Optimal Control

Department of Computer, Control, and
Management Engineering Antonio Ruberti

Sapienza
Università di Roma

Lecture 6

Prof. Daniela Iacoviello

THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

Course outline

- Introduction to optimal control
- Hamilton-Jacobi equation
- Pontryagin minimum principle
- **The regulator problem**
- The regulator problem on infinite time interval
- The optimal tracking problem
- The optimal regulator problem with null final error
- The optimal regulator problem with limited control
- The minimum time problem for steady state system

The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

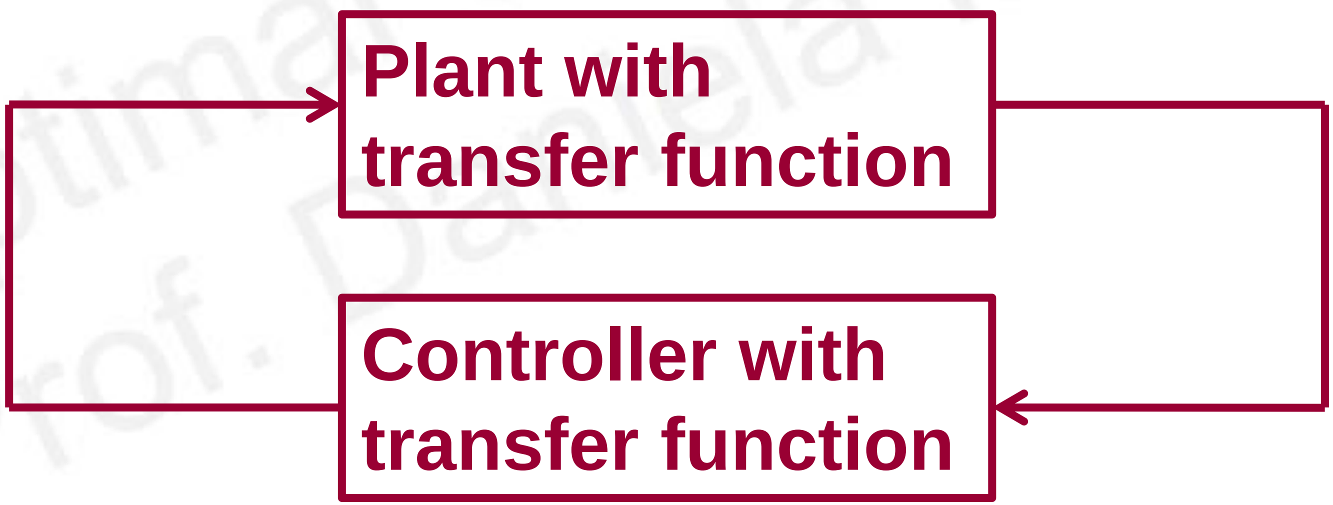
The optimal regulator problem with null final error

The optimal regulator problem with limited control

The regulator problem

The classical description of a system is normally in terms of its transfer function matrix W(s).

A classical **feedback arrangement** is:



The regulator problem

The controller output is assumed to be an **instantaneous function** of the plant state x(t):

$$u(t) = H\big(x(t), t\big)$$

Interesting is the case of **linear control law:**

$$u(t) = H(t)x(t)$$

The regulator problem

It is necessary to keep some measure of control magnitude bounded or even small during the course of a control action.

In defining our optimal regulator task we seek one that has **engineering sense** and **yields an optimal controller suitable for implementation**, preferably linear

The regulator problem

Nonquadratic measures do not lead,

in general,

to a linear feedback laws

The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

The regulator problem

Let us consider the following **linear system**:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

with $[t_i, t_f]$ fixed, and $x(t_i) = x_i$ fixed

$A(t),\ B(t),\ C(t)$ are matrix functions of time with continuous entries; their dimensions are respectively

$n \times n,\ n \times m,\ n \times p$

The regulator problem

A linear control law is obtained if we seek to minimize the **quadratic performance index:**

$$\boxed{\begin{aligned} J(x,u) &= \frac{1}{2}\int_{t_0}^{t_f}\left(x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right)dt \\ &+ \frac{1}{2}x^T(t_f)Fx(t_f) \end{aligned}}$$

where *Q(t)* and *R(t)* have continuous entries, symmetric, nonnegative and positive definite respectively;

F is nonnegative definite matrix simmetric

The Riccati equation (Riccati, 1676-1754)

Definition:

The **Riccati differential equation** associated at the regulator problem is:

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^{T}(t)K(t) - K(t)A(t) - A^{T}(t)K(t) - Q(t)$$

$$K(t_f) = F$$

The Riccati equation

Theorem: The Riccati equation

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^T(t)K(t) - K(t)A(t) - A^T(t)K(t) - Q(t)$$
$$K(t_f) = F$$

admits a **unique** solution, semidefinite positive,
in the control interval.

Proof:

The main steps of the proof are the following:

- The **existence and uniqueness** theorem for differential equation may be applied **only locally**. Therefore there exists a subinterval $(t,t_f]$ in which the Riccati equation admids a unique solution

- It must be shown that **it doesn't exist** an element k_{ij} of the matrix K such that for any $t' < t_f$ it results in:

$$\lim_{t \to t'^{+}} \left| k_{ij}(t) \right| = \infty$$

For the proof it is useful to take into account that the transition matrix of the matrix A is limited.

The Riccati equation

Remark

- The solution of the Riccati equation is symmetric
- Usually the Riccati equation could be solved only numerically

Solution of the regulator problem

Theorem: The optimal control for the regulator problem is given by the **linear feedback law**:

$$\boxed{u^o(t) = -R^{-1}(t)B^T(t)K(t)x^o(t)}$$

where:

$$\dot{x}(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x^o(t)$$

$$x^o(t_i) = x_i$$

and

$$J\left(x^o, u^o\right) = \frac{1}{2}x^{iT}K(t_i)x^i$$

Proof: We apply the results of the Pontryagin principle to our linear system . Let us define the Hamiltonian:

$$H(x,u,\lambda,t)=\frac{1}{2}x^T Qx+\frac{1}{2}u^T Ru+\lambda^T Ax+\lambda^T Bu$$

The necessary and sufficient conditions are:

$$\dot{\lambda}^o=-\left.\frac{\partial H}{\partial x}\right|^{oT}=-Qx^o-A^T\lambda^o$$

$$\left.\frac{\partial H}{\partial u}\right|^{o}=Ru^o+B^T\lambda^o=0$$

$$\lambda^o(t_f)=\left.\frac{dG}{dx(t_f)}\right|^{oT}$$

Taking into account the equation of Riccati and *defining the costate*

$$\boxed{\lambda^o = Kx^o}$$

the result is proved.

In fact:

1.

$$\dot{\lambda}^o = \dot{K}x^o + K\dot{x}^o$$

$$= KBR^{-1}B^T Kx^o - KAx^o - A^T Kx^o - Qx^o + KAx^o - KBR^{-1}B^T Kx^o$$

$$= -A^T Kx^o - Qx^o = -A^T \lambda^o - Qx^o$$

☺

2. $$R\left(-R^{-1}B^{T}Kx^{o}\right)+B^{T}\lambda^{o}=-B^{T}Kx^{o}+B^{T}\lambda^{o}=0$$

3. $$\lambda^{o}(t_f^{o})=K(t_f^{o})x^{o}(t_f^{o})=Fx^{o}(t_f^{o})$$

4. $$J\left(x^{o},u^{o}\right)=\frac{1}{2}x^{oT}(t_f^{o})Fx^{o}(t_f^{o})+\frac{1}{2}\int_{t_i}^{t_f}x^{oT}\left(Q+KBR^{-1}B^{T}K\right)x^{o}dt$$

$$=\frac{1}{2}x^{oT}(t_f^{o})Fx^{o}(t_f^{o})+\frac{1}{2}\int_{t_i}^{t_f}x^{oT}\left(-\dot{K}+2KBR^{-1}B^{T}K-KA-A^{T}K\right)x^{o}dt$$

$$=\frac{1}{2}x^{oT}(t_f^{o})Fx^{o}(t_f^{o})-\frac{1}{2}\int_{t_i}^{t_f}\left(2x^{oT}K\dot{x}^{o}+x^{oT}\dot{K}x^{o}\right)dt$$

$$=\frac{1}{2}x^{oT}(t_f^{o})Fx^{o}(t_f^{o})-\frac{1}{2}\int_{t_i}^{t_f}\frac{d}{dt}\left(x^{oT}Kx^{o}\right)dt=\frac{1}{2}x^{iT}K(t_i)x^{i}$$

The Riccati equation

Theorem: The Riccati equation

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^T(t)K(t) - K(t)A(t) - A^T(t)K(t) - Q(t)$$
$$K(t_f) = F$$

admits a **unique** solution, semidefinite positive,
in the control interval.

