Optimal Control

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Lecture

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Grading
Project + oral exam

Example of project:
Read a paper on an optimal control problem

1) Study: background, motivations, model, optimal control, solution, results
2) **Simulations**

You must give me, **al least ten days before the date of the exam**:
- A .doc document
- A power point presentation
- Matlab simulation files

Example of project:

-Read a paper on an optimal control problem (any field will do)

You can choose a topic proposed in the papers I sent you, or from the examples proposed in the books suggested (i.e. Evans) , or some specific problem of your interest.

1) Study: background, motivations, model, optimal control, solution, results

You have to indicate clearly:

-the **background** of the problem you are studying by a literature research, motivations (**why the problem is important?**),

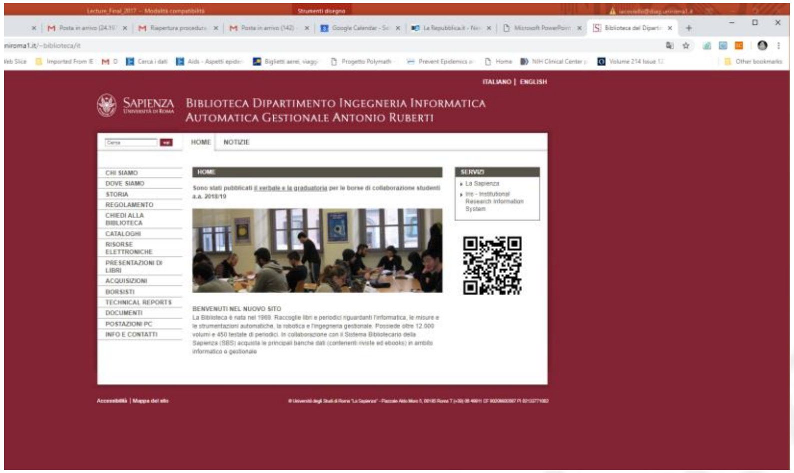
-the **model assumed** (indicate variables, parameters, their meaning, etc),

-the **cost index** and the **optimal control introduced** (the meaning, what they represent, different possible choices, etc).

2) Simulations: You have to implement the model and the control by matlab

Describe the program, propose the results and discuss them.

YOU MUST STUDY ON THE BOOKS



THE SLIDES ARE NOT SUFFICIENT

Schedule

Lecture 1:

Introduction to optimal control: motivations, examples.

Definitions

Unconstrained optimization (first order necessary conditions, second order necessary conditions)

Weierstrass theorem

Constrained optimization, introduction of Lagrangian function

First order necessary conditions, Second order sufficient conditions.

Function spaces: First order necessary condition, second order necessary condition, necessary and sufficient condition

Definitions

Consider a function $f : R^n :\rightarrow R$

And $D \subseteq R^n$

$\left| \bullet \right|$ denotes the Eucledian norm

A point $x^* \in D$ is a **local minimum** of f over $D \subseteq R^n$

If

$$\exists \;\; \varepsilon > 0 \;\; such \; that \;\; for \; all \;\; x \in D \; satisfying \; \left| x - x^* \right| < \varepsilon$$

$$\Rightarrow$$

$$f(x^*) \leq f(x)$$

Definitions

Consider a function $f : R^n :\to R$

And $D \subseteq R^n$

$|\bullet|$ denotes the Eucledian norm

A point $x^* \in D$ is a **<u>strict</u> local minimum** of f over $D \subseteq R^n$

If

$$\exists \ \varepsilon > 0 \ \ such \ that \ \ for \ all \ \ x \in D \ satisfying \ |x - x^*| < \varepsilon$$

$$\Rightarrow$$

$$f(x^*) < f(x), \ \ \forall x \neq x^*$$

Definitions

Consider a function $f : R^n :\to R$

And $D \subseteq R^n$

$|\bullet|$ denotes the Eucledian norm

A point $x^* \in D$ is a **global minimum** of f over $D \subseteq R^n$

If

$$\exists \ \varepsilon > 0 \ \ such \ that \ \ for \ all \ \ x \in D$$

$$\Rightarrow$$

$$f(x^*) \leq f(x)$$

Unconstrained optimization - first order necessary conditions

All points x sufficiently near x^* in R^n are in D

Assume $f \in C^1$ and x^* its local minimum. Let $d \in R^n$ an arbitrary vector.

Being in the unconstrained case:

$$x^* + \alpha d \in D \quad \forall \alpha \in R \; close \; enough \; to \; 0$$

Let's consider:

$$g(\alpha) := f(x^* + \alpha d)$$

⇒ 0 is a minimum of g

$\nabla f(x^*) = 0$ **First order necessary condition for optimality**

Unconstrained optimization- second order conditions

Assume $f \in C^2$ and x^* its local minimum. Let $d \in R^n$ an arbitrary vector.

Second order Taylor expansion of g around $\alpha = 0$

$$g(\alpha) = g(0) + g'(0)\alpha + \frac{1}{2} g''(0) + o(\alpha^2), \qquad \lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0$$

Since $g'(0) = 0$

$$\Rightarrow \boxed{g''(0) \geq 0}$$

$\nabla^2 f(x^*) \geq 0$ **Second order necessary condition for optimality**

Unconstrained optimization- second order conditions

Let $f \in C^2$ and $\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) > 0$

$\Rightarrow$ x^* is a **strict local minimum** of *f*

Definitions- Remarks

A vector $d \in R^n$ is a **feasible direction** at x^* if

$$x^* + \alpha d \in D \ \ for \ \ small \ \ enough \ \alpha > 0$$

If D in not the entire R^n then D is the **constraint set** over which f is being minimized

Global minimum

Weierstrass Theorem

Let f be a **continuous function** and D a **compact set**

there exist a **global minimum** of f over D

Constrained optimization

Let $D \subset R^n, \quad f \in C^1$

Equality constraints $h(x) = 0, \; h : R^p \to R, \; h \in C^1$

Inequality constraints $g(x) \le 0, \; g : R^q \to R, \; g \in C^1$

Regularity condition:

$$rank\left\{ \left. \frac{\partial(h, g_a)}{\partial x} \right|_{x^*} \right\} = p + q_a$$

$where \;\; g_a \;\; are\ the\ active\ constrain\ of$

g with dimension q_a

Lagrangian function $L(x, \lambda_0, \lambda, \mu) = \lambda_0 f(x) + \lambda^T h(x) + \eta^T g(x)$

If $\lambda_0 \neq 0$ the stationary point x^* is called **normal** and we can assume $\lambda_0 = 1$.

Constrained optimization

From now on $\lambda_0 = 1$ and therefore the Lagrangian is

$$L(x,\lambda,\mu) = f(x) + \lambda^T h(x) + \eta^T g(x)$$

If there are only equality constraints the λ_i are called **Lagrange multipliers**

If there are both equality and inequality constrains we have **Kuhn – Tucker multipliers**

Constrained optimization

First order <u>necessary conditions</u> for constrained optimality:

Let $x^* \in D$ and $f, h, g \in C^1$

The **necessary conditions** for x^* to be a constrained local minimum are

$$\boxed{\begin{aligned} &\left.\frac{\partial L}{\partial x}\right|_{x^*}^{T} = 0^T \\ &\eta_i g_i(x^*) = 0, \ \forall i \\ &\eta_i \geq 0 \quad \forall i \end{aligned}}$$

If the functions f and g are convex and the functions h are linear these conditions are necessary and sufficient!!!

