NONLINEAR OBSERVERS AND SEPARATION PRINCIPLE

1 The Observability Canonical Form

In this Chapter we discuss the design of observers for nonlinear systems modelled by equations of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}$. In particular, we focus our discussion on the design of the so-called *high-gain observers*, which have been thoroughly investigated by Gauthier and Kupca in [**?**].

A key requirement for the existence of observers of this kind is the existence of a global diffeomorphism

$$\begin{array}{cccc} \Phi : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & z \end{array}$$

carrying system (1) into a system of the form

$$\begin{aligned} \dot{z}_1 &= \tilde{f}_1(z_1, z_2, u) \\ \dot{z}_2 &= \tilde{f}_2(z_1, z_2, z_3, u) \\ &\cdots \\ \dot{z}_{n-1} &= \tilde{f}_{n-1}(z_1, z_2, \ldots, z_n, u) \\ \dot{z}_n &= \tilde{f}_n(z_1, z_2, \ldots, z_n, u) \\ y &= \tilde{h}(z_1, u) \end{aligned} \tag{2}$$

in which the $\tilde{h}(z_1, u)$ and $f_i(z_1, z_2, \ldots, z_{i+1}, u)$ satisfy

$$\frac{\partial \tilde{h}}{\partial z_1} \neq 0\,, \quad \text{and} \quad \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \neq 0\,, \quad \text{for all} \quad i = 1, \ldots, n-1 \tag{3}$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$. This form will be referred to as the *Gauthier-Kupca's observability canonical form.*

For the sake of completeness, it is useful to review how the existence of canonical forms of this kind can be checked and the canonical form itself can be constructed. We begin with the description of a set of *necessary* conditions for the existence of this canonical form.

Consider again system (1), suppose that $f(0,0) = 0$, $h(0,0) = 0$, and define – recursively – a sequence of real-valued functions $\varphi_i(x,u)$ as follows

$$\varphi_1(x,u) := h(x,u)\,, \qquad \varphi_i(x,u) := \frac{\partial \varphi_{i-1}}{\partial x} f(x,u)\,.$$

for $i = 1, \ldots, n$. From these functions, define a sequence of i-vector-valued functions $\Phi_i(x,u)$ as follows

$$\Phi_i(x,u) = \begin{pmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{pmatrix}$$

for $i = 1, \ldots, n$. Finally, with each of the $\Phi_i(x,u)$'s, associate the subspace

$$K_i(x,u) = \ker \left[\frac{\partial \Phi_i}{\partial x} \right]_{(x,u)} .$$

Note that the map

$$D_i(u) \ : \ x \ \mapsto \ K_i(x,u)$$

identifies a *distribution* on $\mathbb{R}^n$. The collection of all these distributions has been called, by Gauthier and Kupca, the *canonical flag* of (1). The notation chosen stresses the fact that the map in question, in general, depends on the parameter $u \in \mathbb{R}^m$. The canonical flag is said to be *uniform* if:

(i) for all $i = 1, \ldots, n$, for all $u \in \mathbb{R}^m$ and for all $x \in \mathbb{R}^n$

$$\dim K_i(x,u) = n - i .$$

(ii) for all $i = 1, \ldots, n$ and for all $x \in \mathbb{R}^n$

$$K_i(x,u) = \text{independent of } u .$$

In other words, condition (i) says that the distribution $D_i(u)$ has constant dimension $n - i$. This condition is referred to as the "regularity" condition. Condition (ii) says that, for each x, the subspace $K_i(x,u)$ is always the same, regardless of what $u \in \mathbb{R}^m$ is. This condition is referred to as the"u-independency" condition.

Proposition 1 *System (1) is globally diffeomorphic to a system in Gauthier-Kupca's observability canonical form only if its canonical flag is uniform.*

Proof. (Sketch of) Suppose a system is already in observability canonical form and compute its canonical flag. A simple calculation shows that the each of the functions $\varphi_i(x,u)$ are functions of the form

$$\tilde{\varphi}_i(z_1, \ldots, z_i, u) ,$$

and

$$\frac{\partial \tilde{\varphi}_i}{\partial z_i} \neq 0, \qquad \text{for all } z_1, \ldots, z_i, u.$$

Thus, for each $i = 1, \ldots, n$

$$K_i(z,u) = \text{span} \begin{pmatrix} 0 \\ I_{n-i} \end{pmatrix} .$$

This shows that the canonical flag of a system in observability canonical form is uniform. This property is not altered by a diffeomorphism and hence the condition in question is a necessary condition for an observability canonical form to exist. $\triangleleft$

Remark. These necessary condition thus identified is also sufficient for the existence of a *local* diffeomorphism carrying system (1) into a system in observability canonical form (as proven in [**?**], Chapter 3, Theorem 2.1). ◁

We describe now a set of *sufficient* conditions for a system to be globally diffeomorphic to a system in Gauthier-Kupca's observability canonical form.

Proposition 2 *Consider the nonlinear system (1) and define a map*

$$\begin{array}{rcccl}\Phi & : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & & x & \mapsto & z = \Phi(x)\end{array}$$

as

$$\Phi(x) = \begin{pmatrix} \varphi_1(x,0) \\ \varphi_2(x,0) \\ \cdots \\ \varphi_n(x,0) \end{pmatrix}.$$

Suppose that:

(i) *the canonical flag of (1) is uniform,*

(i) $\Phi(x)$ *is a global diffeomorphism.*

Then, system (1) is globally diffeomorphic, via $\Phi(x)$*, to a system in Gauthier-Kupca's observability canonical form.*

Proof. By assumption,

$$\ker\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{(x,u)}$$

has constant dimension $n-i$ and it is independent of u. Now denote by $T(z)$ the inverse of the diffeomorphism $\Phi(x)$. Since $\Phi(x) = \Phi_n(x,0)$, $T(z)$ is a globally defined mapping which satisfies

$$\Phi_n(T(z),0) = z\,.$$

Then

$$\Big[\frac{\partial \Phi_n}{\partial x}\Big]_{\substack{x=T(z)\\u=0}} \frac{\partial T}{\partial z} = I$$

for all $z \in \mathbb{R}^n$. This implies, for all $j > i$,

$$\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{\substack{x=T(z)\\u=0}} \frac{\partial T}{\partial z_j} = 0\,,$$

or, what is the same,

$$\frac{\partial T}{\partial z_j} \in \ker\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{\substack{x=T(z)\\u=0}}\,, \qquad \forall z \in \mathbb{R}^n$$

