Appendix A
Background Material in Linear Systems Theory

A.1 Quadratic forms

In this section, a few fundamental facts about symmetric matrices and quadratic forms are reviewed.

Symmetric matrices. Let P be a $n \times n$ *symmetric* matrix of real numbers (that is, a matrix of real numbers satisfying $P = P^{\mathrm{T}}$). Then there exist an *orthogonal* matrix Q of real numbers[1] and a diagonal matrix Λ of real numbers

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

such that

$$Q^{-1}PQ = Q^{\mathrm{T}}PQ = \Lambda\,.$$

Indeed, the numbers $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of P. Thus, a symmetric matrix P of real numbers has real eigenvalues and a purely diagonal Jordan form.

Note that the previous identity can be rewritten as

$$PQ = Q\Lambda$$

from which it is seen that the i-th column q_i of Q is an eigenvector of P, associated with the i-th eigenvalue λ_i. If P is invertible, so is the matrix Λ, and

$$P^{-1}Q = Q\Lambda^{-1}\,,$$

from which it is seen that Λ^{-1} is a Jordan form of P^{-1} and the i-th column q_i of Q is also an eigenvector of P^{-1}, associated with the i-th eigenvalue λ_i^{-1}.

[1] That is, matrix Q of real numbers satisfying $QQ^{\mathrm{T}} = I$, or – what is the same – satisfying $Q^{-1} = Q^{\mathrm{T}}$.

Quadratic forms. Let P be a $n \times n$ matrix of real numbers and $x \in \mathbb{R}^n$. The expression

$$V(x) = x^{\mathrm{T}} P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

is called a *quadratic form* in x. Without loss of generality, in the expression above we may assume that P is *symmetric*. In fact,

$$V(x) = x^{\mathrm{T}} P x = \frac{1}{2}[x^{\mathrm{T}} P x + x^{\mathrm{T}} P x] = x^{\mathrm{T}} [\frac{1}{2}(P + P^{\mathrm{T}})] x$$

and $\frac{1}{2}(P + P^{\mathrm{T}})$ is by construction symmetric.

Let P be symmetric, express it as $P = Q \Lambda Q^{\mathrm{T}}$ (see above) with eigenvalues sorted so that $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\begin{aligned} x^{\mathrm{T}} P x &= x^{\mathrm{T}} (Q \Lambda Q^{\mathrm{T}}) x = (Q^{\mathrm{T}} x)^{\mathrm{T}} \Lambda (Q^{\mathrm{T}} x) = \textstyle\sum_{i=1}^{n} \lambda_i (Q^{\mathrm{T}} x)_i^2 \\ &\leq \lambda_1 \textstyle\sum_{i=1}^{n} (Q^{\mathrm{T}} x)_i^2 = \lambda_1 (Q^{\mathrm{T}} x)^{\mathrm{T}} Q^{\mathrm{T}} x = \lambda_1 x^{\mathrm{T}} Q Q^{\mathrm{T}} x = \lambda_1 x^{\mathrm{T}} x \\ &= \lambda_1 \|x\|^2 . \end{aligned}$$

With a similar argument we can show that $x^{\mathrm{T}} P x \geq \lambda_n \|x\|^2$. Usually, λ_n is denoted as $\lambda_{\min}(P)$ and λ_1 is denoted as $\lambda_{\max}(P)$. In summary, we conclude that

$$\lambda_{\min}(P) \|x\|^2 \leq x^{\mathrm{T}} P x \leq \lambda_{\max}(P) \|x\|^2 .$$

Note that the inequalities are tight (*hint*: pick, as x, the last and, respectively, the first column of Q).

Sign-definite symmetric matrices. Let P be symmetric. The matrix P is said to be *positive semi-definite* if

$$x^{\mathrm{T}} P x \geq 0 \qquad \text{for all } x .$$

The matrix P is said to be *positive definite* if

$$x^{\mathrm{T}} P x > 0 \qquad \text{for all } x \neq 0 .$$

We see from the above that P is positive semi-definite if and only if $\lambda_{\min} \geq 0$ and is positive definite if and only if $\lambda_{\min} > 0$ (which in turn implies the nonsingularity of P).

There is another criterion for a matrix to be positive definite, that does not require the computation of the eigenvalues of P, known as Sylvester's criterion. For a *symmetric* matrix P, the n minors

$$D_1 = \det(p_{11}), \quad D_2 = \det \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad D_3 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \ldots$$

are called the *leading principal minors*. Note that $D_n = \det(P)$.

Lemma A.1. *A symmetric matrix is positive definite if and only if all leading principal minors are positive, i.e.* $D_1 > 0,\ D_2 > 0, \dots,\ D_n > 0$.

Another alternative criterion, suited a for block-partitioned matrix, is the criterion due to Schur.

Lemma A.2. *The symmetric matrix*

$$\begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix} \tag{A.1}$$

is positive definite if and only if [2]

$$R > 0 \qquad \text{and} \qquad Q - SR^{-1}S^{\mathrm{T}} > 0\,. \tag{A.2}$$

Proof. Observe that a necessary condition for (A.1) to be positive definite is $R > 0$. Hence R is nonsingular and (A.1) can be transformed, by congruence, as

$$\begin{pmatrix} I & 0 \\ -R^{-1}S^{\mathrm{T}} & I \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix} \begin{pmatrix} I & 0 \\ -R^{-1}S^{\mathrm{T}} & I \end{pmatrix} = \begin{pmatrix} Q - SR^{-1}S^{\mathrm{T}} & 0 \\ 0 & R \end{pmatrix}.$$

from which the condition (A.2) follows. ◁

A symmetric matrix P is said to be *negative semi-definite* (respectively, *negative definite*) if $-P$ is *positive semi-definite* (respectively, *positive definite*). Usually, to express the property (of a matrix P) of being positive definite (respectively, positive semi-definite) the notation $P > 0$ (respectively, $P \geq 0$) is used. [3] Likewise, the notation $P < 0$ (respectively $P \leq 0$) is used to express the property that P is negative definite (respectively, negative semi-definite). If P and R are symmetric matrices, the notations

$$P \geq R \qquad \text{and} \qquad P > R$$

stand for "*the matrix $P - R$ is positive semi-definite*" and, respectively, for "*the matrix $P - R$ is positive definite*".

Any matrix P that can be written in the form $P = M^{\mathrm{T}}M$, in which M is a possibly non-square matrix, is positive semi-definite. In fact $x^{\mathrm{T}}Px = x^{\mathrm{T}}M^{\mathrm{T}}Mx = \|Mx\|^2 \geq 0$.

Conversely, any matrix P which is positive semi-definite can always be expressed as $P = M^{\mathrm{T}}M$. In fact, if P is positive semi-definite all its eigenvalues are nonnegative. Let r denote the number of nonzero eigenvalues and let the eigenvalues be sorted so that $\lambda_{r+1} = \cdots = \lambda_n = 0$. Then

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[2] The matrix $Q - SR^{-1}S^{\mathrm{T}}$ is called the *Schur's complement* of R in (A.1).

[3] Note that this *is not* the same as $p_{ij} > 0$ for all i, j. For example, the matrix

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

in which the off-diagonal elements are negative, is positive definite (use Sylvester's criterion above).

$$P = Q\Lambda Q^{\mathrm{T}} = Q \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{\mathrm{T}}$$

in which Λ_1 is an $r \times r$ diagonal matrix, whose diagonal elements are all positive. For $i = 1, \ldots, r$, let σ_i denote the positive square root of λ_i, let Q_1 be the $n \times r$ matrix whose columns coincide with the first r columns of Q and set

$$M = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_r \end{pmatrix} Q_1^{\mathrm{T}}$$

Then, the previous identity yields

$$P = M^{\mathrm{T}} M$$

in which M is a $n \times r$ matrix of rank r.

Finally, recall that, if P is symmetric and invertible, the eigenvalues of P^{-1} are the inverse of the eigenvalues of P. Thus, in particular, if P is positive definite, so is also P^{-1}.

A.2 Linear matrix equations

In this section, we discuss the existence of solutions of two relevant linear matrix equations that arise in the analysis of linear systems. One of such equation is the so-called *Sylvester's equation*

$$AX - XS = R \tag{A.3}$$

in which $A \in \mathbb{R}^{n \times n}$ and $S \in \mathbb{R}^{d \times d}$, in the unknown $X \in \mathbb{R}^{n \times d}$. An equation of this kind arises, for instance, when it is desired to transform a given block-triangular matrix into a (purely) block-diagonal one, by means of a similarity transformation, as in

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & R \\ 0 & S \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Another instance in which an equation of this kind arises is the analysis of the stability of a linear system, where this equation assumes the special form $AX + XA^{\mathrm{T}} = Q$, known as *Lyapunov's equation*.

Another relevant linear matrix equation is the so-called *regulator* or *Francis's equation*

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C\Pi + Q \end{aligned} \tag{A.4}$$

in which $A \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$, in the unknowns $\Pi \in \mathbb{R}^{n \times d}$ and $\Psi \in \mathbb{R}^{m \times d}$. This equation arises in the study of the problem of output regulation of linear systems.

The two equations considered above are special cases of an equation of the form

$$A_1 X q_1(S) + \cdots + A_k X q_k(S) = R \tag{A.5}$$

in which, for $i = 1, \ldots, k$, $A_i \in \mathbb{R}^{\bar{n}\times\bar{m}}$ and $q_i(\lambda)$ is a polynomial in the indeterminate λ, $S \in \mathbb{R}^{\bar{d}\times\bar{d}}$, $R \in \mathbb{R}^{\bar{n}\times\bar{d}}$, in the *unknown* $X \in \mathbb{R}^{\bar{m}\times\bar{d}}$. In fact, the Sylvester's equation corresponds to the case in which $k = 2$ and

$$A_1 = A, \quad q_1(\lambda) = 1, \quad A_2 = I, \quad q_2(\lambda) = -\lambda,$$

while the Francis' equation corresponds to the case in which $k = 2$ and

$$A_1 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad q_1(\lambda) = 1, \quad A_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad q_2(\lambda) = -\lambda, \quad R = \begin{pmatrix} -P \\ -Q \end{pmatrix}.$$

Equations of the form (A.5) are also known as *Hautus' equations*.[4] Noting that the left-hand side of (A.5) can be seen as a linear map

$$\begin{aligned} \mathscr{H} &: \mathbb{R}^{\bar{m}\times\bar{d}} \to \mathbb{R}^{\bar{n}\times\bar{d}} \\ &: \quad X \quad \mapsto \mathscr{H}(X) := A_1 X q_1(S) + \cdots + A_k X q_k(S), \end{aligned}$$

to say that (A.5) has a solution is to say that $R \in \mathrm{Im}(\mathscr{H})$.

In what follows, we are interested in the case in which (A.5) has solutions for all R, i.e. in the case in which the map $\mathscr{H}$ is *surjective*. [5]

Theorem A.1. *The map $\mathscr{H}$ is surjective if and only if the $\bar{n}$ rows of the matrix*

$$A(\lambda) = A_1 q_1(\lambda) + \cdots + A_k q_k(\lambda)$$

are linearly independent for each λ which is an eigenvalue of S. If this is the case and $\bar{n} = \bar{m}$, the solution X of (A.5) is unique.

From this, it is immediate to deduce the following Corollaries.

Corollary A.1. *The Sylvester's equation (A.3) has a solution for each R if and only if $\sigma(A) \cap \sigma(S) = \emptyset$. If this is the case, the solution X is unique.*

Corollary A.2. *The Francis' equation (A.4) has a solution for each pair (P, Q) if and only if the rows of the matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

are linearly independent for each λ which is an eigenvalue of S. If this is the case and $m = p$, the solution pair (Π, Ψ) is unique.

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[4] See [3].

[5] Note that, if this is the case and $\bar{n} = \bar{m}$, tha map is also *injective*, i.e. it is an invertible linear map. In this case the solution X of (A.5) is *unique*.

A.3 The theorems of Lyapunov for linear systems

In this section we describe a powerful criterion useful to determine when a $n \times n$ matrix of real numbers has all eigenvalues with negative real part.[6]

Theorem A.2. [Direct Theorem] *Let $A \in \mathbb{R}^{n\times n}$ be a matrix of real numbers. Let $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix of real numbers and suppose that the matrix*

$$PA + A^{\mathrm{T}}P$$

is negative definite. Then, all eigenvalues of the matrix A have negative real part.

Proof. Let λ be an eigenvalue of A and x an associated eigenvector. Let x_R and x_I denote the real and, respectively, imaginary part of x, i.e. set $x = x_R + jx_I$ and let $x^* = x_R^{\mathrm{T}} - jx_I^{\mathrm{T}}$. Then

$$x^*Px = (x_R)^{\mathrm{T}}Px_R + (x_I)^{\mathrm{T}}Px_I.$$

Since P is positive definite and x_R and x_I cannot be both zero (because $x \neq 0$), we deduce that

$$x^*Px > 0. \tag{A.6}$$

With similar arguments, since $A^{\mathrm{T}}P + PA$ is negative definite, we deduce that

$$x^*(A^{\mathrm{T}}P + PA)x < 0. \tag{A.7}$$

Using the definition of x and λ (i.e. $Ax = x\lambda$, that implies $x^*A^{\mathrm{T}} = \lambda^*x^*$), obtain

$$x^*(A^{\mathrm{T}}P + PA)x = \lambda^*x^*Px + x^*Px\lambda = (\lambda + \lambda^*)x^*Px$$

from which, using (A.6) and (A.7) we conclude

$$\lambda + \lambda^* = 2\mathrm{Re}[\lambda] < 0. \qquad \triangleleft$$

Remark A.1. The criterion described in the previous Theorem provides a sufficient condition under which all the eigenvalues of a matrix A have negative real part. In the analysis of linear systems, this criterion is used as a sufficient condition to determine whether the equilibrium $x = 0$ of the autonomous system

$$\dot{x} = Ax \tag{A.8}$$

is (globally) asymptotically stable. In this context, the previous proof – which only uses algebraic arguments – can be replaced by the linear version of the proof of Theorem B.1, which can be summarized as follows. Let $V(x) = x^{\mathrm{T}}Px$ denote the positive definite quadratic function associated with P, let $x(t)$ denote a generic trajectory of system (A.8) and consider the composite function $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$. Observe that

[6] For further reading, see e.g. [1].

$$\frac{\partial V}{\partial x} = 2x^{\mathrm{T}}P.$$

Therefore, using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\partial V}{\partial x}\Big|_{x=x(t)}\frac{\mathrm{d}x}{\mathrm{d}t} = 2x^{\mathrm{T}}(t)PAx(t) = x^{\mathrm{T}}(t)(PA+A^{\mathrm{T}}P)x(t).$$

If P is positive definite and $PA+A^{\mathrm{T}}P$ is negative definite, there exists positive numbers a_1, a_2, a_3 such that

$$a_1\|x\|^2 \le V(x) \le a_2\|x\|^2 \qquad \text{and} \qquad x^{\mathrm{T}}(PA+A^{\mathrm{T}}P)x \le -a_3\|x\|^2.$$

From this, it is seen that $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le -\lambda V(x(t))$$

with $\lambda = a_3/a_2 > 0$ and therefore

$$a_1\|x(t)\|^2 \le V(x(t)) \le e^{-\lambda t}V(x(0)) \le e^{-\lambda t}a_2\|x(0)\|^2.$$

Thus, for any initial condition $x(0)$, $\lim_{t\to\infty} x(t) = 0$. This proves that all eigenvalues of A have negative real part. $\triangleleft$

Theorem A.3. [Converse Theorem] *Let $A \in \mathbb{R}^{n\times n}$ be a matrix of real numbers. Suppose all eigenvalues of A have negative real part. Then, for any choice of a symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P such that*

$$PA + A^{\mathrm{T}}P = -Q. \tag{A.9}$$

Proof. Consider (A.9). This is a Sylvester equation, and – since the spectra of A and $-A^{\mathrm{T}}$ are disjoint – a unique solution P exists. We compute it explicitly. Define

$$M(t) = e^{A^{\mathrm{T}}t}Qe^{At}$$

and observe that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At} + e^{A^{\mathrm{T}}t}Qe^{At}A = A^{\mathrm{T}}M(t) + M(t)A.$$

Integrating over $[0,T]$ yields

$$M(T) - M(0) = A^{\mathrm{T}}\int_0^T M(t)dt + \int_0^T M(t)dt\, A.$$

Since the eigenvalues of A have negative real part,

$$\lim_{T\to\infty} M(T) = 0$$

and

$$P := \lim_{T\to\infty} \int_0^T M(t)dt < \infty\,.$$

We have shown in this way that P satisfies (A.9). It is the unique solution of this equation.

To complete the proof it remains to show that P is positive definite, if so is Q. By contradiction, suppose is not. Then there exists $x_0 \neq 0$ such that

$$x_0^{\mathrm{T}} P x_0 \leq 0\,,$$

which, in view of the expression found for P, yields

$$\int_0^\infty x_0^{\mathrm{T}} e^{A^{\mathrm{T}}t} Q e^{At} x_0 dt \leq 0\,.$$

Setting

$$x(t) = e^{At} x_0$$

this is equivalent to

$$\int_0^\infty x^{\mathrm{T}}(t) Q x(t) dt \leq 0\,,$$

which, using the estimate $x^{\mathrm{T}} Q x \geq \lambda_{\min}(Q)\|x\|^2$ in turn yields

$$\int_0^\infty \|x(t)\|^2 dt \leq 0$$

which then yields

$$x(t) = 0, \qquad \text{for all } t \in [0,\infty).$$

Bearing in mind the expression of $x(t)$, this implies $x_0 = 0$ and completes the proof.

A.4 Stabilizability, detectability and separation principle

In this section, a few fundamental facts about the stabilization of linear systems are reviewed.[7] Consider a linear system modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{A.10}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$ and we summarize the properties that determine the existence of a (dynamic) output feedback controller of the form

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= A_{\mathrm{c}} x_{\mathrm{c}} + B_{\mathrm{c}} y \\ u &= C_{\mathrm{c}} x_{\mathrm{c}} + D_{\mathrm{c}} y, \end{aligned} \tag{A.11}$$

[7] For further reading, see e.g. [1].

with state $x_c \in \mathbb{R}^{n_c}$, that stabilizes the resulting closed-loop system

$$\begin{aligned} \dot{x} &= (A+BD_cC)x + BC_cx_c \\ \dot{x}_c &= B_cCx + A_cx_c. \end{aligned} \tag{A.12}$$

Definition A.1. The pair (A,B) is *stabilizable* if there exists a matrix F such that $(A+BF)$ has all the eigenvalues in $\mathbb{C}^-$.

Definition A.2. The pair (A,C) is *detectable* if there exists a matrix G such that $(A-GC)$ has all the eigenvalues in $\mathbb{C}^-$.

Noting that the closed-loop system (A.12) can be written as $\dot{x}_{c\ell} = A_{c\ell}x_{c\ell}$, with $x_{c\ell} = \mathrm{col}(x, x_c)$ and

$$A_{c\ell} = \begin{pmatrix} (A+BD_cC) & BC_c \\ B_cC & A_c \end{pmatrix}, \tag{A.13}$$

we have the following fundamental result.

Theorem A.4. *There exists matrices A_c, B_c, C_c, D_c such that (A.13) has all the eigenvalues in $\mathbb{C}^-$ if and only if the pair (A,B) is stabilizable* and *pair (A,C) is detectable.*

Proof. [Necessity] Suppose all eigenvalues of (A.13) have negative real part. Then, by the converse Lyapunov's Theorem, there exists a unique, symmetric, and positive definite solution $P_{c\ell}$ of the matrix equation

$$P_{c\ell}A_{c\ell} + A_{c\ell}^{\mathrm{T}}P_{c\ell} = -I\,. \tag{A.14}$$

Let $P_{c\ell}$ be partitioned as in

$$P_{c\ell} = \begin{pmatrix} P & S \\ S^{\mathrm{T}} & P_c \end{pmatrix}$$

consistently with the partition of $A_{c\ell}$ (note, in this respect, that the two diagonal blocks may have different dimensions n and n_c). Note also that P and P_c are necessarily positive definite (and hence also nonsingular) because so is $P_{c\ell}$. Consider the matrix

$$T = \begin{pmatrix} I & 0 \\ -P_c^{-1}S^{\mathrm{T}} & I \end{pmatrix}.$$

Define $\tilde{P} := T^{\mathrm{T}}P_{c\ell}T$ and note that

$$\tilde{P} = \begin{pmatrix} P - SP_c^{-1}S^{\mathrm{T}} & 0 \\ 0 & P_c \end{pmatrix}. \tag{A.15}$$

Define $\tilde{A} := T^{-1}A_{c\ell}T$ and, by means of a simple computation, observe that

$$\tilde{A} = \begin{pmatrix} A + B(D_cC - C_cP_c^{-1}S^{\mathrm{T}}) & * \\ * & * \end{pmatrix}$$

in which we have denoted by an asterisk blocks whose expression is not relevant in the sequel.

From (A.14) it is seen that

$$\begin{aligned}\tilde{P}\tilde{A}+\tilde{A}^{\mathrm{T}}\tilde{P} &= (T^{\mathrm{T}}P_{\mathrm{c}\ell}T)(T^{-1}A_{\mathrm{c}\ell}T)+(T^{\mathrm{T}}A_{\mathrm{c}\ell}^{\mathrm{T}}(T^{-1})^{\mathrm{T}})(T^{\mathrm{T}}P_{\mathrm{c}\ell}T)\\ &= T^{\mathrm{T}}(P_{\mathrm{c}\ell}A_{\mathrm{c}\ell}+A_{\mathrm{c}\ell}^{\mathrm{T}}P_{\mathrm{c}\ell})T = -T^{\mathrm{T}}T.\end{aligned} \tag{A.16}$$

This shows that the matrix $\tilde{P}\tilde{A}+\tilde{A}^{\mathrm{T}}\tilde{P}$ is negative definite (because T is nonsingular) and hence so is its upper-left block. The latter, if we set

$$P_0 = P - SP_{\mathrm{c}}^{-1}S^{\mathrm{T}}, \qquad F = D_{\mathrm{c}}C - C_{\mathrm{c}}P_{\mathrm{c}}^{-1}S^{\mathrm{T}}$$

can be written in the form

$$P_0(A+BF)+(A+BF)^{\mathrm{T}}P_0\,. \tag{A.17}$$

The matrix P_0 is positive definite, because it is the upper-left block of the positive definite matrix (A.15). The matrix (A.17) is negative definite, because it is the upper-left block of the negative definite matrix (A.16). Thus, by the direct criterion of Lyapunov, it follows that the eigenvalues of $A+BF$ have negative real part. This completes the proof that, if $A_{\mathrm{c}\ell}$ has all eigenvalues in $\mathbb{C}^-$, there exists a matrix F such that $A+BF$ has all eigenvalues in $\mathbb{C}^-$, i.e. the pair (A,B) is stabilizable. In a similar way it is proven that the pair (A,C) is detectable.

[Sufficiency] Assuming that (A,B) is stabilizable and that (A,C) is detectable, pick F and G so that $(A+BF)$ has all eigenvalues in $\mathbb{C}^-$ and $(A-GC)$ has all eigenvalues in $\mathbb{C}^-$. Consider the controller

$$\begin{aligned}\dot{x}_{\mathrm{c}} &= (A+BF-GC)x_{\mathrm{c}}+Gy\\ u &= Fx_{\mathrm{c}},\end{aligned} \tag{A.18}$$

i.e. set

$$A_{\mathrm{c}} = A+BF-GC, \quad B_{\mathrm{c}} = G, \quad C_{\mathrm{c}} = F, \quad D_{\mathrm{c}} = 0.$$

This yields a closed-loop system

$$\begin{aligned}\dot{x} &= Ax+BFx_{\mathrm{c}}\\ \dot{x}_{\mathrm{c}} &= GCx+(A+BF-GC)x_{\mathrm{c}}\,.\end{aligned}$$

The change of variables $z = x-x_{\mathrm{c}}$ changes the latter into the system

$$\begin{pmatrix}\dot{x}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}A+BF & -BF\\ 0 & A-GC\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}$$

This system is in block-triangular form and both diagonal blocks have all eigenvalues in $\mathbb{C}^-$. Thus the controller (A.18) guarantees that the matrix $A_{\mathrm{c}\ell}$ has all eigenvalues in $\mathbb{C}^-$. [8]

[8] Observe that the choices of F and G are *independent* of each other, i.e. F is only required to place the eigenvalues of $(A+BF)$ in $\mathbb{C}^-$ and G is only required to place the eigenvalues of $(A-GC)$

To check whether the two fundamental properties in question hold, the following tests are useful.