Constrained optimization

Second order <u>sufficient conditions</u> for constrained optimality:

Let $x^* \in D$ and $f, h, g \in C^2$ and assume the conditions

$$\left.\frac{\partial L}{\partial x}\right|_{x^*}^T = 0^T \quad \eta_i g_i(x^*) = 0,\ \eta_i \geq 0 \quad \forall i$$

x^* is a strict constrained local minimum if

$$d^T \left.\frac{\partial^2 L}{\partial x^2}\right|_{x^*} d > 0 \quad \forall d \ \ such \ \ that \ \ \left.\frac{dh_i(x)}{dx}\right|_{x^*} \cdot d = 0,\ \ i = 1,\ldots,p$$

Function spaces

Functional $J : V \to R$

Vector space V, $A \subseteq V$

$z^* \in A$ is a local minimum of J over A if there exists an

$\varepsilon > 0 \; such \; that \;\; for \; all \;\; z \in A \;\; satisfying \; \left\| z - z^* \right\| < \varepsilon$

$\Rightarrow J(z^*) \leq J(z)$

Function spaces

Consider function in *V* of the form $z + \alpha\eta,\ \eta \in V,\ \alpha \in R$

The first variation of *J* at *z* is the linear function $\delta J\big|_z : V \to R$

such that $\forall \alpha \ \ and \ \ \forall \eta$

$$J(z+\alpha\eta) = J(z) + \delta J\big|_z(\eta)\alpha + o(\alpha)$$

First order necessary condition for optimality:

For all admissible perturbation we must have:

$$\delta J\big|_{z^*}(\eta) = 0$$

Function spaces

A quadratic form $\delta^2 J\big|_z : V \to R$ is the second variation of *J* at *z* if $\forall \alpha \;\; and \;\; \forall \eta$

we have:

$$J(z+\alpha\eta) = J(z) + \delta J\big|_z(\eta)\alpha + \delta^2 J\big|_z(\eta)\alpha^2 + o(\alpha^2)$$

second order necessary condition for optimality:

If $z^* \in A$ is a local minimum of J over $A \subset V$

for all admissible perturbation we must have:

$$\delta^2 J\big|_{z^*}(\eta) \geq 0$$

Function spaces

The Weierstrass Theorem is still valid

If J is a convex functional and $A \subset V$ is a convex set
A local minimum is automatically a global one and the first order condition are **necessary and sufficient condition** for a minimum

Schedule

Lecture 2:

Calculus of variations- Motivations

The Lagrange problem: Euler equation, Weierstrass Erdmann condition, transversality conditions

The Lagrange problem

Problem 1

Let us consider the linear space $\bar{C}^1(R)\times R\times R$ and define the **admissible set**:

$$D=\left\{(z,t_i,T)\in\bar{C}^1(R)\times R\times R:(z(t_i),t_i)\in D_i\subset R^{\nu+1},(z(T),T)\in D_f\subset R^{\nu+1}\right\}$$

Introduce the norm: $\|(z,t_i,T)\|=\sup_t\|z(t)\|+\sup_t\|\dot z(t)\|+|t_i|+|T|$

and consider the **cost index**:

$$\boxed{J(z,t_i,T)=\int_{t_i}^{T}L[z(t),\dot z(t),t]dt}$$

with L function of C^2 class.

Find the global minimum (optimum) for J over D, (z^o,t_i^o,T^o) :

$$J(z^o,t_i^o,T^o)\le J(z,t_i,T)\quad\forall(z,t_i,T)\in D$$

SCHEME of the theorems

Theorem (Lagrange). If $\left(z^*, t_i^*, t_f^*\right) \in D$ is a local minimum then

1) $$\left.\frac{\partial L}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in \left[t_i, t_f\right]$$ **Euler equation**

2) In any discontinuity point $\bar{t}$ of $\dot{z}^*$

Weierstrass-Erdmann condition

3) Transversality conditions $\left\{\right.$ different cases depending on the nature of the boundary conditions

Theorem (Lagrange). If $\left(z^*, t_i^*, T^*\right) \in D$ is a local minimum then

$$\boxed{\left.\frac{\partial L}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{z}}\right|^* = 0^T \qquad \forall t \in \left[t_i, T\right]}$$

Euler equation

In any discontinuity point $\bar{t}$ of $\dot{z}^*$ the following conditions are verified:

$$\left.\frac{\partial L}{\partial \dot{z}}\right|^*_{\bar{t}^-} = \left.\frac{\partial L}{\partial \dot{z}}\right|^*_{\bar{t}^+} \qquad \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}^-} = \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}^+}$$

Weierstrass-Erdmann condition

Moreover, **transversality conditions** are satisfied:

- If $D_i \; D_f$ are **open subset** we have:

$$\left.\frac{\partial L}{\partial \dot z}\right|_{t_i^*}^{*} = 0^T \quad \left.\frac{\partial L}{\partial \dot z}\right|_{T^*}^{*} = 0^T \quad L\Big|_{t_i^*}^{*} = 0 \quad L\Big|_{T^*}^{*} = 0$$

- If $D_i \; D_f$ **are closed subsets** defined respectively by

$$\gamma\left(z(t_i), t_i\right) = 0 \quad \aleph\left(z(t_f), t_f\right) = 0$$

such that
$$rg\left\{\left.\frac{\partial \gamma}{\partial\left(z(t_i), t_i\right)}\right|^{*}\right\} = \sigma_i \qquad rg\left\{\left.\frac{\partial \aleph}{\partial\left(z(T), T\right)}\right|^{*}\right\} = \sigma_f$$

for $\xi \in R^{\sigma_i} \qquad \varsigma \in R^{\sigma_f}$

$$\left.\frac{\partial L}{\partial \dot z}\right|_{t_i^*}^{*} = \xi^T \left.\frac{\partial \gamma}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial L}{\partial \dot z}\right|_{T^*}^{*} = \varsigma^T \left.\frac{\partial \aleph}{\partial z(T)}\right|^{*}$$

$$\left(L - \frac{\partial L}{\partial \dot z}\dot z\right)_{t_i^*} = \xi^T \left.\frac{\partial \gamma}{\partial t_i}\right|^{*}, \quad \left(L - \frac{\partial L}{\partial \dot z}\dot z\right)_{T^*} = \varsigma^T \left.\frac{\partial \aleph}{\partial T}\right|^{*}$$

- If the sets D_i *and* D_f are defined by the **function** $w(z(t_i), z(T))$ affine with respect to $z(t_i), z(T)$ such that

$$rg\left\{ \left. \frac{\partial w}{\partial \big(z(t_i), z(T)\big)} \right|^{*} \right\} = \sigma$$

$$\left. \frac{\partial L}{\partial \dot{z}} \right|_{t_i^*}^{*} = \vartheta^T \left. \frac{\partial w}{\partial z(t_i)} \right|^{*}, \quad \left. \frac{\partial L}{\partial \dot{z}} \right|_{T^*}^{*} = -\vartheta^T \left. \frac{\partial w}{\partial z(T)} \right|^{*}$$

$$\vartheta \in R^{\sigma}$$

The Lagrange problem

Problem 2

Consider Problem 1 with

- ✓ t_i *and* t_f **fixed**
- ✓ **If** D_i D_f are **closed sets** in R^{v+1} defined by the C[1] functions

$$\gamma\big(z(t_i),t_i\big)=0, \text{ of dimension } \sigma_{\mathrm{i}} \le v+1$$

$$\chi\big(z(T),T\big)=0, \text{ of dimension } \sigma_{\mathrm{f}} \le v+1$$

with γ *and* χ affine functions and

$$rg\left\{\left.\frac{\partial\gamma}{\partial\big(z(t_i)\big)}\right|^{o}\right\}=\sigma_i \qquad rg\left\{\left.\frac{\partial\chi}{\partial\big(z(t_f)\big)}\right|^{o}\right\}=\sigma_f$$

✓ If the sets D_i D_f are defined by the function $w(z(t_i), z(t_f))$ with σ components of C^1 class affine with respect to $z(t_i), z(t_f)$ such that

$$rg\left\{ \frac{\partial w}{\partial\left(z(t_i), z(t_f)\right)}\Bigg|^{o} \right\} = \sigma$$

✓ The function *L* must be ***convex*** with respect to $\left(z(t), \dot{z}(t)\right)$

Find the ***global minimum (optimum)*** z^o for *J* over *D:*

$$J\left(z^o\right) \le J\left(z\right) \quad \forall z \in D$$

Theorem 2. $z^o \in D$ is the optimum **if and only if**

$$\boxed{\left.\frac{\partial L}{\partial z}\right|^o - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{z}}\right|^o = 0^T \quad \forall t \in \left[t_i, t_f\right]}$$

Euler equation

In any discontinuity point $\bar{t}$ of $\dot{z}^*$ the following conditions are verified:

$$\left.\frac{\partial L}{\partial \dot{z}}\right|^o_{\bar{t}^-} = \left.\frac{\partial L}{\partial \dot{z}}\right|^o_{\bar{t}^+} \qquad \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)^o_{\bar{t}^-} = \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)^o_{\bar{t}^+}$$

Weierstrass-Erdmann condition

Moreover, **transversality conditions** are satisfied

…..