Proof:

The main steps of the proof are the following:

- The **existence and uniqueness** theorem for differential equation may be applied **only locally.** Therefore there exists a subinterval $(t,t_f]$ in which the Riccati equation admits a unique solution

Proof:

The main steps of the proof are the following:

- The **existence and uniqueness** theorem for differential equation may be applied **only locally**. Therefore there exists a subinterval $(t, t_f]$ in which the Riccati equation admids a unique solution
- We want to extend the solution to the whole time interval $[t_i, t_f]$

Proof:

The main steps of the proof are the following:

- The **existence and uniqueness** theorem for differential equation may be applied **only locally.** Therefore there exists a subinterval $(t,t_f]$ in which the Riccati equation admids a unique solution
- We want to extend the solution to the whole time interval $[t_i, \ t_f]$
- It must be shown that **it doesn't exist** an element k_{ij} of the matrix K such that for any $t' < t_f$ it results in:

$$\lim_{t \to t'^+} \left| k_{ij}(t) \right| = \infty$$

Proof:

The main steps of the proof are the following:

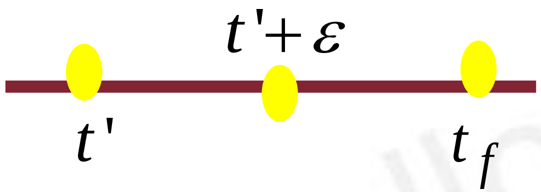
- The **existence and uniqueness** theorem for differential equation may be applied **only locally**. Therefore there exists a subinterval $(t, t_f]$ in which the Riccati equation admids a unique solution
- We want to extend the solution to the whole time interval $[t_i, t_f]$
- It must be shown that **it doesn't exist** an element k_{ij} of the matrix K such that for any $t' < t_f$ it results in:

$$\lim_{t \to t'^{+}} \left| k_{ij}(t) \right| = \infty$$

By contraddiction: assume that $\boxed{\lim_{t \to t'^{+}} \left| k_{ij}(t) \right| = \infty}$

By contraddiction: assume that $\lim\limits_{t \to t'^{+}} \left| k_{ij}(t) \right| = \infty$

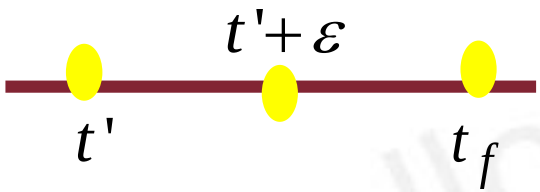
Consider an $\varepsilon > 0$ such that $t' + \varepsilon < t_f$



In $[t' + \varepsilon,\ t_f]$ Riccati equation admits a solution.

By contraddiction: assume that $\lim_{t\to t'^+} \left| k_{ij}(t) \right| = \infty$

Consider an $\varepsilon > 0$ such that $t' + \varepsilon < t_f$



In $[t' + \varepsilon,\ t_f]$ Riccati equation admits a solution.

There exists a solution of the problem, for any $x(t' + \varepsilon)$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t)$$

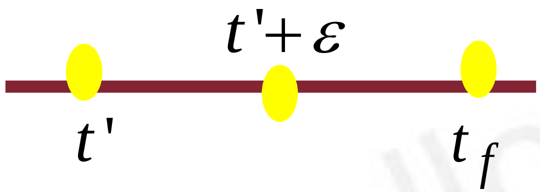
$$\boldsymbol{x(t_i) = x_i}$$

$$J(x,u) = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right) dt$$

$$+ \frac{1}{2} x^T(t_f)Fx(t_f)$$

By contraddiction: assume that $\lim_{t \to t'^{+}} \left| k_{ij}(t) \right| = \infty$

Consider an $\varepsilon > 0$ such that $t' + \varepsilon < t_f$



In $[t' + \varepsilon,\ t_f]$ Riccati equation admits a solution.

There exists a solution of the problem, for any $x(t' + \varepsilon)$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$
$$x(t_i) = x_i$$
$$J(x,u) = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right) dt + \frac{1}{2} x^T(t_f) F x(t_f)$$

In particular:

$$J_{t'+\varepsilon}(x^o, u^o) = \frac{1}{2} x^T(t'+\varepsilon) \mathrm{K}(t'+\varepsilon) x(t'+\varepsilon)$$

In particular:

$$\boxed{J_{t'+\varepsilon}(x^o,u^o)=\frac{1}{2}x^T(t'+\varepsilon)\mathrm{K}(t'+\varepsilon)x(t'+\varepsilon)}$$

For the hypothesis on the matrices *F, Q(t), R(t)*

$$\boxed{J_{t'+\varepsilon}(x^o,u^o)\geq 0,\quad \forall x(t'+\varepsilon)\in R^n}$$

In particular: $$\boxed{J_{t'+\varepsilon}(x^o, u^o) = \frac{1}{2} x^T(t'+\varepsilon)\mathrm{K}(t'+\varepsilon)x(t'+\varepsilon)}$$

For the hypothesis on the matrices *F, Q(t), R(t)*

$$\Rightarrow \quad J_{t'+\varepsilon}(x^o, u^o) \geq 0, \quad \forall x(t'+\varepsilon) \in R^n$$

$$\Rightarrow \quad K(t'+\varepsilon) \geq 0, \quad \forall \varepsilon \in (0, \mathrm{t}_f - t']$$

In particular: $$\boxed{J_{t'+\varepsilon}(x^o,u^o) = \frac{1}{2}x^T(t'+\varepsilon)\mathrm{K}(t'+\varepsilon)x(t'+\varepsilon)}$$

For the hypothesis on the matrices *F, Q(t), R(t)*

$$\Rightarrow \quad J_{t'+\varepsilon}(x^o,u^o) \geq 0, \quad \forall x(t'+\varepsilon) \in R^n$$

$$\Rightarrow \quad K(t'+\varepsilon) \geq 0, \quad \forall \varepsilon \in (0, \mathrm{t}_f - t']$$

$$\Rightarrow \quad \boxed{K(t) \geq 0, \quad \forall t \in (\mathrm{t}', \; \mathrm{t}_f]}$$

Consider the i-th column of the identity matrix, e_i

Being K(t) symmetric one has:

$$\boxed{(e_i \pm e_j)^T K(t)(e_i \pm e_j) = k_{ii}(t) + k_{jj}(t) \pm 2k_{ij}(t) \geq 0, \quad \forall t \in (t',\ t_f]}$$

$\Rightarrow$

Consider the i-th column of the identity matrix, e_i

Being K(t) symmetric one has:

$$\boxed{(e_i \pm e_j)^T K(t)(e_i \pm e_j) = k_{ii}(t) + k_{jj}(t) \pm 2k_{ij}(t) \geq 0, \quad \forall t \in (t', \ t_f]}$$

$$\Rightarrow$$

$$\boxed{k_{ii}(t) + k_{jj}(t) \geq 2\left|k_{ij}(t)\right|, \quad \forall t \in (t', \ t_f]}$$

Consider the i-th column of the identity matrix, e_i

Being K(t) symmetric one has:

$$\boxed{(e_i \pm e_j)^T K(t)(e_i \pm e_j) = k_{ii}(t) + k_{jj}(t) \pm 2k_{ij}(t) \geq 0, \quad \forall t \in (t', \ t_f\,]}$$

$$\Rightarrow \quad \boxed{k_{ii}(t) + k_{jj}(t) \geq 2\left|k_{ij}(t)\right|, \quad \forall t \in (t', \ t_f\,]}$$

$\Rightarrow$ from $\boxed{\lim_{t \to t'^+} \left|k_{ij}(t)\right| = \infty}$ one has that at least one of the two elements $k_{ii}(t), \ k_{jj}(t)$ goes to infinite.

