Optimal Control

DEPARTMENT OF COMPUTER, CONTROL, AND
MANAGEMENT ENGINEERING ANTONIO RUBERTI

SAPIENZA
UNIVERSITÀ DI ROMA

Lecture

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Double integrator (from Bruni et al.1993)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t) \qquad x(t_i) = x_i\,, \quad x(t_f) = 0$$
$$|u(t)| \le 1$$

(1)

Goal: Minimize

$$J = \int_{t_i}^{t_f} dt = t_f - t_i$$

Eigenvalues: $\{0;\ 0\}$

The couple $(A\ B)$ verifies the condition of controllability:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \det(B\ AB) = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$$

$$\Rightarrow \quad \exists\ \left(x^o u^o, t_f^o\right)$$

Unique, non singular and
the optimal control is bang-bang
with a number of discontinuity points

$$\nu^o \le n - 1 = 1$$

Description of the trajectories when $u = \pm 1$

From system (1), it results:

$$\boxed{\begin{aligned} x_2(t) &= \pm(t - t_i) + x_2^i \\ x_1(t) &= x_1^i + x_2^i(t - t_i) \pm \frac{1}{2}(t - t_i)^2 \end{aligned}}$$

$$\boxed{(t - t_i) = \pm\left[x_2(t) - x_2^i\right]}$$

$$\boxed{\begin{aligned} x_1(t) - x_1^i &= \pm x_2^i\left[x_2(t) - x_2^i\right] \pm \frac{1}{2}\left[x_2(t) - x_2^i\right]^2 \\ &= \pm\left[x_2(t) - x_2^i\right]\left[x_2^i + \frac{1}{2}x_2(t) - \frac{1}{2}x_2^i\right] \\ &= \pm\frac{1}{2}\left[x_2(t) - x_2^i\right]\left[x_2(t) + x_2^i\right] \\ &= \pm\frac{1}{2}\left[x_2^2(t) - x_2^{i2}\right] \end{aligned}}$$

Description of the trajectories when $u = \pm 1$

From system (1), it results:

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$$x_1(t) = x_1^i + x_2^i (t - t_i) \pm \frac{1}{2}(t - t_i)^2$$

$$(t - t_i) = \pm \left[x_2(t) - x_2^i \right]$$

$$\underline{x_1(t) - x_1^i} = \pm x_2^i \left[x_2(t) - x_2^i \right] \pm \frac{1}{2} \left[x_2(t) - x_2^i \right]^2$$

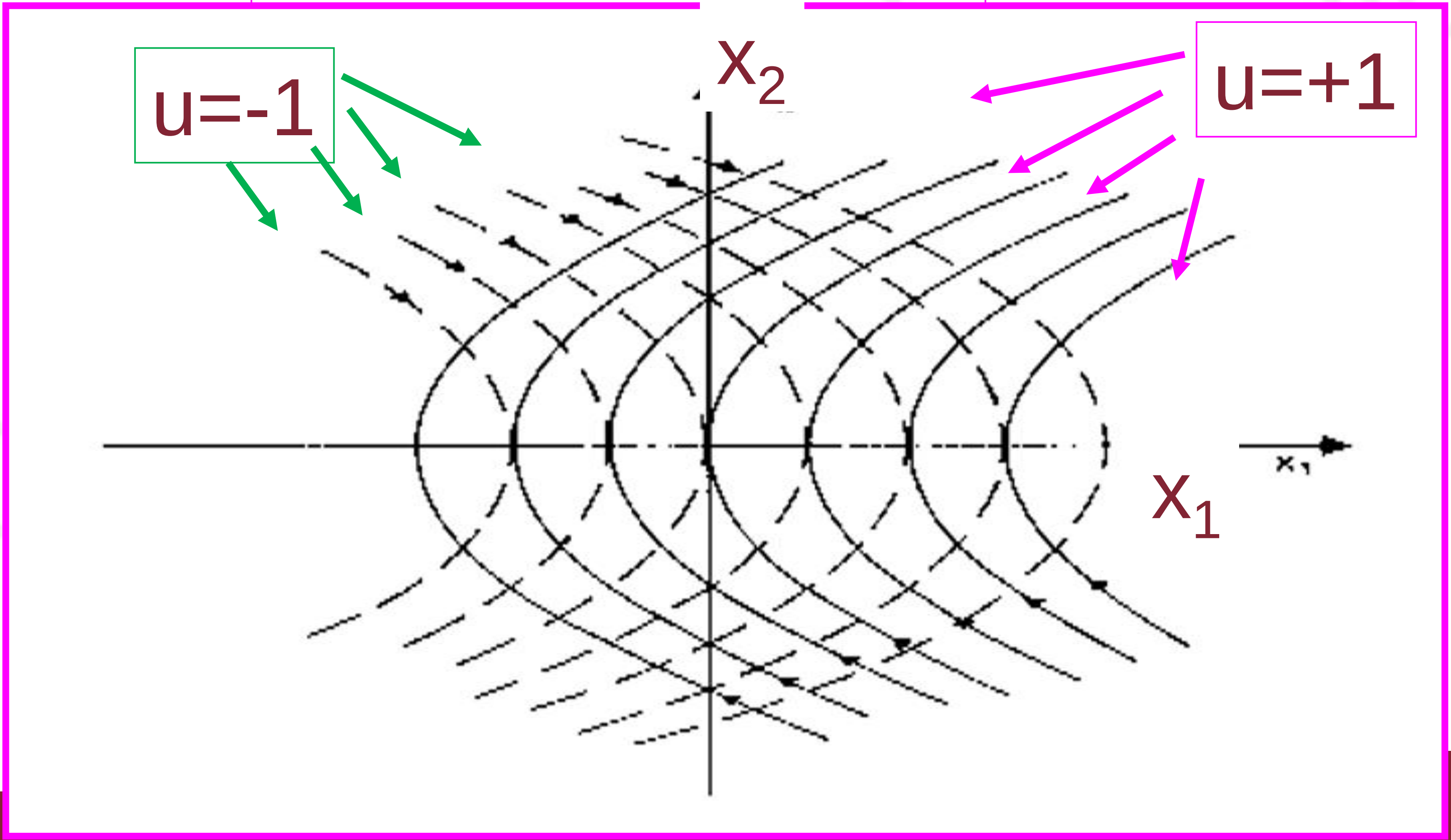
$$= \pm \left[x_2(t) - x_2^i \right] \left[x_2^i + \frac{1}{2} x_2(t) - \frac{1}{2} x_2^i \right]$$

$$= \pm \frac{1}{2} \left[x_2(t) - x_2^i \right] \left[x_2(t) + x_2^i \right]$$

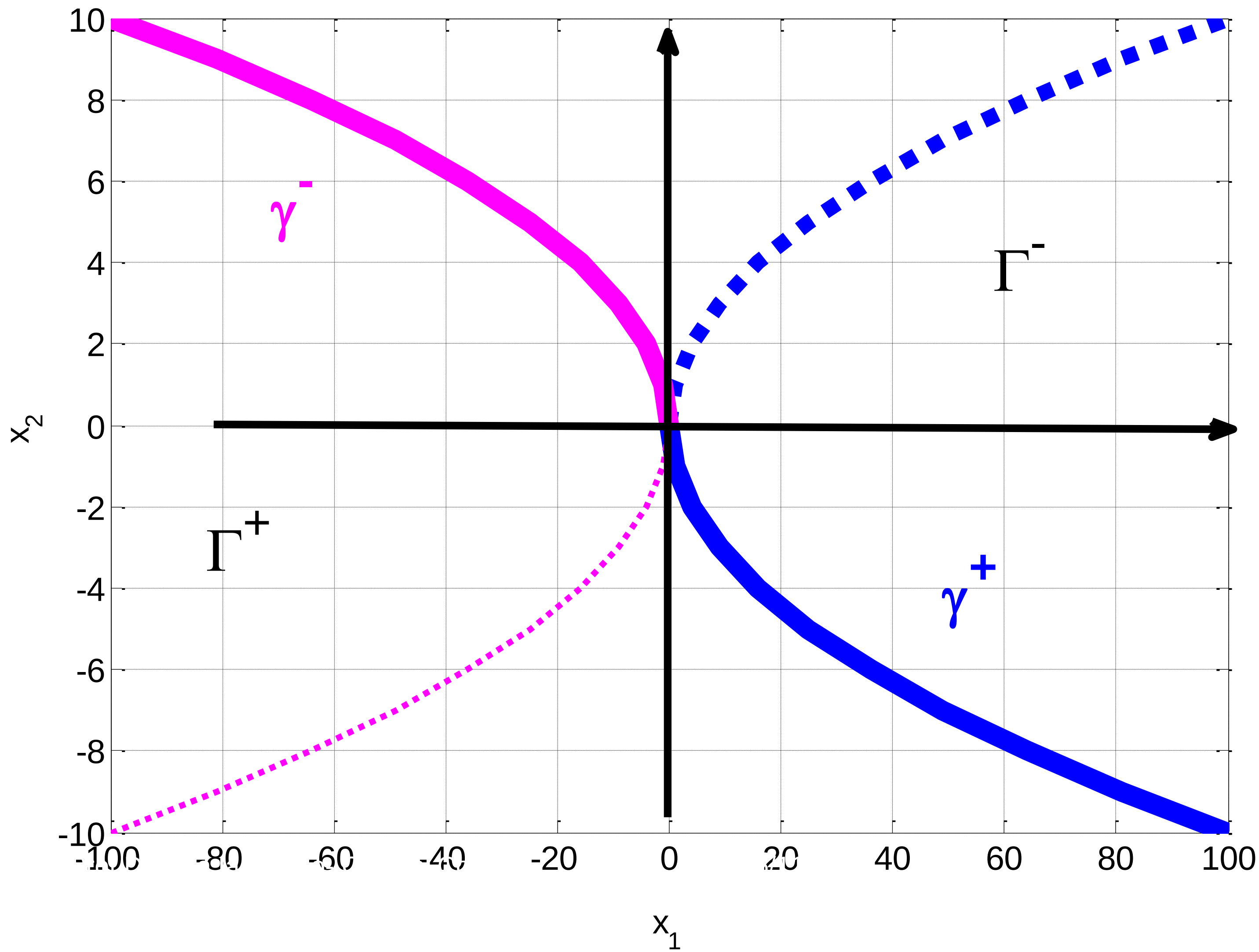
$$\underline{= \pm \frac{1}{2} \left[x_2^2(t) - x_2^{i2} \right]}$$

The optimal trajectory is described by parabolic arcs with equation:

$$x_1(t) - x_1^i = \pm\frac{1}{2}\left[x_2^2(t) - x_2^{i2}\right]$$



Given the initial point x_i the problem is to bring that point to the origin along a trajectory constituted by parabolic arcs with 1 or 0 switching points



Let's define:

$$\gamma^{+} = \left\{ x \in R^{2} : x_1 = \frac{1}{2} x_2^2 ,\ \ x_2 < 0 \right\}$$

$$\gamma^{-} = \left\{ x \in R^{2} : x_1 = -\frac{1}{2} x_2^2 ,\ \ x_2 > 0 \right\}$$

$$\gamma = \gamma^{+} \bigcup \gamma^{-} = \left\{ x \in R^{2} : x_1 = -\frac{1}{2} x_2 |x_2| ,\ \ x_2 \neq 0 \right\}$$

$$\Gamma^{+} = \left\{ x \in R^{2} : x_1 < -\frac{1}{2} x_2 |x_2| \right\}$$

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$$\Gamma^{+} \cup \Gamma^{-} \cup \gamma^{+} \cup \gamma^{-} = R^{2} \setminus \{0\}$$

x_i belongs to one of these regions

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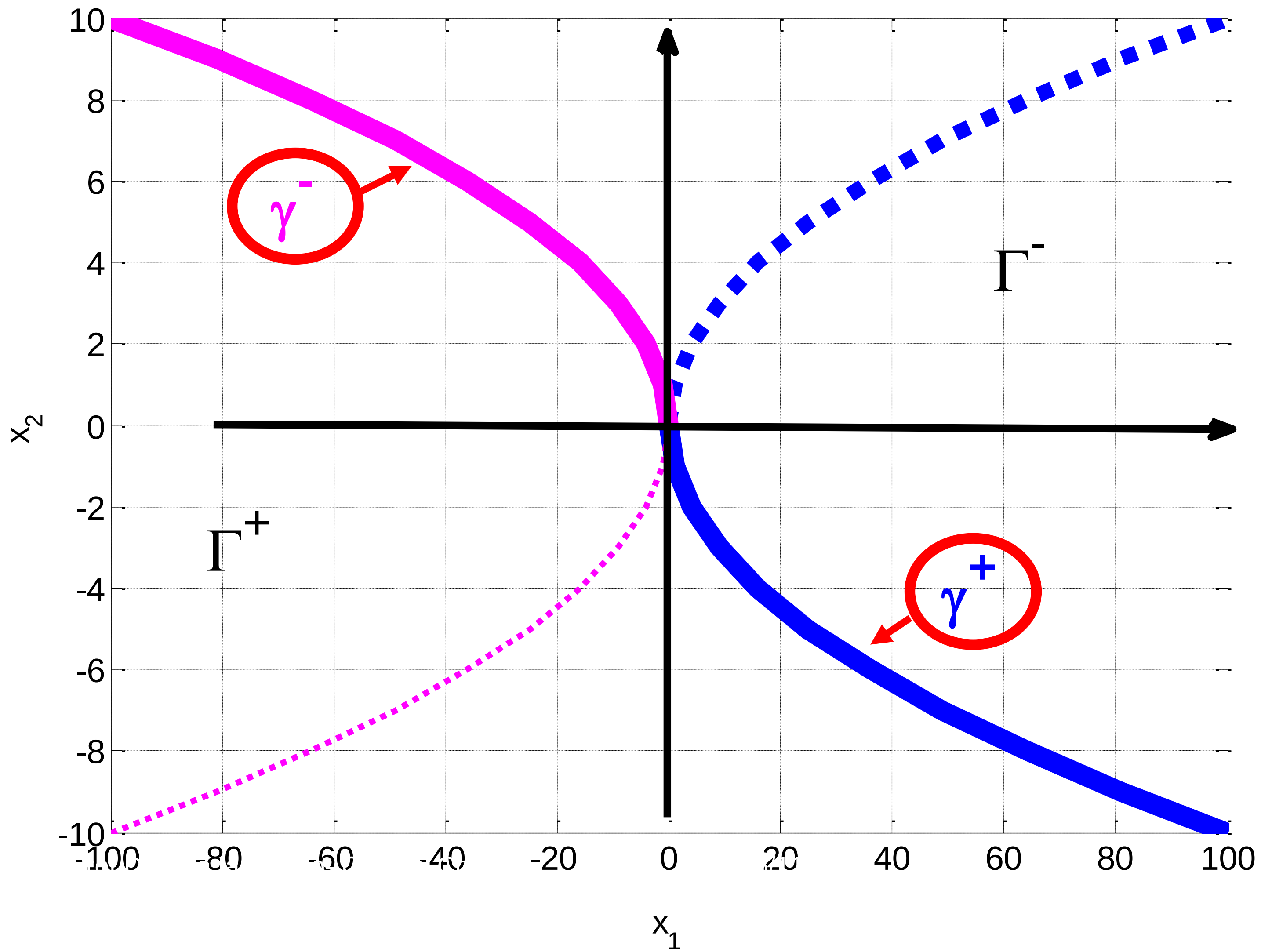

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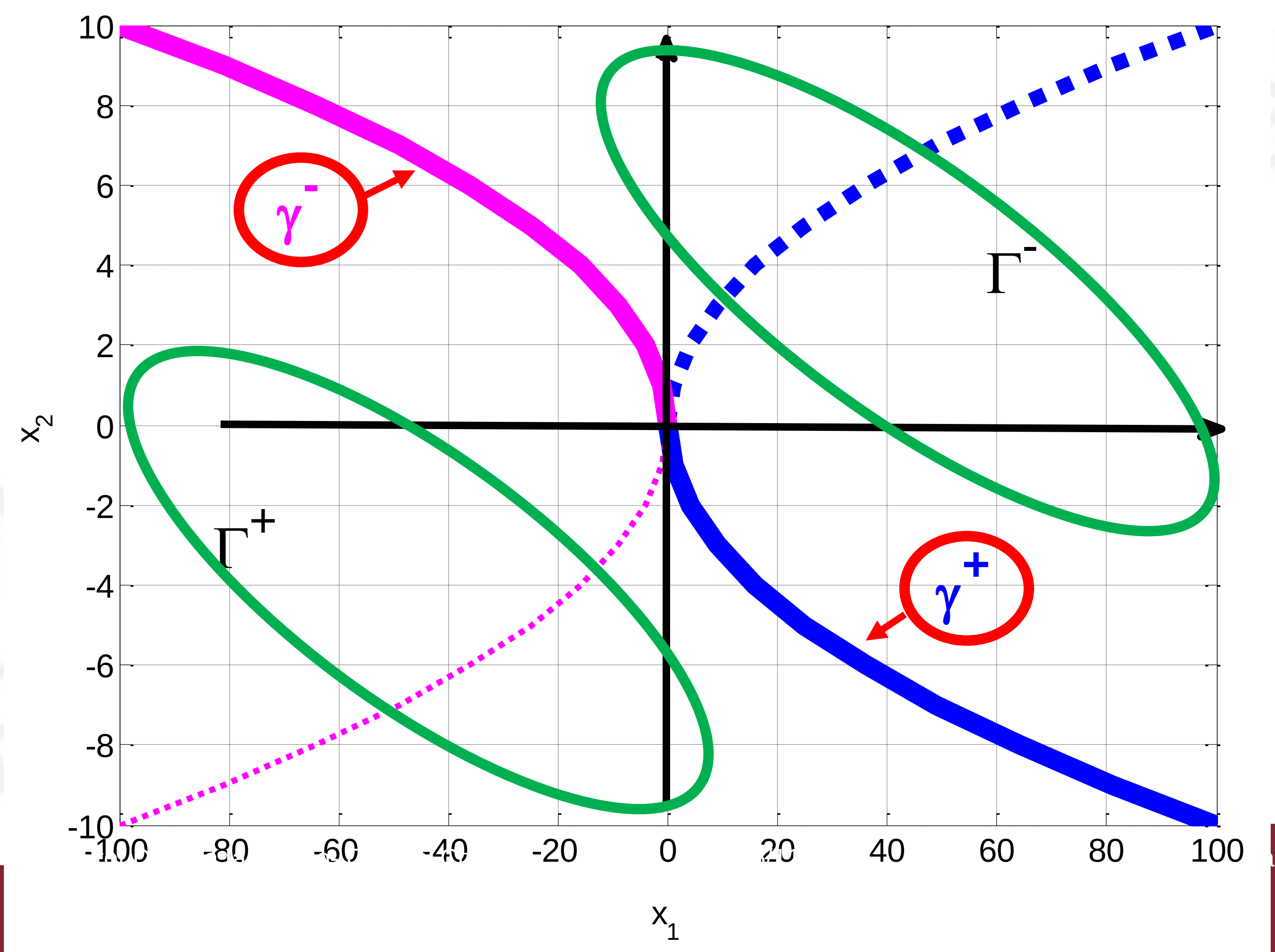
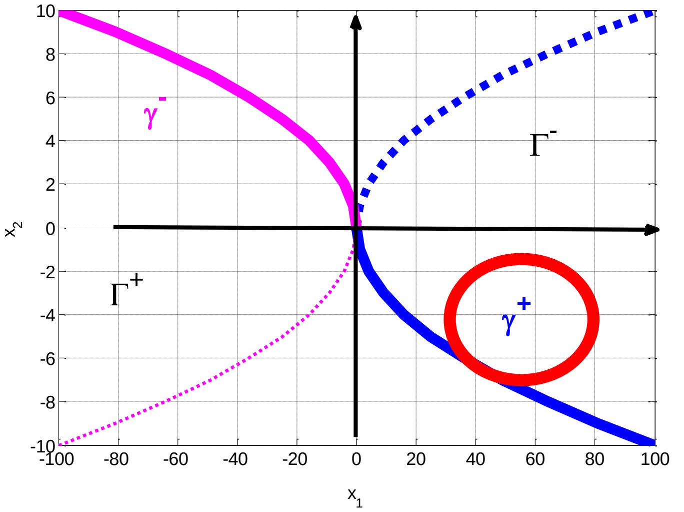
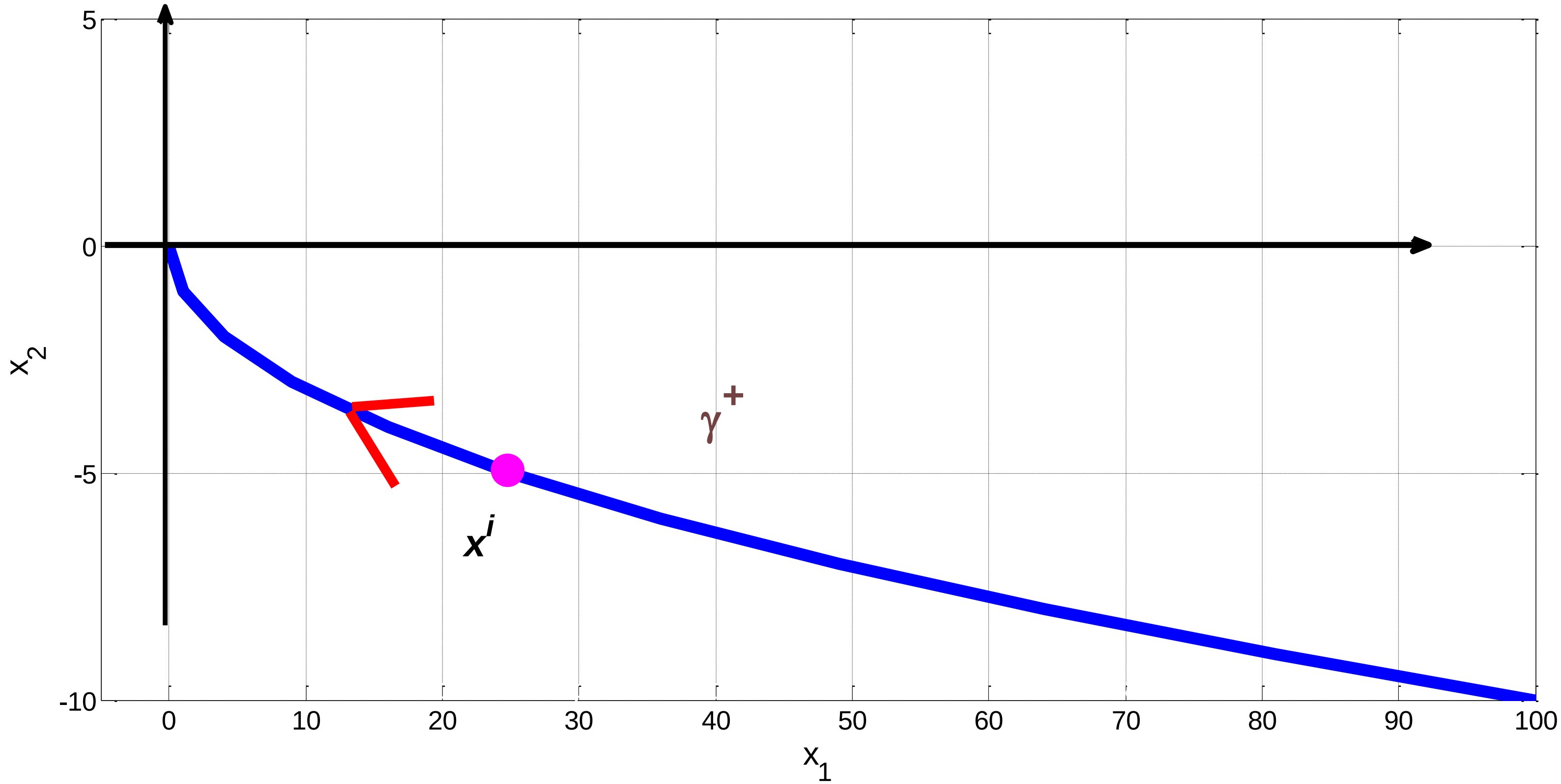


