48

Nonlinear Output Regulation

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48.1 The Problem

A classical problem in control theory is to impose, via feedback, a prescribed steady-state response to every external command in a given family. This may include, for instance, the problem of having the output of a controlled plant asymptotically track any prescribed reference signal in a certain class of functions of time, as well as the problem of having this output asymptotically reject any undesired disturbance in a certain class of disturbances. In both cases, the issue is to force a suitably defined *tracking error* to zero, as time tends to infinity, for every reference output and every undesired disturbance ranging over prescribed families of functions of time.

Generally speaking, the problem can be cast as follows. Consider a finite-dimensional, time-invariant, nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{x} &= F(w, x, u), \\ e &= H(w, x), \end{aligned} \tag{48.1}$$

in which $x \in \mathbb{R}^n$ is a vector of state variables, $u \in \mathbb{R}$ is a vector of inputs used for *control* purposes, $w \in \mathbb{R}^s$ is a vector of inputs that cannot be controlled and include *exogenous* commands, exogenous disturbances, and model uncertainties, and $e \in \mathbb{R}$ is a vector of *regulated* outputs that include tracking errors and any other variable that needs to be steered to 0. The problem is to design a controller, which receives $e(t)$ as input and produces $u(t)$ as output, able to guarantee that, in the resulting closed-loop system, $x(t)$ remains bounded and $e(t) \to 0$ as $t \to \infty$, regardless of what the exogenous input $w(t)$ actually is.

The ability to successfully address this problem very much depends on how much the controller is allowed to know about the exogenous disturbance $w(t)$. In the ideal situation in which $w(t)$ is available to the controller in real time, the design problem indeed looks much simpler. This is, however, only an extremely optimistic situation which does not represent, in any circumstance, a realistic scenario. The other extreme situation is the one in which nothing is known about $w(t)$. In this, pessimistic, scenario the best result one could hope for is the fulfillment of some prescribed ultimate bound for $|e(t)|$, but certainly not a sharp goal such as the convergence of $e(t)$ to 0. A more comfortable, intermediate, situation is the one in which $w(t)$ is only known *to belong to a fixed family* of functions of time, for instance, the family of all solutions obtained from a fixed ordinary differential equation of the form

$$\dot{w} = s(w) \tag{48.2}$$

as the corresponding initial condition $w(0)$ is allowed to vary on a prescribed set. This situation is in fact sufficiently distant from the ideal but unrealistic case of perfect knowledge of $w(t)$ and from the realistic but conservative case of totally unknown $w(t)$. But, above all, this way of thinking about the exogenous inputs covers a number of cases of major practical relevance. There is, in fact, an abundance of design problems in which parameter uncertainties, reference command, and/or exogenous disturbances can be modeled as functions of time that satisfy an ordinary differential equation.

The control law is to be provided by a system modeled by equations of the form

$$\begin{aligned} \dot{x}_c &= F_c(x_c, e), \\ u &= H_c(x_c, e), \end{aligned} \tag{48.3}$$

with state $x_c \in \mathbb{R}^\nu$. The initial conditions $x(0)$ of the *controlled plant* (Equation 48.1), $w(0)$ of the *exosystem* (Equation 48.2), and $x_c(0)$ of the *controller* (Equation 48.3) are allowed to range over a fixed *compact* sets $X \subset \mathbb{R}^n$, $W \subset \mathbb{R}^s$, and $X_c \subset \mathbb{R}^\nu$ respectively. All maps characterizing the model of the controlled plant, of the exosystem, and of the controller are assumed to be sufficiently differentiable.

The problem that is analyzed in this chapter, known as the *problem of output regulation* (or *generalized tracking problem* or also *generalized servomechanism problem*), is to design a feedback controller of the form (Equation 48.3) so as to obtain a closed-loop system in which all trajectories are bounded and the regulated output $e(t)$ asymptotically decays to 0 as $t \to \infty$. More precisely, it is required that the composition of Equations 48.1, 48.2, and 48.3, that is the *autonomous* system

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{x} &= F(w, x, H_c(x_c, H(w, x)), \\ \dot{x}_c &= F_c(x_c, H(w, x)), \end{aligned} \tag{48.4}$$

with output

$$e = H(w, x),$$

be such that

- The positive orbit of $W \times X \times X_c$ is bounded, that is, there exists a bounded subset S of $\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^\nu$ such that, for any $(w_0, x_0, x_{c,0}) \in W \times X \times X_c$, the integral curve $(w(t), x(t), x_c(t))$ of Equation 48.4 passing through $(w_0, x_0, x_{c,0})$ at time $t = 0$ remains in S for all $t \geq 0$.
- $\lim_{t\to\infty} e(t) = 0$, uniformly in the initial condition, that is, for every $\varepsilon > 0$ there exists a time $\bar{t}$, depending only on ε and *not on* $(w_0, x_0, x_{c,0}) \in W \times X \times X_c$, such that the integral curve $(w(t), x(t), x_c(t))$ of Equation 48.4 passing through $(w_0, x_0, x_{c,0})$ at time $t = 0$ satisfies $|e(t)| \leq \varepsilon$ for all $t \geq \bar{t}$.

Cast in these terms, the problem is readily seen to be equivalent to a problem to design a controller yielding a closed-loop system that possesses a steady-state locus* entirely immersed in the set of all (w, x) at which the regulated output $e = H(w, x)$ is 0. This being the case, there is no loss of generality in assuming from the very beginning that *the exosystem is in steady state*, which is the case when the compact set W is *invariant* under the dynamics of Equation 48.2. This will be assumed throughout the entire chapter.

* See Chapter 47 for a definition of steady state in a nonlinear system.

Note also that an approach of this kind covers also the case in which the exosystem dynamics admit a decomposition of the form

$$\begin{aligned}\dot{w}_1 &= s_1(w_1, w_2),\\ \dot{w}_2 &= 0,\end{aligned}$$

in which some of the components of the exogenous input have a trivial dynamics, that is, are constant. The elements of w_2 comprise any uncertain constant parameters (assumed to range on a compact set) affecting the model of the controlled plant (Equation 48.1), as well as the dynamics of the time-varying components of w. Thus, solving a design problem cast in these terms provides *robustness* with respect to structured parametric uncertainties in the model of the plant, as well as in the model of the exogenous inputs to be tracked and/or rejected.

