

Optimal Control

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Lecture 3

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THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References
indicated below

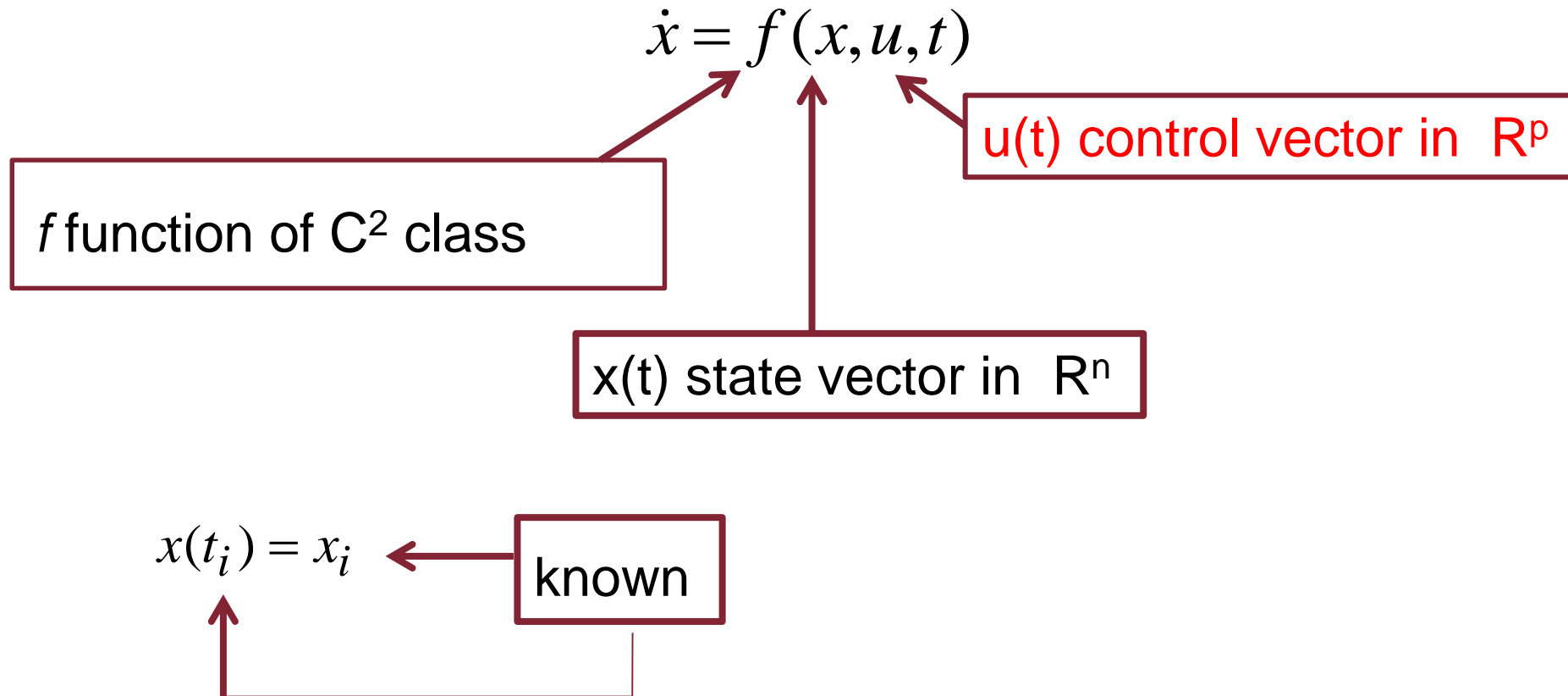
Course outline

- Introduction to optimal control
- Nonlinear optimization
- Dynamic programming
- Calculus of variations
- Calculus of variations and optimal control
- LQ problem
- Minimum time problem

Calculus of variation and optimal control

Problem 1

Let us consider the **dynamical system** described by:



$$\chi(x(t_f), t_f) = 0$$

Vectorial function of C^1 class
of dimension $\sigma_f \leq n+1$

$$q(x, u, t) \leq 0$$

Vectorial function of C^2 class
of dimension β

Assume the norm:

$$\|(x, u, t_f)\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^t u(\tau) d\tau \right\| + \sup_t \|u(t)\| + |t_f|$$

Define the cost index $J(x, u, t_f) = \int_{t_i}^{t_f} L(x, u, t) dt$

with L function of C^2 class

AIM: Find (if it exists)