Lemma A.3. *The pair* (A,B) *is stabilizable if and only if*

$$\operatorname{rank}\begin{pmatrix} A-\lambda I & B \end{pmatrix} = n \tag{A.19}$$

for all $\lambda \in \sigma(A)$ *having non-negative real part.*

Lemma A.4. *The pair* (A,C) *is detectable if and only if*

$$\operatorname{rank}\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = n \tag{A.20}$$

for all $\lambda \in \sigma(A)$ *having non-negative real part.*

Remark A.2. For the sake of completeness, we recall how the two properties of stabilizability and detectability invoked above compare with the properties of reachability and observability. To this end, we recall that the linear system (A.25) is *reachable* if and only if

$$\operatorname{rank}\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = n \tag{A.21}$$

or, what is the same, if and only if the condition (A.19) holds *for all* $\lambda \in \sigma(A)$ (and not just for all such λ's having non-negative real part). The linear system (A.25) is *observable* if and only if

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} = n \tag{A.22}$$

or, what is the same, if and only if the condition (A.20) holds *for all* $\lambda \in \sigma(A)$ (and not just for all such λ's having non-negative real part).

It is seen from this that, in general, reachability is a property stronger than stabilizability and observability is a property stronger than detectability. The two (pairs of) properties coincide when all eigenvalues of A have non-negative real part. If the rank of the matrix on the left-hand side of (A.21) is $n_1 < n$, the system is *not* reachable and there exists a nonsingular matrix T such that [9]

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad TB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

in which $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ and the pair (A_{11}, B_1) is reachable. This being the case, it is easy to check that the pair (A,B) is stabilizable if and only if all eigenvalues of A_{22} have negative real part. A similar criterion determines the relation between

in $\mathbb{C}^-$. For this reason, the controller (A.18) is said to be a controller inspired by a *separation principle*.

[9] This is the well-know *Kalman's* decomposition of a system into reachable/unreachable parts.

detectability and observability. If the rank of the matrix on the left-hand side of (A.22) is $n_1 < n$, the system is *not* observable and there exists a nonsingular matrix T such that

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 0 & C_2 \end{pmatrix}$$

in which $A_{22} \in \mathbb{R}^{n_1 \times n_1}$ and the pair (A_{22}, C_2) is observable. This being the case, it is easy to check that the pair (A, C) is detectable if and only if all eigenvalues of A_{11} have negative real part.

A.5 Steady-state response to harmonic inputs

Invariant subspaces. Let A be a fixed $n \times n$ matrix. A subspace $\mathscr{V}$ of $\mathbb{R}^n$ is *invariant* under A if

$$v \in \mathscr{V} \quad \Rightarrow \quad Av \in \mathscr{V}\,.$$

Let $d < n$ denote the dimension of $\mathscr{V}$ and let $\{v_1, v_2, \ldots, v_d\}$ be a basis of $\mathscr{V}$, that is a set of d linearly independent vectors $v_i \in \mathbb{R}^n$ such that

$$\mathscr{V} = \mathrm{Im}(V)$$

where V is the $n \times d$ matrix

$$V = \begin{pmatrix} v_1 & v_2 & \cdots & v_d \end{pmatrix}.$$

Then, it is an easy matter to check that $\mathscr{V}$ is invariant under A if and only if there exists a $d \times d$ matrix $A_{\mathscr{V}}$ such that

$$AV = VA_{\mathscr{V}}\,.$$

The map $z \mapsto A_{\mathscr{V}} z$ characterizes the *restriction* to $\mathscr{V}$ of the map $x \mapsto Ax$. This being the case, observe that if (λ_0, z_0) is a pair eigenvalue-eigenvector for $A_{\mathscr{V}}$ (i.e. a pair satisfying $A_{\mathscr{V}} z_0 = \lambda_0 z_0$), then $(\lambda_0, V z_0)$ is a pair eigenvalue-eigenvector for A.

Let the matrix A have n_s eigenvalues in $\mathbb{C}^-$, n_a eigenvalues in $\mathbb{C}^+$ and n_c eigenvalues in $\mathbb{C}^0 = \{\lambda \in \mathbb{C} : \mathrm{Re}[\lambda] = 0\}$, with (obviously) $n_\mathrm{s} + n_\mathrm{a} + n_\mathrm{c} = n$. Then (passing for instance through the Jordan form of A) it is easy to check that there exist three invariant subspaces of A, denoted $\mathscr{V}_\mathrm{s}$, $\mathscr{V}_\mathrm{a}$, $\mathscr{V}_\mathrm{c}$, of dimension $n_\mathrm{s}, n_\mathrm{a}, n_\mathrm{c}$ that are complementary in $\mathbb{R}^n$, i.e. satisfy

$$\mathscr{V}_\mathrm{s} \oplus \mathscr{V}_\mathrm{a} \oplus \mathscr{V}_\mathrm{c} = \mathbb{R}^n\,, \tag{A.23}$$

with the property that the restriction of A to $\mathscr{V}_\mathrm{s}$ is characterized by a $n_\mathrm{s} \times n_\mathrm{s}$ matrix A_s whose eigenvalues are precisely the n_s eigenvalues of A that are in $\mathbb{C}^-$, the restriction of A to $\mathscr{V}_\mathrm{a}$ is characterized by a $n_\mathrm{a} \times n_\mathrm{a}$ matrix A_a whose eigenvalues are precisely the n_a eigenvalues of A that are in $\mathbb{C}^+$ and the restriction of A to $\mathscr{V}_\mathrm{c}$ is characterized by a $n_c \times n_c$ matrix A_c whose eigenvalues are precisely the n_c eigenvalues of A that are in $\mathbb{C}^0$. These three subspaces are called the *stable eigenspace*, the *antistable eigenspace* and the *center eigenspace*.

Consider now the autonomous linear system

$$\dot{x} = Ax \tag{A.24}$$

with $x \in \mathbb{R}^n$. It is easy to check that if subspace $\mathscr{V}$ is invariant under A, then for any $x^\circ \in \mathscr{V}$, the integral curve $x(t)$ of (A.24) passing through x° at time $t = 0$ is such that $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}$. [10] Because of (A.23), any trajectory $x(t)$ of (A.24) can be uniquely decomposed as

$$x(t) = x_s(t) + x_a(t) + x_c(t)$$

with $x_s(t) \in \mathscr{V}_s$, $x_a(t) \in \mathscr{V}_a$, $x_c(t) \in \mathscr{V}_c$. Moreover, since the restriction of A to $\mathscr{V}_s$ is characterized by a matrix A_s whose eigenvalues are all in $\mathbb{C}^-$,

$$\lim_{t\to\infty} x_s(t) = 0$$

and, since the restriction of A to $\mathscr{V}_a$ is characterized by a matrix A_a whose eigenvalues are all in $\mathbb{C}^+$,

$$\lim_{t\to-\infty} x_a(t) = 0\,.$$

A geometric characterization of the steady-state response. It is well known that a stable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

subject to a (harmonic) input of the form

$$u(t) = u_0 \cos(\omega_0 t) \tag{A.25}$$

exhibits a well-defined *steady state response*, which is itself a harmonic function of time. The response in question can be easily characterized by means of a simple geometric construction. Observe that the input defined above can be viewed as generated by an autonomous system of the form

$$\begin{aligned} \dot{w} &= Sw \\ u &= Qw \end{aligned} \tag{A.26}$$

in which $w \in \mathbb{R}^2$ and

[10] Note that also the converse of such implication holds. If $\mathscr{V}$ is a subspace with the property that, for any $x^\circ \in \mathscr{V}$, the integral curve $x(t)$ of (A.24) passing through x° at time $t = 0$ satisfies $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}$, then $\mathscr{V}$ is invariant under A. The property in question is sometimes referred to as the *integral version* of the notion of invariance, while the property indicated in the text above is referred to as the *infinitesimal version* on the notion of invariance.

$$S = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}, \qquad Q = u_0 \begin{pmatrix} 1 & 0 \end{pmatrix},$$

set in the initial state [11]

$$w(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A.27}$$

In this way, the *forced* response of the given linear system, from any initial state $x(0)$, to the input (A.25) can be identified with the *free* response of the composite system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} \tag{A.28}$$

from the initial state $(x(0), w(0))$ with $w(0)$ given by (A.27).

Since A has all eigenvalues with negative real part and S has eigenvalues on the imaginary axis, the Sylvester equation

$$\Pi S = A\Pi + BQ \tag{A.29}$$

has a unique solution Π. The composite system (A.28) possesses two complementary invariant eigenspaces: a *stable eigenspace* and a *center eigenspace*, which can be respectively expressed as

$$\mathscr{V}^{\mathrm{s}} = \operatorname{span} \begin{pmatrix} 0 \\ I \end{pmatrix}, \qquad \mathscr{V}^{\mathrm{c}} = \operatorname{span} \begin{pmatrix} I \\ \Pi \end{pmatrix}.$$

The latter, in particular, shows that the center eigenspace is the set of all pairs (w, x) such that $x = \Pi w$.

Consider now the change of variables $\tilde{x} = x - \Pi w$ which, after a simple calculation which uses (A.29), yields

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{x}} &= A\tilde{x}. \end{aligned}$$

Since the matrix A is Hurwitz, for any initial condition

$$\lim_{t\to\infty} \tilde{x}(t) = 0,$$

which shows that the (unique) projection of the trajectory along the stable eigenspace asymptotically tends to zero. In the original coordinates, this reads as

$$\lim_{t\to\infty} [x(t) - \Pi w(t)] = 0,$$

---

[11] To check that this is the case, simply bear in mind that the solution $w(t)$ of (A.26) is given by

$$w(t) = e^{St} w(0) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$$

.

from which we see that the *steady-state response* of the system to any input generated by (A.26) can be expressed as

$$x_{ss}(t) = \Pi w(t)\,. \tag{A.30}$$

It is worth observing that the steady-state response $x_{ss}(t)$ thus defined can also be identified with an *actual* forced response of the system to the input (A.25), provided that the initial state $x(0)$ is appropriately chosen. In fact, since the center eigenspace is invariant for the composite system (A.28), if the initial condition of the latter is taken on $\mathscr{V}^c$, i.e. if $x(0) = \Pi w(0)$, the motion of such system remains confined to $\mathscr{V}^c$ for all t, i.e. $x(t) = \Pi w(t)$ for all t. Thus, if $x(0) = \Pi w(0)$, the actual forced response $x(t)$ of the system to the input (A.25) coincides with the steady-state response $x_{ss}(t)$. Note that, in view of the definition of $w(0)$ given by (A.27), this initial state $x(0)$ is nothing else than the first column of the matrix Π.

The calculation of the solution Π of the Sylvester equation (A.29) is straightforward. Set

$$\Pi = (\,\Pi_1 \quad \Pi_2\,)$$

and observe that the equation in question reduces to

$$\Pi \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} = A\Pi + Bu_0\,(\,1 \quad 0\,)\ .$$

An elementary calculation (multiply first both sides on the right by the vector $(\,1 \quad j\,)^{\mathrm{T}}$) yields

$$\Pi_1 + j\Pi_2 = (j\omega_0 I - A)^{-1}Bu_0\ ,$$

i.e.

$$\Pi = (\,\mathrm{Re}[(j\omega_0 I - A)^{-1}B]u_0 \quad \mathrm{Im}[(j\omega_0 I - A)^{-1}B]u_0\,)\ .$$

As shown above, the steady state response has the form (A.30). Hence, in particular, the periodic input

$$u(t) = u_0 \cos(\omega_0 t)$$

produces the periodic state response

$$x_{ss}(t) = \Pi w(t) = \Pi_1 \cos(\omega_0 t) - \Pi_2 \sin(\omega_0 t)\ , \tag{A.31}$$

and the periodic output response

$$\begin{aligned} y_{ss}(t) &= Cx_{ss}(t) + Du_0 \cos(\omega_0 t) \\ &= \mathrm{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \mathrm{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t)\ , \end{aligned} \tag{A.32}$$

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D\ .$$

A.6 Hamiltonian matrices and algebraic Riccati equations

In this section, a few fundamental facts about algebraic Riccati equations are reviewed.[12] An algebraic Riccati equation is an equation of the form

$$A^{\mathrm{T}}X + XA + Q + XRX = 0 \tag{A.33}$$

in which all matrices involved are $n \times n$ matrices and R, Q are *symmetric* matrices. Such equation can also be rewritten the equivalent form as

$$\begin{pmatrix} X & -I \end{pmatrix} \begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = 0\,.$$

From either one of these expressions, it is easy to deduce the following identity

$$\begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix} (A + RX) \tag{A.34}$$

and to conclude that X is a solution of the Riccati equation (A.33) if and only if the subspace

$$\mathscr{V} = \mathrm{Im} \begin{pmatrix} I \\ X \end{pmatrix} \tag{A.35}$$

is an (n-dimensional) invariant subspace of the matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix}. \tag{A.36}$$

In particular, (A.34) also shows that if X is a solution of (A.33), the matrix $A + RX$ characterizes the restriction of H to its invariant subspace (A.35). A matrix of the form (A.36), with real entries and in which R and Q are symmetric matrices, is called a Hamiltonian matrix. Some relevant features of the Hamiltonian matrix (A.36) and their relationships with the Riccati equation (A.33) are reviewed in what follows.

Lemma A.5. *The spectrum of the Hamiltonian matrix (A.36) is symmetric with respect to the imaginary axis.*

Proof. Set

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and note that

$$J^{-1}HJ = \begin{pmatrix} -A^{\mathrm{T}} & Q \\ -R & A \end{pmatrix} = -H^{\mathrm{T}}\,.$$

Hence H and -H^{T} are similar. As a consequence, if λ is an eigenvalue of H so is also $-\lambda$. Since the entries of H are real numbers, and therefore the spectrum of this matrix is symmetric with respect to the real axis, the result follows. ◁

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[12] For further reading, see e.g. [2].

Suppose now that the matrix (A.36) has no eigenvalues on the imaginary axis. Then, the matrix in question has exactly n eigenvalues in $\mathbb{C}^-$ and n eigenvalues in $\mathbb{C}^+$. As a consequence, there exist two complementary n-dimensional invariant subspaces of H: a subspace $\mathscr{V}^{\mathrm{s}}$ characterized by property that restriction of H to $\mathscr{V}^{\mathrm{s}}$ has all eigenvalues in $\mathbb{C}^-$, the *stable eigenspace*, and a subspace $\mathscr{V}^{\mathrm{a}}$ characterized by property that restriction of H to $\mathscr{V}^{\mathrm{a}}$ has all eigenvalues in $\mathbb{C}^+$, the *antistable eigenspace*. A situation of special interest in the subsequent analysis is the one in which the stable eigenspace (respectively, the antistable eigenspace) of the matrix (A.36) can be expressed in the form (A.35); in this case in fact, as observed before, it is possible to associate with this subspace a particular solution of the Riccati equation (A.33).

If there exists a matrix X^- such that

$$\mathscr{V}^{\mathrm{s}} = \mathrm{Im}\begin{pmatrix} I \\ X^- \end{pmatrix},$$

this matrix satisfies

$$A^{\mathrm{T}}X^- + X^-A + Q + X^-RX^- = 0$$

and the matrix $A+RX^-$ has all eigenvalues in $\mathbb{C}^-$. This matrix is the *unique* [13] solution of the Riccati equation (A.33) having the property that $A+RX$ has all eigenvalues in $\mathbb{C}^-$ and for this reason is called *the stabilizing* solution of the Riccati equation (A.33).

Similarly, if there exists a matrix X^+ such that

$$\mathscr{V}^{\mathrm{a}} = \mathrm{Im}\begin{pmatrix} I \\ X^+ \end{pmatrix},$$

this matrix satisfies

$$A^{\mathrm{T}}X^+ + X^-A + Q + X^+RX^+ = 0$$

and the matrix $A+RX^+$ has all eigenvalues in $\mathbb{C}^+$. This matrix is the unique solution of the Riccati equation (A.33) having the property that $A+RX$ has all eigenvalues in $\mathbb{C}^+$ and for this reason is called *the antistabilizing* solution of the Riccati equation (A.33).

The existence of such matrices X^- and X^+ is discussed in the following statement.

Proposition A.1. *Suppose the Hamiltonian matrix (A.36) has no eigenvalues on the imaginary axis and R is a (either positive or negative) semidefinite matrix.*

If the pair (A,R) is stabilizable, the stable eigenspace $\mathscr{V}^{\mathrm{s}}$ of (A.36) can be expressed in the form

$$\mathscr{V}^{\mathrm{s}} = \mathrm{Im}\begin{pmatrix} I \\ X^- \end{pmatrix}$$

[13] If $\mathscr{V}$ is an n-dimensional subspace of $\mathbb{R}^{2n}$, and $\mathscr{V}$ can be expressed in the form (A.35), the matrix X is necessarily unique.

in which X^- is a symmetric matrix, the (unique) stabilizing solution of the Riccati equation (A.33).

If the pair (A,R) is antistabilizable, the antistable eigenspace $\mathscr{V}^{\mathrm{a}}$ of (A.36) can be expressed in the form

$$\mathscr{V}^{\mathrm{a}} = \mathrm{Im}\begin{pmatrix} I \\ X^+ \end{pmatrix}$$

in which X^- is a symmetric matrix, the (unique) antistabilizing solution of the Riccati equation (A.33).

The following Proposition describes the relation between solutions of the algebraic Riccati equation (A.33) and of the algebraic Riccati *inequality*

$$A^{\mathrm{T}}X + XA + Q + XRX > 0\,. \tag{A.37}$$

Proposition A.2. *Suppose R is negative semidefinite. Let X^- (respectively X^+) be a solution of the Riccati equation (A.33) having the property that $\sigma(A+RX^-) \in \mathbb{C}^-$ (respectively, $\sigma(A+RX^+) \in \mathbb{C}^+$). Then, the set of solutions of*

$$A^{\mathrm{T}}X + XA + Q + XRX > 0\,,$$

is not empty and any X in this set satisfies $X < X^-$ (respectively, $X > X^+$).

References

1. P.S. Antsaklis, A.N. Michel, *Linear Systems*, McGraw Hill, 1997.
2. P. Lancaster, L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, 1995.
3. M. Hautus. Linear matrix equations with applications to the regulator problem. In *I. Landau (ed.), Outils et modèles mathematiques pour lautomatique, lanalyse de systèmes et le traitement du signal*, C.N.R.S. Paris, **3**, pp. 399-412, 1983.

Chapter 2
Stabilization of Minimum-phase Linear Systems

Abstract It is well-known from the elementary theory of servomechanisms that a single-input single-output linear system whose transfer function has all zeros in the left-half complex plane can be stabilized via output feedback. If the transfer function of the system has n poles and m zeros, the feedback in question is a dynamical system of dimension $n-m-1$ whose eigenvalues (in case $n-m>1$) are far away in the left-half complex plane. In this chapter, this result is reviewed using a state-space approach. This makes it possible to systematically handle the case of systems whose coefficients depend on uncertain parameters and serves as a preparation to a similar set of results that will be presented in Chapter 6 for nonlinear systems.

2.1 Normal form and system zeroes

The purpose of this section is to show how single-input single-output linear systems can be given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which certain fundamental properties of the system are highlighted, that plays a relevant role in the method for robust stabilization discussed in the following sections.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally defined as follows. Given a single-input single-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,, \end{aligned} \tag{2.1}$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}$, consider the sequence of real numbers CB, CAB, $CA^2B,\ldots,$ $CA^kB,\ldots$ Let r denote the least integer for which $CA^{r-1}B \neq 0$. This integer is called the *relative degree* of system (2.1). In other words, r is the integer uniquely characterized by the conditions

$$\begin{aligned} &CB = CAB = \ldots = CA^{r-2}B = 0 \\ &CA^{r-1}B \neq 0\,. \end{aligned} \tag{2.2}$$

The relative degree of a system can be easily identified with an integer associated with the transfer function of the system. In fact, consider the transfer function of (2.1)

$$T(s) = C(sI-A)^{-1}B$$

and recall that $(sI-A)^{-1}$ can be expanded, in negative powers of s, as

$$(sI-A)^{-1} = \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots$$

This yields, for $T(s)$, the expansion

$$T(s) = \frac{1}{s}CB + \frac{1}{s^2}CAB + \frac{1}{s^3}CA^2B + \cdots$$

Using the definition of r, it is seen that the expansion in question actually reduces to

$$\begin{aligned} T(s) &= \frac{1}{s^r}CA^{r-1}B + \frac{1}{s^{r+1}}CA^rB + \frac{1}{s^{r+2}}CA^{r+1}B + \cdots \\ &= \frac{1}{s^r}[CA^{r-1}B + \frac{1}{s}CA^rB + \frac{1}{s^2}CA^{r+1}B + \cdots] \end{aligned}$$

Thus

$$s^rT(s) = CA^{r-1}B + \frac{1}{s}CA^rB + \frac{1}{s^2}CA^{r+1}B + \cdots$$

from which it is deduced that

$$\lim_{s\to\infty} s^rT(s) = CA^{r-1}B\,. \tag{2.3}$$

Recall now that $T(s)$ is a rational function of s, the ratio between a numerator polynomial $N(s)$ and a denominator polynomial $D(s)$

$$T(s) = \frac{N(s)}{D(s)}\,.$$

Since the limit (2.3) is finite and (by definition of r) *nonzero*, it is concluded that r is necessarily the difference between the degree of $D(s)$ and the degree of $N(s)$. This motivates the terminology “relative degree”. Finally, note that, if $T(s)$ is expressed (as it is always possible) in the form

$$T(s) = K'\frac{\Pi_{i=1}^{n-r}(s-z_i)}{\Pi_{i=1}^{n}(s-p_i)}\,,$$

it necessarily follows that [1]
$$K' = CA^{r-1}B\,.$$

We use now the concept of relative degree to derive a change of variables in the state space, yielding a form of special interest. The following facts are easy consequences of the definition of relative degree.[2]

Proposition 2.1. *The r rows of the $r \times n$ matrix*
$$T_1 = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} \tag{2.4}$$
are linearly independent. As a consequence, $r \leq n$.

We see from this that, if r is strictly less than n, it is possible to find – in many ways – a matrix $T_0 \in \mathbb{R}^{(n-r)\times n}$ such that the resulting $n \times n$ matrix
$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = \begin{pmatrix} T_0 \\ C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} \tag{2.5}$$
is *nonsingular*. Moreover, the following result also holds.

Proposition 2.2. *It is always possible to pick T_0 in such a way that the matrix (2.5) is nonsingular and $T_0 B = 0$.*

We use now the matrix T introduced above to define a change of variables. To this end, in view of the natural partition of the rows of T in two blocks (the upper block consisting of the $n-r$ rows of T_0 and the lower block consisting of the r rows of the matrix (2.4)) it is natural to choose different notations for the first $n-r$ new state variables and for the last r new state variables, setting
$$z = T_0 x, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} x\,.$$

To determine the equations describing the system in the new coordinates, we take the derivatives of z and ξ with respect to time. For the former, no special structure is found and we simply obtain
$$\dot{z} = T_0 \dot{x} = T_0(Ax + Bu) = T_0 A x + T_0 B u\,. \tag{2.6}$$

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[1] The parameter K' is sometimes referred to as the *high-frequency gain* of the system.

[2] For a proof see, e.g., [2, pp. 142-144]

On the contrary, for the latter, a special structure can be displayed. In fact, observing that $\xi_i = CA^{i-1}x$, for $i = 1, \dots, r$, and using the defining properties (2.2), we obtain

$$\begin{aligned}
\dot{\xi}_1 &= C\dot{x} = C(Ax+Bu) = CAx = \xi_2 \\
\dot{\xi}_2 &= CA\dot{x} = CA(Ax+Bu) = CA^2x = \xi_3 \\
&\cdots \\
\dot{\xi}_{r-1} &= CA^{r-2}\dot{x} = CA^{r-2}(Ax+Bu) = CA^{r-1}x = \xi_r
\end{aligned}$$

and

$$\dot{\xi}_r = CA^{r-1}\dot{x} = CA^{r-1}(Ax+Bu) = CA^r x + CA^{r-1}Bu\,.$$

The equations thus found can be cast in a compact form. To this end, it is convenient to introduce a special triplet of matrices, $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times 1}$, $\hat{C} \in \mathbb{R}^{1\times r}$, which are defined as [3]

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.7}$$

With the help of such matrices, it is easy to obtain

$$\begin{aligned}
\dot{\xi} &= \hat{A}\xi + \hat{B}(CA^r x + CA^{r-1}Bu) \\
y &= Cx = \xi_1 = \hat{C}\xi\,.
\end{aligned} \tag{2.8}$$

Note that the right-hand sides of (2.6) and (2.8) are still expressed in terms of the "original" set of state variables x. To complete the change of coordinates, x should be expressed as a (linear) function of the new state variables z and ξ, that is as a function of the form

$$x = M_0 z + M_1 \xi$$

in which M_0 and M_1 are partitions of the inverse of T, implicitly defined by

$$\begin{pmatrix} M_0 & M_1 \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = I\,.$$

Setting

$$\begin{aligned}
A_{00} &= T_0 A M_0\,, \qquad & A_{01} &= T_0 A M_1\,, \qquad & B_0 &= T_0 B\,, \\
A_{10} &= CA^r M_0\,, \qquad & A_{11} &= CA^r M_1, \qquad & b &= CA^{r-1}B\,,
\end{aligned}$$

the equations in question can be cast in the form

[3] A triplet of matrices of this kind is often referred to as a triplet in *prime form*.

$$\begin{aligned}\dot{z} &= A_{00}z + A_{01}\xi + B_0 u \\ \dot{\xi} &= \hat{A}\xi + \hat{B}(A_{10}z + A_{11}\xi + bu) \\ y &= \hat{C}\xi\,.\end{aligned} \tag{2.9}$$

These equations characterize the so-called *normal form* of the equations (2.1) describing the system. Note that the matrix B_0 can be made equal to 0 if the option described in Proposition 2.2 is used. If this is the case, the corresponding normal form is said to be *strict*. On the contrary, the coefficient b, which is equal to the so-called high-frequency gain of the system, is always nonzero.