But this, because the $K_i(x,u)$'s are independent of u, implies

$$\frac{\partial T}{\partial z_j} \in \ker\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{\substack{x=T(z)\\u=u}}\,, \qquad \forall j > i\,, \forall z \in \mathbb{R}^n\,, \forall u \in \mathbb{R}^m\,. \tag{4}$$

Suppose the map $z = \Phi(x)$ is used to change coordinates in (1) and consider the system in the new coordinates

$$\begin{aligned} \dot{z} &= \tilde{f}(z,u) \\ y &= \tilde{h}(z,u) \end{aligned}$$

where

$$\tilde{h}(z,u) = h(T(z),u)\,, \qquad \tilde{f}(z,u) = \left[\frac{\partial \Phi_n}{\partial x}\right]_{\substack{x=T(z)\\u=0}} f(T(z),u)\,.$$

Define

$$\tilde{\varphi}_1(z,u) := \tilde{h}(z,u)\,, \qquad \tilde{\varphi}_i(z,u) := \frac{\partial \tilde{\varphi}_{i-1}}{\partial z}\tilde{f}(z,u) \qquad \text{for } i = 1,\ldots,n.$$

and note that

$$\tilde{\varphi}_1(z,u) = \varphi_1(T(z),u)\,, \qquad \tilde{\varphi}_i(z,u) = \varphi_i(T(z),u)\,,$$

which implies

$$\tilde{\Phi}_i(z,u) := \begin{pmatrix} \tilde{\varphi}_1(z,u) \\ \vdots \\ \tilde{\varphi}_i(z,u) \end{pmatrix} = \Phi_i(T(z),u)\,.$$

Using (4) for $i = 1$, we obtain

$$\frac{\partial \tilde{\varphi}_1(z,u)}{\partial z_j} = \left[\frac{\partial \varphi_1}{\partial x}\right]_{\substack{x=T(z)\\u=u}} \frac{\partial T}{\partial z_j} = 0$$

for all $j > 1$ which means that $\tilde{h}(z,u) = \tilde{\varphi}_1(z,u)$ only depends on z_1. Note also that

$$\frac{\partial \tilde{\varphi}_1}{\partial z} = \begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1}{\partial z_1} & 0 & \cdots & 0 \end{pmatrix}$$

and hence, by the uniformity assumption,

$$\frac{\partial \tilde{h}}{\partial z_1} = \frac{\partial \tilde{\varphi}_1}{\partial z} \neq 0\,.$$

This being the case,

$$\tilde{\varphi}_2(z,u) = \frac{\partial \tilde{h}}{\partial z_1}\tilde{f}_1(z,u)\,.$$

Use now (4) for $i = 2$, to obtain (in particular)

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z_j} = \left[\frac{\partial \varphi_2}{\partial x}\right]_{\substack{x=T(z)\\u=u}} \frac{\partial T}{\partial z_j} = 0$$

for all $j > 2$ which means that $\tilde{\varphi}_2(z,u)$ only depends on z_1, z_2. Looking at the form of $\tilde{\varphi}_2(z,u)$, we see that

$$\frac{\partial \tilde{f}_1}{\partial z_j} = 0$$

for all $j > 2$ which means that $\tilde{f}_1(z,u)$ only depends on z_1, z_2. Moreover, an easy check shows that

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z} = \begin{pmatrix} * & \dfrac{\partial \tilde{h}}{\partial z_1}\dfrac{\partial \tilde{f}_1}{\partial z_2} & 0 & \cdots & 0 \end{pmatrix}$$

and hence

$$\frac{\partial \tilde{f}_1}{\partial z_2} \neq 0$$

because otherwise the uniformity assumption would be contradicted. Continuing in the same way, the result follows. $\triangleleft$

2 The case of input-affine systems

Consider and input-affine system, namely described by equations of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \tag{5}$$

It is easy to check that the functions $\varphi_i(x,u)$ have the following expressions

$$\begin{array}{rcl} \varphi_1(x,u) & = & h(x) \\ \varphi_2(x,u) & = & L_f h(x) + L_g h(x) u \\ \varphi_3(x,u) & = & L_f^2 h(x) + [L_g L_f h(x) + L_f L_g h(x)]u + L_g^2 h(x) u^2 \\ \varphi_4(x,u) & = & L_f^3 h(x) + [L_g L_f^2 h(x) + L_f L_g L_f h(x) + L_f^2 L_g h(x)]u + \\ & & \quad +[L_g^2 L_f h(x) + L_g L_f L_g h(x) + L_f L_g^2 h(x)]u^2 + L_g^3 h(x) u^3 \\ \varphi_5(x,u) & = & \cdots \end{array}$$

Hence

$$\Phi_n(x,0) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{pmatrix} := \Phi(x).$$

If the canonical flag of (5) is uniform and if $\Phi_n(x,0)$ is a global diffeomorphism, the system is transformable into “uniform observability” canonical form. The form in question is

$$\begin{array}{rcl} \dot{z} & = & \tilde{f}(z) + \tilde{g}(z)u \\ y & = & \tilde{h}(z) \end{array}$$

in which

$$\tilde{f}(z) = \Big[\frac{\partial \Phi(x)}{\partial x} f(x)\Big]_{x=\Phi^{-1}(z)}, \qquad \tilde{g}(z) = \Big[\frac{\partial \Phi(x)}{\partial x} g(x)\Big]_{x=\Phi^{-1}(z)}$$

$$\tilde{h}(z) = h(\Phi^{-1}(z))$$

By construction,

$$\tilde{h}(z) = z_1$$

Moreover, looking at the special structure of $\Phi(x)$, it is easy to check that $\tilde{f}(z)$ has the following form

$$\tilde{f}(z) = \begin{pmatrix} z_2 \\ z_3 \\ \cdots \\ z_n \\ \tilde{f}_n(z_1, \ldots, z_n) \end{pmatrix}$$

