4 A Unified Approach to Problems of Asymptotic Tracking and Disturbance Rejection

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4.1 Introduction

In this Chapter, we address the problem of *output regulation* (sometimes also known as *generalized tracking* problem, or *generalized servomechanism* problem) for nonlinear systems. Formally, the problem of output regulation is cast in the following terms. The controlled plant is a finite-dimensional, time-invariant, nonlinear system modelled by equations of the form

$$\begin{array}{rcl} \dot{x} & = & F(w,x,u) \\ e & = & H(w,x) \\ y & = & K(w,x) \end{array} \tag{4.1}$$

in which $x \in \mathbb{R}^n$ is a vector of state variables, $u \in \mathbb{R}$ is the input used for *control* purposes, $w \in \mathbb{R}^s$ is a vector of inputs which cannot be controlled and include *exogenous* commands, exogenous disturbances and model uncertainties, $e \in \mathbb{R}$ is a vector of *regulated* outputs which include tracking errors and any other variable that needs to be steered to 0, $y \in \mathbb{R}^p$ is a vector of outputs that are available for *measurement.* The exogenous input $w(t)$ is assumed to be a (undefined) member of the family of all solutions of a fixed ordinary differential equation of the form

$$\dot{w} = s(w) \tag{4.2}$$

obtained when the initial condition $w(0)$ is allowed to vary on a prescribed set W. This system is usually referred to as the *exosystem.* The initial state of (4.1) and of (4.2) are assumed to range over fixed compact sets X and W, the latter being invariant under the dynamics of (4.2). The problem of output regulation is to design a controller

$$\begin{array}{rcl} \dot{\xi} & = & \varphi(\xi,y) \\ u & = & \gamma(\xi,y) \end{array}$$

with initial state in a compact set Ξ, yielding a closed-loop system in which

- the positive orbit of $W \times X \times \Xi$ is bounded,
- $\lim\limits_{t\to\infty} e(t) = 0$, uniformly in the initial condition (on $W \times X \times \Xi$).

We observe that, in the general setup presented above, the vector w of exogenous inputs may well include (constant) uncertain parameters, which are hence assumed to range on a given compact set. Thus, if a controller solves the problem at issue, the goal of asymptotic regulation is achieved robustly with respect to (constant) parameter uncertainties.

The theory of output regulation of nonlinear systems, which uses a combination of geometry and nonlinear dynamical systems theory, was initiated by pioneering works of [10, 9] who showed how to design a controller that provides a local solution near an equilibrium point, in the presence of exogenous signals which were produced by a neutrally stable system. Since these early contributions, the theory has experienced a tremendous growth, culminating in the recent development of design methods able to handle issues of global convergence (as in [3]), the case of parametric uncertainties affecting the autonomous (linear) system which generates the exogenous signals (such as in [15]), the case of nonlinear exogenous systems (such as in [2]), or a combination thereof (as in [13]). A thorough presentation of several recent advances in this area can also be found in the recent books [12, 8, 14].

4.2 The plant and the basic assumptions

In what follows, we consider a nonlinear system having relative degree r between control input $u \in \mathbb{R}$ and regulated output $e \in \mathbb{R}$ described in normal form as

$$\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, \xi, \zeta) \\
\dot{\xi} &= A\xi + B\zeta \\
\dot{\zeta} &= q(w, z, \xi, \zeta) + b(w, z, \xi, \zeta)u \\
e &= C\xi
\end{aligned} \tag{4.3}$$

in which $z \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{r-1}$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

As indicated above, the functions characterizing the model (4.3) are assumed to be smooth functions of their arguments. In addition, we assume the existence of a pair of real numbers (b_0, b_1) such that

$$0 < b_0 \le b(w, z, \xi, \zeta) \le b_1 . \tag{4.4}$$

Note that

$$\xi = \mathrm{col}(e, e^{(1)}, \ldots, e^{(r-2)}), \qquad \zeta = e^{(r-1)} .$$

Motivated by well-known standard design procedures (see e.g. [6]), we assume throughout that the entire *partial state* $(\xi_1, \ldots, \xi_{r-1}, \zeta)$ is available for measurement, i.e.

$$y = \mathrm{col}(\xi_1, \ldots, \xi_{r-1}, \zeta) .$$

The states w and z are, on the contrary, not available for measurement.

The point of departure for the solution of the problem is, as usual, the assumption of the existence of a (smooth) manifold which can be rendered invariant by feedback and on which the regulated output vanishes (see [10]). In the case of system (4.3), this amounts to the assumption of the existence of a smooth map $\pi : W \quad \to \quad \mathbb{R}^m$ satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(w, \pi(w), 0, 0) \qquad \forall w \in W . \tag{4.5}$$

This being the case, it is readily seen that the set

$$\mathcal{S}^* = \{(w, z, \xi, \zeta) : w \in W, z = \pi(w), \xi = 0, \zeta = 0\}$$

is rendered invariant by the control

$$u^*(w) = -\frac{q(w, \pi(w), 0, 0)}{b(w, \pi(w), 0, 0)} \tag{4.6}$$

and, indeed, the regulated variable $e = \xi_1$ vanishes on this set. The input $u^*(w)$ is the input which forces e to remain identically zero.

The second step in the solution of the problem usually consists in making assumptions that make it possible to build an "internal" model for the control $u^*(w)$. In a series of recent papers, it was shown how these assumptions could be progressively weakened, moving from the so-called assumption of "immersion into a linear observable system" (as in [9]) to "immersion into a nonlinear uniformly observable system" (as in [2]) to the recent results of [13], in which it was shown that no assumption is in fact needed for the construction of an internal model if only continuous (thus possibly not locally Lipschitz) controllers are acceptable. Motivated by this, we assume, in what follows, the existence of $d \in \mathbb{N}$, a map $F : \mathbb{R}^d \to \mathbb{R}^d$, a $d \times 1$ column vector G_0, a map $\gamma : \mathbb{R}^d \to \mathbb{R}$ and a map $\tau : W \to \mathbb{R}^d$ satisfying

$$\begin{array}{rcll} \dfrac{\partial \tau}{\partial w} s(w) & = & F(\tau(w)) + G_0 \gamma(\tau(w)) & \forall w \in W \\ u^*(w) & = & \gamma(\tau(w)) & \forall w \in W\,. \end{array} \tag{4.7}$$

Coherently with the assumptions on (4.3), $F(\cdot)$, $\gamma(\cdot)$ and $\tau(\cdot)$ are assumed to be smooth maps.