48.2 The Case of Linear Systems as a Design Paradigm

As an introduction, we describe in this section how the problem of output regulation can be analyzed and solved for linear systems. This provides in fact an instructive design paradigm that can be successfully followed in handling the corresponding general problem. The first step is the analysis of the steady state, which in turn entails the derivation of certain *necessary conditions*.

Consider the case in which the composition of exosystem (Equation 48.2) and controlled plant (Equation 48.1) is modeled by equations of the form

$$\begin{aligned}\dot{w} &= Sw,\\ \dot{x} &= Pw + Ax + Bu,\\ e &= Qw + Cx,\end{aligned} \tag{48.5}$$

and the controller (Equation 48.3) is modeled by equations of the form

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c e,\\ u &= C_c x_c + D_c e.\end{aligned} \tag{48.6}$$

The associated closed-loop system is the autonomous linear system

$$\begin{pmatrix}\dot{w}\\ \dot{x}\\ \dot{x}_c\end{pmatrix} = \begin{pmatrix} S & 0 & 0\\ P + BD_cQ & A + BD_cC & BC_c\\ B_cQ & B_cC & A_c\end{pmatrix}\begin{pmatrix} w\\ x\\ x_c\end{pmatrix}. \tag{48.7}$$

If the controller (Equation 48.6) solves the problem of output regulation, the trajectories of Equation 48.7 are bounded and, necessarily, all the eigenvalues of the matrix

$$\begin{pmatrix} A + BD_cC & BC_c\\ B_cC & A_c\end{pmatrix}$$

have negative real part. Since by assumption S has all eigenvalues on the imaginary axis, system (Equation 48.7) possesses two complementary invariant subspaces: a *stable invariant subspace* and a *center invariant subspace*. The latter, in particular, is the graph of a linear map

$$w \;\mapsto \begin{pmatrix} x\\ x_c\end{pmatrix} = \begin{pmatrix}\Pi\\ \Pi_c\end{pmatrix} w$$

in which Π and Π_c are solutions of the Sylvester equation

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_cQ \\ B_cQ \end{pmatrix}. \tag{48.8}$$

Any trajectory of Equation 48.7 has a unique decomposition into a component entirely contained in the stable invariant subspace and a component entirely contained in the center invariant subspace. The former, which asymptotically decays to 0 as $t \to \infty$ is the *transient component* of the trajectory. The latter, in which $x(t)$ and $x_c(t)$ have, respectively, the form

$$x(t) = \Pi w(t), \quad x_c(t) = \Pi_c w(t) \tag{48.9}$$

is the *steady-state component* of the trajectory.

If the controller (Equation 48.6) solves the problem of output regulation, the steady-state component of any trajectory must be contained in the kernel of the map $e = Qw + Cx$ and hence the solution (Π, Π_c) of the Sylvester equation 48.8 necessarily satisfies $Q + C\Pi = 0$. Entering this constraint into Equation 48.8 it is concluded that if the controller (Equation 48.6) solves the problem of output regulation, necessarily there exists a pair (Π, Π_c) satisfying

$$\begin{aligned} \Pi S &= A\Pi + BC_c\Pi_c + P, \\ \Pi_c S &= A_c\Pi_c, \\ 0 &= C\Pi + Q. \end{aligned}$$

Setting $\Psi = C_c\Pi_c$ the first and third equations are more conveniently rewritten in the (controller-independent) form

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P, \\ 0 &= C\Pi + Q, \end{aligned} \tag{48.10}$$

in which, of course, Ψ is a matrix satisfying

$$\begin{aligned} \Psi &= C_c\Pi_c, \\ \Pi_c S &= A_c\Pi_c, \end{aligned} \tag{48.11}$$

for some choice of Π_c, A_c, C_c. The linear equations 48.10 are known as Francis' equations and the existence of a solution pair (Π, Ψ) is—as shown—a *necessary condition* for the solution of the problem of output regulation [1,10,11].

Equations 48.11, from a general viewpoint, could be regarded as a constraint on the component Ψ of the solution of Francis' equations 48.10. However, as an easy calculation shows, this constraint is actually irrelevant. In fact, given any pair S, Ψ it is always possible to fulfill conditions like (Equation 48.11). Let

$$d(\lambda) = s_0 + s_1\lambda + \cdots + s_{d-1}\lambda^{d-1} + \lambda^d$$

denote the minimal polynomial of S and set

$$T = \begin{bmatrix} \Psi \\ \Psi S \\ \cdots \\ \Psi S^{d-2} \\ \Psi S^{d-1} \end{bmatrix}, \quad \Phi = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0 & -s_1 & -s_2 & \cdots & -s_{d-1} \end{pmatrix}, \quad \Gamma = (1 \quad 0 \quad 0 \quad \cdots \quad 0).$$

Then, it is immediate to check that

$$\begin{aligned} \Psi &= \Gamma T, \\ TS &= \Phi T, \end{aligned} \tag{48.12}$$

which is precisely a constraint of the form (Equation 48.11). Note, in particular, that the pair (Φ, Γ) thus defined is observable.

The relevant role, however, of Equations 48.11 is that they interpret the ability, of the controller, to generate the *feedforward input* necessary to keep the regulated variable identically zero in steady state. In steady state—as shown—the state $x(t)$ of the plant and $x_c(t)$ of the controller evolve as in Equation 48.9 and $e(t) = 0$. Consequently, in steady state the controller is driven by input which is identically zero and generates, as output, a control of the form

$$u_{ss}(t) = C_c x_c(t) = C_c \Pi_c w(t) = \Psi w(t).$$

The latter, as predicated by Francis' equations, is a control able to force a steady state trajectory of the form $x(t) = \Pi w(t)$ and consequently to keep $e(t)$ identically zero. The property thus described is usually referred to as the *internal model property*: any controller that solves the problem of output regulation necessarily embeds a model of the feedforward inputs needed to keep $e(t)$ identically zero [2].

