Optimal Control

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Lecture 2

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THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

Lagrange (Torino 1736, Paris 1813)

Lagrange education was completed at the university of Torino, Berlin and Paris.

He was one of the major Mathematician of the 17th, important for his studies in calculus of variations and in astronomy.

The Lagrange problem

Problem 1

Let us consider the linear space $\bar{C}^1(R)\times R\times R$

and define the **admissible set**:

$$D=\left\{\left(z,t_i,t_f\right)\in\bar{C}^1(R)\times R\times R:\left(z(t_i),t_i\right)\in D_i\subset R^{\nu+1},\left(z(t_f),t_f\right)\in D_f\subset R^{\nu+1}\right\}$$

Introduce **the norm**: $\left\|\left(z,t_i,t_f\right)\right\|=\sup_t\|z(t)\|+\sup_t\|\dot{z}(t)\|+|t_i|+|t_f|$

and consider the **cost index**:

$$J(z,t_i,t_f)=\int_{t_i}^{t_f}L\left[z(t),\dot{z}(t),t\right]dt$$

with L function of C^2 class.

Find the global minimum (optimum) $\left(z^{o}, t_{i}^{o}, t_{f}^{o}\right)$

for *J* over *D:*

$$J\left(z^{o}, t_{i}^{o}, t_{f}^{o}\right) \leq J\left(z, t_{i}, t_{f}\right) \quad \forall (z, t_{i}, t_{f}) \in D$$

SCHEME of the theorems

Theorem (Lagrange). If $\left(z^*, t_i^*, t_f^*\right) \in D$ is a local minimum **then**

1) $$\left.\frac{\partial L}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in \left[t_i, t_f\right]$$ **Euler equation**

2) In any discontinuity point $\bar{t}$ of $\dot{z}^*$ **Weierstrass-Erdmann condition**

3) Transversality conditions

different cases
depending on the nature
of the boundary conditions

Theorem 1 (Lagrange)

If $\left(z^*, t_i^*, t_f^*\right) \in D$ is a local minimum then

$$\left.\frac{\partial L}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in \left[t_i^*, t_f^*\right]$$

Euler equation

✓ In any discontinuity point $\bar{t}$ of $\dot{z}^*$

the following conditions are verified:

$$\left.\frac{\partial L}{\partial \dot{z}}\right|_{\bar{t}^-}^* = \left.\frac{\partial L}{\partial \dot{z}}\right|_{\bar{t}^+}^* \qquad \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)_{\bar{t}^-}^* = \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)_{\bar{t}^+}^*$$

Weierstrass-Erdmann condition

✓ Moreover, **transversality conditions** are satisfied:

- If $D_i \;\; D_f$ are **open subset** we have:

$$\left.\frac{\partial L}{\partial \dot{z}}\right|_{t_i^*}^{*} = 0^T \quad \left.\frac{\partial L}{\partial \dot{z}}\right|_{t_f^*}^{*} = 0^T \quad L\Big|_{t_i^*}^{*} = 0 \quad L\Big|_{t_f^*}^{*} = 0$$

- If $D_i \;\; D_f$ **are closed subsets** defined respectively by

$$\gamma\left(z(t_i), t_i\right) = 0 \quad \aleph\left(z(t_f), t_f\right) = 0$$

such that

$$rg\left\{\left.\frac{\partial \gamma}{\partial\left(z(t_i), t_i\right)}\right|^{*}\right\} = \sigma_i \quad rg\left\{\left.\frac{\partial \aleph}{\partial\left(z(t_f), t_f\right)}\right|^{*}\right\} = \sigma_f$$

for $\xi \in R^{\sigma_i}$ $\varsigma \in R^{\sigma_f}$

$$\left.\frac{\partial L}{\partial \dot z}\right|_{t_i^*}^{*} = \xi^T \left.\frac{\partial \gamma}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial L}{\partial \dot z}\right|_{t_f^*}^{*} = \varsigma^T \left.\frac{\partial \aleph}{\partial z(t_f)}\right|^{*}$$

$$\left(L - \frac{\partial L}{\partial \dot z}\dot z\right)_{t_i^*} = \xi^T \left.\frac{\partial \gamma}{\partial t_i}\right|^{*}, \quad \left(L - \frac{\partial L}{\partial \dot z}\dot z\right)_{t_f^*} = \varsigma^T \left.\frac{\partial \aleph}{\partial t_f}\right|^{*}$$