The Lagrange problem

Problem 1_bis

Let us consider the **admissible set**:

$$D=\left\{z\in\overline{C}^1[t_i,t_f]:\ z(t_i)=z_i,\ z(t_f)=z_f\right\}$$

consider the **cost index**:

$$J(z,t_i,T)=\int_{t_i}^{T}L\left[z(t),\dot{z}(t),t\right]dt$$

with *L* function of C^2 class.

Find the global minimum (optimum) for *J* over *D*

The Lagrange problem

Theorem (Legendre)

Necessary condition for $z^* \in D$ to be local minimum is that

$$\left.\frac{\partial^2 L}{\partial \dot{z}^2}\right|^* \geq 0 \quad \forall t \in [t_i, t_f]$$

The Lagrange problem

Problem 3

Let us consider the linear space $\overline{C}^1(R) \times R \times R$ and define the admissible set

$$D = \left\{ \left(z, t_i, T\right) \in \overline{C}^1(R) \times R \times R : \left(z(t_i), t_i\right) \in D_i \subset R^{\nu+1}, \right.$$

$$\left. \left(z(T), T\right) \in D_f \subset R^{\nu+1}, g\left(z(t), \dot{z}(t), t\right) = 0 \; \int_{t_i}^{T} h\left(z(t), \dot{z}(t), t\right) dt = k \right\}$$

g of dimension $\mu < \nu$

Consider the cost index:

$$J(z, t_i, T) = \int_{t_i}^{T} L\left[z(t), \dot{z}(t), t\right] dt$$

Define the **augmented lagrangian:**

$$\ell\left(z(t),\dot{z}(t),t,\lambda_0,\lambda(t),\rho\right)=\lambda_0 L\left(z(t),\dot{z}(t),t\right)$$
$$+\lambda^T(t)g\left(z(t),\dot{z}(t),t\right)+\rho^T(t)h\left(z(t),\dot{z}(t),t\right)$$

Theorem 3 (Lagrange). Let $\left(z^*, t_i^*, t_f^*\right) \in D$ be such that

$$rank\left\{ \left.\frac{\partial g}{\partial \dot{z}(t)}\right|^* \right\} = \mu \quad \forall t\left[t_i^*, t_f^*\right]$$

If $\left(z^*, t_i^*, t_f^*\right)$ is a local minimum for *J* over *D*, **then** there exist $\lambda_0^* \in R,\ \lambda^* \in \overline{C}^0[t_i, t_f^*],\ \rho^* \in R^\sigma$ **not simultaneously null** in $[t_i, t_f^*]$ such that:

- $$\left.\frac{\partial \ell}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in \left[t_i^*, t_f^*\right]$$

- $$\left.\frac{\partial \ell}{\partial \dot{z}}\right|^*_{\bar{t}_k^-} = \left.\frac{\partial \ell}{\partial \dot{z}}\right|^*_{\bar{t}_k^+} \qquad \left(\ell - \frac{\partial \ell}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}_k^-} = \left(\ell - \frac{\partial \ell}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}_k^+}$$

where $\bar{t}_k$ are cuspid points for z^*

with TRANSVERSALITY CONDITIONS

The Lagrange problem

Problem 4 Let us consider the linear space $\overline{C}^1(R) \times R \times R$ and define the admissible set

$$D = \left\{ z \in \overline{C}^1 \left[t_i, t_f \right], \; z(t_i) \in D_i, z(t_i) \in D_f, g\big(z(t), \dot{z}(t), t\big) = 0, \; \int_{t_i}^{t_f} h\big(z(t), \dot{z}(t), t\big) dt = k, \; \forall t \in [t_i, t_f] \right\}$$

- g of dimension $\mu < v$
- t_i t_f fixed
- g and h **affine** functions in $z(t), \dot{z}(t), \; \forall t \in [t_i, t_f]$
- L C^2 function convex with respect to $z(t), \dot{z}(t), \; \forall t \in [t_i, t_f]$

Consider the cost index: $J(z, t_i, t_f) = \int_{t_i}^{t_f} L\big[z(t), \dot{z}(t), t\big] dt$

Define the **augmented lagrangian:**

$$\boxed{\begin{aligned}&\ell\left(z(t),\dot z(t),t,\lambda_0,\lambda(t),\rho\right)=L\left(z(t),\dot z(t),t\right)\\&+\lambda^T(t)g\left(z(t),\dot z(t),t\right)+\rho^T(t)h\left(z(t),\dot z(t),t\right)\end{aligned}}$$

Theorem 4 (Lagrange). Let $z^o \in D$ such that

$$rank\left\{ \left. \frac{\partial g}{\partial \dot{z}(t)} \right|^o \right\} = \mu \quad \forall t \in \left[t_i , t_f \right]$$

$z^o \in D$ is an ***optimal normal solution***

if and only if

- $$\left. \frac{\partial \ell}{\partial z} \right|^* - \frac{d}{dt} \left. \frac{\partial \ell}{\partial \dot{z}} \right|^* = 0^T \quad \forall t \in \left[t_i , t_f \right]$$

- in the instants t_i and/or t_f for which D_i and/or D_f are open we have: $$\left. \frac{\partial \ell}{\partial \dot{z}} \right|^o = 0^T$$

Schedule

Lecture 3:

Calculus of variations and optimal control:

The Hamiltonian function- Necessary conditions- Necessary and sufficient conditions

Problem 1

Let us consider the **dynamical sistem** described by:

$$\dot{x} = f(x,u,t)$$

f function of C^2 class.

u(t) control vector in R^p

x(t) state vector in R^n

$$x(t_i) = x_i$$

known

$$\chi\left(x(t_f),t_f\right)=0$$

$$q(x,u,t)\leq 0$$

Vectorial function of C^1 class of dimension $\sigma_f \leq n+1$

Vectorial function of C^2 class of dimension β

Assume the norm:

$$\left\|\left(x,u,t_f\right)\right\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^{t} u(\tau)d\tau \right\| + \sup_t \|u(t)\| + \left|t_f\right|$$

Define the cost index $J(x,u,t_f) = \int_{t_i}^{t_f} L\left(x,u,t\right)dt$

with L function of C^2 class

AIM: Find (if it exists)

- the instant t_f^o
- the control $u^o \in \overline{C}^0(R)$
- the state $x^o \in \overline{C}^1(R)$

that satisfy the previous equations and minimize the cost index

DEFINE the scalar function

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 L(x,u,t) + \lambda^T(t) f(x,u,t)$$

the Hamiltonian function

Theorem 1

Let (x^*,u^*,t_f^*) be an admissible solution such that

dimension of

$$rk\left\{\left.\frac{\partial \chi}{\partial(x(t_f),t_f)}\right|^*\right\}=\sigma_f \qquad rk\left\{\left.\frac{\partial q_{active}}{\partial u}\right|^*\right\}=\beta_a(t),\ \ \forall t\in\left[t_i,t_f^*\right]$$

IF (x^*,u^*,t_f^*) is a local minimum

⇒

there exist $\lambda_0^*\geq 0,\ \ \lambda^*\in\overline{C}^1[t_i,t_f^*],\ \eta^*\in\overline{C}^0[t_i,t_f^*]$

<u>not simultaneously null i</u>n $[t_i,t_f^*]$ such that:

- $$\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T} - \left.\frac{\partial q}{\partial x}\right|^{*T} \eta^*$$

