^n \times U), \; i = 1,2,\ldots, n}$$

Assume fixed the initial control instant and the initial and final values :

$$\boxed{x(t_i) = x^i \quad x(t_f) = x^f}$$

Define the performance index :

$$J(x,u,t_f) = \int_{t_i}^{t_f} L(x(\tau),u(\tau))d\tau$$

with

$$\boxed{L, \frac{\partial L}{\partial x_i} \in C^0(R^n \times U), \; i = 1,2,\ldots, n}$$

Determine:

- the value $t_f \in (t_i, \infty)$,
- the control $u^o \in \overline{C}^0(R)$
- the state $x^o \in \overline{C}^1(R)$

that satisfy:

- ✓ the dynamical system,
- ✓ the constraint on the control,
- ✓ the initial and final conditions
- ✓ and minimize the cost index

Hamiltonian function

$$H(x,u,\lambda_0,\lambda)=\lambda_0 L(x,u)+\lambda^T(t)f(x,u)$$

The Pontryagin principle

Theorem 1 (necessary condition):

Assume the admissible solution $\left(x^*, u^*, t_f^*\right)$ is a minimum

➡ there exist a constant $\lambda_0 \geq 0$

and a n-dimensional vector $\lambda^* \in \overline{C}^1\left[t_i, t_f^*\right]$

<u>not simultaneously null</u> such that :

$$\boxed{\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T}}$$

$$\boxed{H\big|^* = 0}$$

$$\boxed{\begin{array}{l} H\left(x^*(t), \omega, \lambda_0^*, \lambda^*(t)\right) \geq H\left(x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)\right), \\ \forall \omega \in U \end{array}}$$

The Pontryagin principle

Remark- If t_f is fixed, the condition

$$\boxed{H\big|^* = 0}$$

Is substituted by

$$\boxed{H\big|^* = k, \quad \forall t \in \left[t_i,\ t_f\right]}$$

The Pontryagin principle

Problem 2: Consider the dynamical system: $\dot{x} = f(x,u)$

with:

$$x(t) \in R^n, \quad u(t) \in U \subset R^p \quad f, \frac{\partial f}{\partial x_i} \in C^0\left(R^n \times U\right), \; i = 1,2,\dots, n$$

Assume fixed the initial control instant and the initial state $x(t_i) = x^i$

while for final values assume: $\boxed{\aleph\left(x(t_f)\right) = 0}$

where $\aleph$ is a function of dimension $\sigma_f \le n$ of C^1 class.

Define the performance index :

$$J\left(x,u,t_f\right) = \int_{t_i}^{t_f} L\left(x(\tau),u(\tau)\right) d\tau + G(x(t_f))$$

with

$$L, \frac{\partial L}{\partial x_i} \in C^0\left(R^n \times U\right), \; i = 1,2,\dots, n, \; G \in C^2$$

Determine:

- the value $t_f \in (t_i, \infty)$
- the control $u^o \in \overline{C}^0(R)$
- and the state $x^o \in \overline{C}^1(R)$

that satisfy

- ✓ the dynamical system,
- ✓ the constraint on the control,
- ✓ the initial and final conditions
- ✓ and minimize the cost index .

Theorem 2 (necessary condition):

Consider an admissible solution $\left(x^*, u^*, t_f^*\right)$ such that

$$rank\left\{ \left.\frac{d\aleph}{dx(t_f)}\right|^* \right\} = \sigma_f$$

If it is a minimum $\Rightarrow$

there exist a constant $\lambda_0 \geq 0$ and an n-dimensional vector $\lambda^* \in \overline{C}^1\left[t_i, t_f^*\right]$ *not simultaneously null* such that :

$$\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T}$$

$$H|^* = 0$$

$$H\left(x^*(t), \omega, \lambda_0^*, \lambda^*(t)\right) \geq H\left(x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)\right),$$
$$\forall \omega \in U$$

Moreover there exists a vector $\varsigma \in R^{\sigma_f}$ such that:

$$\lambda(t_f) = \left.\frac{d\aleph}{dx(t_f)}\right|^{*T} \varsigma$$

The Pontryagin principle

Remark- If t_f is fixed condition

$$\boxed{H\Big|^{*} = 0}$$

Is substituted by

$$\boxed{H\Big|^{*} = k, \quad \forall t \in \left[t_i,\ t_f\right]}$$

The Pontryagin principle

Problem 3: Consider the dynamical system: $\dot{x} = f(x,u,t)$

with: $x(t) \in R^n, \quad u(t) \in U \subset R^p \quad f, \dfrac{\partial f}{\partial x_i} \in C^0(R^n \times U), \; i = 1,2,\ldots,n$

Assume fixed the initial control instant and the initial state $x(t_i) = x^i$

while for final values assume: $\boxed{\aleph(x(t_f), t_f) = 0}$ **where** $\aleph$

is a function of dimension $\sigma_f \leq n+1$ **of C¹ class**.