We proceed now with the design of a control that ensures asymptotic convergence to the required steady state. In view of the above analysis, to solve the problem of output regulation it is natural to assume that Francis' equations 48.10 have a solution, and to consider a controller of the form

$$\begin{aligned} u &= \Gamma\eta + v, \\ \dot{\eta} &= \Phi\eta + v', \end{aligned} \tag{48.13}$$

where Γ and Φ satisfy (Equation 48.12) for some T (which, as shown, is always possibile) and where v, v' are additional controls. If these controls vanish in steady state, the graph of the linear map

$$w \mapsto \begin{pmatrix} x \\ \eta \end{pmatrix} = \begin{pmatrix} \Pi \\ T \end{pmatrix} w \tag{48.14}$$

is, by construction, an invariant subspace of the composite system

$$\begin{aligned} \dot{w} &= Sw, \\ \dot{x} &= Pw + Ax + B(\Gamma\eta + v), \\ \dot{\eta} &= \Phi\eta + v', \\ e &= Qw + Cx. \end{aligned} \tag{48.15}$$

The regulated variable e vanishes on the graph of Equation 48.14. Thus, if the additional controls v and v' are able to (robustly) steer all trajectories to this invariant subspace, the problem of output regulation is solved.

Let now the states x and η of Equation 48.15 be replaced by the differences

$$\tilde{x} = x - \Pi w, \quad \tilde{\eta} = \eta - Tw,$$

in which case the equations describing the system, by virtue of Equations 48.10 and 48.12, become

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\Gamma\tilde{\eta} + Bv, \\ \dot{\tilde{\eta}} &= \Phi\tilde{\eta} + v', \\ e &= C\tilde{x}. \end{aligned} \tag{48.16}$$

To steer all trajectories of Equation 48.15 to the graph of Equation 48.14 is the same as to stabilize the equilibrium $(\tilde{x}, \tilde{\eta}) = (0, 0)$ of Equation 48.16.

A simple design option, at this point, is to set $v' = Gv$, and to seek a (possibly dynamic) controller, with input e and output v, which (robustly) stabilizes the equilibrium $(\tilde{x}, \tilde{\eta}) = (0, 0)$ of

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\Gamma\tilde{\eta} + Bv, \\ \dot{\tilde{\eta}} &= \Phi\tilde{\eta} + Gv, \\ e &= C\tilde{x}. \end{aligned} \tag{48.17}$$

A sufficient condition under which such a robust stabilizer exists is that *all zeros of* Equation 48.17 *have negative real part*. The zeros of system (Equation 48.17), on the other hand, are the roots of the equation

$$\begin{aligned} 0 &= \det\begin{pmatrix} A - \lambda I & B\Gamma & B \\ 0 & \Phi - \lambda I & G \\ C & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} A - \lambda I & 0 & B \\ 0 & \Phi - G\Gamma - \lambda I & G \\ C & 0 & 0 \end{pmatrix} \\ &= \det\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \det(\Phi - G\Gamma - \lambda I), \end{aligned}$$

and hence it is readily seen that a sufficient condition for the existence of the desired robust stabilizer is that all roots of the equation

$$0 = \det\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \tag{48.18}$$

and all eigenvalues of the matrix $\Phi - G\Gamma$ have negative real part. The latter condition can always be fulfilled. In fact, as observed earlier, it is always possible to find an *observable* pair (Φ, Γ) which render (Equation 48.12) fulfilled for some T. Hence, there always exists a vector G which makes $\Phi - G\Gamma$ a Hurwitz matrix. In view of this, it is concluded that a sufficient condition for the existence of a robust stabilizer for Equation 48.17, once the vector G has been chosen in this way, is simply that all roots of Equation 48.18, or—what is the same—all zeros of

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ e &= Cx \end{aligned} \tag{48.19}$$

have negative real part. This condition, on the other hand, also guarantees that the Francis' equation 48.10 have a solution.

48.3 Steady-State Analysis

Controlling the nonlinear plant (Equation 48.1) by means of the nonlinear controller (Equation 48.3) yields a closed-loop system modeled by Equations 48.4. If the problem of output regulation is solved, the positive orbit of the set $W \times X \times X_c$ of initial conditions is bounded and hence all trajectories asymptotically approach a steady-state locus $\omega(W \times X \times X_c)$. This set is the graph of a (possibly set-valued) map defined on W.* To streamline the analysis, we assume that this map is *single-valued*, that is, that there exists a pair of maps $x = \pi(w)$ and $x_c = \pi_c(w)$, defined on W, such that

$$\omega(W \times X \times X_c) = \{(w, x, x_c) : w \in W, x = \pi(w), x_c = \pi_c(w)\}. \tag{48.20}$$

This is equivalent to assume that, in the closed-loop system, for each given exogenous input function $w(t)$, there exists a *unique* steady state response, which therefore can be expressed as $x(t) = \pi(w(t))$ and $x_c(t) = \pi_c(w(t))$. Moreover, for convenience, we also assume that the maps $\pi(w)$ and $\pi_c(w)$ are

* See Chapter XXX for details.

continuously differentiable. This enables us to characterize in simple terms the property that the steady state locus is invariant under the flow of the closed-loop system (Equation 48.4). If this is the case, in fact, to say that the locus (Equation 48.20) is invariant under the flow of Equation 48.4 is the same as to say that $\pi(w)$ and $\pi_c(w)$ satisfy

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= F(w, \pi(w), H_c(\pi_c(w), H(w, \pi(w)))), \\ \frac{\partial \pi_c}{\partial w} s(w) &= F_c(\pi_c(w), H(w, \pi(w))), \end{aligned} \qquad \forall w \in W. \tag{48.21}$$

These equations are the nonlinear counterpart of the Sylvester equations 48.8. If the controller solves the problem of output regulation, the steady state locus, which is asymptotically approached by the trajectories of the closed-loop system, must be a subset of the set of all pairs (w, x) for which $H(w, x) = 0$ and hence the map $\pi(w)$ necessarily satisfies $H(w, \pi(w)) = 0$. Entering this constraint into Equation 48.21 it follows that

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= F(w, \pi(w), H_c(\pi_c(w), 0)), \\ \frac{\partial \pi_c}{\partial w} s(w) &= F_c(\pi_c(w), 0), \\ 0 &= H(w, \pi(w)). \end{aligned} \tag{48.22}$$

Proceeding as in the case of linear systems and setting $\psi(w) = H_c(\pi_c(w), 0)$, the first and third Equations of 48.22 can be rewritten in controller-independent form as

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= F(w, \pi(w), \psi(w)), \\ 0 &= H(w, \pi(w)), \end{aligned} \qquad \forall w \in W. \tag{48.23}$$

These equations, introduced in [3] and known as the *nonlinear regulator equations*, are the nonlinear counterpart of the Francis' equations 48.10.

