Optimal Control

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Lecture 7

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THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

SINGULAR SOLUTIONS

Definition:

Let $\left(x^o, u^o, t_f^o\right)$ be an optimal solution of the above problem,

and $\lambda_0^o\ \lambda^o$ the corresponding multipliers.

The solution is **singular** if there exists a subinterval

$\left(t', t''\right),\ t'' > t'$ in which the Hamiltonian $H\left(x^o(t), \omega, \lambda_0^o, \lambda^o(t), t\right)$

is **independent** from at least one component of ω in $\left(t', t''\right)$

SINGULAR SOLUTIONS

Theorem:

Assume the Hamiltonian of the form

$$H(x,u,\lambda_0,\lambda,t)=H_1(x,\lambda_0,\lambda,t)+H_2(x,\lambda_0,\lambda,t)N(x,u,\lambda_0,\lambda,t)$$

Let $\left(x^*,u^*,t_f^*\right)$ be an extremum and λ_0^* λ^* the corresponding multipliers such that $N(x^*,\omega,\lambda^*{}_0,\lambda^*,t)$ **is dependent on any component of** ω **in any subinterval of** $[t_i, t_f]$

A necessary condition for $\left(x^*,u^*,t_f^*\right)$ to be a singular extremum is that there exists a subinterval $[t', t'']\subset[t_i, t_f], t''>t'$ such that:

$$H_2(x,\lambda_0,\lambda,t)=0,\ \ \forall t\in[t',t'']$$

The linear minimum time optimal control

Let us consider the problem of optimal control of a **linear system**

- ✓ with **fixed initial and final state**,
- ✓ **constraints on the control variables** and
- ✓ with cost index equal to the **length of the time interval**

Problem: Consider the linear dynamical system:

$$\boxed{\dot{x}(t) = A(t)x(t) + B(t)u(t)}$$

With $x(t) \in R^n, \quad u(t) \in R^p$ and the constraint

$$\boxed{\left|u_j(t)\right| \le 1, \quad j = 1,2,\ldots,p, \quad \forall t \in R}$$

Matrices A and B have entries with elements of class

C^{n-2} and C^{n-1} respectively, at least of C^1 class .

The initial instant t_i is fixed and also:

$$\boxed{x(t_i) = x^i, \quad x(t_f) = 0}$$

The aim is to determine the **final instant** $t_f^o \in R$

the control $\boxed{u^o \in \bar{C}^o(R)}$

and the state $\boxed{x^o \in \bar{C}^1(R)}$

that minimize the cost index:

$$\boxed{J(t_f) = \int_{t_i}^{t_f} dt = t_f - t_i}$$

Theorem: Necessary conditions for $\left(x^*, u^*, t_f^*\right)$ to be an optimal solution are that there exist a constant $\lambda_0^* \geq 0$ and an n-dimensional function $\lambda^* \in \overline{C}^1\left[t_i, t_f\right]$ not simultaneously null and such that:

$$\boxed{\begin{aligned} &\dot{\lambda}^* = -A^T \lambda^* \\ &\lambda^{*T} B\omega \geq \lambda^{*T} Bu^* \quad \forall \omega \in R^p : \left|\omega_j\right| \leq 1, \quad j = 1,2,\ldots, p \end{aligned}}$$

Possible discontinuities in $\dot{\lambda}^*$ can appear only in the points in which u^* has a discontinuity.

Moreover the associated Hamiltonian is a continuous function with respect to *t* and it results: $\boxed{H\Big|_{t_f^*}^{*} = 0}$

Proof: The Hamiltonian associated to the problem is:

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 + \lambda^T Ax + \lambda^T Bu$$

Applying the **minimum principle** the theorem is proved.

STRONG CONTROLLABILITY

Strong controllability corresponds to the controllability in

any instant t_i,

in any time interval and

by <u>any component</u> of the control vector

STRONG CONTROLLABILITY

Let us indicate with $b_j(t)$ the *j*-th column of the matrix *B* .