- If the sets D_i D_f are defined by the function of σ components $w(z(t_i), t_i, z(t_f), t_f) = 0$ of C^1 class such that

$$rg\left\{\left.\frac{\partial w}{\partial\left(z(t_i), t_i, z(t_f), t_f\right)}\right|^{*}\right\} = \sigma$$

$$\left.\frac{\partial L}{\partial \dot z}\right|_{t_i^*}^{*} = \vartheta^T \left.\frac{\partial w}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial L}{\partial \dot z}\right|_{t_f^*}^{*} = -\vartheta^T \left.\frac{\partial w}{\partial z(t_f)}\right|^{*}, \quad \vartheta \in R^{\sigma}$$

$$\left(L - \frac{\partial L}{\partial \dot z}\dot z\right)_{t_i^*} = \vartheta^T \left.\frac{\partial w}{\partial t_i}\right|^{*}, \quad \left(L - \frac{\partial L}{\partial \dot z}\dot z\right)_{t_f^*} = \vartheta^T \left.\frac{\partial w}{\partial t_f}\right|^{*}$$

Definition: An extremum is **NON-singular** if

$$\left.\frac{\partial^2 L}{\partial \dot{z}(t)^2}\right|^{*} \text{ is non singular in } [t_i^*, t_f^*]$$

A non-singular extremum is a C^2 function

The Lagrange problem

Problem 2

Consider Problem 1 with

- ✓ $t_i \; and \; t_f$ **fixed**
- ✓ **If** $D_i \; D_f$ are **closed sets** in R^{v+1} defined by the C[1] functions

$$\gamma\left(z(t_i),t_i\right)=0, \text{ of dimension } \sigma_\mathrm{i} \le v+1$$

$$\chi\left(z(T),T\right)=0, \text{ of dimension } \sigma_\mathrm{f} \le v+1$$

with $\gamma \; and \; \chi$ affine functions and

$$rg\left\{\left.\frac{\partial \gamma}{\partial\left(z(t_i)\right)}\right|^{o}\right\}=\sigma_i \quad rg\left\{\left.\frac{\partial \chi}{\partial\left(z(t_f)\right)}\right|^{o}\right\}=\sigma_f$$

✓ If the sets D_i D_f are defined by the function $w(z(t_i), z(t_f))$ with σ components of C^1 class affine with respect to $z(t_i), z(t_f)$ such that

$$rg\left\{ \left. \frac{\partial w}{\partial \left(z(t_i), z(t_f) \right)} \right|^{o} \right\} = \sigma$$

✓ The function *L* must be **convex** with respect to $\left(z(t), \dot{z}(t) \right)$

Find the **global minimum (optimum)** z^o for *J* over *D:*

$$J\left(z^o \right) \le J\left(z \right) \quad \forall z \in D$$

Theorem 2. $z^o \in D$ is the optimum **if and only if**

$$\boxed{\left.\frac{\partial L}{\partial z}\right|^{o} - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{z}}\right|^{o} = 0^T \quad \forall t \in \left[t_i, t_f\right]}$$

Euler equation

In any discontinuity point $\bar{t}$ of $\dot{z}^*$ the following conditions are verified:

$$\left.\frac{\partial L}{\partial \dot{z}}\right|^{o}_{\bar{t}^-} = \left.\frac{\partial L}{\partial \dot{z}}\right|^{o}_{\bar{t}^+} \qquad \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)^{o}_{\bar{t}^-} = \left(L - \frac{\partial L}{\partial \dot{z}}\dot{z}\right)^{o}_{\bar{t}^+}$$

Weierstrass-Erdmann condition

Moreover, **transversality conditions** are satisfied:

- If D_i D_f **are closed subsets** defined respectively by

$$\gamma\big(z(t_i)\big)=0 \quad \aleph\big(z(t_f)\big)=0$$

such that

$$rg\left\{\left.\frac{\partial\gamma}{\partial\big(z(t_i)\big)}\right|^{o}\right\}=\sigma_i \qquad rg\left\{\left.\frac{\partial\aleph}{\partial\big(z(t_f)\big)}\right|^{o}\right\}=\sigma_f$$

then

$$\xi\in R^{\sigma_i} \qquad \varsigma\in R^{\sigma_f}$$

$$\left.\frac{\partial L}{\partial\dot z}\right|_{t_i^*}^{*}=\xi^T\left.\frac{\partial\gamma}{\partial z(t_i)}\right|^{o}, \quad \left.\frac{\partial L}{\partial\dot z}\right|_{t_f}^{o}=\varsigma^T\left.\frac{\partial\aleph}{\partial z(t_f)}\right|^{o}$$

$$\left(L-\frac{\partial L}{\partial\dot z}\dot z\right)_{t_i}=0^T, \quad \left(L-\frac{\partial L}{\partial\dot z}\dot z\right)_{t_f}=0^T$$

