Optimal Control

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Lecture 1

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Schedule
Tuersday : 10:00-12:00 (A6)
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Subject: Optimal Control 2018

Grading
Project+ oral exam

Example of project (1-3 Students):
-Read a paper on an optimal control problem
-Study: background, motivations, model, optimal control, solution, results
-Simulations
-Conclusions
-References

You must give me, **before the date of the exam**:
-A .doc document
-A power point presentation
-Matlab simulation files

Oral exam: Discussion of the project AND on the topics of the lectures

Some projects proposed in 2014-15, 2015-16, 2016-17- 2017-18

Application Of Optimal Control To Malaria: Strategies And Simulations

Performance Compare Between Lqr And Pid Control Of Dc Motor

Optimal Low-thrust Leo (Low-earth Orbit) To Geo (Geosynchronous-earth Orbit) Circular Orbit Transfer

Controllo Ottimo Di Una Turbina Eolica A Velocità Variabile Attraverso Il Metodo dell'inseguimento Ottimo A Regime Permanente

Optimalcontrol In Dielectrophoresis

On The Design Of P.I.D. Controllers Using Optimal Linear Regulator Theory

Rocket Railroad Car

Optimal Control Of Quadrotor Altitude Using Linear Quadratic Regulator

Optimal Control Of An Inverted Pendulum

Glucose Optimal Control System In Diabetes Treatment ....

Some projects proposed in 2014-15, 2015-16, 2016-17- 2017-18

Optimal Control Of Shell And Tube Heat Exchanger

Optimal Control Analysis Of A Mathematical Model For Unemployment

Time optimal control of an automatic Cableway

Glucose Optimal Control System In Diabetes Treatment

Optimal Control Of Shell And Tube Heat Exchanger

Optimal Control Analysis Of A Mathematical Model For Unemployment

Time Optimal Control Of An Automatic Cableway

Optimal Control Project On Arduino Managed Module For Automatic Ventilation Of Vehicle Interiors

Optimal Control For A Suspension Of A Quarter Car Model

………

THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

References

B.D.O.Anderson, J.B.Moore, *Optimal control*, Prentice Hall, 1989

E.R.Pinch, Optimal control and the calculus of variations, Oxford science publications, 1993

C.Bruni, G. Di Pillo, Metodi Variazionali per il controllo ottimo, Masson , 1993

A.Calogero, *Notes on optimal control theory*, 2017

L. Evans, *An introduction to mathematical optimal control theory*, 1983

H.Kwakernaak , R.Sivan, *Linear Optimal Control Systems*, Wiley Interscience, 1972

D. E. Kirk, "Optimal Control Theory: An Introduction, New York, NY: Dover, 2004

D. Liberzon, "Calculus of Variations and Optimal Control Theory: A Concise Introduction", Princeton University Press, 2011

How, Jonathan, Principles of optimal control, Spring 2008. (MIT OpenCourseWare: Massachusetts Institute of Technology). License: Creative Commons BY-NC-SA.

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Course outline

- Introduction to optimal control
- Nonlinear optimization
- Dynamic programming
- Calculus of variations
- Calculus of variations and optimal control
- LQ problem
- Minimum time problem

Background

First course on linear systems

(free evolution, transition matrix, gramian matrix,…)

Notations

$x(t) \in R^n$ State variable

$u(t) \in R^p$ Control variable

$f : R^n \times R^p \times R :\rightarrow R$

Function $\overline{C}^{\,k}$ (function with k-th derivative continuous a.e.)

Introduction

Optimal control is one particular branch of modern control that sets out to provide analytical designs of a special appealing type.

The system, that is the end result of an optimal design, is supposed to be **the best possible system of a particular type**

A cost index is introduced

Introduction

Linear optimal control is a special sort of optimal control:

- ✓ the plant that is controlled is assumed linear
- ✓ the controller is constrained to be linear

Linear controllers are achieved by working with **quadratic cost indices**

Introduction

Advantages of linear optimal control

- ✓ Linear optimal control may be applied to nonlinear systems
- ✓ Nearly all linear optimal control problems have computational solutions
- ✓ The computational procedures required for linear optimal design may often be carried over to nonlinear optimal problems