The **strong controllability** **<u>is guaranteed</u>** by the condition:

$$\det\{G_j(t)\} = \det\left\{\begin{pmatrix} b_j^{(1)}(t) & b_j^{(2)}(t) & \cdots & b_j^{(n)}(t) \end{pmatrix}\right\} \neq 0$$

$$j = 1,2,\ldots,p, \quad \forall t \geq t_i$$

where :

Generic column of matrix *B(t)*

$$b_j^{(1)}(t) = b_j(t)$$

$$b_j^{(k)}(t) = \dot{b}_j^{(k-1)}(t) - A(t)b_j^{(k-1)}(t) \quad k = 2,3,\ldots,n$$

Remark:

In the NON-steady state case the above condition is **sufficient** for strong controllability.

In the **steady case** it is **necessary and sufficient** and may be written in the usual way:

$$\boxed{\det\left\{\begin{pmatrix} b_j & Ab_j & \cdots & A^{n-1}b_j \end{pmatrix}\right\} \neq 0 \qquad j = 1,2,\dots,p}$$

Characterization of the optimal solution

Characterization of the optimal solution

Theorem: Let us consider the minimal time optimal problem .
If the strong controllability condition is satisfied and if an optimal solution exists

it is **non singular**.

Every component of the optimal control is **piecewise constant** assuming only the extreme values ± 1

and the **number of discontinuity instants is limited**.

Proof:

✓ the **non singularity** of the solution is proved by **contradiction arguments**.

If the solution were singular, from the necessary condition of the previous theorem:

$$\lambda^{*T} B\omega \geq \lambda^{*T} Bu^{*}$$

$$\forall \omega \in R^{p} : \left|\omega_{j}\right| \leq 1, \quad j = 1,2,\ldots, p$$

there would exist a value $j \in \{1,2,\ldots, p\}$

and a subinterval

$$[t', t''] \subset [t_i, t_f^o]$$

such that:

$$\lambda^{oT}(t) b_j(t) = 0, \quad \forall t \in [t', t'']$$

Deriving successively this expression and using $\dot{\lambda}^* = -A^T \lambda^*$
we obtain:

$$\frac{d^i\left(\lambda^{oT}(t) b_j(t)\right)}{dt^i} = \lambda^{oT}(t) b_j^{(i+1)}(t) = 0, \quad \forall t \in [t', t''], i = 1, 2, \ldots, n-1$$

$$\Rightarrow \quad \lambda^{oT}(t) G_j(t) = 0, \quad \forall t \in [t', t'']$$

On the other hand $\lambda^o(t)$ **must be different from zero in every instant of the interval** , otherwise, from the necessary condition $\dot{\lambda}^* = -A^T \lambda^*$

it should be null in the whole interval and in particular $\lambda^o(t_f) = 0$

From the necessary condition:

$$H\Big|_{t_f^*}^{*} = 0$$

it would result that also $\lambda_0^o = 0$ which is absurd.

Therefore, from the condition:

$$\lambda^{oT}(t) G_j(t) = 0, \quad \forall t \in [t', t'']$$

it results that

$$\det\{G_j(t)\} = 0 \quad \forall t \in [t', t'']$$

which contradicts the hypothesis of strong controllability.

$\Rightarrow$ the **solution is non singular**.

- ✓ From the non singularity of the optimal solution it results that the quantity

$$\lambda^{oT}(t)b_j(t)$$

can be null only on isolated points .

Therefore the necessary condition

$$\lambda^{*T}B\omega \geq \lambda^{*T}Bu^{*} \quad \forall \omega \in R^{p}: \left|\omega_j\right| \leq 1, \quad j=1,2,\dots,p$$

implies

$$\boxed{\begin{array}{l} u_j^{o}(t) = -sign\left\{\lambda^{oT}(t)b_j(t)\right\}, \\ j=1,2,\dots,p \quad \forall t \in [t_i,\, t_f^{o}] \end{array}}$$

✓ To demonstrate that the number of discontinuity instants is finite, we can proceed by **contradiction argument** that would easily imply

$$\det\{G_j(t)\}=0 \quad \forall t\in[t',t'']$$

which contradicts the hypothesis of strong controllability.

In fact, let us assume that at least for one component of the control $u_j^o(t)$ it is not true.