Define the **performance index** :

$$J(x,u,t_f) = \int_{t_i}^{t_f} L(x(\tau), u(\tau), \tau)\, d\tau$$

with $\boxed{L, \dfrac{\partial L}{\partial x_i}, \dfrac{\partial L}{\partial t} \in C^0(R^n \times U \times R), \; i = 1,2,\ldots,n}$

Determine:

- the value $t_f \in (t_i, \infty)$
- the control $u^o \in \overline{C}^0(R)$
- and the state $x^o \in \overline{C}^1(R)$

that satisfy

the dynamical system,

the constraint on the control,

the initial and final conditions

and minimize the cost index .

Theorem 3: Consider an admissible solution $\left(x^*, u^*, t_f^*\right)$ such that

$$rank\left\{\left.\frac{\partial \aleph}{\partial\left(x(t_f), t_f\right)}\right|^*\right\} = \sigma_f$$

IF it is a minimum $\Rightarrow$ there exist a constant $\lambda_0 \geq 0$ and an n-dimensional vector $\lambda^* \in \overline{C}^1\left[t_i, t_f^*\right]$ *not simultaneously null*

such that : $\boxed{\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T},}$ $\boxed{\left.H\right|^* + \int_t^{t_f^*} \left.\frac{\partial H}{\partial \tau}\right|^* d\tau = k, \quad k \in R}$

$$\boxed{H\left(x^*(t), \omega, \lambda_0^*, \lambda^*(t)\right) \geq H\left(x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)\right), \quad \forall \omega \in U}$$

Moreover there exists a vector $\varsigma \in R^{\sigma_f}$ such that:

$$\lambda^*(t_f) = \left.\frac{\partial \aleph}{\partial x(t_f)}\right|^{*T} \varsigma \qquad \left.H\right|_{t_f^*}^* = -\left.\frac{\partial \aleph}{\partial t_f}\right|^{*T} \varsigma$$

The Pontryagin principle

Problem 4: Consider the dynamical system: $\dot{x} = f(x,u,t)$

with: $x(t) \in R^n, \quad u(t) \in U \subset R^p \quad f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial t} \in C^0\left(R^n \times U \times R\right)$

and initial instant and state fixed $x(t_i) = x^i$

For the final values assume: $\boxed{\aleph\left(x(t_f), t_f\right) = 0}$ where $\aleph$ is a function of dimension $\sigma_f \leq n+1$ of C[1] class.

Assume the constraint $\boxed{\int_{t_i}^{t_f} h\left(x(\tau), u(\tau), \tau\right) d\tau = k}$ with

$$h, \frac{\partial h}{\partial x(t)}, \frac{\partial h}{\partial t} \in C^0\left(R^n \times U \times R\right), \; i = 1,2,\ldots,n$$

Define the **performance index** :

$$\boxed{J\left(x,u,t_f\right) = \int_{t_i}^{t_f} L\left(x(\tau), u(\tau), \tau\right) d\tau}$$

with $L, \dfrac{\partial L}{\partial x}, \dfrac{\partial L}{\partial t} \in C^0\left(R^n \times U \times R\right)$

Determine

- the value $t_f \in (t_i, \infty)$
- the control $u^o \in \overline{C}^0(R)$
- and the state $x^o \in \overline{C}^1(R)$

that satisfy

✓ the dynamical system,

✓ the constraint on the control,

✓ the initial and final conditions

✓ and minimize the cost index .