- $$0 = \left.\frac{\partial H}{\partial u}\right|^{*T} + \left.\frac{\partial q}{\partial u}\right|^{*T} \eta^*$$

- $$\eta_j^*(t) q_j(x^*, u^*, t) = 0, \quad \eta_j^*(t) \geq 0, \quad j = 1,2,\ldots,\beta$$

- $$\lambda^*(t_f^*) = -\left.\frac{\partial \chi}{\partial (x(t_f))}\right|_{t_f^*}^{*T} \varsigma, \ \varsigma \in R^{\sigma_f}$$

- $$\left.H\right|_{t_f^*}^{*} = \left.\frac{\partial \chi}{\partial t_f}\right|_{t_f^*}^{*T} \varsigma$$

The discontinuity of $\dot{\lambda}^*$ *and* η^* may occurr only in the points $\bar{t}$

where u^* has a discontinuity and $\left.H\right|_{\bar{t}^-}^{*} = \left.H\right|_{\bar{t}^+}^{*}$ ♦

Problem 3

Let us consider the **dynamical system** described by:

$$\dot{x} = f(x,u,t)$$

f function of C^2 class

u(t) control vector in R^p

x(t) state vector in R^n

$$x(t_i) = x_i$$

known

$$\chi\left(x(t_f),t_f\right)=0$$

$$\int_{t_i}^{t_f} h\left(x(t),u(t),t\right)dt = K$$

Vectorial function of C^1 class of dimension $\sigma_f \le n+1$

Vectorial function of C^2 class of dimension σ

Assume the norm:

$$\left\|\left(x,u,t_f\right)\right\| = \sup_t\|x(t)\| + \sup_t\|\dot{x}(t)\| + \sup_t\left\|\int_{t_i}^{t} u(\tau)d\tau\right\| + \sup_t\|u(t)\| + \left|t_f\right|$$

Define the cost index

$$J(x,u,t_f) = \int_{t_i}^{t_f} L\left(x,u,t\right)dt$$

with L function of C^2 class

AIM: Find

- the instant t_f^o
- the control $u^o \in \overline{C}^0(R)$
- the state $x^o \in \overline{C}^1(R)$

that satisfy the previous equations and minimize the cost index

DEFINE the scalar function

$$H(x,u,\lambda_0,\lambda,t)=\lambda_0 L(x,u,t)+\lambda^T(t)f(x,u,t)+\boxed{\rho^T h(x(t),u(t),t)}$$

the Hamiltonian function

Theorem 3

Let (x^*, u^*, t_f^*) be an admissible solution such that $rk\left\{ \left. \frac{\partial \chi}{\partial (x(t_f), t_f)} \right|^* \right\} = \sigma_f$

IF (x^*, u^*, t_f^*) is a local minimum

$\Rightarrow$ there exist $\lambda_0^* \in R,\ \lambda^* \in \bar{C}^1[t_i, t_f^*],\ \rho^* \in R$ not simultaneously null in $[t_i, t_f^*]$ such that:

- $$\dot{\lambda}^* = -\left. \frac{\partial H}{\partial x} \right|^{*T}$$
- $$0 = \left. \frac{\partial H}{\partial u} \right|^{*T}$$
- $$\lambda^*(t_f^*) = -\left. \frac{\partial \chi}{\partial (x(t_f))} \right|_{t_f^*}^{*T} \varsigma, \quad \varsigma \in R^{\sigma_f}$$
- $$H\big|_{t_f^*}^{*} = \left. \frac{\partial \chi}{\partial t_f} \right|_{t_f^*}^{*T} \varsigma$$

The discontinuity of $\dot{\lambda}^*$ may occurr only in the points $\bar{t}_k$ where u^* has a discontinuity

and

$$H\Big|_{\bar{t}_k^-}^{*} = H\Big|_{\bar{t}_k^+}^{*}$$

Problem 4:

Consider the linear system $\dot{x} = A(t)x + B(t)u$

Functions of C^2 class

Assume:

- $t_i \quad t_f \qquad x(t_i) = x^i$ fixed
- $x(t_f) \in D_f$ being D_f a fixed point or R^n
- $q(x,u,t) \le 0$ Vectorial function of C^2 class of dimension β **CONVEX**

Define the cost index

Functions of C^3 class- **CONVEX**

$$J(x,u) = \int_{t_i}^{t_f} L(x,u,t)dt + G(x(t_f))$$

Functions of C^2 class **CONVEX**

Theorem 4

Let (x^o, u^o) be an admissible solution such that

$$rk\left\{ \left.\frac{\partial q_{active}}{\partial u}\right|^{o} \right\} = \beta_a(t), \quad \forall t \in \left[t_i, t_f\right]$$

(x^o, u^o)

is a **normal** optimal solution

$\Leftrightarrow$

$$\dot{\lambda}^o = -\left.\frac{\partial H}{\partial x}\right|^{oT} - \left.\frac{\partial q}{\partial x}\right|^{oT} \eta^o$$

$$0 = \left.\frac{\partial H}{\partial u}\right|^{oT} + \left.\frac{\partial q}{\partial u}\right|^{oT} \eta^o$$

$$\eta_j^o(t) q_j(x^o, u^o, t) = 0,$$

$$j = 1,2,\ldots,\beta$$

$$\eta^o(t) \geq 0$$

AND IF $D_f = R^n$ $\quad \lambda^o(t_f) = \left.\frac{dG}{dx(t_f)}\right|^{oT}$

Schedule

Lecture 4:

Pontryagin minimum principle-

Necessary conditions- The convex case- Example

The Pontryagin principle

Problem 1: Consider the dynamical system: $\boxed{\dot{x} = f(x,u)}$

with:

$$x(t) \in R^n, \quad u(t) \in U \subset R^p \quad f, \frac{\partial f}{\partial x_i} \in C^0(R^n \times U), \; i = 1,2,\ldots,n$$

Assume fixed the initial control instant and the initial and final values :

$$x(t_i) = x^i \quad x(t_f) = x^f$$

Define the performance index :

$$J(x,u,t_f) = \int_{t_i}^{t_f} L(x(\tau),u(\tau))d\tau + G(x(t_f))$$

with

$$L, \frac{\partial L}{\partial x_i} \in C^0(R^n \times U), \; i = 1,2,\ldots,n, \;\; G \in C^2$$

Determine:

- the value $t_f \in (t_i, \infty)$,
- the control $u^o \in \overline{C}^0(R)$
- the state $x^o \in \overline{C}^1(R)$

that satisfy:

- ✓ the dynamical system,
- ✓ the constraint on the control,
- ✓ the initial and final conditions
- ✓ and minimize the cost index

Hamiltonian function

$$H(x,u,\lambda_0,\lambda)=\lambda_0 L(x,u)+\lambda^T(t)\,f(x,u)$$

The Pontryagin principle

Theorem 1 (necessary condition):

Assume the admissible solution $\left(x^*, u^*, t_f^*\right)$ is a minimum

→ there exist a constant $\lambda_0 \geq 0$

and a n-dimensional vector $\lambda^* \in \overline{C}^1\left[t_i, t_f^*\right]$

not simultaneously null such that :

$$\boxed{\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^{*T}}$$

$$\boxed{\left.H\right|^* = 0}$$

$$\boxed{\begin{aligned}&H\left(x^*(t), \omega, \lambda_0^*, \lambda^*(t)\right) \geq H\left(x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)\right),\\ &\forall \omega \in U\end{aligned}}$$

The Pontryagin principle

Problem 2

Theorem 2

Problem 3

Theorem 3

The Pontryagin principle

Problem 4: Consider the dynamical system: $\dot{x} = f(x,u,t)$

with: $x(t) \in R^n, \quad u(t) \in U \subset R^p \quad f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial t} \in C^0\left(R^n \times U \times R\right)$

and initial instant and state fixed $x(t_i) = x^i$

For the final values assume: $\boxed{\aleph\left(x(t_f), t_f\right) = 0}$ where $\aleph$ is a function of dimension $\sigma_f \le n+1$ of C[1] class.