Finally, since we know that the i-th entry of

$$\tilde{f}(z) + \tilde{g}(z)u$$

can only depend on $z_1, z_2, \ldots, z_{i+1}$, we deduce that $\tilde{g}(z)$ must necessarily be of the form

$$\tilde{g}(z) = \begin{pmatrix} \tilde{g}_(z_1, z_2) \\ \tilde{g}_2(z_1, z_2, z_3) \\ \cdots \\ \tilde{g}_{n-1}(z_1, z_2, \ldots, z_n) \\ \tilde{g}_n(z_1, z_2, \ldots, z_n) \end{pmatrix}$$

However, it is also possible to show that g_i actually is independent of z_{i+1}. We check this for $i = 1$ (the other checks are similar). Observe that

$$\begin{array}{rcl} \tilde{\varphi}_1(z,u) & = & z_1 \\ \tilde{\varphi}_2(z,u) & = & z_2 + \tilde{g}_1(z_1, z_2)u \end{array}$$

Computing Jacobians we obtain

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1(z,u)}{\partial z} \\ \dfrac{\partial \tilde{\varphi}_2(z,u)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \dfrac{\partial \tilde{g}_1}{\partial z_1}u & (1 + \dfrac{\partial \tilde{g}_1}{\partial z_2}u) & 0 & \cdots & 0 \end{pmatrix}.$$

If, at some point (z_1, z_2)

$$\frac{\partial \tilde{g}_1}{\partial z_2} \neq 0$$

it is possible to find u such that

$$1 + \frac{\partial \tilde{g}_1}{\partial z_2}u = 0$$

in which case

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1(z,u)}{\partial z} \\ \dfrac{\partial \tilde{\varphi}_2(z,u)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

At this value of (z,u) the subspace $K_2(z,u)$ has dimension $n-1$ and not $n-2$ as prescribed. Hence the uniformity conditions are violated. We conclude that

$$\frac{\partial \tilde{g}_1}{\partial z_2} = 0$$

for all (z_1, z_2), i.e. that $\tilde{g}_1$ is independent of z_2.

The conclusion is that the "uniform observability" canonical form of an input-affine system has the following structure

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 + \tilde{g}_1(z_1)u \\ \dot{z}_2 & = & z_3 + \tilde{g}_2(z_1, z_2)u \\ & & \cdots \\ \dot{z}_{n-1} & = & z_n + \tilde{g}_{n-1}(z_1, z_2, \ldots, z_{n-1})u \\ \dot{z}_n & = & \tilde{f}_n(z_1, \ldots, z_n) + \tilde{g}_n(z_1, z_2, \ldots, z_n)u \\ y & = & z_1 \end{array}$$

3 High-gain Nonlinear Observers

In this section, we describe how to design a *global* asymptotic state observer for a system in Gauthier-Kupca's observability canonical form. Letting $\mathbf{z}_i$ denote the vector

$$\mathbf{z}_i = \mathrm{col}(z_1, \ldots, z_i)$$

the canonical form in question can be rewritten in more concise form as

$$\begin{array}{rcl}
\dot{z}_1 &=& f_1(\mathbf{z}_1, z_2, u) \\
\dot{z}_2 &=& f_2(\mathbf{z}_2, z_3, u) \\
&& \cdots \\
\dot{z}_{n-1} &=& f_{n-1}(\mathbf{z}_{n-1}, z_n, u) \\
\dot{z}_n &=& f_n(\mathbf{z}_n, u) \\
y &=& h(z_1, u)\,.
\end{array} \tag{6}$$

The construction described below reposes on the follwing two additional technical assumptions:

(i) each of the maps $f_i(\mathbf{z}_i, z_{i+1}, u)$, for $i = 1, \ldots, n$, is globally Lipschitz with respect to $\mathbf{z}_i$, uniformly in z_{i+1} and u,

(ii) there exist two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \le \left|\frac{\partial \tilde{h}}{\partial z_1}\right| \le \beta\,, \quad \text{and} \quad \alpha \le \left|\frac{\partial \tilde{f}_i}{\partial z_{i+1}}\right| \le \beta\,, \quad \text{for all} \quad i = 1, \ldots, n-1$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$.

Remark. Note that the assumption in question is automatically satisfied if it is known – a priori – that $z(t)$ remains in a compact set Z. $\triangleleft$

The observer for (6) consist in a *copy* of the dynamics of (6) corrected by an *innovation* term proportional to the difference between the output of (6) and the output of the copy. More precisely, the observer in question is a system of the form

$$\begin{array}{rcl}
\dot{\hat{z}}_1 &=& f_1(\hat{\mathbf{z}}_1, \hat{z}_2, u) + \kappa c_{n-1}(y - h(\hat{z}_1, u)) \\
\dot{\hat{z}}_2 &=& f_2(\hat{\mathbf{z}}_2, \hat{z}_3, u) + \kappa^2 c_{n-2}(y - h(\hat{z}_1, u)) \\
&& \cdots \\
\dot{\hat{z}}_{n-1} &=& f_{n-1}(\hat{\mathbf{z}}_{n-1}, \hat{z}_n, u) + \kappa^{n-1} c_1(y - h(\hat{z}_1, u)) \\
\dot{\hat{z}}_n &=& f_n(\hat{\mathbf{z}}_n, u) + \kappa^n c_0(y - h(\hat{z}_1, u))\,,
\end{array} \tag{7}$$

in which κ and $c_{n-1}, c_{n-2}, \ldots, c_0$ are design parameters.

Letting the "observation error" be defined as

$$e_i = \hat{z}_i - z_i, \qquad i = 1, 2, \ldots, n\,,$$

an estimate of this error can be obtained as follows. Observe that, using the mean value theorem, one can write

$$\begin{aligned} f_i(\hat{\mathbf{z}}_i, \hat{z}_{i+1}, u) - f_i(\mathbf{z}_i, z_{i+1}, u) &= f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) \\ &\quad + f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), z_{i+1}(t), u(t)) \\ &= f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) \\ &\quad + \frac{\partial f_i}{\partial z_{i+1}}(\mathbf{z}_i(t), \delta_i(t), u(t))\, e_{i+1}(t) \end{aligned}$$

in which $\delta_i(t)$ is a number in the interval $[\hat{z}_{i+1}(t), z_{i+1}(t)]$. Note also that

$$h(\hat{z}_1, u) - y = h(\hat{z}_1(t), u(t)) - h(z_1(t), u(t)) = \frac{\partial h}{\partial z_1}(\delta_0(t), u(t))\, e_1$$

in which $\delta_0(t)$ is a number in the interval $[\hat{z}_1(t), z_1(t)]$. Setting

$$g_{i+1}(t) = \frac{\partial f_i}{\partial z_{i+1}}(\mathbf{z}_i(t), \delta_i(t), u(t))$$

and

$$g_1(t) = \frac{\partial h}{\partial z_1}(\delta_0(t), u(t))$$

the relation above yields

$$\dot{e}_i = f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) + g_{i+1}(t) e_{i+1} - \kappa^i c_{n-i} g_1(t) e_1\,.$$