Hamiltonian function

$$H(x,u,\lambda_0,\lambda)=\lambda_0 L(x,u)+\lambda^T(t)\,f(x,u)+\rho^T h(x(t),u(t),t)$$

Theorem 4 (necessary condition):

Consider an admissible solution $\left(x^*, u^*, t_f^*\right)$ such that

$$rank\left\{\left.\frac{\partial \aleph}{\partial\left(x(t_f), t_f\right)}\right|^*\right\} = \sigma_f$$

IF it is a local minimum

there exist a constant $\lambda_0 \geq 0$, $\rho^* \in R^{\sigma}$, $\lambda^* \in \overline{C}^1\left[t_i, t_f^*\right]$

not simultaneously null such that :

$$\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T},$$

$$H\left(x^*(t), \omega, \lambda_0^*, \lambda^*(t)\right) \geq H\left(x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)\right), \; \forall \omega \in U$$

$$H\Big|^* + \int_t^{t_f^*} \left.\frac{\partial H}{\partial \tau}\right|^* d\tau = k, \;\; k \in R$$

Moreover there exists a vector $\varsigma \in R^{\sigma_f}$ such that:

$$\lambda^*(T) = \left.\frac{\partial \aleph}{\partial x(t_f)}\right|^{*T} \varsigma \qquad H\Big|_{t_f^*}^* = -\left.\frac{\partial \aleph}{\partial t_f}\right|^{*T} \varsigma$$

The discontinuities of $\dot{\lambda}^*$ may occur only in the instants in which u has a discontinuity and in these instants the Hamiltonian is continuous

Remark

If the set U coincides with R^p the minimum condition reduces to :

$$\boxed{\frac{\partial H}{\partial u} = 0}$$

The Pontryagin principle - convex case

Problem 5: Consider the dynamical **linear system:**

$$\dot{x} = A(t)x + B(t)u$$

with A and B of function of C^1 class; assume fixed the initial

and final instants and the initial state, and

$$x(t_f) = x_f \quad fixed \quad or \quad x(t_f) \in R^n$$

Assume $u(t) \in U \subset R^p \quad \forall t \in [t_i, t_f]$

where *U* is a **convex set.**

Define the **performance index** :

$$J(x,u) = \int_{t_i}^{t_f} L(x(\tau), u(\tau), \tau) d\tau + G(x(t_f))$$

with

$$L, \frac{\partial L}{\partial x_i}, \frac{\partial L}{\partial t} \in C^0\left(R^n \times U \times [t_i, t_f]\right), \; i = 1,2,\dots,n$$

L **convex function** with respect to *x(t), u(t)* in $R^n \times U$

per ogni $t \in [t_i, t_f]$

G is a scalar function of C^2 class and **convex with respect to *x(tf)*.**

Determine:

the control $u^o \in \overline{C}^0[t_i, t_f]$

and the state $x^o \in \overline{C}^1[t_i, t_f]$

that satisfy

the dynamical system,

the constraint on the control,

the initial and final conditions

and minimize the cost index .

Theorem 5 (necessary and sufficient condition):

Consider an admissible solution (x^o, u^o) such that

$$rank\left\{ \frac{\partial \aleph}{\partial\left(x(t_f), t_f\right)}\Bigg|^o \right\} = \sigma_f$$

It is a minimum **normal** (i.e.λ_0 =1)

if and only if there exists an n-dimensional vector $\lambda^o \in \overline{C}^1[t_i, t_f]$

such that : $$\boxed{\dot{\lambda}^o = -\frac{\partial H(x,u,\lambda,t)}{\partial x}\bigg|^{oT}}$$

$$\boxed{H\left(x^o(t), \omega, \lambda^o(t)\right) \geq H\left(x^o(t), u^o(t), \lambda^o(t)\right), \quad \forall \omega \in U}$$

Moreover, if $x(t_f) \in R^n$ $\quad \lambda^o(t_f) = \frac{dG}{dx(t_f)}\bigg|^{oT}$

Example 3 (from L.C.Evans)

Control of production and consumption

Consider a factory whose output can be controlled.

Let's set:

$x(t)$ **the amount of output produced at time** t , $0 \le t$.

Assume we consume some fraction of the output at each time

and likewise reinvest the **remaining fraction *u(t)* .**

It is our control, subject to the constraint

$$0 \le u(t) \le 1$$

The corresponding dynamics are:

$$\dot{x}(t) = ku(t)x(t), \quad k > 0$$
$$x(0) = x_0$$

The positive constant k represents the *growth rate* of our reinvestment. We will chose K=1.

