ROBUST STABILIZATION BY DYNAMIC OUTPUT FEEDBACK

1 A robust "observer"

Consider now a system having relative degree $r > 1$, written in normal form as

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)u] \\ y &= \hat{C}\xi\,, \end{aligned} \tag{1}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^r$ and $\hat{A}, \hat{B}, \hat{C}$ are matrices defined as in (XXX). About this system we assume that

$$\begin{aligned} f(0,0) &= 0 \\ q(0,0) &= 0 \end{aligned} \tag{2}$$

and that the coefficient $b(z,\xi)$ satisfies

$$0 < b_{\min} \leq b(z,\xi) \leq b_{\max} \qquad \text{for all } (z,\xi) \tag{3}$$

for some $b_{\min}, b_{\max}$. We assume also that the system is strongly minimum phase, i.e. that

$$\dot{z} = f(z,\xi)\,,$$

viewed as a system with input ξ and state z, is input-to-state stable.

We know that the feedback law

$$u = \frac{1}{b(z,\xi)}\,(-q(z,\xi) + K\xi)\,, \tag{4}$$

if K is such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix, globally asymptotically stabilizes the equilibrium $(z,\xi) = (0,0)$ of the resulting closed-loop system. However, the implementation of this law requires accurate knowledge of $b(z,\xi)$ and $q(z,\xi)$ and availability of the full state (z,ξ). We see in what follows how a suitable "asymptotic proxy" of this law can be designed, which does not suffer such limitations.

The idea is to use the measured output y to drive an appropriate dynamical system to the purpose of estimating the components of ξ as well as to overcome the necessity of knowing the functions $b(z,\xi)$ and $q(z,\xi)$. To this end, let $\psi(\xi,\sigma)$ be the function defined as

$$\psi(\xi,\sigma) = b_0^{-1}[K\xi - \sigma]\,,$$

in which $\xi \in \mathbb{R}^r$, $\sigma \in \mathbb{R}$, b_0 is a design parameter and K a vector with the properties indicated above (i.e. such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix), and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth "saturation" function, characterized as follows: $g(s) = s$ if $|s| \leq L$, $g(s)$ is odd and monotonically increasing, with $0 < g'(s) \leq 1$, and $\lim_{s\to\infty} g(s) = L(1 + c)$ with $0 < c \ll 1$. The "saturation level" L, which in order to simplify the notation is not explicitly indicated the in the symbol used to denote the function in question, is a design parameter that will be determined later.

System (1) will be controlled by a control law of the form

$$u = g(\psi(\hat{\xi}, \sigma)) \tag{5}$$

in which $\hat{\xi} \in \mathbb{R}^r$ and σ are states of the dynamical system [1]

$$\begin{array}{rcl} \dot{\hat{\xi}}_1 & = & \hat{\xi}_2 + \kappa\alpha_1(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 & = & \hat{\xi}_3 + \kappa^2\alpha_2(y - \hat{\xi}_1) \\ & & \cdots \\ \dot{\hat{\xi}}_{r-1} & = & \hat{\xi}_r + \kappa^{r-1}\alpha_{r-1}(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_r & = & \sigma + b_0 g(\psi(\hat{\xi}, \sigma)) + \kappa^r\alpha_r(y - \hat{\xi}_1) \\ \dot{\sigma} & = & \kappa^{r+1}\alpha_{r+1}(y - \hat{\xi}_1)\,. \end{array} \tag{6}$$

The coefficients κ and $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are design parameters.

The dynamical system thus defined has the typical structure of an "observer". In the analysis of the asymptotic properties of the resulting closed-loop system, it is convenient to replace $\hat{\xi}_1, \ldots, \hat{\xi}_r, \sigma$ by means of (scaled) "error" variables, defined as follows

$$\begin{array}{rcl} e_1 & = & \kappa^r(\xi_1 - \hat{\xi}_1) \\ e_2 & = & \kappa^{r-1}(\xi_2 - \hat{\xi}_2) \\ & & \cdots \\ e_r & = & \kappa(\xi_r - \hat{\xi}_r) \\ e_{r+1} & = & q(z, \xi) + [b(z, \xi) - b_0]g(\psi(\xi, \sigma)) - \sigma\,. \end{array} \tag{7}$$

The first r of these relations can be trivially inverted, to recover each $\hat{\xi}_i$, as function of e_i and ξ_i. To recover σ form the latter, we need to choose b_0 appropriately. To this end, bearing in mind the expression of $\psi(\xi, \sigma)$, observe that the relation in question is equivalent to the following one

$$\frac{K\xi - q(z, \xi) + e_{r+1}}{b(z, \xi)} = \frac{b_0}{b(z, \xi)}\Big[\Big(\frac{b(z, \xi) - b_0}{b_0}\Big)g(\psi(\xi, \sigma)) + \psi(\xi, \sigma)\Big]. \tag{8}$$

If we set

$$\psi^*(z, \xi, e_{r+1}) = \frac{K\xi - q(z, \xi) + e_{r+1}}{b(z, \xi)}$$

and we define a function $F : \mathbb{R} \to \mathbb{R}$ as

$$F(s) = \frac{b_0}{b(z, \xi)}\Big[\Big(\frac{b(z, \xi) - b_0}{b_0}\Big)g(s) + s\Big] \tag{9}$$

the relation (8) can be simply rewritten as

$$\psi^* = F(\psi)\,.$$

[1] In the expression that follows we use $\hat{\xi}$ to denote the vector $\mathrm{col}(\hat{\xi}_1, \hat{\xi}_2, \ldots \hat{\xi}_r)$.

Since $b(z,\xi)$, by assumption, is bounded as in (3), it is always possible to pick b_0 so as to make

$$\left|\frac{b(z,\xi)-b_0}{b_0}\right| \leq \delta_0 < 1 \tag{10}$$

for some δ_0. Thus, since $0 < g'(s) \leq 1$ by hypothesis, if b_0 is chosen in this way, $F'(s)$ is strictly positive, i.e. $F(s)$ is a strictly increasing (odd) function. Moreover, since $\lim_{s\to\infty} g(s) = L(1+c)$, it is seen that $\lim_{s\to\infty} F(s) = \infty$, and consequently $F(\mathbb{R}) = \mathbb{R}$. In summary, $F(s)$ is globally invertible. It is also worth noting that, so long as $|s| \leq L$, the function $F(s)$ is an identity, i.e. $F(s) = s$.

Hence, if b_0 is chosen to satisfy (10), we have

$$\psi = F^{-1}(\psi^*)$$

and this – bearing in mind the expressions of ψ and ψ^* – shows that σ can always be recovered, from the last of (7), as a smooth function of z, ξ, e_{r+1}. This makes the change (7) a diffeomorphism.

We return now to the change of variables (7). The expressions that will be derived involve functions of $z, \xi, \hat{\xi}, \sigma$. In order to simplify the notations we find it prop1, whenever appropriate, to replace the pair of variables (z, ξ) with

$$x = \mathrm{col}(z, \xi)$$

to let e denote the vector

$$e = \mathrm{col}(e_1, e_2, \ldots, e_{r+1})\,,$$

and to bear in mind that $\hat{\xi}$ and σ are functions of (x, e).