Consider the i-th column of the identity matrix, e_i

Being K(t) symmetric one has:

$$\boxed{(e_i \pm e_j)^T K(t)(e_i \pm e_j) = k_{ii}(t) + k_{jj}(t) \pm 2k_{ij}(t) \geq 0, \quad \forall t \in (t',\ t_f]}$$

$$\Rightarrow \quad \boxed{k_{ii}(t) + k_{jj}(t) \geq 2\left|k_{ij}(t)\right|, \quad \forall t \in (t',\ t_f]}$$

$\Rightarrow$ from $\boxed{\lim_{t \to t'^+} \left|k_{ij}(t)\right| = \infty}$ one has that at least one of the two elements $k_{ii}(t),\ k_{jj}(t)$ goes to infinite.

Let us assume, for example, that $\lim_{t \to t'^+} k_{jj}(t) = \infty$

Consider the optimal control problem in the interval $[t'+\varepsilon, t_f]$ with initial condition $x(t'+\varepsilon) = e_j$

Consider the optimal control problem in the interval $[t'+\varepsilon, t_f]$ with initial condition $x(t'+\varepsilon) = e_j$

there exists an optimal solution and one has:

$$\boxed{\lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(x^o, u^o) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T \mathrm{K}(t'+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} k_{jj}(t'+\varepsilon) = \infty}$$

Consider the optimal control problem in the interval $[t'+\varepsilon, t_f]$ with initial condition $x(t'+\varepsilon) = e_j$

there exists an optimal solution and one has:

$$\lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(x^o, u^o) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T \mathrm{K}(t'+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} k_{jj}(t'+\varepsilon) = \infty$$

Now consider the null control over $[t'+\varepsilon, t_f]$ and the corresponding **free evolutior** $\hat{x}$ of the state.

Indicate with $\Phi(t,\tau)$ the transition matrix of the state corresponding to *A*

Indicate with $\Phi(t,\tau)$ the transition matrix of the state corresponding to *A*

$$J_{t'+\varepsilon}(\hat{x},0)=\frac{1}{2}\hat{x}^T(t_f)F\,\hat{x}(t_f)+\frac{1}{2}\int_{t'+\varepsilon}^{t_f}\hat{x}^T(t)Q(t)\hat{x}(t)dt$$

$$=\frac{1}{2}e_j^T\Phi^T(t_f,t'+\varepsilon)F\Phi(t_f,t'+\varepsilon)e_j$$

$$+\frac{1}{2}\int_{t'+\varepsilon}^{t_f}e_j^T\Phi^T(t,t'+\varepsilon)Q(t)\Phi(t,t'+\varepsilon)e_j dt$$

Indicate with $\Phi(t,\tau)$ the transition matrix of the state corresponding to *A*

$$\Rightarrow \quad J_{t'+\varepsilon}(\hat{x},0) = \frac{1}{2}\hat{x}^T(t_f)F\,\hat{x}(t_f) + \frac{1}{2}\int_{t'+\varepsilon}^{t_f} \hat{x}^T(t)Q(t)\hat{x}(t)dt$$

$$= \frac{1}{2}e_j^T\Phi^T(t_f,t'+\varepsilon)F\Phi(t_f,t'+\varepsilon)e_j$$

$$+\frac{1}{2}\int_{t'+\varepsilon}^{t_f} e_j^T\Phi^T(t,t'+\varepsilon)Q(t)\Phi(t,t'+\varepsilon)e_j dt$$

Since $\Phi(t,\tau)$ is bounded (in norm) $\Rightarrow$ there exists M>0 such that

$$\lim_{\varepsilon\to 0^+} J_{t'+\varepsilon}(\hat{x},0) < \mathrm{M}$$

$$J_{t'+\varepsilon}(x^o, u^o) \le J_{t'+\varepsilon}(\hat{x}, 0), \quad \forall \varepsilon \in \left(0, t_f - t'\right]$$

By doing the $\lim\limits_{\varepsilon \to 0^+}$

and recalling $\lim\limits_{\varepsilon \to 0^+} J_{t'+\varepsilon}(\hat{x}, 0) < \mathrm{M}$

$$\Rightarrow \quad J_{t'+\varepsilon}(x^o, u^o) \le J_{t'+\varepsilon}(\hat{x}, 0), \quad \forall \varepsilon \in \left(0, t_f - t'\right]$$

By doing the $\lim\limits_{\varepsilon \to 0^+}$

and recalling $\lim\limits_{\varepsilon \to 0^+} J_{t'+\varepsilon}(\hat{x}, 0) < \mathrm{M}$

$$\Rightarrow \quad \boxed{\lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(x^o, u^o) \le \lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(\hat{x}, 0) < \mathrm{M}}$$

$$\Rightarrow \quad J_{t'+\varepsilon}(x^o, u^o) \le J_{t'+\varepsilon}(\hat{x}, 0), \quad \forall \varepsilon \in \left(0, t_f - t'\right]$$

By doing the $\lim\limits_{\varepsilon \to 0^+}$

and recalling $\lim\limits_{\varepsilon \to 0^+} J_{t'+\varepsilon}(\hat{x}, 0) < \mathrm{M}$

$$\Rightarrow \quad \boxed{\lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(x^o, u^o) \le \lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(\hat{x}, 0) < \mathrm{M}}$$

That is in contrast with

$$\lim_{\varepsilon \to 0^+} J_{t'+\varepsilon}(x^o, u^o) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T \,\mathrm{K}(t'+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} k_{jj}(t'+\varepsilon) = \infty$$

Condition $\boxed{\lim_{t \to t'^{+}} \left| k_{ij}(t) \right| = \infty}$ is not possible

Equation of Riccati admits a unique solution semidefinite positive in $[t_i, \ t_f]$

The Riccati equation

Remark

- The solution of the Riccati equation is symmetric

- Usually the Riccati equation could be solved only numerically

The regulator problem

Theorem: The regulator problem admits a **unique** solution.

Proof: the uniqueness property is shown by contraddiction arguments.

Remarks

- The hypothesis of symmetry of the matrices Q, R and F is not restrictive

Remarks

- The hypothesis of symmetry of the matrices Q, R and F is not restrictive
- The solution of the Riccati equation does not depend on the initial state.

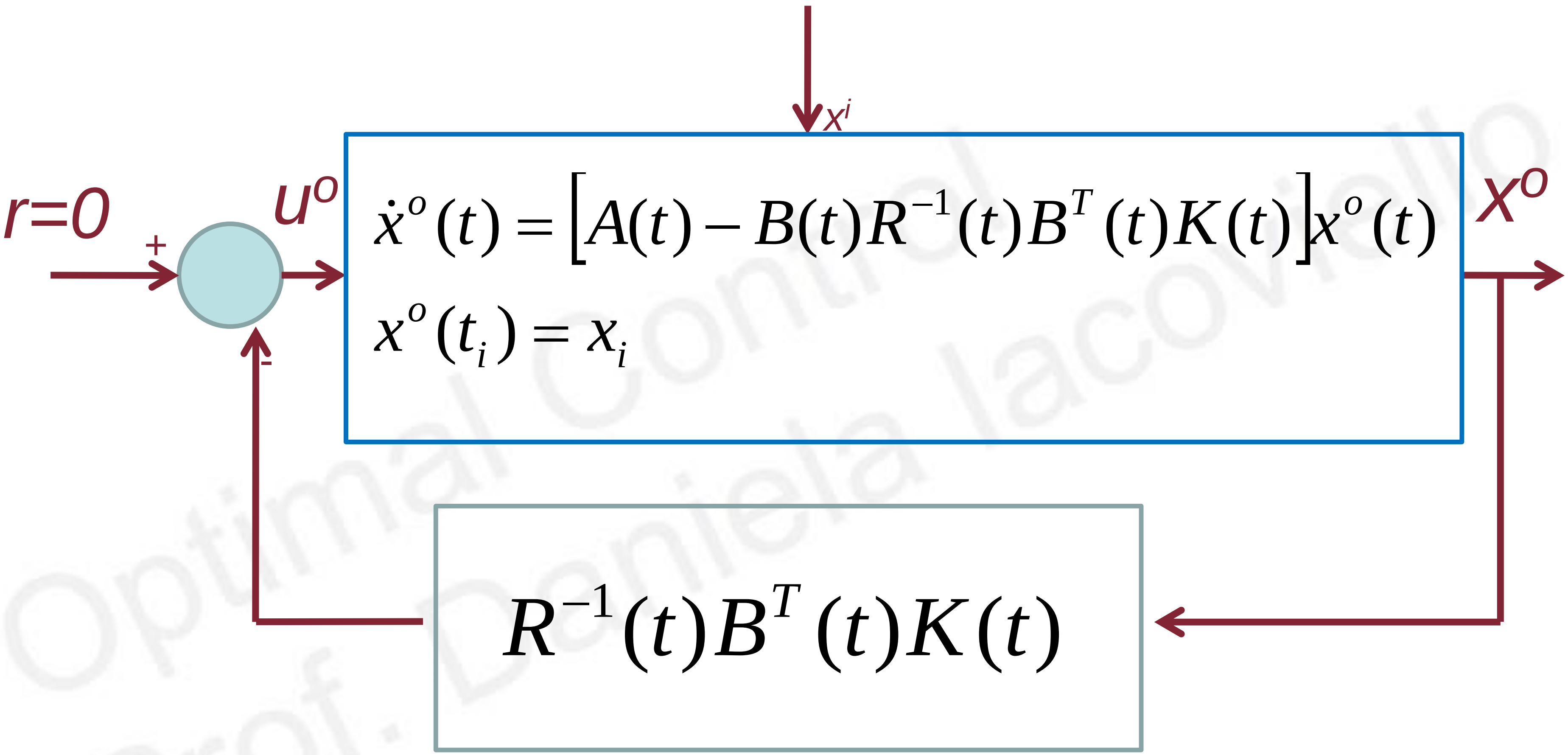
Therefore its solution could be found off line.