In summary, by means of the change of variables indicated above, the original system is transformed into a system described by matrices having the following structure

$$TAT^{-1} = \begin{pmatrix} A_{00} & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} \end{pmatrix}, \qquad TB = \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 0 & \hat{C} \end{pmatrix}. \tag{2.10}$$

One of the most relevant features of the normal form of the equations describing the system is the possibility of establishing a relation between the zeros of the transfer function of the system and certain submatrices appearing in (2.9).

We begin by observing that

$$T(s) = \frac{\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix}}{\det(A - sI)}\,. \tag{2.11}$$

This is a simple consequence of a well-know formula for the determinant of a partitioned matrix. [4] Using the latter, we obtain

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det(A - sI)\det(-C(A - sI)^{-1}B)$$

from which, bearing in mind the fact that $C(A - sI)^{-1}B$ is a scalar quantity, the identity (2.11) immediately follows. Note that in this way we have identified simple and appealing expressions for the numerator and denominator polynomial of $T(s)$.

Next, we determine an expansion of the numerator polynomial.

Proposition 2.3. *The following expansion holds*

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = (-1)^r CA^{r-1}B\det([A_{00} - \frac{1}{b}B_0A_{10}] - sI)\,. \tag{2.12}$$

[4] The formula in question is

$$\det\begin{pmatrix} S & P \\ Q & R \end{pmatrix} = \det(S)\det(R - QS^{-1}P)\,.$$

Proof. Observe that the left-hand side of (2.12) remains unchanged if A, B, C is replaced by TAT^{-1}, TB, CT^{-1}. In fact

$$\begin{pmatrix} TAT^{-1} - sI & TB \\ CT^{-1} & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

from which, using the fact that the determinant of a product is the product of the determinants and that

$$\det \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \det \begin{pmatrix} T^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} TT^{-1} & 0 \\ 0 & 1 \end{pmatrix} = 1$$

we obtain

$$\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det \begin{pmatrix} TAT^{-1} - sI & TB \\ CT^{-1} & 0 \end{pmatrix}.$$

Furthermore, observe also that the left-hand side of (2.12) remains unchanged if A is replaced by $A + BF$, regardless of how the matrix $F \in \mathbb{R}^{1 \times n}$ is chosen. This derives from the expansion

$$\begin{pmatrix} A + BF - sI & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ F & 1 \end{pmatrix}$$

and from the fact that the determinant of the right-hand factor in the product above is simply equal to 1.

With these observations in mind, use for T the transformation that generates the normal form (2.10), to arrive at

$$\begin{aligned} \det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} &= \det \begin{pmatrix} \begin{pmatrix} A_{00} - sI & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} - sI \end{pmatrix} & \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{C} \end{pmatrix} & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} \begin{pmatrix} A_{00} - sI & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} - sI \end{pmatrix} + \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} F & \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{C} \end{pmatrix} & 0 \end{pmatrix}. \end{aligned}$$

The choice

$$F = \frac{1}{b} \begin{pmatrix} -A_{10} & -A_{11} \end{pmatrix}$$

yields

$$\begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} F = \begin{pmatrix} -\frac{1}{b}B_0A_{10} & -\frac{1}{b}B_0A_{11} \\ -\hat{B}A_{10} & -\hat{B}A_{11} \end{pmatrix}.$$

and hence it follows that

$$\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det \begin{pmatrix} [A_{00} - \frac{1}{b}B_0A_{10}] - sI & A_{01} - \frac{1}{b}B_0A_{11} & 0 \\ 0 & \hat{A} - sI & \hat{B}b \\ 0 & \hat{C} & 0 \end{pmatrix}.$$

The matrix on the right-hand side is block-triangular, hence

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det([A_{00} - \frac{1}{b}B_0 A_{10}] - sI)\,\det\begin{pmatrix} \hat{A} - sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix}.$$

Finally, observe that

$$\begin{pmatrix} \hat{A} - sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix} = \begin{pmatrix} -s & 1 & 0 & \cdots & 0 & 0 \\ 0 & -s & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -s & b \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Expanding the determinant according to the entries of last row we obtain

$$\det\begin{pmatrix} \hat{A} - sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix} = (-1)^{r+2}\det\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -s & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -s & b \end{pmatrix} = (-1)^r b$$

which, bearing in mind the definition of b, yields (2.12). ◁

With this expansion in mind, we return to the formula (2.11), from which we deduce that

$$T(s) = \frac{(-1)^r CA^{r-1}B\,\det([A_{00} - \frac{1}{b}B_0 A_{10}] - sI)}{\det(A - sI)}.$$

Changing the signs of both matrices in the numerator and denominator yields the final expression

$$T(s) = CA^{r-1}B\,\frac{\det(sI - [A_{00} - \frac{1}{b}B_0 A_{10}])}{\det(sI - A)}. \tag{2.13}$$

If the triplet A, B, C is a minimal realization of its transfer function, i.e. if the pair (A, B) is reachable and the pair (A, C) is observable , the numerator and denominator polynomial of this fraction cannot have common factors.[5] Thus, we can conclude that if the pair (A, B) is reachable and the pair (A, C) is observable, the $n - r$ eigenvalues of the matrix

$$[A_{00} - \frac{1}{b}B_0 A_{10}].$$

can be identified with *zeros of the transfer function* $T(s)$.

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[5] Otherwise, $T(s)$ could be written as strictly proper rational function in which the denominator is a polynomial of degree *strictly less* than n. This would imply the existence of a realization of dimension strictly less than n, contradicting the minimality of A, B, C.

Remark 2.1. Note that if, in the transformation T used to obtain the normal form, the matrix T_0 has been chosen so as to satisfy $TB = 0$, the structure of the normal form (2.10) is simplified, and $B_0 = 0$. In this case, the zeros of the transfer function coincide with the eigenvalues of the matrix A_{00}. $\triangleleft$

2.2 The hypothesis of minimum-phase

We consider in this chapter the case of a linear single-input single-output system of fixed dimension n, whose coefficient matrices may depend on a *vector* μ of (possibly) *uncertain parameters*. The value of μ is not known, nor is available for measurement (neither directly nor indirectly through an estimation filter) but it is assumed to be *constant* and to range over a fixed, and *know*, *compact set* $\mathbb{M}$.

Accordingly, the equations (2.1) will be written in the form

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u \\ y &= C(\mu)x\,. \end{aligned} \tag{2.14}$$

The theory described in what follows is based on the following basic hypothesis.

Assumption 2.1 *$A(\mu), B(\mu), C(\mu)$ are matrices of continuous functions of μ. For every $\mu \in \mathbb{M}$, the pair $(A(\mu), B(\mu))$ is reachable and the pair $(A(\mu), C(\mu))$ is observable. Moreover:*

(i) The relative degree of the system is the same for all $\mu \in \mathbb{M}$.
(ii) The zeros of the transfer function $C(\mu)(sI - A(\mu))^{-1}B(\mu)$ have negative real part for all $\mu \in \mathbb{M}$.

In the classical theory of servomechanisms, systems whose transfer function has zeros only in the (closed) left-half complex plane have been often referred to as *minimum-phase systems*. This a terminology that dates back more or less to the works of H.W. Bode.[6] For convenience, we keep this terminology to express the property that a system satisfies condition (ii) of Assumption 2.1, even though the latter (which, as it will be seen, is instrumental in the proposed robust stabilization strategy) excludes the occurrence of zeros on the imaginary axis. Thus, in what follows, a (linear) system satisfying condition (ii) of Assumption 2.1 will be referred to as a minimum-phase system.

Letting r denote the relative degree, system (2.14) can be put in *strict* normal form, by means of a change of variables

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0(\mu) \\ T_1(\mu) \end{pmatrix} x$$

in which

[6] See [1] and also [3, p. 283].

$$T_1(\mu) = \begin{pmatrix} C(\mu) \\ C(\mu)A(\mu) \\ \cdots \\ C(\mu)A^{r-1}(\mu) \end{pmatrix}$$

and $T_0(\mu)$ satisfies $T_0(\mu)B(\mu) = 0$. Note that $T_1(\mu)$ is by construction a continuous function of μ and $T_0(\mu)$ can always be chosen as a continuous function of μ.

The normal form in question is written as

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u \\ y &= \xi_1 , \end{aligned}$$

with $z \in \mathbb{R}^{n-r}$ or, alternatively, as

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u] \\ y &= \hat{C}\xi \end{aligned}$$

in which $\hat{A}, \hat{B}, \hat{C}$ are the matrices introduced in (2.7). Moreover,

$$b(\mu) = C(\mu)A^{r-1}(\mu)B(\mu) .$$

As a consequence of Assumption 2.1:

(i) $b(\mu) \neq 0$ for all $\mu \in \mathbb{M}$. By continuity, $b(\mu)$ it is either positive or negative for all $\mu \in \mathbb{M}$. In what follows, without loss of generality, it will be assumed that

$$b(\mu) > 0 \qquad \text{for all } \mu \in \mathbb{M}. \tag{2.15}$$

(ii) The eigenvalues of $A_{00}(\mu)$ have negative real part for all $\mu \in \mathbb{M}$. This being the case, it is known from the converse Lyapunov Theorem [7] that there exists a unique, symmetric and *positive definite*, matrix $P(\mu)$, of dimension $(n-r) \times (n-r)$, of *continuous* functions of μ such that

$$P(\mu)A_{00}(\mu) + A_{00}^{\mathrm{T}}(\mu)P(\mu) = -I \qquad \text{for all } \mu \in \mathbb{M}. \tag{2.16}$$

2.3 The case of relative degree 1

Perturbed systems that belong to the class of systems characterized by Assumption 2.1 can be *robustly stabilized* by means of a very simple-minded feedback strategy,

[7] See Theorem A.3 in Appendix A.

as it will be described in what follows. The equations (2.14) define a *set* of systems, one for each value of μ. A *robust stabilizer* is a feedback law that stabilizes each member of this set. The feedback law in question is not allowed to know which one is the individual member of the set that is being controlled, i.e. it is has to be the same for each member of the set. In the present context of systems modeled as in (2.14), the robustly stabilizing feedback must be a fixed dynamical system not depending on the value of μ.

For simplicity, we address first in this section the case of a system having relative degree is 1. In this case ξ is a vector of dimension 1 (i.e. a scalar quantity) and the normal form reduces to

$$\begin{aligned}\dot{z} &= A_{00}(\mu)z + A_{10}(\mu)\xi \\ \dot{\xi} &= A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u \\ y &= \xi\,.\end{aligned} \tag{2.17}$$

Consider the control law [8]

$$u = -ky\,. \tag{2.18}$$

This yields a closed loop system of the form

$$\begin{aligned}\dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi} &= A_{10}(\mu)z + [A_{11}(\mu) - b(\mu)k]\xi\,,\end{aligned}$$

or, what is the same

$$\begin{pmatrix}\dot{z} \\ \dot{\xi}\end{pmatrix} = \begin{pmatrix}A_{00}(\mu) & A_{01}(\mu) \\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix}\begin{pmatrix}z \\ \xi\end{pmatrix}. \tag{2.19}$$

We want to prove that, under the standing assumptions, if k is large enough this system is stable for all $\mu \in \mathbb{M}$. To this end, consider the positive definite $n \times n$ matrix

$$\begin{pmatrix}P(\mu) & 0 \\ 0 & 1\end{pmatrix}$$

in which $P(\mu)$ is the matrix defined in (2.16). If we are able to show that the matrix

$$\begin{aligned}Q(\mu) = &\begin{pmatrix}P(\mu) & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}A_{00}(\mu) & A_{01}(\mu) \\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix} \\ &+ \begin{pmatrix}A_{00}(\mu) & A_{01}(\mu) \\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}P(\mu) & 0 \\ 0 & 1\end{pmatrix}\end{aligned}$$

is *negative definite*, then by the direct Lyapunov Theorem,[9] we can assert that system (2.19) has all eigenvalues with negative real part. A simple calculation shows that, because of (2.16),

[8] The negative sign is a consequence of the standing hypothesis (2.15). If $b(\mu) < 0$, the sign must be reversed.

[9] See Theorem A.2 in Appendix A.

$$Q(\mu) = \begin{pmatrix} -I & [P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)] \\ [P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]^{\mathrm{T}} & 2[A_{11}(\mu) - b(\mu)k] \end{pmatrix} \tag{2.20}$$

To check positive definiteness, we change sign to $Q(\mu)$ and appeal to Sylvester's criterion for positive definiteness, i.e. we check the sign of all leading principal minors. Because of the special form of $-Q(\mu)$, all its leading principal minors of order $1, 2, \ldots, n-1$ are equal to 1 (and hence positive). Thus the matrix in question is positive definite if (and only if) its determinant is positive. To compute the determinant, we observe that the matrix in question has the form

$$-Q(\mu) = \begin{pmatrix} I & d(\mu) \\ d^{\mathrm{T}}(\mu) & q(\mu) \end{pmatrix}.$$

Hence, thanks to the formula for the determinant of a partitioned matrix

$$\det[-Q(\mu)] = \det[I]\det[q(\mu) - d^{\mathrm{T}}(\mu)I^{-1}d(\mu)] = q(\mu) - d^{\mathrm{T}}(\mu)d(\mu) = q(\mu) - \|d(\mu)\|^2.$$

Thus, the conclusion is that $Q(\mu)$ is negative definite for all $\mu \in \mathbb{M}$ if (and only if)

$$q(\mu) - \|d(\mu)\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Reverting to the notations of (2.20) we can say that $Q(\mu)$ is negative definite, or – what is the same – system (2.19) has all eigenvalues with negative real part, for all $\mu \in \mathbb{M}$ if

$$-2[A_{11}(\mu) - b(\mu)k] - \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Since $b(\mu) > 0$, the inequality is equivalent to

$$k > \frac{1}{2b(\mu)}\Big(2A_{11}(\mu) + \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2\Big) \qquad \text{for all } \mu \in \mathbb{M}.$$

Now, set

$$k^* := \max_{\mu\in\mathbb{M}}\Big(\frac{2A_{11}(\mu) + \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2}{2b(\mu)}\Big).$$

This maximum exists because the functions are continuous functions of μ and $\mathbb{M}$ is a compact set. Then we are able to conclude that if

$$k > k^*$$

the control law

$$u = -ky$$

stabilizes the closed loop system, regardless of what the particular value of $\mu \in \mathbb{M}$ is. In other words, this control *robustly* stabilizes the given set of systems.

Remark 2.2. The control law $u = -ky$ is usually referred to as a *high-gain* output feedback. As it is seen from the previous analysis, a large value of k makes the matrix $Q(\mu)$ negative definite. The matrix in question has the form

$$Q(\mu) = \begin{pmatrix} -I & -d(\mu) \\ -d^{\mathrm{T}}(\mu) & -2b(\mu)k + q_0(\mu) \end{pmatrix}.$$

The role of a large k is to render the term $-2b(\mu)k$ *sufficiently negative*, so as: (i) to overcome the uncertain term $q_0(\mu)$, and (ii) to overcome the effect of the uncertain off-diagonal terms. Reverting to the equations (2.19) that describe the closed-loop system, one may observe that the upper equation can be seen as a stable subsystem with state z and input ξ, while the lower equation can be seen as a subsystem with state ξ and input z. The role of a *large* k is: (i) to render the lower subsystem stable, and (ii) to *lower* the effect of the *coupling* between the two subsystems. This second role, which is usually referred to as a *small-gain property*, will be described and interpreted in full generality in the next Chapter. $\triangleleft$

2.4 The case of higher relative degree: partial state feedback

Consider now the case of a system having higher relative degree $r > 1$. This system can be "artificially" reduced to a system to which the stabilization procedure described in the previous section is applicable, by means of a simple strategy. Let the variable ξ_r of the normal form be replaced by a new state variable defined as

$$\theta = \xi_r + a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} \tag{2.21}$$

in which $a_0, a_1, \ldots, a_{r-2}$ are design parameters. With this change of variable (it's only a change of variables, no control has been chosen yet !), a system is obtained which has the form

$$\begin{aligned}
\dot z &= A_{00}(\mu)z + \tilde a_{01}(\mu)\xi_1 + \cdots + \tilde a_{0,r-1}(\mu)\xi_{r-1} + \tilde a_{0r}(\mu)\theta \\
\dot\xi_1 &= \xi_2 \\
&\cdots \\
\dot\xi_{r-1} &= -(a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1}) + \theta \\
\dot\theta &= A_{10}(\mu)z + \tilde a_{11}(\mu)\xi_1 + \cdots + \tilde a_{1,r-1}(\mu)\xi_{r-1} + \tilde a_{1r}(\mu)\theta + b(\mu)u \\
y &= \xi_1\,,
\end{aligned}$$

in which the $\tilde a_{0i}(\mu)$'s and $\tilde a_{1i}(\mu)$'s are appropriate coefficients.[10]

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[10] These coefficients can be easily derived as follows. Let

$$A_{01}(\mu) = \begin{pmatrix} a_{01,1} & a_{01,2} & \cdots & a_{01,r} \end{pmatrix}.$$

Hence

$$A_{01}(\mu)\xi = a_{01,1}\xi_1 + a_{01,2}\xi_2 + \cdots + a_{01,r}\xi_r\,.$$

This system can be formally viewed as a system having relative degree 1, with input u and output θ. To this end, in fact, it suffices to set

$$\zeta = \begin{pmatrix} z \\ \xi_1 \\ \cdots \\ \xi_{r-1} \end{pmatrix}$$

and rewrite the system as

$$\dot\zeta = \begin{pmatrix} A_{00}(\mu) & \tilde a_{01}(\mu) & \tilde a_{02}(\mu) & \cdots & \tilde a_{0,r-1}(\mu) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix} \zeta + \begin{pmatrix} \tilde a_{0r}(\mu) \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \theta$$

$$\dot\theta = \begin{pmatrix} A_{10}(\mu) & \tilde a_{11}(\mu) & \tilde a_{12}(\mu) & \cdots & \tilde a_{1,r-1}(\mu) \end{pmatrix} \zeta + \tilde a_{1r}(\mu)\theta + b(\mu)u\,.$$

The latter has the structure of a system in normal form

$$\begin{aligned} \dot\zeta &= F_{00}(\mu)\zeta + F_{01}(\mu)\theta \\ \dot\theta &= F_{10}(\mu)\zeta + F_{11}(\mu)\theta + b(\mu)u \end{aligned} \tag{2.22}$$

in which $F_{00}(\mu)$ is a $(n-1)\times(n-1)$ block-triangular matrix

$$F_{00}(\mu) = \begin{pmatrix} A_{00}(\mu) & \tilde a_{01}(\mu) & \tilde a_{02}(\mu) & \cdots & \tilde a_{0,r-1}(\mu) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix} := \begin{pmatrix} A_{00}(\mu) & * \\ 0 & A_0 \end{pmatrix}$$

with

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix}.$$

Since

$$\xi_r = \theta - (a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1})$$

we see that

$$A_{01}(\mu)\xi = [a_{01,1} - a_{01,r}a_0]\xi_1 + [a_{01,2} - a_{01,r}a_1]\xi_2 + \cdots + [a_{01,r-1} - a_{01,r}a_{r-2}]\xi_{r-1} + a_{01,r}\theta$$

The latter can be rewritten as

$$\tilde a_{01}(\mu)\xi_1 + \cdots + \tilde a_{0,r-1}(\mu)\xi_{r-1} + \tilde a_{0r}(\mu)\theta\,.$$

A similar procedure is followed to transform $A_{11}(\mu)\xi + a_0\dot\xi_1 + \cdots + a_{r-1}\dot\xi_{r-1}$.

By Assumption 2.1, all eigenvalues of the submatrix $A_{00}(\mu)$ have negative real part for all μ. On the other hand, the characteristic polynomial of the submatrix A_0, which is a matrix in companion form, coincides with the polynomial

$$p_a(\lambda) = a_0 + a_1\lambda + \cdots + a_{r-2}\lambda^{r-2} + \lambda^{r-1}\,. \tag{2.23}$$

The design parameters $a_0, a_1, \ldots, a_{r-2}$ can be chosen in such a way that all eigenvalues of A_0 have negative real part. If this is the case, we can conclude that *all* the $n-1$ eigenvalues of $F_{00}(\mu)$ have negative real part, for all μ.

Thus, system (2.22) can be seen as a system having relative degree 1 which satisfies all the assumptions used in the previous section to obtain robust stability. In view of this, it immediately follows that there exists a number k^* such that, if $k > k^*$, the control law

$$u = -k\theta$$

robustly stabilizes such system.

Note that the control thus found, expressed in the original coordinates, reads as

$$u = -k[a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} + \xi_r]$$

that is as a linear combination of the components of the vector ξ. This is a *partial state* feedback, which can be written, in compact form, as

$$u = H\xi\,. \tag{2.24}$$

Remark 2.3. It it worth to observe that, by definition,

$$\xi_1(t) = y(t), \quad \xi_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \quad \ldots \quad , \quad \xi_r(t) = \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}\,.$$

Thus, the variable θ is seen to satisfy

$$\theta(t) = a_0 y(t) + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \cdots + a_{r-2}\frac{\mathrm{d}^{r-2}y(t)}{\mathrm{d}t^{r-2}} + \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}\,.$$

In other words, θ can be seen as output of a system with input y and transfer function

$$D(s) = a_0 + a_1 s + \cdots + a_{r-2}s^{r-2} + s^{r-1}\,.$$

It readily follows that, if $T(s,\mu) = C(\mu)(sI - A(\mu))^{-1}B(\mu)$ is the transfer function of (2.14), the transfer function of system (2.22), seen as a system with input u and output θ, is equal to $D(s)T(s,\mu)$. As confirmed by the state space analysis and in particular from the structure of $F_{00}(\mu)$, this new system has $n-1$ zeros, $n-r$ of which coincide with the original zeros of (2.14), while the additional $r-1$ zeros coincide with the roots of the polynomial (2.23). In other words, the indicated design method can be interpreted as an addition of $r-1$ zeros having negative real part (so as to lower the relative degree to the value 1 while keeping the property that all zeros have negative real part) followed by high-gain output feedback on the resulting output.

Example 2.1. Consider the problem of robustly stabilizing a rocket's upright orientation in the initial phase of the launch. The equation describing the motion on a vertical plane is similar to those that describes the motion of an inverted pendulum (see Figure 2.1) and has the form [11]

$$J_t \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = mg\ell\sin(\varphi) - \gamma\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \ell\cos(\varphi)u\,.$$

in which ℓ is the length of the pendulum, m is the mass concentrated at the tip of the pendulum, $J_t = J + m\ell^2$ is the total moment of inertia, γ is a coefficient of rotational viscous friction and u is a force applied at the base.

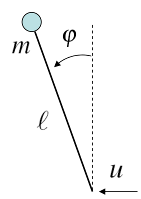

Fig. 2.1 An inverted pendulum

If the angle φ is sufficiently small, one can use the approximations $\sin(\varphi) \approx \varphi$ and $\cos(\varphi) \approx 1$ and obtain a linear model. Setting $\xi_1 = \varphi$ and $\xi_2 = \dot{\varphi}$, the equation can be put in state space form as

$$\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= q_1\xi_1 + q_2\xi_2 + bu\end{aligned}$$

in which

$$q_1 = \frac{mg\ell}{J_t}\,, \qquad q_2 = -\frac{\gamma}{J_t}\,, \qquad b = \frac{\ell}{J_t}\,.$$

Note that the system is unstable (because q_1 is positive) and that, if φ is considered as output, the system has relative degree 2, and hence is trivially minimum phase.