It is readily seen that $e_1, \ldots, e_{r-1}$ satisfy the following identities

$$\begin{array}{rcl} \dot{e}_1 &=& \kappa(e_2 - \alpha_1 e_1) \\ \dot{e}_2 &=& \kappa(e_3 - \alpha_2 e_1) \\ && \cdots \\ \dot{e}_{r-1} &=& \kappa(e_r - \alpha_{r-1} e_1)\,. \end{array}$$

Moreover

$$\begin{array}{rcl} \dot{e}_r &=& \kappa[q(z,\xi) + b(z,\xi)g(\psi(\hat{\xi},\sigma)) - \sigma - b_0 g(\psi(\hat{\xi},\sigma)) - \kappa^r \alpha_r (y - \hat{\xi}_1)] \\ &=& \kappa(e_{r+1} - \alpha_r e_1 + [b(z,\xi) - b_0][g(\psi(\hat{\xi},\sigma)) - g(\psi(\xi,\sigma))] \\ &:=& \kappa(e_{r+1} - \alpha_r e_1) + \Delta_1\,, \end{array}$$

in which we have set

$$\Delta_1(x,e) = \kappa[b(z,\xi) - b_0][g(\psi(\hat{\xi},\sigma) - g(\psi(\xi,\sigma)]\,.$$

In this respect, it is worth observing that $\Delta_1(x,e)$ is a smooth function that vanishes when $e_1 = e_2 = \cdots = e_r = 0$, because in that case $\hat{\xi} = \xi$, and is globally bounded independently of κ (provided that $\kappa > 1$). In fact, $[b(z,\xi) - b_0]$ is globally bounded by assumption. Moreover $g(\cdot)$ is globally Lipschitz and hence from some $\hat{L}$ we have

$$\begin{array}{rcl} |g(\psi(\hat{\xi},\sigma)) - g(\psi(\xi,\sigma))| &\leq& \hat{L}|\psi(\hat{\xi},\sigma) - \psi(\xi,\sigma)| \leq b_0^{-1}|K||\hat{\xi} - \xi| \\ &\leq& b_0^{-1}|K|\Big(\sum_{i=1}^r(\kappa^{-r-1+i}e_i)^2\Big)^{1/2} \leq b_0^{-1}|K|\kappa^{-1}\Big(\sum_{i=1}^r(e_i)^2\Big)^{1/2}, \end{array}$$

in the last of which we have used the fact that $\kappa > 1$. In summary, we have

$$|\Delta_1(x,e)| \le \delta_1 |e|, \tag{11}$$

for some $\delta_1 > 0$.

Finally, we obtain for the time derivative of e_{r+1} an expression of the form

$$\dot{e}_{r+1} = -\kappa\alpha_{r+1}e_1 + \dot{q} + \dot{b}g(\psi(\xi,\sigma)) + [b-b_0]g'(\psi(\xi,\sigma))b_0^{-1}[K\dot{\xi} - \dot{\sigma}]$$

in which, for convenience, we have omitted the indication of some arguments. This expression can be further elaborated as follows. Bearing in mind the dynamics of σ, we can write

$$[b(z,\xi) - b_0]g'(\psi(\xi,\sigma))b_0^{-1}\dot{\sigma} = [b(z,\xi) - b_0]g'(\psi(\xi,\sigma))b_0^{-1}\kappa\alpha_{r+1}e_1 := \Delta_0\kappa\alpha_{r+1}e_1$$

in which

$$\Delta_0(x,e) = [b(z,\xi) - b_0]g'(\psi(\xi,\sigma))b_0^{-1}\,,$$

Hence, we can write

$$\dot{e}_{r+1} = -\kappa\alpha_{r+1}(1+\Delta_0)e_1 + \Delta_2$$

in which

$$\Delta_2(x,e) = \dot{q} + \dot{b}g(\psi) + [b-b_0]g'(\psi)b_0^{-1}K\dot{\xi}\,.$$

The functions $\Delta_0(x,e)$ and $\Delta_2(x,e)$ introduced here have the following properties. Regarding $\Delta_0(x,e)$, recall that $g'(\cdot)$ is by assumption bounded by 1 and that b_0 is chosen so as to satisfy (10). Thus, we may conclude that

$$|\Delta_0(x,e)| \le \delta_0 < 1\,. \tag{12}$$

Regarding $\Delta_2(x,e)$, a closer look reveals that

$$\begin{aligned}
\Delta_2(x,e) \quad = \quad & \tfrac{\partial q}{\partial z}f(z,\xi) + \tfrac{\partial q}{\partial \xi}[\hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)g(\psi(\hat{\xi},\sigma))]] \\
& + \tfrac{\partial b}{\partial z}f(z,\xi)g(\psi(\hat{\xi},\sigma)) + \tfrac{\partial b}{\partial \xi}[\hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)g(\psi(\hat{\xi},\sigma))]]g(\psi(\hat{\xi},\sigma)) \\
& + [b(z,\xi) - b_0]b_0^{-1}g'(\psi(\hat{\xi},\sigma))K[\hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)g(\psi(\hat{\xi},\sigma))]]\,.
\end{aligned}$$

Since $g(\cdot)$ and $g'(\cdot)$ are bounded functions, it is observed that so long as x remains in a bounded region, the function $\Delta_2(x,e)$ remains bounded regardless of the value of e, with a bound which is independent of κ.

Putting the dynamics of the e_i's all together yields a system of the form

$$\dot{e} = \kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]e + \mathbf{B}_1\Delta_1(x,e) + \mathbf{B}_2\Delta_2(x,e) \tag{13}$$

in which

$$\mathbf{A} = \begin{pmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ -\alpha_r & \cdot & \cdot & \cdots & 1 \\ -\alpha_{r+1} & \cdot & \cdot & \cdots & 0 \end{pmatrix} \qquad \mathbf{B} = \mathbf{B}_2 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{B}_1 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{C} = (\, \alpha_{r+1} \quad 0 \quad \cdots \quad 0 \quad 0\,)$$

and $\Delta_0, \Delta_1, \Delta_2$ are the functions indicated above.

2 Convergence analysis

We return now to the equations describing the (controlled) plant (1). Adding and subtracting $\hat{B}K\xi$, the last equation can be rewritten as

$$\dot{\xi} = (\hat{A} + \hat{B}K)\xi + \hat{B}\Delta_3(x, e)$$

in which

$$\Delta_3(x, e) = q(z, \xi) + b(z, \xi)g(\frac{K\hat{\xi} - \sigma}{b_0}) - K\xi\,,$$

where, consistently with the notation used in the previous section, as arguments of Δ_3 we have used x, e instead of $z, \xi, \hat{\xi}, \sigma$,

Accordingly, the entire controlled plant (1) can be written as

$$\dot{x} = \mathbf{F}(x) + \mathbf{G}\Delta_3(x, e) \tag{14}$$

in which

$$\mathbf{F}(x) = \begin{pmatrix} f(z, \xi) \\ (\hat{A} + \hat{B}K)\xi \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 0 \\ \hat{B} \end{pmatrix},$$

highlighting the structure of a globally asymptotically stable system affected by a perturbation.

We proceed now with the asymptotic analysis of the entire closed-loop system, which is a system described by equations of the form

$$\begin{array}{rcl} \dot{x} & = & \mathbf{F}(x) + \mathbf{G}\Delta_3(x, e) \\ \dot{e} & = & \kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x, e)]e + \mathbf{B}_1\Delta_1(x, e) + \mathbf{B}_2\Delta_2(x, e)\,. \end{array} \tag{15}$$

Inspection of the various expressions of $\Delta_1(x, e), \Delta_2(x, e), \Delta_3(x, e)$ reveals that $(x, e) = (0, 0)$ is an equilibrium point. In what follows, we will see how the free design parameters can be adjusted in order to obtain asymptotic stability of this equilibrium, with a region of attraction that contains a fixed (but otherwise arbitrary large) compact set $\mathcal{C}$. Note that the set in question must be assigned in terms of the *original* state variables, namely the state x of the controlled plant (1) and the states $\hat{\xi}$ and σ that characterize the dynamic controller (5)–(6). In this respect, it is observed that in the equations (15), resulting from the change of variables (7), the vector e replaces $(\hat{\xi}, \sigma)$. Thus, looking at the change (7), it should be borne in mind that the (compact) set to which the initial values of x and e belong is influenced by the choice of κ. We will return on this issue later on.