Four cases:

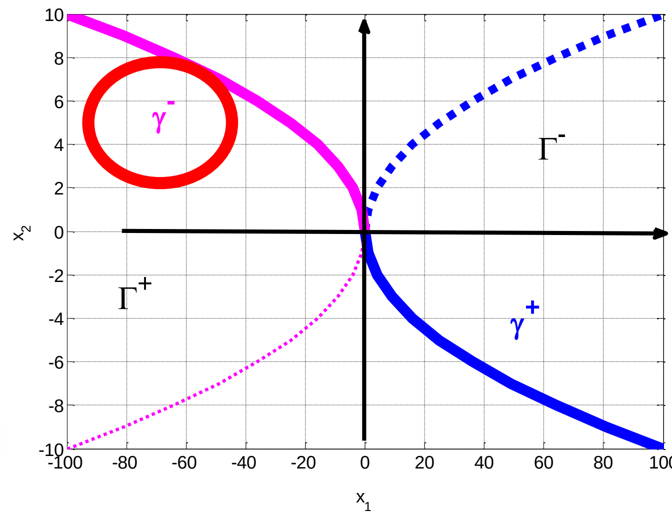
1) $$x^i \in \gamma^+ = \left\{ x \in R^2 : x_1 = \frac{1}{2} x_2^2,\ \ x_2 < 0 \right\}$$



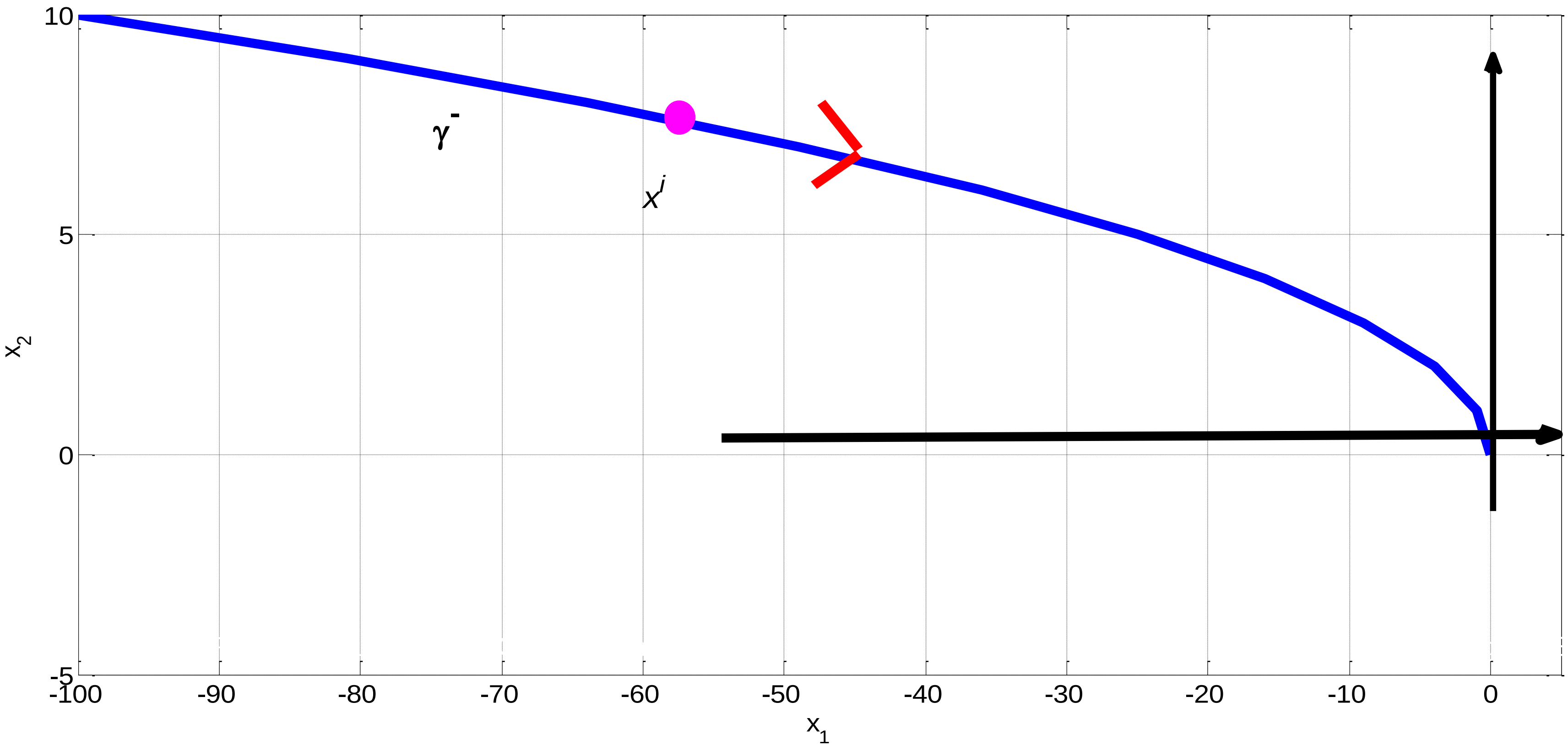
with a control u=1 and without switching you reach the origin



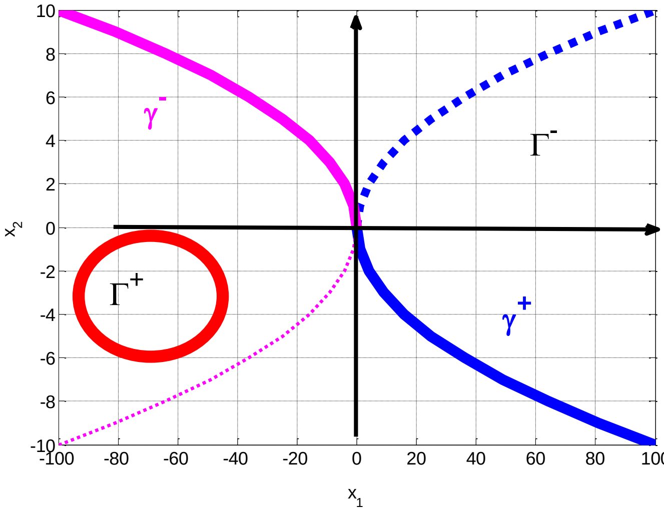
2) $x^i \in \gamma^- = \left\{ x \in R^2 : x_1 = -\frac{1}{2} x_2^2, \;\; x_2 > 0 \right\}$



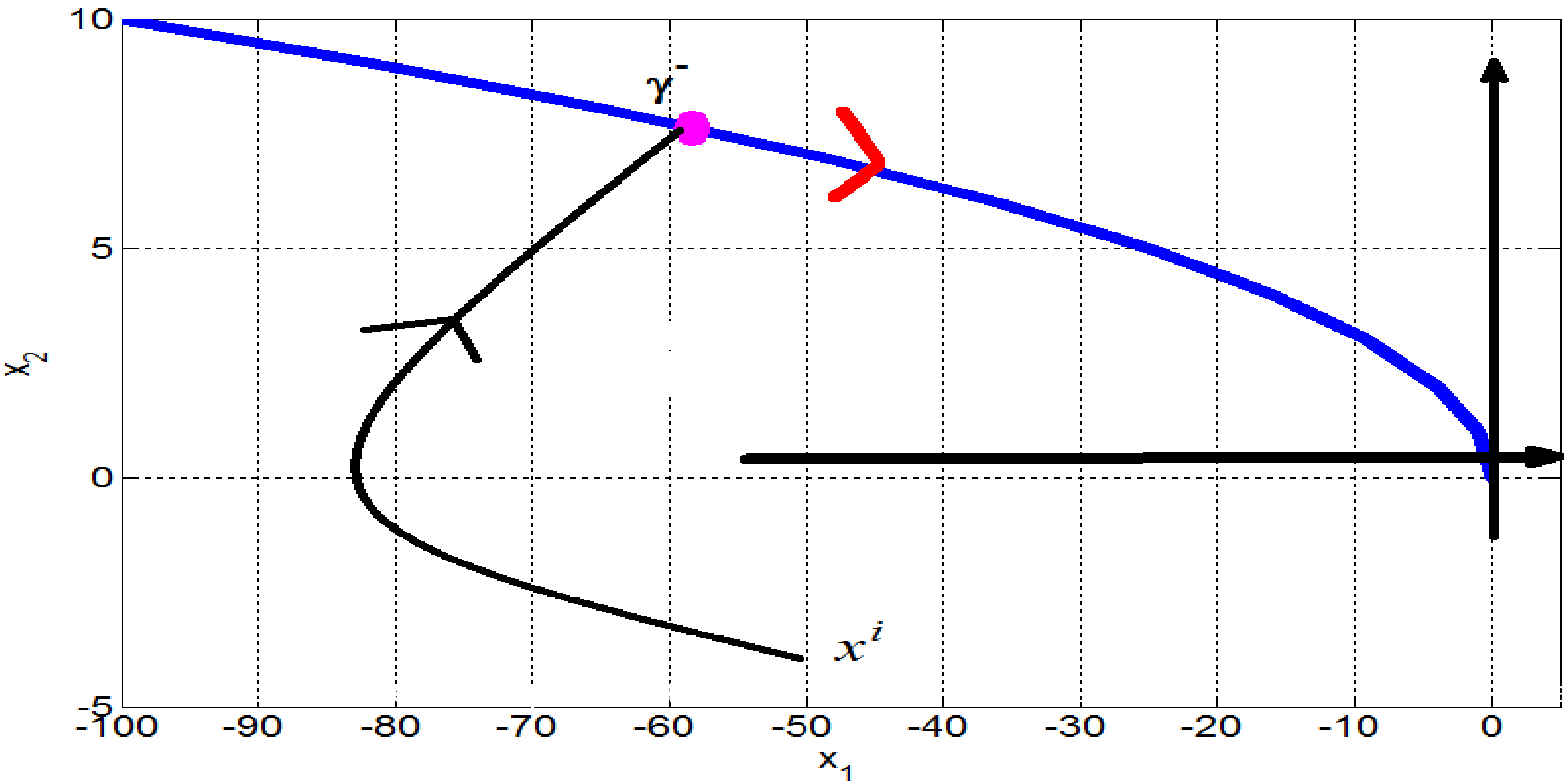
with a control u= -1 and without switching you reach the origin



3) $x^i \in \Gamma^+ = \left\{ x \in R^2 : x_1 < -\frac{1}{2} x_2 |x_2| \right\}$

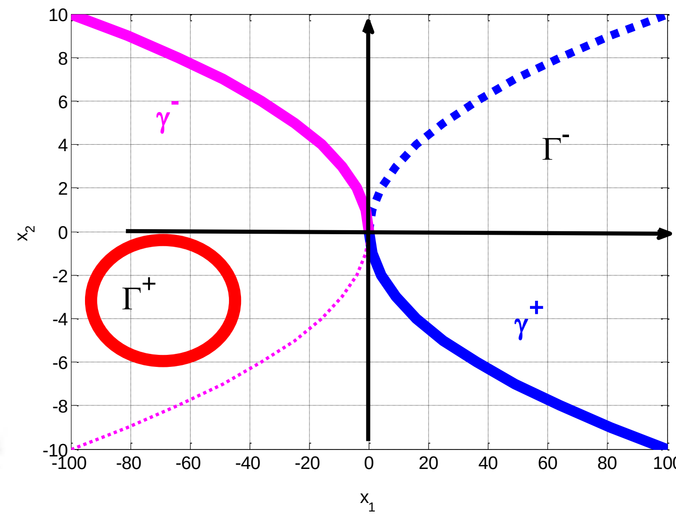


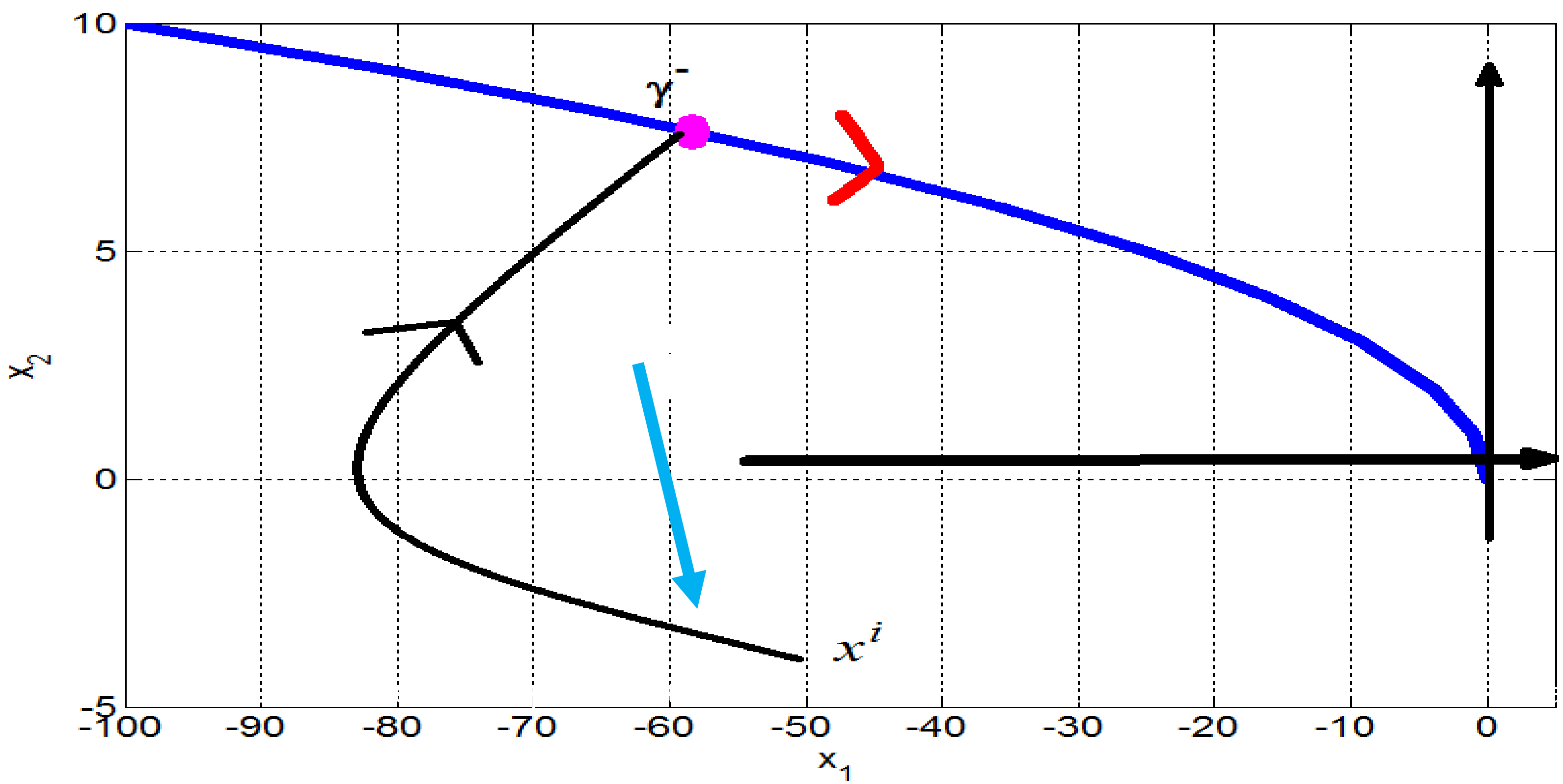
First use a control with u= +1
and you get curve γ^-
then you switch to control u=- 1
and you get to the origin



3) $x^i \in \Gamma^+ = \left\{ x \in R^2 : x_1 < -\frac{1}{2} x_2 |x_2| \right\}$

First use a control with u= +1
and you get curve γ⁻
then you switch to control u=- 1
and you get to the origin





