NORMAL FORMS AND FEEDBACK STABILIZATION

1 Relative degree and local normal forms

The purpose of this Section is to show how single-input single-output nonlinear systems can be locally given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which several important properties can be elucidated.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally described in the following way. The single-input single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

is said to have *relative degree* r at a point x° if [1]

(i) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r-1$

(ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

Note that there may be points where a relative degree cannot be defined. This occurs, in fact, when the first function of the sequence

$$L_g h(x), L_g L_f h(x), \ldots, L_g L_f^k h(x), \ldots$$

which is not identically zero (in a neighborhood of x°) has a zero exactly at the point $x = x^\circ$. However, the set of points where a relative degree can be defined is clearly an open and dense subset of the set U where the system (1) is defined.

Remark. In order to compare the notion thus introduced with a familiar concept, let us calculate the relative degree of a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \,. \end{aligned}$$

In this case, since $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$, it easily seen that

$$L_f^k h(x) = CA^k x$$

and therefore

$$L_g L_f^k h(x) = CA^k B \,.$$

[1] Let λ be real-valued function and f an n-vector-valued vector, both defined on a subset U of $\mathbb{R}^n$. The function $L_f \lambda$ is the real-valued function defined as

$$L_f \lambda(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x) := \frac{\partial \lambda}{\partial x} f(x) \,.$$

This function is sometimes called *derivative of* λ *along* f.

Thus, the integer r is characterized by the conditions

$$\begin{array}{rcll} CA^kB & = & 0 & \text{for all } k < r-1 \\ CA^{r-1}B & \neq & 0\,. & \end{array}$$

It is well-known that the integer satisfying these conditions is exactly equal to the *difference* between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function

$$T(s) = C(sI - A)^{-1}B$$

of the system. ◁

We illustrate now a simple interpretation of the notion of relative degree, which is not restricted to the assumption of linearity considered in the previous Remark. Assume the system at some time t° is in the state $x(t^\circ) = x^\circ$ and suppose we wish to calculate the value of the output $y(t)$ and of its derivatives with respect to time $y^{(k)}(t)$, for $k = 1, 2, \ldots$, at $t = t^\circ$. We obtain

$$\begin{array}{rcl} y(t^\circ) & = & h(x(t^\circ)) = h(x^\circ) \\ y^{(1)}(t) & = & \dfrac{\partial h}{\partial x}\dfrac{dx}{dt} = \dfrac{\partial h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ & = & L_f h(x(t)) + L_g h(x(t))u(t)\,. \end{array}$$

If the relative degree r is larger than 1, for all t such that $x(t)$ is near x°, i.e. for all t near t°, we have $L_g h(x(t)) = 0$ and therefore

$$y^{(1)}(t) = L_f h(x(t))\,.$$

This yields

$$\begin{array}{rcl} y^{(2)}(t) = \dfrac{\partial L_f h}{\partial x}\dfrac{dx}{dt} & = & \dfrac{\partial L_f h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ & = & L_f^2 h(x(t)) + L_g L_f h(x(t))u(t)\,. \end{array}$$

Again, if the relative degree is larger than 2, for all t near t° we have $L_g L_f h(x(t)) = 0$ and

$$y^{(2)}(t) = L_f^2 h(x(t))\,.$$

Continuing in this way, we get

$$\begin{array}{rcll} y^{(k)}(t) & = & L_f^k h(x(t)) & \text{for all } k < r \text{ and all } t \text{ near } t^\circ \\ y^{(r)}(t^\circ) & = & L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ)u(t^\circ)\,. & \end{array}$$

Thus, the relative degree r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t^\circ$ in order to have the value $u(t^\circ)$ of the input explicitly appearing.

Note also that if

$$L_g L_f^k h(x) = 0 \qquad \text{for all } x \text{ in a neighborhood of } x^\circ \text{ and all } k \geq 0$$

(in which case no relative degree can be defined at any point around x°) then the output of the system is not affected by the input, for all t near t°. As a matter of fact, if this is the case, the previous calculations show that the Taylor series expansion of $y(t)$ at the point $t = t^\circ$ has the form

$$y(t) = \sum_{k=0}^{\infty} L_f^k h(x^\circ) \frac{(t - t^\circ)^k}{k!}$$

i.e. that $y(t)$ is a function depending only on the initial state and not on the input.

These calculations suggest that the functions $h(x)$, $L_f h(x), \ldots, L_f^{r-1} h(x)$ must have a special importance. As a matter of fact, it is possible to show that they can be used in order to define, at least partially, a local coordinates transformation around x°(recall that x° is a point where $L_g L_f^{r-1} h(x^\circ) \neq 0$). This fact is based on the following property.

Lemma 1 *The row vectors* [2]

$$dh(x^\circ), dL_f h(x^\circ), \ldots, dL_f^{r-1} h(x^\circ)$$

are linearly independent.

Lemma 1 shows that necessarily $r \leq n$ and that the r functions $h(x), L_f h(x), \ldots, L_f^{r-1} h(x)$ qualify as a partial set of new coordinate functions around the point x°. As we shall see in a moment, the choice of these new coordinates entails a particularly simple structure for the equations describing the system. However, before doing this, it is convenient to summarize the results discussed so far in a formal statement, that also illustrates a way in which the set of new coordinates can be completed in case the relative degree r is strictly less than n.