Remarks

- The hypothesis of symmetry of the matrices Q, R and F is not restrictive
- The solution of the Riccati equation does not depend on the initial state.

  Therefore its solution could be found off line.
- **The Riccati matrix *K* is a function of current time t even if the matrices *A, B, Q* and *R* are constant.**

The regulator problem



Example 1 (from Bruni et al. 1993)

Let us consider the following **linear system:**

$$\dot{x} = Ax + Bu = \frac{1}{2}x + u, \quad x(0) = x^i$$

and the following **cost index**:

$$J(x,u) = \int_0^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right] dt =$$

$$= \int_0^{t_f} \left[\frac{1}{2}e^{-t}x^2(t) + 2e^{-t}u^2(t) \right] dt$$

To determine the optimal control we must find the solution of the **Riccati equation**:

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^{T}(t)K(t) - K(t)A(t) - A^{T}(t)K(t) - Q(t)$$

$$K(t_f) = F$$

$$\dot{K}(t) = \frac{1}{2}e^{t}K^{2}(t) - K(t)A(t) - \frac{1}{2}e^{-t},\; K(t_f) = 0$$

whose solution is:

$$K(t) = \frac{1 - e^{t - t_f}}{e^{t} + e^{2t - t_f}}$$

OFF LINE !!!

whose solution is:

$$K(t) = \frac{1 - e^{t - t_f}}{e^t + e^{2t - t_f}}$$

OFF LINE !!!

Therefore the **optimal control** is:

$$u^o(t) = -R^{-1}(t)B^T(t)K(t)x^o(t) = \boxed{-\frac{1 - e^{t - t_f}}{2\left(e^t + e^{2t - t_f}\right)}}\, x^o(t)$$

where the **optimal state** is given by:

$$\dot{x}^o(t) = \frac{e^t + e^{2t - t_f} + e^{t - t_f} - 1}{2\left(e^t + e^{2t - t_f}\right)} x^o(t), \quad x(0) = x^i$$

And **the optimal value** for the cost index is:

$$J(x^o, u^o) = \frac{1 - e^{-t_f}}{1 + e^{t_f}} \left(x^i\right)^2$$

The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

Solution of the deterministic linear optimal regulator problem on $[t_i, \infty)$

Problem: let us consider the regulator problem over the **infinite time interval** $[t_i, \infty)$.

Assume:

- The matrices *A* and *B* are **bounded** and with elements in C^1 class
- The dynamical system is **completely controllable** or **exponentially stable**
- The matrices *Q* and *R* are symmetric, semidefinite and positive definite respectively, with elements in C^1 class and **bounded**

Solution of the deterministic linear optimal regulator problem on $[t_i, \infty)$

Find:

the control $u^o \in \overline{C}^0[t_i, \infty)$

and the state $x^o \in \overline{C}^1[t_i, \infty)$

satisfying the dynamical system , the initial condition and

minimizing the cost index

$$J(x,u) = \frac{1}{2}\int_{t_i}^{\infty}\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]dt$$

Solution of the deterministic linear optimal regulator problem on $[t_i, \infty)$

Theorem: The problem admits a unique optimal solution that can be expressed as follows:

$$u^o(t) = -R^{-1}(t)B^T(t)\overline{K}(t)x^o(t)$$

$$\dot{x}^o(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)\overline{K}(t)\right]x^o(t), \qquad x^o(t_i) = x^i$$

where $\overline{K}(t)$ is the solution of the Riccati equation

$$\dot{\overline{K}}(t) = \overline{K}(t)B(t)R^{-1}(t)B^T(t)\overline{K}(t) - \overline{K}(t)A(t) - A^T(t)\overline{K}(t) - Q(t)$$

with final condition: $\boxed{\lim_{t_f \to \infty} \overline{K}(t_f) = 0}$

and

$$J\left(x^o, u^o\right) = \frac{1}{2} x^{iT} \overline{K}(t_i) x^i$$

Proof: Let us consider an arbitrary finite final instant t_f. From the previous result the minimum value of the cost index is:

$$J_{t_f}\left(x^o, u^o\right) = \frac{1}{2} x^{iT} K(t_i) x^i$$

This value depends on the value of t_f.

- Let us show that the value is superiorly limited. $J_{t_f}\left(x^o, u^o\right) = \frac{1}{2} x^{iT} K(t_i) x^i$

In fact, **if the system is completely controllable,** whatever the initial state is, there exists a control *u'* able to transfer it to the origin in a finite time t_f', with an evolution *x'*.

We can assume that the couple *(x',u')* is defined for all $t>t_f'$, assuming their values equal to zero for $t>t_f'$. Therefore, for any $t_f>t_f'$:

$$\frac{1}{2}\int_{t_i}^{t_f'}\left(x'^T Qx'+u'^T Ru'\right)dt=\frac{1}{2}\int_{t_i}^{t_f}\left(x'^T Qx'+u'^T Ru'\right)dt$$

$$\geq J_{t_f}\left(x^o,u^o\right)$$

If <u>the system is exponentially stable</u>, from any initial state with null entry the corresponding free evolution of the state $\bar{x}$ satisfies this inequality:

$$\left\|\bar{x}(t)\right\|\leq\alpha e^{-\beta t},\quad \beta>0$$

Therefore we have:

$$J_{t_f}\left(x^o,u^o\right) \le J_{t_f}\left(\bar{x},0\right) \le \frac{1}{2}\int_{t_i}^{\infty} \bar{x}^T Q \bar{x}\, dt$$

$$\le \frac{1}{2}\int_{t_i}^{\infty} \|Q(t)\| \|\bar{x}(t)\|^2 dt \le c\int_{t_i}^{\infty} e^{-2\beta t} dt = \frac{c}{2\beta} e^{-2\beta t_i}$$

- By simple arguments it can be shown that the function $J_{t_f}\left(x^o,u^o\right)$ is monotonically non decreasing in t_f

In fact, consider $\left(x^{o'}, u^{o'}\right)$ $\left(x^{o''}, u^{o''}\right)$ two optimal solutions corresponding to t_f' and t_f'' respectively.

Assume $t_f'' > t_f'$.

Then we have:

$$J_{t_f''}(x^{o''}, u^{o''}) \geq \frac{1}{2}\int_{t_i}^{t_f'} \left(x^{o''T} Q x^{o''} + u^{o''T} R u^{o''} \right) dt \geq J_{t_f'}(x^{o'}, u^{o'})$$

Therefore the function : $$J_{t_f}\left(x^o, u^o\right) = \frac{1}{2} x^{iT} K(t_i) x^i$$

has a limit as t_f tends to infinity. Being x^i arbitrary, it follows that any matrix $K(t_i)$ has a limit as t_f tends to infinity.

Therefore the function : $$J_{t_f}\left(x^o, u^o\right) = \frac{1}{2} x^{iT} K(t_i) x^i$$

has a limit as t_f tends to infinity. Being x^i arbitrary, it follows that any matrix $K(t_i)$ has a limit as t_f tends to infinity.

As t_i is arbitrary, the solution K of the Riccati equation with final condition $K(t_f)=0$ admits a limited solution $K'(t)$ for any finite t as t_f tends to infinity.