VARIATIONAL METHODS

- Early Greeks – Max. area/perimeter
- Hero of Alexandria – Equal angles of incidence /reflection
- Fermat - least time principle (Early 17th Century)
- Newton and Leibniz – Calculus (Mid 17th Century)
- **Johann Bernoulli - brachistochrone problem (1696)**
- Euler - calculus of variations (1744)
- Joseph-Louis Lagrange – Euler-Lagrange Equations (??)
- William Hamilton – Hamilton's Principle (1835)
- Raleigh-Ritz method – VA for linear eigenvalue problems (late 19th Century)
- Quantum Mechanics - Computational methods – (early 20th Century)
- Morse & Feshbach – technology of variational methods (1953)
- Solid State Physics, Chemistry, Engineering – (mid-late 20th Century)
- Personal computers – new computational power (1980's)
- Technology of variational methods essentially lost (1980-2000)
- D. Anderson – VA for perturbations of solitons (1979)
- Malomed, Kaup – VA for solitary wave solutions (1994 – present)

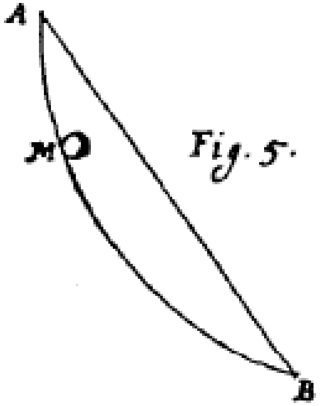
History

1696: THE BIRTH OF OPTIMAL CONTROL

Jan C. Willems
Department of Mathematics
University of Groningen

Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

History



In 1696 Bernoulli posed the

Brachistochrone problem to his contemporaries:

"*it seems to be the first problem which explicitely dealt with optimally controlling the path or the behaviour of a dynamical system*»

Suppose a particle of mass M moves from A to B along smooth wire joining the two fixed points (not on the same vertical line),under the influence of gravity.

What shape the wire should be in order that the bead, when released from rest at A should slide to B in the minimum time?

E.R.Pinch, Optimal control and calculus of variations, Oxford Science Publications, 1993

Motivations

Example 1 (Evans 1983)

Reproductive strategies in social insects

Let us consider the model describing how social insects (for example bees) interact:

$w(t)$ represents the **number of workers** at time t

$q(t)$ represents the **number of queens**

$u(t)$ represents the **fraction of colony** effort devoted to increasing work force

T **lenght of the season**

Death rate

Known rate at which each worker contributes to the bee economy

$$\dot{w}(t) = -\mu\, w(t) + b\, s(t) u(t) w(t)$$
$$w(0) = w_0$$

Evolution of the worker population

$$\dot{q}(t) = -\nu q(t) + c\big(1 - u(t)\big) s(t) w(t)$$
$$q(0) = q_0$$

Evolution of the Population of queens

$$0 \le u(t) \le 1$$

Constraint for the control

The bees goal is to find the control that maximizes the number of queens at time *T:*

$$J\big(u(t)\big) = q(T)$$

The solution is a *bang- bang control*

Motivations

Example 2 (Evans 1983…and everywhere!)   A moon lander

Aim: bring a spacecraft to a soft landing on the lunar surface, **using the least amount of fuel**