Observe that the map $\psi(w)$ appearing in Equation 48.23 satisfies

$$\begin{aligned} \psi(w) &= H_c(\pi_c(w), 0), \\ \frac{\partial \pi_c}{\partial w} s(w) &= F_c(\pi_c(w), 0). \end{aligned} \tag{48.24}$$

These constraints also can be formally rewritten in controller-independent form. In fact, the constraint in question simply expresses the existence of an integer d, of an autonomous dynamical system

$$\dot{\eta} = \varphi(\eta), \quad \eta \in \mathbb{R}^d \tag{48.25}$$

with output

$$u = \gamma(\eta), \tag{48.26}$$

and of a map $\tau : W \to \mathbb{R}^d$ such that

$$\begin{aligned} \psi(w) &= \gamma(\tau(w)), \\ \frac{\partial \tau}{\partial w} s(w) &= \varphi(\tau(w)), \end{aligned} \qquad \forall w \in W. \tag{48.27}$$

These are nonlinear counterparts of the constraints (Equation 48.12). However, while in the case of linear systems constraints of this form are irrelevant (i.e., can always be satisfied), in the case of nonlinear systems some technical problems arise and the existence of a triplet $\{\varphi(\cdot), \gamma(\cdot), \tau(\cdot)\}$ satisfying Equation 48.27 may require extra hypotheses. Issues associated with the existence and the design of such a triplet will be discussed later in Section 48.4. For the time being, we observe that the constraints (Equation 48.24) still interpret the ability, of the controller, to generate the feedforward input necessary to keep $e(t) = 0$ in steady state. In steady state, in fact, a controller that solves the problem generates a control of the form

$$u_{\text{ss}}(t) = H_{\text{c}}(x_{\text{c}}(t), 0) = H_{\text{c}}(\pi_{\text{c}}(w(t)), 0) = \psi(w(t)),$$

which, as predicated by the nonlinear regulator equations, is a control able to force a steady state trajectory of the form $x(t) = \pi(w(t))$ and consequently to keep $e(t)$ identically zero.

The nonlinear regulator equations can be given a more tangible form if the model (Equation 48.1) of the plant is affine in the input u and, viewed as a system with input u and output e, has a globally defined *normal form*. * This means that, in suitable (globally defined) coordinates, the composition of plant (Equation 48.1) and exosystem (Equation 48.2) can be modeled by equations of the form

$$\begin{aligned}
\dot{w} &= s(w),\\
\dot{z} &= f(w, z, \xi_1, \ldots, \xi_r),\\
\dot{\xi}_1 &= \xi_2\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r,\\
\dot{\xi}_r &= a(w, z, \xi_1, \ldots, \xi_r) + b(w, z, \xi_1, \ldots, \xi_r)u,\\
e &= \xi_1
\end{aligned} \tag{48.28}$$

in which r is the relative degree of the system, $z \in \mathbb{R}^{n-r}$ and $b(w, z, \xi_1, \ldots, \xi_r)$, the so-called *high-frequency gain*, is nowhere zero.

If the model of the plant is available in normal form, the nonlinear regulator equations 48.23 can be dealt with as follows. Let $\pi(w)$ be partitioned, consistently with the partition of the state $(z, \xi_1, \ldots, \xi_r)$ of Equation 48.28, into

$$\pi(w) = \text{col}(\pi_0(w), \pi_1(w), \ldots, \pi_r(w))$$

in which case the equations in question become

$$\begin{aligned}
\frac{\partial \pi_0}{\partial w} s(w) &= f(w, \pi_0(w), \pi_1(w), \ldots, \pi_r(w)),\\
\frac{\partial \pi_i}{\partial w} s(w) &= \pi_{i+1}(w) \quad i = 1, \ldots, r-1,\\
\frac{\partial \pi_r}{\partial w} s(w) &= a(w, \pi_0(w), \pi_1(w), \ldots, \pi_r(w)) + b(w, \pi_0(w), \pi_1(w), \ldots, \pi_r(w))\psi(w),\\
0 &= \pi_1(w).
\end{aligned}$$

From these, we deduce that

$$\pi_1(w) = \cdots = \pi_r(w) = 0,$$

while $\pi_0(w)$ satisfies

$$\frac{\partial \pi_0}{\partial w} s(w) = f_0(w, \pi_0(w)), \tag{48.29}$$

in which

$$f_0(w, z) = f(w, z, 0, \ldots, 0).$$

Moreover,

$$\psi(w) = -\,q_0(w, \pi_0(w)), \tag{48.30}$$

* See Section 57.1 of Chapter 57 for the definition of *relative degree* and *normal form.*

in which

$$q_0(w, z) = \frac{a(w, z, 0, \ldots, 0)}{b(w, z, 0, \ldots, 0)}.$$

The autonomous system

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{z} &= f_0(w, z), \end{aligned} \tag{48.31}$$

characterizes the so-called *zero dynamics* of Equation 48.28. Thus, to say that the nonlinear regulator equations 48.23 have a solution is to say that the zero dynamics of Equation 48.28 possess an invariant manifold expressible as the graph of a map $z = \pi_0(w)$ defined on W.

48.4 Convergence to the Required Steady State

Mimicking the design philosophy chosen in the case of linear systems, it is natural to look at a controller of the form

$$\begin{aligned} u &= \gamma(\eta) + v, \\ \dot{\eta} &= \varphi(\eta) + v', \end{aligned} \tag{48.32}$$

where $\gamma(\cdot)$ and $\varphi(\cdot)$ satisfy (Equation 48.27) for some $\tau(\cdot)$ and where v, v' are additional controls. If these controls vanish in steady state, the graph of the nonlinear map

$$w \in W \mapsto \begin{pmatrix} x \\ \eta \end{pmatrix} = \begin{pmatrix} \pi(w) \\ \tau(w) \end{pmatrix} \tag{48.33}$$

is by construction an invariant manifold in the composite system

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{x} &= f(w, x, \gamma(\eta) + v), \\ \dot{\eta} &= \varphi(\eta) + v'. \end{aligned} \tag{48.34}$$

The regulated variable e vanishes on the graph of Equation 48.33. Thus, if the additional controls v and v' are able to steer all trajectories to this invariant manifold, the problem of output regulation is solved. In the case of linear systems, the success of a similar design philosophy was made possible by the additional assumption that all zeros of the controlled plant had negative real part. In what follows, we show how the approach in question can be extended to the case of nonlinear systems.