Proposition 1 *Suppose the system has relative degree r at x°. Then $r \leq n$. If r is strictly less than n, it is always possible to find $n - r$ more functions $\psi_1(x), \ldots, \psi_{r-r}(x)$ such that the mapping*

$$\Phi(x) = \begin{pmatrix} \psi_1(x) \\ \cdots \\ \psi_{n-r}(x) \\ h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

has a jacobian matrix which is nonsingular at x° and therefore qualifies as a local coordinates transformation in a neighborhood of x°. The value at x° of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\psi_1(x), \ldots, \psi_{n-r}(x)$ in such a way that

$$L_g \psi_i(x) = 0 \qquad \text{for all } 1 \leq i \leq n - r \text{ and all } x \text{ around } x^\circ.$$

[2] Let λ be a real-valued function defined on a subset U of $\mathbb{R}^n$. Its *differential*, denote $d\lambda(x)$ is the row vector

$$d\lambda(x) = \begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \cdots & \frac{\partial \lambda}{\partial x_n} \end{pmatrix} := \frac{\partial \lambda}{\partial x}.$$

The description of the system in the new coordinates is found very easily. Set

$$z = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \cdots \\ \psi_{n-r}(x) \end{pmatrix}, \quad \xi = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

and

$$\tilde{x} = (z, \xi) .$$

Looking at the calculations already carried out at the beginning, we obtain for $\xi_1, \ldots, \xi_r$

$$\begin{aligned} \frac{d\xi_1}{dt} &= \frac{\partial h}{\partial x}\frac{dx}{dt} = L_f h(x(t)) = \xi_2(t) \\ &\cdots \\ \frac{d\xi_{r-1}}{dt} &= \frac{\partial (L_f^{r-2}h)}{\partial x}\frac{dx}{dt} = L_f^{r-1} h(x(t)) = \xi_r(t) . \end{aligned}$$

For ξ_r we obtain

$$\frac{d\xi_r}{dt} = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t).$$

On the right-hand side of this equation we must now replace $x(t)$ with its expression as a function of $\tilde{x}(t)$, which will be written as $x(t) = \Phi^{-1}(z(t), \xi(t))$. Thus, setting

$$\begin{aligned} q(z, \xi) &= L_f^r h(\Phi^{-1}(z, \xi)) \\ b(z, \xi) &= L_g L_f^{r-1} h(\Phi^{-1}(z, \xi)) \end{aligned}$$

the equation in question can be rewritten as

$$\frac{d\xi_r}{dt} = q(z(t), \xi(t)) + b(z(t), \xi(t)) u(t) .$$

Note that at the point $(z^\circ, \xi^\circ) = \Phi(x^\circ)$, $b(z^\circ, \xi^\circ) \neq 0$ by definition. Thus, the coefficient $a(z, \xi)$ is nonzero for all (z, ξ) in a neighborhood of (z°, ξ°).

As far as the other new coordinates are concerned, we cannot expect any special structure for the corresponding equations, if nothing else has been specified. However, if $\psi_1(x), \ldots, \psi_{n-r}(x)$ have been chosen in such a way that $L_g \psi_i(x) = 0$, then

$$\frac{dz_i}{dt} = \frac{\partial \psi_i}{\partial x}(f(x(t)) + g(x(t))u(t)) = L_f \psi_i(x(t)) + L_g \psi_i(x(t)) u(t) = L_f \psi_i(x(t)).$$

Setting

$$f_0(z, \xi) = \begin{pmatrix} L_f \psi_1(\Phi^{-1}(z, \xi)) \\ \cdots \\ L_f \psi_{n-r}(\Phi^{-1}(z, \xi)) \end{pmatrix}$$

the latter can be rewritten as

$$\frac{dz}{dt} = f_0(z(t), \xi(t)) .$$

Thus, in summary, the state-space description of the system in the new coordinates will be as follows

$$
\begin{array}{rcl}
\dot{z} & = & f_0(z,\xi) \\
\dot{\xi}_1 & = & \xi_2 \\
\dot{\xi}_2 & = & \xi_3 \\
 & & \cdots \\
\dot{\xi}_{r-1} & = & \xi_r \\
\dot{\xi}_r & = & q(z,\xi) + b(z,\xi)u \ .
\end{array}
\tag{2}
$$

In addition to these equations one has to specify how the output of the system is related to the new state variables. But, being $y = h(x)$, it is immediately seen that

$$
y = \xi_1 \ . \tag{3}
$$

The equations thus defined are said to be in *normal form.* They are useful in understanding how certain control problems can be solved. The equations in question can be given a compact expression if we introduce three matrices $\hat{A} \in \mathbb{R}^r \times \mathbb{R}^r$, $\hat{B} \in \mathbb{R}^r \times \mathbb{R}$ and $\hat{C} \in \mathbb{R} \times \mathbb{R}^r$ defined as follows

$$
\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} .
$$

In fact, it is readily seen that, with the aid of these notations, the equations (2) and (2) can be re-written as

$$
\begin{array}{rcl}
\dot{z} & = & f_0(z,\xi) \\
\dot{\xi} & = & \hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)u] \\
y & = & \hat{C}\xi \ .
\end{array}
\tag{4}
$$

2 Global Normal Forms

We address, in this section, the problem of deriving the global version of the coordinates transformation and normal form introduced in section 1. Consider a single-input single-output system described by equations of the form

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned}
\tag{5}
$$

in which $f(x)$ and $g(x)$ are smooth vector fields, and $h(x)$ is a smooth function, defined on $\mathbb{R}^n$. Assume, as usual, that $f(0) = 0$ and $h(0) = 0$. This system is said to have *uniform* relative degree r if it has relative degree r at each $x^\circ \in \mathbb{R}^n$.

If system (5) has uniform relative degree r, the r differentials

$$
dh(x), dL_f h(x), \ldots, dL_f^{r-1} h(x)
$$

are linearly independent at each $x \in \mathbb{R}^n$ and therefore the set

$$Z^* = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \ldots = L_f^{r-1} h(x) = 0\}$$

(which is nonempty in view of the hypothesis that $f(0) = 0$ and $h(0) = 0$) is a smooth embedded submanifold of $\mathbb{R}^n$, of dimension $n - r$. In particular, each connected component of Z^* is a maximal integral manifold of the (nonsingular and involutive) distribution

$$\Delta^* = (\text{span}\{dh, dL_f h, \ldots, dL_f^{r-1} h\})^\perp .$$

The submanifold Z^* is the point of departure for the construction a globally defined version of the coordinates transformation considered in section 1.