Assume the constraint $\boxed{\int_{t_i}^{t_f} h\left(x(\tau), u(\tau), \tau\right) d\tau = k}$ with

$$h, \frac{\partial h}{\partial x(t)}, \frac{\partial h}{\partial t} \in C^0\left(R^n \times U \times R\right), \; i = 1,2,\dots,n$$

Define the **performance index** : $\boxed{J\left(x,u,t_f\right) = \int_{t_i}^{t_f} L\left(x(\tau), u(\tau), \tau\right) d\tau}$

with $L, \dfrac{\partial L}{\partial x}, \dfrac{\partial L}{\partial t} \in C^0\left(R^n \times U \times R\right)$

Determine

- the value $t_f \in (t_i, \infty)$
- the control $u^o \in \overline{C}^0(R)$
- and the state $x^o \in \overline{C}^1(R)$

that satisfy

- ✓ the dynamical system,
- ✓ the constraint on the control,
- ✓ the initial and final conditions
- ✓ and minimize the cost index .

Hamiltonian function

$$H(x,u,\lambda_0,\lambda)=\lambda_0 L(x,u)+\lambda^T(t)f(x,u)+\boxed{\rho^T h(x(t),u(t),t)}$$

Theorem 4 (necessary condition):

Consider an admissible solution $\left(x^*, u^*, t_f^*\right)$ such that

$$rank\left\{ \frac{\partial \aleph}{\partial\left(x(t_f), t_f\right)}\bigg|^* \right\} = \sigma_f$$

IF it is a local minimum

there exist a constant $\lambda_0 \geq 0$, $\rho^* \in R^{\sigma}$, $\lambda^* \in \overline{C}^1\left[t_i, t_f^*\right]$

not simultaneously null such that :

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x}\bigg|^{*T},$$

$$H\left(x^*(t), \omega, \lambda_0^*, \lambda^*(t)\right) \geq H\left(x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)\right), \quad \forall \omega \in U$$

$$H\big|^* + \int_t^{t_f^*} \frac{\partial H}{\partial \tau}\bigg|^* d\tau = k, \quad k \in R$$

Moreover there exists a vector $\varsigma \in R^{\sigma_f}$ such that:

$$\lambda^*(T) = \frac{\partial \aleph}{\partial x(t_f)}\bigg|^{*T} \varsigma \qquad H\big|_{t_f^*}^* = -\frac{\partial \aleph}{\partial t_f}\bigg|^{*T} \varsigma$$

The discontinuities of $\dot{\lambda}^*$ may occur only in the instants in which u has a discontinuity and in these instants the Hamiltonian is continuous

Remark

If the set U coincides with R^p the minimum condition reduces to :

$$\boxed{\frac{\partial H}{\partial u} = 0}$$

The Pontryagin principle - convex case

Problem 5: Consider the dynamical **linear system:**

$$\dot{x} = A(t)x + B(t)u$$

with A and B of function of C^1 class; assume fixed the initial and final instants and the initial state, and

$$x(t_f) = x_f \quad fixed \quad or \quad x(t_f) \in R^n$$

Assume $u(t) \in U \subset R^p \quad \forall t \in [t_i, t_f]$

where *U* is a **convex set.**

Define the **performance index** :

$$J(x,u) = \int_{t_i}^{t_f} L(x(\tau), u(\tau), \tau) d\tau + G(x(t_f))$$

with

$$L,\ \frac{\partial L}{\partial x_i},\ \frac{\partial L}{\partial t} \in C^0\left(R^n \times U \times [t_i, t_f]\right),\ i = 1,2,\ldots,n$$

L **convex function** with respect to *x(t), u(t)* in $R^n \times U$

per ogni $t \in [t_i, t_f]$

G is a scalar function of C^2 class and **convex with respect to** $\boldsymbol{x(t_f)}$**.**

Determine:

the control $u^o \in \overline{C}^0[t_i, t_f]$

and the state $x^o \in \overline{C}^1[t_i, t_f]$

that satisfy

the dynamical system,

the constraint on the control,

the initial and final conditions

and minimize the cost index .

Theorem 5 (necessary and sufficient condition):

Consider an admissible solution $\left(x^o, u^o\right)$ such that

$$rank\left\{\left.\frac{\partial \aleph}{\partial\left(x(t_f), t_f\right)}\right|^o\right\} = \sigma_f$$

It is a minimum **normal** (i.e.λ_0 =1)

if and only if there exists an n-dimensional vector $\lambda^o \in \bar{C}^1\left[t_i, t_f\right]$

such that : $\boxed{\dot{\lambda}^o = -\left.\frac{\partial H(x,u,\lambda,t)}{\partial x}\right|^{oT}}$

$$\boxed{H\left(x^o(t), \omega, \lambda^o(t)\right) \geq H\left(x^o(t), u^o(t), \lambda^o(t)\right), \quad \forall \omega \in U}$$

Moreover, if $x(t_f) \in R^n$ $\quad \lambda^o(t_f) = \left.\frac{dG}{dx(t_f)}\right|^{oT}$

Schedule

Lecture :

The Hamilton- Jacobi equation- Derivation of the H-J equation-

Principle of optimality- Example

Principle of optimality

A control policy optimal over the interval $[t, t_f]$ is optimal over all subintervals $[t_1, t_f]$

If the shortest path from Rome to Paris passes through Milan THEN the OBTAINED path between Milan to Paris is the shortest one between Milan to Paris

Schedule

Lecture 8:

The optimal regulator problem:

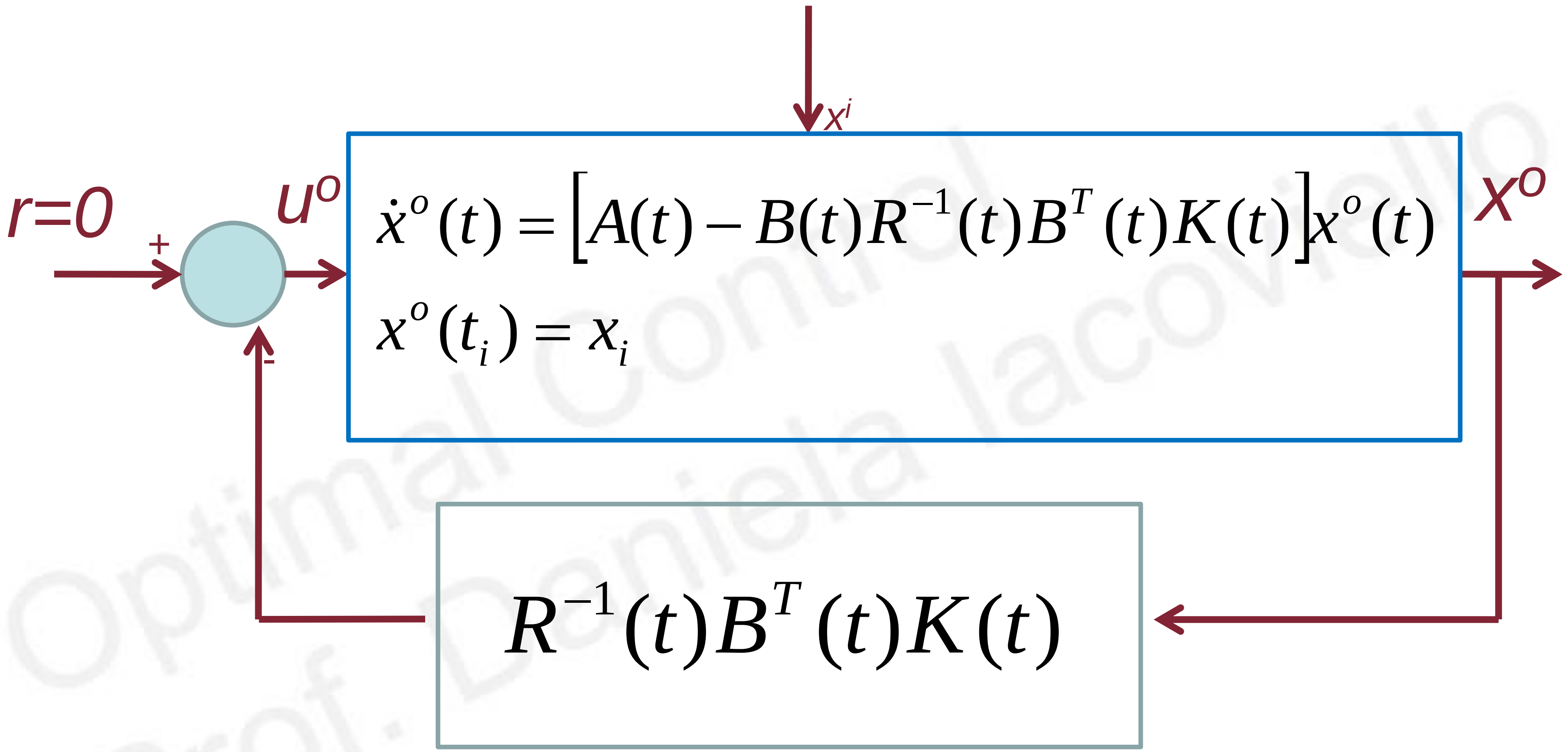
The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