For convenience, we begin by addressing the special case in which $r = 1$ and $b = 1$, deferring to Section 48.6 the discussion of more general cases. Picking, as in the case of linear systems, $v' = Gv$, system (Equation 48.34) reduces to a system of the form

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{z} &= f(w, z, \xi_1), \\ \dot{\xi}_1 &= a(w, z, \xi_1) + \gamma(\eta) + v, \\ \dot{\eta} &= \varphi(\eta) + Gv, \\ e &= \xi_1, \end{aligned} \tag{48.35}$$

which, in what follows, will be referred to as the *augmented system*. The (compact) set of admissible initial conditions of Equation 48.35 is a set of the form $W \times Z \times \Xi \times H$.

System (Equation 48.35) still has relative degree $r = 1$ between input v and output e, with a normal form which can be revealed by simply changing η into

$$\chi = \eta - G\xi_1,$$

which yields

$$\begin{aligned}\dot{w} &= s(w),\\ \dot{z} &= f(w, z, \xi_1),\\ \dot{\chi} &= \varphi(\chi + G\xi_1) - G\gamma(\chi + G\xi_1) - Ga(w, z, \xi_1),\\ \dot{\xi}_1 &= a(w, z, \xi_1) + \gamma(\chi + G\xi_1) + v,\\ e &= \xi_1.\end{aligned} \tag{48.36}$$

Observe, in this respect, that the zero dynamics of this system, which will be referred to as the *augmented zero-dynamics*, obtained by entering the constraint $e = 0$, are those of the autonomous system

$$\begin{aligned}\dot{w} &= s(w),\\ \dot{z} &= f_0(w, z),\\ \dot{\chi} &= \varphi(\chi) - G\gamma(\chi) - Gq_0(w, z).\end{aligned} \tag{48.37}$$

By virtue of Equation 48.27 through 48.30, it is readily seen that the manifold

$$\mathcal{M} = \{(w, z, \chi) : w \in W, z = \pi_0(w), \chi = \tau(w)\} \tag{48.38}$$

is an invariant manifold of Equation 48.37.

It is now convenient to regard system (Equation 48.36) as feedback interconnection of a system with input ξ_1 and state (w, z, χ) and of a system with inputs (w, z, χ) and v and state ξ_1. In particular, setting

$$p = \mathrm{col}(w, z, \chi)$$

the system in question can be regarded as a system of the form

$$\begin{aligned}\dot{p} &= M(p) + N(p, \xi_1),\\ \dot{\xi}_1 &= H(p) + J(p, \xi_1) + b(p, \xi_1)v,\end{aligned} \tag{48.39}$$

in which $M(p)$ and $H(p)$ are defined as

$$M(p) = \begin{pmatrix} s(w) \\ f_0(w, z) \\ \varphi(\chi) - G\gamma(\chi) - Gq_0(w, z) \end{pmatrix}$$

and

$$H(p) = q_0(w, z) + \gamma(\chi),$$

while $N(p, \xi_1)$ and $J(p, \xi_1)$ are residual functions satisfying $N(p, 0) = 0$ and $J(p, 0) = 0$, and $b(p, \xi_1) = 1$.

The advantage of seeing system (Equation 48.36) in this form is that, under appropriate hypotheses, the control

$$v = -k\xi_1 \tag{48.40}$$

keeps all admissible trajectories bounded and forces $\xi_1(t)$ to zero as $t \to \infty$. In fact, the following result holds (see, e.g., [4,9]).

Theorem 48.1:

Consider a system of the form (Equation 48.39) with v as in Equation 48.40. Suppose that $M(p)$, $N(p,\xi_1)$, $H(p)$, $J(p,\xi_1)$, and $b(p,\xi_1)$ are at least locally Lipschitz and $b(p,\xi_1) > 0$. Let the initial conditions of the system range in a compact set $P \times \Xi$. Suppose there exists a set $\mathcal{A}$ which is locally exponentially stable for $\dot{p} = M(p)$, with a domain of attraction that contains the set P. Suppose also that $H(p) = 0$ for all $p \in \mathcal{A}$. Then, there is a number k^* such that, for all $k > k^*$, the set $\mathcal{A} \times \{0\}$ is locally exponentially stable for the interconnection (Equations 48.39 through 48.40), with a domain of attraction that contains $P \times \Xi$.

Applying this result to system (Equation 48.36), it is observed that the system $\dot{p} = M(p)$ coincides with Equation 48.37, that is with the zero dynamics of the augmented system (Equation 48.36). The set (Equation 48.38) is an invariant manifold of these dynamics and, by construction, the map $H(p)$ vanishes on this set. Thus, it is concluded that if the set (Equation 48.38) is locally exponentially stable for Equation 48.37, with a domain of attraction that contains the set of all admissible initial conditions, the choice of a high-gain control as in Equation 48.40 suffices to steer ξ_1 to zero and hence to solve the problem of output regulation.

For convenience, we summarize the result obtained so far as follows.

Corollary 48.1:

Consider a system in normal form (Equation 48.28) with $r = 1$ and $b = 1$. Suppose (Equation 48.29) holds for some $\pi_0(w)$. Let $\psi(w)$ be defined as in Equation 48.30 and $\varphi(\eta)$ and $\gamma(\eta)$ be such that (Equation 48.27) hold for some $\tau(w)$. Consider a controller of the form

$$\begin{aligned} u &= \gamma(\eta) - ke, \\ \dot{\eta} &= \varphi(\eta) - Gke. \end{aligned} \tag{48.41}$$

If the manifold (Equation 48.38) is locally exponentially stable for Equation 48.37, with a domain of attraction that contains the set of all admissible initial conditions, there exists k^* such that, for all $k > k^*$, the positive orbit of the set of admissible initial conditions in bounded and $e(t) \to 0$ as $t \to \infty$.

