Optimal Control

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Lecture
Moon landing

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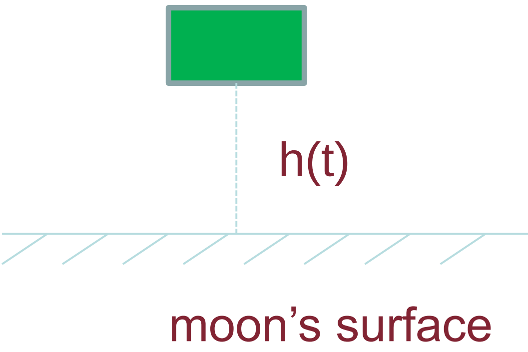
Moon lander (from Evans)

Notations:

m(t)= mass of spacecraft
h(t)=height ay time t
v(t)= velocity
α(t)= thrust at time t with $0 \le \alpha(t) \le 1$



Newton's law

$$m(t)\ddot{h}(t) = -gm(t) + \alpha(t)$$

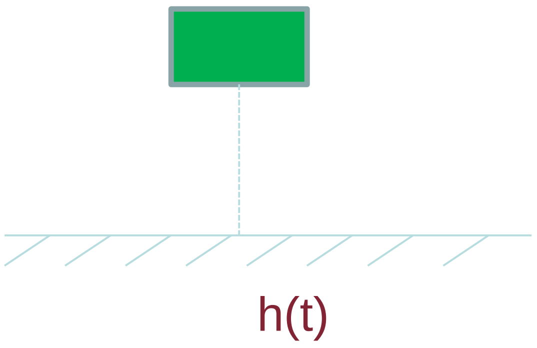
$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \dfrac{\alpha(t)}{m(t)} \\ \dot{m}(t) = -k\alpha(t) \end{cases}$$

$$h(0) = h_0 > 0$$
$$v(0) = v_0$$
$$m(0) = m_0 > 0$$

$$x(t) = \left(v(t) \quad h(t) \quad m(t)\right)^T$$

$$\dot{x}(t) = f\,(x(t), \alpha(t))$$

$$h(t) \geq 0, \quad m(t) \geq 0$$



Aim:

Minimize the amount of fuel used up,

that is to maximize the amount of the remaining one

once we have landed: $J(\alpha(t)) = m(\tau)$

where τ is the first instant in which $h(\tau) = v(\tau) = 0$

The goal is to land safely, maximizing the remaining fuel

$\Longleftrightarrow$

Minimize the total applied thrust before landing

$$J(\alpha(t)) = -\int_0^{\tau} \alpha(t)dt = -\frac{m_0 - m(\tau)}{k}$$

$$\boxed{\dot{m}(t) = -k\alpha(t) \;\Rightarrow\; \alpha(t) = -\frac{\dot{m}(t)}{k}}$$

Hamiltonian (normal case)

$$\boxed{H = -\alpha(t) + \lambda_1(t)v(t) + \lambda_2(t)\left(-g + \frac{\alpha(t)}{m(t)}\right) + \lambda_3(t)(-k\alpha(t))}$$

$$\dot{h}(t) = v(t) \qquad \dot{v}(t) = -g + \frac{\alpha(t)}{m(t)} \qquad \dot{m}(t) = -k\alpha(t)$$

$$\begin{cases} \dot{\lambda}_1 = -\dfrac{\partial H}{\partial h} = 0 \\ \dot{\lambda}_2 = -\dfrac{\partial H}{\partial v} = -\lambda_1 \\ \dot{\lambda}_3 = -\dfrac{\partial H}{\partial m} = \dfrac{\lambda_2 \alpha}{m^2} \end{cases}$$

Pontryagin Principle

$$\boxed{\alpha(t)\left(-1+\frac{\lambda_2(t)}{m(t)}-\lambda_3(t)k\right) \geq \omega(t)\left(-1+\frac{\lambda_2(t)}{m(t)}-\lambda_3(t)k\right), \ \forall admissible\, \omega}$$

$$\boxed{\varphi(t) = -1+\frac{\lambda_2(t)}{m(t)}-\lambda_3(t)k}$$

$$\boxed{\alpha(t)\varphi(t) \geq \omega(t)\varphi(t), \ \forall admissible \quad \omega}$$

$$\boxed{\alpha(t)\varphi(t) \geq \omega(t)\varphi(t),\ \forall admissible\ \ \omega}$$

$$\boxed{\alpha(t) = \begin{cases} 1 & if\ \ \varphi(t) > 0 \\ 0 & if\ \varphi(t) < 0 \end{cases}}$$

$$\boxed{\varphi(t) = -1 + \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k}$$

FORM OF THE SOLUTION ?