$h(t)$ represents the **height** at time t

$v(t)$ represents the **velocity =** $\dot{h}(t)$

$m(t)$ represents **the mass of spacecraft**

$u(t)$ represents **thrust at time t**

We assume $0 \le u(t) \le 1$

Consider the **Newton's law**: $m\ddot{h}(t) = -gm + u$

$$\dot{v}(t) = -g + \frac{u(t)}{m(t)}$$

$$\dot{h}(t) = v(t)$$

$$\dot{m}(t) = -ku(t)$$

$$h(t) \geq 0 \quad m(t) \geq 0$$

We want to **minimize** the amount of fuel

that is maximize the amount remaining once we have landed

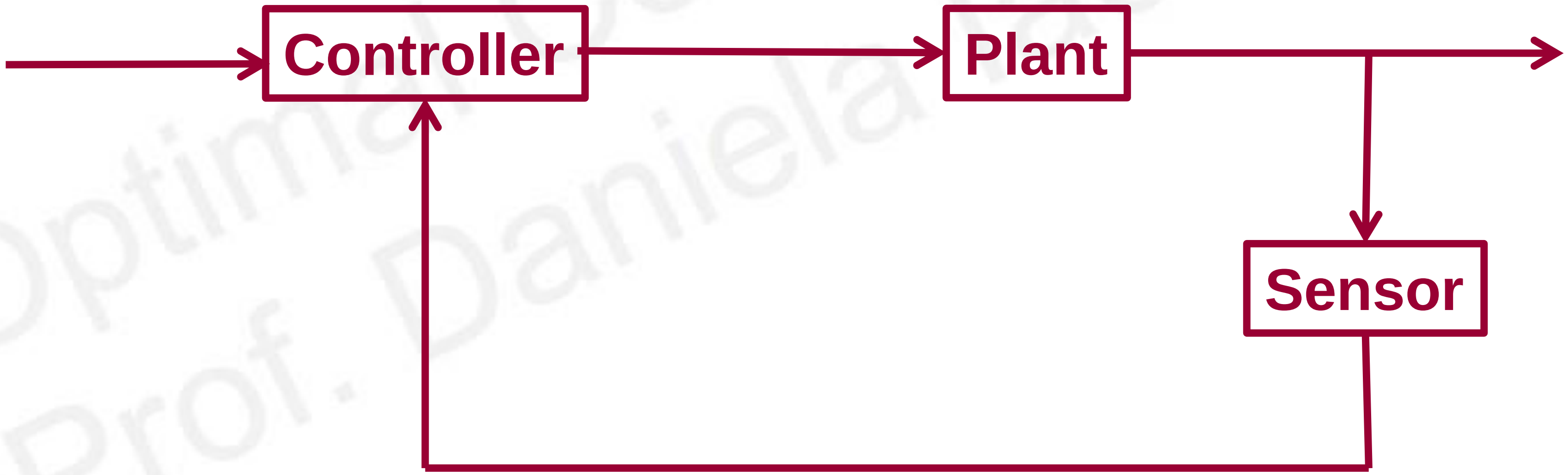
$$J(u(\cdot)) = m(\vartheta)$$

where ϑ is the first time in which $h(\vartheta) = 0 \;\; v(\vartheta) = 0$

Analysis of linear control systems

Essential components of a control system

- ✓ The plant
- ✓ One or more sensors
- ✓ The controller



Analysis of linear control systems

Feedback:

the actual operation of the control system *is compared to the desired operation* and the input to the plant is adjusted on the basis of this comparison.

Feedback control systems are able to operate satisfactorily despite adverse conditions, such as disturbances and variations in plant properties

Definitions

Consider a function $f : R^n :\to R$

and $D \subseteq R^n$

$|\bullet|$ denotes the Eucledian norm

A point $x^* \in D$ is a **local minimum** of f over $D \subseteq R^n$

If $\exists \ \varepsilon > 0 \ \ such \ that \ \ for \ \ all \ \ x \in D \ satisfying \left| x - x^* \right| < \varepsilon$

$\Rightarrow$

$$\boxed{f(x^*) \leq f(x)}$$

Definitions

Consider a function $f : R^n :\to R$

and $D \subseteq R^n$

$|\bullet|$ denotes the Eucledian norm

A point $x^* \in D$ is a **strict local minimum** of f over $D \subseteq R^n$

If $\exists\ \varepsilon > 0\ \ such\ that\ \ for\ \ all\ \ x \in D\ satisfying\ |x - x^*| < \varepsilon$

$\Rightarrow$

$$\boxed{f(x^*) < f(x) \quad \forall x \neq x^*}$$

Definitions

Consider a function $f : R^n :\to R$

and $D \subseteq R^n$

$|\bullet|$ denotes the Eucledian norm

A point $x^* \in D$ is a **global minimum** of f over $D \subseteq R^n$

If

$$for\ all\ \ x \in D$$
$$\Rightarrow$$
$$f(x^*) \leq f(x)$$

Definitions

The notions of a **local/strict/global maximum** are defined similarly

If a point is either a maximum or a minimum

is called an extremum

Unconstrained optimization - first order necessary conditions

All points x sufficiently near x^* in R^n are in D

Assume $f \in C^1$ **and** x^* **its local minimum.**
Let $\delta \in R^n$ be an arbitrary vector.

