Optimal Control

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Lecture 5

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THESE SLIDES ARE NOT SUFFICIENT
FOR THE EXAM:
YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below

Course outline

- Introduction to optimal control
- Nonlinear optimization
- Dynamic programming
- Calculus of variations
- Calculus of variations and optimal control
- LQ problem
- Minimum time problem

- **Hamilton-Jacobi- Bellman equation**

William Rowan Hamilton
(4 August 1805 – 2 September 1865)

He was an Irish physicist, astronomer, and mathematician, who made important contributions to classical mechanics, optics and algebra.
His studies of mechanical and optical systems led him to discover new mathematical concepts and techniques.
His greatest contribution is perhaps the reformulation of Newtonian mechanics, now called **Hamiltonian mechanics**.

This work has proven central to the modern study of classical field theories such as electromagnetism and to the development of quantum mechanics. In mathematics, he is perhaps best known as the **inventor of quaternions.**

Jacobi (Posdam 1804- Berlin 1851)

He was still only 12 years old yet he had reached the necessary standard to enter University in 1817
Around 1825 Jacobi changed from the Jewish faith to become a Christian which now made university teaching possible for him.
By the academic year 1825-26 he was teaching at the University of Berlin.
Jacobi carried out important research in partial differential equations of the first order and applied them to the differential equations of dynamics.
He also worked on determinants and studied the functional determinant now called the Jacobian.

Richard Bellman (1920- 1984, USA)

He was interested in dynamic programming, with applications in biology and medicine.

The Hamilton-Jacobi equation

It is an approach in some sense alternative to the Minimum Principle of Pontryagin, combined with the Euler Lagrange equation.

Actually the Hamilton – Jacobi equation has so far rarely proved useful except for linear regulator problems, to which it seems particularly well suited.

The Hamilton-Jacobi equation

The Hamilton-Jacobi (H-J) equation is satisfied by the optimal performance index under suitable differentiability and continuity assumptions.

If a solution to the H-J equation has certain differentiability properties, then this solution is the *desired performance index*.

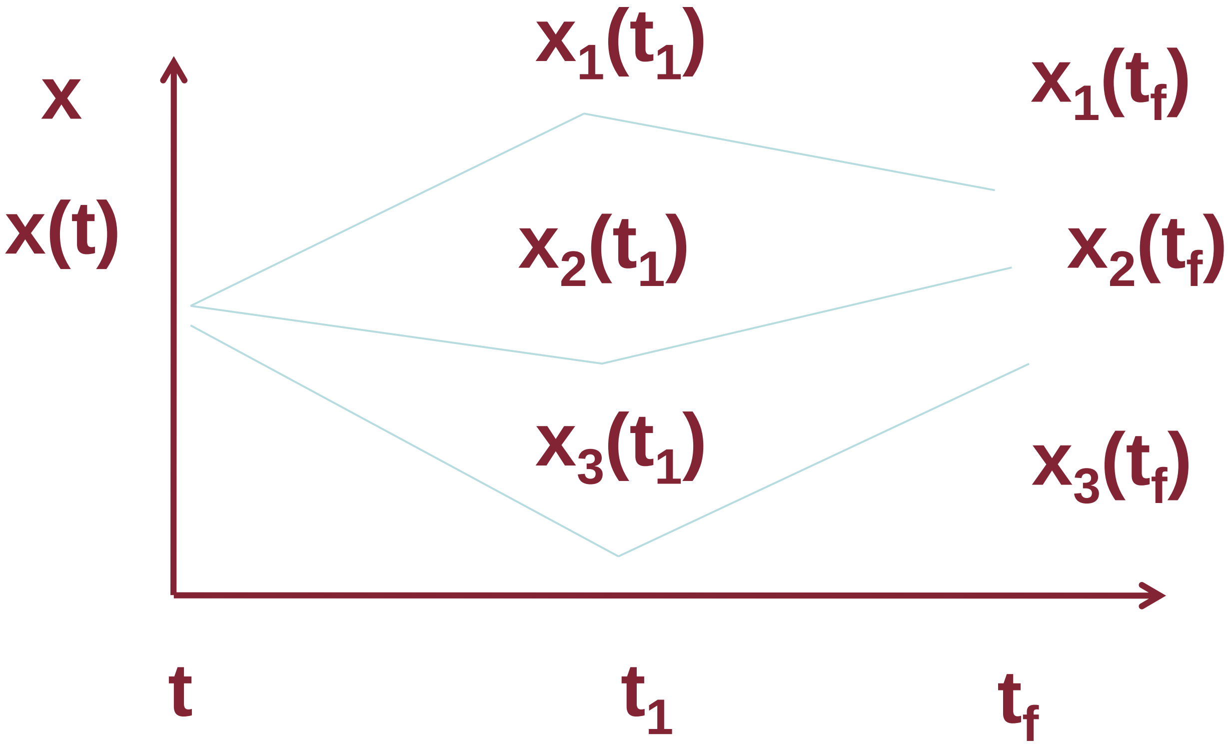
The Hamilton-Jacobi equation

- Such a solution need not exist
- Not every optimal performance index satisfies the Hamilton – Jacobi equation