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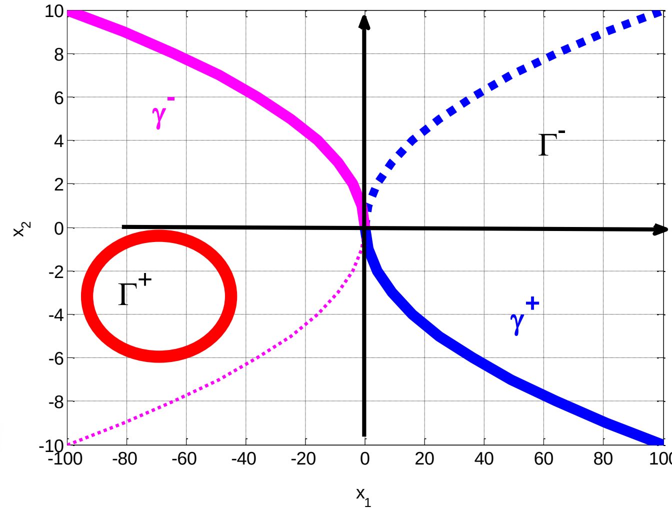
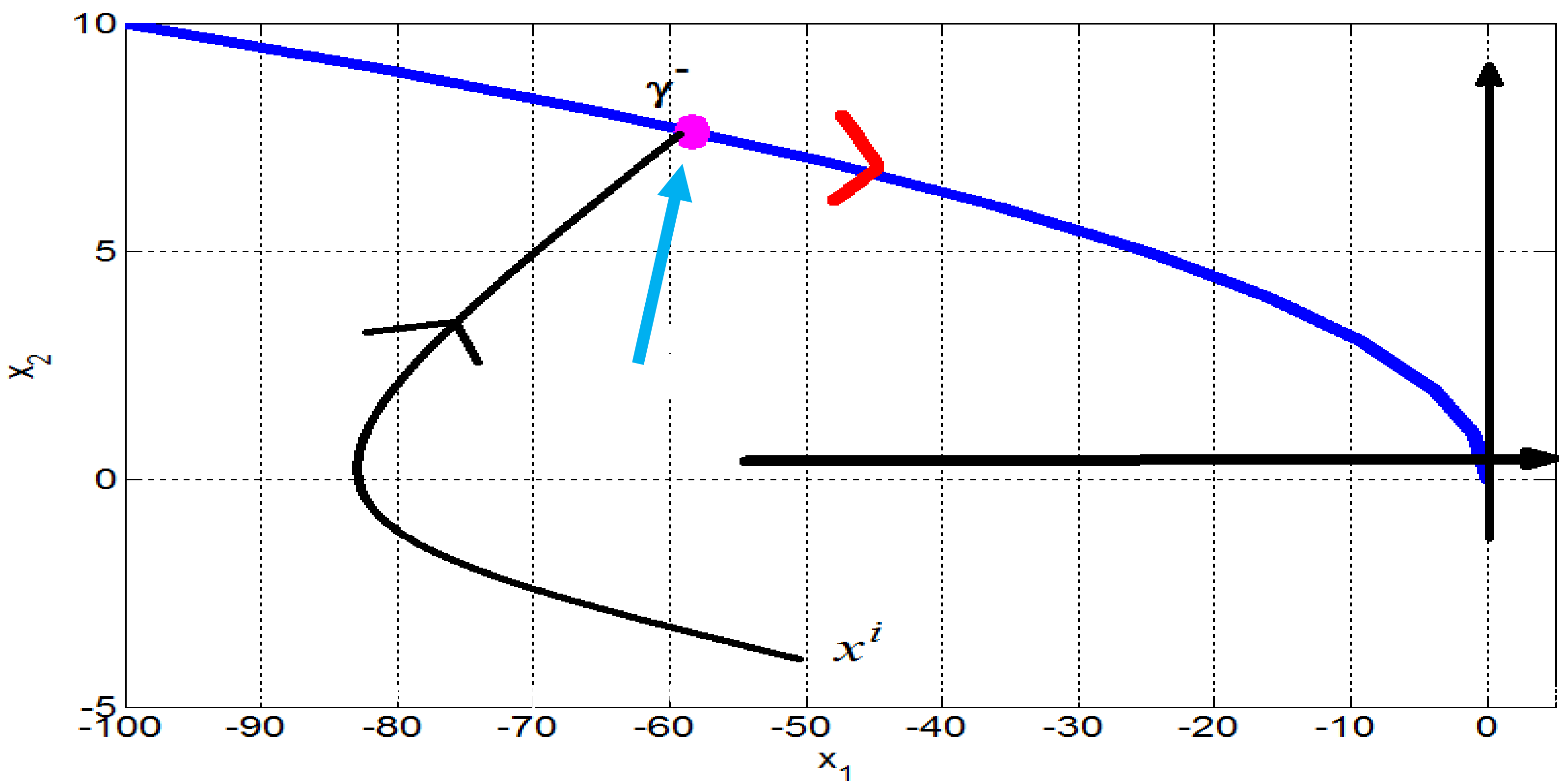


First use a control with u= +1

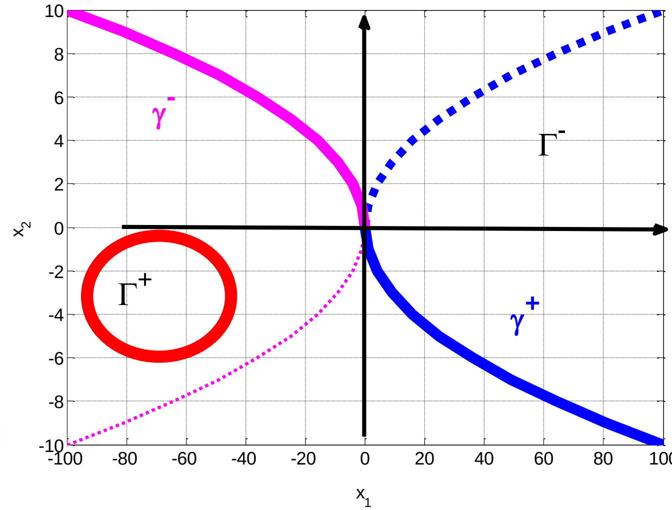
and you get curve γ-

then you switch to control u=- 1

and you get to the origin



3) $$x^i \in \Gamma^+ = \left\{ x \in R^2 : x_1 < -\frac{1}{2} x_2 |x_2| \right\}$$



First use a control with u= +1
and you get curve γ-
then you switch to control u=- 1
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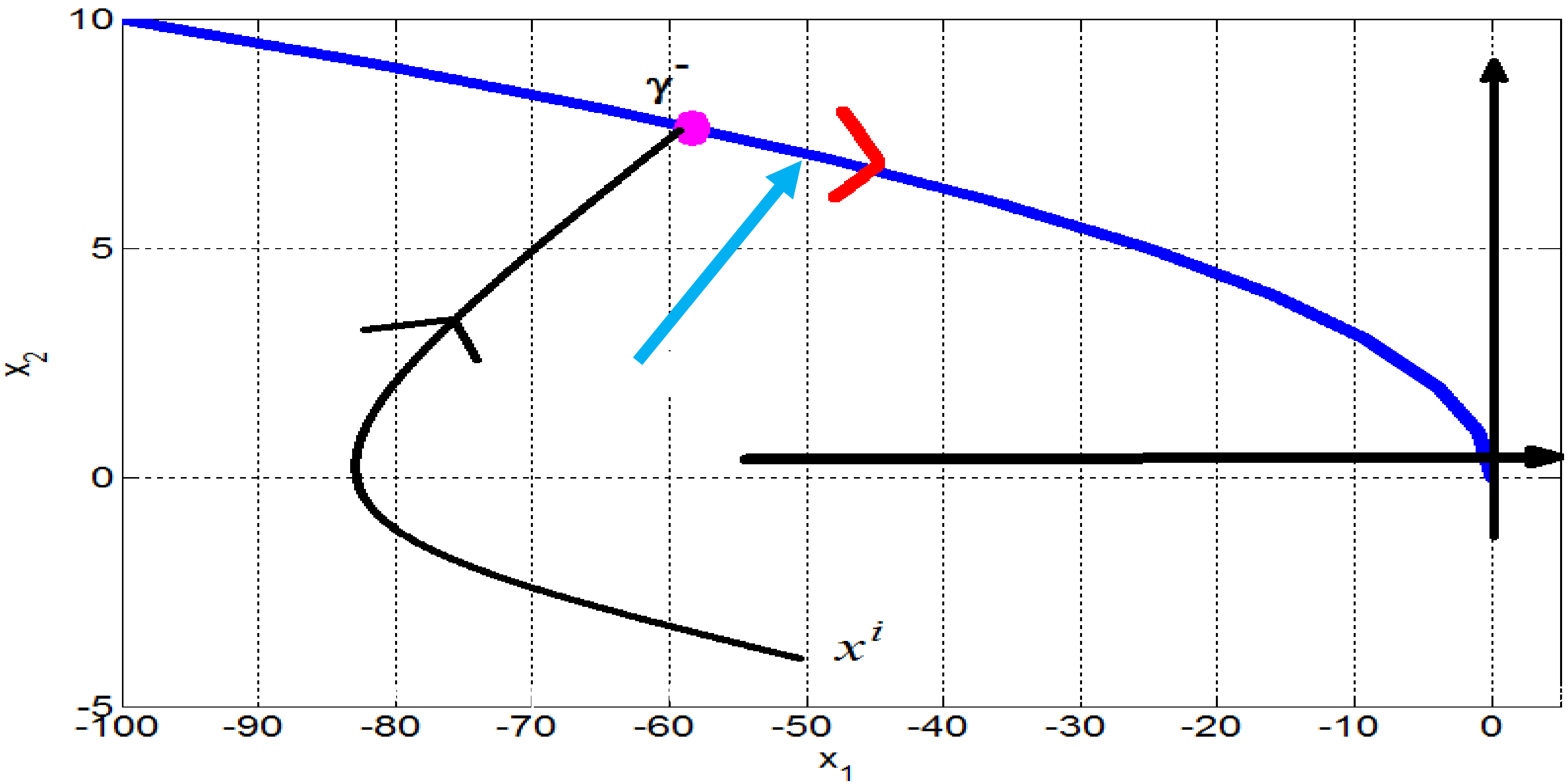
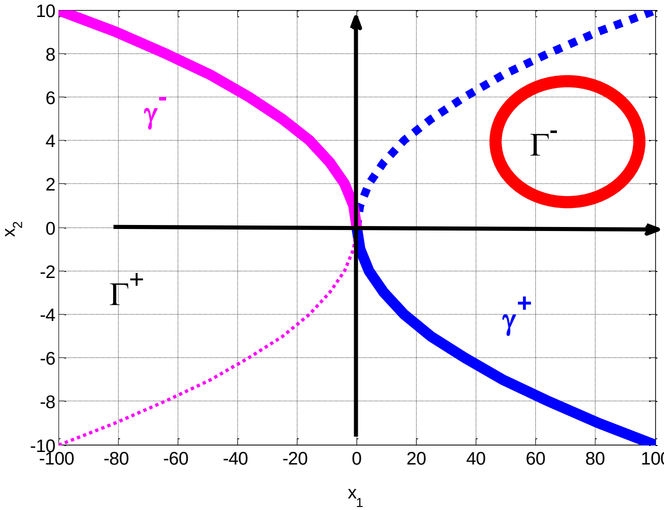
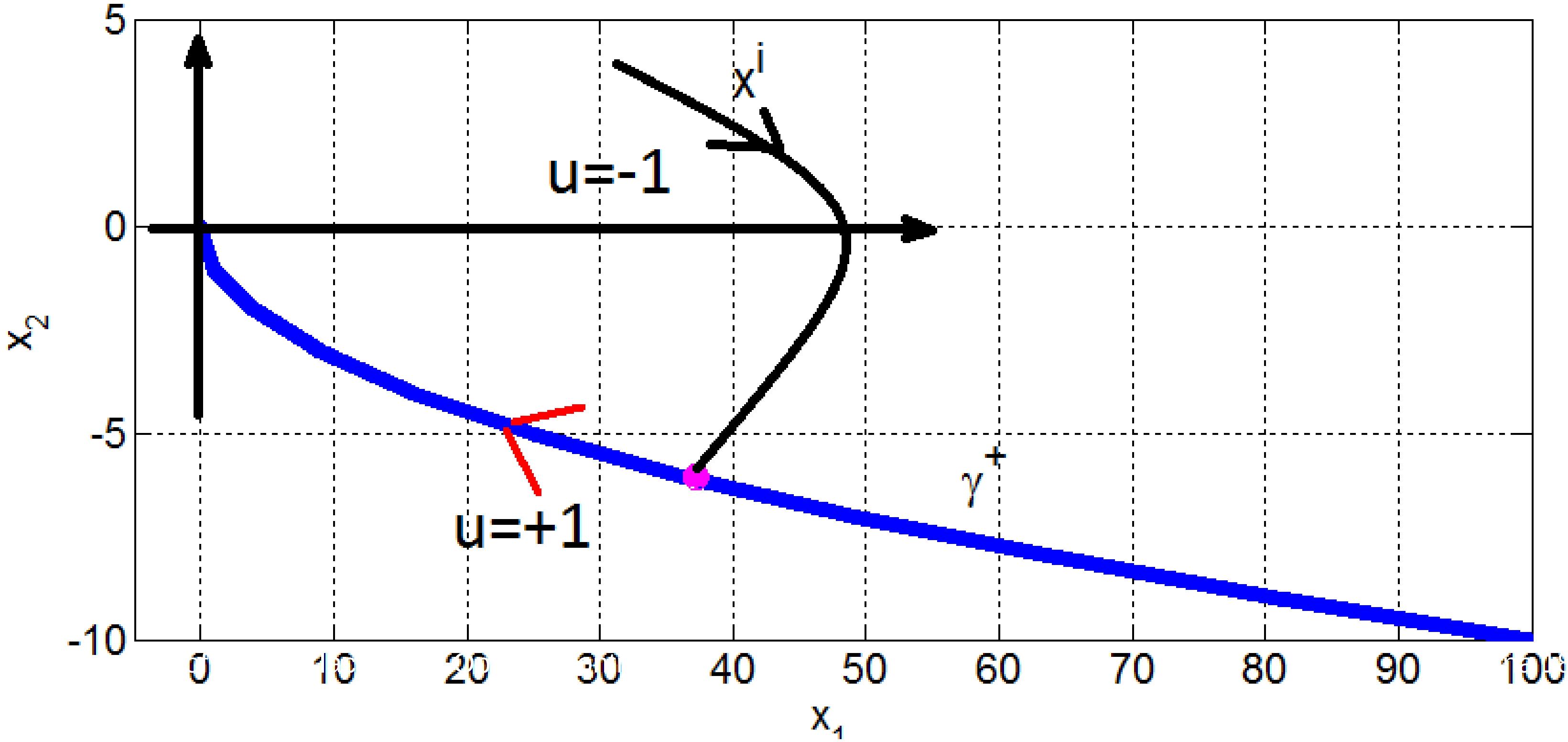


4) $x^i \in \Gamma^- = \left\{ x \in R^2 : x_1 > -\frac{1}{2} x_2 |x_2| \right\}$

First use a control with u= -1
and you get curve γ+
then you switch to control u=+ 1
and you get to the origin





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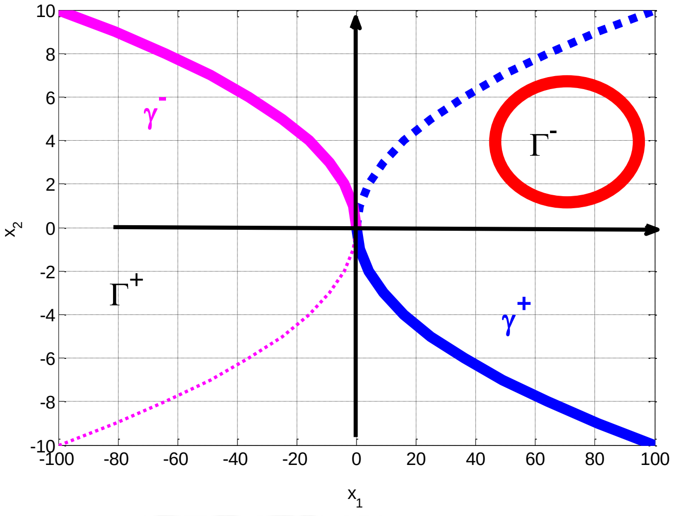
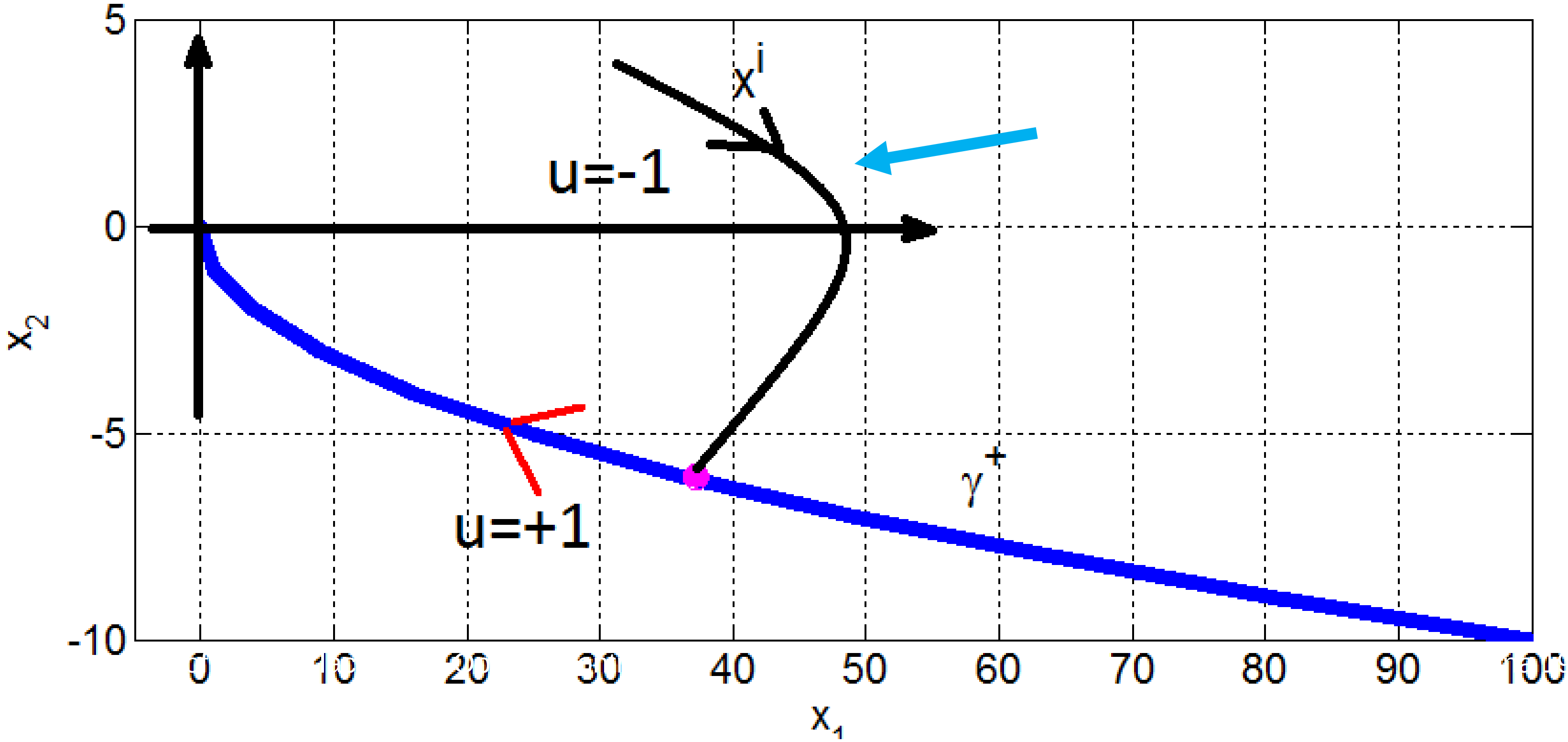


First use a control with u= -1
and you get curve γ^+
then you switch to control u=+ 1
and you get to the origin



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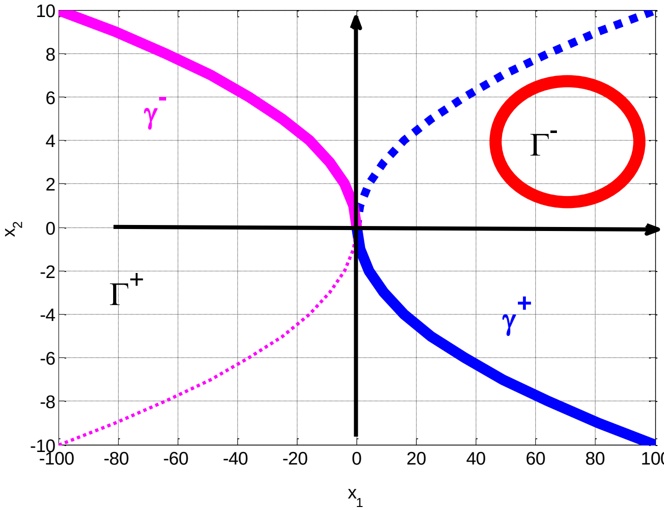