Proposition 2 *Suppose (5) has uniform relative degree r. Set*

$$\alpha(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \qquad \beta(x) = \frac{1}{L_g L_f^{r-1} h(x)}$$

and consider the (globally defined) vector fields

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \qquad \tilde{g}(x) = g(x)\beta(x) .$$

Suppose the vector fields

$$\tau_i = (-1)^{i-1} ad_{\tilde{f}}^{i-1} \tilde{g}(x), \qquad 1 \le i \le r \tag{6}$$

are complete.

Then Z^* is connected. Moreover, the smooth mapping

$$\begin{array}{rccl} \Phi & : & Z^* \times \mathbb{R}^r & \rightarrow \mathbb{R}^n \\ & & (z, (\xi_1, \ldots, \xi_r)) & \mapsto \Phi_{\xi_r}^{\tau_1} \circ \Phi_{\xi_{r-1}}^{\tau_2} \circ \cdots \circ \Phi_{\xi_1}^{\tau_r}(z) , \end{array} \tag{7}$$

in which – as usual – $\Phi_t^\tau(x)$ denotes the flow of the vector field τ, has a globally defined smooth inverse

$$(z, (\xi_1, \ldots, \xi_r)) = \Phi^{-1}(x) \tag{8}$$

in which

$$\begin{array}{rcl} z & = & \Phi_{-h(x)}^{\tau_r} \circ \cdots \circ \Phi_{-L_{\tilde{f}}^{r-2} h(x)}^{\tau_2} \circ \Phi_{-L_{\tilde{f}}^{r-1} h(x)}^{\tau_1}(x) \\ \xi_i & = & L_{\tilde{f}}^{i-1} h(x) \qquad 1 \le i \le r . \end{array}$$

The globally defined diffeomorphism (8) changes system (5) into a system described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & f_0(z, \xi_1, \ldots, \xi_r) \\ \dot{\xi}_1 & = & \xi_2 \\ & & \cdots \\ \dot{\xi}_{r-1} & = & \xi_r \\ \dot{\xi}_r & = & q(z, \xi_1, \ldots, \xi_r) + b(z, \xi_1, \ldots, \xi_r) u \\ y & = & \xi_1 \end{array} \tag{9}$$

where

$$\begin{aligned} q(z,\xi_1,\ldots,\xi_r) &= L_f^r h \circ \Phi(z,(\xi_1,\ldots,\xi_r)) \\ b(z,\xi_1,\ldots,\xi_r) &= L_g L_f^{r-1} h \circ \Phi(z,(\xi_1,\ldots,\xi_r))\,. \end{aligned}$$

If, and only if, the vector fields (6) are such that [3]

$$[\tau_i,\tau_j]=0 \qquad \text{for all } \ 1\le i,j\le r\,,$$

then the globally defined diffeomorphism (8) changes system (5) into a system described by equations of the form

$$\begin{aligned} \dot z &= f_0(z,\xi_1) \\ \dot\xi_1 &= \xi_2 \\ &\cdots \\ \dot\xi_{r-1} &= \xi_r \\ \dot\xi_r &= q(z,\xi_1,\ldots,\xi_r)+b(z,\xi_1,\ldots,\xi_r)u \\ y &= \xi_1\,. \end{aligned} \tag{10}$$

Note also that, if $r<n$, the submanifold Z^* is the largest (with respect to inclusion) smooth submanifold of $h^{-1}(0)$ with the property that, at each $x\in Z^*$, there is $u^*(x)$ such that $f^*(x)=f(x)+g(x)u^*(x)$ is tangent to Z^*. Actually, for each $x\in Z^*$ there is only one $u^*(x)$ rendering this condition satisfied, namely,

$$u^*(x)=\frac{-L_f^r h(x)}{L_g L_f^{r-1}h(x)}\,.$$

The submanifold Z^* is an *invariant* manifold of the (autonomous) system

$$\dot x = f^*(x) \tag{11}$$

and the restriction of this system to Z^* can be identified with $(n-r)$-dimensional system

$$\dot z = f_0(z,0,\ldots,0)$$

of Z^*.

3 The Zero Dynamics

In this section we introduce and discuss an important concept, that in many instances plays a role exactly similar to that of the "zeros" of the transfer function in a linear system. We have already seen that the relative degree r of a linear system can be interpreted as the difference between the number of poles and the number of zeros in the transfer function. In particular,

[3] Let f and g be vector fields on $\mathbb{R}^n$. Their *Lie bracket*, denoted $[f,g]$, is the vector field of $\mathbb{R}^n$ defined as

$$[f,g](x)=\frac{\partial g}{\partial x}f(x)-\frac{\partial f}{\partial x}g(x)\,.$$

any linear system in which r is strictly less than n has zeros in its transfer function. On the contrary, if $r = n$ the transfer function has no zeros; thus, a nonlinear system having relative degree $r = n$ in some sense analogue to a linear systems without zeros. We shall see in this section that this kind of analogy can be pushed much further.