- the instant t_f^o
- the control $u^o \in \bar{C}^0(R)$
- the state $x^o \in \bar{C}^1(R)$

that satisfy the previous equations and minimize the cost index

DEFINE the scalar function

$$H(x, u, \lambda_0, \lambda, t) = \lambda_0 L(x, u, t) + \lambda^T(t) f(x, u, t)$$

the Hamiltonian function

Theorem 1

Let (x^*, u^*, t_f^*) be an admissible solution such that

$$rk \left\{ \frac{\partial \chi}{\partial (x(t_f), t_f)} \Big|_* \right\} = \sigma_f \quad rk \left\{ \frac{\partial q_{active}}{\partial u} \Big|_* \right\} = \beta_a(t), \quad \forall t \in [t_i, t_f^*]$$

dimension of

IF (x^*, u^*, t_f^*) is a local minimum



there exist $\lambda_0^* \geq 0, \lambda^* \in \bar{C}^1[t_i, t_f^*], \eta^* \in \bar{C}^0[t_i, t_f^*]$

not simultaneously null in $[t_i, t_f^*]$ such that:

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x}\bigg|^{*T} - \frac{\partial q}{\partial x}\bigg|^{*T} \eta^*$$

$$0 = \frac{\partial H}{\partial u}\bigg|^{*T} + \frac{\partial q}{\partial u}\bigg|^{*T} \eta^*$$

$$\eta_j^*(t) q_j(x^*, u^*, t) = 0, \quad \eta_j^*(t) \geq 0, \quad j = 1, 2, \dots, \beta$$

$$\lambda^*(t_f^*) = -\frac{\partial \chi}{\partial (x(t_f))}\bigg|_{t_f^*}^{*T} \zeta, \quad \zeta \in R^{\sigma_f}$$

$$H\big|_{t_f^*}^* = \frac{\partial \chi}{\partial t_f}\bigg|_{t_f^*}^{*T} \zeta$$

The discontinuity of $\dot{\lambda}^*$ and η^* may occur only in the points \bar{t}

where u^* has a discontinuity and $H\big|_{\bar{t}^-}^* = H\big|_{\bar{t}^+}^*$



Problem 2

Consider Problem 1 assuming L, f, q not depending on t

Theorem 2

Let (x^*, u^*, t_f^*) be an admissible solution

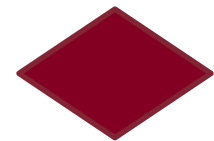


for any continuity point of u^*

$$\left. \frac{dH}{dt} \right|^* = \left. \frac{\partial H}{\partial t} \right|^* + \eta^{*T} \left. \frac{\partial q}{\partial t} \right|^*$$

Moreover for stationary problems:

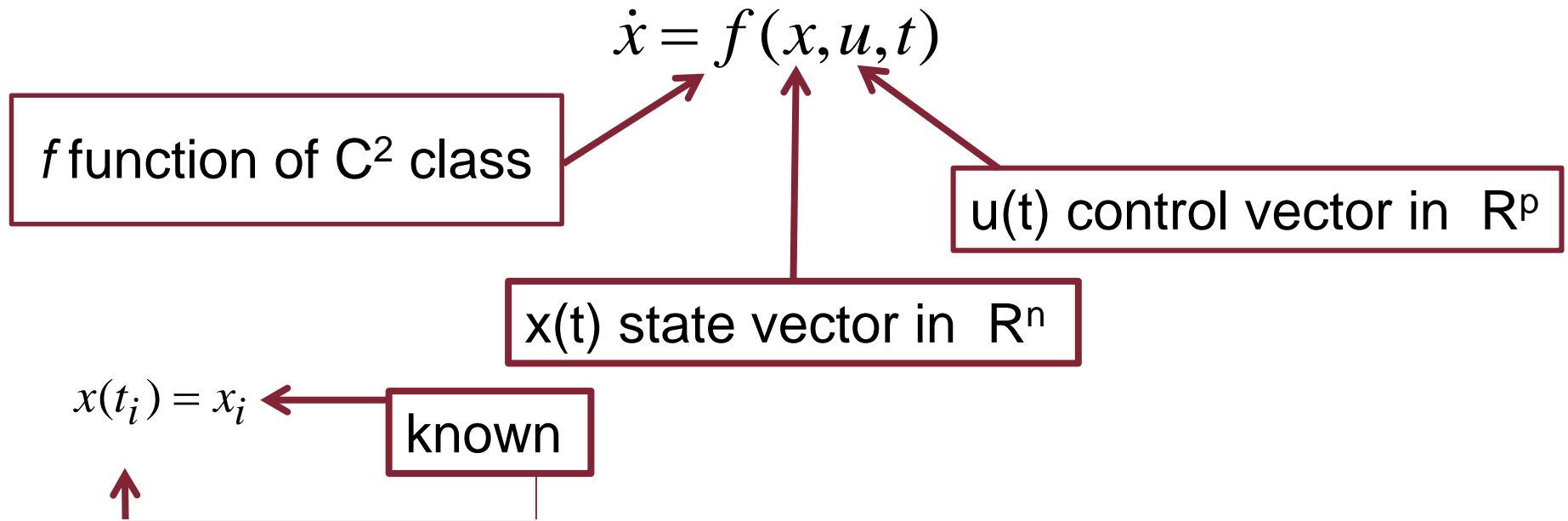
$$H|^* = c, \quad \forall t \in [t_i, t_f^*]$$