Existence, uniqueness and the characterization of the solution can be justified as in the previous theorem, considering the limit for the solution herein identified and recalling that

$$\lim_{t_f \to \infty} K(t_f) = K'(t)$$

Remark:

If the matrix Q(t) has eigenvalues greater or equal to a positive number α for each $t \in [t_i, \infty)$
the optimal regulator problem is **asymptotically stable**; in fact, for the optimal solution it results in:

$$\frac{\alpha}{2}\int_{t_i}^{\infty}\left\|x^o(t)\right\|^2 dt \le \frac{1}{2}\int_{t_i}^{\infty}\left[x^{oT}(t)Q(t)x^o(t)+u^{oT}(t)R(t)u^o(t)\right]dt$$

$$=\frac{1}{2}x^{iT}\overline{K}(t_i)x^i$$

Therefore:

$$\lim_{t\to\infty} x^o(t)=0 \quad \forall x^i \in R^n$$

The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

The steady state solution of the deterministic linear optimal regulator problem

Theorem: Let us consider the previous problem with the additive hypotheses :

- The matrices A, B, Q, R are constant
- The matrix Q is positive definite

Then there exists a unique optimal solution:

$$u^o(t) = -R^{-1}B^T K_r x^o$$

$$\dot{x}^o(t) = \left[A - BR^{-1}B^T K_r\right]x^o(t), \qquad x^o(t_i) = x^i$$

where:

K_r is the constant matrix, unique solution definite positive of the **algebraic Riccati equation**:

$$K_r B R^{-1} B^T K_r - K_r A - A^T K_r - Q = 0$$

The minimum value for the cost index is:

$$J\left(x^o, u^o\right) = \frac{1}{2} x^{iT} K_r x^i$$

Proof:

From the previous theorem we have that the value of the cost index for the considered steady state problem is given by

$$J\left(x^o,u^o\right)=\frac{1}{2}x^{iT}\overline{K}(t_i)x^i$$

Due to the stationarity of the problem defined on infinite time interval it results that **its value is independent from t_i** .

Therefore the unique solution of the equation:

$$\dot{\overline{K}}(t) = \overline{K}(t)B(t)R^{-1}(t)B^{T}(t)\overline{K}(t) - \overline{K}(t)A(t) - A^{T}(t)\overline{K}(t) - Q(t)$$

is constant K_r .

The existence and uniqueness of the optimal solution follow from the previous theorem

Remarks

Consider the system:

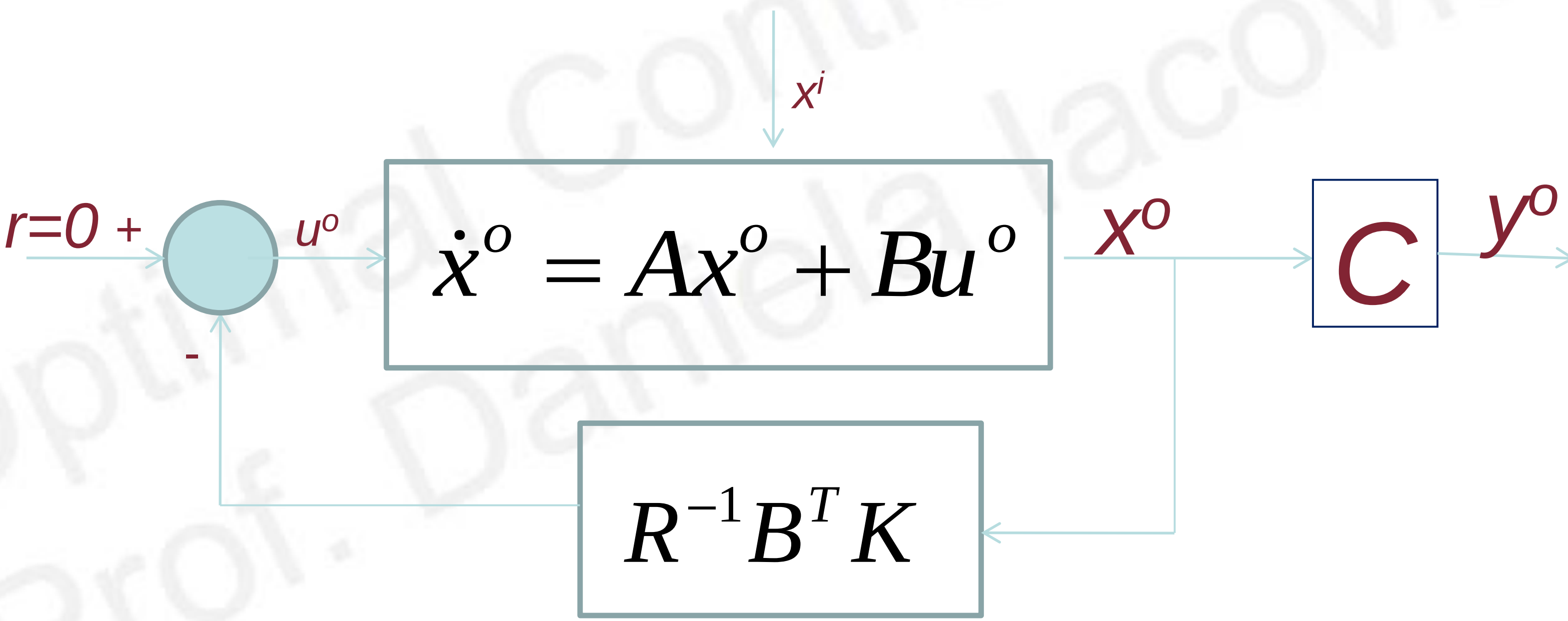
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

It can be defined an **optimal regulation problem from the output *y***, considering the cost index

$$\bar{J}(y,u) = \frac{1}{2} y^T(t_f)\bar{F}y(t_f) + \frac{1}{2}\int_{t_i}^{t_f} \left[y^T(t)\bar{Q}(t)y(t) + u^T(t)R(t)u(t) \right] dt$$

The present problem is similar to the previous one, being satisfied the hypotheses over the matrices involved.

Anyway **the state must be accessible**, **if one want to realize the feedback action:**



The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

The optimal tracking problem

Problem: let us consider the linear system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_i) = x^i$$

where $\boldsymbol{A(t), \ B(t)}$ are matrices with elements functions of time with entries of C^1 class.

Consider the **reference variable** $r \in C^1[t_i, t_f]$

Determine the optimal control $u^o \in \overline{C}^0[t_i, t_f]$

and the state $x^o \in \overline{C}^1[t_i, t_f]$ satisfying the dinamical constraint and minimizing the cost index:

$$J(x,u)=\frac{1}{2}\int_{t_i}^{t_f}\left\{[r(t)-x(t)]^T Q(t)[r(t)-x(t)]+u^T(t)R(t)u(t)\right\}dt$$

where:

- *Q(t)* is a symmetric semi-positive definite matrix
- *R(t)* is a symmetric positive definite matrix
- The elements of *Q(t)* and *R(t)* are functions of C^1 class

The optimal tracking problem

To the optimal tracking problem it can be associated the Riccati equation

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^{T}(t)K(t) - K(t)A(t) - A^{T}(t)K(t) - Q(t)$$
$$K(t_f) = 0$$

Theorem: the optimal tracking problem admits a unique optimal solution:

$$u^o(t) = R^{-1}(t)B^T(t)\left[g(t) - K(t)x^o(t)\right]$$

$$\dot{x}^o(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x^o(t)$$
$$+ B(t)R^{-1}(t)B^T(t)g(t) \qquad x^o(t_i) = x^i$$

where g is the solution of the differential equation

$$\dot{g}(t) = \left[K(t)B(t)R^{-1}(t)B^T(t) - A^T(t)\right]g(t) \boxed{- Q(t)r(t)}$$

$$\boxed{g(t_f) = 0}$$

The minimum value of the cost index is:

$$J\left(x^o, u^o\right) = \frac{1}{2} x^{oT}(t_i)K(t_i)x^o(t_i) - x^{oT}(t_i)g(t_i) + v(t_i)$$

where v is the solution of the equation:

$$\dot{v}(t) = \frac{1}{2} g^T(t)B(t)R^{-1}(t)B^T(t)g(t) \boxed{- \frac{1}{2} r^T(t)Q(t)r(t)}$$

$$\boxed{v(t_f) = 0}$$

Proof: It can be applied Pontryagin Theorem (convex case). Let us define the Hamiltonian:

$$H(x,u,\lambda,t)=\frac{1}{2}(r-x)^T Q(r-x)+\frac{1}{2}u^T Ru+\lambda^T Ax+\lambda^T Bu$$

The necessary and sufficient conditions are:

$$\dot{\lambda}^o=-Qx^o-A^T\lambda^o+Qr$$

$$Ru^o+B^T\lambda^o=0$$

$$\lambda^o(t_f)=0$$

With the following choice for the costate

$$\boxed{\lambda^o = Kx^o - g}$$

the theorem is proved.