The main issue that remains to be addressed, in this framework, is to determine whether or not the desired asymptotic properties of the invariant manifold (Equation 48.38) of Equation 48.37 can be obtained. In this respect, the asymptotic properties of the zero dynamics (Equation 48.31) of the controlled plant—on the one hand—and the choice of $\{\varphi(\cdot), \gamma(\cdot), G\}$—on the other hand—indeed play a major role. This issue is addressed in the next section. For the time being, we observe that, in the case of linear systems, the dynamics of Equation 48.37 reduce to linear dynamics, modeled by

$$\begin{aligned} \dot{w} &= Sw, \\ \dot{z} &= B_0 w + A_0 z, \\ \dot{\chi} &= (\Phi - G\Gamma)\chi - G(D_0 w + C_0 z), \end{aligned}$$

in which A_0 is a matrix whose eigenvalues coincide with the zeros of Equation 48.19. The maps $\pi_0(w)$ and $\tau(w)$ are linear maps, $\pi_0(w) = \Pi_0 w$ and $\tau(w) = Tw$, satisfying

$$\Pi_0 S = B_0 + A_0 \Pi_0, \quad TS = \Phi T, \quad \Gamma T = -(D_0 + C_0 \Pi_0).$$

Changing z, χ into $\tilde{z} = z - \Pi_0 w$ and $\tilde{\chi} = \chi - Tw$, respectively, these dynamics can be rewritten as

$$\dot{w} = Sw,$$

$$\dot{\tilde{z}} = A_0\tilde{z},$$
$$\dot{\tilde{\chi}} = (\Phi - G\Gamma)\tilde{\chi} - GC_0\tilde{z},$$

from which it is readily seen, as expected, that if all zeros of Equation 48.19 and all eigenvalues of $(\Phi - G\Gamma)$ have negative real part, the dynamics in question have the desired asymptotic properties.

48.5 The Design of the Internal Model

System (Equation 48.37) can be interpreted as the cascade of

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{z} &= f_0(w, z), \\ y &= q_0(w, z), \end{aligned} \tag{48.42}$$

and of

$$\dot{\chi} = \varphi(\chi) - G\gamma(\chi) - Gy. \tag{48.43}$$

We are interested in finding hypotheses under which there exists a triplet $\{\varphi(\cdot), \gamma(\cdot), G\}$ such that Equation 48.27, with $\psi(w) = -q_0(w, \pi_0(w))$, holds for some $\tau(\cdot)$ and such that the resulting invariant manifold (Equation 48.38) is locally exponentially stable, with a domain of attraction that contains the set of all admissible initial conditions. The first obvious hypothesis is that the set $z = \pi_0(w)$ is a locally exponentially stable (invariant) manifold of Equation 48.42, with a domain of attraction that contains the set $W \times Z$. This assumption is the nonlinear analogue of the assumption that system (Equation 48.19) has all zeros with negative real part and is usually referred to, with some abuse of terminology, as the *minimum-phase* assumption.

Once this is assumed, the matter is to determine the existence of a triplet $\{\varphi(\cdot), \gamma(\cdot), G\}$ with the appropriate properties. As we have seen, this is always possible for a linear system. The basic argument behind the construction of the pair (Φ, Γ) that makes conditions (Equation 48.12) fulfilled (recall that these are the linear version of Equation 48.27) is that, by Cayley–Hamilton's theorem,

$$\Psi S^d = -(s_0\Psi + s_1\Psi S + \cdots + s_{d-1}\Psi S^{d-1}).$$

It is seen from this that the function $u_{ss}(t) = \Psi w(t)$, with $w(t)$ solution of $\dot{w} = Sw$, satisfies the linear differential equation

$$u_{ss}^{(d)}(t) = -s_0 u_{ss}(t) - s_1 u_{ss}^{(1)}(t) - \cdots - s_{d-1} u_{ss}^{(d-1)}(t).$$

Motivated by this interpretation, we may *assume*, in the nonlinear case, the existence of an integer d, of a (locally Lipschitz) function $\phi : \mathbb{R}^d \to \mathbb{R}$ with the property that, for any $w_0 \in W$, the solution $w(t)$ of $\dot{w} = s(w)$ passing through w_0 at time $t = 0$ is such that the function $u_{ss}(t) = \psi(w(t))$, in which $\psi(w) = -q_0(w, \pi_0(w))$, satisfies the *nonlinear* differential equation

$$u_{ss}^{(d)}(t) = \phi(u_{ss}(t), u_{ss}^{(1)}(t), \ldots, u_{ss}^{(d-1)}(t)). \tag{48.44}$$

If this property is assumed, it is easy to find a pair $\varphi(\cdot), \gamma(\cdot)$ such that Equation 48.27 holds for some $\tau(\cdot)$. In fact, it suffices to set

$$\tau(w) = \mathrm{col}(\psi(w), L_s\psi(w), \ldots, L_s^{d-1}\psi(w)),$$

to pick any function $\phi_c : \mathbb{R}^d \to \mathbb{R}$ which is globally Lipschitz and agrees with $\phi(\cdot)$ on $\tau(W)$, and set

$$\varphi(\eta) = \begin{pmatrix} \eta_2 \\ \cdots \\ \eta_d \\ \phi_c(\eta_1, \ldots, \eta_d), \end{pmatrix}, \quad \gamma(\eta) = \eta_1. \tag{48.45}$$

A simple calculation shows that conditions (Equation 48.27) hold.

Once the $\varphi(\cdot), \gamma(\cdot)$ have been determined, it remains to show that a vector G can be found yielding the desired asymptotic properties. To this end, set

$$\tilde{\chi} = \chi - \tau(w), \quad \tilde{y} = q_0(w,z) - q_0(w, \pi_0(w))$$

and observe that

$$\dot{\tilde{\chi}} = \varphi(\tilde{\chi} + \tau(w)) - \varphi(\tau(w)) - G\tilde{\chi}_1 - G\tilde{y}. \tag{48.46}$$

Since $z = \pi_0(w)$ is an asymptotically stable (invariant) manifold of Equation 48.42, the input $\tilde{y}$ of Equation 48.46 decays to zero as $t \to \infty$. Moreover, by construction, $\tilde{\chi} = 0$ is an equilibrium of Equation 48.46 when $\tilde{y} = 0$.