The equations thus found can be organized in matrix form as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} -\kappa c_{n-1} g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -\kappa^2 c_{n-2} g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -\kappa^{n-1} c_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -\kappa^n c_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_{n-1} \\ e_n \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ \cdots \\ F_{n-1} \\ F_n \end{pmatrix} \tag{8}$$

in which

$$F_i = f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t))$$

for $i = 1, 2, \ldots, n-1$ and

$$F_n = f_n(\hat{\mathbf{z}}_n(t), u(t)) - f_i(\mathbf{z}_i(t), u(t))\,.$$

Consider now a "rescaled" observation error defined as

$$\tilde{e}_i = \frac{1}{\kappa^i}\, e_i, \qquad i = 1, 2, \ldots, n.$$

Simple calculations show that

$$\begin{pmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \\ \cdots \\ \dot{\tilde{e}}_{n-1} \\ \dot{\tilde{e}}_n \end{pmatrix} = \kappa \begin{pmatrix} -c_{n-1} g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -c_{n-2} g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -c_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \cdots \\ \tilde{e}_{n-1} \\ \tilde{e}_n \end{pmatrix} + \begin{pmatrix} \kappa^{-1} F_1 \\ \kappa^{-2} F_2 \\ \cdots \\ \kappa^{-n+1} F_{n-1} \\ \kappa^{-n} F_n \end{pmatrix}. \tag{9}$$

The right-hand side of this equation consists of a term which is linear in the vector

$$\tilde{e} = \mathrm{col}(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n)$$

and of a nonlinear term. The nonlinear term, though, can be bounded by a quantity which is linear in $\|\tilde{e}\|$. In fact, observe that, because of Assumption (i), there is a number L such that

$$\|F_i\| \le L\|\hat{\mathbf{z}}_i(t) - \mathbf{z}_i(t)\| .$$

Clearly,

$$\|\hat{\mathbf{z}}_i - \mathbf{z}_i\| = \sqrt{e_1^2 + e_2^2 + \ldots + e_i^2} = \sqrt{\kappa\tilde{e}_1^2 + \kappa^2\tilde{e}_2^2 + \ldots + \kappa^i\tilde{e}_i^2}$$

from which we see that, if $\kappa > 1$,

$$\|\kappa^{-i}F_i\| \le L\sqrt{\tilde{e}_1^2 + \tilde{e}_2^2 + \ldots + \tilde{e}_i^2} \le L\,\|\tilde{e}\| . \tag{10}$$

As far as the properties of the linear part are concerned, the following useful Lemma can be invoked (see [**?**], Chapter 6, Lemma 2.1).

Lemma 1 *Consider a matrix of the form*

$$A(t) = \begin{pmatrix} 0 & g_2(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_n(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} c_{n-1} \\ c_{n-2} \\ \\ \cdots \\ c_1 \\ c_0 \end{pmatrix} \begin{pmatrix} g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and suppose there exists two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \le g_i(t) \le \beta \qquad \text{for all } t \ge 0 \text{ and } i = 1, 2, \ldots, n . \tag{11}$$

Then, there is a set of real numbers $c_0, c_1, \ldots, c_{n-1}$, a real number $\lambda > 0$ and a symmetric positive definite $n \times n$ matrix S, with λ and S only depending on α and β, such that

$$A(t)S + SA^{\mathrm{T}}(t) \le -\lambda I . \tag{12}$$

Note that the hypothesis that the $g_i(t)$'s satisfy (11) is in the present case fulfilled as a straightforward consequence of Assumption (ii). With this result in mind consider, for system (9), a candidate Lyapunov function

$$V(\tilde{e}) = \tilde{e}^{\mathrm{T}}S\tilde{e} .$$

Then, we have

$$\dot{V}(\tilde{e}(t)) = \kappa\tilde{e}^{\mathrm{T}}(t)[A(t)S + SA^{\mathrm{T}}(t)]\tilde{e}(t) + 2\tilde{e}^{\mathrm{T}}(t)S\tilde{F}(t)$$

having denoted by $\tilde{F}(t)$ the vector

$$\tilde{F}(t) = \mathrm{col}(\kappa^{-1}F_1, \kappa^{-2}F_2, \ldots, \kappa^{-n}F_n) .$$

Using the estimates (10) and (12), it is seen that

$$\dot{V}(\tilde{e}(t)) \leq -\kappa\lambda\|\tilde{e}\|^2 + 2\|S\|\, L\sqrt{n}\,\|\tilde{e}\|^2\,.$$

Set

$$\kappa^* = \frac{2\|S\|\, L\sqrt{n}}{\lambda}\,.$$

Then, if $\kappa > \kappa^*$, the estimate

$$\dot{V}(\tilde{e}(t)) \leq -cV(\tilde{e}(t))$$

holds, for some $c > 0$. As a consequence, by standard results, it follows that

$$\lim_{t\to\infty} \tilde{e}(t) = 0\,.$$

In summary, the observer (7) asymptotically tracks the state of system (6) if the coefficients $c_0, c_1, \ldots, c_{n-1}$ are such that the property indicated in Lemma 1 holds (which is always possible by virtue of Assumption (ii)) and if the number κ is sufficiently large. This is why the observer in question is called a "high-gain" observer.

4 The Gains of the Nonlinear Observer

In this section, we prove Lemma 1. The result is obviously true in case $n = 1$. In this case, in fact, the matrix $A(t)$ reduces to the scalar quantity

$$A(t) = -c_0 g_1(t)\,.$$

Taking $S = 1$ we need to prove that

$$-2c_0 g_1(t) \leq -\lambda$$

for some $\lambda > 0$, which is indeed possible if the sign[1] of c_0 is the same as that of $g_1(t)$.