Remark 4.1 Under the (mild) assumption that

$$L_s^d u^*(w) = \phi(u^*(w), L_s u^*(w), \ldots, L_s^{d-1} u^*(w))\,, \tag{4.8}$$

for some smooth $\phi(x_1, \ldots, x_d)$ and all $w \in W$, a smooth internal model can be obtained taking

$$F(x) = \begin{pmatrix} x_2 \\ \cdots \\ x_d \\ \phi(x_1, \ldots, x_d) \end{pmatrix} - G_0 x_1 \qquad \gamma(x) = x_1\,, \tag{4.9}$$

in which case (4.7) hold with

$$\tau(w) = \mathrm{col}(u^*(w), \ldots, L_s^{d-1} u^*(w))\,.$$

This includes the (classical) case of linear internal models. Recent advances in the theory of nonlinear observers (see e.g. [13]) show that, if d is large enough, and $F(x) = F_0 x$ with F_0 Hurwitz and (F_0, G_0) controllable, a C^1 map $\tau(\cdot)$ and a C^0 map $\gamma(\cdot)$ which do fulfill (4.7) always exist. $\triangleleft$

4.3 The control

Consider, for the plant (4.3), a controller of the form

$$\begin{array}{rcl} u & = & \gamma(\eta) + \beta \dot{N}(\varphi) + v \\ \dot{\varphi} & = & L(\varphi) - M v \\ \dot{\eta} & = & F(\eta) + G_0[\gamma(\eta) + v] \\ v & = & -k[\zeta - H\xi - N(\varphi)] \end{array} \tag{4.10}$$

which is a dynamic controller, with internal state (φ, η), "driven" only by the measured variables (ξ, ζ). Assume (without loss of generality) that

$$\frac{\partial N}{\partial \varphi} M = 0 \tag{4.11}$$

in which case

$$\dot{N}(\varphi) = \frac{\partial N}{\partial \varphi} L(\varphi)\,.$$

Changing ζ into

$$\theta = \zeta - H\xi - N(\varphi)$$

yields a closed-loop system of the form

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f(w, z, \xi, H\xi + N(\varphi) + \theta)\\
\dot{\xi} &= A\xi + B[H\xi + N(\varphi) + \theta]\\
\dot{\varphi} &= L(\varphi) - Mv\\
\dot{\eta} &= F(\eta) + G_0[\gamma(\eta) + v]\\
\dot{\theta} &= Q(w, z, \xi, H\xi + N(\varphi) + \theta) + b(w, z, \xi, H\xi + N(\varphi) + \theta)[\gamma(\eta) + v]\\
&\quad + \Delta(w, z, \xi, H\xi + N(\varphi) + \theta)\dot{N}(\varphi)\,,
\end{aligned}$$

in which we have set

$$\begin{aligned}
Q(w, z, \xi, \zeta) &= q(w, z, \xi, \zeta) - H(A\xi + B\zeta)\\
\Delta(w, z, \xi, \zeta) &= b(w, z, \xi, \zeta)\beta - 1\,,
\end{aligned}$$

with control

$$v = -k\theta\,.$$

Remark 4.2 Note that, in case the coefficient $b(w, z, \xi, \zeta)$ only depends on the measured variables (ξ, ζ), one can choose $\beta = 1/b$, obtaining in this way $\Delta(w, z, \xi, \zeta) = 0$. $\triangleleft$

This system can be regarded as a system with input v and output θ, having relative degree 1, with input v to be chosen as $v = -k\theta$, that is as a negative output feedback. To facilitate the analysis, we bring this system in normal form. Since $b(w, z, \xi, \zeta)$ is bounded as in (4.4), by the method of the characteristics one can obtain a globally defined change of coordinates

$$X \;:\; \eta \;\mapsto\; x = X(w, z, \xi, \varphi, \eta, \theta)$$

in which X satisfies

$$\frac{\partial X}{\partial \eta} G_0 + \frac{\partial X}{\partial \theta} b(w, z, \xi, H\xi + N(\varphi) + \theta) = 0\,.$$

At $\theta = 0$, the map X is the identity map, namely $X(w, z, \xi, \varphi, \eta, 0) = \eta$ which in turn implies

$$\left[\frac{\partial X}{\partial \eta}\right]_{\theta=0} = I\,.$$

Actually, it is not difficult to find a closed form for X, which turns out to be

$$X(w, z, \xi, \varphi, \eta, \theta) = \eta - G_0 \int_0^\theta \frac{1}{b(w, z, \xi, H\xi + N(\varphi) + t)}\, dt\,.$$

From this, using our earlier assumption (4.11), it is readily seen that

$$\frac{\partial X}{\partial \varphi} M = 0\,.$$

Likewise, by the method of the characteristics one can obtain a globally defined change of coordinates

$$K \;:\; \varphi \;\mapsto\; \chi = K(w, z, \xi, \varphi, \theta)$$

in which K satisfies

$$\frac{\partial K}{\partial \varphi} M - \frac{\partial K}{\partial \theta} b(w, z, \xi, H\xi + N(\varphi) + \theta) = 0\,.$$

At $\theta = 0$, the map K is the identity map, namely $K(w, z, \xi, \varphi, 0) = \varphi$ which in turn implies

$$\left[\frac{\partial K}{\partial \varphi}\right]_{\theta=0} = I\,.$$

The inverses of K and X define a pair of maps

$$\begin{array}{rcl} \varphi & = & \hat{K}(w, z, \xi, \chi, \theta) \\ \eta & = & \hat{X}(w, z, \xi, \chi, x, \theta) \end{array}$$

which, at $\theta = 0$, are identities in χ and – respectively – in x, that is

$$\hat{K}(w, z, \xi, \chi, 0) = \chi\,, \qquad \hat{X}(w, z, \xi, \chi, x, 0) = x\,.$$

Changing coordinates in this way yields a system of the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w, z, \xi, H\xi + N(\hat{K}) + \theta) \\ \dot{\xi} & = & A\xi + B[H\xi + N(\hat{K}) + \theta] \\ \dot{\chi} & = & \dfrac{\partial K}{\partial \varphi}\Big[L(\hat{K}) + M\Big(\dfrac{Q}{b}(w, z, \xi, \theta + N(\hat{K}) + H\xi) + \dfrac{\Delta}{b}\,\dot{N}(\hat{K}) + \gamma(\hat{X})\Big)\Big] + R_\chi \\ \dot{x} & = & F(\hat{X}) - G_0\Big(\dfrac{Q}{b}(w, z, \xi, \theta + N(\hat{K}) + H\xi) + \dfrac{\Delta}{b}\,\dot{N}(\hat{K})\Big) + R_x \\ \dot{\theta} & = & Q(w, z, \xi, H\xi + N(\hat{K}) + \theta) + b(w, z, \xi, H\xi + N(\hat{K}) + \theta)[\gamma(\hat{X}) + v] \\ & & \quad + \Delta(w, z, \xi, H\xi + N(\hat{K}) + \theta)\dot{N}(\hat{K})\,, \end{array} \tag{4.12}$$

in which, for readability, we have omitted the indication of the arguments of $\hat{K}$, $\hat{X}$ and Δ/b, and we have set

$$R_\chi = \frac{\partial K}{\partial w}s(w) + \frac{\partial K}{\partial z}f(w, z, \xi, H\xi + N(\hat{K}) + \theta) + \frac{\partial K}{\partial \xi}(A\xi + B[H\xi + N(\varphi) + \theta])$$

$$R_x = \frac{\partial X}{\partial w}s(w) + \frac{\partial X}{\partial z}f(w, z, \xi, H\xi + N(\hat{K}) + \theta) + \frac{\partial X}{\partial \xi}(A\xi + B[H\xi + N(\varphi) + \theta]) + \frac{\partial X}{\partial \varphi}L(\hat{K})\,.$$

Note that at $\theta = 0$ both these terms vanish, because at $\theta = 0$ the map K is simply an identity in φ and the map X is simply an identity in η.