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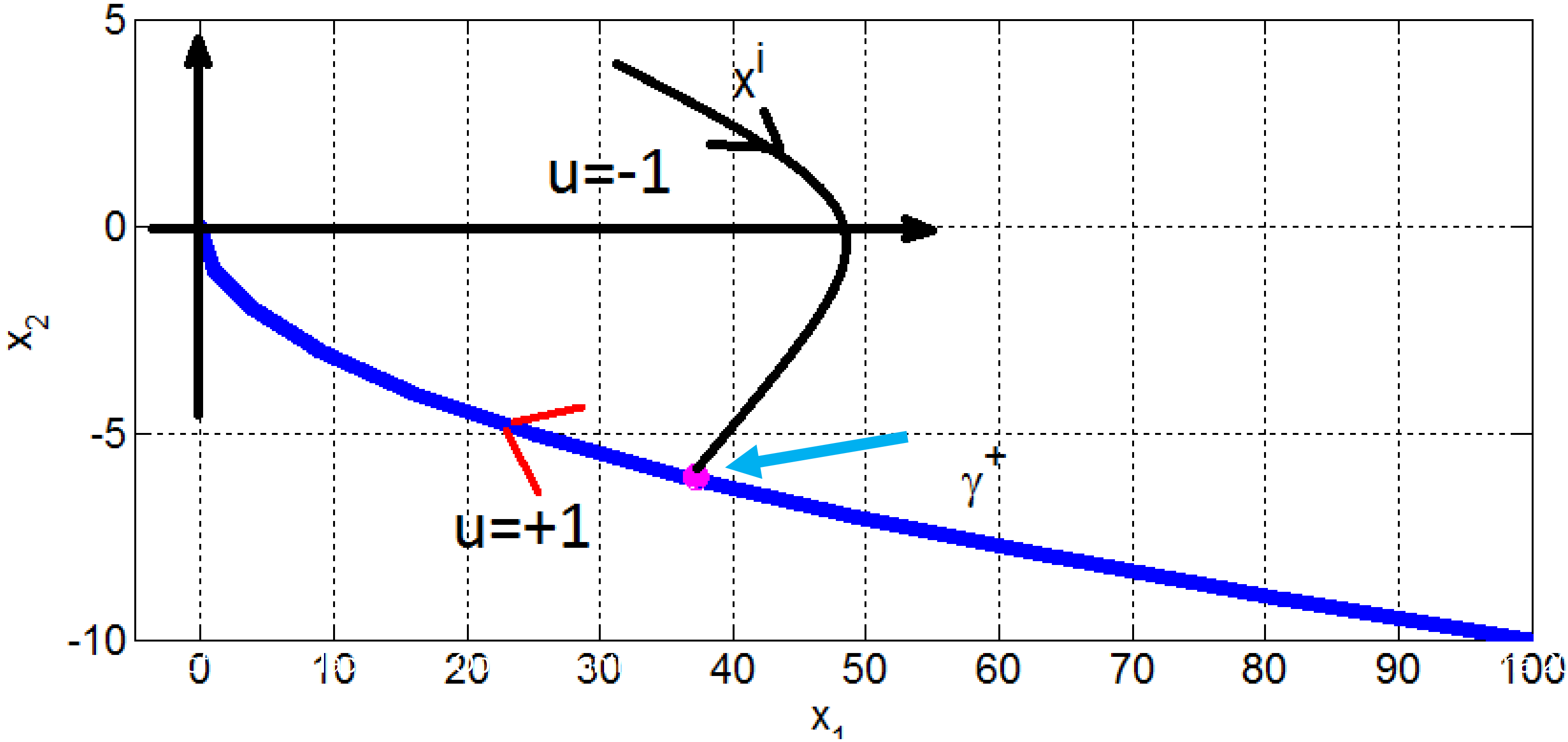

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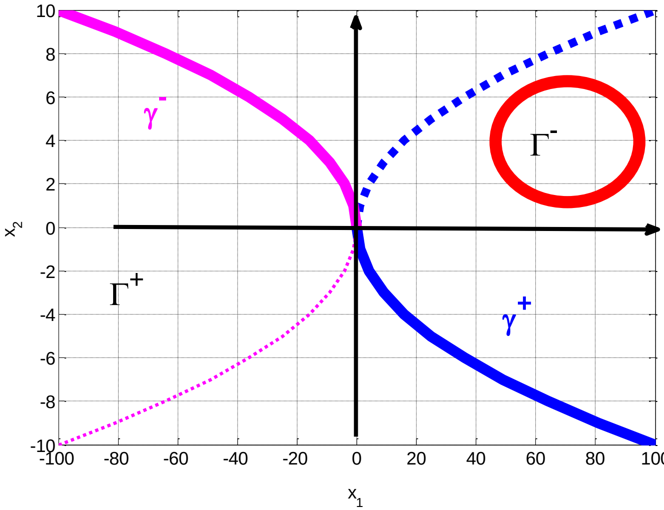
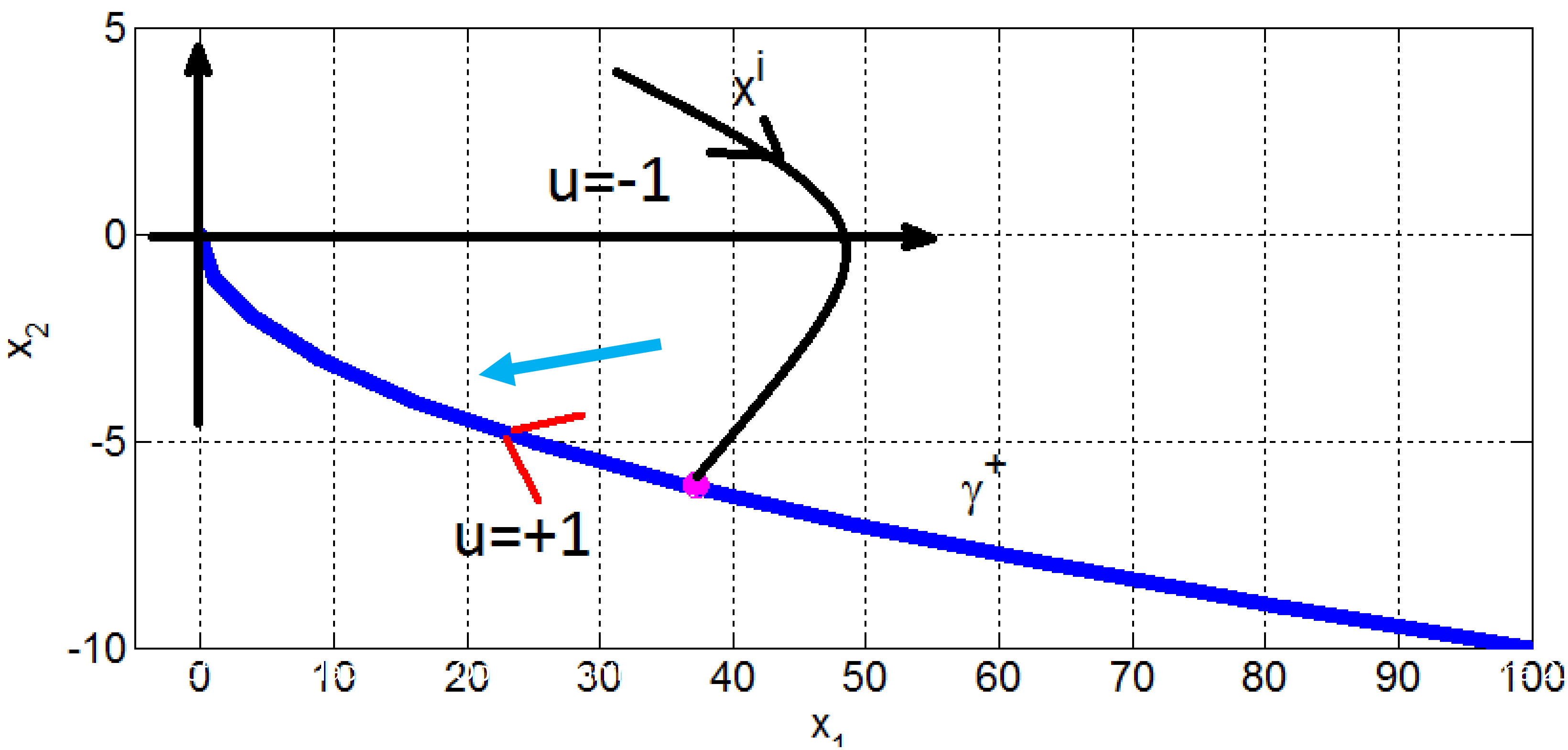




$$\boxed{u^o\left(x^o(t)\right)=\begin{cases}1 & if \quad x^o(t)\in\Gamma^+\cup\gamma^+\\ -1 & f \quad x^o(t)\in\Gamma^-\cup\gamma^-\end{cases}}$$

$$u^o\left(x^o(t)\right)=\begin{cases}1 & if \quad x^o(t)\in\Gamma^+\cup\gamma^+\\ -1 & f \quad x^o(t)\in\Gamma^-\cup\gamma^-\end{cases}$$

Calculus of the minimum time:
it depends on the location of the initial point x^i

A) $x^i\in\gamma=\gamma^-\cup\gamma^+$

The control doesn't switch; from $(t-t_i)=\pm\left[x_2(t)-x_2^i\right]$

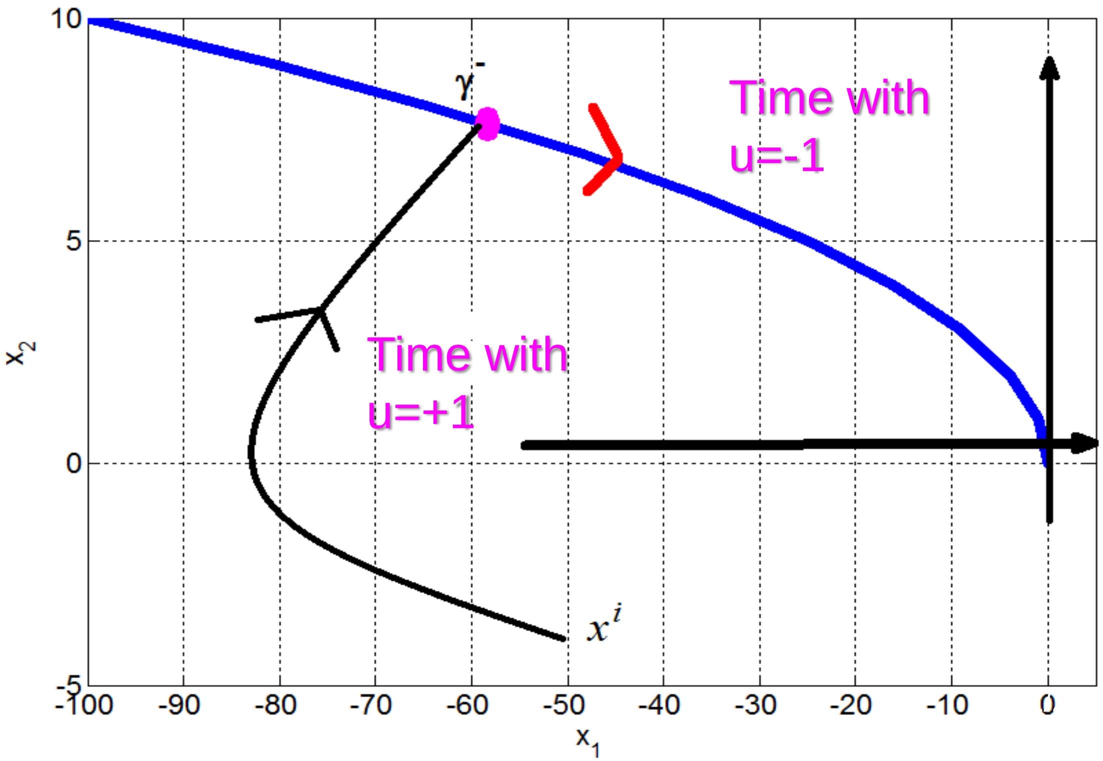
$$\left(t_f^o-t_i\right)=\pm\left[0-x_2^i\right]=\mp x_2^i=\left|x_2^i\right|$$

B) $\boxed{x^i \in \Gamma^+}$ The control switches at an instant $\bar{t}$ at the position $\bar{x}_2^i$

$$\boxed{t_f - t_i = \left(t_f - \bar{t}\right) + \left(\bar{t} - t_i\right)}$$

Time with u=-1

Time with u=+1



From

$$\boxed{\left(t - t_i\right) = \pm\left[x_2(t) - x_2^i\right]}$$

$$t_f^o - \bar{t} = \bar{x}_2^i$$

$$\bar{t} - t_i = \bar{x}_2^i - x_2^i$$

$$\boxed{t_f^o - t_i = 2\bar{x}_2^i - x_2^i}$$

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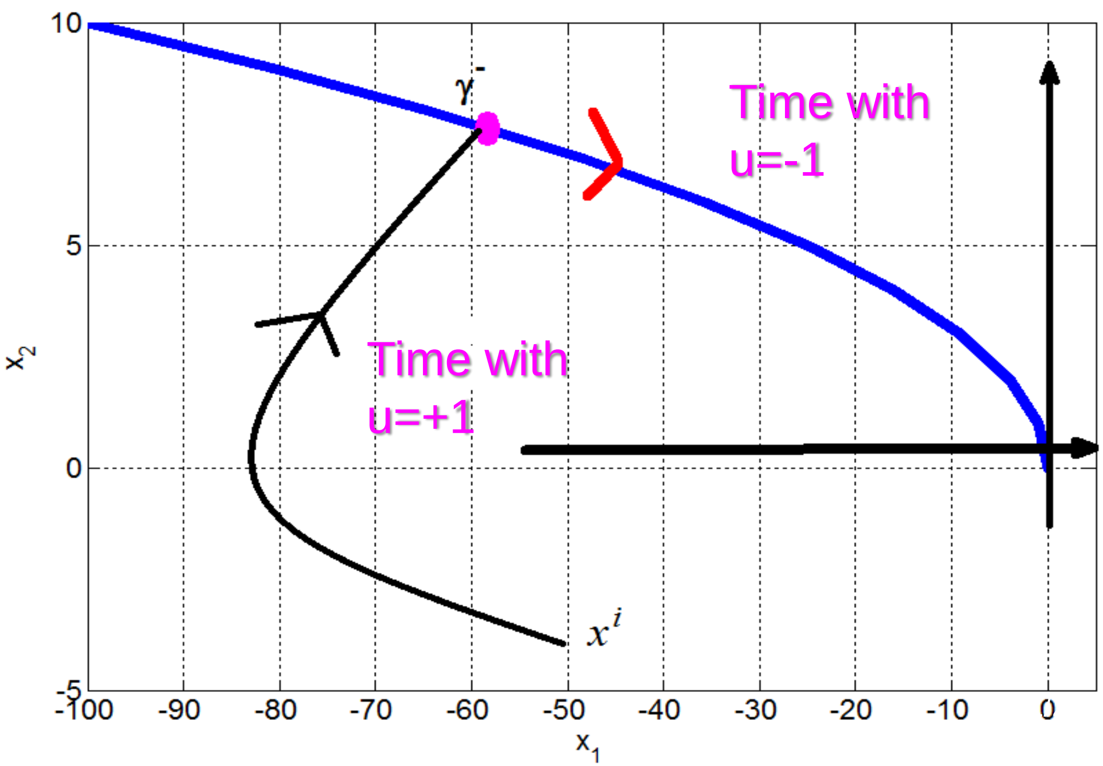

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$$\bar{t} - t_i = \bar{x}_2^i - x_2^i$$

$$\boxed{t_f^o - t_i = 2\bar{x}_2^i - x_2^i}$$

The position $\bar{x}_2^i$ must belong to the two parabolic arcs

$$x_1(t) - x_1^i = \pm \frac{1}{2}\left[x_2^2(t) - x_2^{i2}\right]$$

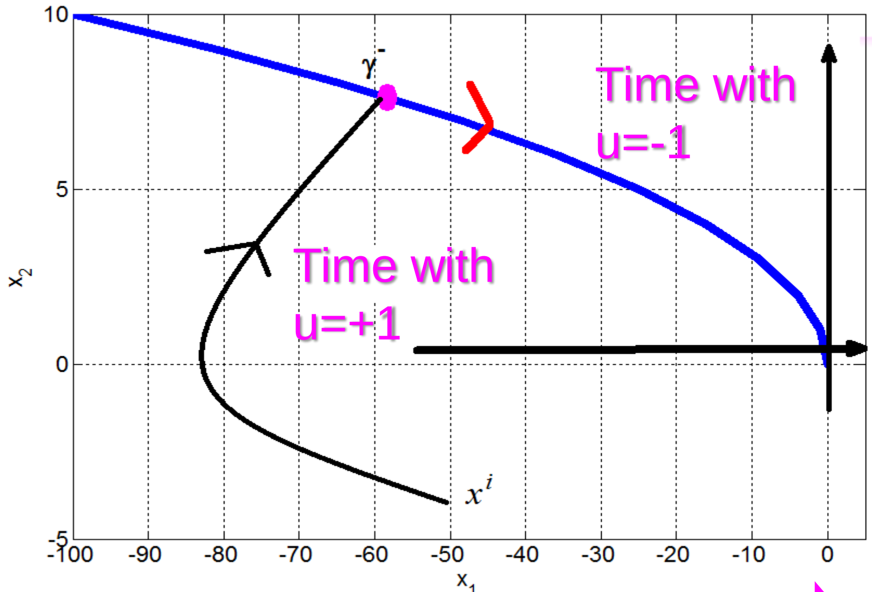
$$\bar{x}_1 = x_1^i + \frac{1}{2}\bar{x}_2^2 - \frac{1}{2}x_2^{i2}$$

$$\bar{x}_1 = -\frac{1}{2}\bar{x}_2^2$$

$$x_1^i + \frac{1}{2}\bar{x}_2^2 = -\frac{1}{2}\bar{x}_2^2 + \frac{1}{2}x_2^{i2}$$

$$\bar{x}_2^2 = -x_1^i + \frac{1}{2}x_2^{i2}, \quad x_2 > 0$$

$$\bar{x}_2 = \sqrt{-x_1^i + \frac{1}{2}x_2^{i2}}$$



By substituting in

$$\boxed{t_f^o - t_i = 2\bar{x}_2^i - x_2^i}$$

$$\boxed{t_f^o - t_i = \sqrt{-4x_1^i + 2x_2^{i2}} - x_2^i}$$

C) $\boxed{x^i \in \Gamma^-}$

The same calculations yield:

$$\boxed{t_f^o - t_i = \sqrt{4x_1^i + 2x_2^{i2}} + x_2^i}$$

The commutation curve is the γ-curve

$$\boxed{\varphi(x) = x_1 + \frac{1}{2} x_2 |x_2|}$$

$$\boxed{u^o(t) = -sign\{\varphi(x)\} = -sign\left\{x_1 + \frac{1}{2} x_2 |x_2|\right\}}$$

Harmonic oscillator (from Bruni et al.1993)

$$\dot{x}_1(t) = \omega x_2(t)$$
$$\dot{x}_2(t) = -\omega x_1(t) + u(t) \qquad \omega > 0$$

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenvalues: $\pm j\omega$

When u=0 $\quad \ddot{x}_1(t) + \omega^2 x_1(t) = 0$

$$\ddot{x}_2(t) + \omega^2 x_2(t) = 0$$

The natural modes are oscillatory

Problem

Determine a control such that

$$x(t_i) = x_i \quad x(t_f) = 0$$

$$|u(t)| \le 1$$

That minimizes the cost index: $J(t_f) = t_f - t_i$

- $\left(b \quad Ab\right) = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \Rightarrow \text{not} \quad \text{singular}$
- Eigenvalues of *A*: $\pm j\omega$

There exists a unique, non singular solution $\left(x^o, u^o, t_f^o\right)$
and the control is bang-bang for any initial condition

BUT

Since the eigenvalues are complex **we can't apply the theorem about the maximum number of commutation points**

Alternative procedure:

Introduce the Hamiltonian:

$$\boxed{H = 1 + \lambda_1(t)\omega x_2(t) - \lambda_2(t)\omega x_1(t) + \lambda_2(t)u(t)}$$

Necessary conditions:

$$\dot{\lambda}^o(t) = -A^T\lambda^o(t)$$

$$1 + \lambda_1^o(t)\omega x_2^o(t) - \lambda_2^o(t)\omega x_1^o(t) + \lambda_2^o(t)u^o(t) \le$$

$$1 + \lambda_1^o(t)\omega x_2^o(t) - \lambda_2^o(t)\omega x_1^o(t) + \lambda_2^o(t)\upsilon(t),$$

$$\forall \upsilon : |\upsilon(t)| \le 1$$

$$\dot{\lambda}^o(t) = -A^T \lambda^o(t)$$

$$1 + \lambda_1^o(t)\omega x_2^o(t) - \lambda_2^o(t)\omega x_1^o(t) + \lambda_2^o(t)u^o(t) \le$$
$$1 + \lambda_1^o(t)\omega x_2^o(t) - \lambda_2^o(t)\omega x_1^o(t) + \lambda_2^o(t)\upsilon(t),$$
$$\forall \upsilon : |\upsilon(t)| \le 1$$

$$\boxed{\begin{aligned} \dot{\lambda}_1^o(t) &= \omega \lambda_2^o(t) \\ \dot{\lambda}_2^o(t) &= -\omega \lambda_1^o(t) \end{aligned}}$$

$$\boxed{\lambda_2^o(t)u^o(t) \le \lambda_2^o(t)\upsilon(t), \quad \forall \upsilon : |\upsilon(t)| \le 1}$$

$$\boxed{\ddot{\lambda}_2^o(t) = -\omega^2 \lambda_2^o(t) \quad \Rightarrow \quad \lambda_2^o(t) = K \sin\left(\omega(t - t_i) + \alpha\right)}$$

$$\boxed{\begin{aligned} u^o(t) &= -sign\left\{\lambda^o(t)b\right\} = -sign\left\{\lambda_2^o(t)\right\} \\ &= -sign\left\{K \sin\left(\omega(t - t_i) + \alpha\right)\right\} \end{aligned}}$$