In what follows, occasionally, we use the notation $\hat{\xi}_{\rm ext}$ to indicate the vector

$$\hat{\xi}_{\rm ext} = \mathrm{col}(\hat{\xi}, \sigma)$$

that characterizes the state of the dynamic controller (5)–(6). With $\mathcal{C}$ being the fixed compact set of initial conditions of the system, pick a number $R > 0$ such that

$$(x, \hat{\xi}_{\rm ext}) \in \mathcal{C} \qquad \Rightarrow \qquad \begin{array}{c} x \in B_R \\ \hat{\xi}_{\rm ext} \in B_R \end{array}$$

in which B_R denotes a ball of radius R in $\mathbb{R}^n$

$$B_R = \{x \in \mathbb{R}^n : |x| \le R\}\,.$$

The first thing is to fix the "saturation" level L that characterizes the function $g(\cdot)$. To this end we begin with the observation that, since the "unperturbed" system

$$\dot{x} = \mathbf{F}(x)$$

is by assumption globally asymptotically stable, there exists a positive definite and proper smooth function $V(x)$ satisfying

$$\frac{\partial V}{\partial x}\mathbf{F}(x) \le -\alpha(|x|)$$

for some class $\mathcal{L}_\infty$ function $\alpha(\cdot)$.

Pick a number $c > 0$ such that

$$\Omega_c = \{x : V(x) \le c\} \supset B_R.$$

As we will see, among other things, one of the purposes of the design is to make sure that, for all times $t \ge 0$, the vector $x(t)$ remains in Ω_{c+1}. With this goal in mind, we pick for L the value

$$L = \max_{x\in\Omega_{c+1}} \Big[\frac{K\xi - q(z,\xi)}{b(z,\xi)}\Big] + 1\,.$$

We begin by analyzing what happens to $x(t)$ for small values of $t \ge 0$. Looking at the expression of the perturbation term $\Delta_3(x,e)$ in the upper system of (15) it is seen that, since $g(\cdot)$ is bounded, there exist a number δ_3, such that

$$|\Delta_3(x,e)| \le \delta_3 \qquad \text{for all } x \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^{r+1}\,.$$

This number δ_3 is independent of the choice of κ and depends only on the choice of the number R and hence, indirectly, only on the choice of the set $\mathcal{C}$ in which the initial conditions are taken.

Let $x(0) \in B_R \subset \Omega_c$. Regardless of what $e(t)$ is, so long as $x(t) \in \Omega_{c+1}$, we have

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}\Delta_3(x,e)] \le -\alpha(|x|) + \Big|\frac{\partial W}{\partial x}\Big|\delta_3$$

Setting

$$M = \max_{x\in\Omega_{c+1}} \Big|\frac{\partial W}{\partial x}\Big|$$

we obtain

$$\dot{V}(x(t)) \le (M\delta_3)t$$

which in turn yields

$$V(t) - V(0) \le (M\delta_3)t\,.$$

Thus, since $V(0) \le c$, we see that $x(t)$ remains in Ω_{c+1} at least until time $T_0 = 1/M\delta_3$. This time may be very small but, because of the presence of the saturation function $g(\cdot)$, it is independent of κ. It rather only depends only on the choice of the number R and hence, indirectly, only on the choice of the set $\mathcal{C}$ in which the initial conditions are taken.

We proceed now to the analysis of what happens to $e(t)$. To this end, the following results are relevant.

Lemma 1 *There exist a choice of the coefficients $\alpha_1, \dots, \alpha_{r+1}$, a positive definite and symmetric $(r+1) \times (r+1)$ matrix P and a number $\lambda > 0$ such that*

$$P[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)] + [\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]^{\mathrm{T}} P \le -\lambda I \,. \tag{16}$$

Proof. Observe that the system

$$\dot{e} = [\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]e$$

can be seen as resulting from the interconnection of

$$\begin{array}{rcl} \dot{e} & = & \mathbf{A}e + \mathbf{B}u \\ y & = & \mathbf{C}e \end{array} \tag{17}$$

with

$$u = -\Delta_0(x,e)y \,. \tag{18}$$

Looking at the expressions of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ it is seen that the transfer function $T(s)$ of (17) is

$$T(s) = \frac{\alpha_{r+1}}{s^{r+1} + \alpha_1 s^r + \cdots \alpha_r s + \alpha_{r+1}} \,.$$

If the α_i's are such that the poles of $T(s)$ are all real and with negative real part, $\|T(\cdot)\|_\infty = 1$. As a consequence

$$\|T(\cdot)\|_\infty < \delta_0^{-1}$$

in which $\delta_0 < 1$ is any number for which (10) holds. By the Bounded Real Lemma, there is a positive definite and symmetric matrix P and a number $\lambda > 0$ such that

$$2e^{\mathrm{T}} P[\mathbf{A}e + \mathbf{B}u] \le -\lambda |e|^2 + (\delta_0^{-1})^2 |u|^2 - |y|^2$$

for all e and u. Using (18) and bearing in mind (12), which implies $|u|^2 \le \delta_0^2 |y|^2$, we obtain

$$2e^{\mathrm{T}} P[\mathbf{A}e - \mathbf{B}\mathbf{C}\Delta_0(x,e)e] \le -\lambda |e|^2 + (\delta_0^{-1})^2 \delta_0^2 |y|^2 - |y|^2 \le -\lambda |e|^2$$

which proves the Lemma. $\triangleleft$

Lemma 2 *Let the α_i's be chosen so as to make (16) satisfied. Suppose $x(t) \in \Omega_{c+1}$ for all $t \in [0, T_{\max})$ and suppose that $|\hat{\xi}_{\mathrm{ext}}(0)| \le R$. Then, for every $T < T_{\max}$ and every $\epsilon > 0$, there is a κ^* such that, for all $\kappa \ge \kappa^*$,*

$$|e(t)| \le 2\epsilon \qquad \textit{for all } t \in [T, T_{\max}).$$

Proof. Set $U(e) = e^{\mathrm{T}} P e$, and observe that

$$\begin{array}{rcl} \dot{U}(e(t)) & = & 2e^{\mathrm{T}} P[\kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]e + \mathbf{B}_1\Delta_1 + \mathbf{B}_2\Delta_2] \le -\kappa\lambda |e|^2 + 2|e||P||\Delta_1| + 2|e||P||\Delta_2| \\ & \le & -(\kappa\lambda - 2\delta_1 |P|)|e|^2 + 2|e||\Delta_2| \le -(\kappa\lambda - 2\delta_1|P| - \mu)|e|^2 + \frac{1}{\mu}|\Delta_2| \end{array}$$

where we have taken advantage of the bound (11) for Δ_1 and used Young's inequality.

Recalling the expression $\Delta_2(x,e)$, observe that, since $g(\cdot)$ is bounded, there exist a number $\delta_2 > 0$, such that

$$|\Delta_2(x,e)| \le \delta_2 \qquad \text{for all } x \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^{r+1}\,.$$

This number δ_2 is independent of the choice of κ and depends only on the choice of the number R and hence, indirectly, only on the choice of the set $\mathcal{C}$ in which the initial conditions are taken.

Now, let $\epsilon > 0$ be fixed and pick μ so that

$$\frac{\delta_2}{\mu} \le \epsilon^2\,.$$

Using the standard estimates

$$a_1|e|^2 \le U(e) \le a_2|e|^2$$

and setting, for convenience

$$\alpha = \frac{\kappa\lambda - 2\delta_1|P| - \mu}{a_2}$$

(in which we take a large κ, so that $\alpha > 0$), we end-up with the inequality

$$\dot U(e(t)) \le -\alpha U(e(t)) + \epsilon^2\,,$$

which is guaranteed to hold so long as $x(t) \in \Omega_{c+1}$. Standard arguments show that

$$|e(t)| \le A e^{-\alpha t}|e(0)| + \frac{\epsilon}{\sqrt{\alpha a_1}} \qquad \text{where } A = \left(\frac{a_2}{a_1}\right)^{\frac{1}{2}}.$$