Consider a nonlinear system with r strictly less than n and look at its normal form. Recall that, if x° is such that $f(x^\circ) = 0$ and $h(x^\circ) = 0$, then necessarily the set ξ of the last r new coordinates is 0 at x°. Note also that it is always possible to choose arbitrarily the value at x° of the first $n-r$ new coordinates, thus in particular being 0 at x°. Therefore, without loss of generality, one can assume that $\xi = 0$ and $z = 0$ at x°. Thus, if x° was an equilibrium for the system in the original coordinates, its corresponding point $(z, \xi) = (0, 0)$ is an equilibrium for the system in the new coordinates and from this we deduce that

$$\begin{aligned} q(0,0) &= 0 \\ f_0(0,0) &= 0 \,. \end{aligned}$$

Suppose now we want to analyze the following problem, called the *Problem of Zeroing the Output*. Find, if any, pairs consisting of an initial state x° and of an input function $u^\circ(\cdot)$, defined for all t in a neighborhood of $t = 0$, such that the corresponding output $y(t)$ of the system is identically zero for all t in a neighborhood of $t = 0$. Of course, we are interested in finding *all* such pairs (x°, u°) and not simply in the trivial pair $x^\circ = 0, u^\circ = 0$ (corresponding to the situation in which the system is initially at rest and no input is applied). We perform this analysis on the normal form of the system.

Recalling that in the normal form

$$y(t) = \xi_1(t) \,,$$

we see that the constraint $y(t) = 0$ for all t implies

$$\xi_1(t) = \xi_2(t) = \ldots = \xi_r(t) = 0 \,,$$

that is $\xi(t) = 0$ for all t.

Thus, we see that when the output of the system is identically zero its state is constrained to evolve in such a way that also $\xi(t)$ is identically zero. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = q(z(t), 0) + b(z(t), 0)u(t)$$

(recall that $b(z(t), 0) \neq 0$ if $z(t)$ is close to 0). As far as the variable $z(t)$ is concerned, it is clear that, being $\xi(t)$ identically zero, its behavior is governed by the differential equation

$$\dot{z}(t) = f_0(z(t), 0) \,. \tag{12}$$

From this analysis we deduce the following facts. If the output $y(t)$ has to be zero, then necessarily the initial state of the system must be set to a value such that $\xi(0) = 0$, whereas $z(0) = z^\circ$ can be chosen arbitrarily. According to the value of z°, the input must be set as

$$u(t) = -\frac{q(z(t), 0)}{b(z(t), 0)}$$

where $z(t)$ denotes the solution of the differential equation

$$\dot{z}(t) = f_0(z(t), 0) \qquad \text{with initial condition } z(0) = z^\circ.$$

Note also that for each set of initial data $\xi = 0$ and $z = z^\circ$ the input thus defined is the *unique* input capable to keep $y(t)$ identically zero for all times.

The dynamics of (12) correspond to the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. These dynamics, which are rather important in many of our developments, are called the *zero dynamics* of the system.

Remark. In the case of a linear system, functions $f_0(z, \xi)$ and $q(z, \xi)$ are linear functions, and $b(z, \xi)$ is a constant. The normal form (in the notation (4) can be written as

$$\begin{array}{rcl} \dot{z} &=& Fz + G\xi \\ \dot{\xi} &=& \hat{A}\xi + \hat{B}[Hz + K\xi + bu] \\ y &=& \hat{C}\xi\,. \end{array}$$

Using

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

the system can be rewritten as

$$\begin{array}{rcl} \dot{\tilde{x}} &=& A\tilde{x} + Bu \\ y &=& C\tilde{x} \end{array}$$

in which

$$A = \begin{pmatrix} F & G \\ \hat{B}H & \bar{A} + \hat{B}K \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 \\ \hat{B}b \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & \hat{C} \end{pmatrix}$$

A simple calculation shows that

$$\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = b \det(sI - F)\,.$$

On the other hand, it is known that the transfer function $T(s) = C(sI - A)^{-1}B$ of the system can be expressed as

$$T(s) = \frac{\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix}}{\det(sI - A)}\,.$$

Hence,

$$T(s) = b\,\frac{\det(sI - F)}{\det(sI - A)}\,,$$

from which it is concluded that in a (controllable and observable) linear system, the zeros of the transfer function $T(s)$ coincide with the eigenvalues of F. In other words, in a linear system the zero dynamics are linear dynamics with eigenvalues coinciding with the zeros of the transfer function of the system. $\triangleleft$

As we will see in the sequel of this Chapter, the asymptotic properties of the system identified by the upper equation in (9) play a role of paramount importance in the design of stabilizing feedback laws. In this context, the properties expressed by the following two definitions are considered.

Definition. Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal. The system is *globally minimum phase* if the equilibriun $z = 0$ of

$$\dot{z} = f_0(z, 0)$$

is globally asymptotically stable.

According to the well-known criterion of Lyapunov, a system is minimum phase if there exists a positive definite and proper smooth real-valued function $V(z)$ satisfying [4]

$$\frac{\partial V}{\partial z} f(z, 0) \leq -\alpha(|z|) \qquad \text{for all } z \in \mathbb{R}^n \,,$$

in which $\alpha(|z|)$ is a class $\mathcal{K}_\infty$ function.

Definition. Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal. The system is *strongly minimum phase* if system

$$\dot{z} = f_0(z, \xi) \,, \tag{13}$$

viewed as a system with input ξ and state z, is input-to-state stable.

By definition – then – a system is strongly minimum phase if there exist a class $\mathcal{K}L$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{L}$ function $\gamma_z(\cdot)$ such that the following property holds: the response $z(t)$ of (13)from the initial state $z(0) = z_0$ to a piecewise-continuous bounded input $\xi_0(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies, for any z_0 and for any such $\xi_0(\cdot)$,

$$|z(t)| \leq \max\{\beta(|z_0|, t), \gamma_z(\|\xi_0(\cdot)\|_\infty)\} \qquad \text{for all } t \geq 0.$$

The function $\gamma_z(\cdot)$ is called a *gain function* of (13).