All the switching subintervals have length equal to $\dfrac{\pi}{\omega}$

with the exception of the first and the last whose length is less or equal than $\dfrac{\pi}{\omega}$

$$\dot{\lambda}^o(t) = -A^T \lambda^o(t)$$

$$1 + \lambda_1^o(t)\omega x_2^o(t) - \lambda_2^o(t)\omega x_1^o(t) + \lambda_2^o(t)u^o(t) \le$$
$$1 + \lambda_1^o(t)\omega x_2^o(t) - \lambda_2^o(t)\omega x_1^o(t) + \lambda_2^o(t)\upsilon(t),$$
$$\forall \upsilon : |\upsilon(t)| \le 1$$

$$\dot{\lambda}_1^o(t) = \omega\lambda_2^o(t)$$
$$\dot{\lambda}_2^o(t) = -\omega\lambda_1^o(t)$$

$$\lambda_2^o(t)u^o(t) \le \lambda_2^o(t)\upsilon(t), \quad \forall \upsilon : |\upsilon(t)| \le 1$$

$$\ddot{\lambda}_2^o(t) = -\omega^2 \lambda_2^o(t) \quad \Rightarrow \quad \lambda_2^o(t) = K \sin\left(\omega(t - t_i) + \alpha\right)$$

$$u^o(t) = -sign\left\{\lambda^o(t)b\right\} = -sign\left\{\lambda_2^o(t)\right\}$$
$$= -sign\left\{K \sin\left(\omega(t - t_i) + \alpha\right)\right\}$$

All the switching subintervals have length equal to $\frac{\pi}{\omega}$

with the exception of the first and the last whose length is less or equal than $\frac{\pi}{\omega}$

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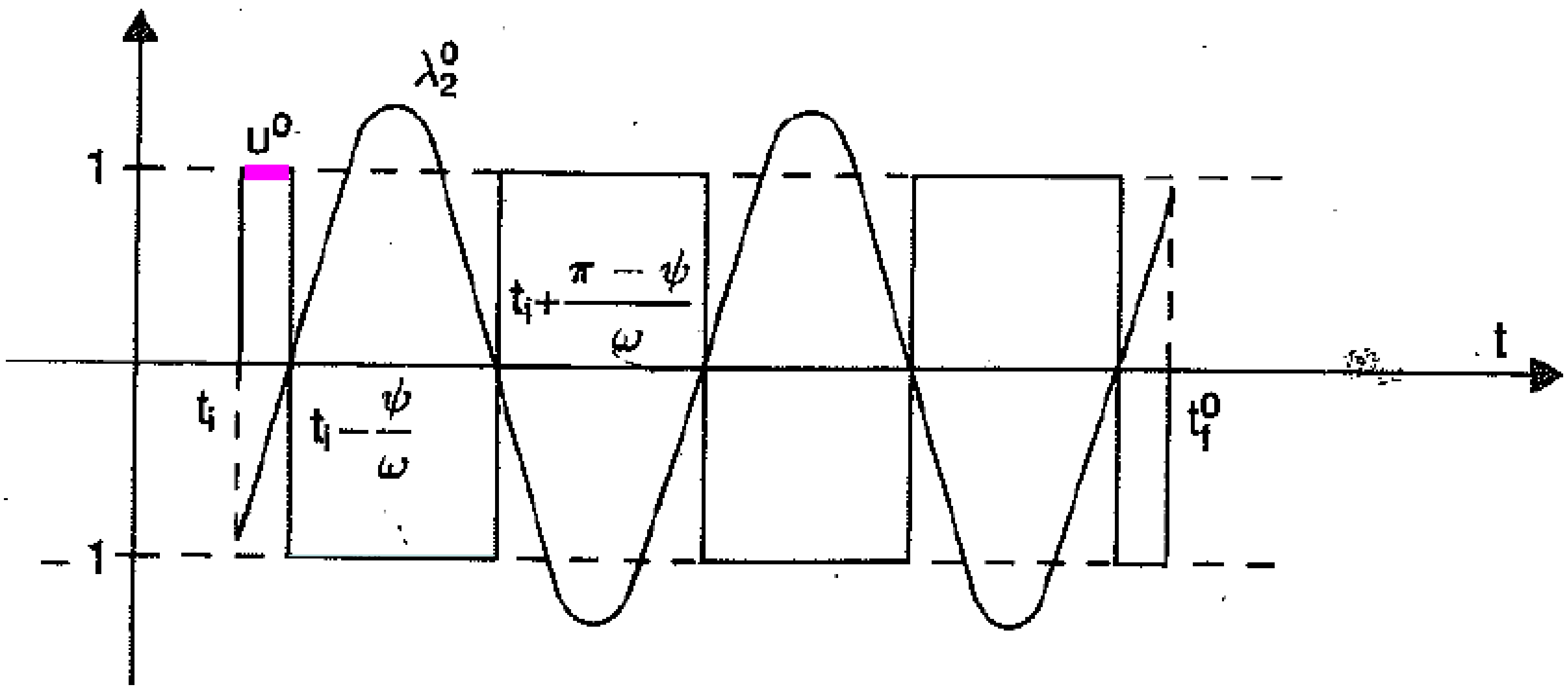

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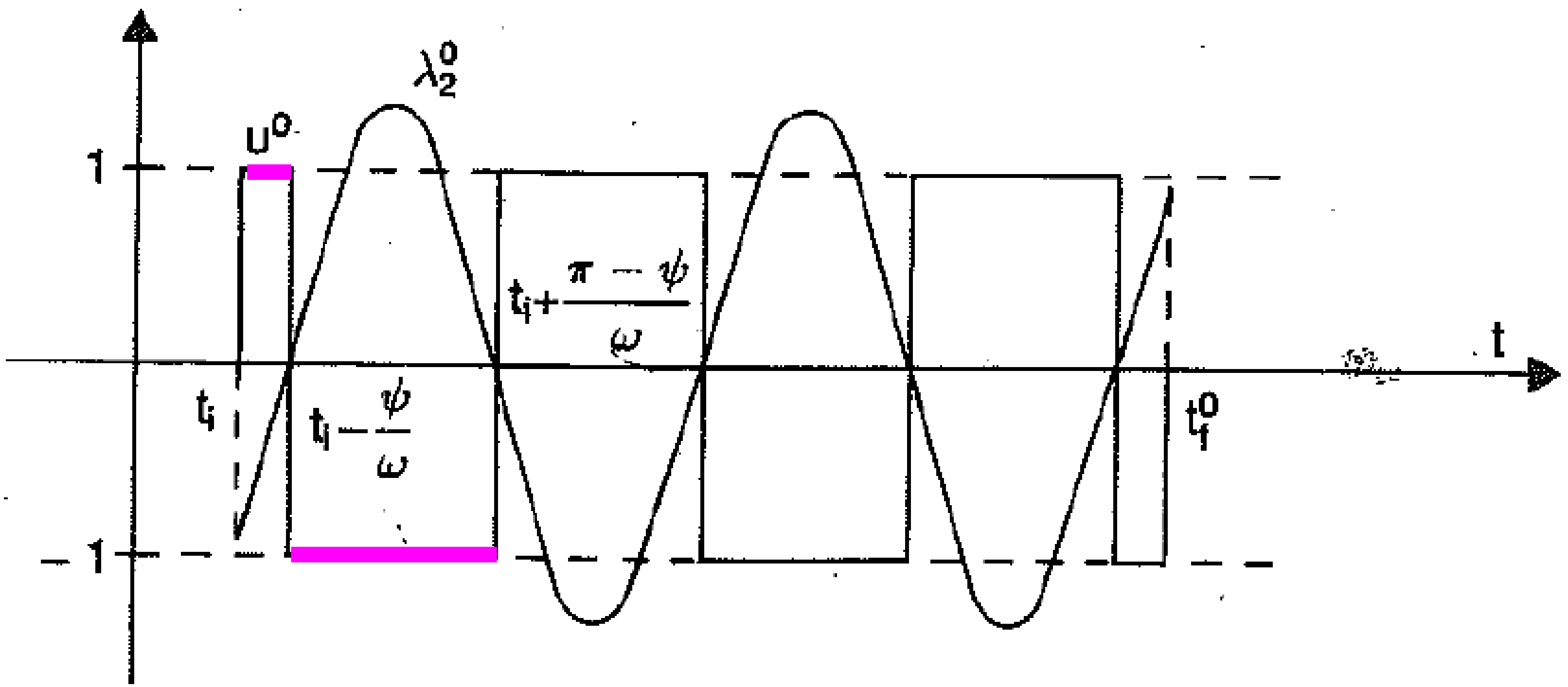

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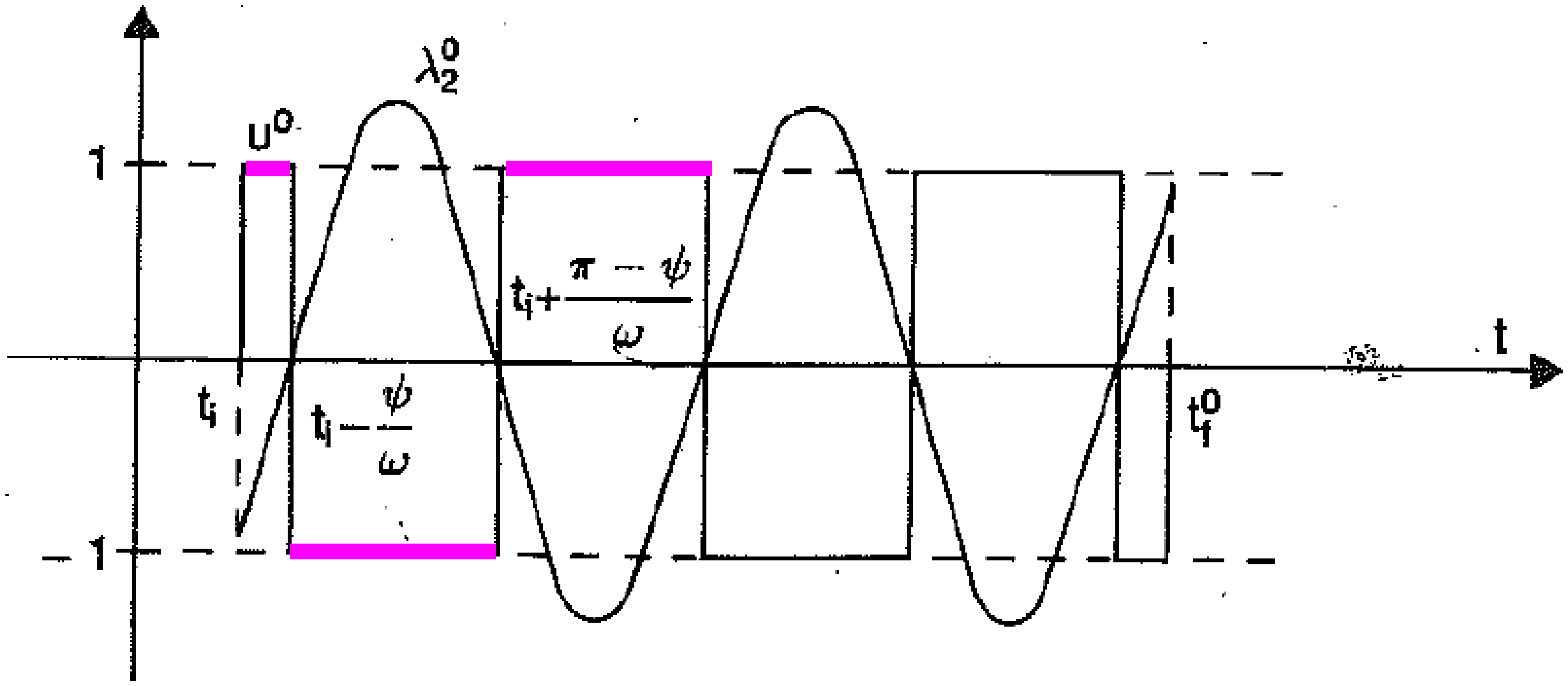

Prof.Daniela Iacoviello- Optimal Control

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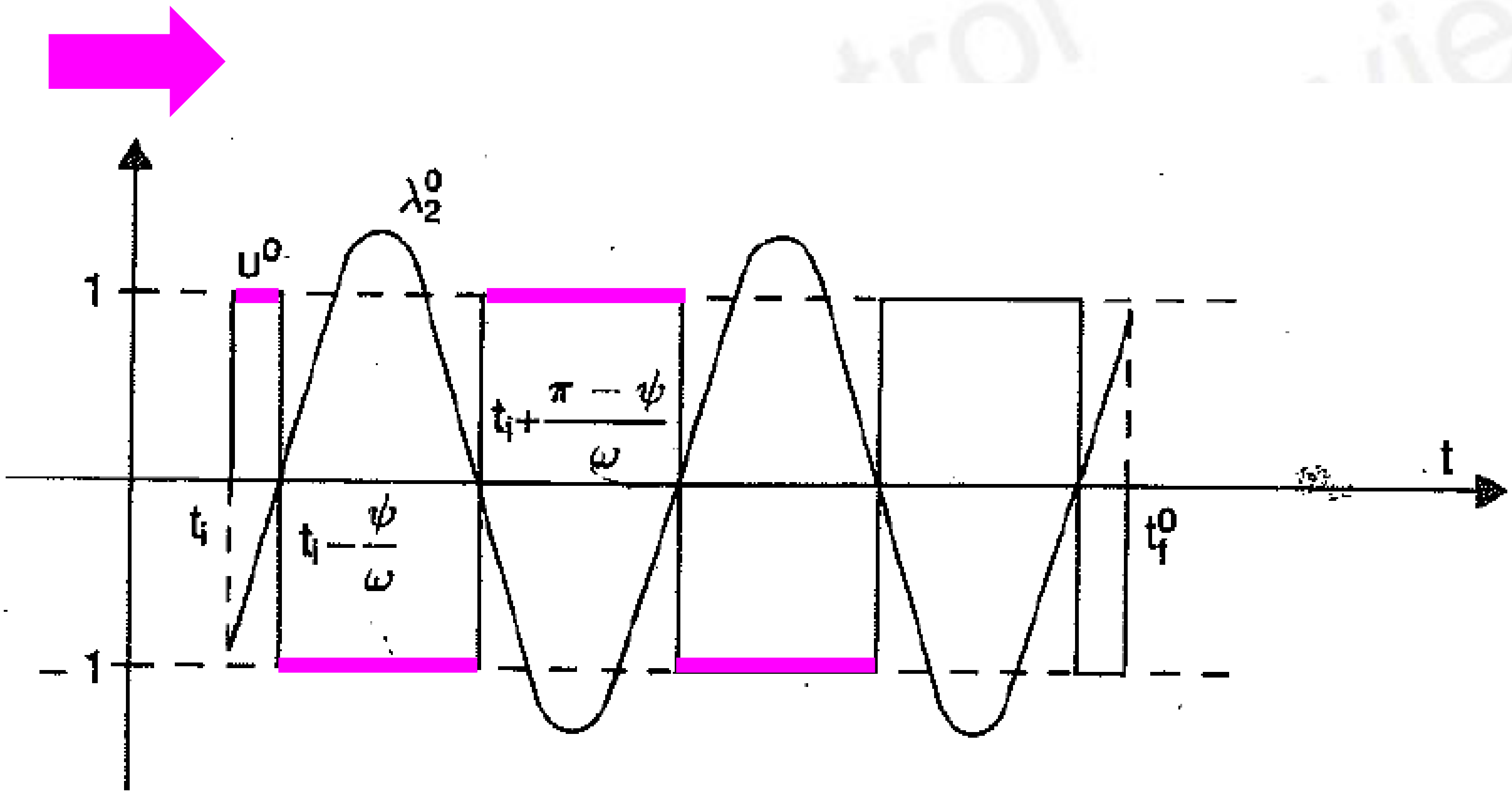

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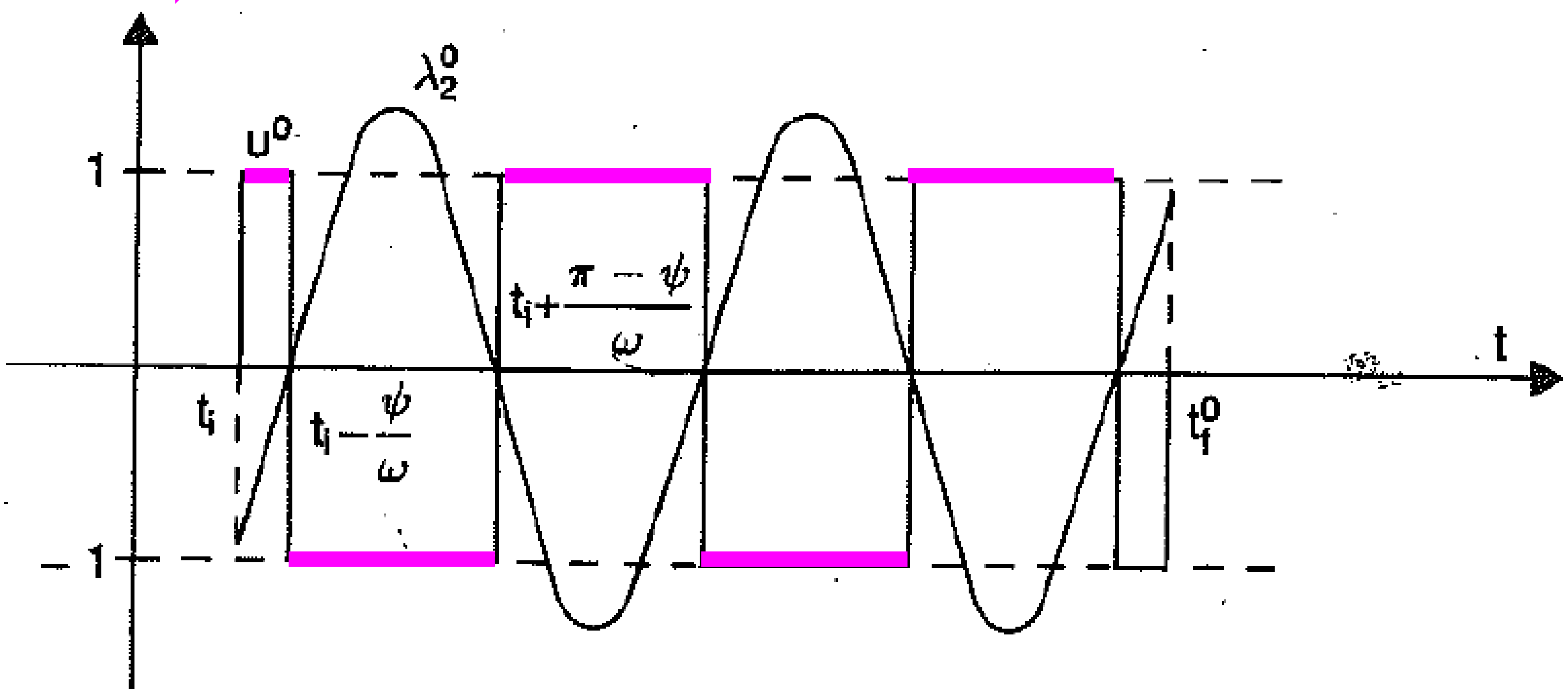