In this expression, we have now to take into account an estimate of $|e(0)|$. Looking at (7) and assuming $\kappa > 1$, it is seen that so long as $|x(0)| \le R$ and $|\hat{\xi}_{\text{ext}}(0)| \le R$, we have

$$|e(0)| \le \kappa^r R'$$

in which R' is a number only depending on R and not on κ. Entering this in the previous inequality, and picking κ so that $\alpha a_1 > 1$, we obtain an estimate of the form

$$|e(t)| \le A' e^{-\kappa t}\kappa^r + \epsilon\,, \qquad \text{where } A' = AR' e^{\frac{2\delta_1|P|+\mu}{a_2}}\,,$$

which, we stress, is valid so long as $x(t) \in \Omega_{c+1}$. Pick $0 < T < T_{\max}$ and let κ^* be such that

$$A' e^{-\kappa^* T}(\kappa^*)^r = \epsilon\,,$$

as it is always possible. Then, we see that, for all $\kappa \ge \kappa^*$,

$$|e(T)| \le 2\epsilon\,.$$

Moreover, since $x(t) \in \Omega_{c+1}$ for $t \in [T, T_{\max})$, we have

$$|e(t)| \le A' e^{-\kappa^* T}(\kappa^*)^r e^{-\kappa^*(t-T)} + \epsilon = \epsilon e^{-\kappa^*(t-T)} + \epsilon \le 2\epsilon \qquad \text{for all } t \in [T, T_{\max})\,.$$

This concludes the proof. $\triangleleft$

We return now to the examine the motion of $x(t)$, which we know is in Ω_{c+1} for $t \le T_0$. Suppose that, for some $T_{\max} > T_0$,

$$x(t) \in \Omega_{c+1} \qquad \text{for all } t \in [T_0, T_{\max}).$$

We know from Lemma 2 that, for any choice of ϵ, there is a value of κ^* such that, if $\kappa \ge \kappa^*$, $|e(t)| \le 2\epsilon$ for all $t \in [T_0, T_{\max})$. If $4\epsilon < b_{\min}$, we see in particular that for all $t \in [T_0, T_{\max})$

$$\left|\frac{e_{r+1}}{b(z,\xi)}\right| < \frac{1}{2},$$

which, in view of the choice of L, implies

$$\left|\frac{K\xi - q(z,\xi) + e_{r+1}}{b(z,\xi)}\right| \le L - \frac{1}{2}.$$

From this, using (8) and taking advantage of the fact that function $F(s)$ defined in (9) is strictly increasing, with $F(s) = s$ when $|s| \le L$, we deduce that

$$|\psi(\xi,\sigma)| \le L - \frac{1}{2}.$$

Recall now

$$\hat{\xi} = \xi - D(\kappa)e$$

in which $D(\kappa)$ is $r \times (r+1)$ a matrix in which $d_{ii}(\kappa) = \kappa^{-r-1+i}$, for $i = 1, \ldots, r$, while all other entries are 0. Note that $|D(\kappa)| \le 1$, if $\kappa > 1$. Thus

$$\psi(\hat{\xi},\sigma) = \psi(\xi,\sigma) - b_0^{-1} K D(\kappa) e$$

and

$$|\psi(\hat{\xi},\sigma)| \le |\psi(\xi,\sigma)| + b_0^{-1}|K||e|.$$

If $4|K|\epsilon < b_0$, we have

$$|\psi(\hat{\xi},\sigma)| \le L,$$

and consequently

$$g(\psi(\hat{\xi},\sigma)) = \psi(\hat{\xi},\sigma).$$

This being the case, we have for all $t \in [T_0, T_{\max})$

$$\begin{aligned}
\Delta_3(x,e) &= q(z,\xi) + b(z,\xi) g(\psi(\hat{\xi},\sigma)) - K\xi = q(z,\xi) + b(z,\xi)\psi(\hat{\xi},\sigma) - K\xi \\
&= q(z,\xi) + b(z,\xi)[\psi(\xi,\sigma) - b_0^{-1} K D(\kappa) e] - K\xi \\
&= -e_{r+1} - b(z,\xi) b_0^{-1} K D(k) e
\end{aligned}$$

where we have used again the property that the function (9) is an identity if $|s| \le L$, which yields

$$\psi(\xi,\sigma) = F(\psi(\xi,\sigma)) = \frac{K\xi - q(z,\xi) + e_{r+1}}{b(x,\xi)}.$$

As a consequence of this we can conclude that

$$|\Delta_3(x,e)| \le (1 + b_{\max} b_0^{-1} |K|) 2\epsilon \qquad \text{for all } t \in [T_0, T_{\max}).$$

From all of the above it is seen that, for all $t \in [T_0, T_{\max})$,

$$\frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}\Delta_3(x,e)] \le -\alpha(|x|) + M(1 + b_{\max} b_0^{-1} |K|) 2\epsilon\,.$$

Pick now any number $d \ll c$ and consider the "annular" compact set

$$S_d^{c+1} = \{z : d \le V(x) \le c+1\}\,.$$

Let ρ be

$$\rho = \min_{x \in S_d^{c+1}} |x|$$

By construction

$$\alpha(|x|) \ge \alpha(\rho) \qquad \text{for all } x \in S_d^{c+1}.$$

If ϵ is such

$$(1 + b_{\max} b_0^{-1} |K|) 2\epsilon \le \frac{1}{2}\,\alpha(\rho)\,,$$

there follows that

$$\dot V(x(t)) \le -\frac{1}{2}\,\alpha(\rho)\,,$$

so long as $x(t) \in S_d^{c+1}$. This, in turn, implies

$$V(x(t)) \le V(x(T_0)) - \frac{1}{2}\,\alpha(\rho) \le c + 1 - \frac{1}{2}\,\alpha(\rho)$$

so long as $x(t) \in S_d^{c+1}$. Clearly, there is a time $T_1 > T_0$ such that $x(t) \in \Omega_{c+1}$ for all $t \in [T_0, T_1]$ and $V(x(T_1)) = d$. Since $\dot V(x(t))$ is negative on the boundary of Ω_d, it is concluded that $x(t) \in \Omega_d$ for all $t \ge T_1$ and $T_{\max} = \infty$.

In summary, we have shown that, by tuning the design parameter κ, all trajectories of the closed-loop system with initial conditions in $\mathcal{C}$ remain bounded and enter, in finite time, an arbitrarily small compact set. As a matter of fact, the following result has been just proven.

Proposition 1 *Consider system (1), controlled by (5)–(6. Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Suppose (1) is strongly minimum phase. Let $\hat K$ such that $\hat A + \hat B \hat K$ is Hurwitz. For every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \ge \kappa^*$ there is a finite time T such that all trajectories of the closed-loop system with initial conditions $(x(0), \hat\xi_{\rm ext}(0)) \in \mathcal{C}$ remain bounded and satisfy $|(x(t)| \le \varepsilon$ and $|\hat\xi_{\rm ext}(t)| \le \varepsilon$ for all $t \ge T$.*

If asymptotic stability is sought, it suffices to strengthen the assumption of strong minimum phase by requiring that the equilibrium $z = 0$ of $\dot z = f(z,0)$ is also locally exponentially stable.

Proposition 2 *Consider system (1), controlled by (5)–(6). Suppose that (2) hold and that $b(x, \xi)$ is bounded as in (3). Suppose (1) is strongly minimum phase. Let $\hat{K}$ such that $\hat{A}+\hat{B}\hat{K}$ is Hurwitz. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(x, \hat{\xi}_{\text{ext}}) = (0, 0)$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.*

Remark. In both these results, to streamline the analysis, we have assumed that the controlled system is strongly minimum phase, but this assumption can be weakened, to some extent. In fact, in both results, due to the necessity of using the "rough" observer (6), it has been possible to prove convergence (to a small neighborhood of the origin or to the origin) only from a *compact set* of initial conditions. In other words, we have not proven global convergence, but only convergence with a guaranteed region of attraction. This being the case, it should be observed that identical results hold if the assumption that the system is strongly minimum phase is replaced by the assumption that the system possesses a globally defined normal form in which the first equation of (1) has the special structure

$$\dot{z} = f(z, \xi_1)$$

and is only minimum phase (rather than being strongly minimum phase). The proof, in which $\hat{K}$ has to be chosen of the special form

$$\hat{K} = \begin{pmatrix} -a_0 k^r & -a_1 k^{r-1} & \cdots & -a_{r-2} k^2 & -a_{r-1} k \end{pmatrix}$$

with $k > 0$ a design parameter to be tuned in accordance with choice of the set $\mathcal{C}$, is not conceptually different from the one described above and is not covered here. $\triangleleft$

3 Extensions

The design procedure described in the previous sections applies to systems that are strongly minimum phase (or, under a special assumption regarding the structure of the normal form, minimum phase). We discuss in this section how these methods might be used – to some extent – to handle also systems that are not minimum phase.