According to the procedure described above, we pick (compare with (2.21))

$$\theta = \xi_2 + a_0\xi_1$$

with $a_0 > 0$, and obtain (compare with (2.22))

[11] See [3, pp. 36-37].

$$\begin{aligned}\dot{\xi}_1 &= -a_0\xi_1 + \theta \\ \dot{\theta} &= (q_1 - a_0q_2 - a_0^2)\xi_1 + (q_2 + a_0)\theta + bu\,.\end{aligned}$$

This system is going to be controlled by

$$u = -k\theta\,,$$

which results in (compare with (2.19))

$$\begin{pmatrix}\dot{\xi}_1 \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}-a_0 & 1 \\ (q_1 - a_0q_2 - a_0^2) & (q_2 + a_0) - bk\end{pmatrix}\begin{pmatrix}\xi_1 \\ \theta\end{pmatrix}.$$

It is known from the theory described above that, if k is sufficiently large, the system is stable. To determine the minimal value of k as a function of the parameters, we impose that the matrix

$$\begin{pmatrix}-a_0 & 1 \\ (q_1 - a_0q_2 - a_0^2) & (q_2 + a_0) - bk\end{pmatrix}$$

has eigenvalues with negative real part. This is the case if

$$\begin{aligned}&(q_2 + a_0) - bk < 0 \\ &(-a_0)(q_2 + a_0 - bk) - (q_1 - a_0q_2 - a_0^2) > 0\,,\end{aligned}$$

that is

$$k > \max\{\frac{q_2 + a_0}{b}, \frac{q_1}{a_0b}\}\,.$$

Reverting to the original parameters, this yields

$$k > \max\{\frac{a_0J_t - \gamma}{\ell}, \frac{mg}{a_0}\}\,.$$

A conservative estimate is obtained if the term $-\gamma$ is neglected (which is reasonable, since the value of γ, the coefficient of viscous friction, may be subject to large variations), obtaining

$$k > \max\{\frac{a_0(J + m\ell^2)}{\ell}, \frac{mg}{a_0}\}\,.$$

Once the ranges of the parameters J, m, ℓ are specified, this expression can be used to determine the design parameters a_0 and k. Expressed in the original variable the stabilizing control is

$$u = -k(\xi_2 + a_0\xi_1) = -k\dot{\varphi} - ka_0\varphi\,, \tag{2.25}$$

that is, the classical "proportional-derivative" feedback. ◁

2.5 The case of higher relative degree: output feedback

We have seen in the previous section that a system satisfying Assumption 2.1 can be robustly stabilized by means of a feedback law which is a linear form in the states $\xi_1,\ldots,\xi_r$ that characterize its normal form. In general, the components of the state ξ are not directly available for feedback, nor they can be retrieved from the original state x, since the transformation that defines ξ in terms of x depends on the uncertain parameter μ. We see now how this problem can be overcome, by designing a dynamic controller that provides appropriate "replacements" for the components of ξ in the control law (2.24). Observing that these variables coincide, by definition, with the measured output y and with its first $r-1$ derivatives with respect to time, it seems reasonable to try to generate the latter by means of a dynamical system of the form

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \kappa c_{r-1}(y-\hat{\xi}_1)\\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \kappa^2 c_{r-2}(y-\hat{\xi}_1)\\
&\cdots\\
\dot{\hat{\xi}}_{r-1} &= \hat{\xi}_r + \kappa^{r-1} c_1(y-\hat{\xi}_1)\\
\dot{\hat{\xi}}_r &= \kappa^r c_0(y-\hat{\xi}_1)\,.
\end{aligned} \tag{2.26}$$

In fact, if $\hat{\xi}_1(t)$ where identical to $y(t)$, it would follow that $\hat{\xi}_i(t)$ coincides with $y^{(i-1)}(t)$, that is with $\xi_i(t)$, for all $i = 1,2,\ldots,r$. In compact form, the system thus defined can be rewritten as

$$\dot{\hat{\xi}} = \hat{A}\hat{\xi} + D_\kappa G_0(y-\hat{C}\hat{\xi})\,,$$

in which

$$G_0 = \begin{pmatrix} c_{r-1}\\ c_{r-2}\\ \cdots\\ c_1\\ c_0 \end{pmatrix}, \qquad D_\kappa = \begin{pmatrix} \kappa & 0 & \cdots & 0\\ 0 & \kappa^2 & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdot & \kappa^r \end{pmatrix},$$

and $\hat{A},\hat{C}$ are the matrices defined in (2.7).

Let now ξ be replaced by $\hat{\xi}$ in the expression of the control law (2.24). In this way, we obtain a dynamic controller, described by equations of the form

$$\begin{aligned}
\dot{\hat{\xi}} &= \hat{A}\hat{\xi} + D_\kappa G_0(y-\hat{C}\hat{\xi})\\
u &= H\hat{\xi}\,.
\end{aligned} \tag{2.27}$$

It will be shown in what follows that, if the parameters κ and $c_0,\ldots,c_{r-1}$ which characterize (2.27) are chosen appropriately, this *dynamic – output feedback – control* law does actually robustly stabilize the system.

Controlling the system (assumed to be expressed in strict normal form) by means of the control (2.27) yields a closed loop system

$$\begin{aligned}
\dot z &= A_{00}(\mu)z + A_{01}(\mu)\xi\\
\dot\xi &= \hat A\xi + \hat B[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H\hat\xi]\\
\dot{\hat\xi} &= \hat A\hat\xi + D_\kappa G_0(y - \hat C\hat\xi)\,.
\end{aligned}$$

To analyze this closed-loop system, we first perform a change of coordinates, letting the $\hat\xi_i$'s be replaced by variables e_i's defined as

$$e_i = \kappa^{r-i}(\xi_i - \hat\xi_i), \qquad i = 1,\dots,r\,.$$

According to the definition of the matrix D_κ, it is observed that

$$e = \kappa^r D_\kappa^{-1}(\xi - \hat\xi)\,,$$

that is

$$\hat\xi = \xi - \kappa^{-r}D_\kappa e\,.$$

The next step in the analysis is to determine the differential equations for the new variables e_i, for $i = 1,\dots,r$. Setting

$$e = \mathrm{col}(e_1,\dots,e_r)\,.$$

a simple calculation yields

$$\dot e = \kappa(\hat A - G_0\hat C)e + \hat B[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H\hat\xi]\,.$$

Replacing also $\hat\xi$ with its expression in terms of ξ and e we obtain, at the end, a description of the closed loop system in the form

$$\begin{aligned}
\dot z &= A_{00}(\mu)z + A_{01}(\mu)\xi\\
\dot\xi &= \hat A\xi + \hat B[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H(\xi - \kappa^{-r}D_\kappa e)]\\
\dot e &= \kappa(\hat A - G_0\hat C)e + \hat B[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H(\xi - \kappa^{-r}D_\kappa e)]\,.
\end{aligned}$$

To simplify this system further, we set

$$\tilde x = \begin{pmatrix} z\\ \xi \end{pmatrix}$$

and define the matrices

$$\begin{aligned}
F_{00}(\mu) &= \begin{pmatrix} A_{00}(\mu) & A_{01}(\mu)\\ \hat B A_{10}(\mu) & \hat A + \hat B[A_{11}(\mu) + b(\mu)H] \end{pmatrix}\\
F_{01}(\mu) &= -\begin{pmatrix} 0\\ \hat B b(\mu)H \end{pmatrix}\\
F_{10}(\mu) &= (\,\hat B A_{10}(\mu) \quad \hat B[A_{11}(\mu) + b(\mu)H]\,)\\
F_{11}(\mu) &= -\hat B b(\mu)H
\end{aligned}$$

in which case the equations of the closed-loop system will be rewritten as

$$\begin{aligned}\dot{\tilde{x}} &= F_{00}(\mu)\tilde{x} + F_{01}(\mu)\kappa^{-r}D_\kappa e\\ \dot{e} &= F_{10}(\mu)\tilde{x} + [\kappa(\hat{A}-G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa]e\,.\end{aligned}$$

The advantage of having the system written in this form is that we know that the matrix $F_{00}(\mu)$, if H has been chosen as described in the earlier section, has eigenvalues with negative real part for all μ. Hence, there is a positive definite symmetric matrix $P(\mu)$ such that

$$P(\mu)F_{00}(\mu) + F_{00}(\mu)^{\mathrm{T}}P(\mu) = -I\,.$$

Moreover, it is readily seen that the characteristic polynomial of the matrix $(\hat{A} - G_0\hat{C})$ coincides with the polynomial

$$p_c(\lambda) = c_0 + c_1\lambda + \cdots + c_{r-1}\lambda^{r-1} + \lambda^r\,. \tag{2.28}$$

Thus, the coefficients $c_0, c_1, \ldots, c_{r-1}$ can be chosen in such a way that all eigenvalues of $(\hat{A} - G_0\hat{C})$ have negative real part. If this is done, there exists a positive definite symmetric matrix $\hat{P}$ such that

$$\hat{P}(\hat{A}-G_0\hat{C}) + (\hat{A}-G_0\hat{C})^{\mathrm{T}}\hat{P} = -I\,.$$

This being the case, we proceed now to show that the direct criterion of Lyapunov is fulfilled, for the positive definite matrix

$$\begin{pmatrix} P(\mu) & 0\\ 0 & \hat{P}\end{pmatrix}$$

if the number κ is large enough. To this end, we need to check that the matrix

$$\begin{aligned}Q = &\begin{pmatrix} P(\mu) & 0\\ 0 & \hat{P}\end{pmatrix}\begin{pmatrix} F_{00}(\mu) & F_{01}(\mu)\kappa^{-r}D_\kappa\\ F_{10}(\mu) & \kappa(\hat{A}-G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa\end{pmatrix}\\ &+\begin{pmatrix} F_{00}(\mu) & F_{01}(\mu)\kappa^{-r}D_\kappa\\ F_{10}(\mu) & \kappa(\hat{A}-G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} P(\mu) & 0\\ 0 & \hat{P}\end{pmatrix}\end{aligned}$$

is negative definite. In view of the definitions of $P(\mu)$ and $\hat{P}$, we see that $-Q$ has the form

$$-Q = \begin{pmatrix} I & -d(\mu,\kappa)\\ -d^{\mathrm{T}}(\mu,\kappa) & \kappa I - q(\mu,\kappa)\end{pmatrix}.$$

in which

$$\begin{aligned}d(\mu,\kappa) &= [P(\mu)F_{01}(\mu)\kappa^{-r}D_\kappa + F_{10}^{\mathrm{T}}(\mu)\hat{P}]\\ q(\mu,\kappa) &= [\hat{P}F_{11}(\mu)\kappa^{-r}D_\kappa + \kappa^{-r}D_\kappa F_{11}^{\mathrm{T}}(\mu)\hat{P}]\,.\end{aligned}$$

We want the matrix $-Q$ to be positive definite. According to Schur's Lemma,[12] this is the case if and only if the Schur's complement

$$\kappa I - q(\mu,\kappa) - d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa) \tag{2.29}$$

is positive definite. This is actually the case if κ is large enough. To check this claim, assume, without loss of generality, that $\kappa \geq 1$ and observe that in this case the diagonal matrix

$$\kappa^{-r}D_\kappa = \mathrm{diag}(\kappa^{-r+1},\ldots,\kappa^{-1},1)\,,$$

has norm 1. If this is the case, the positive number

$$\kappa^* = \sup_{\substack{\mu\in\mathbb{M}\\ \kappa\geq 1}} \|q(\mu,\kappa) + d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa)\|$$

is well-defined. To say that the quadratic form (2.29) is positive definite is to say that, for any nonzero $z \in \mathbb{R}^r$,

$$\kappa z^{\mathrm{T}} z > z^{\mathrm{T}}[q(\mu,\kappa) + d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa)]z\,.$$

Clearly, if $\kappa > \max\{1,\kappa^*\}$, this inequality holds and the matrix (2.29) is positive definite.

It is therefore concluded that if system (2.14) is controlled by (2.27), with H chosen as indicated in the previous section, and $\kappa > \max\{1,\kappa^*\}$, the resulting closed loop system has all eigenvalues with negative real part, for any $\mu \in \mathbb{M}$.

In summary, we have shown that the uncertain system (2.1), under Assumption 2.1, can be *robustly stabilized* by means of a dynamic output-feedback control law of the form

$$\dot{\hat{\xi}} = \begin{pmatrix} -\kappa c_{r-1} & 1 & 0 & \cdots & 0 \\ -\kappa^2 c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\kappa^{r-1} c_1 & 0 & 0 & \cdots & 1 \\ -\kappa^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} \kappa c_{r-1} \\ \kappa^2 c_{r-2} \\ \cdots \\ \kappa^{r-1} c_1 \\ \kappa^r c_0 \end{pmatrix} y$$

$$u = -k\,(a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{r-2} \quad 1)\,\hat{\xi}\,,$$

in which $c_0, c_1, \ldots, c_{r-1}$ and, respectively, $a_0, a_1, \ldots, a_{r-2}$ are such that the polynomials (2.28) and (2.23) have negative real part, and κ and k are large (positive) parameters.

Remark 2.4. Note the striking similarity of the arguments used to show the negative definiteness of Q with those used in section 2.3. The large value of κ is instrumental in overcoming the effects of an additive term in the bottom-right block and of the off-diagonal terms. Both these terms depend now on κ (this was not the case in

[12] See (A.1) in Appendix A.

section 2.3) but fortunately, if $\kappa \geq 1$, such terms have bounds that are independent of κ. Also in this case, we can interpret the resulting system as interconnection of a stable subsystem with state x and input e, connected to a subsystem with state e and input z. The role of a large κ is: (i) to render the lower subsystem stable, and (ii) to lower the effect of the coupling between the two subsystems. ◁

Example 2.2. Consider again the system of Example 2.1 and suppose that only the variable φ is available for feedback. In this case, we use a control

$$u = -k(a_0\hat{\xi}_1 + \hat{\xi}_2)\,,$$

in which $\hat{\xi}_1, \hat{\xi}_2$ are provided by the dynamical system

$$\dot{\hat{\xi}} = \begin{pmatrix} -\kappa c_1 & 1 \\ -\kappa^2 c_0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} \kappa c_1 \\ \kappa^2 c_0 \end{pmatrix} \varphi\,.$$

For convenience, we can take $c_0 = 1$ and $c_1 = 2$, in which case the characteristic polynomial of this system becomes $(s+\kappa)^2$. Setting $\varepsilon = 1/\kappa$ and computing the transfer function $T_{\mathrm{c}}(s)$ of the controller, between φ and u, it is seen that

$$T_{\mathrm{c}}(s) = -k\,\frac{(1+2a_0\varepsilon)s + a_0}{(\varepsilon s + 1)^2}\,.$$

Or course, as $\varepsilon \to 0$, the control approaches the proportional-derivative control (2.25) . ◁

References

1. H. W. Bode, *Network Analysis and Feedback Amplifier Design*, Van Nostrand, New York (1945)
2. A. Isidori: Sistemi di Controllo, Vol. II, Siderea, Roma (1993), *in Italian.*
3. K.J. Astrom, R.M. Murray: Feedback Systems. Princeton University Press, Princeton (2008)

Chapter 3
The Small-gain Theorem for Linear Systems and its Applications to Robust Stability

Abstract In a system consisting of the interconnection of several component subsystems, some of which could be only poorly modeled, stability analysis and feedback design might not be easy tasks. Thus, methods allowing to understand the influence of interconnections on stability and asymptotic behavior are important. The methods in question are based on the use of a concept of *gain*, which can take alternative forms and can be evaluated by means of a number of alternative methods. This Chapter describes the various alternative forms of such concept of gain, and shows why this is useful in the analysis of stability of interconnected systems. A major consequence is the development of a systematic method for stabilization in the presence of a (general class of) model uncertainties.

3.1 The $\mathscr{L}_2$ gain of a stable linear system

In this section, we analyze some properties of the forced response of a linear system to piecewise continuous input functions – defined on the time interval $[0,\infty)$ – which have the following property

$$\lim_{T\to\infty}\int_0^T \|u(t)\|^2 dt < \infty .$$

The space of all such functions, endowed with the so-called $\mathscr{L}_2$ norm, which is defined as

$$\|u(\cdot)\|_{\mathscr{L}_2} := \Big(\int_0^\infty \|u(t)\|^2 dt\Big)^{\frac{1}{2}},$$

is denoted by $\mathscr{U}_{\mathscr{L}_2}$. The main purpose of the analysis is to show that, if the system is *stable*, the forced output response from the initial state $x(0) = 0$ has a similar property, i.e.

$$\lim_{T\to\infty}\int_0^T \|y(t)\|^2 dt < \infty . \tag{3.1}$$

This makes it possible to compare the $\mathscr{L}_2$ norms of input and output functions and define a concept of "gain" accordingly.

Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.2}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$. Suppose the matrix A has all eigenvalues with negative real part. Let α be a positive number. According to the direct criterion of Lyapunov, the equation

$$PA + A^{\mathrm{T}}P = -\alpha I \tag{3.3}$$

has a unique solution P, which is a symmetric and positive definite matrix.

Let $V(x) = x^{\mathrm{T}}Px$ be the associated positive definite quadratic form. If $x(t)$ is any trajectory of (2.2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}(t)Px(t)] &= \frac{\partial V}{\partial x}\Big|_{x=x(t)}\dot{x}(t) = 2x^{\mathrm{T}}(t)P(Ax(t)+Bu(t)) \\ &= x^{\mathrm{T}}(t)(PA+A^{\mathrm{T}}P)x(t) + 2x^{\mathrm{T}}(t)PBu(t)\,. \end{aligned}$$

Adding and subtracting $d^2u(t)^{\mathrm{T}}u(t)$ to the right-hand side, and dropping – for convenience – the dependence on t, we obtain successively

$$\begin{aligned} \frac{\partial V}{\partial x}(Ax+Bu) &= x^{\mathrm{T}}(PA+A^{\mathrm{T}}P)x + d^2u^{\mathrm{T}}u - d^2u^{\mathrm{T}}u + 2x^{\mathrm{T}}PBu \\ &= -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u - d^2(u - \tfrac{1}{d^2}B^{\mathrm{T}}Px)^{\mathrm{T}}(u - \tfrac{1}{d^2}B^{\mathrm{T}}Px) + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px \\ &\le -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px\,. \end{aligned}$$

Subtracting and adding to the right-hand side the quantity $y^{\mathrm{T}}y$ and using the inequality

$$y^{\mathrm{T}}y = (Cx+Du)^{\mathrm{T}}(Cx+Du) \le 2x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2u^{\mathrm{T}}D^{\mathrm{T}}Du$$

it is seen that

$$\begin{aligned} \frac{\partial V}{\partial x}(Ax+Bu) &\le -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px - y^{\mathrm{T}}y + 2x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2u^{\mathrm{T}}D^{\mathrm{T}}Du \\ &= x^{\mathrm{T}}(-\alpha I + \tfrac{1}{d^2}PBB^{\mathrm{T}}P + 2C^{\mathrm{T}}C)x + u^{\mathrm{T}}(d^2I + 2D^{\mathrm{T}}D)u - y^{\mathrm{T}}y\,. \end{aligned}$$

Clearly, for any choice of $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$-\alpha I + 2C^{\mathrm{T}}C \le -2\varepsilon I\,.$$

Pick one of such α and let P be determined accordingly as a solution of (3.3). With P fixed in this way, it is seen that, if the number d is sufficiently large, the inequality

$$\frac{1}{d^2}PBB^{\mathrm{T}}P \le \varepsilon I$$

holds, and hence

$$x^{\mathrm{T}}(-\alpha I + \frac{1}{d^2}PBB^{\mathrm{T}}P + 2C^{\mathrm{T}}C)x \leq -\varepsilon x^{\mathrm{T}}x\,.$$

Finally, let $\bar{\gamma} > 0$ be any number satisfying

$$d^2 I + 2D^{\mathrm{T}}D \leq \bar{\gamma}^2 I\,.$$

Using the last two inequalities, it is concluded that the derivative of $V(x(t))$ *along the trajectories of* (3.2) satisfies an inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\partial V}{\partial x}\Big|_{x=x(t)}(Ax(t)+Bu(t)) \leq -\varepsilon\|x(t)\|^2 + \bar{\gamma}^2\|u(t)\|^2 - \|y(t)\|^2\,. \quad (3.4)$$

This inequality is called a *dissipation inequality*. [1] Summarizing the discussion up to this point, one can claim that for *any stable linear system*, given any positive real number ε, it is always possible to find a positive definite symmetric matrix P and a coefficient $\bar{\gamma}$ such that a dissipation inequality of the form (3.4), which – dropping for convenience the dependence on t – can be simply written as

$$\frac{\partial V(x)}{\partial x}(Ax+Bu) \leq -\varepsilon\|x\|^2 + \bar{\gamma}^2\|u\|^2 - \|y\|^2\,, \quad (3.5)$$

is satisfied.

The inequality thus established plays a fundamental role in characterizing a parameter, associated with a stable linear system, which is called the $\mathscr{L}_2$ *gain*. As a matter of fact, suppose the input $u(\cdot)$ of (3.2) is a function in $\mathscr{U}_{\mathscr{L}_2}$. Integration of the inequality (3.4) on the interval $[0,t]$ yields, for any initial state $x(0)$,

$$\begin{aligned} V(x(t)) &\leq V(x(0)) + \bar{\gamma}^2\int_0^t \|u(\tau)\|^2 d\tau - \int_0^t \|y(\tau)\|^2 d\tau \\ &\leq V(x(0)) + \bar{\gamma}^2\int_0^\infty \|u(\tau)\|^2 d\tau = V(x(0)) + \bar{\gamma}^2\Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2\,, \end{aligned}$$

from which it is deduced that the response $x(t)$ of the system is defined for all $t \in [0,\infty)$ and bounded. Now, suppose $x(0) = 0$ and observe that the previous inequality yields

$$V(x(t)) \leq \bar{\gamma}^2\int_0^t \|u(\tau)\|^2 d\tau - \int_0^t \|y(\tau)\|^2 d\tau$$

for any $t > 0$. Since $V(x(t)) \geq 0$, it is seen that

$$\int_0^t \|y(\tau)\|^2 d\tau \leq \bar{\gamma}^2\int_0^t \|u(\tau)\|^2 dt \leq \bar{\gamma}^2\Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2$$

[1] The concept of dissipation inequality was introduced by J.C. Willems in [1], to which the reader is referred for more details.

for any $t > 0$ and therefore the property (3.1) holds. In particular,

$$\left[\|y(\cdot)\|_{\mathscr{L}_2}\right]^2 \leq \bar{\gamma}^2 \left[\|u(\cdot)\|_{\mathscr{L}_2}\right]^2 ,$$

i.e.

$$\|y(\cdot)\|_{\mathscr{L}_2} \leq \bar{\gamma}\|u(\cdot)\|_{\mathscr{L}_2} .$$

In summary, for any $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$, the response of a stable linear system from the initial state $x(0) = 0$ is defined for all $t \geq 0$ and produces an output $y(\cdot)$ which has the property (3.1). Moreover the *ratio* between the $\mathscr{L}_2$ norm of the output and the $\mathscr{L}_2$ norm of the input is bounded by the number $\bar{\gamma}$ which appears in the dissipation inequality (3.5).

Having seen that, in stable linear system, an input having finite $\mathscr{L}_2$ norm produces, from the initial state $x(0) = 0$, an output response which also has a finite $\mathscr{L}_2$ norm, suggests to look at the ratios between such norms, for all possible $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$, and to seek the least upper bound of such ratios. The quantity thus defined is called the $\mathscr{L}_2$ *gain* of the (stable) linear system. Formally the gain in question is defined as follows: pick any $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$ and let $y_{0,u}(\cdot)$ be the resulting response from the initial state $x(0) = 0$; the $\mathscr{L}_2$ gain of the system is the quantity

$$\mathscr{L}_2 \; gain = \sup_{\|u(\cdot)\|_{\mathscr{L}_2} \neq 0} \frac{\|y_{0,u}(\cdot)\|_{\mathscr{L}_2}}{\|u(\cdot)\|_{\mathscr{L}_2}} .$$

With this definition in mind, return to the dissipation inequality (3.5). Suppose that a system of the form (3.2) is given and that an inequality of the form (3.5) holds for a positive definite matrix P. We have in particular (set $u = 0$)

$$\frac{\partial V(x)}{\partial x} Ax \leq -\varepsilon\|x\|^2 ,$$

from which it is seen that the system is stable. Since the system is stable, the $\mathscr{L}_2$ gain can be defined and, as a consequence of the previous discussion, it is seen that

$$\mathscr{L}_2 \; gain \leq \bar{\gamma}. \tag{3.6}$$

Thus, in summary, if a inequality of the form (3.5) holds for a positive definite matrix P, the system is stable and *its* $\mathscr{L}_2$ *gain is bounded from above by the number* $\bar{\gamma}$.

We will see in the next section that the fulfillment of an inequality of the form (3.5) is equivalent to the fulfillment of a *linear matrix inequality* involving the system data A, B, C, D.

3.2 An LMI characterization of the $\mathscr{L}_2$ gain

In this section, we derive alternative characterizations of the inequality (3.5).