Being in the unconstrained case: $x^* + \alpha\delta \in D$

$\forall \alpha \in R \; close \; enough \; to \; 0$

Let's consider:

$$g(\alpha) := f(x^* + \alpha\delta)$$

⇒ **0 is a minimum of** g

Unconstrained optimization - first order necessary conditions

First order Taylor expansion of g around $\alpha = 0$

$$g(\alpha) = g(0) + g'(0)\alpha + o(\alpha), \qquad \lim_{\alpha \to 0} \frac{o(\alpha)}{\alpha} = 0$$

$\Rightarrow$ $g'(0) = 0$

Proof: assume $g'(0) \neq 0$ $\Rightarrow$ $\exists \;\; \varepsilon > 0 \;\; small \;\; enough \; so \;\; that$

$for \;\; |\alpha| < \varepsilon \qquad |o(\alpha)| < |g'(0)\alpha|$

For these values of α $\Rightarrow$ $g(\alpha) - g(0) < g'(0)\alpha + |g'(0)\alpha|$

If we restrict α to have the opposite sign of $g'(0)$

$g(\alpha) - g(0) < g'(0)\alpha + |g'(0)\alpha|$ $\Rightarrow$ $g(\alpha) - g(0) < 0$

Contraddiction $\Rightarrow$ $g'(0) = 0$

Unconstrained optimization - first order necessary conditions

$$g'(\alpha) = \nabla f(x^* + \alpha\delta) \cdot \delta \quad where\ \nabla f := \begin{pmatrix} f_{x_1} & \cdots & f_{x_n} \end{pmatrix}^T$$

$$is\ the\ gradient\ of\ \ f$$

$$g'(0) = \nabla f(x^*) \cdot \delta = 0$$

Δ is arbitrary

First order necessary condition for optimality

$$\nabla f(x^*) = 0$$

Unconstrained optimization - first order necessary conditions

A point x^* satisfying this condition is a **stationary point**

Unconstrained optimization – second order conditions

Assume $f \in C^2$ **and** x^* **its local minimum**. Let $\delta \in R^n$ be an arbitrary vector.

Second order Taylor expansion of g around $\alpha = 0$

$$g(\alpha) = g(0) + g'(0)\alpha + \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2), \quad \lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0$$

Since $g'(0) = 0$

$$\Rightarrow \quad \boxed{g''(0) \geq 0}$$

Unconstrained optimization – second order conditions

Proof: suppose $g''(0) < 0$

$$\exists\ \varepsilon > 0\ \ small\ \ enough\ so\ \ that$$

$$for\ \ |\alpha| < \varepsilon \qquad |o(\alpha^2)| < \frac{1}{2}|g''(0)|\alpha^2$$

For these values of α

$$g(\alpha) - g(0) < 0$$

Contraddiction $\quad g''(0) \geq 0$

Unconstrained optimization – second order conditions

$$g'(\alpha) = \sum_{i=1}^{n} f_{x_i}(x^* + \alpha\delta)\delta_i$$

By differentiating both sides with respect to α

$$g''(\alpha) = \sum_{i,j=1}^{n} f_{x_i x_j}(x^* + \alpha\delta)\delta_i\delta_j$$

$$\Rightarrow g''(0) = \sum_{i,j=1}^{n} f_{x_i x_j}(x^*)\delta_i\delta_j = \delta^T \nabla^2 f(x^*)\delta$$

$$\nabla^2 f = \begin{pmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_n} \end{pmatrix}$$

Hessian matrix

Second order necessary condition for optimality

$$\nabla^2 f(x^*) \geq 0$$

Unconstrained optimization – second order conditions

Remark:

The second order condition distinguishes minima from maxima:

At a local maximum the Hessian must be negative semidefinite

At a local minimum the Hessian must be positive semidefinite

Unconstrained optimization- second order conditions

Let $f \in C^2$ and $\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) > 0$

$\Rightarrow$ x^* is a **strict local minimum** of *f*

Global minimum – Existence result

Weierstrass Theorem

Let *f* be a **continuous function** and *D* a **compact set**

$\Rightarrow$ there exist a **global minimum** of *f* over *D*

Constrained optimization

Let $D \subset R^n, \quad f \in C^1$

Equality constraints $h(x) = 0, \;\; h : R^p \to R, \;\; h \in C^1$

Inequality constraints $g(x) \le 0, \;\; g : R^q \to R, \;\; g \in C^1$

Regularity condition: $$rank\left\{ \left. \frac{\partial (h, g_a)}{\partial x} \right|_{x^*} \right\} = p + q_a$$

where g_a are the active constraint of g with dimension q_a

Constrained optimization

Lagrangian function

$$L(x,\lambda_0,\lambda,\mu)=\lambda_0 f(x)+\lambda^T h(x)+\eta^T g(x)$$

If $\lambda_0 \neq 0$ the stationary point x^* is called **normal**

and we can assume $\lambda_0 = 1$

Constrained optimization

From now on $\lambda_0 = 1$ and therefore the Lagrangian is

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \eta^T g(x)$$

If there are only equality constraints the λ_i are called **Lagrange multipliers**