Consider again a plant modeled by equations of the form (1), in which we assume $q(0,0) = 0$ and $b(z, \xi)$ bounded as in (3), and let its dynamics be extended by that of a system

$$\dot{\varrho} = L(\varrho, \xi_1, \ldots, \xi_{r-1}) + Mu\,,$$

with state $\varrho \in \mathbb{R}^\nu$. Setting

$$x_{\text{e}} = \text{col}(z, \xi, \varrho)$$

the resulting *extended* system defined in this way can be seen as a system of the form

$$\dot{x}_{\text{e}} = f_{\text{e}}(x_{\text{e}}) + g_{\text{e}}(x_{\text{e}})u \tag{19}$$

in which

$$f_{\text{e}}(x_{\text{e}}) = \begin{pmatrix} f(z, \xi) \\ \hat{A}\xi + \hat{B}q(z, \xi) \\ L(\varrho, \xi_1, \ldots, \xi_{r-1}) \end{pmatrix}, \qquad g_{\text{e}}(x_{\text{e}}) = \begin{pmatrix} 0 \\ \hat{B}b(z, \xi) \\ M \end{pmatrix}. \tag{20}$$

Let now $h_e(x_e)$ be a function defined as

$$h_e(x_e) = \xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1}) \,, \tag{21}$$

in which we assume that

$$\frac{\partial N}{\partial \varrho} M = 0 \,. \tag{22}$$

If this property holds, the derivative of $N(\varrho, \xi_1, \ldots, \xi_{r-1})$ along the trajectories of the extended system is a function of ϱ and ξ only, which we will write as

$$\dot{N}(\varrho, \xi) := \frac{\partial N}{\partial \varrho} L(\varrho) + \sum_{i=1}^{r-1} \frac{\partial N}{\partial \xi_i} \xi_{i+1} \,. \tag{23}$$

In what follows we assume also that this function $\dot{N}(\varrho, \xi)$ is globally Lipschitz in the argument ξ, uniformly in ϱ, i.e. that

$$|\dot{N}(\varrho, \hat{\xi}) - \dot{N}(\varrho, \xi)| \leq L_N |\hat{\xi} - \xi| \,, \tag{24}$$

for some fixed $L_N > 0$,

Using (22) it is seen that

$$L_{g_e} h_e(x_e) = b(z, \xi)$$

and therefore system (19)–(20) with output (21) has relative degree $r = 1$. Since $b(z, \xi)$ is bounded as in (3), the vector field

$$\tilde{g}_e(x_e) = \frac{1}{L_{g_e} h_e(x_e)} g_e(x_e)$$

is complete. As a consequence, system (19)–(20) with output (21) possesses a globally defined normal form.

The stabilization techniques described in the previous sections are applicable if the dynamically extended system introduced in this way is strongly minimum phase. In view of this we assume the following.

Assumption 1 *There exist an integer ν and a triplet*

$$L(\varrho, \xi_1, \ldots, \xi_{r-1}), \; M, \; N(\varrho, \xi_1, \ldots, \xi_{r-1})$$

in which $L(\cdot)$ and $N(\cdot)$ are smooth functions satisfying (22)–(24) and in which $L(0, 0, \ldots, 0) = 0$, $N(0, 0, \ldots, 0) = 0$*, such that system (19)–(20) with output (21) is strongly minimum phase.*

We will return later on the interpretation of this Assumption. For the time being, we observe that if this Assumption holds and if the full state x_e of this dynamically extended system is available for measurement, global stabilization can be achieved by means of a feedback law of the form

$$u = u(x_e) = \frac{-L_{f_e} h_e(x_e) + k h_e(x_e)}{L_{g_e} h_e(x_e)}$$

in which k is any negative number. In fact, this is precisely the equivalent, in the current context, of the feedback law XXX).

The control $u(x_\mathrm{e})$ can easily be expressed in the (z, ξ, ϱ) coordinates. To this end, it suffices to take into account the definition (21) of $h_\mathrm{e}(x_\mathrm{e})$ and to observe that (recall (23))

$$L_{f_\mathrm{e}} h_\mathrm{e}(x_\mathrm{e}) = q(w, \xi) - \dot{N}(\varrho, \xi) .$$

In this way, we arrive at the following preliminarily conclusion: if Assumption 1 holds, a *dynamic* control law of the form

$$\begin{array}{rcl} \dot{\varrho} & = & L(\varrho, \xi_1, \dots, \xi_{r-1}) + Mu \\ u & = & [b(z, \xi)]^{-1}[-\, q(z, \xi) + \dot{N}(\varrho, \xi) + k(\xi_r - N(\varrho, \xi_1, \dots, \xi_{r-1}))] \end{array} \tag{25}$$

(in which k is a negative number) globally asymptotically stabilizes the equilibrium $(z, \xi, \varrho) = (0, 0, 0)$ of the closed loop system (1)–(25).

If only the output $y = \xi_1$ of system (1) is available for feedback, this law cannot be directly implemented, because $\xi_2, \dots, \xi_r$ and $q(z, \xi)$ and $b(z, \xi)$ are not directly available. But these quantities are precisely the same quantities that needed to be estimated for the implementation of the feedback law (4) in the context of the design problem considered in the previous sections. Thus, it is expected that the very same procedure used earlier to estimate such quantities can be successfully used also in the present context. With this in mind, let $\psi_\mathrm{e}(\xi, \sigma, \varrho)$ be the function defined as

$$\psi_\mathrm{e}(\xi, \sigma, \varrho) = b_0^{-1}[\dot{N}(\varrho, \xi) + k(\xi_r - N(\varrho, \xi_1, \dots, \xi_{r-1})) - \sigma] \tag{26}$$

in which b_0 is a design parameter, $\dot{N}(\varrho, \xi)$ is the function defined above, k is a negative number and $\sigma \in \mathbb{R}$. Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a smooth "saturation" function characterized by the properties: $g(s) = s$ if $|s| \le L$, $g(s)$ is odd and monotonically increasing, with $0 < g'(s) \le 1$, and $\lim_{s\to\infty} g(s) = L(1 + c)$ with $0 < c \ll 1$.

Plant (1) will be controlled by a law of the form

$$\begin{array}{rcl} \dot{\varrho} & = & L(\varrho, \hat{\xi}_1, \dots, \hat{\xi}_{r-1}) + Mu \\ u & = & g(\psi_\mathrm{e}(\hat{\xi}, \sigma, \varrho)) \end{array} \tag{27}$$

in which $\hat{\xi} \in \mathbb{R}^r$ and $\sigma \in \mathbb{R}$ are states of the dynamical system

$$\begin{array}{rcl} \dot{\hat{\xi}}_1 & = & \hat{\xi}_2 + \kappa\alpha_1(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 & = & \hat{\xi}_3 + \kappa^2\alpha_2(y - \hat{\xi}_1) \\ & \cdots & \\ \dot{\hat{\xi}}_{r-1} & = & \hat{\xi}_r + \kappa^{r-1}\alpha_{r-1}(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_r & = & \sigma + b_0 g(\psi_\mathrm{e}(\hat{\xi}, \sigma, \varrho)) + \kappa^r\alpha_r(y - \hat{\xi}_1) \\ \dot{\sigma} & = & \kappa^{r+1}\alpha_{r+1}(y - \hat{\xi}_1) . \end{array} \tag{28}$$

Note that the first $r - 1$ and the $(r + 1)$-th equations in this set are exactly the same as in (6), while the r-th one differs from the r-th of (6) in the replacement of $\psi(\hat{\xi}, \sigma)$ by $\psi_\mathrm{e}(\hat{\xi}, \sigma, \varrho)$.