The system obtained in this way can be seen as feedback interconnection of a system with input θ and state (w, z, ξ, χ, x) and of a system with input (w, z, ξ, χ, x) and state θ. In fact, setting

$$\mathbf{p} = \mathrm{col}\{w, z, \xi, \chi, x\}$$

the system cam be viewed as a system of the form

$$\begin{array}{rcl} \dot{\mathbf{p}} & = & \mathbf{F}(\mathbf{p}) + \mathbf{G}(\mathbf{p}, \theta)\theta \\ \dot{\theta} & = & \mathbf{H}(\mathbf{p}) + \mathbf{H}(\mathbf{p}, \theta)\theta - \mathbf{b}(\mathbf{p}, \theta)v \end{array} \tag{4.13}$$

with control to be chosen as

$$v = -k\theta\,. \tag{4.14}$$

The advantage of seeing system (4.12) in this form is that we can appeal to the following result (see e.g. [13]).

Theorem 4.1 *Consider a system of the form (4.13) with v as in (4.14). The functions $\mathbf{F}(\mathbf{p})$ and $\mathbf{H}(\mathbf{p})$ are locally Lipschitz and the functions $\mathbf{G}(\mathbf{p},\theta)$ and $\mathbf{H}(\mathbf{p},\theta)$ are continuous. Let $\mathbf{P}$ be an arbitrary fixed compact set. Suppose that $\mathbf{b}(\mathbf{p},\theta) > 0$ for all $(\mathbf{p},\theta)$. Suppose there exists a set $\mathcal{A}$ which is locally exponentially stable for*

$$\dot{\mathbf{p}} = \mathbf{F}(\mathbf{p}),$$

with a domain of attraction that contains the set $\mathbf{P}$. Suppose also that

$$\mathbf{H}(\mathbf{p}) = 0, \qquad \forall \mathbf{p} \in \mathcal{A}.$$

Then, for any choice of a compact set Θ, there is a number k^* such that, for all $k > k^*$, the set $\mathcal{A} \times \{0\}$ is locally exponentially stable, with a domain of attraction that contains $\mathbf{P} \times \Theta$.

If the assumption of this Theorem are fulfilled and, in addition, the regulated variable $e = \xi_1$ vanishes $\mathcal{A}$, we conclude that the proposed controller is able to solve the problem of output regulation.

4.4 The structure of the core subsystem

All of the above suggests the use of the degrees of freedom in the choice of the parameters of the controller in order to impose appropriate asymptotic properties on the subsystem obtained by setting $\theta = 0$ in (4.12). The latter, in view of the fact that at $\theta = 0$

$$\hat{K} = \chi\,, \quad \hat{X} = x\,, \quad \frac{\partial K}{\partial \varphi} = 0\,, \quad R_\chi = 0\,, \quad R_x = 0\,,$$

reads as

$$\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f(w,z,\xi,H\xi + N(\chi)) \\
\dot{\xi} &= A\xi + B(H\xi + N(\chi)) \\
\dot{\chi} &= L(\chi) + M\Big(\frac{Q}{b}(w,z,\xi,H\xi + N(\chi)) + \frac{\Delta}{b}\dot{N}(\chi) + \gamma(x)\Big) \\
\dot{x} &= F(x) - G_0\Big(\frac{Q}{b}(w,z,\xi,H\xi + N(\chi)) + \frac{\Delta}{b}\dot{N}(\chi)\Big)
\end{aligned} \tag{4.15}$$

Theorem 4.1 above identifies an auxiliary problem which, if solved, makes it possible to use the controller (4.10) for the solution of the problem of output regulation for the original plant: shape the internal model $\{F(x), G_0, \gamma(x)\}$ and find, if possible, a triplet $\{L(\chi), M, N(\chi)\}$ in such a way that system (4.15) possesses a compact invariant set $\mathcal{A}$ which is locally exponentially stable and attracts all admissible initial conditions, and that both ξ_1 and the map

$$Q(w,z,\xi,H\xi + N(\chi)) + b(w,z,\xi,H\xi + N(\chi))\gamma(x) + \Delta(w,z,\xi,H\xi + N(\chi))\dot{N}(\chi)\,. \tag{4.16}$$

vanish on this set.

Recall now that, by assumption, there exists $\pi(w)$ and $\tau(w)$ satisfying (4.5) and (4.7). Hence, it is readily seen that if $L(0) = 0$ and $N(0) = 0$, the set

$$\mathcal{A} = \{(w,z,\xi,\chi,x) : w \in W, z = \pi(w), \xi = 0, \chi = 0, x = \tau(w)\}$$

is a compact invariant set of (4.15). Moreover, by construction, the map (4.16) vanishes on this set. Trivially, also ξ_1 vanishes on this set. Thus, it is concluded if the set $\mathcal{A}$ can be made local exponentially stable, with a domain of attraction that contains the compact set of all admissible initial conditions, the proposed controller, with large k solves the problem of output regulation.

System (4.15) is not terribly difficult to handle. As a matter of fact, it can be regarded as interconnection of three much simpler subsystems. To see this, set

$$\begin{aligned} z_{\mathrm{a}} &= z - \pi(w) \\ \tilde{x} &= x - \tau(w) \end{aligned}$$

and define

$$f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta) = f(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta) - f(w, \pi(w), 0, 0)$$

$$h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta) = \frac{Q}{b}(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta) - \frac{Q}{b}(w, \pi(w), 0, 0)$$

and

$$\Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta) = \frac{\Delta}{b}(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta) = \beta - \frac{1}{b(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta)}\,.$$

In the new coordinates thus introduced, the invariant manifold $\mathcal{A}$ is simply the set

$$\mathcal{A} = \{(w, z_{\mathrm{a}}, \xi, \chi, \tilde{x}) : w \in W, (z_{\mathrm{a}}, \xi, \chi, \tilde{x}) = (0, 0, 0, 0)\}\,.$$

Bearing in mind (4.5), (4.7) and (4.6), it is readily seen that

$$\dot{z}_{\mathrm{a}} = f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi))$$

and

$$\frac{Q}{b}(w, z, \xi, H\xi + N(\chi)) = h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) - \gamma(\tau(w))\,.$$