According to the well-known criterion of Sontag, a system is strongly minimum phase if there exists a positive definite and proper smooth real-valued function $V(z)$ and a class $\mathcal{K}$ function $\chi(\cdot)$ satisfying

$$\frac{\partial V}{\partial z} f(z, 0) \leq -\alpha(|z|) \qquad \text{for all } z \in \mathbb{R}^n \,, \qquad \text{for all } (z, \xi) \text{ such that } |z| \geq \chi(|\xi|) \,,$$

in which $\alpha(|z|)$ is a class $\mathcal{K}_\infty$ function.

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[4] To say that a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is proper and positive definite is equivalent to say that there exist class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \,, \qquad \text{for all } z \in \mathbb{R}^n \,.$$

4 Stabilization via full-state feedback

A typical setting in which normal forms are useful is the derivation of systematic method for stabilization *in the large* of certain classes of nonlinear system, even in the presence of parameter uncertainties. We begin this analysis with the observation that, if the system is *strongly minumum phase*, it is quite easy to design a globally stabilizing *state feedback* law. To be precise, consider again a system in normal form (4), which we assume to be globally defined, and assume that the system is strongly minimum phase, i.e. assume that $f_0(0,0) = 0$ and that

$$\dot{z} = f_0(z, \xi)\,,$$

viewed as a system with input ξ and state z, is input-to-state stable. Assume also that and that the coefficient $b(z, \xi)$ satisfies

$$b(z,\xi) \geq b_0 > 0 \qquad \text{for all } (x, \xi)$$

for some b_0. Consider the feedback law

$$u = \frac{1}{b(z,\xi)}\,(-q(z,\xi) - \hat{K}\xi)\,, \tag{14}$$

in which $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is a vector of design parameters. Under this feedback law, the system becomes

$$\begin{array}{rcl} \dot{z} & = & f_0(z,\xi) \\ \dot{\xi} & = & (\hat{A} + \hat{B}\hat{K})\xi\,. \end{array} \tag{15}$$

Since the pair $(\hat{A}, \hat{B})$ is controllable, it is possible to pick $\hat{K}$ so that the matrix $(\hat{A}+\hat{B}\hat{K})$ is a Hurwitz matrix. If this is the case, system (15) appears as a cascade-connection in which a globally asymptotically stable system (the lower sub-system) drives an input-to-stable system (the upper-system). By known properties, such cascade-connection is globally asymptotically stable. In other words, if $\hat{K}$ is chosen in this way, the feedback law (14) *globally asymptotically* stabilizes the equilibrium $(z, \xi) = (0, 0)$ of the closed-loop system.

The feedback law (14) is expressed in the (z, ξ) coordinates that characterize the normal form (4). To express it in the original coordinates that characterize the model (1), it suffices to bear in mind that

$$b(z,\xi) = L_g L_f^{r-1} h(x)\,, \qquad q(z,\xi) = L_f^r h(x)$$

observe that

$$\hat{K}\xi = \sum_{i=1}^{r} \hat{k}_i L_f^i h(x)$$

in which $\hat{k}_1, \ldots, \hat{k}_r$ are the entries of the row vector $\hat{K}$. Thus, can conclude what follows.

Lemma 2 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal. Suppose the system is strongly minimum phase. If $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is any vector such that $\sigma(\hat{A} + \hat{B}\hat{K}) \in \mathbb{C}^-$, the* state feedback *law*

$$u(x) = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) - \sum_{i=1}^{r} \hat{k}_i L_f^i h(x)\right), \tag{16}$$

globally asymptotically stabilizes the equilibrium $x = 0$.

This feedback strategy, although very intuitive and elementary, is not useful in a practical context because it relies upon exact cancelation of certain nonlinear function and, as such, possibly *non-robust.* Uncertainties in $q(z, \xi)$ and $b(z, \xi)$ would make this strategy unapplicable. Moreover, it relies upon the assumption that the system is *strongly* minimum phase, which is a somewhat stronger assumption. Finally, the implementation of such control law requires the availability, for feedback purposes, of the *full* state (z, ξ) of the system, a condition that might be hard to ensure. Thus, motivated by these considerations, we readdress the problem in what follows, by seeking feedback law depending on fewer measurements (hopefully only on the measurement output y), requiring less a stringent assumption (the property of being simply *minimum-phase*, other than being strongly minimum-phase) and possibly robust with respect to model uncertainties. Of course, in return, some price has to be paid.

We conduct this analysis in the subsequent sections. For the time being we conclude by showing how, in the context of full state feedback, the assumption that the system is *strongly* minimum-phase can be weakened. This is possible, to some extent, if the normal form of the system has the special structure (10). To this end, we use again a feedback of the form (16) but in which now $\hat{K}$ has the following structure

$$\hat{K} = \begin{pmatrix} -a_0 k^r & -a_1 k^{r-1} & \cdots & -a_{r-2} k^2 & -a_{r-1} k \end{pmatrix}$$

where $k > 0$ is a design parameter and the a_i's are such that

$$d(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1 \lambda + a_0$$

is a Hurwitz polynomial. In fact, it is possible to show that, with a feedback law of this kind, it is possible to asymptotically stabilize the equilibrium $x = 0$ of the closed-loop system, with a domain of attraction that includes a fixed (but otherwise arbitrary) compact set.

Lemma 3 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal form having the special structure (10). Suppose the zero dynamics are globally asymptotically stable. Let the control be provided by the state feedback law*

$$u_k(x) = \frac{1}{L_g L_f^{r-1} h(x)} \Big(-L_f^r h(x) - \sum_{i=1}^{r} k^{r+1-i} a_{i-1} L_f^i h(x) \Big). \tag{17}$$

Then, for every choice of a compact set $\mathcal{C}$ there is a number k^* such that, for all $k \geq k^*$ the feedback law (17) asymptotically stabilizes the equilibrium $x = 0$ of the resulting closed-loop system, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$. If, in addition, the zero dynamics are also locally exponentially stable, then the equilibrium $x = 0$ of the resulting closed-loop system is also locally expoentially stable.