With the "observer" (28), associate "error" variables $e_1, \dots, e_r, e_{r+1}$, the first r of which are defined exactly as in (7), while e_{r+1} has the same structure as the last one of (7), with $\psi(\xi, \sigma)$ now replaced by the function $\psi_{\rm e}(\xi, \sigma, \varrho)$ defined by (26), that is

$$e_{r+1} = q(z, \xi) + [b(z, \xi) - b_0] g(\psi_{\rm e}(\xi, \sigma, \varrho)) - \sigma\,.$$

The new variables defined in this way characterize a global diffeomorphism. In particular, the possibility of recovering σ from e_{r+1} relies upon arguments identical to those used to prove this property in the previous context. In fact, setting

$$\psi_{\rm e}^*(z, \xi, \varrho, e_{r+1}) = \frac{\dot N(\varrho, \xi) + k(\xi_r - N(\varrho, \xi_1, \dots, \xi_{r-1})) - q(z, \xi) + e_{r+1}}{b(z, \xi)}$$

and using the function $F(s)$ defined as in (9), it is seen that

$$\psi_{\rm e}^* = F(\psi_{\rm e})$$

and therefore, assuming that b_0 is chosen to fulfill (10), one can recover σ via

$$\psi_{\rm e} = F^{-1}(\psi_{\rm e}^*)\,.$$

In these new coordinates, we have

$$\dot e_i = \kappa(e_{i+1} - \alpha_i e_1) \qquad \text{for } i = 1, \dots, r-1$$

and

$$\dot e_r = \kappa(e_{r+1} - \alpha_r e_1) + \Delta_1(x_{\rm e}, e)$$

in which

$$\Delta_1(x_{\rm e}, e) = \kappa[b(z, \xi) - b_0][g(\psi_{\rm e}(\hat\xi, \sigma, \varrho) - g(\psi_{\rm e}(\xi, \sigma, \varrho)]\,.$$

As in the previous section, bearing in mind the definition of $\psi_{\rm e}(\xi, \sigma, \varrho)$, using property (24), and picking $\kappa > 1$, it can be proven that

$$|\Delta_1(x_{\rm e}, e) \le \delta_1 |e|$$

for some $\delta_1 > 0$.

Finally, we obtain for the time derivative of e_{r+1} an expression of the form

$$\dot e_{r+1} = -\kappa\alpha_{r+1} e_1 + \dot q + \dot b g(\psi_{\rm e}(\xi, \sigma, \varrho)) + [b - b_0] g'(\psi_{\rm e}(\xi, \sigma, \varrho)) b_0^{-1} [\ddot N(\varrho, \xi) + k(\dot\xi_r - \dot N(\varrho, \xi)) - \dot\sigma]$$

in which, for convenience, we have omitted the indication of some arguments. Bearing in mind the expression of $\dot\sigma$, the latter can be rewritten as

$$\dot e_{r+1} = -\kappa\alpha_{r+1}(1 + \Delta_0) e_1 + \Delta_2$$

in which

$$\Delta_0(x_{\rm e}, e) = [b(z, \xi) - b_0] g'(\psi_{\rm e}(\xi, \sigma, \varrho)) b_0^{-1}\,.$$

$$\Delta_2(x_{\rm e}, e) = \dot q + \dot b g(\psi_{\rm e}) + [b - b_0] g'(\psi_{\rm e}) b_0^{-1} [\ddot N(\varrho, \xi) + k\dot\xi_r]\,.$$

The function $\Delta_0(x_e, e)$ is bounded as in (12), i.e.

$$|\Delta_0(x_e, e)| \leq \delta_0 < 1\,,$$

if b_0 is chosen so as to satisfy (10). Regarding $\Delta_2(x_e, e)$, observe in particular that

$$\begin{aligned}\ddot{N}(\varrho, \xi) + k(\dot{\xi}_r - \dot{N}(\varrho, \xi)) \;&=\; \tfrac{\partial \dot{N}}{\partial \varrho}[L(\varrho) + Mg(\psi_e(\hat{\xi}, \sigma, \varrho))] \\ &\quad + \tfrac{\partial \dot{N}}{\partial \xi}[\hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)g(\psi_e(\hat{\xi}, \sigma, \varrho))] \\ &\quad + k[q(z, \xi) + b(z, \xi)g(\psi_e(\hat{\xi}, \sigma, \varrho)) - \dot{N}(\varrho, \xi)]\,.\end{aligned}$$

With this in mind, arguments identical to those used in the previous section can be invoked to claim that, since $g(\cdot)$ and $g'(\cdot)$ are bounded functions, so long as x_e remains in a bounded region, the function $\Delta_2(x_e, e)$ remains bounded regardless of the value of e, with a bound which is independent of κ.

Putting the dynamics of the e_i's all together yields a system of the form

$$\dot{e} = \kappa[\mathbf{A} - \mathbf{BC}\Delta_0(x_e, e)]e + \mathbf{B}_1\Delta_1(x_e, e) + \mathbf{B}_2\Delta_2(x_e, e) \tag{29}$$

in which $\mathbf{A}, \mathbf{B}, \mathbf{B}_2, \mathbf{B}_2, \mathbf{C}$ have exactly the same form as in the previous section, and $\Delta_0, \Delta_1, \Delta_2$ are the functions indicated above.

Passing to the equations that describe the controlled plant, note that

$$\dot{x}_e = f_e(x_e) + g_e(x_e)g(\psi_e(\hat{\xi}, \sigma, \varrho)) = f_e(x_e) + g_e(x_e)u_e(x_e) + g_e(x_e)[g(\psi_e(\hat{\xi}, \sigma, \varrho)) - u_e(x_e)]$$

which we will write as

$$\dot{x}_e = \mathbf{F}(x_e) + g_e(x_e)\Delta_3(x_e, e)$$

where $\mathbf{F}(x_e) = f_e(x_e) + g_e(x_e)u_e(x_e)$ and

$$\Delta_3(x_e, e) = g(\psi_e(\hat{\xi}, \sigma, \varrho)) - [b(z, \xi)]^{-1}[-\,q(z, \xi) + \dot{N}(\varrho, \xi) + k(\xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1}))] \tag{30}$$

highlighting the structure of a globally asymptotically stable system affected by a perturbation.

It is realized at this point that the controlled system has a *structure identical* to that of (15), i.e.

$$\begin{aligned}\dot{x}_e \;&=\; \mathbf{F}(x_e) + g_e(x_e)\Delta_3(x_e, e) \\ \dot{e} \;&=\; \kappa[\mathbf{A} - \mathbf{BC}\Delta_0(x_e, e)]e + \mathbf{B}_1\Delta_1(x_e, e) + \mathbf{B}_2\Delta_2(x_e, e)\end{aligned}$$

and that *identical hypotheses* hold.[2] Therefore, the proposed controller - if Assumption 1 holds - yields identical asymptotic properties.

Proposition 3 *Consider system (1), controlled by (27)–(28). Suppose that (2) hold and that $b(x, \xi)$ is bounded as in (3). Let $L(\varrho, \xi_1, \ldots, \xi_{r-1}), M, N(\varrho, \xi_1, \ldots, \xi_{r-1})$ be such that Assumption 1 holds. For every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$ there is a finite time T such that all trajectories of the closed-loop system with initial conditions $(x_e(0), \hat{\xi}_{\rm ext}(0)) \in \mathcal{C}$ remain bounded and satisfy $|(x_e(t)| \leq \varepsilon$ and $|\hat{\xi}_{\rm ext}(t)| \leq \varepsilon$ for all $t \geq T$.*

[2] Note that, while in (15) the vector field that multiplies Δ_3 is a constant vector, in the present context it is an x_e-dependent vector. However, $g_e(x_e)$ is globally bounded and this suffices to use arguments identical to those used earlier to establish the desired asymptotic properties.

If asymptotic stability is sought, it is convenient to strengthen Assumption 1 as follows.

Assumption 2 *There exist an integer ν and a triplet*

$$L(\varrho, \xi_1, \dots, \xi_{r-1}),\ M,\ N(\varrho, \xi_1, \dots, \xi_{r-1})$$

in which $L(\cdot)$ and $N(\cdot)$ are smooth functions satisfying (22)–(24) and in which $L(0,0,\dots,0) = 0$, $N(0,0,\dots,0) = 0$, such that system (19)–(20) with output (21) is strongly minimum phase and the zero dynamics is locally exponentially stable.