- If the sets D_i *and* D_f are defined by the function $w(z(t_i), z(t_f))$ affine with respect to $z(t_i), z(t_f)$ such that

$$rg\left\{ \left.\frac{\partial w}{\partial \big(z(t_i), z(t_f)\big)}\right|^{o} \right\} = \sigma$$

$$\left.\frac{\partial L}{\partial \dot{z}}\right|_{t_i^o}^{o} = \vartheta^T \left.\frac{\partial \omega}{\partial z(t_i)}\right|^{o}, \quad \left.\frac{\partial L}{\partial \dot{z}}\right|_{t_f^o}^{o} = -\vartheta^T \left.\frac{\partial \omega}{\partial z(t_f)}\right|^{o}$$

$$\vartheta \in R^{\sigma}$$

If the function *L* is strictly ***convex*** if the solution exists is unique

The Lagrange problem

Problem 3

Let us consider the linear space $\overline{C}^{1}(R)\times R\times R$ and define the admissible set

$$D=\Big\{\big(z,t_i,t_f\big)\in \overline{C}^{1}(R)\times R\times R:\big(z(t_i),t_i\big)\in D_i\subset R^{\nu+1},$$

$$\big(z(t_f),t_f\big)\in D_f\subset R^{\nu+1},\ \boxed{g\big(z(t),\dot z(t),t\big)=0}\ \boxed{\int_{t_i}^{t_f} h\big(z(t),\dot z(t),t\big)dt=k}\Big\}$$

g of dimension $\mu<\nu$

h of dimension σ

consider the cost index:

$$\boxed{J(z,t_i,t_f)=\int_{t_i}^{t_f} L\big[z(t),\dot z(t),t\big]dt}$$

with L scalar function of C^2 class.

Define the **augmented lagrangian:**

$$\boxed{\begin{aligned}\ell\big(z(t),\dot z(t),t,\lambda_0,\lambda(t),\rho\big) &= \lambda_0 L\big(z(t),\dot z(t),t\big)\\ &\quad+\lambda^T(t)g\big(z(t),\dot z(t),t\big)+\rho^T h\big(z(t),\dot z(t),t\big)\end{aligned}}$$

Theorem 3 (Lagrange). Let $\left(z^*, t_i^*, t_f^*\right) \in D$ be such that

$$rank\left\{ \left.\frac{\partial g}{\partial \dot{z}(t)}\right|^* \right\} = \mu \quad \forall t \left[t_i^*, t_f^*\right]$$

If $\left(z^*, t_i^*, t_f^*\right)$ is a local minimum for *J* over *D*, **then** there exist $\lambda_0^* \in R,\ \lambda^* \in \overline{C}^0[t_i, t_f^*],\ \rho^* \in R^\sigma$ **not simultaneously null** in $[t_i, t_f^*]$ such that:

- $$\left.\frac{\partial \ell}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in \left[t_i^*, t_f^*\right]$$
- $$\left.\frac{\partial \ell}{\partial \dot{z}}\right|^*_{\bar{t}_k^-} = \left.\frac{\partial \ell}{\partial \dot{z}}\right|^*_{\bar{t}_k^+} \qquad \left(\ell - \frac{\partial \ell}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}_k^-} = \left(\ell - \frac{\partial \ell}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}_k^+}$$

where $\bar{t}_k$ are cuspid points for z^*

❑ Moreover, the **transversality conditions** are satisfied:

- **If D_i D_f are open subset we have:**

$$\left.\frac{\partial \ell}{\partial \dot{z}}\right|_{t_i^*}^{*} = 0^T \quad \left.\frac{\partial \ell}{\partial \dot{z}}\right|_{t_f^*}^{*} = 0^T \quad \left.\ell\right|_{t_i^*}^{*} = 0 \quad \left.\ell\right|_{t_f^*}^{*} = 0$$