For $n > 2$ we proceed by induction. The induction hypothesis is that, having defined $A_0(t)$ as

$$A_0(t) = \begin{pmatrix} 0 & g_3(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_4(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_n(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} d_{n-2} \\ d_{n-3} \\ \\ \cdots \\ d_1 \\ d_0 \end{pmatrix} \begin{pmatrix} g_2(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

there is a choice of $d_{n-2}, \ldots, d_0$ and a matrix S_0, which depend only on α and β, such that

$$S_0 A_0(t) + A_0^{\mathrm{T}}(t) S_0 \leq -\lambda_0 I$$

for some $\lambda_0 > 0$.

[1]The $g_i(t)$'s are continuous functions that never vanish. Thus, each of them has a well-defined sign.

This being the case, set

$$F_0(t) = \begin{pmatrix} 0 & g_3(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_4(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_n(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad C_0 = -\begin{pmatrix} d_{n-2} \\ d_{n-3} \\ \\ \cdots \\ d_1 \\ d_0 \end{pmatrix}$$

$$K = \begin{pmatrix} c_{n-2} \\ c_{n-3} \\ \\ \cdots \\ c_1 \\ c_0 \end{pmatrix} \qquad H_0(t) = (\, g_2(t) \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \,)$$

and note that

$$A_0(t) = F_0(t) + C_0 H_0(t)$$

and

$$A(t) = \begin{pmatrix} -c_{n-1} g_1(t) & H_0(t) \\ -K g_1(t) & F_0(t) \end{pmatrix}$$

Change variables as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_0 & I_{n-1} \end{pmatrix} z := Tz\,,$$

to obtain

$$\tilde{A}(t) = TA(t)T^{-1} = \begin{pmatrix} -c_{n-1} g_1(t) - H_0(t) C_0 & H_0(t) \\ -(K + C_0 c_{n-1}) g_1(t) - A_0(t) C_0 & A_0(t) \end{pmatrix}$$

In these new coordinates, we look for a matrix $\tilde{S}$ of the form

$$\tilde{S} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & S_0 \end{pmatrix}$$

and we prove that the quadratic form

$$Q(x) = x^{\mathrm{T}} (\tilde{S} \tilde{A}(t) + \tilde{A}(t)^{\mathrm{T}} \tilde{S}) x$$

satisfies the required inequality. Simple calculations show that

$$\begin{aligned} Q(x) \quad = \quad & (-c_{n-1} g_1(t) - H_0(t) C_0) x_1^2 + H_0(t) x_1 x_2 \\ & + 2 x_2^{\mathrm{T}} S_0 [-(K + C_0 c_{n-1}) g_1(t) - A_0(t) C_0] x_1 + 2 x_2^{\mathrm{T}} S_0 A_0(t) x_2 \end{aligned}$$

Choose

$$K = -c_{n-1} C_0$$

and observe that (recall that $\|H_0(t)\| = |g_2(t)|$)

$$\begin{array}{rcl} Q(x) & \leq & (-c_{n-1}g_1(t) - H_0(t)C_0)x_1^2 \\ & & \quad + (|g_2(t)| + 2\|S_0\|\,\|A_0(t)C_0\|)|x_1|\,\|x_2\| - \lambda_0\|x_2\|^2 \\ & := & (-c_{n-1}g_1(t) - H_0(t)C_0)x_1^2 + \delta(t)|x_1|\,\|x_2\| - \lambda_0\|x_2\|^2\,, \end{array}$$

in which we have set

$$\delta(t) = |g_2(t)| + 2\|S_0\|\,\|A_0(t)C_0\|\,.$$

For any $\epsilon > 0$,

$$|x_1|\,\|x_2\| \leq \frac{\epsilon}{2}\|x_2\|^2 + \frac{1}{2\epsilon}|x_1|^2$$

and this yields

$$Q(x) \leq (-c_{n-1}g_1(t) - H_0(t)C_0 + \frac{\delta(t)}{2\epsilon})|x_1|^2 + (-\lambda_0 + \delta(t)\frac{\epsilon}{2})\|x_2\|^2\,.$$

The function $\delta(t)$ is bounded from above, by a number that only depends on α and β (in fact, $|g_2(t)|$ and the entries of $A_0(t)$ are bounded by β, while S_0 and C_0, determined at the earlier stage of the algorithm, only depend on α and β). There exists a (sufficiently small) number ϵ, that only depends on α and β, such that

$$(-\lambda_0 + \delta(t)\frac{\epsilon}{2}) < -\frac{\lambda_0}{2}.$$

Since $g_1(t)$ is bounded from below (and has a fixed sign) and $|H_0(t)C_0|$ is bounded from above (by a number that only depends on α and β), there exists a (sufficiently large, in magnitude) number c_{n-1} (having the same sign as $g_1(t)$) such that

$$(-c_{n-1}g_1(t) - H_0(t)C_0 + \frac{\delta(t)}{2\epsilon}) < -\frac{\lambda_0}{2}.$$

In this way, we obtain

$$Q(x) \leq -\frac{\lambda_0}{2}\|x\|^2\,.$$

Reverting to the original coordinates proves the Lemma. In fact, it suffices to set

$$S = T^{\mathrm{T}}\tilde{S}T$$

to obtain

$$z^{\mathrm{T}}[A^{\mathrm{T}}(t)S + SA(t)]z = x^{\mathrm{T}}[\tilde{A}^{\mathrm{T}}(t)\tilde{S} + \tilde{S}\tilde{A}(t)]x \leq -\frac{\lambda_0}{2}\|x\|^2 = -\frac{\lambda_0}{2}\|Tz\|^2 \leq -\frac{\lambda_0}{2}\gamma\|z\|^2 := -\lambda\|z\|^2$$

in which $\gamma = \lambda_{min}(T^{\mathrm{T}}T)$.