Lemma 3.1. *Let $V(x) = x^{\mathrm{T}}Px$, with P a positive definite symmetric matrix. Suppose that, for some $\varepsilon > 0$ and some $\bar{\gamma} > 0$, the inequality*

$$\frac{\partial V}{\partial x}(Ax+Bu) \le -\varepsilon\|x\|^2 + \bar{\gamma}^2\|u\|^2 - \|Cx+Du\|^2 \tag{3.7}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Then, for any $\gamma > \bar{\gamma}$,

$$D^{\mathrm{T}}D - \gamma^2 I < 0 \tag{3.8}$$

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0\,. \tag{3.9}$$

Conversely, suppose (3.8) and (3.9) hold for some γ. Then there exists $\varepsilon > 0$ such that, for all $\bar{\gamma}$ satisfying $0 < \gamma - \varepsilon < \bar{\gamma} < \gamma$, the inequality (3.7) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Proof. Suppose that the inequality (3.7), that can be written as

$$2x^{\mathrm{T}}P[Ax+Bu] + \varepsilon x^{\mathrm{T}}x - \bar{\gamma}^2 u^{\mathrm{T}}u + x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2x^{\mathrm{T}}C^{\mathrm{T}}Du + u^{\mathrm{T}}D^{\mathrm{T}}Du \le 0\,, \tag{3.10}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. For $x = 0$ this implies, in particular,

$$-\bar{\gamma}^2 I + D^{\mathrm{T}}D \le 0\,.$$

Since $\gamma > \bar{\gamma}$, it follows that $D^{\mathrm{T}}D < \gamma^2 I$, which is precisely condition (3.8). Moreover, since $\gamma > \bar{\gamma}$, (3.7) implies

$$2x^{\mathrm{T}}P[Ax+Bu] + \varepsilon x^{\mathrm{T}}x - \gamma^2 u^{\mathrm{T}}u + x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2x^{\mathrm{T}}C^{\mathrm{T}}Du + u^{\mathrm{T}}D^{\mathrm{T}}Du \le 0 \tag{3.11}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Now, observe that, for each fixed x, the left-hand side of (3.11) is a quadratic form in u, expressible as

$$M(x) + N(x)u - u^{\mathrm{T}}Wu\,. \tag{3.12}$$

in which

$$W = \gamma^2 I - D^{\mathrm{T}}D$$

and

$$M(x) = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon I)x\,, \qquad N(x) = 2x^{\mathrm{T}}(PB + C^{\mathrm{T}}D)\,.$$

Since W is positive definite, this form has a unique maximum point, at

$$u_{\max}(x) = \frac{1}{2}W^{-1}N^{\mathrm{T}}(x)\,.$$

Hence, (3.11) holds if and only if the value of the form (3.12) at $u = u_{\max}(x)$ is non positive, that is if and only if

$$M(x) + \frac{1}{4}N(x)W^{-1}N(x)^{\mathrm{T}} \leq 0\,.$$

Using the expressions of $M(x), N(x), W$, the latter reads as

$$x^{\mathrm{T}}(PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon I + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}})x \leq 0$$

and this, since $\varepsilon > 0$, implies condition (3.9).

To prove the converse claim, observe that (3.9) implies

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon_1 I + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0 \qquad (3.13)$$

provided that $\varepsilon_1 > 0$ is small enough. The left-hand sides of (3.8) and (3.13), which are negative definite, by continuity remain negative definite if γ is replaced by any $\bar{\gamma}$ satisfying $\gamma - \varepsilon_2 < \bar{\gamma} < \gamma$, provided that $\varepsilon_2 > 0$ is small enough. Take now $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. The resulting form $M(x) + N(x)u - u^{\mathrm{T}}\bar{W}u$, in which $\bar{W} = \bar{\gamma}^2 I - D^{\mathrm{T}}D$, is non positive and (3.7) holds.

Remark 3.1. Note that the inequalities (3.8) and (3.9) take a substantially simpler form in the case of a system in which $D = 0$ (i.e. systems with no direct feed-through between input and output). In this case, in fact, (3.8) becomes irrelevant and (3.9) reduces to

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \frac{1}{\gamma^2}PBB^{\mathrm{T}}P < 0\,.$$

Lemma 3.2. *Let γ be a fixed positive number. The inequality (3.8) holds* and *there exists a positive definite symmetric matrix P satisfying (3.9) if and only if there exists a positive definite symmetric matrix X satisfying*

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{pmatrix} < 0\,. \qquad (3.14)$$

Proof. Consider the matrix inequality

$$\begin{pmatrix} A^{\mathrm{T}}X + XA + \frac{1}{\gamma}C^{\mathrm{T}}C & XB + \frac{1}{\gamma}C^{\mathrm{T}}D \\ B^{\mathrm{T}}X + \frac{1}{\gamma}D^{\mathrm{T}}C & -\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D \end{pmatrix} < 0\,. \qquad (3.15)$$

This inequality holds if and only if the lower right block

$$-\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D$$

is negative definite, which is equivalent to condition (3.8), and so is its the Schur's complement

$$A^{\mathrm{T}}X + XA + \frac{1}{\gamma}C^{\mathrm{T}}C - [XB + \frac{1}{\gamma}C^{\mathrm{T}}D][-\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D]^{-1}[XB + \frac{1}{\gamma}C^{\mathrm{T}}D]^{\mathrm{T}} .$$

This, having replaced X by $\frac{1}{\gamma}P$, is identical to condition (3.9).

Rewrite now (3.15) as

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB \\ B^{\mathrm{T}}X & -\gamma I \end{pmatrix} + \begin{pmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{pmatrix} \frac{1}{\gamma} \begin{pmatrix} C & D \end{pmatrix} < 0$$

and use again (backward) Schur's complement to arrive at (3.14). $\triangleleft$

3.3 The H_∞ norm of a transfer function

Functions having finite $\mathscr{L}_2$ norm may be seen as signals having *finite energy* [2] over the infinite time interval $[0,\infty)$, and therefore the $\mathscr{L}_2$ gain can be given the interpretation of (an upper bound of the) ratio between energies of output and input. Another similar interpretation, in terms of energies associated with input and output, is possible, which does not necessarily require the consideration of the case of finite energy over the infinite time interval $[0,\infty)$. Suppose the input is a *periodic* function of time, with period T, i.e. that

$$u(t+kT) = \bar{u}(t), \quad \text{for all } t \in [0,T), \text{ all integer } k$$

for some piecewise continuous function $\bar{u}(t)$, defined on $[0,T)$. Also, suppose that, for some suitable initial state $x(0) = \bar{x}$, the state response $x(t)$ of the system is defined for all $t \in [0,T]$ and satisfies

$$x(T) = \bar{x} .$$

Then, it is obvious that $x(t)$ exists for all $t \geq 0$, and is a *periodic* function, having the same period T of the input, namely

$$x(t+kT) = x(t), \quad \text{for all } t \in [0,T), k \geq 0$$

and so is the corresponding output response $y(t)$.

For the triplet $\{u(t), x(t), y(t)\}$ thus defined, integration of the inequality (3.4) over an interval $[t_0, t_0+T]$, with arbitrary $t_0 \geq 0$, yields

$$V(x(t_0+T)) - V(x(t_0)) \leq \bar{\gamma}^2 \int_{t_0}^{t_0+T} \|u(\tau)\|^2 d\tau - \int_{t_0}^{t_0+T} \|y(\tau)\|^2 d\tau ,$$

i.e., since $V(x(t_0+T)) = V(x(t_0))$,

---

[2] If, in the actual physical system, the components of the input $u(t)$ are *voltages* (or *currents*), the quantity $\|u(t)\|^2$ can be seen as instantaneous *power*, at time t, associated with such input and its integral over a time interval $[t_0, t_1]$ as *energy* associated with such input over this time interval.

$$\int_{t_0}^{t_0+T} \|y(\tau)\|^2 d\tau \le \bar{\gamma}^2 \int_{t_0}^{t_0+T} \|u(\tau)\|^2 d\tau \,. \tag{3.16}$$

Observe that the integrals on both sides of this inequality are independent of t_0, because the integrands are periodic functions having period T, and recall that the *root mean square* value of any (possibly vector-valued) periodic function $f(t)$ (which is usually abbreviated as "r.m.s." and characterizes the *average power* of the signal represented by $f(t)$ is defined as

$$\|f(\cdot)\|_{\text{r.m.s.}} = \Big(\frac{1}{T}\int_{t_0}^{t_0+T} \|f(\tau)\|^2 d\tau\Big)^{\frac{1}{2}} \,.$$

With this in mind, (3.16) yields

$$\|y(\cdot)\|_{\text{r.m.s.}} \le \bar{\gamma}\|u(\cdot)\|_{\text{r.m.s.}} \,. \tag{3.17}$$

In other words, the number $\bar{\gamma}$ which appears in the inequality (3.5) can be seen also an upper bound for the ratio between the r.m.s. value of the output and the r.m.s value of the input, whenever a periodic input is producing (from an appropriate initial state) a periodic (state and output) response.

Consider now the special case in which the input signal is a *harmonic function* of time, i.e.

$$u(t) = u_0 \cos(\omega_0 t)$$

It is known that, if the state $x(0)$ is appropriately chosen, [3] the output response of the system coincides with the so-called *steady-state response*, which is the harmonic function

$$y_{\text{ss}}(t) = \text{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \text{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t) \,,$$

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D \,.$$

Recall that

$$\int_0^{\frac{2\pi}{\omega_0}} \|u(t)\|^2 dt = \frac{\pi}{\omega_0}\|u_0\|^2$$

and therefore

$$\int_0^{\frac{2\pi}{\omega_0}} \|y_{\text{ss}}(t)\|^2 dt = \frac{\pi}{\omega_0}\|T(j\omega_0)u_0\|^2 \,.$$

In other words

$$\begin{aligned} \|u(\cdot)\|^2_{\text{r.m.s.}} &= \tfrac{1}{2}\|u_0\|^2 \\ \|y_{\text{ss}}(\cdot)\|^2_{\text{r.m.s.}} &= \tfrac{1}{2}\|T(j\omega_0)u_0\|^2 \,. \end{aligned}$$

Thus, from the interpretation illustrated above one can conclude that, if the system satisfies (3.5), then

[3] See Section A.5 in Appendix A. This is the case if $x(0) = \Pi_1$, in which Π_1 is the first column of the solution Π of the Sylvester equation (A.29).

$$\|T(j\omega_0)u_0\|^2 = 2\|y_{ss}(\cdot)\|^2_{\text{r.m.s.}} \leq \bar{\gamma}^2 2\|u(\cdot)\|^2_{\text{r.m.s.}} = \bar{\gamma}^2\|u_0\|^2$$

i.e.

$$\|T(j\omega_0)u_0\| \leq \bar{\gamma}\|u_0\|,$$

or, bearing in mind the definition of norm of a matrix, [4]

$$\|T(j\omega_0)\| \leq \bar{\gamma}.$$

Define now the quantity

$$\|T\|_{H_\infty} := \sup_{\omega \in \mathbb{R}} \|T(j\omega)\|,$$

which is called the H_∞ *norm* of the matrix $T(j\omega)$. Observing that ω_0 in the above inequality is arbitrary, it is concluded

$$\|T\|_{H_\infty} \leq \bar{\gamma}. \tag{3.18}$$

Therefore, a linear system that satisfies (3.5) is stable and *the H_∞ norm of its frequency response matrix is bounded from above by the number $\bar{\gamma}$.*

3.4 The bounded real lemma

We have seen in the previous sections that, for a system of the form (3.2), if there exists a number $\gamma > 0$ and a symmetric positive definite matrix X satisfying (3.14), then there exists a number $\bar{\gamma} < \gamma$ and a positive definite quadratic form $V(x)$ satisfying (3.5) for some $\varepsilon > 0$. This, in view of the interpretation provided above, proves that the fulfillment of (3.14) for some γ (with X positive definite) *implies* that

(i) the system is asymptotically stable,

(ii) its $\mathscr{L}_2$ gain is strictly less than γ,

(iii) the H_∞ norm of its transfer function is strictly less than γ.

However, put in these terms, we have only learned that (3.5) implies both (ii) and (iii) and we have not investigated yet whether converse implications might hold. In this section, we complete the analysis, by showing that the two properties (ii) and (iii) are, in fact, two different manifestations of the same property and both imply (3.14).

This will be done by means of a circular proof involving another equivalent version of the property that the number γ is an upper bound for the H_∞ norm of the

[4] Recall that the norm of a matrix $T \in \mathbb{R}^{p\times m}$ is defined as

$$\|T\| = \sup_{\|u\|\neq 0} \frac{\|Tu\|}{\|u\|} = \max_{\|u\|=1} \|Tu\|.$$

transfer function matrix of the system, which is very useful for practical purposes, since it can be easily checked. More precisely, the fact that γ is an upper bound for the H_∞ norm of the transfer function matrix of the system can be checked by looking at the spectrum of a matrix of the form

$$H = \begin{pmatrix} A_0 & R_0 \\ -Q_0 & -A_0^{\mathrm{T}} \end{pmatrix}, \tag{3.19}$$

in which R_0 and Q_0 are *symmetric* matrices which, together with A_0, depend on the matrices A, B, C, D which characterize the system and on the number γ.[5]

As a matter of fact the following result, known in the literature as *Bounded Real Lemma*, holds.

Theorem 3.1. *Consider the linear system (3.2) and let $\gamma > 0$ be a fixed number. The following are equivalent:*

(i) *there exists $\bar{\gamma} < \gamma$, $\varepsilon > 0$ and a symmetric positive definite matrix P such that (3.7) holds for $V(x) = x^{\mathrm{T}}Px$,*

(ii) *all the eigenvalues of A have negative real part and the frequency response matrix of the system $T(j\omega) = C(j\omega I - A)^{-1}B + D$ satisfies*

$$\|T\|_{H_\infty} := \sup_{\omega\in\mathbb{R}} \|T(j\omega)\| < \gamma, \tag{3.20}$$

(iii) *all the eigenvalues of A have negative real part, the matrix $W = \gamma^2 I - D^{\mathrm{T}}D$ is positive definite, and the Hamiltonian matrix*

$$H = \begin{pmatrix} A + BW^{-1}D^{\mathrm{T}}C & BW^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}C - C^{\mathrm{T}}DW^{-1}D^{\mathrm{T}}C & -A^{\mathrm{T}} - C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}} \end{pmatrix} \tag{3.21}$$

has no eigenvalues on the imaginary axis,

(iv) *there exists a positive definite symmetric matrix X satisfying*

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{pmatrix} < 0. \tag{3.22}$$

Proof. We have already shown, in the previous sections, that if (i) holds, then (3.2) is an asymptotically stable system, with a frequency response matrix satisfying

$$\|T\|_{H_\infty} \le \bar{\gamma}.$$

Thus, (i) $\Rightarrow$ (ii).

To show that (ii) $\Rightarrow$ (iii), first of all that note, since

---

[5] A matrix (of real numbers) with this structure is called an *Hamiltonian matrix* and has the property that its spectrum is symmetric with respect to the imaginary axis (see Lemma A.5 in Appendix A).

$$\lim_{\omega\to\infty} T(j\omega) = D$$

it necessarily follows that $\|Du\| < \gamma$ for all u with $\|u\| = 1$ and this implies $\gamma^2 I > D^{\mathrm{T}}D$, i.e. the matrix W is positive definite.

Now observe that the Hamiltonian matrix (3.21) can be expressed in the form

$$H = L + MN$$

for

$$L = \begin{pmatrix} A & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{pmatrix}, \quad M = \begin{pmatrix} B \\ -C^{\mathrm{T}}D \end{pmatrix},$$

$$N = \begin{pmatrix} W^{-1}D^{\mathrm{T}}C & W^{-1}B^{\mathrm{T}} \end{pmatrix}.$$

Suppose, by contradiction, that the matrix H has eigenvalues on the imaginary axis. By definition, there exist a $2n$-dimensional vector x_0 and a number $\omega_0 \in \mathbb{R}$ such that

$$(j\omega_0 I - L)x_0 = MNx_0.$$

Observe now that the matrix L has no eigenvalues on the imaginary axis, because its eigenvalues coincide with those of A and $-A^{\mathrm{T}}$, and A is by hypothesis stable. Thus $(j\omega_0 I - L)$ is nonsingular. Observe also that the vector $u_0 = Nx_0$ is nonzero because otherwise x_0 would be an eigenvector of L associated with an eigenvalue at $j\omega_0$, which is a contradiction. A simple manipulation yields

$$u_0 = N(j\omega_0 I - L)^{-1}Mu_0. \tag{3.23}$$

It is easy to check that

$$N(j\omega_0 I - L)^{-1}M = W^{-1}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D] \tag{3.24}$$

where $T(s) = C(sI - A)^{-1}B + D$. In fact, it suffices to compute the transfer function of

$$\begin{aligned} \dot{x} &= Lx + Mu \\ y &= Nx \end{aligned}$$

and observe that $N(sI - L)^{-1}M = W^{-1}[T^{\mathrm{T}}(-s)T(s) - D^{\mathrm{T}}D]$.

Multiply (3.24) on the left by $u_0^{\mathrm{T}}W$ and on the right by u_0, and use (3.23), to obtain

$$u_0^{\mathrm{T}}Wu_0 = u_0^{\mathrm{T}}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D]u_0\,,$$

which in turn, in view of the definition of W, yields

$$\gamma^2\|u_0\|^2 = \|T(j\omega_0)u_0\|^2\,.$$

This implies

$$\|T(j\omega_0)\| = \sup_{\|u_0\|\neq 0} \frac{\|T(j\omega_0)u_0\|}{\|u_0\|} \geq \gamma$$

and this contradicts (ii), thus completing the proof.

To show that (iii) $\Rightarrow$ (iv), set

$$\begin{aligned} F &= (A+BW^{-1}D^{\mathrm{T}}C)^{\mathrm{T}} \\ Q &= -BW^{-1}B^{\mathrm{T}} \\ GG^{\mathrm{T}} &= C^{\mathrm{T}}(I+DW^{-1}D^{\mathrm{T}})C \end{aligned}$$

(the latter is indeed possible because $I+DW^{-1}D^{\mathrm{T}}$ is a positive definite matrix: in fact, it is a sum of the positive definite matrix I and of the positive semidefinite matrix $DW^{-1}D^{\mathrm{T}}$; hence there exists a nonsingular matrix M such that $I+DW^{-1}D^{\mathrm{T}} = M^{\mathrm{T}}M$; in view of this, the previous expression holds with $G^{\mathrm{T}} = MC$).

It is easy to check that

$$H^{\mathrm{T}} = \begin{pmatrix} F & -GG^{\mathrm{T}} \\ -Q & -F^{\mathrm{T}} \end{pmatrix}$$

and this matrix by hypothesis has no eigenvalues on the imaginary axis. Moreover, it is also possible to show that the pair $(F, -GG^{\mathrm{T}})$ thus defined is stabilizable. In fact, suppose that this is not the case. Then, there is a vector $x \neq 0$ such that

$$x^{\mathrm{T}}\begin{pmatrix} F-\lambda I & -GG^{\mathrm{T}} \end{pmatrix} = 0$$

for some λ with non-negative real part. Then,

$$0 = \begin{pmatrix} A+BW^{-1}D^{\mathrm{T}}C-\lambda I \\ -C^{\mathrm{T}}M^{\mathrm{T}}MC \end{pmatrix} x\,.$$

This implies in particular $0 = x^{\mathrm{T}}C^{\mathrm{T}}M^{\mathrm{T}}MCx = \|MCx\|^2$ and hence $Cx = 0$, because M is nonsingular. This in turn implies $Ax = \lambda x$, and this is a contradiction because all the eigenvalues of A have negative real part.

Thus [6] there is a unique solution Y^- of the Riccati equation

$$Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q = 0\,, \tag{3.25}$$

satisfying $\sigma(F - GG^{\mathrm{T}}Y^-) \subset \mathbb{C}^-$. Moreover [7], the set of solutions Y of the inequality

$$YF + F^{\mathrm{T}}Y - YGG^{\mathrm{T}}Y + Q > 0 \tag{3.26}$$

is nonempty and any Y in this set is such that $Y < Y^-$.

Observe now that

$$\begin{aligned} &Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q \\ &= Y^-(A^{\mathrm{T}} + C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}}) + (A+BW^{-1}D^{\mathrm{T}}C)Y^- - Y^-C^{\mathrm{T}}(I+DW^{-1}D^{\mathrm{T}})CY^- - BW^{-1}B^{\mathrm{T}} \\ &= Y^-A^{\mathrm{T}} + AY^- - [Y^-C^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY^- - B^{\mathrm{T}}] - Y^-C^{\mathrm{T}}CY^-, \end{aligned}$$

[6] See Proposition A.1 in Appendix A

[7] See Proposition A.2 in Appendix A

and therefore (3.25) yields

$$Y^- A^T + AY^- \geq 0\,.$$

Set now $U(z) = z^T Y^- z$, let $z(t)$ denote (any) integral curve of

$$\dot{z} = A^T z\,, \tag{3.27}$$

and observe that the function $U(z(t))$ satisfies

$$\frac{\partial U(z(t))}{\partial t} = 2z^T(t) Y^- A^T z(t) = z^T(t)[Y^- A^T + AY^-] z(t) \geq 0\,.$$

This inequality shows that the function $V(z(t))$ is non-decreasing, i.e. $V(z(t)) \geq V(z(0))$ for any $z(0)$ and any $t \geq 0$. On the other hand, system (3.27) is by hypothesis asymptotically stable, i.e. $\lim_{t\to\infty} z(t) = 0$. Therefore, necessarily, $V(z(0)) \leq 0$, i.e the matrix Y^- is negative semi-definite. From this, it is concluded that any solution Y of (3.26), that is of the inequality

$$YA^T + AY - [YC^T D - B] W^{-1} [D^T CY - B^T] - YC^T CY > 0\,, \tag{3.28}$$

which necessarily satisfies $Y < Y^- \leq 0$, is a negative definite matrix.

Take any of the solutions Y of (3.28) and consider $P = -Y^{-1}$. By construction, this matrix is a positive definite solution of the inequality in (3.9). Thus, by Lemma 3.2, (iv) holds.

The proof that (iv) $\Rightarrow$ (i) is provided by Lemma 3.2 and 3.1. $\triangleleft$

Example 3.1. As an elementary example of what the criterion described in (iii) means, consider the case of the single-input single-output linear system

$$\begin{aligned} \dot{x} &= -ax + bu \\ y &= x + du \end{aligned} \tag{3.29}$$

in which it is assumed that $a > 0$, so that the system is stable. The transfer function of this system is

$$T(s) = \frac{ds + (ad + b)}{s + a}\,.$$

This function has a pole at $p = -a$ and a zero at $z = -(ad+b)/d$. Thus, bearing in mind the possible Bode plots of a function having one pole and one zero (see Fig. 3.1), it is seen that

$$\begin{aligned} |d| < \left|\frac{ad+b}{a}\right| \qquad &\Rightarrow \qquad \|T\|_{H_\infty} = |T(0)| = \left|\frac{ad+b}{a}\right| \\ |d| > \left|\frac{ad+b}{a}\right| \qquad &\Rightarrow \qquad \|T\|_{H_\infty} = \lim_{\omega\to\infty} |T(j\omega)| = |d|\,. \end{aligned}$$

Thus,

$$\|T\|_{H_\infty} = \max\{|d|, \left|\frac{ad+b}{a}\right|\}\,. \tag{3.30}$$

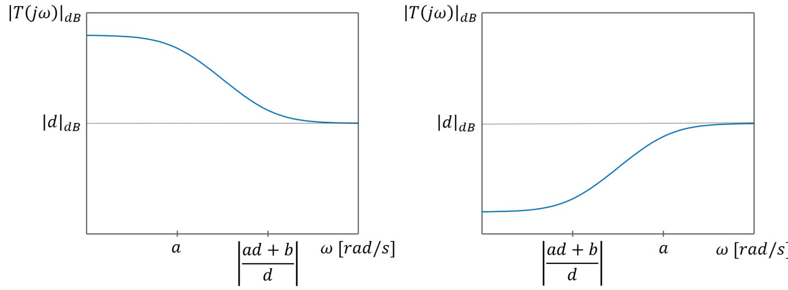

Fig. 3.1 The Bode plots of the transfer function of (3.29). Left: the case $|T(0)| > |d|$. Right: the case $|T(0)| < |d|$

The criterion in the Bounded Real Lemma says that the H_∞ norm of $T(s)$ is strictly less than γ if and only if $\gamma^2 I - D^\mathrm{T} D > 0$, which in this case becomes

$$\gamma^2 > d^2 \tag{3.31}$$

and the Hamiltonian matrix (3.21), which in this case becomes

$$H = \begin{pmatrix} -a + \dfrac{bd}{\gamma^2 - d^2} & \dfrac{b^2}{\gamma^2 - d^2} \\ -1 - \dfrac{d^2}{\gamma^2 - d^2} & a - \dfrac{bd}{\gamma^2 - d^2} \end{pmatrix},$$

has no eigenvalues with zero real part. A simple calculation shows that the characteristic polynomial of H is

$$p(\lambda) = \lambda^2 - a^2 + \frac{2abd}{\gamma^2 - d^2} + \frac{b^2}{\gamma^2 - d^2}\,.$$

This polynomial has no roots with zero real part if and only if

$$a^2 - \frac{2abd}{\gamma^2 - d^2} - \frac{b^2}{\gamma^2 - d^2} > 0$$

which, since $\gamma^2 - d^2 > 0$, is the case if and only if

$$\gamma^2 a^2 > (ad + b)^2\,. \tag{3.32}$$

Using both (3.31) and (3.32), it is concluded that, according to the Bounded Real Lemma, $\|T\|_{H_\infty} < \gamma$ if and only if

$$\gamma > \max\{|d|, |\frac{ad + b}{a}|\}$$

which is exactly what (3.30) shows. ◁

3.5 Small gain theorem and robust stability

The various characterizations of the $\mathscr{L}_2$ gain of a system given in the previous sections provide a powerful tool for the study of the stability properties of feedback interconnected systems.To see why this is the case, consider two systems Σ_1 and Σ_2, described by equations of the form

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i + D_i u_i \end{aligned} \tag{3.33}$$

with $i = 1, 2$, in which we assume that

$$\begin{aligned} \dim(u_2) &= \dim(y_1) \\ \dim(u_1) &= \dim(y_2) \;. \end{aligned}$$

Suppose that the matrices D_1 and D_2 are such that the *interconnections*

$$\begin{aligned} u_2 &= y_1 \\ u_1 &= y_2 \end{aligned} \tag{3.34}$$

makes sense (see Figure 3.2).