In view of this, using again (4.7), the core subsystem (4.15) can be seen as a system with input $\bar{u}$ and output $\bar{y}$ defined as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_{\mathrm{a}} &= f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) \\ \dot{\xi} &= A\xi + B(H\xi + N(\chi)) \\ \dot{\chi} &= L(\chi) + M[h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) + \Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi))\dot{N}(\chi) + \bar{u}] \\ \dot{\tilde{x}} &= F(\tilde{x} + \tau(w)) - F(\tau(w)) - G_0[h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) + \Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi))\dot{N}(\chi)] \\ \bar{y} &= \gamma(\tilde{x} + \tau(w)) - \gamma(\tau(w)) \end{aligned} \tag{4.17}$$

subject to unitary output feedback

$$\bar{u} = \bar{y}\,.$$

System (4.17), in turn, can be seen as the cascade of an "inner loop" consisting of a subsystem, which we call the "auxiliary plant", modelled by equations of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_{\mathrm{a}} &= f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + u_{\mathrm{a}}) \\ \dot{\xi} &= (A + BH)\xi + Bu_{\mathrm{a}} \\ y_{\mathrm{a}} &= h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + u_{\mathrm{a}}) + \Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + u_{\mathrm{a}})v_{\mathrm{a}}\,, \end{aligned} \tag{4.18}$$

controlled by

$$\begin{aligned} \dot{\chi} &= L(\chi) + M[y_{\mathrm{a}} + \bar{u}] \\ u_{\mathrm{a}} &= N(\chi) \\ v_{\mathrm{a}} &= \dot{N}(\chi)\,, \end{aligned} \tag{4.19}$$

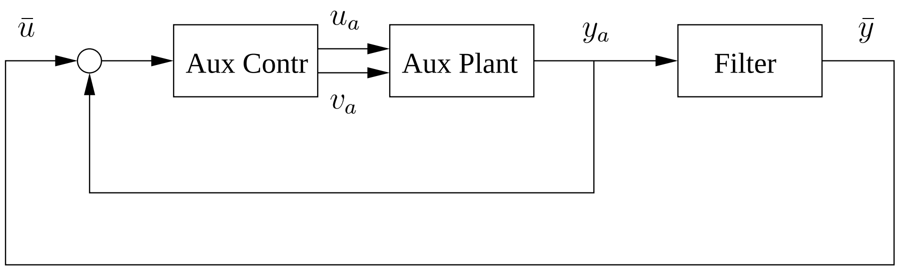

Figure 4.1: The feedback structure of system (4.15)

cascaded with a system, which we call a “weighting filter”, modelled by equations of the form

$$\begin{array}{rcl}\dot{\tilde{x}} & = & F(\tilde x + \tau(w)) - F(\tau(w)) - G_0 y_{\mathrm{a}} \\ \bar y & = & \gamma(\tilde x + \tau(w)) - \gamma(\tau(w))\,.\end{array} \tag{4.20}$$

All of this is depicted in Fig.F 4.1.

With this interpretation in mind, the idea is now to use the small-gain Theorem to enforce the desired asymptotic properties on system (4.15). To this purpose, set

$$\mathbf{x} = \mathrm{col}(z_{\mathrm{a}}, \xi, \chi, \tilde x)$$

and observe that system (4.17) is a system of the form

$$\begin{array}{rcl}\dot w & = & s(w) \\ \dot{\mathbf{x}} & = & \mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\bar u \\ \bar y & = & \mathbf{C}(w,\mathbf{x})\end{array} \tag{4.21}$$

in which

$$\mathbf{A}(w,\mathbf{0}) = \mathbf{0} \qquad \mathbf{C}(w,\mathbf{0}) = \mathbf{0}\,.$$

Let $W \times \mathbf{X}$ be the desired (compact) set of initial conditions for $(w,\mathbf{x})$ in system (4.15). Suppose there exists a smooth positive definite and proper function $\mathbf{V}(\mathbf{x})$, with quadratic bounds for small $|\mathbf{x}|$, namely satisfying

$$a_1|\mathbf{x}|^2 \le \mathbf{V}(\mathbf{x}) \le a_2|\mathbf{x}|^2, \qquad \forall |\mathbf{x}| \le d$$

for some $a_1 > 0, a_2 > 0, d > 0$, such that

$$\begin{array}{c}\dfrac{\partial \mathbf{V}}{\partial \mathbf{x}}\,[\mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\bar u] - \gamma^2 \bar u^2 + \delta^2[\mathbf{C}(w,\mathbf{x})]^2 \le -a|\mathbf{x}|^2 \\ \\ \forall (w,\mathbf{x},\bar u) \in W \times \mathbf{V}^{-1}([0,c]) \times R\end{array} \tag{4.22}$$

for some $a > 0$, with $c > 0$ such that

$$\mathbf{X} \in \mathbf{V}^{-1}([0,c])\,.$$

If $\gamma \le \delta$ in the inequality above, trivially

$$\frac{\partial \mathbf{V}}{\partial \mathbf{x}}\,[\mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\mathbf{C}(w,\mathbf{x})] \le -a|\mathbf{x}|^2 \qquad \forall (w,\mathbf{x}) \in W \times \mathbf{V}^{-1}([0,c])\,.$$

Since system

$$\dot{\mathbf{x}} = \mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\mathbf{C}(w,\mathbf{x})$$

is precisely system (4.15), we conclude that if the inequality above holds for $\gamma \le \delta$, the set $\mathcal{A} = W \times \mathbf{0}$ is a locally exponentially stable invariant set of (4.15), with a domain of attraction that contains the set $W \times \mathbf{X}$ of all admissible initial conditions, as sought.

We have in this way shown that, if it is possible to use the degrees of freedom in the parameters of the controller so as to enforce a *dissipation inequality* of the form (4.22) for some $\gamma \le \delta$, system (4.15) has the desired asymptotic properties and the proposed controller solves the problem of output regulation. In the next sections we will see how this can be achieved.

4.5 The asymptotic properties of the core subsystem

We concentrate now on the issue of enforcing the dissipation inequality (4.22) for some $\gamma \leq \delta$ on system (4.17). The latter is the cascade of two subsystems: the "inner loop", consisting of (4.18) and (4.19), and the "filter" (4.20). An obvious prerequisite of (4.22) is the local exponential stability, with appropriate regions of attraction, of both subsystems of this cascade. Stability of the filter is not an issue, as we will show below.

4.5.1 Stability of the filter

For convenience, assume that $F(x)$ and $\gamma(x)$ have the form (4.9), and consider initially the simpler case in which the function $\phi(\cdot)$ in (4.8) is a linear function. In this case, $F(x)$ is a linear function, which we write as

$$F(x) = F_0 x = (\Phi - G_0 H)x$$

where

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0 & -s_1 & 0 & \cdots & -s_{d-1} \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The pair (Φ, H) is an observable pair and therefore one can always find a vector G_0 assigning to F_0 any prescribed set of eigenvalues. Note also that, if the spectra of F_0 and Φ are disjoint, any such G_0 makes the pair (F_0, G_0) a controllable pair.