5 A Nonlinear Separation Principle

In this section, we show how the theory of nonlinear observers discussed earlier to the purpose of achieving asymptotic stability via dynamic output feedback. Consider a nonlinear system in observability canonical form (2), which we rewrite in compact form as

$$\begin{array}{rcl} \dot{z} & = & f(z,u) \\ y & = & h(z,u)\,, \end{array} \tag{13}$$

with $f(0,0) = 0$ and $h(0,0) = 0$ and suppose there exists a feedback law $u = \alpha(z)$, with $\alpha(0) = 0$, such that the equilibrium $z = 0$ of

$$\dot{z} = f(z, \alpha(z))$$

is globally asymptotically stable. For convenience, let the latter system be rewritten as

$$\dot{z} = F(z)\,. \tag{14}$$

Note that, by the inverse Lyapunov theorem, there exists a smooth function $W(z)$, satisfying

$$\underline{\alpha}(|z|) \le W(z) \le \overline{\alpha}(|z|) \qquad \text{for all } z, \tag{15}$$

and

$$\frac{\partial W}{\partial z} F(z) \le -\alpha(|z|) \qquad \text{for all } z, \tag{16}$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$.

Assume, for the time being, that the hypotheses (i) and (ii) of Section 3 hold (we shall see later how these can be removed) and consider an observer of the form (7), which we rewrite in compact form as

$$\dot{\hat{z}} = f(\hat{z}, u) + G(y - h(\hat{z}, u))\,. \tag{17}$$

An obvious choice to achieve asymptotic stability, suggested by the analogy with linear systems, would be to replace z by its estimate $\hat{z}$ in the map $\alpha(z)$. However, this simple choice may prove to be dangerous, for the following reason. Bearing in mind the analysis carried out in Section 3, define

$$D_\kappa = \mathrm{diag}(\kappa, \kappa^2, \ldots, \kappa^n)$$

and observe that

$$D_\kappa \tilde{e} = z - \hat{z}\,.$$

Thus, feeding the system (13) with a control $u = \alpha(\hat{z})$ would result in a system

$$\dot{z} = f(z, \alpha(z - D_\kappa \tilde{e}))\,.$$

which contains the possibly large parameter κ. This, since the system is nonlinear, may cause finite escape times. To avoid this, as a precautionary measure, it is appropriate to "saturate" the control, by choosing instead a law of the form

$$u = \sigma_L(\alpha(\hat{z}))) \tag{18}$$

in which $\sigma(r)$ is any function that coincides with r when $|r| \leq L$, is strictly increasing and satisfies $|\sigma(r) \leq 2L$ for all $r \in \mathbb{R}$. The consequence of this is that global asymptotic stability is no longer assured. However, as it it will be shown, *semiglobal stability* can still be obtained.

Consider now the aggregate of (13), of (17) and of (18), namely

$$\begin{aligned} \dot{z} &= f(z, \sigma_L(\alpha(\hat{z}))) \\ \dot{\hat{z}} &= f(\hat{z}, \sigma_L(\alpha(\hat{z}))) + G(h(z, \sigma_L(\alpha(\hat{z}))) - h(\hat{z}, \sigma_L(\alpha(\hat{z}))) \,. \end{aligned} \tag{19}$$

Replacing $\hat{z}$ by its expression in terms of $\tilde{e}$, we obtain for the first equation a system that can be written as

$$\begin{aligned} \dot{z} &= f(z, \sigma_L(\alpha(z - D_\kappa \tilde{e}))) \\ &= f(z, \alpha(z)) - f(z, \alpha(z)) + f(z, \sigma_L(\alpha(z - D_\kappa \tilde{e}))) \\ &= F(z) + H(z, \tilde{e}) \end{aligned}$$

in which

$$H(z, \tilde{e}) = f(z, \sigma_L(\alpha(z - D_\kappa \tilde{e}))) - f(z, \alpha(z)) \,.$$

In what follows, we shall show a *semiglobal* stabilizability property, namely the property that, for every compact set $\mathcal{A}$ of initial conditions in the state space, there is a choice of design parameters such that the equilibrium $(z, \hat{z}) = (0, 0)$ of the closed loop system is asymptotically stable, with a domain of attraction that contains $\mathcal{A}$. For simplicity let the set in question be of the form $B_R \times B_R$ in which B_R is a closed ball of radius R in $\mathbb{R}^n$.

The analysis proceeds as follows. Choose a number c such that

$$\Omega_c = \{z : W(z) \leq c\} \supset B_R \,,$$

and then choose the parameter L in the definition of $\sigma_L(\cdot)$ as

$$L = \max_{z \in \Omega_{c+1}} \alpha(z) + 1 \,.$$

With this choice, it is easy to realize that there is a number M_0 such that

$$|H(z, \tilde{e})| \leq M_0, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } \tilde{e} \in \mathbb{R}^n$$

and a pair of numbers M_1, δ such that

$$|H(z, \tilde{e})| \leq M_1 |D_\kappa \tilde{e}|, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } |D_\kappa \tilde{e}| \leq \delta \,.$$

These numbers are independent of κ and only depend on number R that characterizes the radius of the ball B_R in which $z(0)$ is taken.

Let $z(0) \in B_R \subset \Omega_c$. Regardless of what $\tilde{e}(t)$ is, so long as $z(t) \in \Omega_{c+1}$, we have

$$\dot{W}(z(t)) = \frac{\partial W}{\partial z}[F(z) + H(z, \tilde{e})] \leq -\alpha(|z|) + |\frac{\partial W}{\partial z}| M_0$$

Setting

$$M_2 = \max_{z \in \Omega_{c+1}} |\frac{\partial W}{\partial z}|$$

and

$$M = M_2 M_0$$

we obtain

$$\dot{W}(z(t)) \leq Mt$$

which in turn yields

$$W(t) - W(0) \leq Mt$$

and we deduce that $z(t)$ remains in Ω_{c+1} at least until time $T = 1/M$. This time may be very small but, because of the presence of the saturation function $\sigma_L(\cdot)$, it is independent of κ. It rather only depends on the number R that characterizes the radius of the ball B_R in which $z(0)$ is taken.

Recall now that the variable $\tilde{e}$ satisfies the estimate established in Section 3. Letting $V(\tilde{e})$ denote the quadratic form $V(\tilde{e}) = \tilde{e}^{\mathrm{T}} S \tilde{e}$, we know that

$$\dot{V}(\tilde{e}(t)) \leq -cV(\tilde{e})$$

in which

$$c := \kappa\lambda - 2\|S\| L\sqrt{n}$$

is a number that can be made arbitrarily large by increasing κ (recall that λ and $\|S\|$ only depend of the bounds α and β in assumption (ii) and L on the Lipschitz constant in assumption (i)). From this inequality, bearing in mind the fact that

$$a_1|\tilde{e}|^2 \leq V(\tilde{e}) \leq a_2|\tilde{e}|^2$$

in which a_1, a_2 are numbers depending on S and hence only on α, β, we obtain

$$|\tilde{e}(t)| \leq \sqrt{\frac{a_2}{a_1}}\, e^{-\frac{c}{2}t}\, |\tilde{e}(0)|$$

which is valid actually for all t, so long $z(t)$ exists.