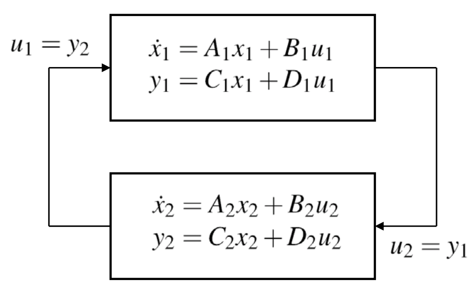

Fig. 3.2 A pure feedback interconnection of two systems Σ_1 and Σ_2.

This will be the case if, for each x_1, x_2, there is a unique pair u_1, u_2 satisfying

$$\begin{aligned} u_1 &= C_2 x_2 + D_2 u_2 \\ u_2 &= C_1 x_1 + D_1 u_1 \;, \end{aligned}$$

i.e. if the system of equations

$$\begin{aligned} u_1 - D_2 u_2 &= C_2 x_2 \\ -D_1 u_1 + u_2 &= C_1 x_1 \end{aligned}$$

has a *unique* solution u_1, u_2. This occurs if and only if the matrix

$$\begin{pmatrix} I & -D_2 \\ -D_1 & I \end{pmatrix}$$

is invertible, i.e. the matrix $I - D_2 D_1$ is nonsingular. [8] The (autonomous) system defined by (3.33) together with (3.34) is the *pure feedback interconnection* of Σ_1 and Σ_2.

Suppose now that both systems Σ_1 and Σ_2 are stable. As shown in section 3.1, there exist two positive definite matrices P_1, P_2, two positive numbers $\varepsilon_1, \varepsilon_2$ and two real numbers $\bar{\gamma}_1, \bar{\gamma}_2$ such that Σ_1 and Σ_2 satisfy inequalities of the form (3.5), namely

$$\frac{\partial V_i}{\partial x_i}(A_i x_i + B_i u_i) \le -\varepsilon_i \|x_i\|^2 + \bar{\gamma}_i^2 \|u_i\|^2 - \|y_i\|^2 \tag{3.35}$$

in which $V_i(x_i) = x_i^{\mathrm{T}} P_i x_i$.

Consider now the quadratic form

$$W(x_1, x_2) = V(x_1) + aV(x_2) = (x_1^{\mathrm{T}} \quad x_2^{\mathrm{T}}) \begin{pmatrix} P_1 & 0 \\ 0 & aP_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

which, if $a > 0$, is positive definite. A simple calculation shows that, for any trajectory $(x_1(t), x_2(t))$ of the pure feedback interconnection of Σ_1 and Σ_2, the function $W(x_1(t), x_2(t))$ satisfies

$$\begin{aligned} \frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 &\le -\varepsilon_1 \|x_1\|^2 - a\varepsilon_2 \|x_1\|^2 + \bar{\gamma}_1^2 \|u_1\|^2 - \|y_1\|^2 + a\bar{\gamma}_2^2 \|u_2\|^2 - a\|y_2\|^2 \\ &\le -\varepsilon_1 \|x_1\|^2 - a\varepsilon_2 \|x_1\|^2 + \bar{\gamma}_1^2 \|y_2\|^2 - \|y_1\|^2 + a\bar{\gamma}_2^2 \|y_1\|^2 - a\|y_2\|^2 \\ &= -\varepsilon_1 \|x_1\|^2 - a\varepsilon_2 \|x_1\|^2 + (y_1^{\mathrm{T}} \quad y_2^{\mathrm{T}}) \begin{pmatrix} (-1 + a\bar{\gamma}_2^2)I & 0 \\ 0 & (\bar{\gamma}_1^2 - a)I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \end{aligned}$$

If

$$\begin{pmatrix} (-1 + a\bar{\gamma}_2^2)I & 0 \\ 0 & (\bar{\gamma}_1^2 - a)I \end{pmatrix} \le 0, \tag{3.36}$$

we have

$$\frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 \le -\varepsilon_1 \|x_1\| - a\varepsilon_2 \|x_2\| = (x_1^{\mathrm{T}} \quad x_2^{\mathrm{T}}) \begin{pmatrix} -\varepsilon_1 I & 0 \\ 0 & -a\varepsilon_2 I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The quadratic form on the right-hand side is negative definite and hence, according to the criterion of Lyapunov, the interconnected system is asymptotically stable.

[8] Note that an equivalent condition is that the matrix $I - D_1 D_2$ is nonsingular.

Condition (3.36), on the other hand, can be fulfilled for some $a > 0$ if (and only if)

$$-1 + a\bar{\gamma}_2^2 \leq 0, \qquad \bar{\gamma}_1^2 - a \leq 0$$

i.e. if

$$\bar{\gamma}_1^2 \leq a \leq \frac{1}{\bar{\gamma}_2^2}.$$

A number $a > 0$ satisfying these inequalities exists if and only if $\bar{\gamma}_2^2\bar{\gamma}_1^2 \leq 1$.

All of the above yields the following important result.

Theorem 3.2. *Consider a pair of systems (3.33) and suppose that the matrix $I - D_2D_1$ is nonsingular. Suppose (3.33) satisfy inequalities of the form (3.35), with P_1, P_2 positive definite and*

$$\bar{\gamma}_1\bar{\gamma}_2 \leq 1.$$

Then, the pure feedback interconnection of Σ_1 and Σ_2 is asymptotically stable.

Remark 3.2. In the proof of the above statement, we have considered the general case in which both component subsystems have an internal dynamics. To cover the special case in which one of the two subsystem is a memoryless system, a slightly different (actually simpler) argument is needed. Suppose the second component subsystem is modeled as

$$y_2 = D_2u_2$$

and set $\bar{\gamma}_2 = \|D_2\|$. Then,

$$\|u_1\|^2 = \|y_2\|^2 \leq \bar{\gamma}_2^2\|u_2\|^2 = \bar{\gamma}_2^2\|y_1\|^2.$$

As a consequence, for the interconnection, we obtain

$$\begin{aligned}\frac{\partial V_1}{\partial x_1}(A_1x_1 + B_1u_1) &\leq -\varepsilon_1\|x_1\|^2 + \bar{\gamma}_1^2\|u_1\|^2 - \|y_1\|^2 \\ &\leq -\varepsilon_1\|x_1\|^2 + (\bar{\gamma}_1^2\bar{\gamma}_2^2 - 1)\|y_1\|^2.\end{aligned}$$

If $\bar{\gamma}_1\bar{\gamma}_2 \leq 1$, the quantity above is negative definite and the interconnected system is stable. $\triangleleft$

Note also that, in view of the Bounded Real Lemma, the Theorem above can be rephrased in terms of H_∞ norms of the transfer functions of the two component subsystems, as follows.

Corollary 3.1. *Consider a pair of systems (3.33) and suppose that the matrix $I - D_2D_1$ is nonsingular. Suppose both systems are asymptotically stable. Let*

$$T_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$$

denote the respective transfer functions. If

$$\|T_1\|_{H_\infty} \cdot \|T_2\|_{H_\infty} < 1 \tag{3.37}$$

the pure feedback interconnection of Σ_1 and Σ_2 is asymptotically stable.

Proof. Suppose that condition (3.37) holds, i.e. that

$$\|T_1\|_{H_\infty} < \frac{1}{\|T_2\|_{H_\infty}}.$$

Then, it is possible to find a positive number γ_1 satisfying

$$\|T_1\|_{H_\infty} < \gamma_1 < \frac{1}{\|T_2\|_{H_\infty}}.$$

Set $\gamma_2 = 1/\gamma_1$. Then

$$\|T_i\|_{H_\infty} < \gamma_i, \qquad i = 1, 2.$$

From the Bounded Real Lemma, it is seen that – for both $i = 1, 2$ – there exists $\varepsilon_i > 0$, $\bar{\gamma}_i < \gamma_i$ and positive definite P_i such that (3.33) satisfy inequalities of the form (3.35), with $V_i(x) = x_i^{\mathrm{T}} P_i x_i$. Moreover, $\bar{\gamma}_1\bar{\gamma}_2 < \gamma_1\gamma_2 = 1$. Hence, from Theorem 3.2 and Remark 3.2, the result follows. $\triangleleft$

The result just proven is known as the *Small Gain Theorem* (of linear systems). In a nutshell, it says that if both component systems are stable, a *sufficient* condition for the stability of their (pure feedback) interconnection is that the product of the H_∞ norms of the transfer functions of the two component subsystems is *strictly less than 1*. This theorem is the point of departure for the study of robust stability via H_∞ methods.

It should be stressed that the "small gain condition" (3.37) provided by this Corollary is only sufficient for stability of the interconnection and that the strict inequality in (3.37) cannot be replaced, in general, by a loose inequality. Both these facts are explained in the example which follows.

Example 3.2. Let the two component subsystems be modeled by

$$\begin{aligned} \dot{x}_1 &= -ax_1 + u_1 \\ y_1 &= x_1 \end{aligned}$$

and

$$y_2 = du_2.$$

Suppose $a > 0$ so that the first subsystem is stable. The stability of the interconnection can be trivially analyzed by direct computation. In fact, the interconnection is modeled as

$$\dot{x}_1 = -ax_1 + dx_1$$

from which is it is seen that a necessary and sufficient condition for the interconnection to be stable is that $d < a$.

On the other hand, the Small Gain Theorem yields a more conservative estimate. In fact

$$T_1(s) = \frac{1}{s+a} \qquad \text{and} \qquad T_2(s) = d$$

and hence

$$\|T_1\|_{H_\infty} = T_1(0) = \frac{1}{a} \qquad \text{and} \qquad \|T_2\|_{H_\infty} = |d|$$

The (sufficient) condition (3.37) becomes $|d| < a$, i.e. $-a < d < a$.

This example also shows that the inequality in (3.37) has to be strict. In fact, the condition $\|T_1\|_{H_\infty} \cdot \|T_2\|_{H_\infty} = 1$ would yield $|d| = a$, which is not admissible because, if $d = a$ the interconnection is not (asymptotically) stable (while it would be asymptotically stable if $d = -a$).

Example 3.3. It is worth observing that a special version of such sufficient condition was implicit in the arguments used in Chapter 2 to prove asymptotic stability. Take, for instance, the case studied in Section 2.3. System (2.19) can be seen as pure feedback interconnection of a subsystem modeled by

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)u_1 \\ y_1 &= A_{10}(\mu)z \end{aligned} \tag{3.38}$$

and of a subsystem modeled by

$$\begin{aligned} \dot{\xi} &= [A_{11}(\mu) - b(\mu)k]\xi + u_2 \\ y_2 &= \xi\,. \end{aligned} \tag{3.39}$$

The first of two such subsystems is a stable system, because $A_{00}(\mu)$ has all eigenvalues in $\mathbb{C}^-$. Let

$$T_1(s) = A_{10}(\mu)[sI - A_{00}(\mu)]^{-1}A_{01}(\mu)\,.$$

denote its transfer function. Its H_∞ norm depends on μ but, since μ ranges over a compact set $\mathbb{M}$, it is possible to find a number $\gamma_1 > 0$ such that

$$\max_{\mu\in\mathbb{M}} \|T_1\|_{H_\infty} < \gamma_1\,.$$

Note that this number γ_1 depends only on the data that characterize the controlled system (2.17) and not on the coefficient k that characterizes the feedback law (2.18).

Consider now the second subsystem. If $b(\mu)k - A_{11}(\mu) > 0$, this is a stable system with transfer function

$$T_2(s) = \frac{1}{s + (b(\mu)k - A_{11}(\mu))}$$

Clearly,

$$\|T_2\|_{H_\infty} = T_2(0) = \frac{1}{b(\mu)k - A_{11}(\mu)}\,.$$

It is seen from this that $\|T_2\|_{H_\infty}$ can be arbitrarily *decreased* by *increasing* the coefficient k. In other words: a *large* value of the output feedback gain coefficient k in (2.18) forces a *small* value of $\|T_2\|_{H_\infty}$.

As a consequence of the small-gain Theorem, the interconnected system (namely, system (2.19)) is stable if

$$\frac{\gamma_1}{b(\mu)k - A_{11}(\mu)} < 1$$

which is indeed the case if

$$k > \max_{\mu \in \mathbb{M}} \frac{\gamma_1 + A_{11}(\mu)}{b(\mu)}.$$

In summary, the result established in Section 2.3 can be re-interpreted in the following terms. Subsystem (3.38) is a stable system, with a transfer function whose H_∞ norm has some fixed bound γ_1 (on which the control has no influence, though). By increasing the value of the gain coefficient k in (2.18), the subsystem (3.39) can be rendered stable, with a H_∞ norm that can be made arbitrarily small, in particular smaller than the (fixed) number $1/\gamma_1$. This makes the small gain condition (3.37) fulfilled and guarantees the (robust) stability of system (2.19). $\triangleleft$

We turn now to the discussion of the problem of robust stabilization. The problem in question can be cast, in rather general terms, as follows. A plant with *control* input u and *measurement* output y whose model is uncertain can be, from a rather general viewpoint, thought of as the interconnection of a *nominal system* modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v \end{aligned} \tag{3.40}$$

in which the "additional" input v and the "additional" output z are seen as output and, respectively, input of a system

$$\begin{aligned} \dot{x}_{\rm p} &= A_{\rm p} x_{\rm p} + B_{\rm p} z \\ v &= C_{\rm p} x_{\rm p} + D_{\rm p} z \end{aligned} \tag{3.41}$$

whose parameters are *uncertain*.[9]

This setup includes the special case of a plant of fixed dimension whose parameters are uncertain, that is a plant modeled as

$$\begin{aligned} \dot{x} &= (A_0 + \delta A)x + (B_0 + \delta B)u \\ y &= (C_0 + \delta C)x \end{aligned} \tag{3.42}$$

in which A_0, B_0, C_0 represent nominal values and $\delta A, \delta B, \delta C$ uncertain perturbations. In fact, the latter can be seen as interconnection of a system of the form (3.40) in which

[9] Note that, for the interconnection (3.40) – (3.41) to be well-defined, the matrix $I - D_{\rm p} D_{11}$ is required to be invertible.

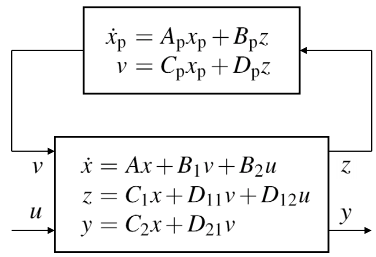

Fig. 3.3 A controlled system seen as interconnection of an accurate model and of a lousy model.

$$
\begin{array}{lll}
A = A_0 & B_1 = \begin{pmatrix} I & 0 \end{pmatrix} & B_2 = B_0 \\
C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix} & D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix} \\
C_2 = C_0 & D_{21} = \begin{pmatrix} 0 & I \end{pmatrix} &
\end{array}
$$

with a system of the form

$$v = \begin{pmatrix} \delta A & \delta B \\ \delta C & 0 \end{pmatrix} z$$

which is indeed a special case of a system of the form (3.41). The interconnection (3.40) – (3.41) of a *nominal model* and of a *dynamic perturbation* is more general, though, because it accommodates for perturbations which possibly include *unmodeled dynamics* (see Example at the end of the section). In this setup, all modeling uncertainties are confined in the model (3.41), including the dimension itself of x_p.

Suppose now that, in (3.41), A_p is a Hurwitz matrix and that the transfer function

$$T_p(s) = C_p(sI - A_p)^{-1} B_p + D_p$$

has an H_∞ norm which is bounded by a known number γ_p. That is, *assume* that, no matter what the perturbations are, the perturbing system (3.41) is a stable system satisfying

$$\|T_p\|_{H_\infty} < \gamma_p \,. \tag{3.43}$$

for some γ_p.

Let

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned} \tag{3.44}$$

be a controller for the nominal plant (3.40) yielding a closed loop system

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{pmatrix} v$$

$$z = \begin{pmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + (D_{11} + D_{12} D_c D_{21}) v$$

which is asymptotically stable and whose transfer function, between the input v and output z has an H_∞ norm bounded by a number γ satisfying

$$\gamma \gamma_p < 1 . \tag{3.45}$$

If this is the case, thanks to the Small Gain Theorem, it can be concluded that the controller (3.44) stabilizes any of the perturbed plants (3.40) – (3.41), so long as the perturbation is such that (3.41) is asymptotically stable and the bound (3.43) holds.

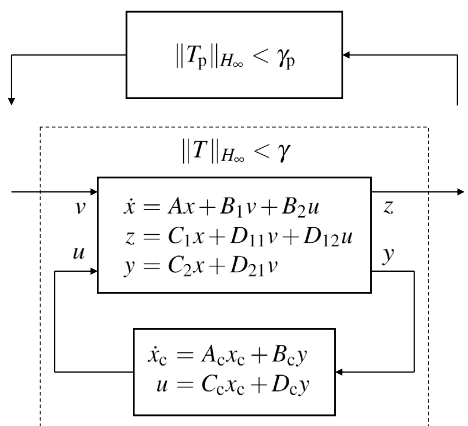

Fig. 3.4 The H_∞ approach to robust stabilization.

In this way, the problem of robust stabilization is reduced to a problem of stabilizing a nominal plant and to simultaneously enforce a bound on the H_∞ norm of its transfer function.

Example 3.4. A simplified model describing the motion of a vertical takeoff and landing (VTOL) aircraft in the vertical-lateral plane can be obtained in the following way. Let y, h and θ denote, respectively, the horizontal and vertical position of the center of mass C and the roll angle of the aircraft with respect to the horizon, as in 3.5. The control inputs are the thrust directed out the bottom of the aircraft, denoted by T, and the rolling moment produced by a couple of equal forces, denoted by F, acting at the wingtips. Their direction is not perpendicular to the horizontal body axis, but tilted by some fixed angle α. Letting M denote the mass of the aircraft, J the moment of inertia about the center of mass, ℓ the distance between the wingtips and

g the gravitational acceleration, it is seen that the motion of the aircraft is modeled by the equations

$$\begin{aligned} M\ddot{y} &= -\sin(\theta)T + 2\cos(\theta)F\sin\alpha \\ M\ddot{h} &= \cos(\theta)T + 2\sin(\theta)F\sin\alpha - gM \\ J\ddot{\theta} &= 2\ell F\cos\alpha \,. \end{aligned}$$

The purpose of the control input T is that of moving the center of mass of the airplane in the vertical-lateral plane, while that of the input F is to control the airplane's attitude, i.e. the roll angle θ.

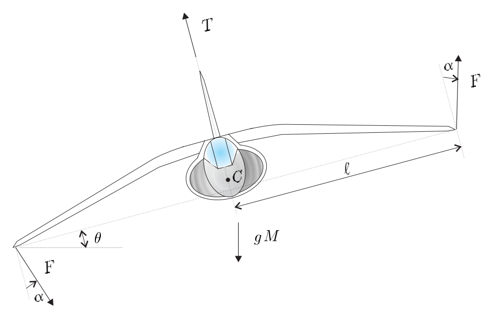

Fig. 3.5 A VTOL moving in the lateral-vertical plane.

In the hovering maneuver, the thrust T is expected to compensate exactly the force of gravity. Thus, T can be expressed as $T = gM + \delta T$ in which δT is a residual input use to control the attitude. If the angle θ is sufficiently small one can use the approximations $\sin(\theta) \simeq \theta$, $\cos(\theta) \simeq 1$ and neglect the nonlinear terms, so as to obtain a linear simplified model

$$\begin{aligned} M\ddot{y} &= -gM\theta + 2F\sin\alpha \\ M\ddot{h} &= \delta T \\ J\ddot{\theta} &= 2\ell F\cos\alpha \,. \end{aligned}$$

In this simplified model, the motion in the vertical direction is totally decoupled, and controlled by δT. On the other hand, the motion in the lateral direction and the roll angle are coupled, and controlled by F. We concentrate on the analysis of the latter.

The system with input F and output y is a four-dimensional system, having relative degree 2. Setting $\zeta_1 = \theta$, $\zeta_2 = \dot{\theta}$, $\xi_1 = y$, $\xi_2 = \dot{y}$ and $u = F$, it can be expressed, in state-space form, as

$$\begin{aligned}
\dot\zeta_1 &= \zeta_2\\
\dot\zeta_2 &= \tfrac{2\ell}{J}(\cos\alpha)u\\
\dot\xi_1 &= \xi_2\\
\dot\xi_2 &= -g\zeta_1 + \tfrac{2}{M}(\sin\alpha)u\,.
\end{aligned}$$

Note that this is not a strict normal form. It can be put in strict normal form, though, changing ζ_2 into

$$\tilde\zeta_2 = \zeta_2 - \frac{\ell M\cos\alpha}{J\sin\alpha}\xi_2\,,$$

as the reader can easily verify. The strict normal form in question is given by

$$\begin{aligned}
\begin{pmatrix}\dot\zeta_1\\ \dot{\tilde\zeta}_2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ \frac{\ell Mg\cos\alpha}{J\sin\alpha} & 0\end{pmatrix}\begin{pmatrix}\zeta_1\\ \tilde\zeta_2\end{pmatrix} + \begin{pmatrix}\frac{\ell M\cos\alpha}{J\sin\alpha}\\ 0\end{pmatrix}\xi_2\\
\begin{pmatrix}\dot\xi_1\\ \dot\xi_2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} + \begin{pmatrix}0\\ -g\end{pmatrix}\zeta_1 + \begin{pmatrix}0\\ 1\end{pmatrix}\tfrac{2}{M}(\sin\alpha)u\,.
\end{aligned} \tag{3.46}$$

It is seen from this that the zeros of the system are the roots of

$$N(\lambda) = \lambda^2 - \frac{\ell Mg\cos\alpha}{J\sin\alpha}\,,$$

and hence the system *is not* minimum phase, because one zero has positive real part. Therefore the (elementary) methods for robust stabilization described in the previous section cannot be applied.