The inequality multipliers are called **Kuhn – Tucker multipliers**

Constrained optimization

First order **necessary conditions** for constrained optimality:

Let $x^* \in D$ and $f, h, g \in C^1$

The necessary conditions for x^* to be a constrained local minimum are

$$\left.\frac{\partial L}{\partial x}\right|_{x^*}^T = 0^T$$

$$\eta_i g_i(x^*) = 0, \ \forall i$$

$$\eta_i \geq 0 \quad \forall i$$

If the functions *f* and *g* are convex and the functions *h* are linear these conditions are **necessary and sufficient**

Constrained optimization

Second order sufficient conditions for constrained optimality:

Let $x^* \in D$ and $f,\ h,\ g \in C^2$ and assume the conditions

$$\left.\frac{\partial L}{\partial x}\right|_{x^*}^{T} = 0^T \quad \eta_i g_i(x^*) = 0,\ \eta_i \geq 0 \quad \forall i \quad \Rightarrow$$

x^* is a strict constrained local minimum **if**

$$\delta^T \left.\frac{\partial^2 L}{\partial x^2}\right|_{x^*} \delta > 0 \quad \forall \delta \ \ such \ \ that \ \ \left.\frac{dh_i(x)}{dx}\right|_{x^*} \cdot \delta = 0, \ \ i = 1, \ldots, p$$

Constrained optimization

Definition:

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{\partial h_j}{\partial x_i} \\ \left(\dfrac{\partial h_j}{\partial x_i}\right)^T & 0 \end{pmatrix}$$ Bordered Hessian

A point x^* in which $\nabla L = 0$ and $\det\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{\partial h_j}{\partial x_i} \\ \left(\dfrac{\partial h_j}{\partial x_i}\right)^T & 0 \end{pmatrix} \neq 0$ is called

a **non-degenerate critical point of the constrained problem.**

Constrained optimization

Theorem:

A necessary and sufficient condition for the non-degenerate critical point x^* to minimize the cost f subjects to the constraints $h_j(x^*) = 0, \quad j = 1, 2, ..., m$ is that $\delta^T \dfrac{\partial^2 L}{\partial x^2} \delta \geq 0$ for all non-zero tangent vector δ

E.R.Pinch, Optimal control and calculus of variations, Oxford Science Publications, 1993

A geometrical interpretation

Let us consider the problem of minimizing function $f(x_1, x_2)$

with one constraint $h(x_1, x_2) = c$

The previous theorem implies that we should look for λ and $\bar{x} = (\bar{x}_1, \bar{x}_2)$ such that:

$\Rightarrow$ $\nabla\left(f + \lambda h\right) = 0 \quad at \quad \bar{x}$ $\Rightarrow$ $\nabla f = -\lambda \nabla h \quad at \quad \bar{x}$

If $\nabla f \neq 0$ $\nabla h \neq 0$ $\lambda \neq 0$ $\Rightarrow$ the normal to the constraint curve $h(x_1, x_2) = c$ at $\bar{x} = (\bar{x}_1, \bar{x}_2)$ has its normal parallel to $\nabla f(\bar{x})$

$\Rightarrow$ the level surface of $f(x_1, x_2)$ that passes through $\bar{x} = (\bar{x}_1, \bar{x}_2)$ has the same normal direction as the constraint curve at $\bar{x} = (\bar{x}_1, \bar{x}_2)$

$\Rightarrow$ In R^2 this means that the two curves touch at $\bar{x} = (\bar{x}_1, \bar{x}_2)$

A geometrical interpretation - Example

Minimize the function

$$f(x_1, x_2) = 1 - x_1^2 - x_2^2$$

subject to

$$h(x_1, x_2) = x_2 - 1 + x_1^2 = 0$$

Introduce a Lagrange multiplier λ and consider:

$$\nabla(f + \lambda h) = \nabla\left(1 - x_1^2 - x_2^2 + \lambda(x_2 - 1 + x_1^2)\right) = 0$$

$$-2x_1 + \lambda x_1 = 0 \qquad -2x_2 + \lambda = 0$$

There are three unknowns and we need another equation: $x_2 - 1 + x_1^2 = 0$

$x_1 = 0,\ x_2 = 1,\ \lambda = 2$ and $x_1 = \pm 1/\sqrt{2},\ x_2 = 1/2,\ \lambda = 1$