Let's start by guessing that we first leave rocket engine (α=0)
and turn the engine on only at the end

We guess there exists a switching time $t_s < \tau$ such that:

$$\alpha(t) = \begin{cases} 0 \ \ if \ \ 0 \le t \le t_s \\ 1 \ \ if \ \ t_s < t \le \tau \end{cases}$$

$$t_s < t \le \tau$$

$$\boxed{\begin{aligned} &\dot{h}(t) = v(t) \\ &\dot{v}(t) = -g + \frac{1}{m(t)} \\ &\dot{m}(t) = -k \\ &h(\tau) = v(\tau) = 0, \;\; m(t_s) = m_0 \end{aligned}}$$

$$\boxed{\begin{aligned} &m(t) = m_0 + k(t_s - t) \\ &v(t) = g(\tau - t) + \frac{1}{k}\log\left[\frac{m_0 + k(t_s - \tau)}{m_0 + k(t_s - t)}\right] \\ &h(t) = \ldots\ldots.. \end{aligned}}$$

$$\boxed{\begin{aligned} &m(t_s) = m_0 \\ &v(t_s) = g(\tau - t_s) + \frac{1}{k}\log\left[\frac{m_0 + k(t_s - \tau)}{m_0}\right] \\ &h(t) = \ldots\ldots.. \end{aligned}}$$

Let m_1 be the total amount of fuel to start

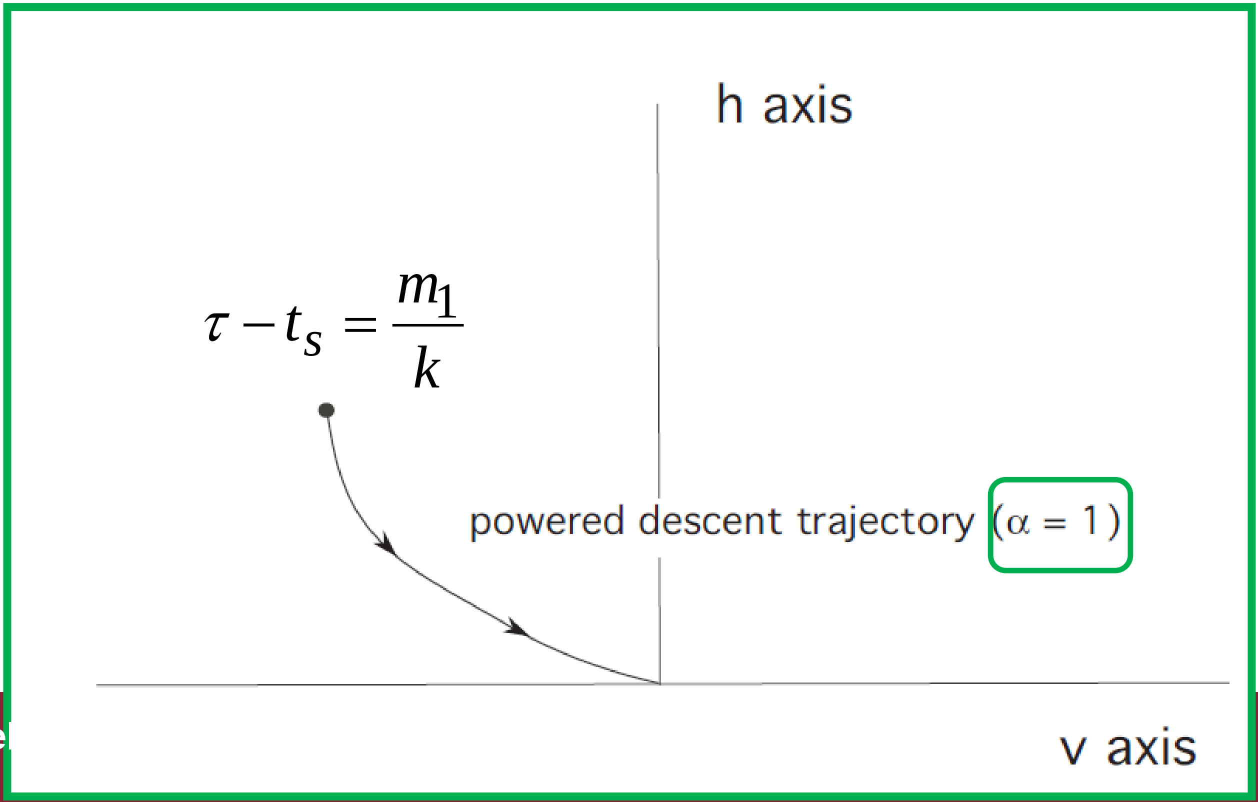
$m_0 - m_1$ is the weight of the empty spacecraft