Examples

The regulator problem



The regulator problem

Let us consider the following **linear system**:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

with $[t_i, t_f]$ fixed, and $x(t_i) = x_i$ fixed

$A(t),\ B(t),\ C(t)$ are matrix functions of time with continuous entries; their dimensions are respectively

$$n \times n,\ n \times m,\ n \times p$$

The regulator problem

A linear control law is obtained if we seek to minimize the quadratic performance index:

$$J(x,u)=\frac{1}{2}\int_{t_0}^{t_f}\left(x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right)dt$$
$$+\frac{1}{2}x^T(t_f)Fx(t_f)$$

where *Q(t)* and *R(t)* have continuous entries, symmetric, nonnegative and positive definite respectively;

F is nonnegative definite matrix simmetric

The Riccati equation

Theorem: The Riccati equation

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^T(t)K(t) - K(t)A(t) - A^T(t)K(t) - Q(t)$$
$$K(t_f) = F$$

admits a **unique** solution, symmetric, semidefinite positive, in the control interval.

Solution of the regulator problem

Theorem: The optimal control for the regulator problem is given by the **linear feedback law**:

$$\boxed{u^o(t) = -R^{-1}(t)B^T(t)K(t)x^o(t)}$$

where:

$$\dot{x}^o(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x^o(t)$$

$$x^o(t_i) = x_i$$

and

$$J\left(x^o, u^o\right) = \frac{1}{2}x^{iT}K(t_i)x^i$$

The regulator problem

Theorem: The regulator problem admits a **unique** solution.

Proof: the uniqueness property is shown by contraddiction arguments.

Solution of the deterministic linear optimal regulator problem on $[t_i, \infty)$

Problem: let us consider the regulator problem over the **infinite time interval** $[t_i, \infty)$.

Assume:

- The matrices A and B are **bounded** and with elements in C^1 class
- The dynamical system is **completely controllable** or **exponentially stable**
- The matrices Q and R are symmetric, semidefinite and positive definite respectively, with elements in C^1 class and **bounded**

Solution of the deterministic linear optimal regulator problem on $[t_i, \infty)$

Find:

the control $u^o \in \overline{C}^0[t_i, \infty)$

and the state $x^o \in \overline{C}^1[t_i, \infty)$

satisfying the dynamical system , the initial condition and

minimizing the cost index

$$J(x,u) = \frac{1}{2}\int_{t_i}^{\infty}\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]dt$$

Solution of the deterministic linear optimal regulator problem on $[t_i, \infty)$

Theorem: The problem admits a unique optimal solution that can be expressed as follows:

$$u^o(t) = -R^{-1}(t)B^T(t)\overline{K}(t)x^o(t)$$

$$\dot{x}^o(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)\overline{K}(t)\right]x^o(t), \qquad x^o(t_i) = x^i$$

where $\overline{K}(t)$ is the solution of the Riccati equation

$$\dot{\overline{K}}(t) = \overline{K}(t)B(t)R^{-1}(t)B^T(t)\overline{K}(t) - \overline{K}(t)A(t) - A^T(t)\overline{K}(t) - Q(t)$$

with final condition: $\lim_{t_f \to \infty} \overline{K}(t_f) = 0$

and $J\left(x^o, u^o\right) = \frac{1}{2} x^{iT}\overline{K}(t_i)x^i$

The steady state solution of the deterministic linear optimal regulator problem

Theorem: Let us consider the previous problem with the additive hypotheses :

- The matrices A, B, Q, R are constant
- The matrix Q is positive definite

Then there exists a unique optimal solution:

$$u^o(t) = -R^{-1}B^T K_r x^o$$

$$\dot{x}^o(t) = \left[A - BR^{-1}B^T K_r\right]x^o(t), \qquad x^o(t_i) = x^i$$

where:

K_r is the constant matrix, unique solution definite positive of the **algebraic Riccati equation:**

$$K_r B R^{-1} B^T K_r - K_r A - A^T K_r - Q = 0$$

The minimum value for the cost index is:

$$J\left(x^o, u^o\right) = \frac{1}{2} x^{iT} K_r x^i$$

Remarks

Consider the system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

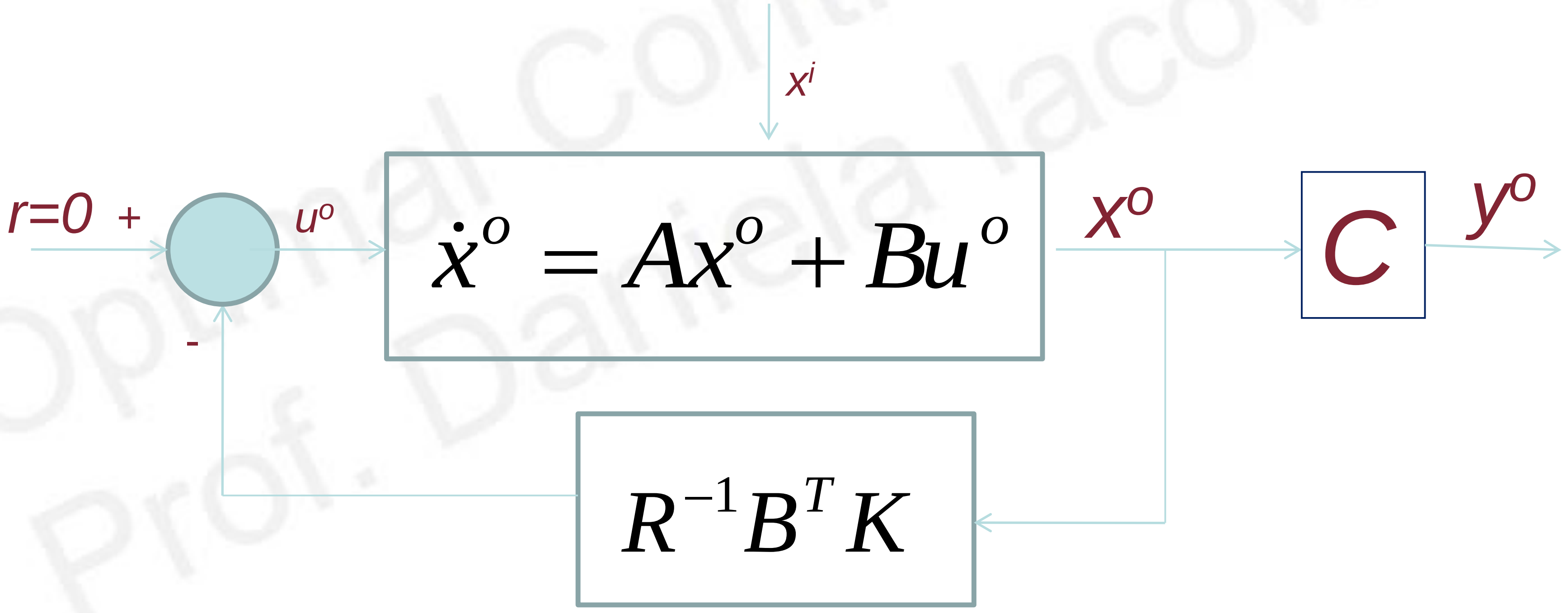
$$y(t) = C(t)x(t)$$

It can be defined an **optimal regulation problem from the output *y***, considering the cost index

$$\bar{J}(y,u) = \frac{1}{2} y^T(t_f)\bar{F}y(t_f) + \frac{1}{2}\int_{t_i}^{t_f} \left[y^T(t)\bar{Q}(t)y(t) + u^T(t)R(t)u(t) \right] dt$$

The present problem is similar to the previous one, being satisfied the hypotheses over the matrices involved.