Proposition 4 *Consider system (1), controlled by (27)–(28). Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Let $L(\varrho, \xi_1, \dots, \xi_{r-1}), M, N(\varrho, \xi_1, \dots, \xi_{r-1})$ be such that Assumption 2 holds. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters b_0, L and $\alpha_1, \dots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(x_\mathrm{e}, \hat{\xi}_\mathrm{ext})$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.*

4 The controlled zero dynamics

The convergence results described Propositions YYY and YYY have been derived under the assumption that the controlled system is strongly minimum phase. Under this assumption, the proof the results in question consists in a straightforward use "off the shelf" the results presented earlier in section XXX, and for this reason has not been repeated. However, it happens that - since system (19)–(20) with output (21) has relative degree 1 - the same conclusions hold under the weaker assumption that this system is just minimum phase. In this case, though, the proof of the results requires a modification of the arguments presented earlier, as it will be explained in what follows.

To this end it is convenient, first of all, to bring the system in question, namely system (19)–(20) with output (21), in normal form. Bearing in mind the result established earlier on the existence of globally defined normal forms and the fact that the system in question has relative degree 1, it is known that the transformation is obtained via a global diffeomorphism

$$\tilde{x}_\mathrm{e} = \Phi(x_\mathrm{e})$$

defined as

$$\tilde{x}_\mathrm{e} = \begin{pmatrix} z_\mathrm{e} \\ \theta \end{pmatrix} = \begin{pmatrix} \Phi^{\tilde{g}_\mathrm{e}}_{-h_\mathrm{e}(x_\mathrm{e})}(x_\mathrm{e}) \\ h_\mathrm{e}(x_\mathrm{e}) \end{pmatrix}$$

in which $\Phi^{\tilde{g}_\mathrm{e}}_t(x_\mathrm{e})$ denotes the flow of the vector field $\tilde{g}_\mathrm{e}$.

A simple calculation shows that

$$z_\mathrm{e} = \mathrm{col}(z, \xi_1, \dots, \xi_{r-1}, \chi)$$

in which

$$\chi = \varrho + M \int_0^{N(\varrho,\xi_1,\dots,\xi_{r-1}) - \xi_r} \frac{1}{b(z, \xi_1, \dots, \xi_{r-1}, \xi_r + s)} ds$$

and, of course,

$$\theta = \xi_r - N(\varrho, \xi_1, \dots, \xi_{r-1}) \,.$$

Note also that the new variable χ can be also expressed as

$$\chi = \varrho + M \int_{\xi_r}^{N(\varrho,\xi_1,\ldots,\xi_{r-1})} \frac{1}{b(z,\xi_1,\ldots,\xi_{r-1},s)} ds := T(z,\xi,\varrho)$$

in which $T(z,\xi,\varrho)$ is an identity at $\xi_r = N(\varrho,\xi_1,\ldots,\xi_{r-1})$, i.e. when $\theta = 0$. In the new coordinates, the system is described by equations of the form

$$\begin{array}{rcl} \dot{z}_{\mathrm{e}} & = & \tilde{f}(z_{\mathrm{e}},\theta) \\ \dot{\theta} & = & \tilde{q}(z_{\mathrm{e}},\theta) + \tilde{b}(z_{\mathrm{e}},\theta)u\,. \end{array} \tag{31}$$

To this system we impose the same control as before, namely the control consisting of (27)–(28). In this respect, note that, by definition, we have

$$\tilde{q}(z_{\mathrm{e}},\theta) + \tilde{b}(z_{\mathrm{e}},\theta)u = q(z,\xi) - \dot{N}(\varrho,\xi) + b(z,\xi)u$$

and therefore if u is chosen as

$$u = g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho))$$

we have

$$\begin{array}{rcl} \dot{\theta} & = & \tilde{q}(z_{\mathrm{e}},\theta) + \tilde{b}(z_{\mathrm{e}},\theta)g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho)) = q(z,\xi) - \dot{N}(\varrho,\xi) + b(z,\xi)g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho)) \\ & = & k\theta + \tilde{\Delta}_3(x_{\mathrm{e}},e) \end{array}$$

where, in analogy to what it was before with (30), we have denoted by $\tilde{\Delta}_3(x_{\mathrm{e}},e)$ the "perturbation"

$$\tilde{\Delta}_3(x_{\mathrm{e}},e) = b(z,\xi)g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho)) + q(z,\xi) - \dot{N}(\varrho,\xi) - k(\xi_r - N(\varrho,\xi_1,\ldots,\xi_{r-1}))\,.$$

In this respect, note that

$$\tilde{\Delta}_3(x_{\mathrm{e}},e) = b(z,\xi)\Delta_3(x_{\mathrm{e}},e)\,.$$

In summary, the controlled closed loop system, in the new coordinates, is modeled as

$$\begin{array}{rcl} \dot{z}_{\mathrm{e}} & = & \tilde{f}(z_{\mathrm{e}},\theta) \\ \dot{\theta} & = & k\theta + \tilde{\Delta}_3(x_{\mathrm{e}},e) \\ \dot{e} & = & \kappa[\mathbf{A} - \mathbf{BC}\Delta_0(x_{\mathrm{e}},e)]e + \mathbf{B}_1\Delta_1(x_{\mathrm{e}},e) + \mathbf{B}_2\Delta_2(x_{\mathrm{e}},e)\,. \end{array} \tag{32}$$

With a forgivable abuse of notation, in the right-hand sides, we have left x_{e} as argument of the various Δ_i's, while – for consistency – we should have replaced it by $x_{\mathrm{e}} = \Phi^{-1}(\tilde{x}_{\mathrm{e}})$. This does not affect the analysis, though, because what matters is only the possibility of establish bounds that are independent of κ, which is in fact the case.

At this point, it is an easy matter to arrive at the following conclusion.

Assumption 3 *There exist an integer ν and a triplet*

$$L(\varrho,\xi_1,\ldots,\xi_{r-1}),\ M,\ N(\varrho,\xi_1,\ldots,\xi_{r-1})$$

in which $L(\cdot)$ and $N(\cdot)$ are smooth functions satisfying (22)–(24) and in which $L(0,0,\ldots,0) = 0$, $N(0,0,\ldots,0) = 0$, such that the zero dynamics of system (19)–(20) with output (21) are globally asymptotically and locally exponentially stable.

Proposition 5 *Consider system (1), controlled by (27)–(28). Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Let $L(\varrho,\xi_1,\ldots,\xi_{r-1}), M, N(\varrho,\xi_1,\ldots,\xi_{r-1})$ be such that Assumption 3 holds. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters k, b_0, L and $\alpha_1,\ldots,\alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(x_\mathrm{e},\hat{\xi}_\mathrm{ext})$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.*

Proof. By assumption, the zero dynamics of system (19)–(20) with output (21) are globally asymptotically and locally exponentially stable. Looking at the normal form (31) of this system, we have that the equilibrium $z_\mathrm{e} = 0$ of

$$\dot{z}_\mathrm{e} = \tilde{f}(z_\mathrm{e}, 0)$$

is globally asymptotically and locally exponentially stable. Thus, there exists a smooth function $W(z_\mathrm{e})$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and $\alpha(\cdot)$, that are quadratic in a neighborhood of the origin, such that

$$\begin{aligned}\underline{\alpha}(|z_\mathrm{e}|) \leq W(z_\mathrm{e}) &\leq \overline{\alpha}(|z_\mathrm{e}|) \\ \frac{\partial W}{\partial z_\mathrm{e}} \tilde{f}(z_\mathrm{e}, 0) &\leq -\alpha(|z_\mathrm{e}|)\,.\end{aligned}$$