$\Rightarrow$

The HJ equation represents only a sufficient condition on the optimal performance index

Principle of optimality



Minimizing over $[t, t_f]$ is equivalent to minimizing over $[t, t_1]$ and $[t_1, t_f]$

The Hamilton-Jacobi equation

Principle of optimality:

$$J^*\left(x(t),t\right)=\min_{u[t,t_1]}\left[\int_t^{t_1} L\left(x(\tau),u(\tau),\tau\right)d\tau+\min_{u[t_1,t_f]}\int_{t_1}^{t_f} L\left(x(\tau),u(\tau),\tau\right)d\tau+G(x(t_f))\right]$$

OR:

$$J^*\left(x(t),t\right)=\min_{u[t,t_1]}\left[\int_t^{t_1} L\left(x(\tau),u(\tau),\tau\right)d\tau+J^*\left(x(t_1),t_1\right)\right]$$

The Hamilton-Jacobi equation

Principle of optimality:

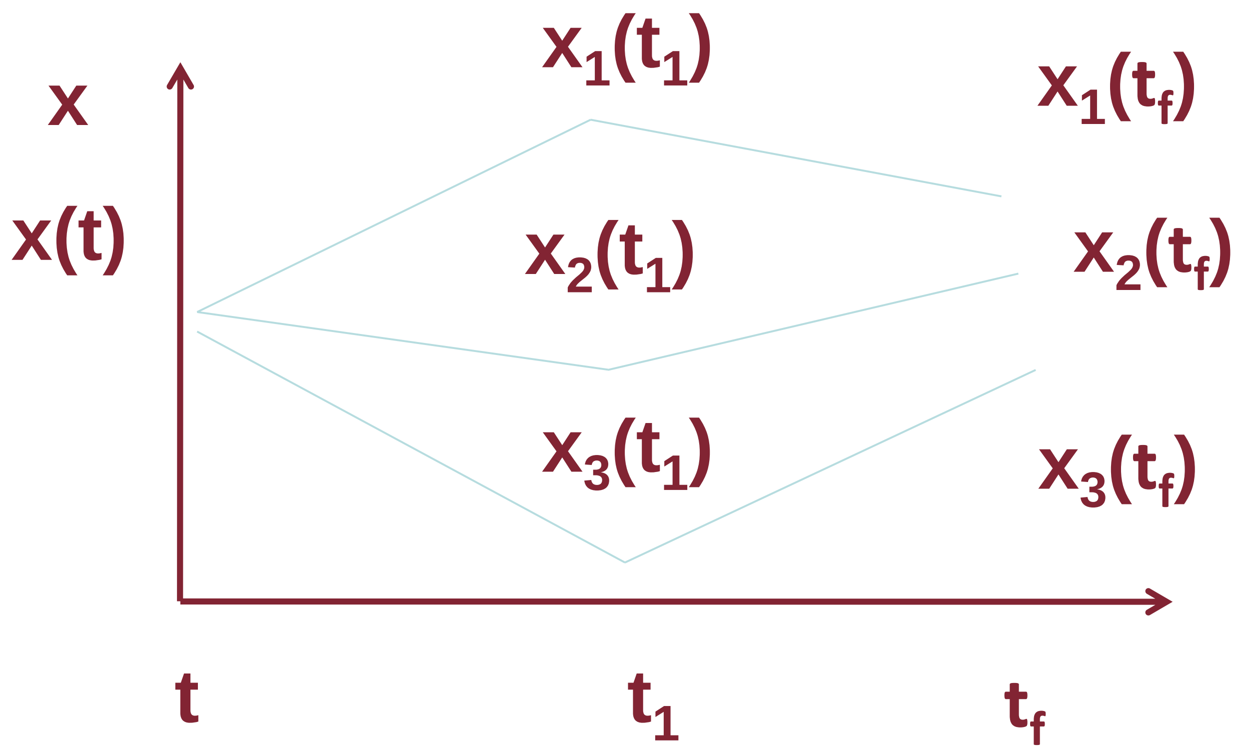
$$J^*\big(x(t),t\big)=\min_{u[t,t_1]}\left[\int_t^{t_1} L\big(x(\tau),u(\tau),\tau\big)d\tau + J^*\big(x(t_1),t_1\big)\right]$$

The optimal cost for trajectories commencing at t and finishing at t_f is incurred by minimizing the sum of the cost in transiting to $x_1(t_1)$ and the optimal cost from there onwards that is:

$$\int_t^{t_1} L\big(x(\tau),u(\tau),\tau\big)d\tau \quad \text{and} \quad J^*\big(x(t),t\big)$$

Principle of optimality

A control policy optimal over the interval $[t, t_f]$ is optimal over <u>all subintervals $[t_1, t_f]$</u>



The Hamilton-Jacobi equation

The Hamilton-Jacobi equation has so far rarely proved useful except for linear regulator problems

The Hamilton-Jacobi equation

Assume the Hamiltonian regular: there exists a unique minimum with respect to *u(t)*.