- **If D_i D_f are closed subset defined respectively by such that**

$$\gamma\left(z(t_i), t_i\right) = 0 \quad \aleph\left(z(t_f), t_f\right) = 0$$

Size of γ

Size of χ

$$rg\left\{\left.\frac{\partial \gamma}{\partial\left(z(t_i), t_i\right)}\right|^{*}\right\} = \sigma_i \qquad rg\left\{\left.\frac{\partial \aleph}{\partial\left(z(t_f), t_f\right)}\right|^{*}\right\} = \sigma_f$$

for

$$\xi \in R^{\sigma_i} \qquad \varsigma \in R^{\sigma_f}$$

$$\left.\frac{\partial \ell}{\partial \dot{z}}\right|_{t_i^*}^{*} = \xi^T \left.\frac{\partial \gamma}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial \ell}{\partial \dot{z}}\right|_{t_f^*}^{*} = \varsigma^T \left.\frac{\partial \aleph}{\partial z(t_f^*)}\right|^{*}$$

$$\left(\ell - \frac{\partial \ell}{\partial \dot{z}}\dot{z}\right)_{t_i^*} = \xi^T \left.\frac{\partial \gamma}{\partial t_i}\right|^{*}, \quad \left(\ell - \frac{\partial \ell}{\partial \dot{z}}\dot{z}\right)_{t_f^*} = \varsigma^T \left.\frac{\partial \aleph}{\partial t_f}\right|^{*}$$

- If **the sets** D_i *and* D_f **are defined by the function** $w(z(t_i), z(t_f))$ affine with respect to $z(t_i), z(t_f)$ such that

$$rg\left\{ \left. \frac{\partial w}{\partial\left(z(t_i), t_i, z(t_f), t_f\right)} \right|^* \right\} = \sigma$$

$$\left. \frac{\partial \ell}{\partial \dot{z}} \right|_{t_i^*}^* = \vartheta^T \left. \frac{\partial w}{\partial z(t_i)} \right|^*, \quad \left. \frac{\partial \ell}{\partial \dot{z}} \right|_{t_f^*}^* = -\vartheta^T \left. \frac{\partial w}{\partial z(t_f)} \right|^*$$

$$\left. \left(\ell - \frac{\partial \ell}{\partial \dot{z}} \dot{z} \right) \right|_{t_i^*}^* = \vartheta^T \left. \frac{\partial w}{\partial t_i} \right|^*, \quad \left. \left(\ell - \frac{\partial \ell}{\partial \dot{z}} \dot{z} \right) \right|_{t_f^*}^* = \vartheta^T \left. \frac{\partial w}{\partial t_f} \right|^*$$

$$\vartheta \in R^\sigma$$

The Lagrange problem

Problem 4 Let us consider the linear space $\overline{C}^1(R)\times R\times R$ and define the admissible set

$$D=\left\{z\in\overline{C}^1\left[t_i,t_f\right],\ z(t_i)\in D_i, z(t_i)\in D_f, g\left(z(t),\dot{z}(t),t\right)=0,\ \int_{t_i}^{t_f} h\left(z(t),\dot{z}(t),t\right)dt=k,\ \forall t\in[t_i,t_f]\right\}$$

- g of dimension $\mu<\nu$
- t_i t_f fixed
- g and h **affine** functions in $z(t),\dot{z}(t),\ \forall t\in[t_i,t_f]$
- L C^2 function convex with respect to $z(t),\dot{z}(t),\ \forall t\in[t_i,t_f]$

Consider the cost index:

$$J(z,t_i,t_f)=\int_{t_i}^{t_f} L\left[z(t),\dot{z}(t),t\right]dt$$

Define the **augmented lagrangian:**

$$\boxed{\begin{aligned}&\ell\big(z(t),\dot z(t),t,\lambda_0,\lambda(t),\rho\big)=L\big(z(t),\dot z(t),t\big)\\&+\lambda^T(t)g\big(z(t),\dot z(t),t\big)+\rho^T h\big(z(t),\dot z(t),t\big)\end{aligned}}$$

Theorem 4 (Lagrange). Let $z^o \in D$ such that

$$rank\left\{ \left.\frac{\partial g}{\partial \dot{z}(t)}\right|^o \right\} = \mu \quad \forall t \in \left[t_i , t_f\right]$$

$z^o \in D$ is an optimal **normal** solution

if and only if

- $$\left.\frac{\partial \ell}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in \left[t_i , t_f\right]$$

- in the instants t_i and/or t_f for which D_i and/or D_f are open we have:

$$\left.\frac{\partial \ell}{\partial \dot{z}}\right|^o = 0^T$$