The equations (4.20) describing the filter reduce to linear equations

$$\begin{aligned} \dot{\tilde{x}} &= F_0 \tilde{x} - G_0 y_{\mathrm{a}} \\ \bar{y} &= H \tilde{x} \end{aligned} \tag{4.23}$$

in which F_0 is a Hurwitz matrix and (F_0, G_0) is controllable. The degrees of freedom in this structure are the entries of G_0, which are subject only to the condition that F_0 is Hurwitz. Alternatively, one may consider changing system (4.23), via state-space isomorphism, into a system

$$\begin{aligned} \dot{\hat{x}} &= F \hat{x} - G y_{\mathrm{a}} \\ \bar{y} &= \Psi \hat{x} \end{aligned} \tag{4.24}$$

in which

$$F = T F_0 T^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & 0 & \cdots & -c_{d-1} \end{pmatrix}, \qquad G = T G_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix},$$

and

$$\Psi = H T^{-1} = \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 & \cdots & \psi_{d-1} \end{pmatrix}.$$

In this case, the degrees of freedom are the entries $c_0, \ldots, c_{d-1}$ of F, subject to the condition that the latter is Hurwitz. Note that the matrix Ψ necessarily satisfies

$$T \Phi T^{-1} = F + G\Psi$$

and hence is the unique matrix which assigns to $F + G\Psi$ the eigenvalues of Φ.

If the function $\phi(\cdot)$ in (4.8) is nonlinear, one can proceed as follows. Let $\phi_c(\cdot)$ be any function which agrees with $\phi(\cdot)$ on $\tau(W)$ and is globally Lipschitz (in which case a condition identical to (4.8) continues to hold with $\phi_c(\cdot)$ replacing $\phi(\cdot)$). Then, proceeding as in [2] (see also [7]), choose

$$G_0 = D_\kappa \bar{G}$$

in which

$$\begin{aligned} D_\kappa &= \mathrm{diag}(\kappa, \kappa^2, \ldots, \kappa^d) \\ \bar{G} &= \mathrm{col}(c_{d-1}, c_{d-2}, \ldots, c_0) \end{aligned}$$

with $c_{d-1}, \ldots, c_0$ coefficients of a Hurwitz polynomial

$$p(\lambda) = c_0 + c_1\lambda + \cdots + c_{d-1}\lambda^{d-1} + \lambda^d .$$

Set $\bar{x} = D_\kappa^{-1}\tilde{x}$ and observe that

$$\dot{\bar{x}} = \kappa\bar{A}\bar{x} - \bar{B}\Delta(\bar{x}, \tau(w), \kappa) - \bar{G}y_{\mathrm{a}}$$

in which $\bar{A}$ is a Hurwitz matrix (only determined by the choice of $c_{d-1}, \ldots, c_0$),

$$\bar{B} = \mathrm{col}(0, 0, \ldots, 0, 1) ,$$

and

$$\Delta(\bar{x}, \tau(w), \kappa) = \frac{1}{\kappa^d}[\phi_c(D_k\bar{x} + \tau(w)) - \phi_c(\tau(w))]$$

Since $\phi_c(\cdot)$ is globally Lipschitz, with a Lipschitz constant L, for any $\kappa > 1$ we have

$$|\Delta(\bar{x}, \tau(w), \kappa)| \leq \frac{1}{\kappa^d}L|D_\kappa\bar{x}| \leq L|\bar{x}| .$$

Let $\bar{P}$ be a positive definite solution of the Lyapunov equation $\bar{P}\bar{A} + \bar{A}^{\mathrm{T}}\bar{P} = -I$ and set $\bar{V}(\bar{x}) = \bar{x}^{\mathrm{T}}\bar{P}\bar{x}$. Then,

$$\dot{\bar{V}} \leq -\kappa|\bar{x}|^2 + 2|\bar{P}|\,L\,|\bar{x}|^2 + 2|\bar{P}\bar{G}|\,|\bar{x}|\,|y_{\mathrm{a}}| \leq -(\kappa - 2|\bar{P}|L - |\bar{P}\bar{G}|^2)|\bar{x}|^2 + |y_{\mathrm{a}}|^2 .$$

From this, it is seen that, if κ is large enough,

$$\dot{\bar{V}} \leq -d|\bar{x}|^2 + |y_{\mathrm{a}}|^2 \tag{4.25}$$

for some $d < 0$, i.e. the filter is globally input-to-state stable, actually with a linear gain function.

4.5.2 Design for minimum-phase plants

The simplest situation in which the design paradigm outlined above can be successfully implemented is the case in which the controlled plant satisfies the following assumptions:

- the function $f(w, z, \xi, \zeta)$ in (4.3) does not depend on $\xi_2, \xi_3, \ldots, \xi_{r-1}, \zeta$, in which case this function will be rewritten as $f(w, z, C\xi)$ and, accordingly, the function $f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta)$ will be rewritten as $f_{\mathrm{a}}(w, z_{\mathrm{a}}, C\xi)$,

- there exists a smooth positive definite and proper function $V(z_{\mathrm{a}})$, with quadratic bounds for small $|z_{\mathrm{a}}|$, satisfying
$$\frac{\partial V}{\partial z_{\mathrm{a}}} f_{\mathrm{a}}(w, z_{\mathrm{a}}, 0) \leq -\alpha(|z_{\mathrm{a}}|)$$
some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ which is quadratic for small values of the argument.

A plant which satisfies this assumption is said to be globally asymptotically and locally exponentially *minimum phase* (see e.g. [1]).

With this in mind, set $M = 0$, $N = 0$ and let $\dot{\chi} = L(\chi)$ any arbitrary globally stable system. In this case system (4.17) reduces to

$$\begin{aligned}
\dot{w} &= s(w) \\
\dot{z_{\rm a}} &= f_{\rm a}(w, z_{\rm a}, C\xi) \\
\dot{\xi} &= (A + BH)\xi \\
\dot{\chi} &= L(\chi) \\
\dot{\tilde{x}} &= F(\tilde{x} + \tau(w)) - F(\tau(w)) - G_0 h_{\rm a}(w, z_{\rm a}, \xi, H\xi) \\
\bar{y} &= \gamma(\tilde{x} + \tau(w)) - \gamma(\tau(w))
\end{aligned} \tag{4.26}$$

In this system, the input $\bar{u}$ is no longer present. Thus, it suffices to fulfill the dissipation inequality (4.22) for $\gamma = \delta = 0$. Neglecting the dynamics of χ, which is totally decoupled, and letting $\mathbf{x} = (z_{\rm a}, \xi, \tilde{x})$, we seek a positive definite and proper function $\mathbf{V}(\mathbf{x})$, with quadratic bounds for small $|\mathbf{x}|$, satisfying

$$\dot{\mathbf{V}}(w, \mathbf{x}) \le -a|\mathbf{x}|^2 \qquad \forall (w, \mathbf{x}) \in W \times \mathbf{V}^{-1}([0, c])\,, \tag{4.27}$$

where $c > 0$ is such that $\mathbf{X} \in \mathbf{V}^{-1}([0, c])$. If the latter holds, the proposed controller solves the problem of output regulation.