The Hamilton-Jacobi equation:

$$\boxed{\frac{\partial V(x,t)}{\partial t} + H\left[x(t), \bar{u}\left(x(t), \left(\frac{\partial V(x,t)}{\partial x}\right)^T, t\right), \left(\frac{\partial V(x,t)}{\partial x}\right)^T t\right] = 0}$$

$$\boxed{\begin{aligned}&\frac{\partial V(x,t)}{\partial t} + L\left[x(t), \bar{u}\left(x(t), \left(\frac{\partial V(x,t)}{\partial x}\right)^T, t\right), \left(\frac{\partial V(x,t)}{\partial x}\right)^T, t\right]\\ &\qquad + \left(\frac{\partial V(x,t)}{\partial x}\right)^T f\left(x(t), \bar{u}\left(x(t), \left(\frac{\partial V(x,t)}{\partial x}\right)^T, t\right), t\right) = 0\end{aligned}}$$

The Hamilton-Jacobi equation

Problem:

Consider the system:

$$\dot{x} = f\left(x, u, t\right), \quad x(t_0) \ given$$

Find the **optimal control**

$$u^*(t),\ t \in \left[t_0, t_f\right]$$

that minimizes the cost:

$$J\left(x(t), u(\cdot), t\right) = \int_{t_0}^{t_f} L\left(x(\tau), u(\tau), \tau\right) d\tau + G\left(x(t_f)\right)$$

Define the Hamiltonian:

$$H(x,u,t,\lambda) = L(x,u,t) + \lambda^T f(x,u,t)$$

Assume:

- *f*, *L* and *G* are continuously differentiable in all arguments
- The Hamiltonian regular: there exists a unique minimum with respect to *u(t)*,

Derivation of the Hamilton-Jacobi equation

Let's consider the system:

$$\dot{x} = f(x,u),\ \ x_0\ \ fixed,\ \ x(t_f),t_f\ \ fixed\ or\ \ free$$

with the cost index:

$$\min\ J = \min\left\{\int_{t_0}^{t_f} L(x(t),u(t),t)dt + G(x(t_f))\right\}$$

Notation:

$$J(t_0,x_0) = \min_{\substack{u(t)\\ t\in[t_0,t_f]}}\left\{\int_{t_0}^{t_f} L(x(t),u(t),t)dt + G(x(t_f))\right\}$$

Initial instant

initial state

$$J(t_0, x_0) = \min_{\substack{u(t) \\ t\in[t_0, t_f]}} \left\{ \int_{t_0}^{t_f} L(x(t), u(t), t)dt + G(x(t_f)) \right\}$$

$$= \min_{\substack{u(t) \\ t\in[t_0, t_f]}} \left\{ \int_{t_0}^{t_1} L(x(t), u(t), t)dt + \int_{t_1}^{t_f} L(x(t), u(t), t)dt + G(x(t_f)) \right\}$$

$$= \min_{\substack{u(t) \\ t\in[t_0, t_f]}} \left\{ \int_{t_0}^{t_1} L(x(t), u(t), t)dt + \left[\int_{t_1}^{t_f} L(x(t), u(t), t)dt + G(x(t_f)) \right] \right\}$$

$$= \min_{\substack{u(t) \\ t\in[t_0,t_f]}} \left\{ \int_{t_0}^{t_1} L(x(t),u(t),t)dt + \left[\int_{t_1}^{t_f} L(x(t),u(t),t)dt + G(x(t_f)) \right] \right\}$$

$$= \min_{\substack{u(t) \\ t\in[t_0,t_f]}} \left\{ \int_{t_0}^{t_1} L(x(t),u(t),t)dt + J^*(t_1,x_1) \right\}, \ \forall t_1 \in (t_0,t_f), \ x_1 = x(t_1,t_0,x_0,u|_{[t_0,t_1]})$$