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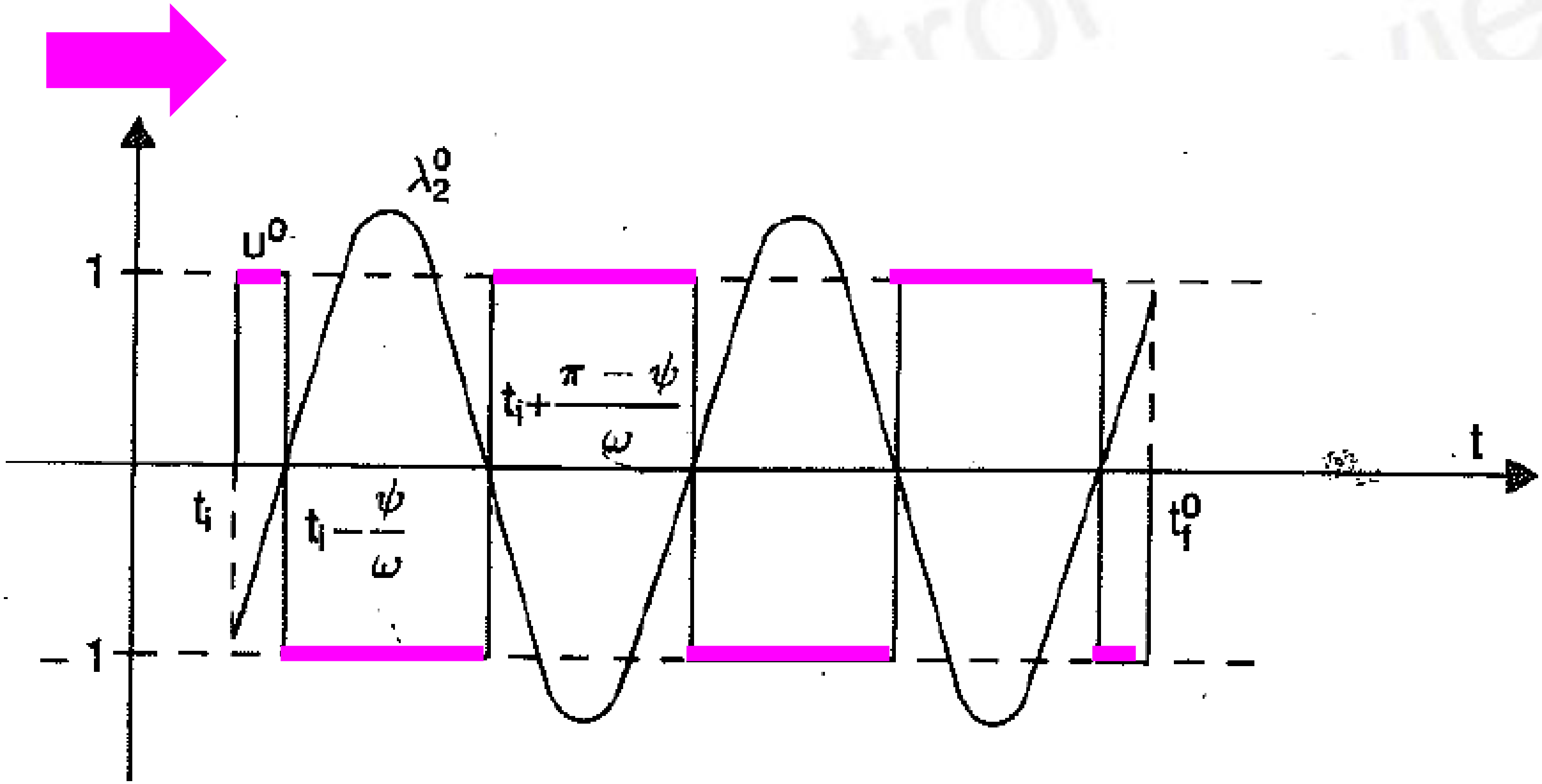


Let's assume $\boxed{u(t) = \pm 1}$

And integrate the system:

$$\boxed{\begin{aligned} &\dot{x}_1(t) = \omega x_2(t) \\ &\dot{x}_2(t) = -\omega x_1(t) + u(t) \qquad \omega > 0 \end{aligned}}$$

$$x_1(t) = \left(x_1^i \mp \frac{1}{\omega} \right) \cos \omega (t - t_i) + x_2^i \sin \omega (t - t_i) \pm \frac{1}{\omega}$$

$$x_2(t) = \left(x_1^i \mp \frac{1}{\omega} \right) \sin \omega (t - t_i) + x_2^i \cos \omega (t - t_i)$$

$$\boxed{\left(x_1(t) \mp \frac{1}{\omega} \right)^2 + x_2^2(t) = \left(x_1^i \mp \frac{1}{\omega} \right)^2 + \left(x_2^i \right)^2}$$

$$\left(x_1(t) \mp \frac{1}{\omega}\right)^2 + x_2^2(t) = \left(x_1^i \mp \frac{1}{\omega}\right)^2 + \left(x_2^i\right)^2$$

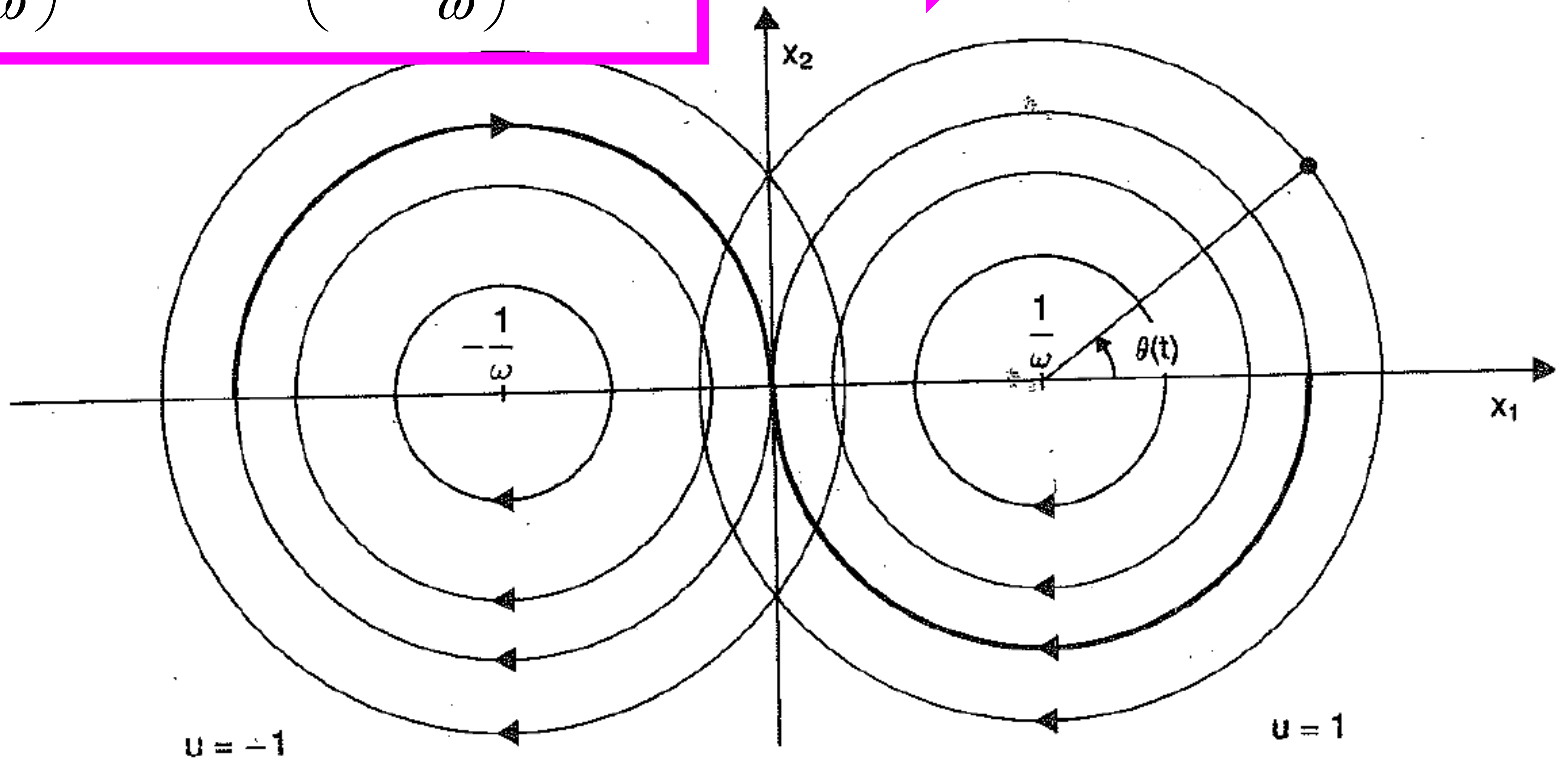
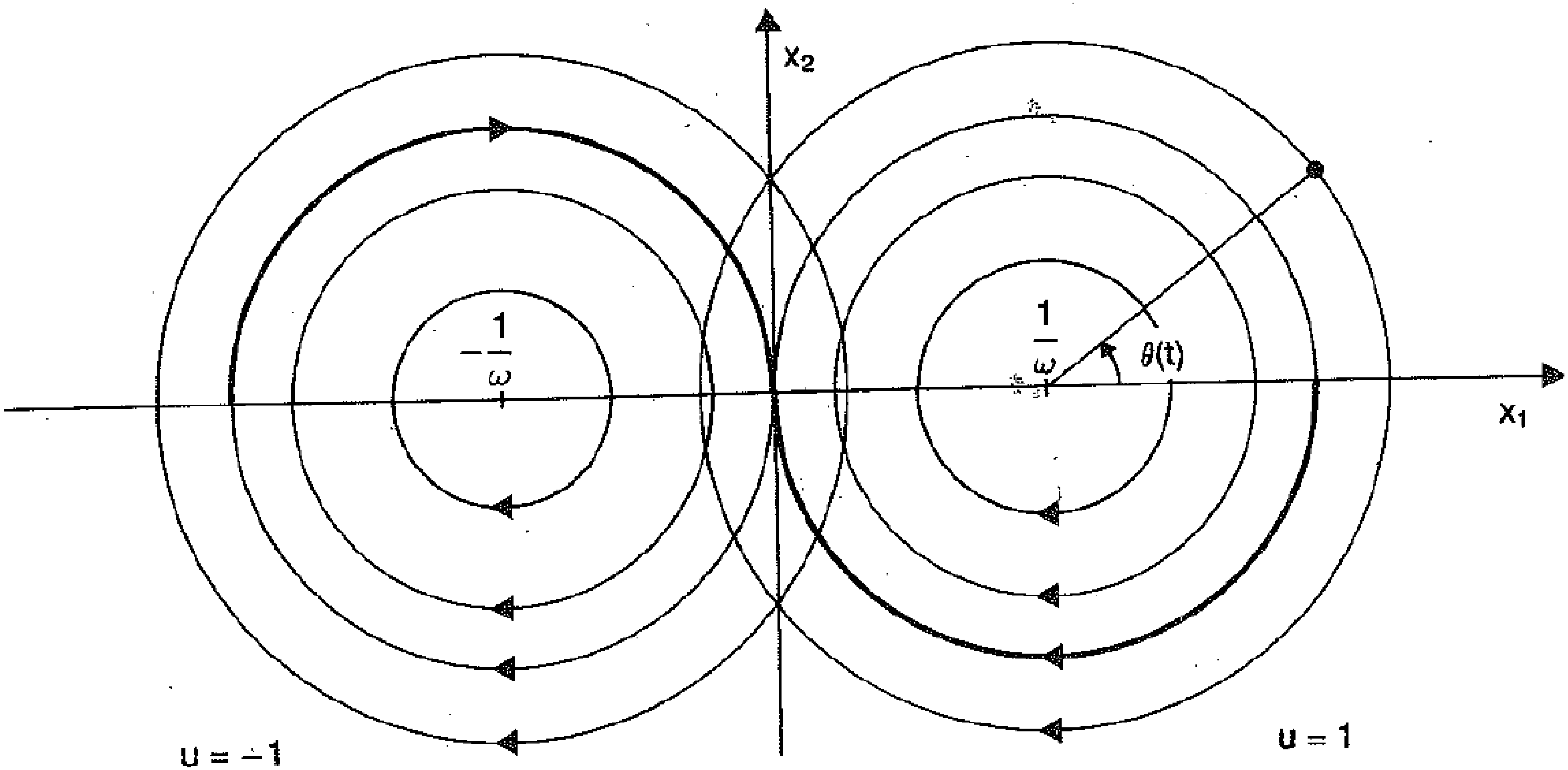


Figura 2. (Bruni et al.1993)

The trajectories with control $u(t) = \pm 1$ are circumferences with center

$\left(\pm\frac{1}{\omega}, 0\right)$ passing through the initial condition x^i

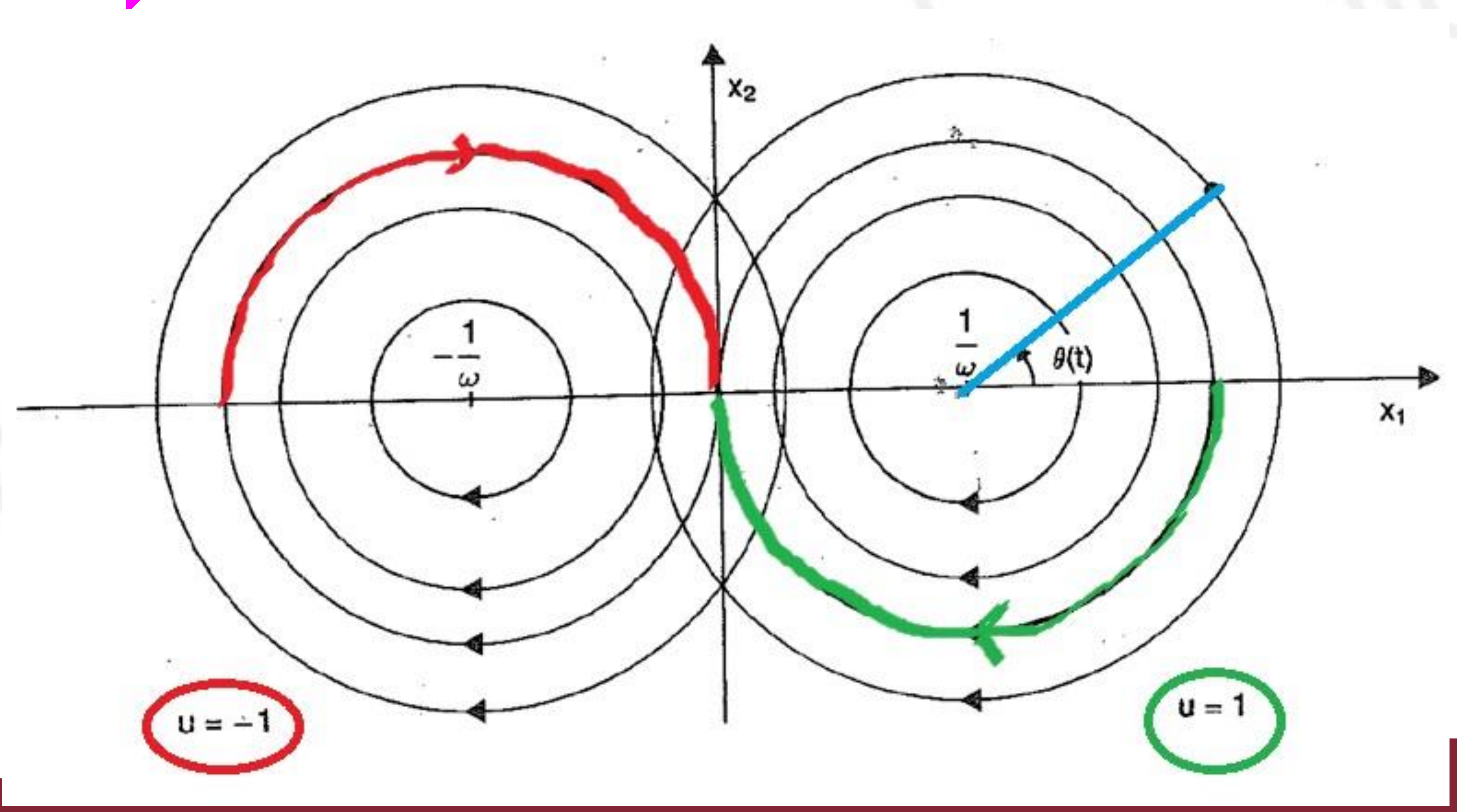
The direction is clockwise.



From Bruni et al. 2003

Only one trajectory for each of these families passes through the origin

$$\left(x_1(t) \mp \frac{1}{\omega}\right)^2 + x_2^2(t) = \frac{1}{\omega}$$



For a generic instant t:

$$\boxed{\vartheta(t) = tg^{-1}\left(\frac{x_2(t)}{x_1(t) \mp \dfrac{1}{\omega}}\right)}$$

$$\dot{\vartheta}(t) = \frac{1}{1 + \dfrac{x_2^2(t)}{\left(x_1(t) \mp \dfrac{1}{\omega}\right)^2}} \frac{\dot{x}_2(t)\left(x_1(t) \mp \dfrac{1}{\omega}\right) - x_2(t)\dot{x}_1(t)}{\left(x_1(t) \mp \dfrac{1}{\omega}\right)^2}$$

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$$\dot{\vartheta}(t) = \cdots = -\omega$$

$$\dot{x}_1(t) = \omega x_2(t)$$
$$\dot{x}_2(t) = -\omega x_1(t) + u(t)$$
$$\omega > 0$$

The trajectories are traversed with constant angular velocity

The trajectories are traversed with constant angular velocity

The time interval needed to move on an arc of length β

with constant control $u(t) = \pm 1$

is given by $\boxed{\Delta t = \dfrac{\beta}{\omega}}$

$$\boxed{\frac{\beta}{\omega} \leq \frac{\pi}{\omega} \Rightarrow \beta \leq \pi}$$

The trajectories are traversed with constant angular velocity

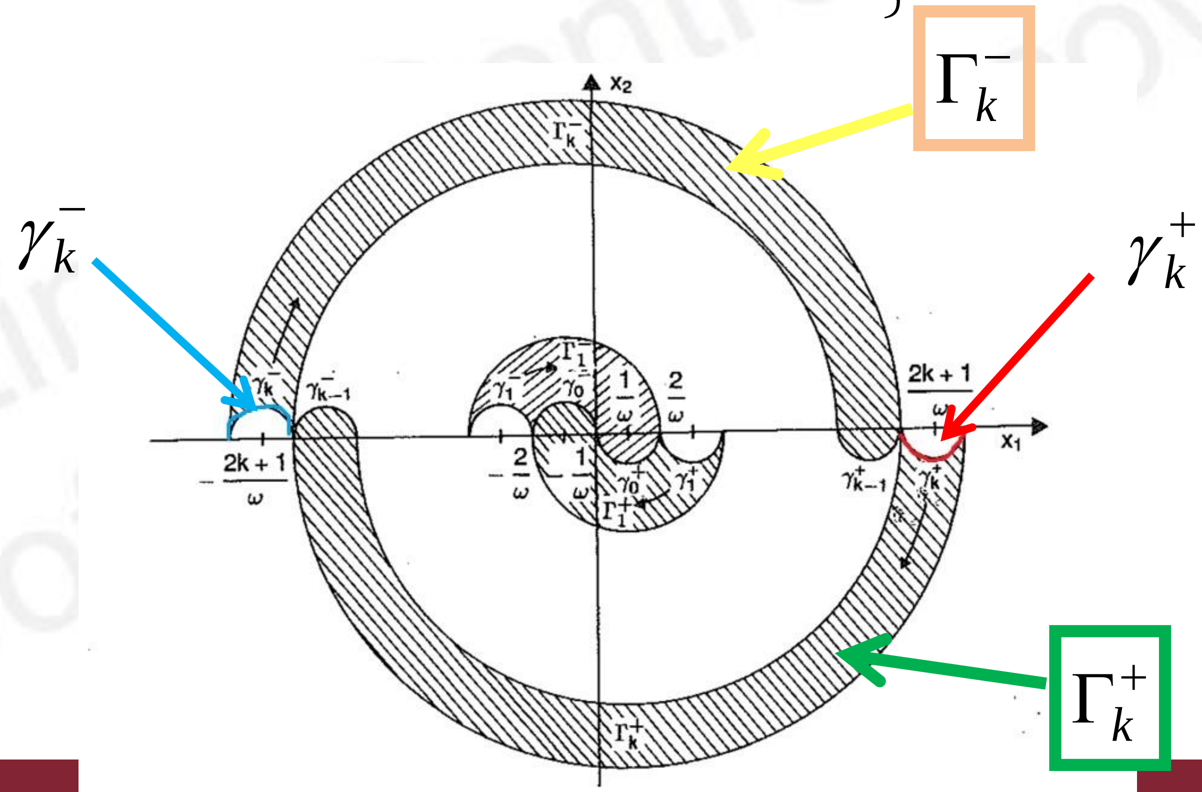
We must reach the origin along arcs with amplitude $\leq \pi$

There exists a unique way to do it!