The only free parameters left in the design of the controller are the coefficients of the vector H. Let $\mathbf{Z}_{\rm a} \times \mathbf{\Xi}$ be a fixed compact set of initial conditions for $(z_{\rm a}, \xi)$. It is well known that (see e.g. [11]), under the assumptions above, it is always possible to find a vector H and a positive definite matrix P such that the positive definite and proper function

$$U(z_{\rm a}, \xi) = V(z_{\rm a}) + \xi^{\rm T} P \xi,$$

satisfies

$$\frac{\partial U}{\partial z_{\rm a}} f_{\rm a}(w, z_{\rm a}, C\xi) + \frac{\partial U}{\partial \xi}(A + BH)\xi \le -a'(|z_{\rm a}|^2 + |\xi|^2)\,, \qquad \forall (w, z_{\rm a}, \xi) \in W \times U^{-1}([0, c+1])$$

for some $a > 0$, where c is such that

$$\mathbf{Z}_{\rm a} \times \mathbf{\Xi} \subset U^{-1}([0, c])\,.$$

From this, using (4.25), the estimate

$$|y_{\rm a}|^2 = |h_{\rm a}(w, z_{\rm a}, \xi, H\xi)|^2 \le a''(|z_{\rm a}|^2 + |\xi|^2), \qquad \forall (w, z_{\rm a}, \xi) \in W \times U^{-1}([0, c+1])$$

and standard arguments, it is easily seen that there exists a number $\lambda > 0$ such that the function

$$\mathbf{V}(z_{\rm a}, \xi, \tilde{x}) = U(z_{\rm a}, \xi) + \lambda \bar{V}(D_\kappa \tilde{x})$$

satisfies an inequality of the form (4.27).

4.5.3 Taking advantage of the degrees of freedom in the filter

In the case of minimum-phase plants, the only property expected, in the design of the filter, is the property of global input-to-state stability and this, as shown above, can always be achieved. If the plant is not minimum phase, a more elaborate analysis is needed, which reposes – among other things – on the exploitation of the degrees of freedom left in the design of the filter. In fact, if the plant is non-minimum phase, a nontrivial controller (4.19) is needed to stabilize the auxiliary plant (4.18), and – because of this – system (4.15) is not anymore the simple cascade-connection of two stable subsystems. Rather, system (4.15) has now the feedback structure depicted in Fig. 4.1: system (4.17) controlled by $\bar{u} = \bar{y}$. In this setting, it is therefore reasonable to exploit the degrees of freedom left in the design of the filter to the purpose of lowering as much as possible the value of the gain of the system (4.17) between input $\bar{u}$ and output $\bar{y}$.

To better understand what the problem is about, it is useful to consider –as a preliminary example– the case in which the controlled plant is a linear system, and the exogenous inputs (to be followed and/or rejected) are sinusoidal functions of time. Accordingly, the auxiliary plant (4.18) is a linear system, modelled as

$$\begin{aligned} \dot{\bar{z}}_{\mathrm{a}} &= A_{\mathrm{a}}\bar{z}_{\mathrm{a}} + B_{\mathrm{a}}u_{\mathrm{a}} \\ y_{\mathrm{a}} &= C_{\mathrm{a}}\bar{z}_{\mathrm{a}} + D_{\mathrm{a}}u_{\mathrm{a}} + \Delta_{\mathrm{a}}v_{\mathrm{a}}\,, \end{aligned} \tag{4.28}$$

in which we have set $\bar{z}_{\mathrm{a}} = \mathrm{col}(z_{\mathrm{a}}, \xi)$. The matrices $A_{\mathrm{a}}, B_{\mathrm{a}}, C_{\mathrm{a}}, D_{\mathrm{a}}, \Delta_{\mathrm{a}}$ are, possibly, continuous functions of a vector μ of uncertain parameters, ranging on a compact set. Also the filter (4.20) is a linear system, as in (4.24), with parameters chosen in such a way that $F + G\Psi$ has purely imaginary eigenvalues at $\pm\, i\,\Omega_k$, $k = 1, \ldots, p$ (the Ω_i's being the frequencies of the exogenous inputs).

If a linear controller

$$\begin{aligned} \dot{\chi} &= L\chi + M[\bar{u} + y_{\mathrm{a}}] \\ u_{\mathrm{a}} &= N\chi \\ v_{\mathrm{a}} &= NL\chi \end{aligned} \tag{4.29}$$

is chosen, the composition of (4.28) and (4.29) is a linear system with input $\bar{u}$ and output y_{a}, characterized by a transfer function $T(s)$. The filter, on the other hand, is a linear system with input y_{a} and output $\bar{y}$, characterized by a transfer function $\Phi(s)$. Thus, in this case, system (4.17) is a linear system with transfer function $W(s) = \Phi(s)T(s)$. In view of the discussion above, the goal is to choose the parameters of the filter and the controller (4.29) in such a way that the composition of (4.28) and (4.29) is a *stable* linear system and

$$\|\Phi(s)T(s)\|_\infty < 1\,. \tag{4.30}$$

As observed above, the vector Ψ in (4.24) is necessarily the unique vector which assigns to $F + \Psi G$ the characteristic polynomial $q(s) = \prod_{k=1}^{p}(s^2 + \Omega_k^2)$. Since the characteristic polynomial of $F + \Psi G$ is precisely the numerator polynomial of the transfer function of

$$\begin{aligned} \dot{x} &= Fx - Gu \\ y &= \Psi x + u \end{aligned}$$

we see that

$$\frac{\prod_{k=1}^{p}(s^2 + \Omega_k^2)}{\det(sI - F)} = 1 - \Psi(sI - F)^{-1}G = 1 + \Phi(s)\,.$$

Therefore, necessarily, $\Phi(\pm\, i\,\Omega_k) = -1$ regardless of how F is chosen. This may seem disappointing, because it implies $\|\Phi(s)\|_\infty \geq 1$. In particular, this shows that (4.30) cannot be approached by trying to enforce the conservative estimate

$$\|\Phi(s)T(s)\|_\infty \leq \|\Phi(s)\|_\infty \|T(s)\|_\infty < 1\,,$$

because it is very unlikely that $\|T(s)\|_\infty$ can be made less than 1. Consider, in this respect, the case in which A_{a} has a pole at the origin and $\Delta_{\mathrm{a}} = 0$, and observe that necessarily $T(0) = 1$ (yielding $\|T(s)\|_\infty \geq 1$) no matter how the controller is chosen.

Despite of the fact that the magnitude of $\Phi(\pm i\Omega_k)$ is necessarily 1, a clever choice of F may still be sought to the purpose of lowering the magnitude of $\Phi(i\omega)$ at frequencies other than Ω_k, in view of the fact that – after all – it is the H_∞ norm of the *product* $\Phi(s)T(s)$ that matters in the basic condition (4.30).