Then an accumulation point $\tau\in\left[t_i,\, t_f^o\right]$ of instants $\bar{t}_k^{(j)}$ exists and, for continuity argument

$$\lambda^{oT}(\tau)b_j(\tau)=0$$

For the continuity of $\lambda^{oT}(t)b_j(t)=0$ between $\bar{t}_k^{(j)}$ and $\bar{t}_{k+1}^{(j)}$

there exists an instant, say $\bar{t}_k^{(j)}$, in which $\frac{d\left(\lambda^{oT}(t)b_j(t)\right)}{dt}=0$

Therefore, for continuity, also $\boxed{\left.\frac{d\left(\lambda^{oT}(t)b_j(t)\right)}{dt}\right|_{\tau}=0}$

Analogously: $\boxed{\left.\frac{d^i\left(\lambda^{oT}(t)b_j(t)\right)}{dt^i}\right|_{\tau}=0, \quad i=0,1,\ldots,(n-1)}$

$$\Rightarrow \qquad \det\{G_j(t)\}=0 \quad \forall t\in[t',t'']$$

That is in contraddiction with the strong controllability hypothesis.

A control function that assumes only the limit values

is called **bang-bang control**

and

the instants of discontinuity are called

commutation instants

UNIQUENESS RESULT

Theorem (UNIQUENESS):

If the hypothesis of strong controllability is satisfied, if an optimal solution exists it is **unique**.

Proof: the theorem is proved by contradiction arguments.

Remark

These results hold also to the case in which the control is in the ***space of measurable function*** defined on the space of real number *R*.

Let $S[t_i, T]$ be the space of piecewise constant functions $s(t)$

If $z(t) = \lim_{k \to \infty} s_k(t)$, for $\{s_k(t)\} \subset S[t_i, T]$ then z(t) is a measurable function on $[t_i, T]$

Existence of the optimal solution

Theorem: if the condition of strong controllability is satisfied and an admissible solution exists, then there exists a unique optimal solution nonsingular and the control is bang-bang.

Proof: the existence theorem is proved on the basis of the following results.

The following result holds:

Theorem: Let assume that the control functions belong to the **space of measurable functions.**

If an admissible solution exists then the optimal solution exists.

Proof: let us consider the non trivial case in which the number of admissible solution is **infinite** and let us indicate with t_f^o the **minimum extremum** of the instants corresponding to the admissible solutions t_f

It is possible to built a sequence of admissible solutions

$\boxed{\left\{\left(x^{(k)}, u^{(k)}, t_f^{(k)}\right)\right\},\ t_f^{(k)} \geq t_f^o}$ such that $\lim_{k\to\infty} t_f^{(k)} = t_f^o$

Let us indicate with $\phi(t,\tau)$ the transition matrix of the linear system. It results:

$$x^{(k)}(t_f^{(k)}) = \phi\left(t_f^{(k)}, t_i\right)x^i + \int_{t_i}^{t_f^{(k)}} \phi\left(t_f^{(k)}, t\right)B(t)u^{(k)}(t)dt$$

$$x^{(k)}(t_f^o) = \phi\left(t_f^o, t_i\right)x^i + \int_{t_i}^{t_f^o} \phi\left(t_f^o, t\right)B(t)u^{(k)}(t)dt$$

$$\lim_{k\to\infty}\left[x^{(k)}(t_f^{(k)}) - x^{(k)}(t_f^o)\right] = \boxed{\lim_{k\to\infty}\left[\phi\left(t_f^{(k)}, t_i\right) - \phi\left(t_f^o, t_i\right)\right]x^i}$$

$$+\boxed{\lim_{k\to\infty}\int_{t_i}^{t_f^o}\left[\phi\left(t_f^{(k)}, t\right) - \phi\left(t_f^o, t\right)\right]B(t)u^{(k)}(t)dt} + \boxed{\lim_{k\to\infty}\int_{t_f^o}^{t_f^{(k)}} \phi\left(t_f^{(k)}, t\right)B(t)u^{(k)}(t)dt} = 0$$

0

It is possible to built a sequence of admissible solutions

$\boxed{\left\{\left(x^{(k)}, u^{(k)}, t_f^{(k)}\right)\right\}, \; t_f^{(k)} \geq t_f^o}$ such that $\lim\limits_{k\to\infty} t_f^{(k)} = t_f^o$