Assume as cost index the function:

$$J(u(\cdot)) = \int_0^{t_f} \left(1 - u(t)\right)x(t)dt$$

The aim is to maximize the total consumption of the output

We apply the Pontryagin Principle; the Hamiltonian is:

$$H(x,u,\lambda)=(1-u)x+\lambda xu$$

The necessary conditions are:

$$\dot{\lambda}(t)=-1-u(t)(\lambda(t)-1) \quad \lambda(t_f)=0$$

$$\dot{x}(t)=u(t)x(t)$$

$$H(x(t),u(t),\lambda(t))=\max_{0\le u\le 1}\{x(t)+u(t)x(t)(\lambda(t)-1)\}$$

$\Rightarrow$

$$x(t)+u(t)x(t)\lambda(t)-u(t)x(t)\ge x(t)+\omega(t)x(t)\lambda(t)-\omega(t)x(t),\ \forall\omega\in[0,1]$$

$\Rightarrow$

$$u(t)[\lambda(t)-1]\ge\omega(t)[\lambda(t)-1],\ \forall\omega\in[0,1]$$ since $x(t)>0$

$\Rightarrow$

$$u(t)[\lambda(t)-1] \geq \omega(t)[\lambda(t)-1], \ \forall \omega \in [0,1]$$

$$\Rightarrow$$

$$\boxed{u(t) = \begin{cases} 1 & if \quad \lambda(t) > 1 \\ 0 & if \quad \lambda(t) \leq 1 \end{cases}}$$

From the equation of the costate, since $\lambda(t_f)=0$,

by continuity we deduce for $t<t_f$, t close to t_f, that $\lambda(t)\le 1$

thus $u(t)=0$ for such values of t.

Therefore $\dot{\lambda}(t)=-1$ and consequently: $\lambda(t)=t_f-t$

More precisely $\lambda(t)=t_f-t$ so long as $\lambda(t)\le 1$

and this holds for: $t_f-1\le t\le t_f$

For times $t\le t_f-1$ with t near t_f we have $u(t)=1$

Therefore the costate equation yields:

$$\dot{\lambda}(t)=-1-\left(\lambda(t)-1\right)=-\lambda(t)$$

Since $\lambda(t_f - 1) = 1$ we have for all $t \leq t_f - 1$

$$\lambda(t) = e^{t_f - 1 - t} > 1$$

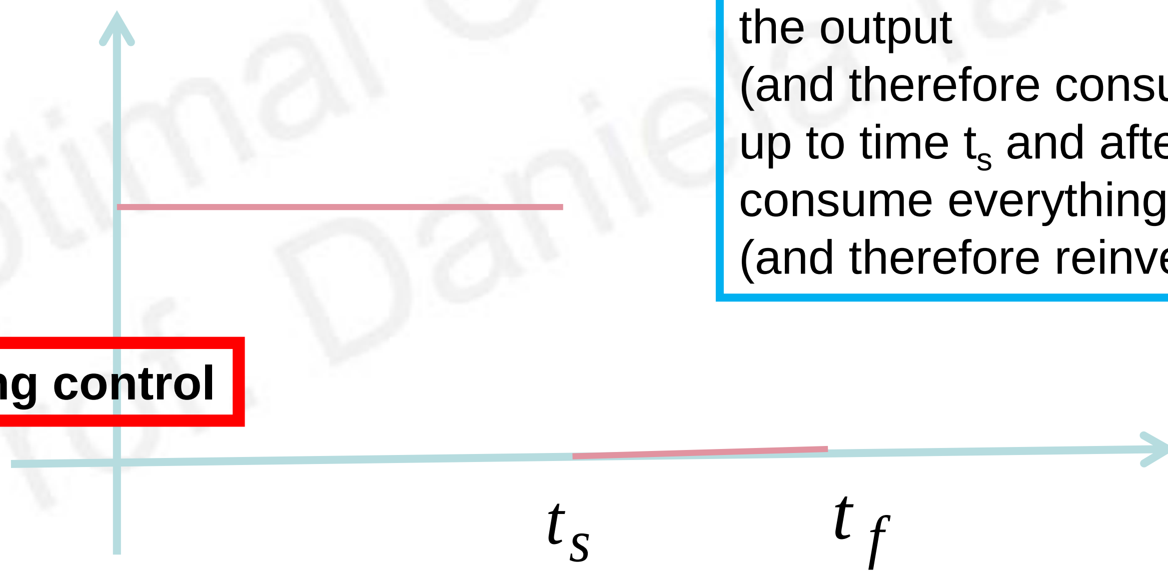
and over this time interval **there are no switchings**

Therefore: $$u^*(t)=\begin{cases}1 & if \quad 0\le t\le t_s\\ 0 & if \quad t_s\le t\le t_f\end{cases}$$

For the switching time $t_s=t_f-1$

Homework: find the switching time

Optimal solution: we should reinvest all the output
(and therefore consume nothing)
up to time t_s and afterwards we should consume everything
(and therefore reinvest nothing)



Bang-bang control