Principle of optimality

Idea: $t_1 = t_0 + dt$

$$J^*(t,x) = \min_{\substack{u(t) \\ [t,t_f]}} \left\{ \int_{t}^{t_f} L(x(t),u(t),t)dt + +G(x(t_f)) \right\}$$

$$= \min_{[t,t+dt]} \left\{ \int_{t}^{t+dt} L(x(t),u(t),t)dt + J^*(t+dt, x(t+dt)) \right\},$$

dt small

$$\boxed{u(\tau) \cong u(t) \quad \tau \in [t, t+dt]}$$

$$\boxed{\int_t^{t\ +dt} L(x(\tau), u(\tau), \tau)d\tau \cong L(x(t), u(t), t)dt}$$

$$\boxed{J^*(t+dt, x(t+dt)) \cong J^*(t, x(t)) + \frac{\partial J^*}{\partial x}dx + \frac{\partial J^*}{\partial d}dt}$$

$$\boxed{J^*(t,x)} = \min_{[t,t+dt]} \left\{ \int_t^{t\ +dt} L(x(t), u(t), t)dt + J^*(t+dt, x(t+dt) \right\}$$

$$= \min \left\{ L(x(t), u(t), t)dt + \boxed{J^*(t,x)} + \frac{\partial J^*(t,x)}{\partial x}dx + \frac{\partial J^*(t,x)}{\partial t}dt \right\}$$

$$-\frac{\partial J^*(t,x)}{\partial t}dt = \min_{u(t)}\left\{L(x(t),u(t),t)dt + \frac{\partial J^*(t,x)}{\partial x}dx\right\}$$

Divide for *dt* and consider $\lim_{t\to 0}$

$$\boxed{-\frac{\partial J^*(t,x)}{\partial t} = \min\left\{L(x(t),u(t),t) + \frac{\partial J^*(t,x)}{\partial x}f(x,u)\right\}}$$

J(x(t),t)* is a solution of the Hamilton-Jacobi (H-J) equation:

$$\boxed{\begin{aligned}\frac{\partial J^*}{\partial t} &= -L\left[x(t),\bar{u}\left(x(t),\frac{\partial J^*}{\partial x},t\right),t\right]\\ &\qquad -\frac{\partial J^{*\prime}}{\partial x} f\left[\left(x(t),\bar{u}\left(x(t),\frac{\partial J^*}{\partial x},t\right),t\right),t\right]\end{aligned}}$$

with boundary conditions:

-*J*(x(t),t) continuous with respect to its arguments*

-the first derivatives of *J*(x(t),t)* continuous with respect to its arguments

- $J^*\left(x(t_f),t_f\right)=G\left(x(t_f)\right)$
- $u^o\left(x(t),t\right)=\bar{u}\left(x(t),\left(\frac{\partial J^*}{\partial x}\left(x(t),t\right)\right)^T,t\right)$

$$J^*\left(x(t),t\right) = \min_{u\left(t,t_f\right)} J\left(x(t),u(\cdot),t\right)$$

is the optimal performance index for

$$J\left(x(t),u(\cdot),t\right) = \int_t^{t_f} L\left(x(\tau),u(\tau),\tau\right)d\tau + G\left(x(t_f)\right)$$

and the control: $$u^o\left(x(t),t\right) = \bar{u}\left(x(t), \frac{\partial J^*}{\partial x}\left(x(t),t\right),t \right)$$

Is the optimal control at time t for the class problems with cost index

$$J\left(x(\sigma),u(\cdot),\sigma\right) = \int_\sigma^{t_f} L\left(x(\tau),u(\tau),\tau\right)d\tau + G\left(x(t_f)\right)$$

The Hamilton-Jacobi equation

Example 2 (from Anderson)

Consider the system: $\dot{x} = u$

With performance index:

$$J\left(x(0), u(\cdot)\right) = \int_{t_0}^{t_f} \left(u^2 + x^2 + \frac{1}{2} x^4 \right) dt, \;\; with\ t_0 = 0$$

Applying the H-J equation we have:

$$\frac{\partial J^*}{\partial t} = -\min_{u(t)} \left\{ u^2 + x^2 + \frac{1}{2}x^4 + \frac{\partial J^*}{\partial x}u \right\}$$

The minimizing control is: $\bar{u} = -\frac{1}{2}\frac{\partial J^*}{\partial x}$

And we have: $\frac{\partial J^*}{\partial t} = \frac{1}{4}\left(\frac{\partial J^*}{\partial x}\right)^2 - x^2 - \frac{1}{2}x^4$

With boundary condition: $J\left(x(t_f), t_f\right) = 0$

In actual fact, it is **rarely possible** to solve a Hamilton-Jacobi equation

Main results

The Hamilton-Jacobi (H-J) equation is satisfied by the optimal performance index under suitable differentiability and continuity assumptions.

If a solution to the H-J equation has certain differentiability properties, then this solution is the desired performance index.

Note that the H-J equation represents only **a sufficient condition** on the optimal performance index