The pair $\{\varphi(\cdot), \gamma(\cdot)\}$ defined in Equation 48.45 is *uniformly observable* (see [5]). Therefore, it is likely that the desired asymptotic properties of Equation 48.46 could be achieved by picking G as in the design of *high-gain observers*. Proceeding in this way, choose

$$G = D_\kappa G_0,$$

in which

$$\begin{aligned} D_\kappa &= \mathrm{diag}(\kappa, \kappa^2, \ldots, \kappa^d),\\ G_0 &= \mathrm{col}(c_{d-1}, c_{d-2}, \ldots, c_0), \end{aligned}$$

with $c_{d-1}, \ldots, c_0$ coefficients of a Hurwitz polynomial

$$p(\lambda) = c_0 + c_1\lambda + \cdots + c_{d-1}\lambda^{d-1} + \lambda^d.$$

It is possible to prove (see, e.g., [6]) that, if κ is large enough, system (Equation 48.46) is globally input-to-state stable, actually with a linear gain function, and that the equilibrium $\tilde{\chi} = 0$—achieved for $\tilde{y} = 0$—is globally exponentially stable. This, coupled with the assumption that $z = \pi_0(w)$ is a locally exponentially stable (invariant) manifold of Equation 48.42, proves that Equation 48.38 is a locally exponentially stable (invariant) manifold of Equation 48.37, with a domain of attraction that contains the set $W \times Z \times \mathbb{R}^d$. This makes the result of Corollary 48.1 applicable.

We have shown, in this way, that under the assumptions that the controlled plant is minimum phase and that the family of all "steady state inputs" $u_{ss}(t) = \psi(w(t))$ obeys a (possibly nonlinear) high-order differential equation of the form (Equation 48.44), the problem of output regulation can be solved. One may wonder whether these assumptions can be weakened, in particular the assumption of the existence of a differential equation of the form (Equation 48.44). There is, in fact, an alternative design strategy, in which the assumption in question is not needed. This strategy is based on seeking the fulfillment of Equation 48.27 with a $\varphi(\eta)$ of the form

$$\varphi(\eta) = F\eta + G\gamma(\eta),$$

which entails, for the system (Equation 48.43), a structure of the form

$$\dot{\chi} = F\chi - Gy, \tag{48.47}$$

with F a Hurwitz matrix. In this case, to say that Equation 48.27 are fulfilled is to say that there exists a pair (F, G), with F a Hurwitz matrix and a map $\gamma(\eta)$ such that

$$\left.\begin{aligned} \psi(w) &= \gamma(\tau(w)),\\ \frac{\partial\tau}{\partial w}s(w) &= F\tau(w) + G\psi(w), \end{aligned}\right. \quad \forall w \in W \tag{48.48}$$

hold for some $\tau(w)$. The relevant result, in this respect, is that such a triplet always exists, if the dimension d of F is sufficiently large and $\gamma(\eta)$ is allowed to be only continuous (and, thus, possibly not locally

Lipschitz). Specifically, note that, regardless of what the dimension of the matrix F is, if the latter is Hurwitz, a map $\tau(w)$ fulfilling the second equation of Equation 48.48 always exists. In fact, if the controlled plant is minimum phase, and hence all trajectories of Equation 48.42 with initial conditions in $W \times Z$ are bounded, and if the matrix F is Hurwitz, also all trajectories of

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{z} &= f_0(w, z), \\ \dot{\chi} &= F\chi - Gq_0(w, z) \end{aligned} \tag{48.49}$$

for any initial conditions in $W \times Z \times \mathbb{R}^d$ are bounded, and converge to a steady state locus. The latter is the graph of a map defined on W, in which $z = \pi_0(w)$ and $\chi = \tau(w)$, where $\tau(w)$, if continuously differentiable, satisfies

$$\frac{\partial \tau}{\partial w} s(w) = F\tau(w) - Gq_0(w, \pi_0(w)) = F\tau(w) + G\psi(w). \tag{48.50}$$

Thus, the real issue is simply when there exists a map $\gamma(\eta)$ such that the map $\tau(w)$ which characterizes the steady state locus of Equation 48.49 satisfies the first condition in Equation 48.48. This problem has been recently answered in [4], in the following terms.

Proposition 48.1:

There is an integer $\ell > 0$ such that, if the eigenvalues of F have real part which is less than $-\ell$, there exists a unique continuously differentiable $\tau(w)$ which satisfies (Equation 48.50). Suppose

$$d \geq 2\dim(w) + 2.$$

Then for almost all choices (see [4] for details) of a controllable pair (F, G), with F a Hurwitz matrix whose eigenvalues have real part which is less than $-\ell$, the map $\tau(w)$ satisfies

$$\tau(w_1) = \tau(w_2) \;\; \Rightarrow \;\; \psi(w_1) = \psi(w_2), \quad \forall (w_1, w_2) \in W \times W.$$

As a consequence, there exists a continuous map $\gamma : \mathbb{R}^d \to \mathbb{R}$ such that the first identity in Equation 48.48 holds.

This shows that the existence of a triplet $\{F, G, \gamma(\cdot)\}$ with the desired properties can always be achieved, so long as the integer d is large enough. As a consequence, the design procedure outlined earlier in Corollary 48.1 is always applicable, so long as the controlled plant satisfies the minimum-phase assumption. From the constructive viewpoint, however, it must be observed that the result indicated in Proposition 48.1 is only an existence result and that the function $\gamma(\cdot)$, whose existence is guaranteed, is only known to be continuous. Obtaining continuous differentiability of such $\gamma(\cdot)$ is likely to require further hypotheses.*

48.6 The Case of Higher Relative Degree

Consider now the case of a system having relative degree higher than 1, but still assume, for simplicity, that $b(w, z, \xi_1, \ldots, \xi_r) = 1$. In addition, *assume* that the function $f(w, z, \xi_1, \ldots, \xi_r)$ is independent of

* Closed-form expressions for $\gamma(\cdot)$ and other relevant constructive aspects are discussed in [7].