Proof. Let the system (5) and the feedback law (17) be expressed in normal form. By assumption, there is a positive definite and proper smooth function $V_0 : \mathbb{R}^{n-r}$ satisfying

$$\begin{aligned} \underline{\alpha}_0(|z|) \leq V_0(z) \leq \overline{\alpha}_0(|z|) \\ \frac{\partial V_0}{\partial z} f_0(z, 0) \leq -\alpha(|z|) \end{aligned} \qquad \text{for all } z \in \mathbb{R}^{n-r}$$

in which $\underline{\alpha}_0(\cdot)$, $\overline{\alpha}_0(\cdot)$ and $\alpha_0(\cdot)$ are class $\mathcal{K}_\infty$ functions. Moreover, by construction, the matrix

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{r-2} & -a_{r-1} \end{pmatrix}.$$

is a Hurwitz matrix and therefore there exists a positive definite symmetric matrix P solution of the Lyapunov equation

$$A^{\mathrm{T}}P + PA = -I\,.$$

Change now the ξ_i's in new variables ζ_i defined as

$$\zeta_i = \frac{1}{k^{i-1}}\xi_i, \qquad 1 \le i \le r$$

The closed loop system, after this transformation of coordinates, is described by equations of the form (recall that the normal form of (5) has the special structure (10)

$$\begin{aligned} \dot{z} &= f(z,\zeta_1) \\ \dot{\zeta} &= kA\zeta \end{aligned} \tag{18}$$

in which

$$\zeta = \mathrm{col}(\zeta_1, \zeta_2, \ldots, \zeta_r)\,.$$

5 Stabilization via partial-state feedback

5.1 Systems having relative degree 1

We discuss, in this section, the case of systems having *relative degree 1*, i.e. we consider systems modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= q(z,\xi) + b(z,\xi)u \\ y &= \xi_1 \end{aligned} \tag{19}$$

in which $z \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$. All functions are smooth functions of their arguments. [5]

About this system we assume that

$$\begin{aligned} f(0,0) &= 0 \\ q(0,0) &= 0 \end{aligned}$$

[5]Note that, in order to simplify the notation, we have dropped the subscript "0" from the symbol f_0 that characterizes the right-hand-side of the first equation in the normal form. This is legitimate because we assume that the system is already given in normal form and hence no conflict of notation may occur with the symbol "f" used earlier in (1).

and that the coefficient $b(z,\xi)$ satisfies

$$b(z,\xi) \geq b_0 > 0 \qquad \text{for all } (x,\xi)$$

for some b_0. We also assume that the system is *minimum-phase*, i.e. the equilibrium $z = 0$ of

$$\dot{z} = f_0(z, 0)$$

is globally asymptotically stable.

The system will be controlled the very simple feedback law

$$u = -ky$$

in which $k > 0$, which yields a closed loop system

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= q(z,\xi) - b(z,\xi)k\xi \,. \end{aligned} \tag{20}$$

For convenience, we set

$$x = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

and rewrite the latter as

$$\dot{x} = F_k(x)$$

in which

$$F_k(x) = \begin{pmatrix} f(z,\xi) \\ q(z,\xi) - b(z,\xi)k\xi \end{pmatrix} .$$

Proposition 3 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree 1 and possesses a globally defined normal form. Suppose the system is globally minimum phase. Let the control be provided by the output feedback $u = ky$. Then, for every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a number k^* such that, for all $k \geq k^*$ there is a finite time T such that all trajectories of the closed-loop system with initial condition $x(0) \in \mathcal{C}$ remain bounded and satisfy $|x(t)| \leq \varepsilon$ for all $t \geq T$.*

Proof. Consider, for this system, the candidate Lyapunov function

$$W(x) = V(z) + \frac{1}{2}\xi^2$$

which is positive definite and proper. For any real number $a \geq 0$, let

$$\Omega_a = \{x \in \mathbb{R}^n : W(x) \leq a\}$$

denote the sublevel set consisting of all points of $\mathbb{R}^n$ at which the value of $W(x)$ is less than or equal to a and let

$$B_a = \{x \in \mathbb{R}^n : |x| \leq a\}$$

denotes the closed ball consisting of all points of $\mathbb{R}^n$ whose norm does not exceed a. Since $W(x)$ is proper, the set Ω_a is a compact set for any a. Pick any arbitrarily large number

R, any arbitrarily small number r. Since $W(x)$ is positive definite and proper, there exist numbers numbers $0 < d < c$ such that

$$\Omega_d \subset B_r \subset B_R \subset \Omega_c \,.$$

Consider also the compact "annular" region

$$S_d^c = \{x \in \mathbb{R}^n : d \le V(x) \le c\} \,.$$

Our goal is to show that – if the gain coefficient k is large enough – the function

$$\dot{W}(x) := \frac{\partial W}{\partial x} F_k(x)$$

is negative at each point of S_d^c. Observe, in this respect, that

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi) k \xi^2 \,.$$

To this end, we proceed as follows. Consider the compact set

$$Z = S_d^c \bigcap \{x \in \mathbb{R}^n : \xi = 0\} \,.$$

At each point of Z

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,0) \le -\alpha(|z|)$$

Since $z \neq 0$ at each point of Z, there is a number $a > 0$ such that

$$\dot{W}(x) \le -a \qquad \forall x \in Z \,.$$

Hence, by continuity, there is an open set Z_ε containing Z such

$$\dot{W}(x) \le -a/2 \qquad \forall x \in Z_\varepsilon \,. \tag{21}$$

Consider now the set

$$\tilde{S} = \{x \in S_d^c : x \notin Z_\varepsilon\} \,.$$

which is a compact set, let

$$M = \max_{x \in \tilde{S}} \{\frac{\partial V}{\partial x} f_0(z,\xi) + \xi q(z,\xi)\}$$