However, the approach presented in this section is applicable. To this end, observe that the characteristic polynomial of the system is the polynomial $D(\lambda) = \lambda^4$ and consequently its transfer function has the form

$$T(s) = \frac{\frac{2}{M}(\sin\alpha)s^2 - \frac{2\ell g}{J}(\cos\alpha)}{s^4}\,.$$

From this – by known facts – it is seen that an equivalent realization of (3.46) is given by

$$\begin{aligned}
\dot x &= \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}x + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}u := A_0 + B_0 u\\
y &= \begin{pmatrix}-\frac{2\ell g}{J}(\cos\alpha) & 0 & \frac{2}{M}(\sin\alpha) & 0\end{pmatrix}x := Cx\,.
\end{aligned} \tag{3.47}$$

The matrices A_0 and B_0 are not subject to perturbations. Thus, the only uncertain parameters are the entries of the matrix C. We may write Cx in the form

$$Cx = C_0 x + D_{\mathrm{p}} z$$

in which $C_0 \in \mathbb{R}^4$ is a vector of nominal values, $z = \mathrm{col}(x_1, x_3)$ and $D_\mathrm{p} \in \mathbb{R}^2$ is an (uncertain) term whose elements represent the deviations, from the assumed nominal values, of the first and third entry of C. [10]

This being the case, it is concluded that the (perturbed) model can be expressed as interconnection of an accurately known system, and of a (memoryless) uncertain system, with the former modeled as

$$\begin{aligned} \dot{x} &= A_0 x + B_0 u \\ z &= \mathrm{col}(x_1, x_3) \\ y &= C_0 x + v\,, \end{aligned}$$

and the latter modeled as

$$v = D_\mathrm{p} z\,. \qquad \triangleleft$$

Example 3.5. Consider a d.c. motor, in which the stator field is kept constant, and the control is provided by the rotor voltage. The system in question can be characterized by equations expressing the mechanical balance and the electrical balance. The mechanical balance, in the hypothesis of viscous friction only (namely friction torque purely proportional to the angular velocity) has the form

$$J\dot{\Omega} + F\Omega = T$$

in which Ω denotes the angular velocity of the motor shaft, J the inertia of the rotor, F the viscous friction coefficient, and T the torque developed at the shaft. The torque T is proportional to the rotor current I, namely,

$$T = k_m I\,.$$

The rotor current, in turn, is determined by the electrical balance of the rotor circuit. This circuit, with reasonable approximation, can be modeled as in Fig. 3.6, in which R is the resistance of the rotor winding, L is the inductance of the rotor winding, R_b is the contact resistance of the brushes, C_s is a stray capacitance. The voltage V is the control input and the voltage E is the so-called "back electromotive force (e.m.f.)" which is proportional to the angular velocity of the motor shaft, namely

$$E = k_e \Omega\,.$$

The equations expressing the electrical balance are

---

[10] Since α is a small angle, it makes sense to take

$$C_0 = \begin{pmatrix} -\frac{2\ell_0 g}{J_0} & 0 & 0 & 0 \end{pmatrix},$$

where ℓ_0 and J_0 are the nominal values of ℓ and J, in which case

$$D_\mathrm{p} = \mathrm{row}(2g(\frac{\ell_0}{J_0} - \frac{\ell}{J}(\cos\alpha)), \frac{2}{M}(\sin\alpha))\,.$$

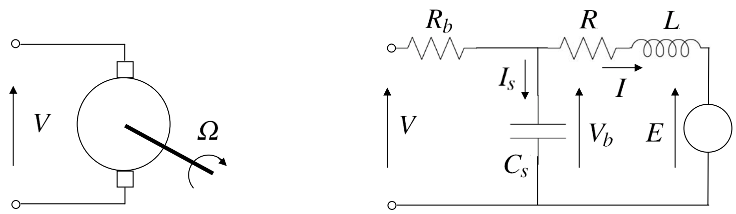

Fig. 3.6 A d.c. motor and its equivalent circuit.

$$
\begin{aligned}
R_b(I+I_s) &= V - V_b \\
C_s \frac{\mathrm{d}V_b}{\mathrm{d}t} &= I_s \\
L\frac{\mathrm{d}I}{\mathrm{d}t} + RI &= V_b - E\,.
\end{aligned}
$$

These can be put in state-space form by setting

$$\xi = \mathrm{col}(V_b, I)\,, \qquad z = \mathrm{col}(V, \Omega)$$

to obtain (recall that $E = k_e\Omega$)

$$
\begin{aligned}
\dot{\xi} &= F\xi + Gz \\
I &= H\xi
\end{aligned}
$$

in which

$$F = \begin{pmatrix} \dfrac{-1}{C_sR_b} & \dfrac{-1}{C_s} \\ \dfrac{1}{L} & \dfrac{-R}{L} \end{pmatrix}, \quad G = \begin{pmatrix} \dfrac{1}{C_sR_b} & 0 \\ 0 & \dfrac{-k_e}{L} \end{pmatrix}, \quad H = (0 \quad 1)\,.$$

In this way, the rotor current I is seen as output of a system with input z. Note that this system is stable, because characteristic polynomial of the matrix F

$$d(\lambda) = \lambda^2 + (\frac{1}{C_sR_b} + \frac{R}{L})\lambda + \frac{1}{LC_s} + \frac{R}{LC_sR_b}$$

has roots with negative real part. A simple calculation shows that the two entries of the transfer function of this system

$$T(s) = (\,T_1(s) \quad T_2(s)\,)$$

have the expressions

$$T_1(s) = \frac{1}{(C_sR_b s+1)(Ls+R)+R_b}, \qquad T_2(s) = \frac{-(C_sR_b s+1)k_e}{(C_sR_b s+1)(Ls+R)+R_b}.$$

Note that, if R_b is negligible, the two functions can be approximated as

$$T_1(s) \cong \frac{1}{(Ls+R)}, \qquad T_2(s) \cong \frac{-k_e}{(Ls+R)}.$$

and also that, if $(L/R) \ll 1$, the functions can be further approximated, over a reasonable range of frequencies, as

$$T_1(s) \cong \frac{1}{R}, \qquad T_2(s) = \frac{-k_e}{R}.$$

This shows that, neglecting the dynamics of the rotor circuit, the rotor current can be approximately expressed as

$$I_0 \cong \frac{1}{R}(V - k_e\Omega) = Kz,$$

in which K is the row vector

$$K = \begin{pmatrix} \frac{1}{R} & \frac{-k_e}{R} \end{pmatrix}.$$

With this in mind, the (full) expression of the rotor current can be written as

$$I = I_0 + v$$

where v is the output of a system

$$\begin{aligned} \dot{\xi} &= F\xi + Gz \\ v &= H\xi - Kz. \end{aligned}$$

Replacing this expression into the equation of the mechanical balance, letting x_1 denote the angular position of the rotor shaft (in which case it is seen that $\dot{x}_1 = \Omega$), setting

$$x_2 = \Omega\,, \qquad u = V$$

one obtains a model of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\Big(\frac{F}{J} + \frac{k_m k_e}{JR}\Big)x_2 + \frac{k_m}{J}v + \frac{k_m}{JR}u. \end{aligned}$$

In summary, letting $y = x_1$ denote the measured output of the system, the (full) model of the system in question can be seen as a system of the form (3.40), with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\left(\frac{F}{J} + \frac{k_m k_e}{JR}\right) \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ \frac{k_m}{J} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ \frac{k_m}{JR} \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C_2 = (1 \quad 0), \qquad D_{21} = 0,$$

interconnected with a system of the form (3.41) in which A_p, B_p, C_p, D_p coincide, respectively, with the matrices $F, G, H, -K$ defined above. The system is modeled as the interconnection of a two subsystems: a low-dimensional subsystem, that models only the dominant dynamical phenomena (the dynamics of the motion of the rotor shaft) and whose parameters are assumed to be known with sufficient accuracy, and a subsystem which may include highly uncertain parameters (the parameters R_b and C_s) and whose dynamics are neglected when a model valid only on a low range of frequencies is sought. The design philosophy described above is that of seeking a feedback law, acting on the system that models the dominant dynamical phenomena, so as to obtain robust stability in spite of parameter uncertainties and un-modeled dynamics.

3.6 The coupled LMIs approach to the problem of γ-suboptimal H_∞ feedback design

Motivated by the results discussed at the end of the previous section, we consider now a design problem which goes under the name of *problem of γ-suboptimal H_∞ feedback design*.[11] Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v\,. \end{aligned} \tag{3.48}$$

Let $\gamma > 0$ be a fixed number. The problem of γ-suboptimal H_∞ feedback design consists in finding a controller

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned} \tag{3.49}$$

yielding a closed loop system

[11] In this section, the exposition closely follows the approach of [9], [11] to the problem of γ-suboptimal H_∞ feedback design. For the numerical implementation of the design methods, see also [10].

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{x}_c\end{pmatrix} &= \begin{pmatrix}A+B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{pmatrix}\begin{pmatrix}x\\ x_c\end{pmatrix} + \begin{pmatrix}B_1+B_2D_cD_{21}\\ B_cD_{21}\end{pmatrix}v\\ z &= (\,C_1+D_{12}D_cC_2 \quad D_{12}C_c\,)\begin{pmatrix}x\\ x_c\end{pmatrix} + (D_{11}+D_{12}D_cD_{21})\,v\end{aligned} \tag{3.50}$$

which is *asymptotically stable* and whose transfer function, between the input v and output z, has an H_∞ norm *strictly less* than γ.

The interest in such problem, in view of the results discussed earlier, is obvious. In fact, if this problem is solved, the controller (3.49) robustly stabilizes any perturbed system that can be seen as pure feedback interconnection of (3.48) and of an uncertain system of the form (3.41), so long as the latter is a stable system having a transfer function whose H_∞ norm is less than or equal to the inverse of γ. Of course, the smaller is the value of γ for which the problem is solvable, the "larger" is the set of perturbations against which robust stability can be achieved.

Rewrite system (3.50) as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathscr{A}\mathbf{x} + \mathscr{B}v\\ z &= \mathscr{C}\mathbf{x} + \mathscr{D}v\end{aligned}$$

where

$$\mathscr{A} = \begin{pmatrix}A+B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix}B_1+B_2D_cD_{21}\\ B_cD_{21}\end{pmatrix}$$
$$\mathscr{C} = (\,C_1+D_{12}D_cC_2 \quad D_{12}C_c\,), \qquad \mathscr{D} = D_{11}+D_{12}D_cD_{21}\,.$$

In view of the Bounded Real Lemma, the closed loop system has the desired properties if and only if there exists a symmetric matrix $\mathscr{X} > 0$ satisfying

$$\mathscr{X} > 0 \tag{3.51}$$

$$\begin{pmatrix}\mathscr{A}^{\mathrm{T}}\mathscr{X} + \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} & \mathscr{C}^{\mathrm{T}}\\ \mathscr{B}^{\mathrm{T}}\mathscr{X} & -\gamma I & \mathscr{D}^{\mathrm{T}}\\ \mathscr{C} & \mathscr{D} & -\gamma I\end{pmatrix} < 0\,. \tag{3.52}$$

Thus, the problem is to try to find a quadruplet $\{A_c, B_c, C_c, D_c\}$ such that (3.51) and (3.52) hold for some symmetric $\mathscr{X}$.

The basic inequality (3.52) will be now transformed as follows. Let

$$x \in \mathbb{R}^n, \quad x_c \in \mathbb{R}^k, \quad v \in \mathbb{R}^{m_1}, \quad u \in \mathbb{R}^{m_2}, \quad z \in \mathbb{R}^{p_1}, \quad y \in \mathbb{R}^{p_2}\,.$$

Set

$$\mathbf{A}_0 = \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix}B_1\\ 0\end{pmatrix}, \quad \mathbf{C}_0 = (\,C_1 \quad 0\,)\,, \tag{3.53}$$

$$\Psi(\mathscr{X}) = \begin{pmatrix}\mathbf{A}_0^{\mathrm{T}}\mathscr{X} + \mathscr{X}\mathbf{A}_0 & \mathscr{X}\mathbf{B}_0 & \mathbf{C}_0^{\mathrm{T}}\\ \mathbf{B}_0^{\mathrm{T}}\mathscr{X} & -\gamma I & D_{11}^{\mathrm{T}}\\ \mathbf{C_0} & D_{11} & -\gamma I\end{pmatrix}, \tag{3.54}$$

$$\mathscr{P} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ B_2^{\mathrm{T}} & 0 & 0_{m_2\times m_1} & D_{12}^{\mathrm{T}} \end{pmatrix},$$

$$\mathscr{Q} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ C_2 & 0 & D_{21} & 0_{p_2\times p_1} \end{pmatrix},$$

and

$$\Xi(\mathscr{X}) = \begin{pmatrix} \mathscr{X} & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{p_1} \end{pmatrix}.$$

Collecting the parameters of the controller (3.49) in the $(n+m_2)\times(n+p_2)$ matrix

$$\mathbf{K} = \begin{pmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{pmatrix} \tag{3.55}$$

the inequality (3.52) can be rewritten as

$$\Psi(\mathscr{X}) + \mathscr{Q}^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}[\mathscr{P}\Xi(\mathscr{X})] + [\mathscr{P}\Xi(\mathscr{X})]^{\mathrm{T}}\mathbf{K}\mathscr{Q} < 0. \tag{3.56}$$

Thus, the problem of γ-suboptimal H_∞ feedback design can be cast as the problem of finding a symmetric matrix $\mathscr{X} > 0$ and a matrix $\mathbf{K}$ such that (3.56) holds. Note that this inequality, is not a linear matrix inequality in the unknowns $\mathscr{X}$ and $\mathbf{K}$. Rather, it is a *bilinear* matrix inequality. [12] However, the problem of finding a matrix $\mathscr{X} > 0$ for which (3.56) is solved by some $\mathbf{K}$ can be cast in terms of linear matrix inequalities. In this context, the following result is very useful. [13]

Lemma 3.3. *Given a symmetric $m\times m$ matrix Ψ and two matrices P,Q having m columns, consider the problem of finding some matrix K of compatible dimensions such that*

$$\Psi + Q^{\mathrm{T}}KP + P^{\mathrm{T}}K^{\mathrm{T}}Q < 0. \tag{3.57}$$

Let W_P and W_Q be two matrices such that [14]

$$\begin{aligned} \mathrm{Im}(W_P) &= \mathrm{Ker}(P) \\ \mathrm{Im}(W_Q) &= \mathrm{Ker}(Q). \end{aligned}$$

Then (3.57) is solvable in K if and only if

$$\begin{aligned} W_Q^{\mathrm{T}}\Psi W_Q &< 0 \\ W_P^{\mathrm{T}}\Psi W_P &< 0. \end{aligned} \tag{3.58}$$

This Lemma can be used to eliminate $\mathbf{K}$ from (3.56) and obtain existence conditions depending only on $\mathscr{X}$ and on the parameters of the plant (3.48). Let $W_{\mathscr{P}\Xi(\mathscr{X})}$

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[12] The inequality (3.56), for each fixed $\mathscr{X}$ is a linear matrix inequality in $\mathbf{K}$ and, for each fixed $\mathbf{K}$ is a linear matrix inequality in $\mathscr{X}$.

[13] The proof of Lemma 3.3 can be found in [9].

[14] Observe that, since $\mathrm{Ker}(P)$ and $\mathrm{Ker}(Q)$ are subspaces of $\mathbb{R}^m$, then W_P and W_Q are matrices having m rows.

be a matrix whose columns span $\mathrm{Ker}(\mathscr{P}\Xi(\mathscr{X}))$ and let $W_{\mathscr{Q}}$ be a matrix whose columns span $\mathrm{Ker}(\mathscr{Q})$. According to Lemma 3.3, there exists $\mathbf{K}$ for which (3.56) holds if and only if

$$\begin{aligned} &W_{\mathscr{Q}}^{\mathrm{T}}\Psi(\mathscr{X})W_{\mathscr{Q}} < 0 \\ &W_{\mathscr{P}\Xi(\mathscr{X})}^{\mathrm{T}}\Psi(\mathscr{X})W_{\mathscr{P}\Xi(\mathscr{X})} < 0\,. \end{aligned} \tag{3.59}$$

The second of these two inequalities can be further manipulated observing that if $W_{\mathscr{P}}$ is a matrix whose columns span $\mathrm{Ker}(\mathscr{P})$, the columns of the matrix $[\Xi(\mathscr{X})]^{-1}W_{\mathscr{P}}$ span $\mathrm{Ker}(\mathscr{P}\Xi(\mathscr{X}))$. Thus, having set

$$\Phi(\mathscr{X}) = \begin{pmatrix} \mathbf{A}_0\mathscr{X}^{-1} + \mathscr{X}^{-1}\mathbf{A}_0^{\mathrm{T}} & \mathbf{B}_0 & \mathscr{X}^{-1}\mathbf{C}_0^{\mathrm{T}} \\ \mathbf{B}_0^{\mathrm{T}} & -\gamma I & D_{11}^{\mathrm{T}} \\ \mathbf{C_0}\mathscr{X}^{-1} & D_{11} & -\gamma I \end{pmatrix}, \tag{3.60}$$

the second of (3.59) can be rewritten as

$$W_{\mathscr{P}}^{\mathrm{T}}[\Xi(\mathscr{X})]^{-1}\Psi(\mathscr{X})[\Xi(\mathscr{X})]^{-1}W_{\mathscr{P}} = W_{\mathscr{P}}^{\mathrm{T}}\Phi(\mathscr{X})W_{\mathscr{P}} < 0\,. \tag{3.61}$$

It therefore concluded that (3.56) is solvable for some $\mathbf{K}$ if and only if the matrix $\mathscr{X}$ satisfies the first of (3.59) and (3.61).

In view of this, the following (intermediate) conclusion can be drawn.

Proposition 3.1. *There exists a k-dimensional controller that solves the problem of γ-suboptimal H_∞ feedback design if and only if there exists a $(n+k)\times(n+k)$ symmetric matrix $\mathscr{X} > 0$ satisfying the following set of linear matrix inequalities*

$$\begin{aligned} &W_{\mathscr{Q}}^{\mathrm{T}}\Psi(\mathscr{X})W_{\mathscr{Q}} < 0 \\ &W_{\mathscr{P}}^{\mathrm{T}}\Phi(\mathscr{X})W_{\mathscr{P}} < 0\,, \end{aligned} \tag{3.62}$$

in which $W_{\mathscr{P}}$ is a matrix whose columns span $\mathrm{Ker}(\mathscr{P})$*, $W_{\mathscr{Q}}$ is a matrix whose columns span* $\mathrm{Ker}(\mathscr{Q})$*, and $\Psi(\mathscr{X})$ and $\Phi(\mathscr{X})$ are the matrices defined in (3.54) and (3.60). For any of such $\mathscr{X}$'s, a solution* $\mathbf{K}$ *of the resulting linear matrix inequality (3.56) exists and provides a solution of the problem of γ-suboptimal H_∞ feedback design.*

The two inequalities (3.62) thus found can be further simplified. To this end, it is convenient to partition $\mathscr{X}$ and $\mathscr{X}^{-1}$ as

$$\mathscr{X} = \begin{pmatrix} S & N \\ N^{\mathrm{T}} & * \end{pmatrix}, \qquad \mathscr{X}^{-1} = \begin{pmatrix} R & M \\ M^{\mathrm{T}} & * \end{pmatrix} \tag{3.63}$$

in which R and S are $n\times n$ and N and M are $n\times k$. With the partition (3.63), the matrix (3.54) becomes

$$\Psi(\mathscr{X}) = \begin{pmatrix} A^{\mathrm{T}}S + SA & A^{\mathrm{T}}N & SB_1 & C_1^{\mathrm{T}} \\ N^{\mathrm{T}}A & 0 & N^{\mathrm{T}}B_1 & 0 \\ B_1^{\mathrm{T}}S & B_1^{\mathrm{T}}N & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & 0 & D_{11} & -\gamma I \end{pmatrix} \tag{3.64}$$

and the matrix (3.60) becomes

$$\Phi(\mathscr{X}) = \begin{pmatrix} AR + RA^{\mathrm{T}} & AM & B_1 & RC_1^{\mathrm{T}} \\ M^{\mathrm{T}}A^{\mathrm{T}} & 0 & 0 & M^{\mathrm{T}}C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1R & C_1M & D_{11} & -\gamma I \end{pmatrix}. \tag{3.65}$$

From the definition of $\mathscr{Q}$, it is readily seen that a matrix $W_{\mathscr{Q}}$ whose columns span $\mathrm{Ker}(\mathscr{Q})$ can be expressed as

$$W_{\mathscr{Q}} = \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \\ Z_2 & 0 \\ 0 & I_{p_1} \end{pmatrix}$$

in which

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

is a matrix whose columns span

$$\mathrm{Ker}\begin{pmatrix} C_2 & D_{21} \end{pmatrix}.$$

Then, an easy calculation shows that the first inequality in (3.62) can be rewritten as

$$\begin{pmatrix} Z_1^{\mathrm{T}} & Z_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}}S + SA & SB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0.$$

Likewise, from the definition of $\mathscr{P}$, it is readily seen that a matrix $W_{\mathscr{P}}$ whose columns span $\mathrm{Ker}(\mathscr{P})$ can be expressed as

$$W_{\mathscr{P}} = \begin{pmatrix} V_1 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \\ V_2 & 0 \end{pmatrix}$$

in which

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

is a matrix whose columns span

$$\mathrm{Ker}\begin{pmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} \end{pmatrix}.$$

Then, an easy calculation shows that the second inequality in (3.62) can be rewritten as

$$\begin{pmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} AR + RA^{\mathrm{T}} & RC_1^{\mathrm{T}} & B_1 \\ C_1R & -\gamma I & D_{11} \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{pmatrix} < 0.$$

In this way, the two inequalities (3.62) have been transformed into two inequalities involving the submatrix S of $\mathscr{X}$ and the submatrix R of $\mathscr{X}^{-1}$. To complete the analysis, it remains to connect these matrices to each other. This is achieved via the following Lemma.

Lemma 3.4. *Let S and R be symmetric $n \times n$ matrices. There exist a $n \times k$ matrix N and $k \times k$ symmetric matrix Z a such that*

$$\begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix} > 0 \tag{3.66}$$

and

$$\begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix}^{-1} = \begin{pmatrix} R & * \\ * & * \end{pmatrix} \tag{3.67}$$

if and only if

$$\mathrm{rank}(I - SR) \le k \tag{3.68}$$

and

$$\begin{pmatrix} S & I \\ I & R \end{pmatrix} \ge 0\,. \tag{3.69}$$

Proof. To prove necessity, write

$$\mathscr{X} = \begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix} \qquad \text{and} \qquad \mathscr{X}^{-1} = \begin{pmatrix} R & M \\ M^{\mathrm{T}} & * \end{pmatrix}.$$

Then, by definition we have

$$\begin{aligned} SR + NM^{\mathrm{T}} &= I \\ N^{\mathrm{T}}R + ZM^{\mathrm{T}} &= 0\,. \end{aligned} \tag{3.70}$$

Thus $I - SR = NM^{\mathrm{T}}$ and this implies (3.68), because N has k columns. Now, set

$$\mathscr{T} = \begin{pmatrix} I & R \\ 0 & M^{\mathrm{T}} \end{pmatrix}.$$

Using (3.70), the first of which implies $MN^{\mathrm{T}} = I - RS$ because S and R are symmetric, observe that

$$\mathscr{T}^{\mathrm{T}} \mathscr{X} \mathscr{T} = \begin{pmatrix} S & I \\ I & R \end{pmatrix}.$$

Pick $y \in \mathbb{R}^{2n}$ and define $x \in \mathbb{R}^{n+k}$ as $x = \mathscr{T} y$. Then, it is seen that

$$y^{\mathrm{T}} \begin{pmatrix} S & I \\ I & R \end{pmatrix} y = x^{\mathrm{T}} \mathscr{X} x \ge 0$$

because the matrix $\mathscr{X}$ is positive definite by assumption. This concludes the proof of the necessity.

To prove sufficiency, let $\hat{k} \le k$ be the rank of $I - SR$ and let $\hat{N}, \hat{M}$ two $n \times \hat{k}$ matrices of rank $\hat{k}$ such that

$$I - SR = \hat{N}\hat{M}^{\mathrm{T}}. \tag{3.71}$$

Using the property (3.71) it is possible to show that the equation

$$\hat{N}^{\mathrm{T}}R + \hat{Z}\hat{M}^{\mathrm{T}} = 0 \tag{3.72}$$

has a solution $\hat{Z}$. In fact, observe that

$$\hat{M}\hat{N}^{\mathrm{T}}R - R\hat{N}\hat{M}^{\mathrm{T}} = (I - RS)R - R(I - SR) = 0\,.$$

Let L be any matrix such that $\hat{M}^{\mathrm{T}}L = I$ (which exists because the $\hat{k}$ rows of $\hat{M}^{\mathrm{T}}$ are independent) and, from the identity above, obtain

$$\hat{M}\hat{N}^{\mathrm{T}}RL + R\hat{N} = 0\,.$$

This shows that the matrix $\hat{Z} = (\hat{N}^{\mathrm{T}}RL)^{\mathrm{T}}$ satisfies (3.72). The matrix $\hat{Z}$ is symmetric. In fact, note that

$$0 = \hat{M}(\hat{N}^{\mathrm{T}}R + \hat{Z}\hat{M}^{\mathrm{T}}) = (I - RS)R + \hat{M}\hat{Z}\hat{M}^{\mathrm{T}} = R - RSR + \hat{M}\hat{Z}\hat{M}^{\mathrm{T}}$$

and hence $\hat{M}\hat{Z}\hat{M}^{\mathrm{T}}$ is symmetric. This yields

$$0 = \hat{M}\hat{Z}\hat{M}^{\mathrm{T}} - \hat{M}\hat{Z}^{\mathrm{T}}\hat{M}^{\mathrm{T}} = \hat{M}(\hat{Z} - \hat{Z}^{\mathrm{T}})\hat{M}^{\mathrm{T}}$$

from which it is seen that $\hat{Z} = \hat{Z}^{\mathrm{T}}$ because $\hat{M}^{\mathrm{T}}$ has independent rows.

As a consequence of (3.71) and(3.72), the symmetric matrix

$$\hat{\mathscr{X}} = \begin{pmatrix} S & \hat{N} \\ \hat{N}^{\mathrm{T}} & \hat{Z} \end{pmatrix} \tag{3.73}$$

is a solution of

$$\begin{pmatrix} S & I \\ \hat{N}^{\mathrm{T}} & 0 \end{pmatrix} = \hat{\mathscr{X}} \begin{pmatrix} I & R \\ 0 & \hat{M}^{\mathrm{T}} \end{pmatrix}. \tag{3.74}$$

The symmetric matrix $\hat{\mathscr{X}}$ thus found is invertible because, otherwise, the independence of the rows of the matrix on the left-hand side of (3.74) would be contradicted. It is easily seen that

$$\hat{\mathscr{X}}^{-1} = \begin{pmatrix} R & * \\ \hat{M}^{\mathrm{T}} & * \end{pmatrix}. \tag{3.75}$$

Moreover, it is also possible to prove that $\hat{\mathscr{X}} > 0$. In fact, letting

$$\hat{\mathscr{T}} = \begin{pmatrix} I & R \\ 0 & \hat{M}^{\mathrm{T}} \end{pmatrix}$$

observe that

$$\hat{\mathscr{T}}^{\mathrm{T}} \hat{\mathscr{X}} \hat{\mathscr{T}} = \begin{pmatrix} S & I \\ I & R \end{pmatrix}.$$

Suppose $x^{\mathrm{T}} \hat{\mathscr{X}} x < 0$ for some $x \neq 0$. Using the fact that the rows of $\hat{\mathscr{T}}$ are independent, find y such that $x = \hat{\mathscr{T}} y$. This would make

$$y^{\mathrm{T}} \begin{pmatrix} S & I \\ I & R \end{pmatrix} y < 0\,,$$

which contradicts (3.69). Thus, $x^{\mathrm{T}} \hat{\mathscr{X}} x \geq 0$ for all x, i.e. $\hat{\mathscr{X}} \geq 0$. But since $\hat{\mathscr{X}}$ is nonsingular, we conclude that $\hat{\mathscr{X}}$ is positive definite.