Calculus of variation and optimal control

Problem 3

Let us consider the **dynamical system** described by:



$$\chi(x(t_f), t_f) = 0$$

Vectorial function of C^1 class
of dimension $\sigma_f \leq n+1$

$$\int_{t_i}^{t_f} h(x(t), u(t), t) dt = K$$

Vectorial function of C^2 class
of dimension σ

Assume the norm:

$$\|(x, u, t_f)\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^t u(\tau) d\tau \right\| + \sup_t \|u(t)\| + |t_f|$$

Define the cost index

$$J(x, u, t_f) = \int_{t_i}^{t_f} L(x, u, t) dt$$

with L function of C^2 class

AIM: Find

- ☐ the instant t_f^o
- ☐ the control $u^o \in \bar{C}^0(R)$
- ☐ the state $x^o \in \bar{C}^1(R)$

that satisfy the previous equations and minimize the cost index

DEFINE the scalar function


$$H(x, u, \lambda_0, \lambda, t) = \lambda_0 L(x, u, t) + \lambda^T(t) f(x, u, t) + \rho^T h(x(t), u(t), t)$$

the Hamiltonian function

Theorem 3

Let (x^*, u^*, t_f^*) be an admissible solution such that $rk \left\{ \left. \frac{\partial \chi}{\partial (x(t_f), t_f)} \right| \right\}^* = \sigma_f$

$IF(x^*, u^*, t_f^*)$ is a local minimum

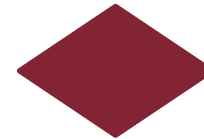
 there exist $\lambda_0^* \in R$, $\lambda^* \in \overline{C}^1[t_i, t_f^*]$, $\rho^* \in R$ not simultaneously null in $[t_i, t_f^*]$ such that:

- $\dot{\lambda}^* = - \left. \frac{\partial H}{\partial x} \right|^{*T}$
- $0 = \left. \frac{\partial H}{\partial u} \right|^{*T}$
- $\lambda^*(t_f^*) = - \left. \frac{\partial \chi}{\partial (x(t_f))} \right|_{t_f^*}^{*T} \zeta, \zeta \in R^{\sigma_f}$
- $H|_{t_f^*}^* = \left. \frac{\partial \chi}{\partial t_f} \right|_{t_f^*}^{*T} \zeta$

The discontinuity of $\dot{\lambda}^*$ may occur only in the points \bar{t}_k where u^* has a discontinuity

and

$$H|_{\bar{t}_k^-}^* = H|_{\bar{t}_k^+}^*$$



Problem 4:

Consider the linear system $\dot{x} = A(t)x + B(t)u$

Assume:

- ☐ $t_i \ t_f \ x(t_i) = x^i$ fixed
- ☐ $x(t_f) \in D_f$ being D_f a fixed point or \mathbb{R}^n
- ☐ $q(x, u, t) \leq 0$ Vectorial function of C^2 class of dimension β

Functions of C^2 class

CONVEX

Define the cost index

$$J(x, u) = \int_{t_i}^{t_f} L(x, u, t) dt + G(x(t_f))$$

Functions of C^3 class- **CONVEX**

Functions of C^2 class **CONVEX**

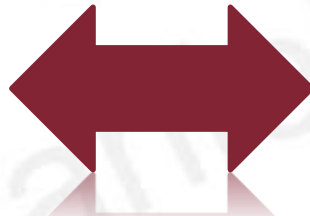
Theorem 4

Let (x^o, u^o) be an admissible solution such that

$$rk \left\{ \left. \frac{\partial q_{active}}{\partial u} \right| ^o \right\} = \beta_a(t), \quad \forall t \in [t_i, t_f]$$

$$(x^o, u^o)$$

is a **normal**
optimal solution



$$\begin{aligned} \dot{\lambda}^o &= - \left. \frac{\partial H}{\partial x} \right|^{oT} - \left. \frac{\partial q}{\partial x} \right|^{oT} \eta^o \\ 0 &= \left. \frac{\partial H}{\partial u} \right|^{oT} + \left. \frac{\partial q}{\partial u} \right|^{oT} \eta^o \\ \eta_j^o(t) q_j(x^o, u^o, t) &= 0, \\ j &= 1, 2, \dots, \beta \\ \eta^o(t) &\geq 0 \end{aligned}$$

AND IF $D_f = R^n$ $\lambda^o(t_f) = \left. \frac{dG}{dx(t_f)} \right|^{oT}$