Definitions:

$$\gamma_k^+ = \left\{ x \in R^2 : \left(x_1 - \frac{2k+1}{\omega} \right)^2 + x_2^2 = \frac{1}{\omega^2},\ x_2 \le 0 \right\}, \quad k = 0,1,2,\ldots.$$

$$\gamma_k^- = \left\{ x \in R^2 : \left(x_1 + \frac{2k+1}{\omega} \right)^2 + x_2^2 = \frac{1}{\omega^2},\ x_2 \ge 0 \right\}, \quad k = 0,1,2,\ldots$$



Let Γ_k^+ and Γ_k^- k=1,2,…., be the sets in Figure,

obtained rotating γ_k^+

and γ_k^- around

$\left(\pm\frac{1}{\omega}, 0,\right)$

respectively

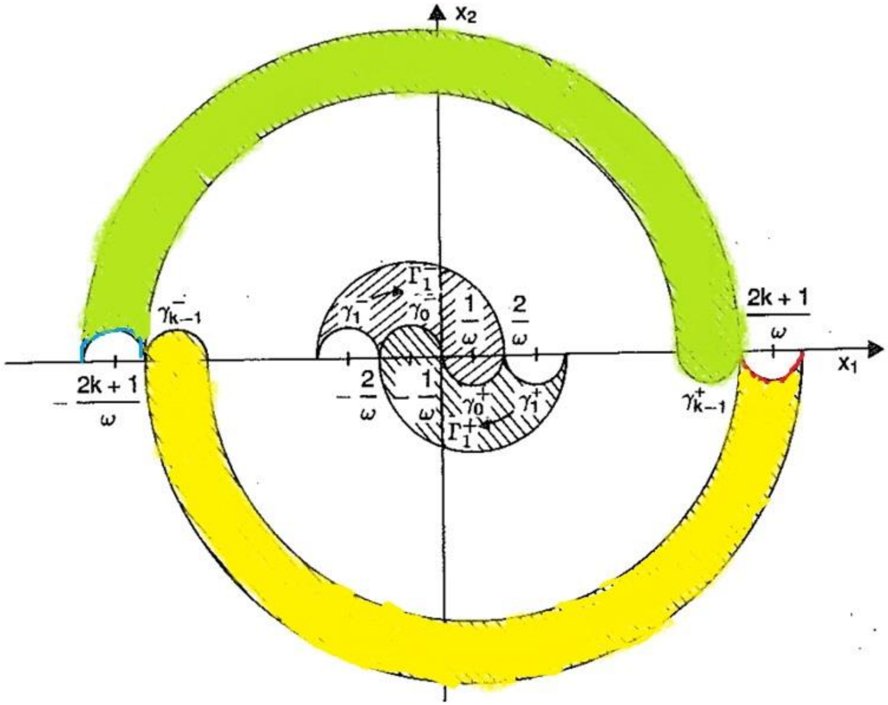


Figura 5. (from Bruni et al. 1993)

Define:

$$\gamma^+ = \bigcup_{k=0}^{\infty} \gamma_k^+ \qquad \gamma^- = \bigcup_{k=0}^{\infty} \gamma_k^- \qquad \gamma = \gamma^+ \cup \gamma^-$$

$$\Gamma^+ = \bigcup_{k=0}^{\infty} \Gamma_k^+ \qquad \Gamma^- = \bigcup_{k=0}^{\infty} \Gamma_k^-$$

The initial states $\boxed{x_i \in \gamma_0^+ \text{ and } x_i \in \gamma_0^-}$ may be moved to the origin with a

constant control equal to +1 and -1 respectively and the corresponding arc

is less or equal than π.

1a 1b

The points belonging to $\gamma_0^+ \cup \gamma_0^-$ are the only ones that can be transfered to the origin with a constant control, without switching

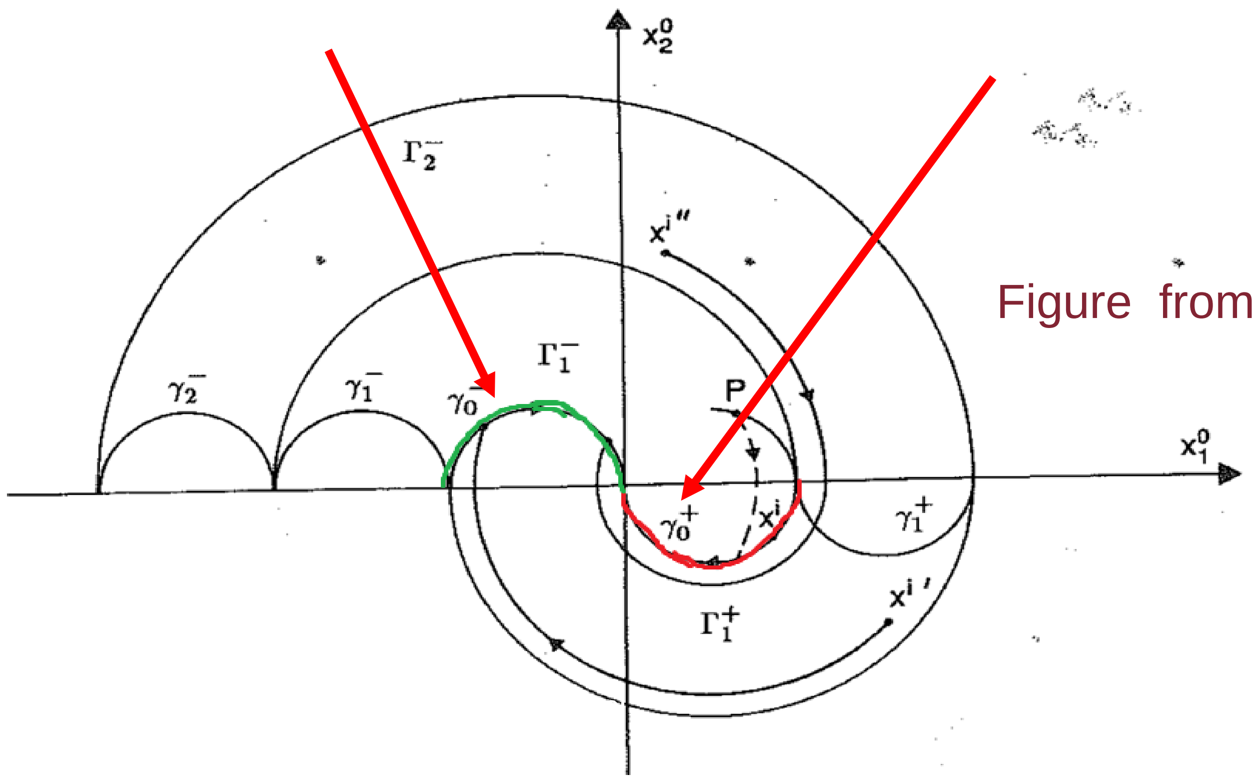


Figure from Bruni et al. 1993

Let's consider initial states $x_i \in \Gamma_1^+ \setminus \left(\gamma_0^+ \cup \gamma_0^-\right)$

2a

They may be moved to the origin with the following path:

- First a constant control +1 up to the arc γ_0^- for an interval time $\leq \frac{\pi}{\omega}$
- Switching to control -1 to reach the origin.

⇒ 1 switching instant

Let’s consider initial states $x_i \in \Gamma_1^+ \setminus \left(\gamma_0^+ \cup \gamma_0^-\right)$

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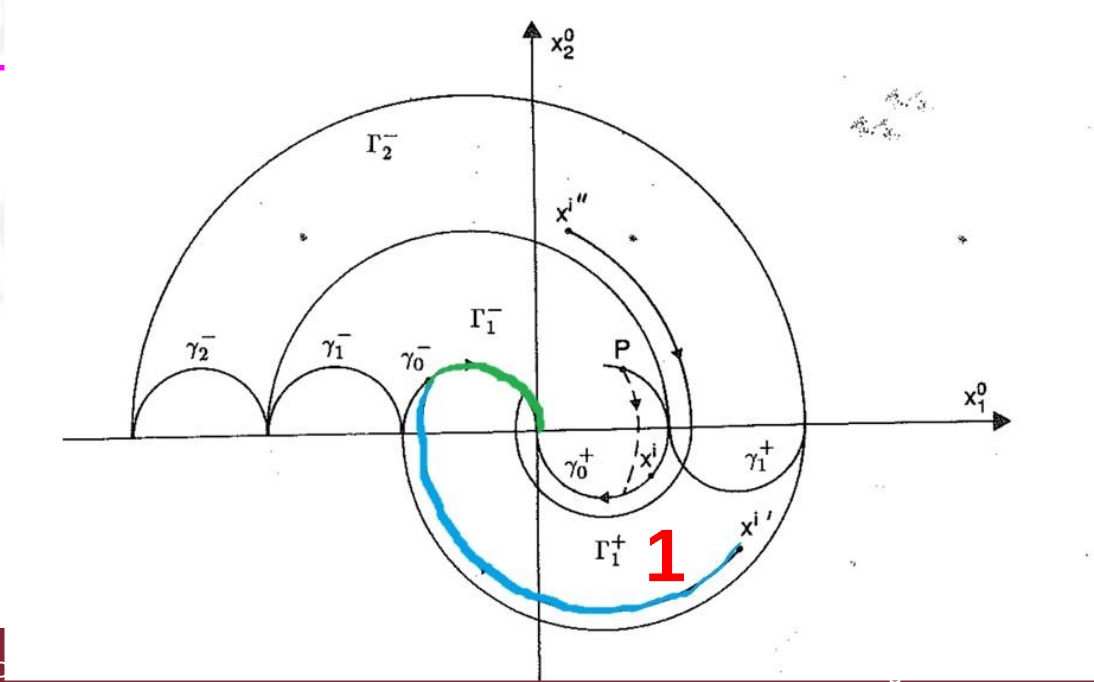

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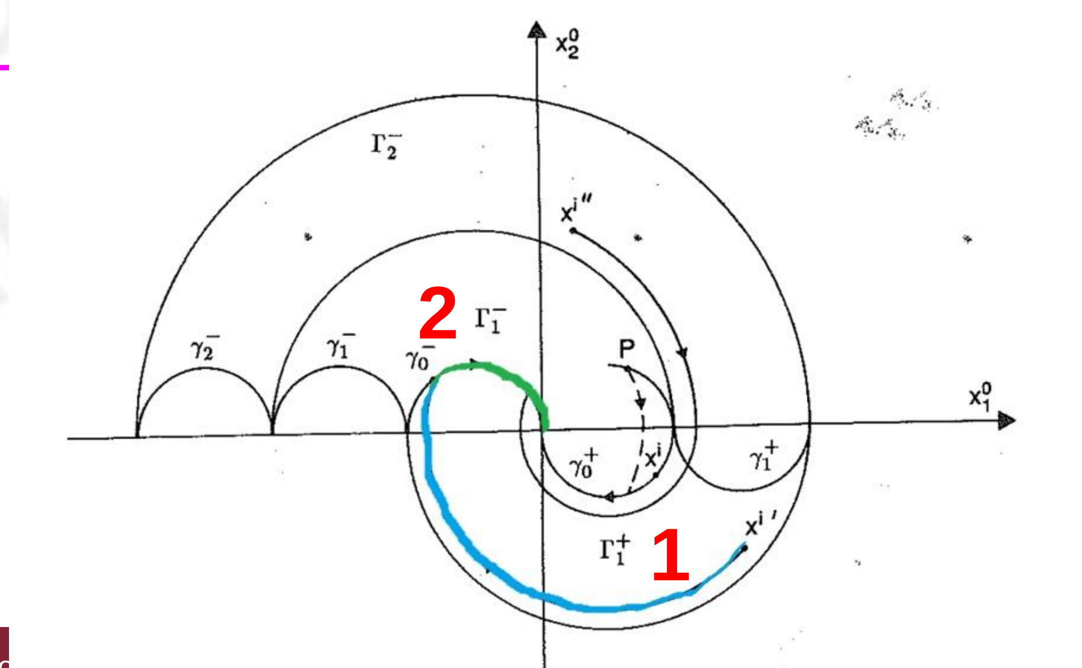
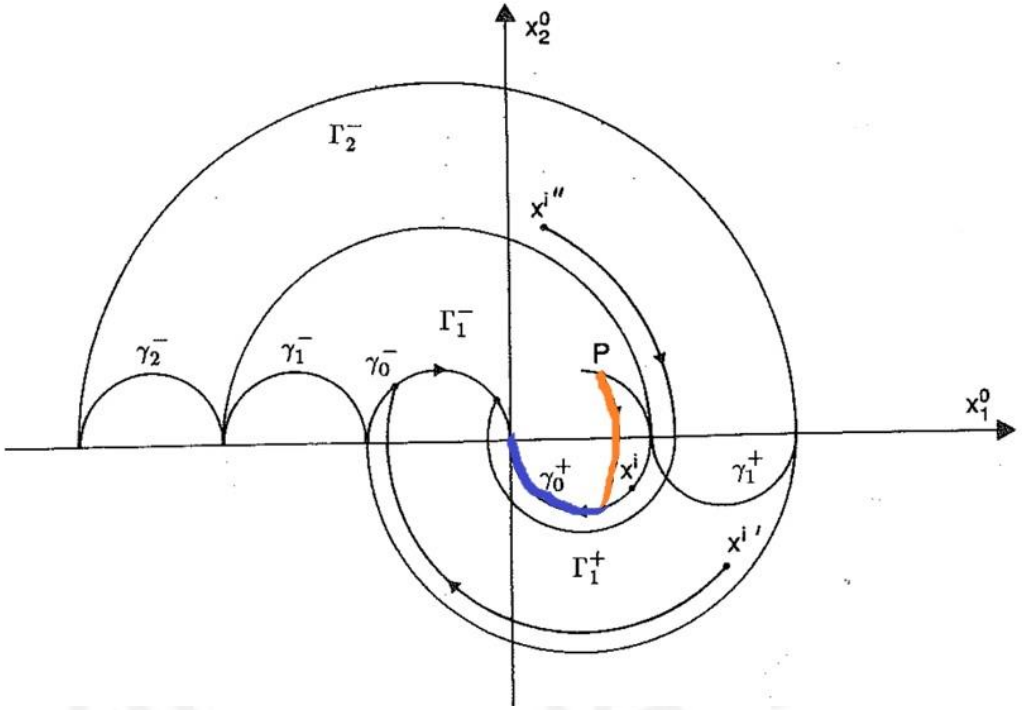




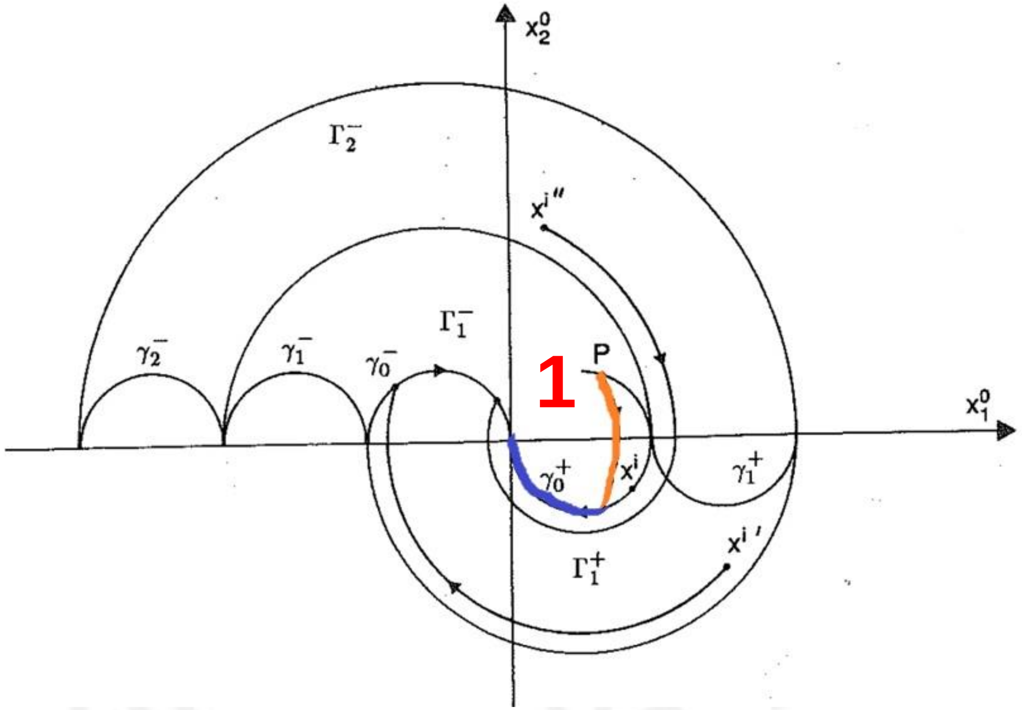
Let's consider initial states $x_i \in \Gamma_1^- \setminus \left(\gamma_0^+ \cup \gamma_0^-\right)$

2b

They may be moved to the origin with the following path:

- First a constant control -1 up to the arc γ_0^+ for an interval time $\leq \dfrac{\pi}{\omega}$
- Switching to control +1 to reach the origin.

1 switching instant



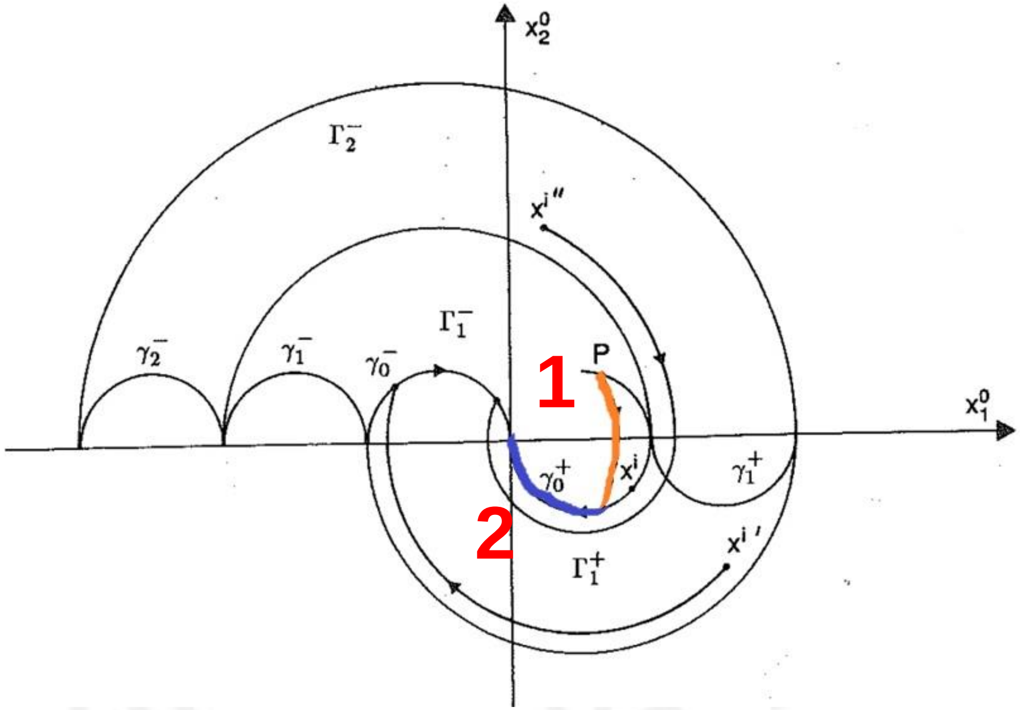
Let's consider initial states $x_i \in \Gamma_1^- \setminus \left(\gamma_0^+ \cup \gamma_0^-\right)$

2b

They may be moved to the origin with the following path:

- First a constant control -1 up to the arc γ_0^+ for an interval time $\leq \dfrac{\pi}{\omega}$
- Switching to control +1 to reach the origin.

1 switching instant



Let's consider initial states $x_i \in \Gamma_1^- \setminus \left(\gamma_0^+ \cup \gamma_0^-\right)$

2b

They may be moved to the origin with the following path:

- First a constant control -1 up to the arc γ_0^+ for an interval time $\leq \dfrac{\pi}{\omega}$
- Switching to control +1 to reach the origin.