$$m = \min_{x \in \tilde{S}} \{b(z,\xi)\xi^2\}$$

and observe that $m > 0$ because $b(z,\xi) \ge b_0 > 0$ and ξ cannot vanish at any point of $\tilde{S}$. Thus, since $k > 0$, we obtain

$$\dot{W}(x) \le M - km \qquad \forall x \in \tilde{S} \,.$$

Let k_1 be such that $M - k_1 m = -a/2$. Then, if $k \ge k_1$,

$$\dot{W}(x) \le -a/2 \qquad \forall x \in \tilde{S} \,. \tag{22}$$

This, together with (21) shows that

$$k \geq k_1 \qquad \Rightarrow \qquad \dot{W}(x) \leq -a/2 \qquad \forall x \in S_d^c\,,$$

as requested. This being the case, we see that for any trajectory with initial condition in S_d^c, so long as $z(t) \in S_d^c$ we have

$$W(x(t)) \leq W(x(0)) - (a/2)t\,.$$

As a consequence, the trajectory in finite time enters the set Ω_d, and remains there (because $\dot{W}(x)$ is negative on the boundary of this set). $\lhd$

This property is commonly known as property of *semi-global practical* stabilizability, of the equilibrium $(z, \xi) = (0, 0)$ of (19).

To obtain *asymptotic* stability, either a nonlinear control law $u = -\kappa(y)$ is needed or, if one insists in using a linear law $u = -ky$, extra assumptions are necessary. Note that in the former case, the closed-loop system becomes (compare with (20))

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= q(z, \xi) - b(z, \xi)\kappa(\xi)\,. \end{aligned}$$

which will be rewritten as

$$\dot{x} = F_{\kappa(\cdot)}(x)$$

in which

$$F_{\kappa(\cdot)}(x) = \begin{pmatrix} f(z, \xi) \\ q(z, \xi) - b(z, \xi)\kappa(\xi) \end{pmatrix}\,.$$

The following results hold.

Proposition 4 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree 1 and possesses a globally defined normal form. Suppose the zero dynamics are globally asymptotically stable. Then, for every choice of a compact set $\mathcal{C}$ there exists a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that the equilibrium $x = 0$ is asymptotically stabilized by the feedback law $u = -\kappa(y)$, with a domain of attraction $\mathcal{A}$ that includes the set $\mathcal{C}$. If, in addition, the zero dynamics are also locally exponentially stable, then for each compact set $\mathcal{C}$ there is a number k^* such that, for all $k \geq k^*$, the same result holds with $\kappa(y) = -ky$.*

Proof. To prove the second part, we observe that to say that the equilibrium $z = 0$ of $\dot{z} = f(z, 0)$ is locally exponentially stable is to say that the eigenvalues of the matrix

$$F := \frac{\partial f}{\partial z}(0, 0) \tag{23}$$

have negative real part. Linear arguments can be invoked to prove the existence of a number k_2 such that, if $k \geq k_2$, the equilibrium $x = 0$ of (19) is locally asymptotically stable. It is also possible to show that there is a number r' such that, for all $k \geq k_2$, the closed ball $B_{r'}$

is always contained in the domain of attraction of $x = 0$. To check that this is the case, let P be the positive definite solution of

$$PF_0 + F_0^{\mathrm{T}} P = -2I\,,$$

which exists because all eigenvalues of F_0 have negative real part and consider, for the system the candidate quadratic Lyapunov function

$$U(x) = z^{\mathrm{T}} \tilde{P} z + \frac{1}{2}\xi^2$$

yielding

$$\dot{U}(x) = 2z^{\mathrm{T}} P f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi) k \xi^2\,.$$

Expand $f_0(z,\xi)$ as

$$f_0(z,\xi) = F_0 z + g(z) + [f_0(z,\xi) - f_0(z,0)]$$

in which

$$\lim_{z\to 0} \frac{|g(z)|}{|z|} = 0\,.$$

Then, there is a number δ such that,

$$|z| \leq \delta \qquad \Rightarrow \qquad |g(z)| \leq \frac{1}{2|P|}|z|$$

and this yields

$$2z^{\mathrm{T}} P[F_0 z + g(z)] = -2|z|^2 + 2z^{\mathrm{T}} P g(z) \leq -|z|^2 \qquad \forall z \in B_\delta\,.$$

Since the function $[f_0(z,\xi) - f_0(z,0)]$ is a continuously differentiable function that vanish at $\xi = 0$, there is a number M_1 such that

$$|2z^{\mathrm{T}} P[f_0(z,\xi) - f_0(z,0)]| \leq M_1 |z||\xi| \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \leq \delta\,.$$

Likewise, since $q(z,\xi)$ is a continuously differentiable function that vanish at $(z,\xi) = (0,0)$, there are numbers N_1 and N_2 such that

$$|\xi q(z,\xi)| \leq N_1|z||\xi| + N_2|\xi|^2 \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \leq \delta\,.$$

Finally, since $b(z,\xi)$ is positive and nowhere zero, there is a number b_0 such that (recall that $k > 0$)

$$-kb(z,\xi) \leq -kb_0 \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \leq \delta\,.$$

Putting all these inequalities together, one finds that, for all for all (z,ξ) such that $z \in B_\delta$ and $|\xi| \leq \delta$

$$\dot{U}(x) \leq -|z|^2 + (M_1 + N_1)|z||\xi| - (kb_0 - N_2)|\xi|^2\,.$$

It is easy to check that if k is such that

$$(2kb_0 - 2N_2 - 1) \geq (M_1 + N_1)^2\,,$$

$$\dot{U}(x) \leq -\frac{1}{2}|x|^2 .$$

This shows that there is a number k_2 such that, if $k \geq k_2$, the function $\dot{U}(x)$ is negative definite for all x satisfying $z \in B_\delta$ and $|\xi| \leq \delta$. Pick now any (nontrivial) sublevel set $\tilde{\Omega}_c$ of $U(x)$ entirely contained in the set of all x satisfying $z \in B_\delta$ and $|\xi| \leq \delta$ and let r' be such that $B_{r'} \subset \tilde{\Omega}_c$. Then, the argument above shows that, for all $k \geq k_2$, the equilibrium $x = 0$ is asymptotically stable with a domain of attraction that contains $B_{r'}$, which is a set independent of the choice of k.