As far as the cost index is concerned, it is useful to observe that:

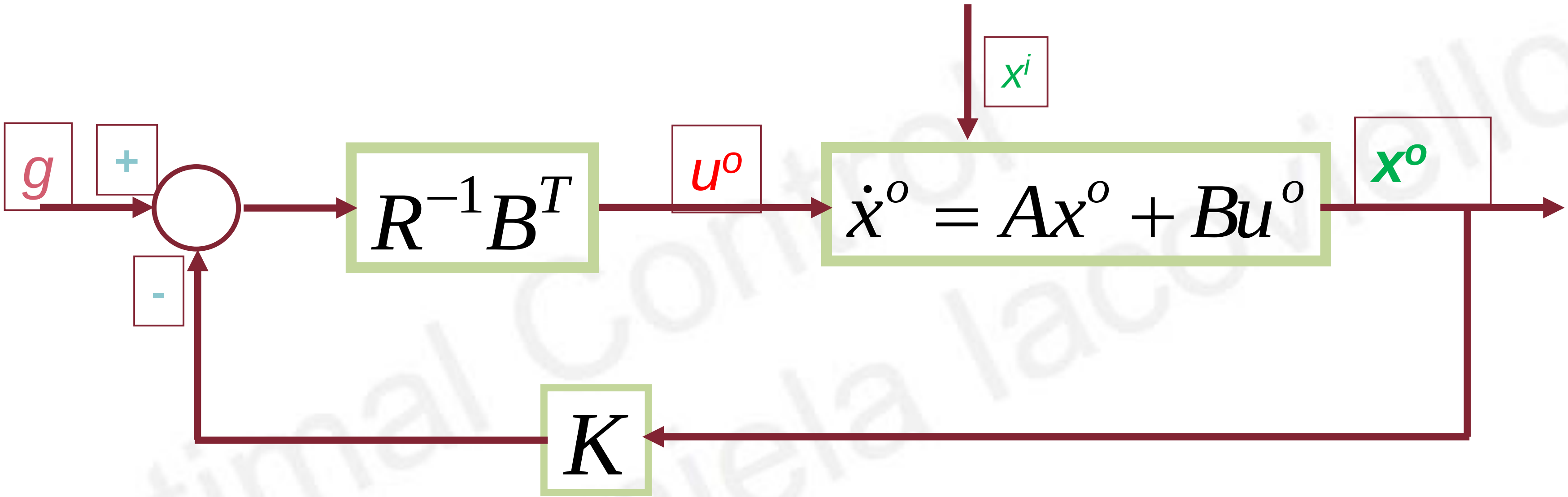
$$\frac{1}{2}x^{oT}(t_i)K(t_i)x^o(t_i) - x^{oT}(t_i)g(t_i) + v(t_i) =$$

$$= \int_{t_i}^{t_f} -\frac{d}{dt}\left(\frac{1}{2}x^{oT}Kx^o - x^{oT}g + v\right)dt$$

Theorem

The optimal control tracking problem admits a **unique** optimal solution.

Proof: The theorem is proved by contraddiction arguments.



g
+
-
$R^{-1}B^T$
u^o
x^i
$\dot{x}^o = Ax^o + Bu^o$
x^o
K

Remarks

- Note that in the cost index the quadratic error on the final instant is not present $r(t_f) - x(t_f)$.

Remarks

- Note that in the cost index the quadratic error on the final instant is not present $r(t_f) - x(t_f)$.

- The realization of the optimal control for the tracking problem needs the solution of the Riccati equation *K* and the solution of the differential equation in *g*

Remarks

- Note that in the cost index the quadratic error on the final instant is not present $r(t_f)-x(t_f)$.

- The realization of the optimal control for the tracking problem needs the solution of the Riccati equation *K* and the solution of the differential equation in *g*

- To solve the tracking problem ***the reference variable r must be known in advance in the interval***

- The **steady state tracking problem over an infinite interval** requires a steady state solution for the differential equation in the *g* function;

for example, for a constant reference value $\bar{r}$:

$$g_r = \left[K_r B R^{-1} B^T - A^T \right]^{-1} Q \bar{r}$$

Example (from C.Bruni, G.Di Pillo)

Let us consider the dynamical system $\boxed{\dot{x}(t) = u(t), \quad x(0) = 0}$

and the cost index

$$\boxed{J(x,u) = \frac{1}{2}\int_0^{t_f} \left[(r(t) - x(t))^2 + \rho u^2(t) \right] dt, \quad \rho > 0}$$

r is **the reference value** in $[0, \ t_f]$.

Applying the previous Theorem we obtain:

$$u^o(t) = \frac{1}{\rho}\left(g(t) - K(t)x^o t(t)\right)$$

$$\dot{x}^o(t) = -\frac{K(t)}{\rho}x^o(t) + \frac{1}{\rho}g(t) \qquad x^o(0) = 0$$

where K and g are, respectively, the solution of the equation:

$$\boxed{\begin{aligned} &\dot{K}(t) = \frac{K^2(t)}{\rho} - 1 && K(t_f) = 0 \\ &\dot{g}(t) = \frac{K(t)}{\rho}g(t) - r(t) && g(t_f) = 0 \end{aligned}}$$

We obtain:

$$\boxed{K(t) = \sqrt{\rho}\, tgh \frac{T-t}{\sqrt{\rho}}}$$

Therefore:

$$\boxed{\dot{g}(t) = \frac{1}{\sqrt{\rho}} \left(tgh \frac{T-t}{\sqrt{\rho}} \right) g(t) - r(t) \qquad g(t_f) = 0}$$

We obtain:

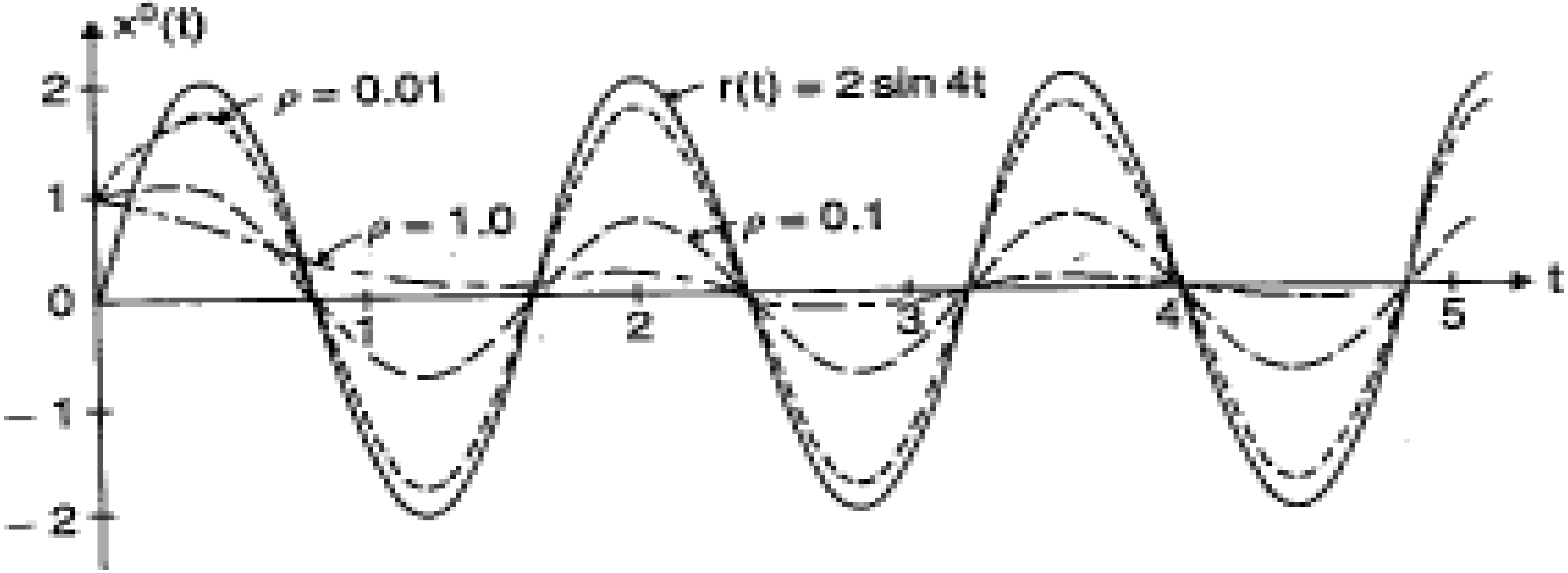
$$\boxed{K(t) = \sqrt{\rho}\, tgh \frac{T-t}{\sqrt{\rho}}}$$

Therefore:

$$\boxed{\dot{g}(t) = \frac{1}{\sqrt{\rho}} \left(tgh \frac{T-t}{\sqrt{\rho}} \right) g(t) - r(t) \qquad g(t_f) = 0}$$

Let's choose for example:

$$x(0) = 1 \qquad t_f = 5 \qquad r(t) = 2\sin 4t$$



Note that the behaviour of x^o follows the oscillatory behaviour of the reference;

the fit error becomes smaller as ρ decreasis.