1 switching instant

Let's consider initial states $\boxed{x_i \in \Gamma_2^+ \setminus \gamma_1^-}$

3a

They may be moved to the origin with the following path:

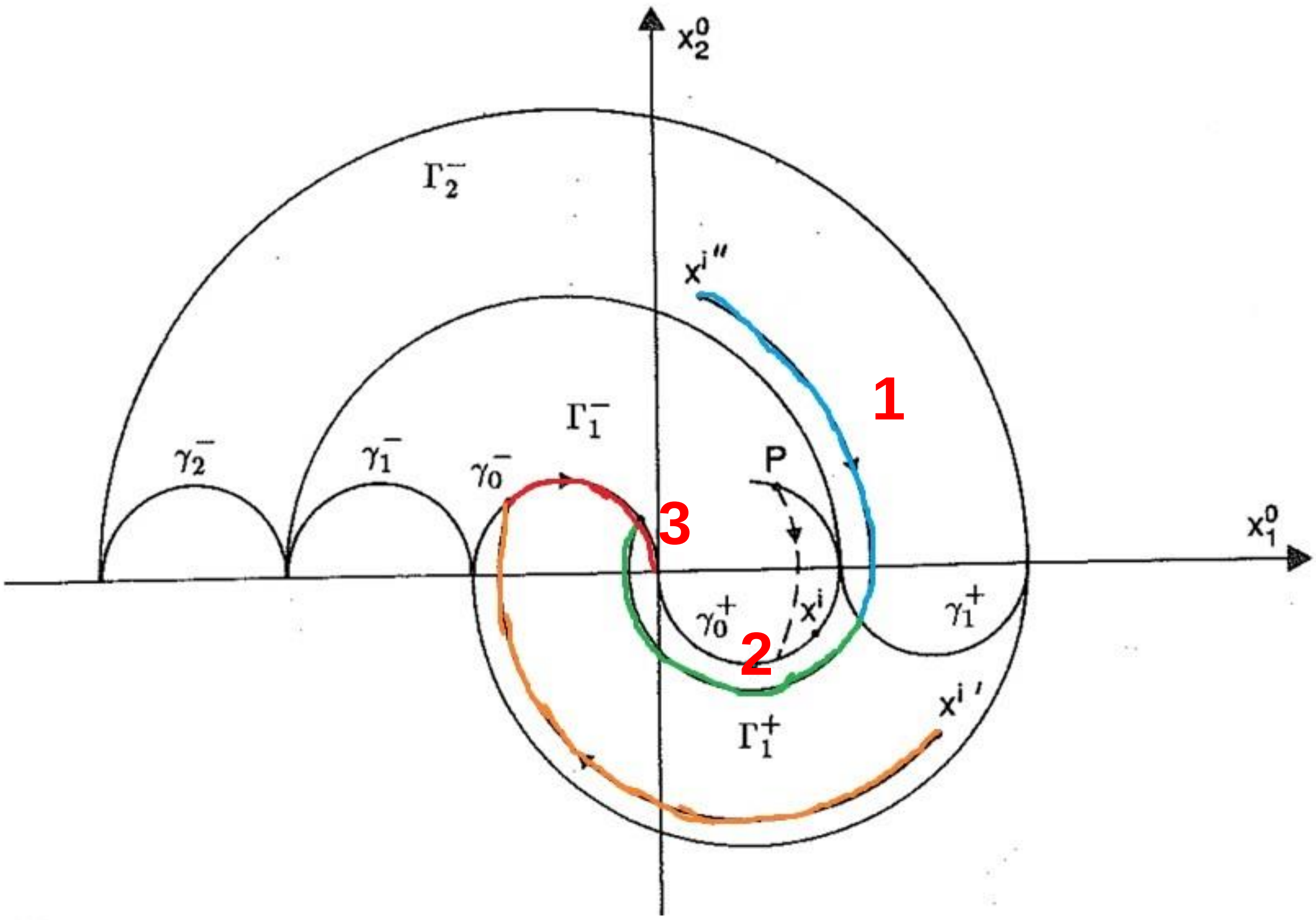
- First a constant control +1 up to the arc γ_1^- for an interval time $\leq \dfrac{\pi}{\omega}$
- Note that: $\gamma_1^- \in \Gamma_1^-$ → apply strategy **2b**
- Switch to control -1 to reach γ_0^+ and then switch to control +1 to reach the origin → two switching instants

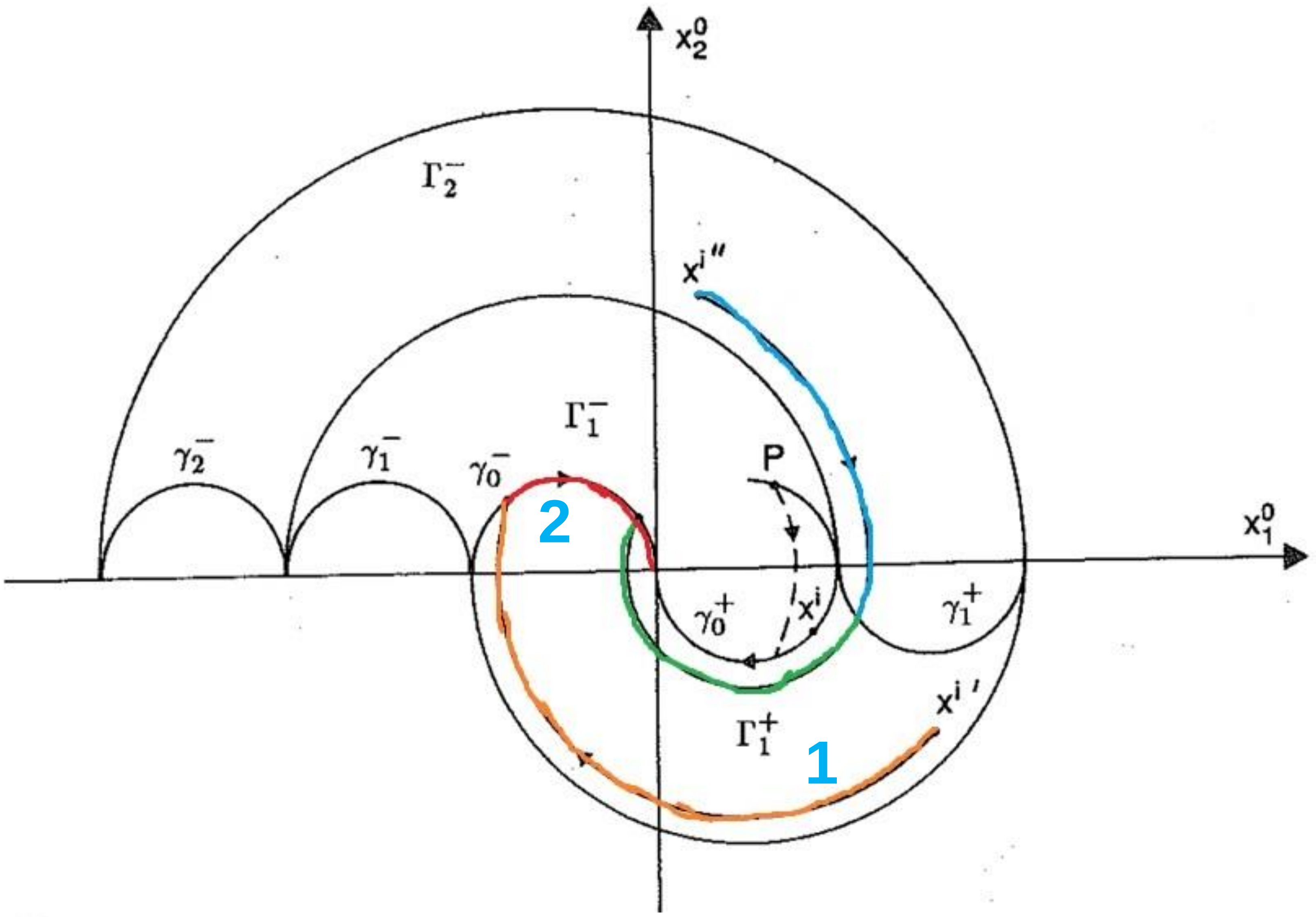
Let's consider initial states $x_i \in \Gamma_2^- \setminus \gamma_1^+$

3b

They may be moved to the origin with the following path:

- First a constant control -1 up to the arc γ_1^+ for an interval time $\leq \dfrac{\pi}{\omega}$
- Note that $\gamma_1^+ \in \Gamma_1^+$ → apply strategy **2a**
- Switch to control +1 to reach γ_0^- and then switch to control -1 to reach the origin: → two switching instants


x_2^0
Γ_2^-
$x^{i''}$
1
Γ_1^-
γ_2^-
γ_1^-
γ_0^-
P
3
x_1^0
γ_0^+
x^i
γ_1^+
2
$x^{i'}$
Γ_1^+


x_2^0
Γ_2^-
$x^{i''}$
Γ_1^-
γ_2^-
γ_1^-
γ_0^-
P
x_1^0
2
γ_0^+
x^i
γ_1^+
$x^{i'}$
Γ_1^+
1

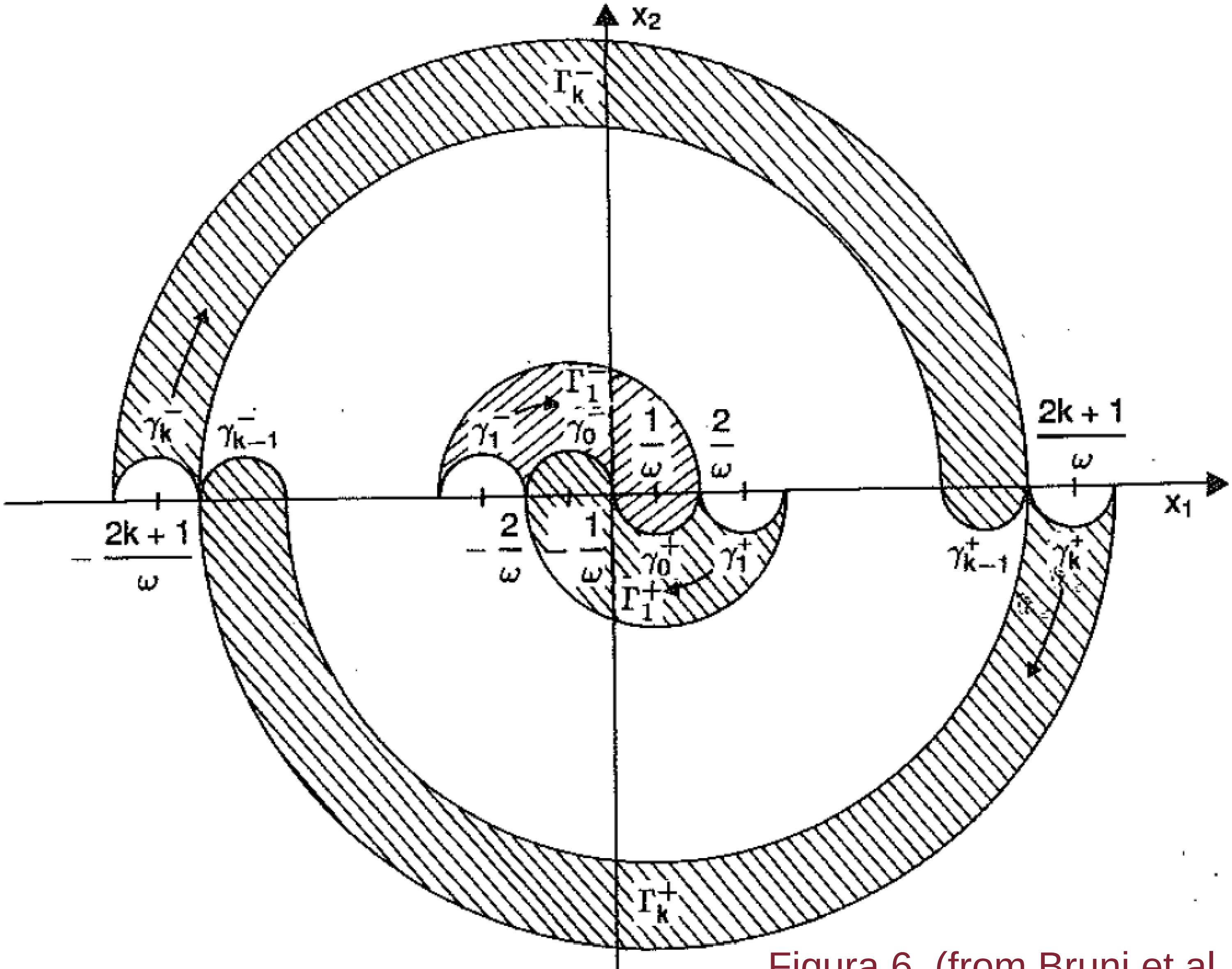
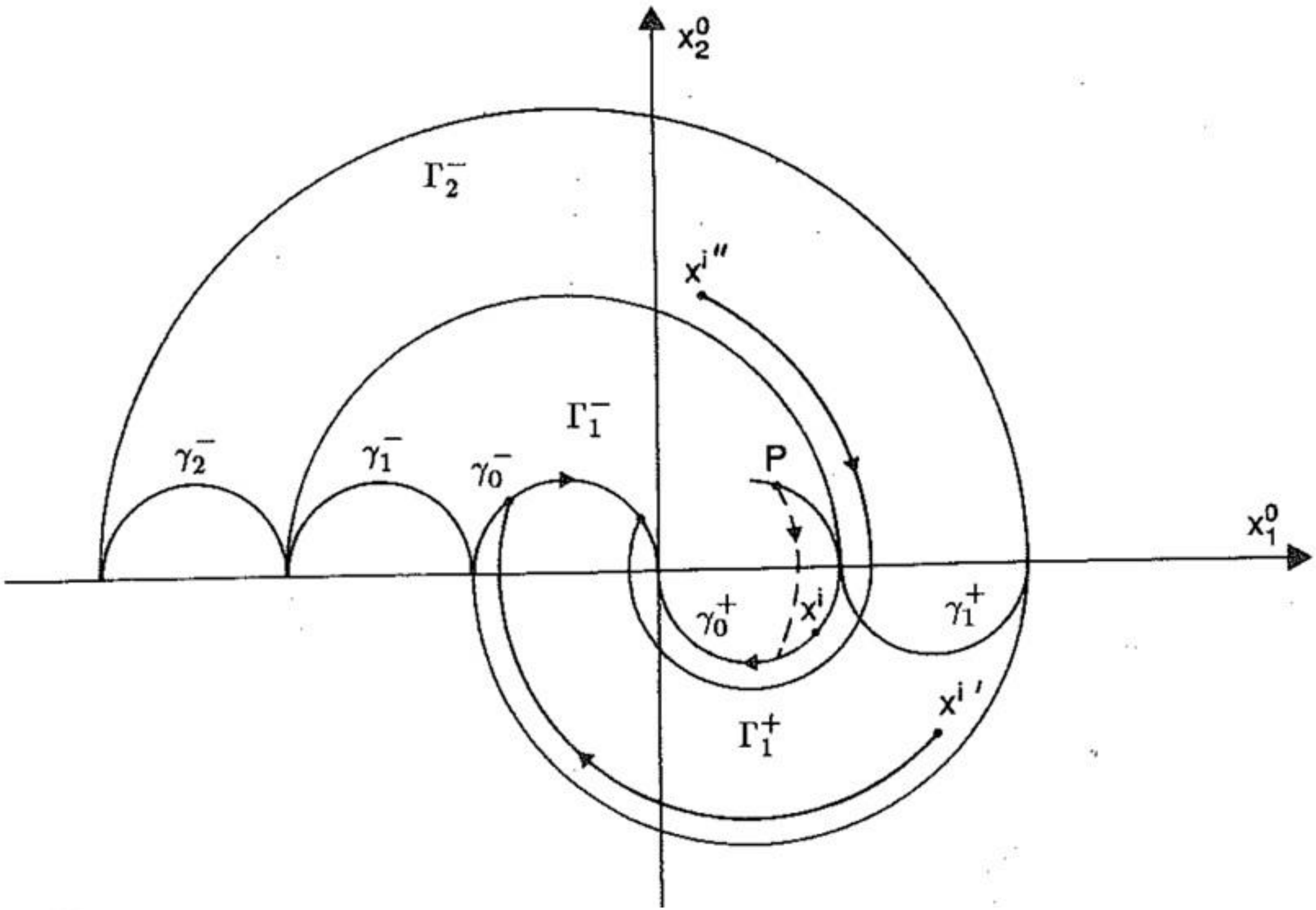


Figura 6. (from Bruni et al. 1993)



x_2^0
x_1^0
Γ_2^-
Γ_1^-
Γ_1^+
γ_2^-
γ_1^-
γ_0^-
γ_0^+
γ_1^+
P
x^i
$x^{i\,\prime}$
$x^{i\,\prime\prime}$

The same happens for any intial condition: $x_i \in \Gamma_k^+ \setminus \gamma_{k-1}^-$ $x_i \in \Gamma_k^- \setminus \gamma_{k-1}^+$

At the generic state $x_i \in \Gamma^+ \setminus \gamma^-$ you must associate u=+1

At the generic state $x_i \in \Gamma^- \setminus \gamma^+$ you must associate u=-1

$$u^o\left(x^o(t)\right) = \begin{cases} 1 & \forall x^o(t) \in \Gamma^+ \setminus \gamma^- \\ -1 & \forall x^o(t) \in \Gamma^- \setminus \gamma^+ \end{cases}$$

The number of switching points is given by the minimum index k among the ones characterizing the sets Γ_k^+ and Γ_k^-

Example: Consider the point $P \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^+$
Then there is only one switching instant.

Time necessary to get to the origin

If ν^o is the number of switching instants:

Rotation of the optimal trajectory to go from the initial point to the first point of commutation

Rotation of the optimal trajectory to go from the final point to the origin

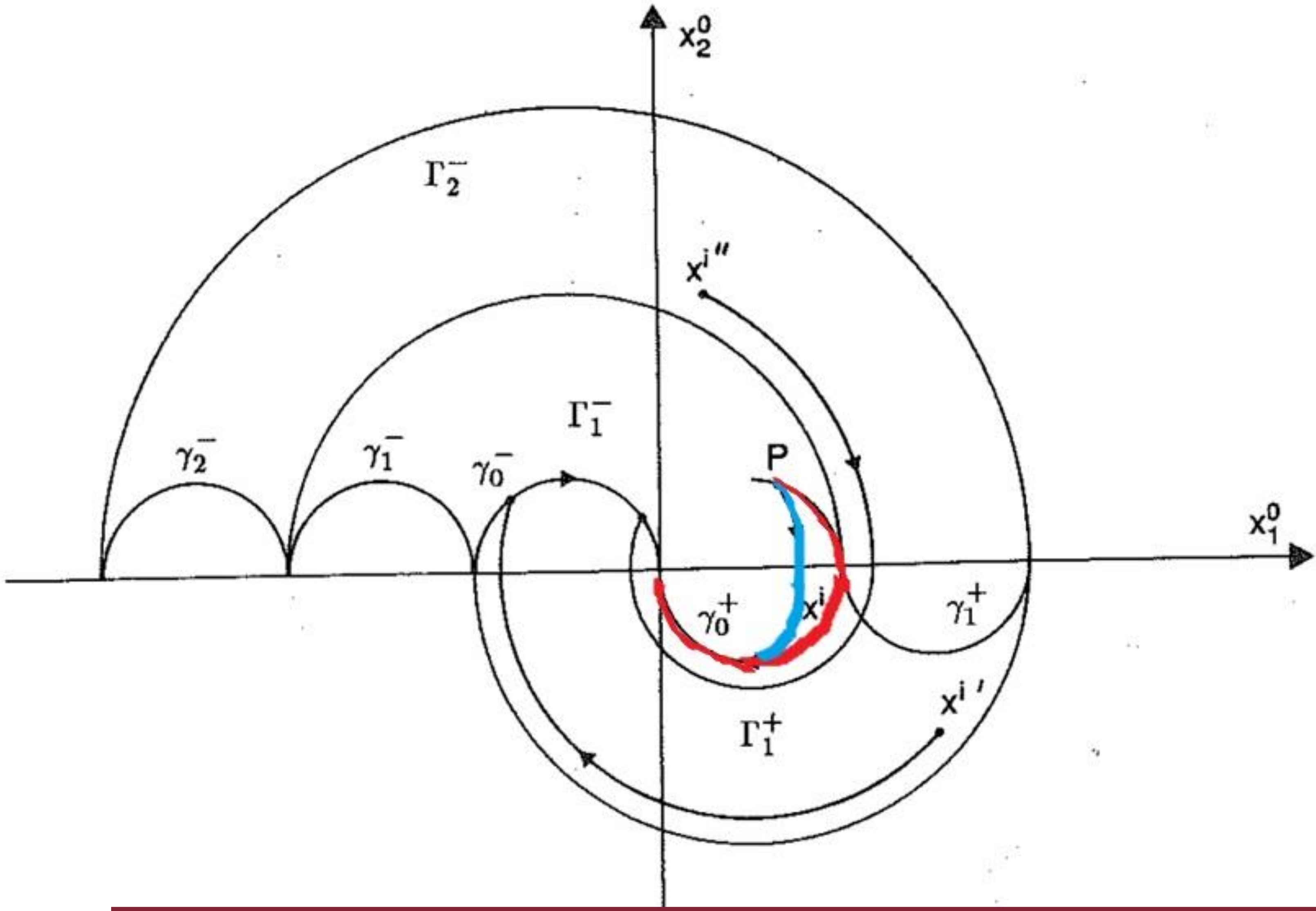
$$\text{if } \nu^o \geq 1 \qquad t_f^o - t_i = \frac{\beta_i}{\omega} + \left(\nu^o - 1\right)\frac{\pi}{\omega} + \frac{\beta_f}{\omega}$$

If $\nu^o = 0$ $\qquad t_f^o - t_i = \dfrac{\beta_i}{\omega}$

where:

if $x_i \in \gamma_o^+$ $\qquad \beta_i = tg^{-1}\left(\dfrac{\omega x_2^i}{1 - \omega x_1^i}\right)$

if $x_i \in \gamma_o^-$ $\qquad \beta_i = tg^{-1}\left(\dfrac{\omega x_2^i}{1 + \omega x_1^i}\right)$


x_2^0
Γ_2^-
$x^{i''}$
Γ_1^-
γ_2^-
γ_1^-
γ_0^-
P
x_1^0
γ_0^+
x^i
γ_1^+
$x^{i'}$
Γ_1^+