$\xi_2, \ldots, \xi_r$. Choose, as in Section 48.4, a control of the form (Equation 48.32) with $v' = Gv$. This yields an augmented system that can be written in the form

$$\begin{aligned}
\dot{w} &= s(w),\\
\dot{z} &= f(w, z, C\bar{z}),\\
\dot{\bar{z}} &= A\bar{z} + B\xi_r,\\
\dot{\xi}_r &= a(w, z, \bar{z}, \xi_r) + \gamma(\eta) + v,\\
\dot{\eta} &= \varphi(\eta) + Gv,\\
e &= C\bar{z},
\end{aligned} \tag{48.51}$$

in which $\bar{z} = \mathrm{col}(\xi_1, \ldots, \xi_{r-1})$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0\\ 0\\ \cdot\\ 0\\ 1 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0 \quad \cdots \quad 0).$$

The idea is to try to transform this system into a system of the form (Equation 48.39) and to try to use again, if possible, the result of Theorem 48.1. Transformation into a system of the form (Equation 48.39) can be achieved by changing ξ_r into

$$\theta = \xi_r - K\bar{z},$$

and η into

$$\chi = \eta - G\theta, \tag{48.52}$$

in which K is a vector of design parameters yet to be determined. In this way, the augmented system (Equation 48.51) can be written as

$$\begin{aligned}
\dot{p} &= M(p) + N(p, \theta),\\
\dot{\theta} &= H(p) + J(p, \theta) + b(p, \theta)v,
\end{aligned} \tag{48.53}$$

in which $p = \mathrm{col}(w, z, \bar{z}, \chi)$. The subsystem $\dot{p} = M(p)$, which coincides with the zero dynamics of Equation 48.51 if the latter is viewed as a system with input v and output θ, is a system of the form

$$\begin{pmatrix} \dot{w}\\ \dot{z}\\ \dot{\bar{z}}\\ \dot{\chi} \end{pmatrix} = \begin{pmatrix} s(w)\\ f_0(w, z) + f_1(w, z, C\bar{z})\\ (A + BK)\bar{z}\\ \varphi(\chi) - G\gamma(\chi) - Gq_0(w, z) - Gq_1(w, z, \bar{z}) \end{pmatrix}, \tag{48.54}$$

in which

$$\begin{aligned}
f_1(w, z, C\bar{z}) &= f(w, z, C\bar{z}) - f_0(w, z),\\
q_1(w, z, \bar{z}) &= a(w, z, \bar{z}, K\bar{z}) - K(A + BK)\bar{z} - q_0(w, z)
\end{aligned}$$

are functions vanishing at $\bar{z} = 0$. The map $H(p)$ is a map of the form

$$H(p) = q_0(w, z) + \gamma(\chi) + q_1(w, z, \bar{z}),$$

the residual functions $N(p, \theta)$ and $J(p, \theta)$ vanish at $\theta = 0$, and $b(p, \theta) = 1$. Observe that the manifold

$$\mathcal{M} = \{(w, z, \bar{z}, \chi) : w \in W, z = \pi_0(w), \bar{z} = 0, \chi = \tau(w)\} \tag{48.55}$$

is invariant for the dynamics of Equation 48.54 and that the map $N(p)$ vanishes on this set.

It follows from Theorem 48.1 that, if the manifold (Equation 48.55) is locally exponentially stable for Equation 48.54, with a domain of attraction that contains the set of all admissible initial conditions, the choice of a high-gain control

$$v = -k\theta = -k(\xi_r - K\bar{z})$$

suffices to steer θ to zero and p to Equation 48.55. Since ξ_1, a component of $\bar{z}$, is zero in Equation 48.55, then ξ_1 also is steered to zero and the problem of output regulation is solved.

The control law proposed in this way is a law which presupposes the availability of $\xi_1, \ldots, \xi_r$, that is, of the regulated variable e and its derivatives $e^{(1)}, \ldots, e^{(r-1)}$. This is not a problem, however, since appropriate substitutes for these variables can be generated, so long as the set of admissible initial conditions is compact, by means of an appropriate r-dimensional system driven by e, as suggested in [8]. The applicability of the methods depends therefore on the ability to choose the design parameters in such a way that the manifold (Equation 48.55) is locally exponentially stable for Equation 48.54, with a domain of attraction that contains the set of all admissible initial conditions. In this respect, it must be observed that the system in question can be seen as a cascade of

$$\begin{aligned}
\dot{w} &= s(w), \\
\dot{z} &= f_0(w,z) + f_1(w,z,C\bar{z}), \\
\dot{\bar{z}} &= (A+BK)\bar{z}, \\
y &= q_0(w,z) + q_1(w,z,\bar{z}),
\end{aligned} \tag{48.56}$$

and of system (Equation 48.43).

It is known that, if the controlled plant is minimum phase, that is, if the set $z = \pi_0(w)$ is a locally exponentially stable (invariant) manifold for the dynamics of Equation 48.42, with a domain of attraction that contains the set $W \times Z$, then given any compact set $\bar{Z}$, there is a matrix K such that the set $(z, \bar{z}) = (\pi_0(w), 0)$ is a locally exponentially stable invariant manifold for the dynamics of Equation 48.56, with a domain of attraction that contains the set $W \times Z \times \bar{Z}$ (see, e.g., [9]). Thus, if K is chosen in this way, either one of the two methods for the design of $\{\varphi(\cdot), \gamma(\cdot), G\}$ suggested in Section 48.5 can be used to complete the design of the regulator.

We conclude by observing that, if the high-frequency gain on the system is not equal to 1, identical results hold, which can be proven using the change of variable

$$\chi = \eta - G\int_0^{\theta} \frac{1}{b(w,z,\xi_1,\ldots,\xi_{r-1},s)}\,\mathrm{d}s,$$

instead of Equation 48.52. On the contrary, the assumption that $f(w,z,\xi_1,\ldots,\xi_r)$ is independent of $\xi_2,\ldots,\xi_r$ can only be removed at the expense of other assumptions, such as the property, of system

$$\begin{aligned}
\dot{w} &= s(w), \\
\dot{z} &= f(w,z,\xi_1,\ldots,\xi_r),
\end{aligned}$$

of being input-to-state stable, in the input $(\xi_1,\ldots,\xi_r)$.

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