MATLAB

lqr

Linear-quadratic (LQ) state-feedback regulator for state-space system

Syntax

[K,S,e] = LQR(A,B,Q,R,N)

Description

[K,S,e] = lqr(SYS,Q,R) calculates the optimal gain matrix K.

For a continuous time system, the state-feedback law

u = –Kx minimizes the quadratic cost function subject to the system dynamics

$$J(x,u)=\frac{1}{2}\int_{t_i}^{\infty}\left[x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right]dt$$

MATLAB

In addition to the state-feedback gain K, lqr returns the solution S of the associated Riccati equation

$$SBR^{-1}B^T S - SA - A^T S - Q = 0$$ attention!!!

and the closed-loop eigenvalues e = eig(A-B*K).

K is derived from S using

$$K = -R^{-1}B^T S$$

MATLAB

```
A=[1 3; -4 2];
B=[1 2; 3 4]
Q=[2 0; 0 3]
R=[1 0;0 1]
[K S ]=LQR(A,B,Q,R)

K =     1.0806    1.0000
        3.1612    1.0000

S =     2.5806   -0.5000
       -0.5000    0.5000

e =  -6.4031
     -5.0000
```

For MATLAB: control gain

For MATLAB:
solution of the RICCATI equation

The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

The optimal regulator problem with null final error

Problem: Let us consider the linear system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

$$x(t_i) = x^i, \quad x(t_f) = 0$$

Determine the control $u^o \in \overline{C}^0[t_i, t_f]$

and the state $x^o \in \overline{C}^1[t_i, t_f]$

in order to satisfy the dynamical system, the initial and final conditions and minimizing the following cost index:

$$J(x,u) = \frac{1}{2}\int_0^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right] dt$$

where:

- *Q(t)* is symmetric semidefinite positive with elements of C^1 class
- *R(t)* is symmetric definite positive with elements of C^1 class

Theorem: Let us introduce the matrix of dimension 2n × 2n

$$\boxed{\Omega(t) = \begin{pmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{pmatrix}}$$

and indicate with

$$\boxed{\Phi(t,\tau) = \begin{pmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{pmatrix}}$$

its transition matrix partitioned in submatrices of dimension n × n.

Assume that the **submatrix** $\phi_{12}(t_f, t_i)$ **is not singular .**

⇒ **The optimal regulator problem with null final error admits a <u>unique optimal solution:</u>**

$$u^o(t) = -R^{-1}(t)B^T(t)\left[\phi_{21}(t,t_i) - \phi_{22}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i$$

$$x^o(t) = \left[\phi_{11}(t,t_i) - \phi_{12}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i$$

$$u^o(t) = -R^{-1}(t)B^T(t)\left[\phi_{21}(t,t_i) - \phi_{22}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i$$

$$x^o(t) = \left[\phi_{11}(t,t_i) - \phi_{12}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i$$

$$\boxed{u^o(t) = -R^{-1}(t)B^T(t)\left[\phi_{21}(t,t_i) - \phi_{22}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i}$$

$$\boxed{x^o(t) = \left[\phi_{11}(t,t_i) - \phi_{12}(t,t_i)\phi_{12}^{-1}(t_f,t_i)\phi_{11}(t_f,t_i)\right]x^i}$$

Proof. We can apply theorem 4 (Pontryaging theorem, convex case). In particular let us define **the Hamiltonian:**

$$H(x,u,\lambda,t) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + \lambda^T Ax + \lambda^T Bu$$

The **necessary and sufficient** conditions are:

$$\dot{\lambda}^o = -Qx^o - A^T\lambda^o$$
$$Ru^o + B^T\lambda^o = 0$$

$$\Rightarrow \quad u^o = -R^{-1}B^T\lambda^o$$

Recalling the definition of Ω,

$$\boxed{\Omega(t) = \begin{pmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{pmatrix}}$$

we can consider the system:

$$\begin{pmatrix} \dot{x}^o \\ \dot{\lambda}^o \end{pmatrix} = \Omega \begin{pmatrix} x^o \\ \lambda^o \end{pmatrix}$$

whose solution, taking into account the initial condition, is:

$$\begin{pmatrix} x^o(t) \\ \lambda^o(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t,t_i) & \phi_{12}(t,t_i) \\ \phi_{21}(t,t_i) & \phi_{22}(t,t_i) \end{pmatrix} \begin{pmatrix} x^i \\ \lambda^o(t_i) \end{pmatrix}$$

Taking into account the condition in t_f and the non singularity of submatrix $\phi_{12}(t_f, t_i)$,

it results in:

$$\lambda^o(t_i) = -\phi_{12}^{-1}(t_f, t_i)\phi_{11}(t_f, t_i)x^i$$

and therefore:

$$\lambda^o(t) = \left[\phi_{21}(t, t_i) - \phi_{22}(t, t_i)\phi_{12}^{-1}(t_f, t_i)\phi_{11}(t_f, t_i)\right]x^i$$

that, substituted into the expression of control

$$u^o = -R^{-1}B^T\lambda^o$$

yields the optimal solution.

Remarks

- The optimal solution depends on the initial state x^i

- Note that **if** the matrix $\phi_{12}(t_f, t_i)$ is singular two situations are possible:

  - no optimal solution exists or

  - there exist infinite ones depending on the number of solution of the system:

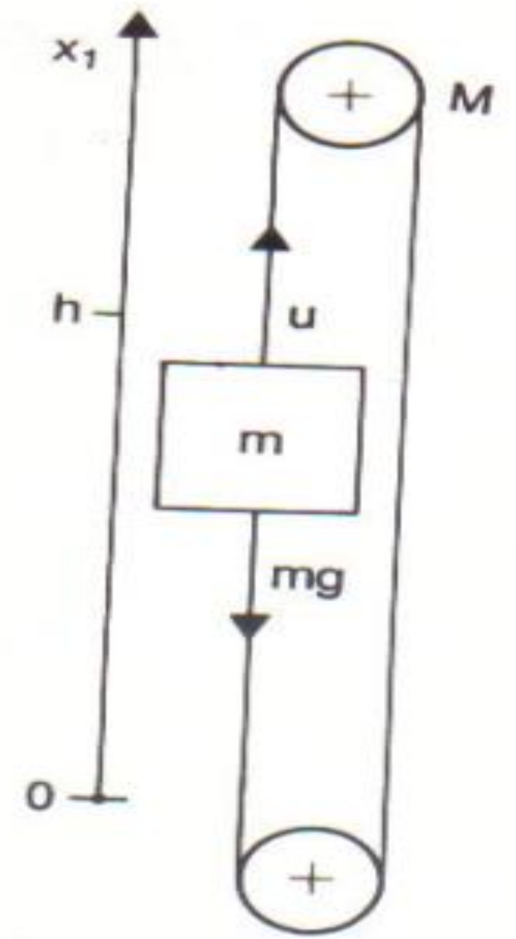
$$\phi_{12}(t_f, t_i)\lambda^o(t_i) = -\phi_{11}(t_f, t_i)x^i$$

with respect to $\lambda^o(t_i)$

Example (from Bruni et al. 1993)

Let us consider an elevator
whose cabin of mass
m must be lowered
from a level ***h*** to the level ***0***,

by minimizing the strenght ***u***
to be applied to
the engine M of the cabin



Let us consider the following modelization:

[0 , t_f] is the control interval

x_1 denotes the level of the cabin

x_2 denotes the velocity of the cabin

g denotes the acceleration do to gravity

u denotes the the strenght to be applied to the engine M of the cabin

Assume null the velocity at the beginning and at the end of the control interval.