Anyway **the state must be accessible**, if one want to realize the feedback action:



The optimal tracking problem

Problem: let us consider the linear system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_i) = x^i$$

where $A(t),\ B(t)$ are matrices with elements functions of time with entries of C^1 class.

Consider the **reference variable** $r \in C^1[t_i, t_f]$

Determine the optimal control $u^o \in \overline{C}^0[t_i, t_f]$ and the state $x^o \in \overline{C}^1[t_i, t_f]$ satisfying the dinamical constraint and minimizing the cost index:

$$J(x,u)=\frac{1}{2}\int_{t_i}^{t_f}\left\{[r(t)-x(t)]^T Q(t)[r(t)-x(t)]+u^T(t)R(t)u(t)\right\}dt$$

where:

- *Q(t)* is a symmetric semi-positive definite matrix
- *R(t)* is a symmetric positive definite matrix
- The elements of *Q(t)* and *R(t)* are functions of C^1 class

The optimal tracking problem

To the optimal tracking problem it can be associated the Riccati equation

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^{T}(t)K(t) - K(t)A(t) - A^{T}(t)K(t) - Q(t)$$
$$K(t_f) = 0$$

Theorem: the optimal tracking problem admits a unique optimal solution:

$$u^o(t) = R^{-1}(t)B^T(t)\left[g(t) - K(t)x^o(t)\right]$$

$$\dot{x}^o(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x^o(t) + B(t)R^{-1}(t)B^T(t)g(t) \qquad x^o(t_i) = x^i$$

where g is the solution of the differential equation

$$\dot{g}(t) = \left[K(t)B(t)R^{-1}(t)B^T(t) - A^T(t)\right]g(t) - \boxed{Q(t)r(t)}$$

$$\boxed{g(t_f) = 0}$$

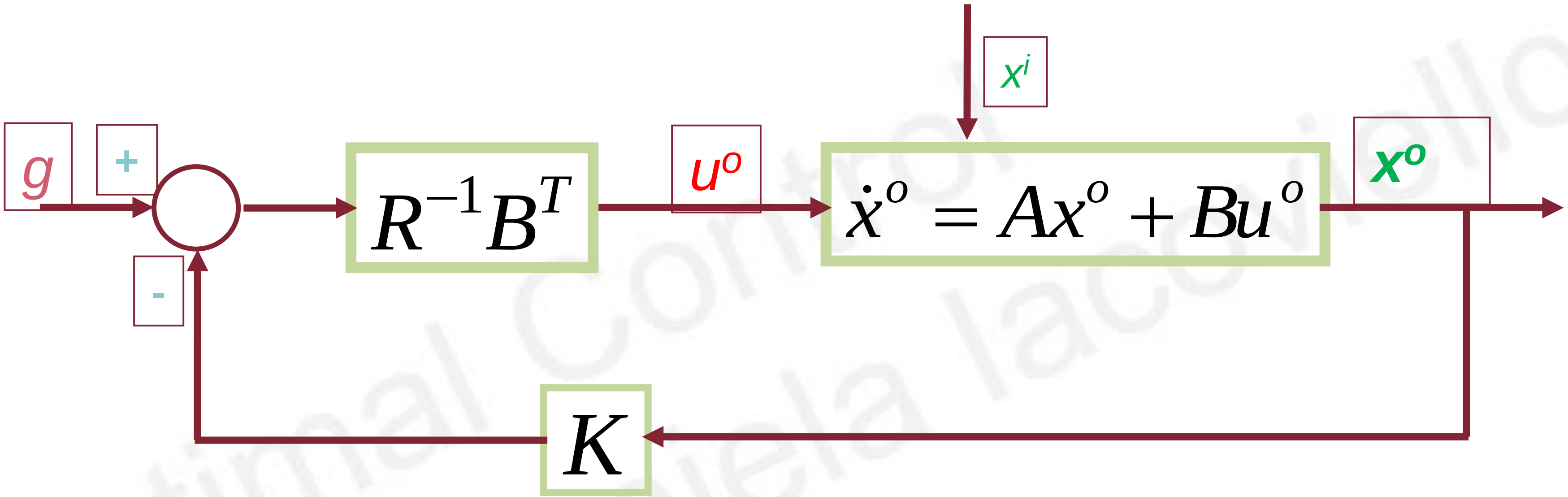
The minimum value of the cost index is:

$$J\left(x^o, u^o\right) = \frac{1}{2}x^{oT}(t_i)K(t_i)x^o(t_i) - x^{oT}(t_i)g(t_i) + v(t_i)$$

where v is the solution of the equation:

$$\dot{v}(t) = \frac{1}{2}g^T(t)B(t)R^{-1}(t)B^T(t)g(t)\boxed{-\frac{1}{2}r^T(t)Q(t)r(t)}$$

$$\boxed{v(t_f) = 0}$$


g
+
-
$R^{-1}B^T$
u^o
x^i
$\dot{x}^o = Ax^o + Bu^o$
x^o
K

The optimal regulator problem with null final error

Problem: Let us consider the linear system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$x(t_i) = x^i, \; x(t_f) = 0$$

Determine the control $u^o \in \overline{C}^0[t_i, t_f]$

and the state $x^o \in \overline{C}^1[t_i, t_f]$

in order to satisfy the dynamical system, the initial and final conditions and minimizing the following cost index:

$$J(x,u)=\frac{1}{2}\int_0^{t_f}\left[x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right]dt$$

where:

- *Q(t)* is symmetric semidefinite positive with elements of C^1 class
- *R(t)* is symmetric definite positive with elements of C^1 class

Theorem: Let us introduce the matrix of dimension 2n × 2n

$$\boxed{\Omega(t) = \begin{pmatrix} A(t) & -B(t)R^{-1}(t)B^{T}(t) \\ -Q(t) & -A^{T}(t) \end{pmatrix}}$$

and indicate with

$$\boxed{\Phi(t,\tau) = \begin{pmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{pmatrix}}$$

its transition matrix partitioned in submatrices of dimension n × n.

Assume that the **submatrix** $\phi_{12}(t_f, t_i)$ **is not singular .**

The optimal regulator problem with null final error admits a <u>unique optimal solution:</u>

$$u^o(t) = -R^{-1}(t)B^T(t)\left[\phi_{21}(t,t_i) - \phi_{22}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i$$

$$x^o(t) = \left[\phi_{11}(t,t_i) - \phi_{12}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i$$

Schedule

Lecture 9:

The minimum time problem- the minimal time optimal problem – characterization of the optimal solution- uniqueness result- Existence- Minimum time problem in case of steady state system

Switching points

The linear time-optimal control

Let us consider the problem of optimal control of a linear system

- ✓ with fixed initial and final state,
- ✓ constraints on the control variables and
- ✓ with cost index equal to the **length of the time interval**

SINGULAR SOLUTIONS

Definition:

Let $\left(x^o, u^o, t_f^o\right)$ be an optimal solution of the above problem, and $\lambda_0^o\ \lambda^o$ the corresponding multipliers.