Observe now that the first two equations of (32) can be written as (compare with (14))

$$\dot{\tilde{x}}_\mathrm{e} = \mathbf{F}(\tilde{x}_\mathrm{e}) + \mathbf{G}\Delta_3(x_\mathrm{e}, e) \tag{33}$$

in which

$$\mathbf{F}(\tilde{x}_\mathrm{e}) = \begin{pmatrix} \tilde{f}(z_\mathrm{e},\theta) \\ k\theta \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and consider the (positive definite and proper) function

$$V(\tilde{x}_\mathrm{e}) = W(z_\mathrm{e}) + \theta^2\,.$$

With $\mathcal{C}$ being the fixed compact set of initial conditions of the system, pick a number $R > 0$ such that

$$(x_\mathrm{e}, \hat{\xi}_\mathrm{ext}) \in \mathcal{C} \qquad \Rightarrow \qquad \tilde{x}_\mathrm{e} \in B_R$$

and pick a number c such that

$$\Omega_c = \{\tilde{x}_\mathrm{e} \in \mathbb{R}^{n+\nu} : V(\tilde{x}_\mathrm{e}) \leq c\} \supset B_R\,.$$

Observe now that, along the trajectories of (33),

$$\frac{\partial V}{\partial z_\mathrm{e}}\mathbf{F}(\tilde{x}_\mathrm{e}) = \frac{\partial W}{\partial z_\mathrm{e}}\tilde{f}(z_\mathrm{e},0) + \frac{\partial W}{\partial z_\mathrm{e}}[\tilde{f}(z_\mathrm{e},\theta) - \tilde{f}(z_\mathrm{e},0)] - 2|k|\theta^2\,.$$

In this respect, it should be noted that $\frac{\partial W}{\partial z_\mathrm{e}}$ is a smooth function vanishing a $z_\mathrm{e} = 0$, while $[\tilde{f}(z_\mathrm{e},\theta) - \tilde{f}(z_\mathrm{e},0)]$ is a smooth function vanishing at $\theta = 0$. Thus, there is a number $K_1 > 0$ such that, for all $\tilde{x}_\mathrm{e} \in \Omega_{c+1}$, we have

$$\left|\frac{\partial W}{\partial z_\mathrm{e}}[\tilde{f}(z_\mathrm{e},\theta) - \tilde{f}(z_\mathrm{e},0)]\right| \leq K_1|z_\mathrm{e}||\theta|\,.$$

Let

$$K_2 = \max_{\tilde{x}_e \in \Omega_{c+1}} |z_e| \,.$$

Since the function $\alpha(s)$ is quadratic in a neighborhood of the origin, it is always possible to find a number a such that

$$\alpha(s) \geq as^2 \qquad \text{for all } 0 \leq s \leq K_2 \,.$$

Hence, we can conclude that

$$\dot{V}(\tilde{x}_e) \leq -a|z_e|^2 + K_1|z_e||\theta| - 2|k|\theta^2 \qquad \text{for all } \tilde{x}_e \in \Omega_{c+1} \,.$$

From this, using standard arguments, we can conclude that, if k is sufficiently large,

$$\frac{\partial V}{\partial z_e}\mathbf{F}(\tilde{x}_e) \leq -d|\tilde{x}_e|^2 \qquad \text{for all } \tilde{x}_e \in \Omega_{c+1} \,,$$

for some $d > 0$.

A property of this kind is precisely the property that was used in the previous discussions and hence we can conclude that, if k is chosen in the way indicated, the remaining design parameters can be chosen so as to obtain the desired convergence result. $\triangleleft$

We conclude the section with some observation regarding the zero dynamics of (19)–(20) with output (21), whose asymptotic properties are used in the proof of this result. As it is well-known, to identify these dynamics it is not really necessary to use the diffeomorphism that brings the system to its normal form (31). In fact, it suffices to determine the unique u that forces $h_e(x_e)$ to be identically zero and to replace it into the dynamics of ϱ, along with the constraint resulting from $h_e(x_e) = 0$. To simplify the notation, let $\xi^*(\varrho, \xi_1, \ldots, \xi_{r-1})$ denote the r-dimensional function of ϱ and $\xi_1, \ldots, \xi_{r-1}$ defined as

$$\xi^*(\varrho, \xi_1, \ldots, \xi_{r-1}) = \begin{pmatrix} \xi_1 \\ \cdots \\ \xi_{r-1} \\ N(\varrho, \xi_1, \ldots, \xi_{r-1}) \end{pmatrix}$$

and note that the unique u forcing $\theta = \xi_{r-1} - N(\varrho, \xi_1, \ldots, \xi_{r-1})$ to be zero is the solution of

$$0 = q(z, \xi^*) + b(z, \xi^*)u - \dot{N}(\varrho, \xi^*) \,.$$

With this in mind, it is readily seen that the zero dynamics of the system can be expressed as

$$\begin{array}{rcl} \dot{z} & = & f(z, \xi^*) \\ \dot{\xi}_1 & = & \xi_2 \\ & & \cdots \\ \dot{\xi}_{r-1} & = & N(\varrho, \xi_1, \ldots, \xi_{r-1}) \\ \dot{\varrho} & = & L(\varrho, \xi_1, \ldots, \xi_{r-1}) + M[b(z, \xi^*)]^{-1}[\dot{N}(\varrho, \xi^*) - q(z, \xi^*)] \,. \end{array} \tag{34}$$

The system thus obtained can be given the following interpretation. Let $b_0(\xi))$ be a fixed function of ξ and define

$$\overline{L}(\varrho, \xi_1, \ldots, \xi_{r-1}) = L(\varrho, \xi_1, \ldots, \xi_{r-1}) + [b_0(\xi^*)]^{-1}M\dot{N}(\varrho, \xi^*) \,,$$

$$\Delta_b(z,\xi) = \frac{b_0(\xi) - b(z,\xi)}{b_0(\xi)b(z,\xi)}\,.$$

Then, the last equation that characterizes the system can be written as

$$\dot{\varrho} = \overline{L}(\varrho, \xi_1, \ldots, \xi_{r-1}) + M\Big(\Delta_b(z,\xi^*)\dot{N}(\varrho,\xi^*) - \frac{q(z,\xi^*)}{b(z,\xi^*)}\Big)\,.$$

In this way, system (34) can be seen as the interconnection of a system, with state $z, \xi_1, \ldots, \xi_{r-}$, inputs $u_{\rm a}, v_{\rm a}$ and outputs $\xi_1, \ldots, \xi_{r-1}, y_{\rm a}$, defined as

$$\begin{array}{rcl} \dot{z} &=& f(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a}) \\ \dot{\xi}_1 &=& \xi_2 \\ && \cdots \\ \dot{\xi}_{r-1} &=& u_{\rm a} \\ y_{\rm a} &=& \dfrac{q(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a})}{b(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a})} - \Delta_b(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a})v_{\rm a} \end{array} \tag{35}$$

and a system with state ϱ, defined as

$$\begin{array}{rcl} \dot{\varrho} &=& \overline{L}(\varrho, \xi_1, \ldots, \xi_{r-1}) - My_{\rm a} \\ u_{\rm a} &=& N(\varrho, \xi_1, \ldots, \xi_{r-1}) \\ v_{\rm a} &=& \dot{N}(\varrho, \xi_1, \ldots, \xi_{r-1}, u_{\rm a})\,. \end{array} \tag{36}$$

In this interpretation, the Assumption XXX above can be simply interpreted as the assumption that system (36) globally asymptotically stabilizes system (35). Note, in particular, that if $b(z,\xi)$ is independent of z, one could pick $b_0(\xi) = b(z,\xi)$ resulting in

$$\Delta_b(z,\xi) = 0\,.$$

In this special case, system (35) reduces to

$$\begin{array}{rcl} \dot{z} &=& f(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a}) \\ \dot{\xi}_1 &=& \xi_2 \\ && \cdots \\ \dot{\xi}_{r-1} &=& u_{\rm a} \\ y_{\rm a} &=& \dfrac{q(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a})}{b(z, \xi_1, \ldots, \xi_{r-1}, u_{\rm a})} \end{array}$$

and system (36) reduces to

$$\begin{array}{rcl} \dot{\varrho} &=& \overline{L}(\varrho, \xi_1, \ldots, \xi_{r-1}) - My_{\rm a} \\ u_{\rm a} &=& N(\varrho, \xi_1, \ldots, \xi_{r-1})\,. \end{array}$$