The problem may be formulated as follows.

Given the **linear system**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)/m - g$$

the aim is to transfer the **initial state**:

$$x_1(0) = h \qquad x_2(0) = 0$$

to the **final one**

$$x_1(t_f) = 0 \qquad x_2(t_f) = 0$$

minimizing the **cost index:**

$$J(u) = \frac{1}{2}\int_0^{t_f} u^2(t)dt$$

The problem may be considered in the framework of **the regulator problems with null final error.**

Let us introduce the **Hamiltonian** in the normal case (i.e. $\lambda_0 = 1$):

$$H(x,u,1,\lambda) = \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \frac{1}{m}\lambda_2(t)u(t) - g\lambda_2(t)$$

The **necessary and sufficient** conditions are:

$$\dot{\lambda}_1^0(t) = 0$$

$$\dot{\lambda}_2^0(t) = -\lambda_1^0(t)$$

$$u^o(t) + \frac{1}{m}\lambda_2^0(t) = 0$$

Integrating the **costate equations** we have:

$$\lambda_1^o(t) = K_1 \qquad \lambda_2^o(t) = -K_1 t + K_2$$

From the **control equation**:

$$u^o(t) = \frac{K_1}{m} t - \frac{K_2}{m}$$

Integrating the **dynamical system**:

$$x_1^o(t) = \frac{K_1}{m^2}\frac{t^3}{6} - \frac{K_2}{m^2}\frac{t^2}{2} - g\frac{t^2}{2} + h$$

$$x_2^o(t) = \frac{K_1}{m^2}\frac{t^2}{2} - \frac{K_2}{m^2} t - gt$$

Taking into account the final conditions:

$$K_1 = \frac{12m^2h}{T^3} \qquad K_2 = \frac{16m^2h}{T^2} - m^2 g$$

Therefore the **unique optimal control** is:

$$u^o(t) = \frac{12mh}{t_f^3}\left(t - \frac{t_f}{2}\right) + mg$$

Note that the optimal control *<u>does not depend con the current state</u>*

The optimal regulator problem

The optimal regulator problem on finite time interval

The optimal regulator problem on infinite time interval

The steady state linear optimal regulator problem

The optimal tracking problem

The optimal regulator problem with null final error

The optimal regulator problem with limited control

The optimal regulator problem with limited control

Problem: Let us consider the optimal regulator problem on **finite time interval** with the **additional constraint:**

$$\boxed{\left|u_j(t)\right| \le 1, \quad j = 1,2,\ldots,p, \quad \forall t \in \left[t_i,\, t_f\right]}$$

and the hypothesis that the weighting matrix R is diagonal

Theorem: All the normal solutions can be found solving the differential system:

$$\dot{x}^o(t) = A(t)x^o(t) - B(t)sat\left\{R^{-1}(t)B^T(t)\lambda^o(t)\right\}$$

$$\dot{\lambda}^o(t) = -Q(t)x^o(t) - A^T(t)\lambda^o(t)$$

$$x^o(t_i) = x^i \qquad \lambda^o(T) = Fx^o(t_f)$$

assuming:

$$\boxed{u^o(t) = -sat\left\{R^{-1}(t)B^T(t)\lambda^o(t)\right\}}$$

Proof: Let's apply Pontryagin theorem (convex case); the necessary and sufficient conditions becomes:

$$\dot{\lambda}^o = -Qx^o - A^T \lambda^o$$

$$\frac{1}{2} u^{oT} R u^o + \lambda^{oT} B u^o \leq \frac{1}{2} \omega^{oT} R \omega^o + \lambda^{oT} B \omega^o$$

$$\forall \omega : \left| \omega_j \right| \leq 1, \quad j = 1,2,\ldots, p$$

$$\lambda^o (t_f) = F x^o (t_f)$$

By adding the quantity $\frac{1}{2}\lambda^{oT} B R^{-1} B^{T} \lambda^{o}$ to both members of the inequality, we can write it as follows:

$$\left(u^{o} + R^{-1} B^{T} \lambda^{o}\right)^{T} R \left(u^{o} + R^{-1} B^{T} \lambda^{o}\right)$$
$$\leq \left(\omega + R^{-1} B^{T} \lambda^{o}\right)^{T} R \left(\omega + R^{-1} B^{T} \lambda^{o}\right)$$
$$\forall \omega : \left|\omega_{j}\right| \leq 1, \quad j = 1,2,\ldots,p$$

Taking into account that the matrix R is diagonal and positive definite, the theorem is proved.

Remarks

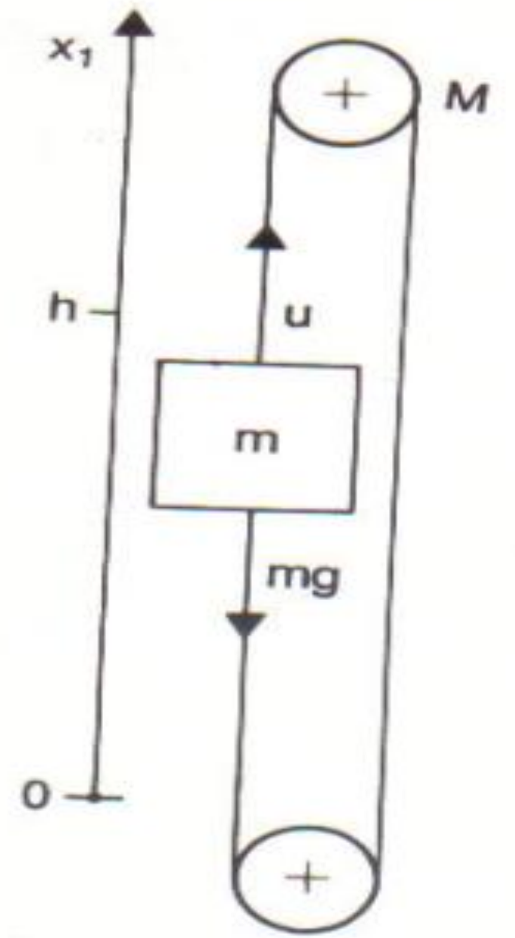
- With the new constraint the system state-costate is non-linear
- The existence of the solution is not guardanteed
- It is possible to consider the same problem with the final condition $x(t_f) = x^f$
- The preceeding theorem still holds without the condition on the costate.

Example (Bruni et al.1993)

Let us consider again the previous example of
the elevator with the additional constraint for the control:

$$u(t) \in [0, M]$$

The following equations hold:



$$\dot{\lambda}_1^0(t) = 0$$

$$\dot{\lambda}_2^0(t) = -\lambda_1^0(t)$$

$$\frac{1}{2} u^{o^2}(t) + \frac{\lambda_2^o(t)}{m} u^o(t) \leq \frac{1}{2}\omega^2(t) + \frac{\lambda_2^o(t)}{m}\omega(t)$$

$$\forall \omega \in [0, M],\ \forall t \in [0, t_f]$$

Note that the last equation is the minimum principle in its original formulation

The minimum principle condition implies:

$$\left(u^o(t) + \frac{\lambda_2^o(t)}{m} \right)^2 \leq \left(\omega(t) + \frac{\lambda_2^o(t)}{m} \right)^2$$

$$\forall \omega \in [0,\ M]\ ,\ \forall t \in [0, t_f\ ]$$

Taking into account the solution of the costate equation:

$$\lambda_1^o(t) = K_1 \qquad\qquad \lambda_2^o(t) = -K_1 t + K_2$$

The **optimal control** is obtained:

$$u^o(t) = \frac{M}{2}\left(1 - sat\left\{\frac{2(K_2 - K_1 t)}{mM} + 1\right\}\right)$$

It must be substituted in the **state equation**

$$\begin{aligned}\dot{x}^o(t) &= A(t)x^o(t) + B(t)u(t) = \\ &= A(t)x^o(t) + B(t)\left[\frac{M}{2}\left(1 - sat\left\{\frac{2(K_2 - K_1 t}{mM} + 1\right\}\right)\right]\end{aligned}$$

All the solutions, if they exist, are found by solving the differential equation.