It is immediate to see that, if the characteristic polynomial of F is chosen as

$$d_0(s) = \prod_{k=1}^{p} (s + \Omega_k)^2 \,, \tag{4.31}$$

the transfer function $\Phi(s)$ of the filter (4.24) has a zero at the origin and, consequently the first entry ψ_0 of Ψ is 0. The fact that $\Phi(s)$ is zero at $s = 0$ may alleviate the task of fulfilling the inequality (4.30). In fact, since $\psi_0 = 0$, the first component $\hat{x}_1$ of the state $\hat{x}$ of the filter (4.24) can be replaced

$$\bar{x}_1 = -c_0\hat{x} - y_{\rm a} \,,$$

yielding equations of the form

$$\begin{array}{rcl} \dot{\bar{x}} & = & \bar{F}\bar{x} - \bar{G}\dot{y}_{\rm a} \\ \bar{y} & = & \Psi\bar{x} \,, \end{array} \tag{4.32}$$

in which the output map has remained the same. Having done this change, system (4.17) can be seen as the cascade of a linear system having transfer function $sT(s)$ and of a (modified) filter having transfer function $Q(s) = -\Psi(sI - \bar{F})^{-1}\bar{G}$. Accordingly, the bound (4.30) becomes

$$\|\Upsilon(s)sT(s)\|_\infty < 1 \,.$$

The advantage of this different viewpoint is that now the (still conservative) estimate

$$\|\Upsilon(s)sT(s)\|_\infty \le \|\Upsilon(s)\|_\infty \|sT(s)\|_\infty < 1 \,,$$

can be more easily handled. As a matter of fact, it is possible to show (see [5]) that

$$\|\Upsilon(s)\|_\infty \le \sum_{k=1}^{p} \frac{2}{\Omega_k} \,, \tag{4.33}$$

and hence the basic design goal is achieved if

$$\|sT(s)\|_\infty \le \Big(\sum_{k=1}^{p} \frac{2}{\Omega_k}\Big)^{-1} . \tag{4.34}$$

We discuss in the next section cases in which the inequality (4.34) can be enforced.

4.5.4 Examples of design for non-minimum-phase plants

As a first example of a non-minimum phase plant to which the design procedure outlined in the previous section can be successfully applied consider again the case in which the controlled plant is a linear system, and the exogenous inputs are sinusoidal functions of time. In this case, the auxiliary plant is an $n_{\rm a}$-dimensional linear system, as in (4.28), whose coefficient matrices are continuous functions of a vector μ of uncertain parameters, ranging on a *compact* set. Suppose that the matrix $A_{\rm a}$ has a fixed number $m \ge 1$ of eigenvalues at the origin, while the remaining $n_{\rm a} - m$ eigenvalues have negative real part. This case can be handled as follows. Set $u_{\rm a} = \chi_0$ with χ_0 generated by a system

$$\dot{\chi}_0 \;=\; -\chi_0 + u'_{\rm a} \tag{4.35}$$

in which case, trivially, $v_{\rm a} = \dot{u}_{\rm a} = -\chi_0 + u'_{\rm a}$. Composing with (4.28) yields a system with input $u'_{\rm a}$ and output $y_{\rm a}$ modelled by

$$\begin{array}{rcl} \dot{z}_{\rm a} & = & A_{\rm a}z_{\rm a} + B_{\rm a}\chi_0 \\ \dot{\chi}_0 & = & -\chi_0 + u'_{\rm a} \\ y_{\rm a} & = & C_{\rm a}z_{\rm a} + [D_{\rm a} - \Delta_{\rm a}]\chi_0 + \Delta_{\rm a}u'_{\rm a} \,. \end{array} \tag{4.36}$$

This is a $(n_a + 1)$-dimensional linear system, having m eigenvalues at the origin and $n_a - m + 1$ eigenvalues with negative real part, for every value of μ. Recall that

$$\Delta_a = \beta - 1/b$$

in which b is a positive coefficient, possibly (continuously) dependent on μ, bounded from below and from above. Indeed, β can be chosen in such a way that $\Delta_a > 0$ for all values of μ. In this case (4.36) has relative degree zero between input u'_a and output y_a, and a high-frequency gain Δ_a which is positive for every μ. As a result, the system has $n_a + 1$ zeros for every value of μ which range, in view of the continuity hypotheses, on some compact set. Let $P'_a(s)$ denote the transfer function of (4.36).

Suppose now that the control u' to (4.36) is provided by another controller, written as

$$\begin{array}{rcl} \dot{\chi}_1 & = & L_1\chi_1 + N_1[\bar{u} + y_a] \\ u' & = & N_1\chi_1\,. \end{array} \tag{4.37}$$

This results in a system, with input $\bar{u}$ and output y_a, characterized by a transfer function of the form

$$T(s) = \frac{R(s)P'_a(s)}{1 - R(s)P'_a(s)} \tag{4.38}$$

in which

$$R(s) = N_1(sI - L_1)^{-1}M_1\,.$$

If we succeed in choosing L_1, M_1, N_1 in such a way that the H_∞ norm of $sT(s)$ satisfies the bound (4.34), the problem is solved, with (4.29) being the cascade of (4.35) and (4.37), namely

$$\begin{array}{rcl} \dot{\chi}_1 & = & L_1\chi_1 + N_1[\bar{u} + y_a] \\ \dot{\chi}_0 & = & -\chi_0 + N_1\chi_1 \\ u_a & = & \chi_0 \\ v_a & = & -\chi_0 + N_1\chi_1\,. \end{array}$$

Enforcing (4.34), in this case, is not difficult at all. In fact, this depends on the following Lemma.

Lemma 4.1 *Let $N_0(s)$ and $D_0(s)$ be polynomial of fixed degrees, whose coefficients are continuous functions of a vector μ of uncertain parameters, ranging on a compact set. Let $D_0(s)$ be monic and Hurwitz for every value of μ and let*

$$\deg[D_0] \geq \deg[N_0]\,.$$

Set, for $k = 1, \ldots,$

$$\begin{array}{rcl} N_1(s) & = & gN_0(s) \\ N_k(s) & = & (s + z_{k-1})N_{k-1}(s) \\ D_1(s) & = & sD_0(s) \\ D_k(s) & = & sD_{k-1}(s) \end{array}$$

and

$$P_k(s) = D_k(s) + N_k(s) \qquad T_k(s) = \frac{N_k(s)}{D_k(s) + N_k(s)}\,.$$

Let $\gamma > 0$ be fixed. For any choice of $a > 1$ there is a choice of real numbers $g, z_1, z_2, \ldots, z_k$ such that $P_{k+1}(s)$ is Hurwitz, $T_{k+1}(0) = 1$ and

$$\|sT_{k+1}(s)\|_\infty \leq a^k\gamma\,.$$

The proof of this Lemma requires elementary arguments and is not included here. It can be found in [5]. In the context above, the Lemma is used as follows. Let $N_a(s)$ and $s^m D_a(s)$ be the numerator and, respectively, monic denominator of $P'_a(s)$ (recall, to this end, that $P'_a(s)$ has m poles at the origin). Set

$$N_0(s) = N_a(s), \qquad D_0(s) = (s + 1)^m D_a(s)\,.$$

The polynomials thus defined meet the assumptions of the Lemma. Thus, using the result of the Lemma for $k = m$, it is immediately seen that, for any $\gamma > 0$ and $a > 0$ there is a choice of parameters $g, z_1, \dots, z_{m-1}$, such that, choosing the transfer function of (4.37) as

$$R(s) = -g\frac{(s + z_{m-1}) \dots (s + z_1)}{(s + 1)^m},$$

one obtains

$$\|sT(s)\|_\infty = \|s\,\frac{R(s)P'_\mathrm{a}(s)}{1 - R(s)P'_\mathrm{a}(s)}\|_\infty = \|sT_m(s)\|_\infty \le a^{m-1}\gamma\,.$$

Since a and γ are arbitrary, this shows that the inequality (4.34) can always be enforced.