Total amount of fuel to start

With α=1 the fuel is used up at rate k: $\dot{m}(t) = -\alpha(t)$

$$k(\tau - t_s) \le m_1$$

$$0 \le \tau - t_s \le \frac{m_1}{k}$$



BEFORE t_s $\alpha=0$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g \\ \dot{m}(t) = 0 \end{cases} \Rightarrow \begin{aligned} h(t) &= -\frac{1}{2}gt^2 + tv_0 + h_0 \\ v(t) &= -gt + v_0 \\ m(t) &= m_0 \end{aligned}$$

$$\Rightarrow \quad h(t) = -\frac{1}{2g}\left(v^2 - v_0^2\right) + h_0 \qquad 0 \le t \le t^*$$

$\Rightarrow$ The freefall trajectory lies on a parabola

$$h(t) = -\frac{1}{2g}\left(v^2 - v_0^2\right) + h_0 \qquad 0 \le t \le t^*$$

If we move along this parabola *until we hit the soft landing curve* we can **turn on** the rocket engine and land softly

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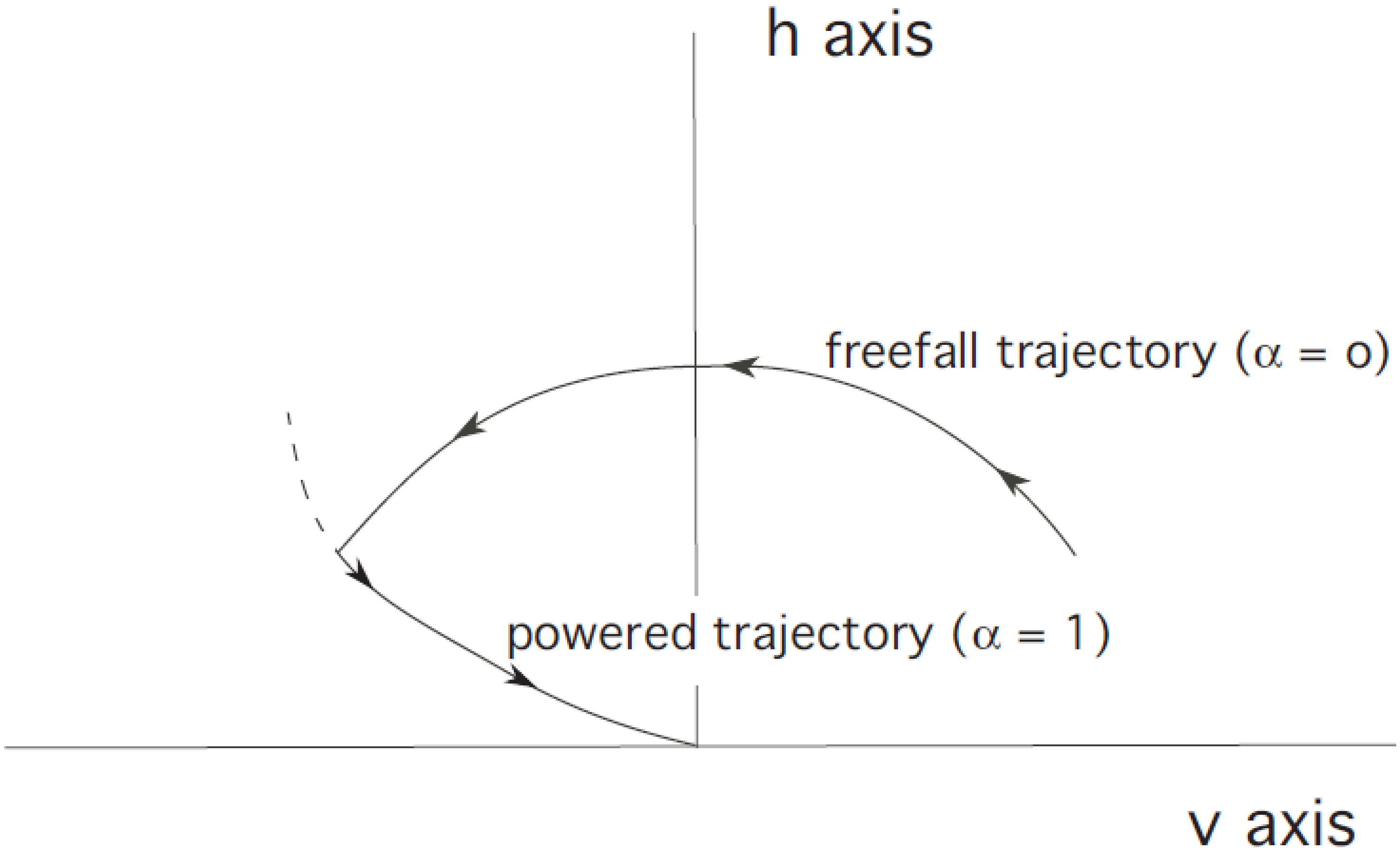