A geometrical interpretation - Example

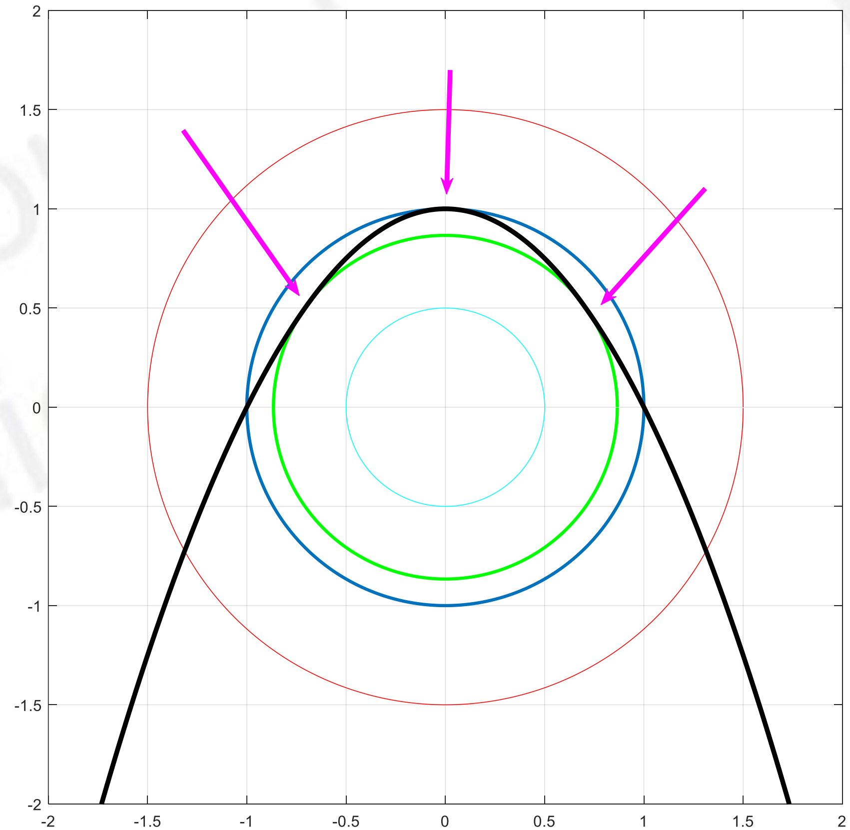
The minimum is at $x_1 = 0,\ x_2 = 1$

You can find it evaluating $f(x_1, x_2) = 1 - x_1^2 - x_2^2$

in the two points

$$x_1 = 0,\ x_2 = 1,\ \lambda = 2$$

$$x_1 = \pm \frac{1}{\sqrt{2}},\ x_2 = 1/2,\ \lambda = 1$$



Note that the level curves of *f* are centered in the origin O and *f* decreases as you move away from the origin.

Function spaces

Functional $J : V \to R$

Vector space V, $A \subseteq V$

$z^* \in A$ is a local minimum of J over A if there exists an

$$\varepsilon > 0 \ such\ that \ \ for\ all\ \ z \in A \ \ satisfying \ \left\| z - z^* \right\| < \varepsilon$$

$$\Rightarrow J(z^*) \leq J(z)$$

Function spaces

Consider function in V of the form $z+\alpha\eta,\ \eta\in V,\ \alpha\in R$

The first variation of J at z is the linear function $\delta J\big|_z : V \to R$ such that $\forall\alpha \ \ and \ \ \forall\eta$

$$J(z+\alpha\eta) = J(z) + \delta J\big|_z(\eta)\alpha + o(\alpha)$$

First order necessary condition for optimality:

For all admissible perturbation we must have:

$$\delta J\big|_{z^*}\eta = 0$$

Function spaces

A quadratic form $\delta^2 J\big|_z : V \to R$ is the second variation of ***J*** at ***z*** if

$\forall \alpha \;\; and \;\; \forall \eta$

we have:

$$J(z+\alpha\eta) = J(z) + \delta J\big|_z(\eta)\alpha + \delta^2 J\big|_z(\eta)\alpha^2 + o(\alpha^2)$$

second order necessary condition for optimality:

If $z^* \in A$ is a local minimum of J over $A \subset V$
for all admissible perturbation we must have:

$$\delta^2 J\big|_{z^*}(\eta) \geq 0$$

Function spaces

The Weierstrass Theorem is still valid

If J is a convex functional and $A \subset V$ is a convex set a local minimum is automatically a global one and the first order condition are **necessary and sufficient condition** for a minimum