Pick now $r \leq r'$, pick any $R > 0$ and use the result of the Proposition above. There is a number k_1 such that, if $k \geq k_1$ all trajectories with initial condition in B_R in finite time enter the region B_r, and hence enter the region of attraction of $x = 0$. It can be concluded, setting $k^* = \max\{k_1, k_2\}$, that for all $k \geq k^*$ the equilibrium $x = 0$ of (20) is asymptotically stable, with a domain of attraction that contains B_R.

5.2 Systems having relative degree $r > 1$

Consider now the general case of a system having relative degree $r > 1$, which in normal form is expressed as

$$\begin{aligned}
\dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, \xi_r) \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= q(z, \xi_1, \ldots, \xi_{r-1}, \xi_r) + b(z, \xi_1, \ldots, \xi_{r-1}, \xi_r)u
\end{aligned} .$$

For this system, choose a control of the form

$$u = -k(\xi_r + d_{r-2}\xi_{r-1} + \cdots + d_1\xi_2 + d_0\xi_1) .$$

To analyze the corresponding closed-loop system, we change the variable ξ_r into a variable ζ defined as

$$\theta = d_0\xi_1 + d_1\xi_2 + \cdots + d_{r-2}\xi_{r-1} + \xi_r .$$

With this change of coordinates, the closed-loop system can be rewritten in the form

$$\begin{aligned}
\dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta) \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-2} &= \xi_{r-1} \\
\dot{\xi}_{r-1} &= -\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta \\
\dot{\theta} &= d_0\xi_2 + d_1\xi_3 + \cdots + d_{r-2}(-\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta) + q(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta) \\
&\quad + b(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta)(-k\theta)
\end{aligned} .$$

This system has a structure which is identical to that of system (20). In fact, if we set

$$\bar{z} = \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}) \in \mathbb{R}^{n-1}$$

and define $\bar{f}_0(\bar{z},\theta)$, $\bar{q}(\bar{z},\theta)$, $\bar{b}(\bar{z},\theta)$ as

$$\bar{f}_0(\bar{z},\theta) = \begin{pmatrix} f_0(z,\xi_1,\ldots,\xi_{r-1},-\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta) \\ \xi_2 \\ \cdots \\ \xi_{r-1} \\ -\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta \end{pmatrix}$$

$$\begin{aligned} \bar{q}(\bar{z},\theta) &= d_0\xi_2 + d_1\xi_3 + \cdots + d_{r-2}(-\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta) + q(z,\xi_1,\ldots,\xi_{r-1},-\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta) \\ \bar{b}(\bar{z},\theta) &= b(z,\xi_1,\ldots,\xi_{r-1},-\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta)\,, \end{aligned}$$

the system cam be rewritten in the form

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}_0(\bar{z},\theta) \\ \dot{\theta} &= \bar{q}(\bar{z},\theta) - \bar{b}(\bar{z},\theta)k\theta\,, \end{aligned}$$

identical to that of (20).

Thus, identical stability results can be obtained, if $\bar{f}_0(\bar{z},\theta)$, $\bar{q}(\bar{z},\theta)$, $\bar{b}(\bar{z},\theta)$ satisfy conditions corresponding to those assumed on $f_0(z,\xi)$, $q(z,\xi)$, $b(z,\xi)$.

From this viewpoint, it is trivial to check that if the functions $f_0(z,\xi)$, $q(z,\xi)$, and $b(z,\xi)$ satisfy

$$\begin{aligned} f_0(0,0) &= 0 \\ q(0,0) &= 0 \\ b(z,\xi) &\geq b_0 > 0 \qquad \text{for all } (x,\xi) \end{aligned}$$

then also

$$\begin{aligned} \bar{f}_0(0,0) &= 0 \\ \bar{q}(0,0) &= 0 \\ \bar{b}(\bar{z},\theta) &\geq b_0 > 0 \qquad \text{for all } (\bar{z},\theta)\,. \end{aligned}$$

Thus, in order to be able to use the stabilization results developed in the previous subsection, it remains to check the input-to-state stability property of the system

$$\dot{\bar{z}} = \bar{f}_0(\bar{z},\theta)\,.$$

This system can be interpreted as a cascade interconnection

$$\begin{aligned} \dot{z} &= f_0(z, C\zeta + D\theta) \\ \dot{\zeta} &= A\theta + B\theta \end{aligned} \tag{24}$$

in which we have used ζ to denote the vector

$$\zeta = \mathrm{col}(\xi_1,\xi_2,\ldots,\xi_{r-1})$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{r-2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ -d_0 & -d_1 & \cdots & -d_{r-3} & -d_{r-2} \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}.$$

If the d_i's are such that the matrix A is Hurwitz, the lower subsystem is input-to-state stable. If the original system is strongly minimum phase, the upper subsystem, viewed as a system with input $C\zeta + D\theta$ and state z is input-to-state stable. Thus, system (24), being the cascade of two input-to-state stable systems, is input-to-state stable.

6 Examples and counterexamples

Finite escape time in a cascade

Practical but not asymptotic stability