Let's justify our guess about the structure of the control:

The costate variables are:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial h} = 0$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial v} = -\lambda_1$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial m} = \frac{\lambda_2 \alpha}{m^2}$$

$$\lambda_1 = \bar{\lambda}_1 \qquad 0 \le t \le \tau$$

$$\lambda_2 = -\bar{\lambda}_1 t + \bar{\lambda}_2 \qquad 0 \le t \le \tau$$

$$\lambda_3 = \begin{cases} \bar{\lambda}_3 & 0 \le t \le t^* \\ \bar{\lambda}_3 + \int_{t^*}^{t} \frac{\lambda_2 - \lambda_1 s}{\left(m_0 + k\left(t^* - s\right)\right)^2} ds & t^* \le t \le \tau \end{cases}$$

The maximum principle is:

$$\alpha(t)\left(-1 + \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k\right) \ge \omega(t)\left(-1 + \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k\right), \forall admissible\, \omega$$

We will have to figure out appropriate *initial conditions* $\bar{\lambda}_1 \quad \bar{\lambda}_2 \quad \bar{\lambda}_3$

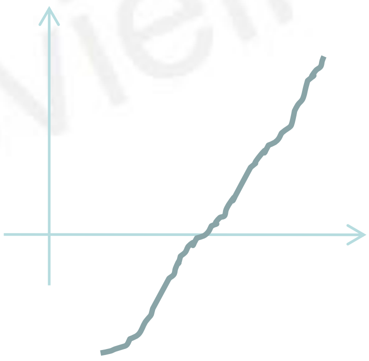
We have defined:

$$\boxed{\varphi(t) = -1 + \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k} \quad \Rightarrow \quad \boxed{\dot{\varphi}(t) = -\frac{\lambda_1(t)}{m(t)} = -\frac{\bar{\lambda}_1}{m(t)}}$$

Choose $\bar{\lambda}_1 < 0$ ⇒ φ is increasing

$$\boxed{\varphi(t) = -1 + \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k \;\Rightarrow\; \varphi(t_s) = -1 + \frac{\bar{\lambda}_2 - \bar{\lambda}_1 t_s}{m_0(t_s)} - \bar{\lambda}_3 k}$$

Adjust $\bar{\lambda}_2, \bar{\lambda}_3$ so that $\varphi(t_s) = 0$



φ is increasing

$\varphi(t_s) = 0$

⇒

$$\boxed{\begin{array}{l} \varphi < 0 \;\; on \;\; [0, t_s) \\ \varphi > 0 \;\; on \;\; (t_s, \tau] \end{array}}$$

We have already found from the minimum principle:

$$\boxed{\alpha(t) = \begin{cases} 1 & if \;\; \varphi(t) < 0 \\ 0 & if \; \varphi(t) > 0 \end{cases}}$$

⇒ Thus the optimal control changes just once from 0 to 1

Homework: *what happen if you choose* $\bar{\lambda}_1 > 0$ *?*