To be able to use the estimates thus obtained, we need to bear in mind that, in the dynamics of z, $\tilde{e}$ is multiplied by D_κ. If, without loss of generality, $\kappa > 1$, we have

$$|D_\kappa \tilde{e}| \leq \kappa^n |\tilde{e}|, \qquad |\tilde{e}(0)| \leq |e(0)| = |z(0) - \hat{z}(0)| \leq 2R\,.$$

Hence we see that

$$|D_\kappa \tilde{e}(t)| \leq \kappa^n \sqrt{\frac{a_2}{a_1}}\, e^{-\frac{c}{2}t}\, 2R\,.$$

For any fixed time, the right-hand side is a polynomial function of κ multiplied by an exponentially decaying function of κ. Thus, bearing in mind the definitions given before of of time T, we see that there for any choice of ε, there is a κ^* such that, for all $\kappa \geq \kappa^*$,

$$|D_\kappa \tilde{e}(t)| \leq \varepsilon\,, \qquad \text{for all } t \geq T.$$

Consider now again the inequality

$$\dot{W}(z(t)) = \frac{\partial W}{\partial z}[F(z) + H(z, \tilde{e})] \leq -\alpha(|z|) + |\frac{\partial W}{\partial z}||H(z, \tilde{e})|\,.$$

We know from the argument above that, if κ is large enough, we can make $|D_\kappa \tilde{e}(t)| \leq \delta$ for all $t \geq T$ and hence, so long as $z(t) \in \Omega_{c+1}$, we have

$$\dot{W}(z(t)) \leq -\alpha(|z(t)|) + M_2 M_1 \delta \,.$$

Pick now any number $d \ll c$ and consider the "annular" compact set

$$S_d^{c+1} = \{z : d \leq W(z) \leq c+1\} \,.$$

Let r be

$$r = \min_{z \in S_d^{c+1}} |z|$$

By construction

$$\alpha(|z|) \geq \alpha(r) \qquad \text{for all } z \in S_d^{c+1}.$$

If δ is small enough

$$M_2 M_1 \delta \leq \frac{1}{2}\alpha(r) \,,$$

and hence

$$\dot{W}(z(t)) \leq -\frac{1}{2}\alpha(r) \,,$$

so long as $z(t) \in S_d^{c+1}$. By standard arguments, this proves that any trajectory which starts in B_R in finite time (which only depends on R) enters the set Ω_d and remain in this set thereafter.

The argument so far have shown that, given any arbitrary number $d \ll c$, there exist a number κ^* such that, if $\kappa \geq \kappa^*$, all trajectories starting in $B_R \times B_R$ are such $z(t)$ enters in finite time the set Ω_d and $\tilde{e}(t)$ decays to zero. To conclude the proof of convergence to the equilibrium, it suffices to argue as follows. We have shown that trajectries with initial conditions in $B_R \times B_R$ are bounded. Thus $\omega(B_R \times B_R)$ exists and, since $\tilde{e}(t)$ decays asymptotically to 0, the set in question is a subset of the set in which $\tilde{e} = 0$. The restriction of the entire system to the set in which $\tilde{e} = 0$ is nothing else than system

$$\dot{z} = F(z)$$

in which $z = 0$ is by assumption a globally asymptotically stable equilibrium. Hence, we conclude that

$$\omega(B_R \times B_R) = \{(0,0)\}$$

and this completes the proof.

It remains to discuss the role of the assumptions (i) and (ii). Having proven that the trajectories of the system starting in $B_R \times B_R$ remain in a bounded region, it suffices to look for numbers α and β and a Lipschitz constant L making assumptions (i) and (ii) valid *only on this bounded region* and these numbers indeed exist (as a simple consequence of the existence of the observability canonical form). Specifically, let c and L be fixed as before and consider a system

$$\begin{array}{rcl} \dot{z} & = & f_c(z,u) \\ y & = & h_c(z,u) \end{array} \tag{20}$$

with $f_c(z,u)$ and $h_c(z,u)$ satisfying

$$\begin{array}{rcll} f_c(z,u) & = & f(z,u), & \forall (z,u) : z \in \Omega_{c+1}, |u| \leq 2L \\ h_c(z,u) & = & h(z,u), & \forall (z,u) : z \in \Omega_{c+1}, |u| \leq 2L\,. \end{array}$$

Also, let $\alpha_c(z)$ be a function satisfying $\alpha_c(z) = \alpha(z)$ for all $z \in \Omega_{c+1}$ and such that the equilibrium $z = 0$ of

$$\dot{z} = f_c(z, \alpha_c(z))$$

is globally asymptotically stable.

Since $f_c(z,u)$ and $h_c(z,u)$ are arbitrary outside a compact set, assumptions (i) and (ii) can be fulfilled [for details, see J.P. Gauthier, I. Kupka: *Deterministic Observation Theory and Applications*, Cambridge University Press, 2001]. As a consequence, the previous result show how to find a controller for (20) steering all trajectories of the closed loop system

$$\begin{array}{rcl} \dot{z} & = & f_c(z, \sigma_L(\alpha_c(\hat{z}))) \\ \dot{\hat{z}} & = & f_c(\hat{z}, \sigma_L(\alpha_c(\hat{z}))) + G(h_c(z, \sigma_L(\alpha_c(\hat{z}))) - h_c(\hat{z}, \sigma_L(\alpha_c(\hat{z}))) \end{array} \tag{21}$$

with initial conditions in $B_R \times B_R$ to the equilibrium $(z, \hat{z}) = (0, 0)$. This controller generates an input always satisfying $|u| \leq 2L$ and induces in (20) a trajectory which always satisfies $z(t) \in \Omega_{c+1}$. Since (20) and the original plant agree on this set, the controller constructed in this way achieves the same result if used for the original plant.