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$$x^{(k)}(t_f^o) = \phi\left(t_f^o, t_i\right)x^i + \int_{t_i}^{t_f^o} \phi\left(t_f^o, t\right)B(t)u^{(k)}(t)dt$$

$$\lim_{k\to\infty}\left[x^{(k)}(t_f^{(k)}) - x^{(k)}(t_f^o)\right] = \boxed{\lim_{k\to\infty}\left[\phi\left(t_f^{(k)}, t_i\right) - \phi\left(t_f^o, t_i\right)\right]x^i} \rightarrow 0$$

$$+ \boxed{\lim_{k\to\infty}\int_{t_i}^{t_f^o}\left[\phi\left(t_f^{(k)}, t\right) - \phi\left(t_f^o, t\right)\right]B(t)u^{(k)}(t)dt} + \boxed{\lim_{k\to\infty}\int_{t_f^o}^{t_f^{(k)}} \phi\left(t_f^{(k)}, t\right)B(t)u^{(k)}(t)dt} = 0$$

Since $x^{(k)}(t_f^{(k)})=0, \quad k=1,2,\ldots.$ it results: $\boxed{\lim_{k\to\infty} x^{(k)}(t_f^o)=0}$

Let us indicate by $\overline{u}^{(k)}$ the function $u^{(k)}$ truncated on the interval $\left[t_i,\, t_f^o\right]$

This function is limited and L_2 , since $\left|u_j^{(k)}(t)\right|\le 1, \; j=1,2,\ldots, p.$

The sequence of functions $\left\{\overline{u}^{(k)}\right\}$ belongs to the space M of measurable function, L_2 and such that

$$\left|u_j(t)\right|\le 1, \;\; j=1,2,\ldots,p, \;\; \forall t\in\left[t_i,t_f^o\right]$$

Therefore this sequence admits a subsequence, indicated still with $\left\{\overline{u}^{(k)}\right\}$, *weakly converging* to a function $u^o\in M$

This implies that for any measurable function h in L_2 it results:

$$\lim_{k\to\infty}\int_{t_i}^{t_f^o} h^T(t)\overline{u}^{(k)}(t)dt=\int_{t_i}^{t_f^o} h^T(t)u^o(t)dt$$

If we indicate by x^o the evolution of the state corresponding to the input u^o from the initial condition we have:

$$\lim_{k\to\infty} x^{(k)}(t_f^o)=\phi\left(t_f^o,t_i\right)x^i+\lim_{k\to\infty}\int_{t_i}^{t_f^o}\phi\left(t_f^o,t\right)B(t)\overline{u}^{(k)}(t)dt$$

$$=\phi\left(t_f^o,t_i\right)x^i+\int_{t_i}^{t_f^o}\phi\left(t_f^o,t\right)B(t)u^o(t)dt=x^o(t_f^o)$$

Since the elements of $\phi\left(t_f^o,t\right)B(t)$ are measurable and L_2

Therefore, taking into account the expression $\boxed{\lim_{k\to\infty} x^{(k)}(t_f^o)=0}$ we deduce : $x^o(t_f^o)=0$

An admissible measurable control L_2 exists able to transfer the initial state to the origin.

The optimal solution is $\left(x^o, u^o, t_f^o\right)$

The results about the characterization of the control may be extended to the case in which the control belong to the space of measurable functions:

Therefore it results in:

If the strong controllability condition is satisfied and

if an optimal control in the **space of measurable function**

exists then this solution is unique,

non singular and the *control is bang bang type*

Remark

The existence of the optimal solution is guaranteed only for the couples (t_i, x^i) for which the admissible solution exists

Existence of the optimal solution

Theorem: if the condition of strong controllability is satisfied and an admissible solution exists
with the control in the space of measurable functions,
then there exists a unique optimal solution
nonsingular and
the control is bang-bang.

Minimum time problem for steady state system

Let us consider the minimum time problem in case of **steady state system**.

In this case it is possible to deduce results about the number of commutation points

Problem: Consider the linear dynamical system:

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)}$$

With $x(t) \in R^n, \;\; u(t) \in R^p$ and the constraint

$$\boxed{\left| u_j(t) \right| \leq 1, \;\; j = 1,2,\ldots,p, \;\; \forall t \in R}$$

The initial instant t_i is fixed and also:

$$\boxed{x(t_i) = x^i, \quad x(t_f) = 0}$$

The aim is to determine the final instant $t_f^o \in R$

the control $u^o \in \overline{C}^o(R)$

and the state $x^o \in \overline{C}^1(R)$

that minimize the cost index:

$$J(t_f) = \int_{t_i}^{t_f} dt = t_f - t_i$$

Theorem:

Let us consider the above time minimum problem and assume that the control function belong to the space of measurable function ;

if the system is controllable

$\Rightarrow$

there exists a neighbor Ω of the origin such that for any

$x^i \in \Omega$ there exists an optimal **solution.**

Proof:

We will show that it is possible to determine a neighbor Ω of the origin such that for any $x^i \in \Omega$ there exists an admissible solution.