The solution is **singular** if there exists a subinterval $\left(t', t''\right),\ t'' > t'$ in which the Hamiltonian $H\left(x^o(t), \omega, \lambda_0^o, \lambda^o(t), t\right)$ is **independent** from at least one component of ω in $\left(t', t''\right)$

SINGULAR SOLUTIONS

Theorem:

Assume the Hamiltonian of the form

$$H(x,u,\lambda_0,\lambda,t) = H_1(x,\lambda_0,\lambda,t) + H_2(x,\lambda_0,\lambda,t)N(x,u,\lambda_0,\lambda,t)$$

Let $\left(x^*, u^*, t_f^*\right)$ be an extremum and $\lambda_0^*\ \lambda^*$ the corresponding multipliers such that $N(x^*, \omega, \lambda^*{}_0, \lambda^*, t)$ **is dependent on any component of** ω **in any subinterval of**

A necessary condition for $\left(x^*, u^*, t_f^*\right)$ to be a singular extremum is that there exists a subinterval $[t_i, t_f]$ such that:

$$[t', t''] \subset [t_i, t_f], \ t'' > t'$$

$$H_2(x,\lambda_0,\lambda,t) = 0, \ \ \forall t \in [t', t'']$$

The linear minimum time optimal control

Let us consider the problem of optimal control of a linear system

- with fixed initial and final state,
- constraints on the control variables and
- with cost index equal to the **length of the time interval**

Problem: Consider the linear dynamical system:

$$\boxed{\dot{x}(t) = A(t)x(t) + B(t)u(t)}$$

With $x(t) \in R^n, \quad u(t) \in R^p$ and the constraint

$$\boxed{|u_j(t)| \le 1, \quad j = 1, 2, \ldots, p, \quad \forall t \in R}$$

Matrices A and B have entries with elements of class

C^{n-2} and C^{n-1} respectively, at least of C^1 class .

The initial instant t_i is fixed and also:

$$\boxed{x(t_i) = x^i, \quad x(t_f) = 0}$$

The aim is to determine the **final instant** $t_f^o \in R$

the control $\boxed{u^o \in \overline{C}^o(R)}$ and the state $\boxed{x^o \in \overline{C}^1(R)}$

that minimize the cost index:

$$\boxed{J(t_f) = \int_{t_i}^{t_f} dt = t_f - t_i}$$

Theorem: Necessary conditions for $\left(x^*, u^*, t_f^*\right)$ to be an optimal solution are that there exist a constant $\lambda_0^* \geq 0$ and an n-dimensional function $\lambda^* \in \overline{C}^1\left[t_i, t_f\right]$, not simultaneously null and such that:

$$\boxed{\begin{aligned} &\dot{\lambda}^* = -A^T \lambda^* \\ &\lambda^{*T} B\omega \geq \lambda^{*T} Bu^* \quad \forall \omega \in R^p : \left|\omega_j\right| \leq 1, \quad j = 1,2,\ldots, p \end{aligned}}$$

Possible discontinuities in $\dot{\lambda}^*$ can appear only in the points in which u^* has a discontinuity.

Moreover the associated Hamiltonian is a continuous function with respect to t and it results:

$$\boxed{H\Big|_{t_f^*}^{*} = 0}$$

Proof: The Hamiltonian associated to the problem is:

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 + \lambda^T Ax + \lambda^T Bu$$

Applying the minimum principle the theorem is proved.

STRONG CONTROLLABILITY

Let us indicate with $b_j(t)$ the *j*-th column of the matrix *B* .

The **strong controllability** **is guaranteed** by the condition:

$$\det\{G_j(t)\} = \det\left\{\begin{pmatrix} b_j^{(1)}(t) & b_j^{(2)}(t) & \cdots & b_j^{(n)}(t) \end{pmatrix}\right\} \neq 0$$

$$j = 1,2,\ldots,p, \quad \forall t \geq t_i$$

where :

Generic column of matrix *B(t)*

$$\boxed{\begin{array}{l} b_j^{(1)}(t) = b_j(t) \\ b_j^{(k)}(t) = \dot{b}_j^{(k-1)}(t) - A(t)b_j^{(k-1)}(t) \quad k = 2,3,\ldots,n \end{array}}$$

Remark:

Strong controllability corresponds to the controllability in

any instant t_i,

in any time interval and

by any component of the control vector

Remark:

In the NON-steady state case the above condition is **sufficient** for strong controllability.

In the **steady case** it is **necessary and sufficient** and may be written in the usual way:

$$\boxed{\det\left\{\begin{pmatrix} b_j & Ab_j & \cdots & A^{n-1}b_j \end{pmatrix}\right\} \neq 0 \qquad j = 1,2,\ldots,p}$$

Characterization of the optimal solution

Theorem: Let us consider the minimal time optimal problem . If the strong controllability condition is satisfied and if an optimal solution exists

⇒ it is **non singular**.

Moreover every component of the optimal control is **piecewise constant** assuming only the extreme values ± 1 and the **number of discontinuity instants is limited**.

A control function that assumes only the limit values is called **bang-bang control**

and

the instants of discontinuity are called **commutation instants**

Theorem (UNIQUENESS):

If the hypothesis of strong controllability is satisfied, if an optimal solution exists it is **unique**.

Proof: the theorem is proved by contradiction arguments.

Remark

This result holds also to the case in which the control is in the *space of measurable function* defined on the space of real number *R*.

Let $S[t_i, T]$ be the space of piecewise constant functions $s(t)$

If $z(t) = \lim_{k \to \infty} s_k(t)$, for $\{s_k(t)\} \subset S[t_i, T]$ then z(t) is a measurable function on $[t_i, T]$

Existence of the optimal solution

Theorem: if the condition of strong controllability is satisfied and an admissible solution exists, then there exists a unique optimal solution nonsingular and the control is bang-bang.

Proof: the existence theorem is proved on the basis of the following results.

The following result holds:

Theorem: Let assume that the control functions belong to the **space of measurable functions.**

If an admissible solution exists then the optimal solution exists.

The results about the characterization of the control may be extended to the case in which the control belong to the space of measurable functions:

Therefore it results in:

If the strong controllability condition is satisfied and

if an optimal control in the **space of measurable function**

exists then this solution is unique,

non singular and the ***control is bang bang type***

Remark

The existence of the optimal solution is guaranteed only for the couples $\left(t_i, x^i\right)$ for which the admissible solution exists

Existence of the optimal solution

Theorem: if the condition of strong controllability is satisfied and an admissible solution exists
with the control in the space of measurable functions,
then there exists a unique optimal solution
nonsingular and
the control is bang-bang.

Minimum time problem for steady state system

Let us consider the minimum time problem in case of **steady state system**.

In this case it is possible to deduce results about the number of commutation points

Problem: Consider the linear dynamical system:

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)}$$

With $x(t) \in R^n, \quad u(t) \in R^p$ and the constraint

$$\boxed{\left| u_j(t) \right| \leq 1, \quad j = 1,2,\ldots,p, \quad \forall t \in R}$$

The initial instant t_i is fixed and also:

$$\boxed{x(t_i) = x^i, \quad x(t_f) = 0}$$

The aim is to determine the final instant $\boxed{t_f^o \in R}$

the control $\boxed{u^o \in \overline{C}^o(R)}$

and the state $\boxed{x^o \in \overline{C}^1(R)}$

that minimize the cost index:

$$\boxed{J(t_f) = \int_{t_i}^{t_f} dt = t_f - t_i}$$

Theorem:

Let us consider the above time minimum problem and assume that the control function belong to the space of measurable function ;

if the system is controllable

$\Rightarrow$

there exists a neighbor Ω of the origin such that for any $x^i \in \Omega$ there exists an optimal **solution.**

Theorem: Let us consider the above time minimum problem in the steady state case and assume that the control function belongs to the space of measurable function *M(R).*

If the system is controllable and the **eigenvalues of matrix A have negative real part**

there exists an optimal solution

whatever the initial state is.

Theorem: **If the controllability condition** is satisfied and if all the eigenvalues of the matrix A are real non positive, then **the number of commutation instants for any components of control is less or equal than n-1**, whatever the initial state is.

Schedule

Lecture The minimum time problem-
examples: double integrator; harmonic oscillator