6 Exercises

Exercise 1. Consider the system

$$\begin{array}{rcl} \dot{x}_1 & = & x_3 + (x_1 + x_1^3 + x_2)^3 u \\ \dot{x}_2 & = & x_1 - 3x_1^2 x_3 + (x_2 + x_1^3)u - 3x_1^2(x_1 + x_1^3 + x_2)^3 u \\ \dot{x}_3 & = & (x_2 + x_1^3)^2 \\ y & = & x_2 + x_1^3 \end{array}$$

We want to see whether or not it is transformable into a system in "uniform observability" canonical form. We compute the $\varphi_i(x,u)$ and obtain

$$\begin{array}{rcl} \varphi_1(x,u) & = & x_2 + x_1^3 \\ \varphi_2(x,u) & = & x_1 + (x_2 + x_1^3)u \\ \varphi_3(x,u) & = & x_3 + (x_1 + x_1^3 + x_2)^3 u + (x_1 - 3x_1^2 x_3 + (x_2 + x_1^3)u - 3x_1^2(x_1 + x_1^3 + x_2)^3 u)u \\ & & \quad + 3x_1^2(x_3 + (x_1 + x_1^3 + x_2)^3 u)u \end{array}$$

Computing the Jacobians, we obtain

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \\ \dfrac{\partial \varphi_3(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 3x_1^2 & 1 & 0 \\ 1 + 3x_1^2 u & u & 0 \\ * & * & 1 \end{pmatrix}$$

from which we see that

$$K_1(x,u) = \text{span}\begin{pmatrix} 1 & 0 \\ -3x_1^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad K_2(x,u) = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad K_3(x,u) = \{0\}\,.$$

Hence, the "uniformity conditions" hold:

- $K_1(x,u)$ has constant dimension 2 and is independent of u.
- $K_2(x,u)$ has constant dimension 1 and is independent of u.
- $K_3(x,u)$ has constant dimension 0 and (trivially) is independent of u.

Moreover

$$z = \Phi_n(x,0) = \begin{pmatrix} x_2 + x_1^3 \\ x_1 \\ x_3 \end{pmatrix}$$

is a global diffemorphism. Its inverse is

$$x = \begin{pmatrix} z_2 \\ z_1 - z_2^3 \\ z_3 \end{pmatrix}.$$

In the new coordinates, the system reads as

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 + z_1 u \\ \dot{z}_2 & = & z_3 + (z_2 + z_1)^3 u \\ \dot{z}_3 & = & z_1^2 \\ y & = & z_1\,. \end{array}$$

Exercise 2. Consider the system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + x_1 x_2 u \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & (x_2 + x_1^3)^2 + u \\ y & = & x_1 \end{array}$$

This system is not transformable into a system in "uniform observability" canonical form. In fact, we have

$$\begin{array}{rcl} \varphi_1(x,u) & = & x_1 \\ \varphi_2(x,u) & = & x_2 + x_1 x_2 u \end{array}$$

and

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_2 u & 1 + x_1 u & 0 \end{pmatrix}.$$

Hence

$$K_1(x,u) = \text{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$K_2(x,u) = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } 1 + x_1 u \neq 0$$

$$K_2(x,u) = \text{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } 1 + x_1 u = 0$$

and the uniformity conditions are not fulfilled. The transformation is not possible.

Exercise 3. Consider the system

$$\begin{aligned} \dot{z} &= z - z_3 + \xi_1 \\ \dot{\xi}_1 &= \xi_1 + \xi_2 \\ \dot{\xi}_2 &= z + \xi_1^2 + u \\ y &= \xi_1 \end{aligned}$$

This system is a relative degree 2 system in normal form, but its zero dynamics

$$\dot{z} = z - z_3$$

is unstable. Hence, stabilization methods based on high gain feedback from ξ_1, ξ_2 (or estimates of $y, y^{(1)}$) cannot be used.

However, the system is (semiglobally) stabilizable by dynamic output feedback. In fact, the system is stabilizable by *state feedback*, and is transformable into "uniform observability" canonical form. Thus, it can be semiglobally stabilized by dynamic output feedback via the separation principle.

Construction of a stabilizing feedback $u(z, \xi_1, \xi_2)$. The z dynamics can be stabilized by the virtual control $\xi_1^*(z) = -2z$. Then, back-step. Change ξ_1 into $\eta_1 = \xi_1 - \xi_1^*(z) = \xi_1 - 2z$, to obtain

$$\begin{aligned} \dot{z} &= -z - z_3 + \eta_1 \\ \dot{\eta}_1 &= 4z + 2z^3 - \eta_1 + \xi_2 \\ \dot{\xi}_2 &= z + (\eta_1 + 2z)^2 + u \end{aligned}$$

Change again ξ_2 into $\eta_2 = 4z + 2z^3 - \eta_1 + \xi_2$ to obtain a system of the form

$$\begin{aligned} \dot{z} &= -z - z_3 + \eta_1 \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= a(z, \eta_1, \eta_2) + u \end{aligned}$$

Since we are interested only in semiglobal stability, we can obtain this results by means of a simple feedback law of the form

$$u = -\gamma(\eta_2 + k a_0 \eta_1)$$

and k, γ positive and sufficiently large (note that, in the original coordinates, this feedback depends on *all* the state variables z, ξ_1, ξ_2). If we were interested in global stability, we would proceed differently (by standard back-stepping).

Construction of the observer. Note that

$$\begin{array}{rcl} \varphi_1(x,u) & = & \xi_1 \\ \varphi_2(x,u) & = & \xi_1 + \xi_2 \\ \varphi_3(x,u) & = & \xi_1 + \xi_2 + z + \xi_1^2 + u \end{array}$$

and therefore

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \\ \dfrac{\partial \varphi_3(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1+2\xi_1 & 1 \end{pmatrix}$$

Thus, the uniformity conditions are fulfilled and the change of coordinates is

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \Phi_n(x,0) = \begin{pmatrix} \xi_1 \\ \xi_1 + \xi_2 \\ \xi_1 + \xi_1^2 + \xi_2 + z \end{pmatrix}$$

whose inverse is given by

$$\begin{pmatrix} z \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} z_3 - z_1 - z_1^2 - z_2 \\ z_1 \\ z_2 - z_1 \end{pmatrix}.$$

Compute now the canonical form and construct the observer.