The peculiar feature of this example was the possibility of finding a controller which does stabilize the auxiliary plant (which is unstable) while, at the same time, keeping an arbitrarily low gain between $\bar{u}$ and the output $\dot{y}_\mathrm{a}$. It is precisely in this respect that the idea of replacing y_a by $\dot{y}_\mathrm{a}$ has helped. This feature may still be exploited for certain classes of nonlinear systems. In [4], for instance, it was shown that the same design result can be achieved in the special case of a nonlinear auxiliary plant in feed-forward form

$$\begin{aligned} \dot{z}_{\mathrm{a}1} &= p(z_{\mathrm{a}1}, z_{\mathrm{a}2}, w)z_{\mathrm{a}2} \\ \dot{z}_{\mathrm{a}2} &= u_\mathrm{a} \\ y_\mathrm{a} &= z_{\mathrm{a}1} \end{aligned} \tag{4.39}$$

in which $p(z_{\mathrm{a}1}, z_{\mathrm{a}2}, w)$ is a continuous function bounded as

$$0 < p_0 \le p(z_{\mathrm{a}1}, z_{\mathrm{a}2}, w) \le p_1\,.$$

As a second example, we consider a case in which $r = 1$ and the zero dynamics of the controlled plant (4.3) are unstable. In order to keep the example elementary, a number of strong simplifying assumptions are made. To begin with, we assume that $\Delta_\mathrm{a} = 0$ and that $f_\mathrm{a}(\cdot)$ and $h_\mathrm{a}(\cdot)$ are affine in u_a, in which case the auxiliary plant (4.18) can be rewritten in the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_\mathrm{a} &= A(w, z_\mathrm{a}) + B(w, z_\mathrm{a})u_\mathrm{a} \\ y_\mathrm{a} &= C(w, z_\mathrm{a}) + D(w, z_\mathrm{a})u_\mathrm{a}\,. \end{aligned} \tag{4.40}$$

Setting, as before, $u_\mathrm{a} = \chi_0$ with χ_0 generated by (4.35), yields a composite system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_\mathrm{a} &= A(w, z_\mathrm{a}) + B(w, z_\mathrm{a})\chi_0 \\ \dot{\chi} &= -\chi_0 + u'_\mathrm{a} \\ y_\mathrm{a} &= C(w, z_\mathrm{a}) + D(w, z_\mathrm{a})\chi_0\,. \end{aligned}$$

If $D(w, z_\mathrm{a})$ is nowhere zero, this system has uniform relative degree 1 between input u'_a and output y_a and can be rewritten in normal form as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_\mathrm{a} &= \hat{A}(w, z_\mathrm{a}) + \hat{B}(w, z_\mathrm{a})\xi \\ \dot{\xi} &= \hat{C}(w, z_\mathrm{a}) + \hat{D}(w, z_\mathrm{a})\xi + D(w, z_\mathrm{a})u'_\mathrm{a} \\ y_\mathrm{a} &= \xi\,, \end{aligned}$$

in which $\hat{A}(w, z_\mathrm{a}), \hat{B}(w, z_\mathrm{a}), \hat{C}(w, z_\mathrm{a}), \hat{D}(w, z_\mathrm{a})$ are appropriate functions. Next, we seek conditions under which the simplest possible control u'_a, namely

$$u'_\mathrm{a} = K[\xi + \bar{u}]$$

yields the required asymptotic properties (if K is appropriately chosen). The easiest case in which this occurs is when there exists a function $V(z_a)$ satisfying, for some choice of positive numbers a_1, a_2, a_3,

$$a_1|z_a|^2 \leq V(z_a) \leq a_2|z_a|^2, \qquad |\frac{\partial V}{\partial z_a}| \leq a_3|z_a|, \qquad \forall z_a \in \mathbb{R}^m$$

such that, for some positive a_4,

$$\frac{\partial V}{\partial z_a}\hat{A}(w, z_a) \leq -a_4|z_a|^2, \qquad \forall (w, z)_a) \in W \times \mathbb{R}^m .$$

If this is the case and, in addition, there are positive numbers $b_0, c_0, d_0, \delta_0, \delta_1$ such that

$$|\hat{B}(w, z_a)| \leq b_0, \quad |\hat{C}(w, z_a)| \leq c_0|z_a|, \quad |\hat{D}(w, z_a)| \leq d_0, \quad \delta_0 \leq |D(w, z_a)| \leq \delta_1,$$
$$\forall (w, z)_a) \in W \times \mathbb{R}^m ,$$

standard arguments show that the closed loop system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z}_a & = & \hat{A}(w, z_a) + \hat{B}(w, z_a)\xi \\ \dot{\xi} & = & \hat{C}(w, z_a) + \hat{D}(w, z_a)\xi + D(w, z_a)K[\xi + \bar{u}] , \end{array}$$

if $D(w, z_a)K < 0$ and $|K|$ is large enough, can be forced to satisfy a dissipation inequality of the form

$$\frac{\partial V}{\partial z_a}\dot{z}_a + \frac{\partial \xi^2}{\partial \xi}\dot{\xi} - K^2\bar{u}^2 \leq -a(|z_a|^2 + \xi^2) ,$$

for some $a > 0$. As a consequence, the same system, with output

$$\dot{y}_a = \hat{C}(w, z_a) + \hat{D}(w, z_a)\xi + D(w, z_a)K[\xi + \bar{u}]$$

satisfies a dissipation inequality of the form

$$\frac{\partial V}{\partial z_a}\dot{z}_a + \frac{\partial \xi^2}{\partial \xi}\dot{\xi} - K^2\bar{u}^2 + \delta^2[\dot{y}_a]^2 \leq -\frac{a}{2}(|z_a|^2 + \xi^2) ,$$

for some small $\delta > 0$. This being the case, from the previous arguments it follows that if the frequencies of the exogenous input satisfy

$$\frac{K}{\delta} \leq \Big(\sum_{k=1}^{p} \frac{2}{\Omega_k}\Big)^{-1}$$

the proposed controller solves the problem of output regulation.

Remark 4.3 It is worth stressing that, while we have implicitly assumed that the zero dynamics of the auxiliary plant (4.40) are globally asymptotically stable, the zero dynamics of the original plant (4.3) may well be unstable. In fact, while the zero dynamics of the original plant in the setting of the previous example are those of

$$\dot{z}_a = A(w, z_a) ,$$

the zero dynamics of the auxiliary plant are those of

$$\dot{z}_a = A(w, z_a) - \frac{B(w, z_a)C(w, z_a)}{D(w, z_a)} .$$

It is precisely in the case of a plant having unstable zero dynamics that this (more elaborate) design procedure may help. ◁

Remark 4.4 We have shown that the method is applicable if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value determined by the gain needed to solve an auxiliary stabilization problem. In other words, the minimal gain needed to stabilize the unstable zero dynamics of the original plant determines a lower limit on the frequencies of the exogenous inputs for which the desired steady state *performance* can be achieved. ◁

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