On the basis of the previous results this implies the existence of the optimal solution.

Let us fix any $t_f > 0$. The existence of a control able to transfer the initial state $x^i \in \Omega$ to the origin at the instant t_f corresponds to the possibility of solving :

$$e^{A(t_f - t_i)}x^i + \int_{t_i}^{t_f} e^{A(t_f - \tau)} Bu(\tau)d\tau = 0$$

with respect to the control function *u*.

Let assume for the control the expression:

$$u(\tau) = B^T e^{-A^T\tau}\upsilon, \quad t_i \le \tau \le t_f, \ \upsilon \in R^p$$

and substitute it in the preceeding equation:

$$-e^{-At_i}x^i = \boxed{\left(\int_{t_i}^{t_f} e^{-A\tau}BB^T e^{-A^T\tau}d\tau\right)\upsilon}$$

Gramian matrix

From the controllability hypothesis, the Gramian matrix is not singular, therefore:

$$u(\tau) = -B^T e^{-A^T \tau} \left(\int_{t_i}^{t^f} e^{-A\sigma} B B^T e^{-A^T \sigma} d\sigma \right)^{-1} e^{-At_i} x^i ,$$

$$t_i \le \tau \le t_f$$

$\Rightarrow$ a measurable admissible control exists if the initial state is sufficiently near the origin.

Remark

From results obtained in the non steady state case it results that **an optimal non singular solution exists and the control is bang-bang type.**

Theorem: Let us consider the above time minimum problem in the steady state case and assume that the control function belongs to the space of measurable function *M(R).*

If the system is controllable and the **eigenvalues of matrix A have negative real part** there exists an optimal solution **whatever the initial state is.**

Proof: Whatever the initial state is chosen, an admissible solution can be defined as follows:

- Let Ω be a neighbor of the origin constituted by the original state for which an admissible solution exists;
- Let Ω’ be a closed subset in Ω
- From the asymptotic stability of the system it is possible to reach Ω’ in a finite time with null input, from any initial state

Remark

From the results obtained in the non steady state case it results that an optimal non singular solution exists and the control is bang-bang type.

Question: how many commutation instants are possible in the optimal time control problem?

Theorem: **If the controllability condition** is satisfied and if all the eigenvalues of the matrix A are real non positive, then **the number of commutation instants for any components of control is less or equal than n-1,** whatever the inititial state is.

Proof: from the previous results, we have:

$$u_j^o(t) = -sign\left\{\lambda^{oT}(t) b_j\right\} \quad j = 1,2,\ldots, p, \ \forall t \in \left[t_i, t_f^o\right]$$

where the costate λ^0 in the steady state case is:

$$\lambda^0(t) = e^{-A^T(t-t_i)}\lambda_i$$

Denoting by α_k and m_k the eigenvalues of A and their molteplicity, the r-th component of λ^0 can be expressed as follows:

$$\lambda_r^0(t) = \sum_{s=1}^{k} p_{rs}(t) e^{-\alpha_s (t-t_i)}, \quad r = 1,2,\ldots,n$$

where p_{rs} is a polynomial function of degree less than m_s.

Substituting this expression in the bang bang control we obtain:

$$u_j^o(t) = -sign\left\{ \sum_{s=1}^{k} \left(\sum_{r=1}^{n} b_{jr} p_{rs}(t) \right) e^{-\alpha_s (t-t_i)} \right\}$$

$$= -sign\left\{ \sum_{s=1}^{k} P_{js}(t) e^{-\alpha_s (t-t_i)} \right\},$$

Polynomial function of degree less than m_s

$$j = 1,2,\ldots, p; \; \forall t \in \left[t_i, \, t_f^o \right]$$

The argument of the sign function has at most

$m_1 + m_{12} + \ldots. + m_{k-1} = n-1$

real solutions.

Therefore the control u_j^o has at most n-1 roots.

Remarks

1. If the eigenvalues of A are not all real the number of switching instants is bounded but not uniformly.
2. In general it is not possible to establish an esplicit relation between the control and the state