If $\hat{k} = k$ the sufficiency is proven. Otherwise, set $\ell = k - \hat{k}$ and

$$\mathscr{X} = \begin{pmatrix} S & \hat{N} & 0 \\ \hat{N}^{\mathrm{T}} & \hat{Z} & 0 \\ 0 & 0 & I_{\ell \times \ell} \end{pmatrix},$$

observe that $\mathscr{X}$ is positive definite and that

$$\mathscr{X}^{-1} = \begin{pmatrix} R & * & 0 \\ \hat{M}^{\mathrm{T}} & * & 0 \\ 0 & 0 & I_{\ell \times \ell} \end{pmatrix}$$

has the required structure. $\triangleleft$

Remark 3.3. Note that condition (3.69) implies $S > 0$ and $R > 0$. This is the consequence of the arguments used in the proof of the previous Lemma, but can also be proven directly as follows. Condition (3.69) implies that the two diagonal blocks S and R are positive semidefinite. Consider the quadratic form associated with the matrix on left-hand side of (3.69),

$$V(x,z) = \begin{pmatrix} x^{\mathrm{T}} & z^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} S & I \\ I & R \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = x^{\mathrm{T}} S x + z^{\mathrm{T}} R z + 2 z^{\mathrm{T}} x\,.$$

We prove that R is nonsingular (and thus positive definite). By contradiction, suppose this is not the case. Then, there is a vector $\bar{z} \neq 0$ such that $\bar{z}^{\mathrm{T}} R \bar{z} = 0$. Pick any vector $\bar{x}$ such that $\bar{z}^{\mathrm{T}} \bar{x} \neq 0$. Then, for any $c \in \mathbb{R}$,

$$V(\bar{x}, c\bar{z}) \leq \lambda_{\max}(S) \|\bar{x}\|^2 + 2c\bar{z}^{\mathrm{T}} \bar{x}\,.$$

Clearly, by choosing an appropriate c, the right hand side can be made strictly negative. Thus, $V(x,z)$ is not positive semidefinite and this contradicts (3.69). The same arguments are used to show that also S is nonsingular. $\triangleleft$

This Lemma establishes a *coupling condition* between the two submatrices S and R identified in the previous analysis, that determines the positivity of the matrix $\mathscr{X}$. Using this Lemma it is therefore possible to arrive at the following conclusion.

Theorem 3.3. *Consider a plant of modeled by equations of the form (3.48). Let V_1, V_2, Z_1, Z_2 be matrices such that*

$$\mathrm{Im}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} C_2 & D_{21} \end{pmatrix}, \qquad \mathrm{Im}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} \end{pmatrix}.$$

The problem of γ-suboptimal H_∞ feedback design has a solution if and only if there exist symmetric matrices S and R satisfying the following system of linear matrix inequalities

$$\begin{pmatrix} Z_1^{\mathrm{T}} & Z_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} A^{\mathrm{T}}S+SA & SB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{pmatrix}\begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{3.76}$$

$$\begin{pmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} AR+RA^{\mathrm{T}} & RC_1^{\mathrm{T}} & B_1 \\ C_1R & -\gamma I & D_{11} \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix}\begin{pmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{3.77}$$

$$\begin{pmatrix} S & I \\ I & R \end{pmatrix} \geq 0. \tag{3.78}$$

In particular, there exists a solution of dimension k if and only if there exist R and S satisfying (3.76), (3.77), (3.78) and, in addition,

$$\mathrm{rank}(I - RS) \leq k. \tag{3.79}$$

The result above describes necessary and sufficient conditions for the existence of a controller that solves the problem of γ-suboptimal H_∞ feedback design. For the actual *construction* of a controller, one may proceed as follows. Assuming that S and R are positive definite symmetric matrices satisfying the system of linear matrix inequalities (3.76), (3.77), (3.78), construct a matrix $\mathscr{X}$ as indicated in the proof of Lemma 3.4, that is, find two $n \times k$ matrices N and M such that

$$I - SR = NM^{\mathrm{T}}$$

with $k = \mathrm{rank}(I - SR)$, and solve for $\mathscr{X}$ the linear equation

$$\begin{pmatrix} S & I \\ N^{\mathrm{T}} & 0 \end{pmatrix} = \mathscr{X}\begin{pmatrix} I & R \\ 0 & M^{\mathrm{T}} \end{pmatrix}.$$

By construction, the matrix in question is positive definite, satisfies (3.62) and their equivalent versions (3.59). Thus, according to Lemma 3.3, there exists a matrix $\mathbf{K}$ satisfying (3.56). This is a linear matrix inequality in the only unknown $\mathbf{K}$. The solution of such inequality provides the required controller as

$$\mathbf{K} = \begin{pmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{pmatrix}.$$

Remark 3.4. It is worth observing that if the problem of γ-suboptimal H_∞ feedback design has *any* solution at all, it does have a solution in which the dimension of the controller, i.e. the dimension of the vector x_c in (3.49), does not exceed n, i.e. the dimension of the vector x in (3.48). This is an immediate consequence of the construction shown above, in view of the fact that the rank of $(I - SR)$ cannot exceed n.

References

1. J. Willems, Dissipative dynamical systems, *Archive for Rational Mechanics and Analysis*, **45**, pp. 321–393, 1972.
2. C.A. Desoer, M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.
3. B. Francis, *A Course in H_∞ Control Theory*, Springer Verlag, Berlin, 1987.
4. B.D.O. Anderson, J.B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice Hall, Englewoof Cliffs NY, 1990.
5. T. Basar, P. Bernhard, *H_∞ Optimal Control and Related Minimaz Design Problems*, Birkäuser, Boston, 1990.
6. J. Doyle, B. Francis and A. Tannenbaum, *Feedback Control Theory*, MacMillan, New York, 1992.
7. S.P. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Studies in Applied and Numerical Mathematics, SIAM, 1994.
8. A.J. Van der Schaft, *L_2 Gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag, London, 1996.
9. P. Gahinet and P. Apkarian, A Linear Matrix Inequality Approach to H_∞ Control, *Int. J. Robust and Nonlinear Control*, **4**, pp. 421–448, 1994.
10. P. Gahinet, A. Nemirovski, A. J. Laub, M. Chilali, *LMI Control Toolbox: for use with MATLAB*. The Mathworks, Inc., 1995.
11. C.W. Scherer, P. Gahinet, M. Chilali, Multiobjective output-feedback control via LMI optimization, *IEEE Trans. Automat. Control*, **42**, pp. 896-911, 1997.
12. J. G. VanAntwerp, R. D. Braatz, A tutorial on linear and bilinear matrix inequalities, *Journal of Process Control*, **10**, pp. 363–385, 2000.
13. C. Scherer, S. Weiland, Linear Matrix Inequalities in Control, in *The Control Handbook: Control Systems Advanced Methods*, W.S. Levine ed., CRC Press, Chapter 24, 2011.

Chapter 4
Regulation and Tracking in Linear Systems

Abstract In this Chapter, we will study in some generality the problem of designing a feedback law to the purpose of making a controlled plant stable, and securing exact asymptotic tracking of external commands (respectively, exact asymptotic rejection of external disturbances) which belong to a fixed family of functions. The problem in question can be seen as a (broad) generalization of the classical set-point control problem in the elementary theory of servomechanisms.

4.1 The problem of asymptotic tracking and disturbance rejection

In the last part of the previous Chapter, we have considered a plant modeled as in (3.48), and – regarding y as an output available for *measurement* and u as an input available for *control* – we have studied the problem of finding a controller of the form (3.49) yielding a *stable* closed loop system having a transfer function – between input v and output z – whose H_∞ norm does not exceed a given number γ. The problem in question was motivated by the interest in solving a problem of robust stability, but it has an independent interest *per se*. In fact, regarding v as a vector of external *disturbances*, affecting the behavior of the controlled plant, and z as set of variables of special interest, the problem considered in section 3.6 can be regarded as the problem of finding a controller that – while keeping the closed loop stable – enforces a prescribed *attenuation* of the effect of the disturbance v on the variable of interest z, the attenuation being expressed in terms of the H_∞ norm (or of the $\mathscr{L}_2$ gain, if desired) of the resulting system.

In the present Chapter, we continue to study a problem of this kind, i.e. the control of a plant affected by external disturbances, from which certain variables of interest have to be "protected", but with some differences. Specifically, while in section 3.6 we have considered the case in which the influence of the disturbances on the variables of interest had to be attenuated by a given factor, we consider in this section the case in which the influence of the disturbances on the variables of interest should ultimately vanish. This is indeed a much stronger requirement and it is unlikely that it might be enforced in general. It can be enforced, though, if the disturbances happen to belong to a specific (well-defined) family of signals. This gives rise to a specific setup, known as *problem of asymptotic disturbance rejection and/or tracking*, or more commonly *problem of output regulation*, that will be explained in more detail hereafter.

For consistency with the notations currently used in the literature dealing with the specific problem addressed in this Chapter, the controlled plant (compare with (3.48)) is assumed to be modeled by equations written in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w \\ y &= Cx + Qw\,. \end{aligned} \tag{4.1}$$

The first equation of this system describes a *plant* with *state* $x \in \mathbb{R}^n$ and *control input* $u \in \mathbb{R}^m$, subject to a set of *exogenous input* variables $w \in \mathbb{R}^{n_w}$ which includes *disturbances* (to be rejected) and/or *references* (to be tracked). The second equation defines a set of *regulated* (or *error*) variables $e \in \mathbb{R}^p$, which are expressed as a linear combination of the plant state x and of the exogenous input w.[1] The third equation defines a set of *measured* variables $y \in \mathbb{R}^q$, which are assumed to be available for feedback, and are also expressed as a linear combination of the plant state x and of the exogenous input w.

The control action to (4.1) is to be provided by a feedback *controller* which processes the measured information y and generates the appropriate control input. In general, the controller in question is a system modeled by equations of the form

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}y \\ u &= C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y\,, \end{aligned} \tag{4.2}$$

with state $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, which yields a closed-loop system modeled by equations of the form

$$\begin{aligned} \dot{x} &= (A + BD_{\mathrm{c}}C)x + BC_{\mathrm{c}}x_{\mathrm{c}} + (P + BD_{\mathrm{c}}Q)w \\ \dot{x}_{\mathrm{c}} &= B_{\mathrm{c}}Cx + A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}Qw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w\,. \end{aligned} \tag{4.3}$$

This system is seen as a system with *input* w and *output* e. The purpose of the control is to make sure that the closed loop system be *asymptotically stable* and that the regulated variable e, viewed as a function of time, *asymptotically decays to zero* as time tends to ∞, for every possible initial state and for every possible *exogenous input* in a prescribed family of functions of time. This requirement is also known as property of *output regulation*. For the sake of mathematical simplicity, and also because in this way a large number of relevant practical situations can be covered, it

[1] If some components of w are (external) commands that certain variables of interest are required to track, then some of the components of e can be seen as *tracking errors*, that is differences between the actual values of those variables of interest and their expected reference values. Overall, the components of e can simply be seen as variables on which the effect of w is expected to vanish asymptotically.

is assumed that the family of the exogenous inputs $w(\cdot)$ which affect the plant, and for which asymptotic decay of the regulated variable is to be achieved, is the family of all functions of time which are solution of a homogeneous linear differential equation

$$\dot{w} = Sw \tag{4.4}$$

with state $w \in \mathbb{R}^{n_w}$, for all possible initial conditions $w(0)$. This system, which is viewed as a mathematical model of a "generator" for all possible exogenous input functions, is called the *exosystem*. In this Chapter, we will discuss this design problem at various levels of generality.

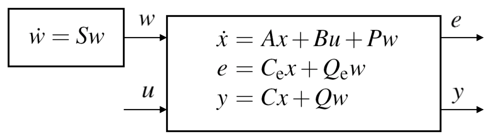

Fig. 4.1 The controlled plant and the exosystem

Note that, without loss of generality, in the analysis of this problem it can be assumed that all the eigenvalues of S have non-negative real part. In fact, if there were eigenvalues having negative real part, system (4.4) could be split (after a similarity transformation) as

$$\begin{aligned} \dot{w}_s &= S_s w_s \\ \dot{w}_u &= S_u w_u \end{aligned}$$

with S_s having *all* eigenvalues in $\mathbb{C}^-$ and S_u having *no* eigenvalue in $\mathbb{C}^-$. The input $w_s(t)$ generated by the upper subsystem is asymptotically vanishing. Thus, once system (4.3) has been rendered stable, the influence of $w_s(t)$ on the regulated variable $e(t)$ asymptotically vanishes as well. In other words, the only exogenous inputs that matter in the problem under consideration are those generated by the lower subsystem.

This being said, it must be observed – though – that if some of the eigenvalues of S have *positive* real part, there will be initial conditions from which the exosystem (4.4) generates signals that grow unbounded as time increases. Thus, also the associated response of the state of (4.3) can grow unbounded as time increases and this is not a reasonable setting. In view of all of this, in the analysis which follows we will proceed under the assumption that all eigenvalues of S have zero real part (and are simple roots of its minimal polynomial), even though all results that will be presented are also valid in the more general setting in which S has eigenvalues with positive real part (and/or has eigenvalues with zero real part which are multiple roots of its minimal polynomial).

4.2 The case of full information and Francis' equations

For expository reasons we consider first the (non realistic) case in which the full state x of the plant and the full state w of the exosystem are available for measurement, i.e. the case in which the measured variable y in (4.1) is $y = \mathrm{col}(x,w)$. This is called the case of "full information". We also assume that all system parameters are known exactly.

In this setup, we consider the special version of the controller (4.2) in which $u = D_c y$, that we rewrite for convenience as

$$u = Fx + Lw\,. \tag{4.5}$$

Proposition 4.1. *The problem of output regulation in the case of full information has a solution if and only if*

(i) the matrix pair (A,B) is stabilizable
(ii) there exists a solution pair (Π,Ψ) of the linear matrix equations

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C_e \Pi + Q_e\,. \end{aligned} \tag{4.6}$$

Proof. [Necessity] System (4.1) controlled by (4.5) can be regarded as an autonomous linear system

$$\begin{pmatrix} \dot w \\ \dot x \end{pmatrix} = \begin{pmatrix} S & 0 \\ P+BL & A+BF \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}. \tag{4.7}$$

Suppose the pair of matrices F,L is such the controller (4.5) solves the problem of output regulation. Then, the matrix $A+BF$ has all eigenvalues with negative real part and, hence, the matrix pair (A,B) *must be stabilizable*.

The eigenvalues of system (4.7) are those of $(A+BF)$ and those of S. The former are all in $\mathbb{C}^-$ while none of the latter, in view of the standing assumption described at the end of the previous section, is in $\mathbb{C}^-$. Hence this system possesses a *stable* eigenspace and a *center* eigenspace. The former can be described as

$$\mathscr{V}^s = \mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix},$$

while the latter, which is complementary to the stable eigenspace, can be described as

$$\mathscr{V}^c = \mathrm{Im}\begin{pmatrix} I \\ \Pi \end{pmatrix},$$

in which Π is a solution of the Sylvester equation

$$\Pi S = (A+BF)\Pi + (P+BL)\,. \tag{4.8}$$

Setting

$$\Psi = L + F\Pi$$

we deduce that the pair Π, Ψ satisfies the first equation of (4.6).

Any trajectory of (4.7) has a unique decomposition into a component entirely contained in the stable eigenspace and a component entirely contained in the center eigenspace. The former, which asymptotically decays to 0 as $t \to \infty$, is the *transient component* of the trajectory. The latter, on the contrary, is persistent: it is the *steady-state component* of the trajectory. We denote it as

$$x_{\mathrm{ss}}(t) = \Pi w(t)\,.$$

As a consequence, the steady-state component of the regulated output $e(t)$ is

$$e_{\mathrm{ss}}(t) = (C_{\mathrm{e}}\Pi + Q_{\mathrm{e}})w(t)\,.$$

If the controller (4.5) solves the problem of output regulation, we must have $e_{\mathrm{ss}}(t) = 0$ and this can only occur if Π satisfies $C_{\mathrm{e}}\Pi + Q_{\mathrm{e}} = 0$. In terms of state trajectories, this is equivalent to say that the steady-state component of any state trajectory *must be contained in the kernel of the map* $e = Q_{\mathrm{e}}w + C_{\mathrm{e}}x$. We have shown in this way that the solution Π of the Sylvester equation (4.8) necessarily satisfies the second equation of (4.6). In summary, if the pair of matrices F, L is such that the controller (4.5) solves the problem of output regulation, the equations (4.6) *must have a solution* pair (Π, Ψ).

[Sufficiency] Suppose that (A, B) is stabilizable and pick F such that the eigenvalues of $A + BF$ have negative real part. Suppose the equations (4.6) have a solution (Π, Ψ) and pick L as

$$L = \Psi - F\Pi\,,$$

that is, consider a control law of the form

$$u = \Psi w + F(x - \Pi w)\,.$$

The corresponding closed loop system is

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P + B(\Psi - F\Pi) & A + BF \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}.$$

Consider the change of variables

$$\tilde{x} = x - \Pi w$$

and, bearing in mind the first equation of (4.6) and the choice of L, observe that, in the new coordinates,

$$\dot{\tilde{x}} = (A + BF)\tilde{x}\,.$$

Hence $\lim_{t\to\infty} \tilde{x}(t) = 0$, because the eigenvalues of $A + BF$ have negative real part.

In the new coordinates,

$$e(t) = C_e\tilde{x}(t) + (C_e\Pi + Q_e)w(t) = C_e\tilde{x}(t)\,.$$

Therefore, using the second equation of (4.6), we conclude that

$$\lim_{t\to\infty} e(t) = 0$$

and we see that the proposed law solves the problem. $\triangleleft$

Equations (4.6) are called the *linear regulator equations* or, also, the *Francis's equations*.[2] The following result is useful to determine the existence of solutions.[3]

Lemma 4.1. *The Francis' equation (4.6) have a solution for any* (P, Q_e) *if and only if*

$$\operatorname{rank}\begin{pmatrix} A-\lambda I & B \\ C_e & 0 \end{pmatrix} = \#\ \text{rows} \qquad \forall\lambda \in \sigma(S)\,. \tag{4.9}$$

If this is the case and the matrix on the left-hand side of (4.9) is square (i.e. the control input u and the regulated output e have the same number of components), the solution (Π, Ψ) *is unique.*

The condition (4.9) is usually referred to as the *non-resonance condition*. Note that the condition is necessary and sufficient if the existence of a solution *for any* (P, Q_e) is sought. Otherwise, it is simply a sufficient condition. Note also that such condition requires $m \geq p$, i.e. that the number of components of the control input u be larger than or equal to the number of components of the regulated output.

Example 4.1. As an example of what the non-resonance condition (4.9) means and why it plays a crucial role in the solution of (4.6), consider the case in which $\dim(e) = \dim(u) = 1$. Observe that the first two equations of (4.1), together with (4.4), can be written in the form of a composite system with input u and output e as

$$\begin{aligned} \begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} &= \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}\begin{pmatrix} w \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \\ e &= \begin{pmatrix} Q_e & C_e \end{pmatrix}\begin{pmatrix} w \\ x \end{pmatrix}. \end{aligned} \tag{4.10}$$

Bearing in mind the conditions (2.2) used to identify the value of the relative degree of a system, let r be such that $C_eA^kB = 0$ for all $k < r-1$ and $C_eA^{r-1}B \neq 0$. A simple calculation shows that, for any $k \geq 0$,

$$\begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^k \begin{pmatrix} 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ A^kB \end{pmatrix}.$$

Therefore

[2] See [2].

[3] See Appendix A.2

$$
\begin{pmatrix} Q_e & C_e \end{pmatrix} \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^k \begin{pmatrix} 0 \\ B \end{pmatrix} = 0
$$

for $k = 1, \ldots, r-2$ and

$$
\begin{pmatrix} Q_e & C_e \end{pmatrix} \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^{r-1} \begin{pmatrix} 0 \\ B \end{pmatrix} = C_e A^{r-1} B .
$$

Thus, system (4.10), viewed as system with input u and output e has relative degree r and can be brought to (strict) normal form, by means of a suitable change of variables. In order to identify such change of variables, let

$$
\xi_1 = Q_e w + C_e x
$$

and define recursively $\xi_2, \ldots, \xi_r$ so that

$$
\xi_{i+1} = \dot{\xi}_i
$$

for $i = 1, \ldots, r-1$, as expected. It is easy to see that ξ_i can be given an expression of the form

$$
\xi_i = R_i w + C_e A^{i-1} x .
$$

in which R_i is a suitable matrix. This is indeed the case for $i = 1$ if we set $R_1 = Q_e$. Assuming that it is the case for a generic i, it is immediate to check that

$$
\begin{aligned}
\xi_{i+1} &= \dot{\xi}_i = R_i \dot{w} + C_e A^{i-1} \dot{x} \\
&= R_i S w + C_e A^{i-1}(Ax + Bu + Pw) = (R_i S + C_e A^{i-1} P) w + C_e A^{i-1} x ,
\end{aligned}
$$

which has the required form with $R_{i+1} = (R_i S + C_e A^{i-1} P)$.

Then, set (compare with (2.4))

$$
T_1 = \begin{pmatrix} C_e \\ C_e A \\ \cdots \\ C_e A^{r-1} \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 \\ R_2 \\ \cdots \\ R_r \end{pmatrix},
$$

let T_0 be a matrix, satisfying $T_0 B = 0$, such that (compare with (2.5))

$$
T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix}
$$

is nonsingular and define the new variables as

$$
\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0 x \\ Rw + T_1 x \end{pmatrix}
$$

(the variable w is left unchanged).

This being the case, it is readily seen that the system, in the new coordinates, reads as

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{z} &= T_0 A x + T_0 P w \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= C_{\mathrm{e}} A^r x + bu + (R_r S + C_{\mathrm{e}} A^{r-1} P) w \\
e &= \xi_1 \,,
\end{aligned}$$

in which

$$b = C_{\mathrm{e}} A^{r-1} B \neq 0 \,.$$

To complete the transformation, it remains to replace x by its expression in terms of z, ξ, w. Proceeding as in section 3.1, let M_0 and M_1 be partitions of the inverse of T and observe that

$$x = M_0 z + M_1 \xi - M_1 R w \,.$$

Then, it can be concluded that the (strict) normal form of such system has the following expression

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{z} &= A_{00} z + A_{01} \xi + P_0 w \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= A_{10} z + A_{11} \xi + bu + P_1 w \\
e &= \xi_1 \,,
\end{aligned} \tag{4.11}$$

in which the four matrices $A_{00}, A_{01}, A_{10}, A_{11}$ are *precisely* the matrices that characterize the (strict) normal form of a system defined as

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
e &= C_{\mathrm{e}} x \,,
\end{aligned}$$

and P_0, P_1 are suitable matrices. Hence, as shown in Section 2.1, the matrix A_{00} is a matrix whose eigenvalues coincide with the zeros of the transfer function

$$T_{\mathrm{e}}(s) = C_{\mathrm{e}} (sI - A)^{-1} B \,. \tag{4.12}$$

The normal form (4.11) can be used to determine a solution of Francis' equations. To this end, let Π be partitioned as

$$\Pi = \begin{pmatrix} \Pi_0 \\ \Pi_1 \end{pmatrix}$$

consistently with the partition of $\tilde{x}$, and let $\pi_{1,i}$ denote the i-th row of Π_1.

Then, it is immediate to check that Francis's equations are rewritten as

$$
\begin{aligned}
\Pi_0 S &= A_{00}\Pi_0 + A_{01}\Pi_1 + P_0 \\
\pi_{1,1} S &= \pi_{1,2} \\
&\cdots \\
\pi_{1,r-1} S &= \pi_{1,r} \\
\pi_{1,r} S &= A_{10}\Pi_0 + A_{11}\Pi_1 + b\Psi + P_1 \\
0 &= \pi_{1,1}\,.
\end{aligned}
\tag{4.13}
$$

The last equation, along with the second, third and r-th, yield $\pi_{1,i} = 0$ for all i, i.e.

$$\Pi_1 = 0\,.$$

As a consequence, the remaining equations reduce to

$$
\begin{aligned}
\Pi_0 S &= A_{00}\Pi_0 + P_0 \\
0 &= A_{10}\Pi_0 + b\Psi + P_1\,.
\end{aligned}
\tag{4.14}
$$

The first of these equations is a Sylvester equation in Π_0, that has a unique solution if and only if the none of the eigenvalues of S is an eigenvalue of the matrix A_{00}. Since the eigenvalues of A_{00} are the zeros of (4.12), this equation has a unique solution if and only if none of the eigenvalues of S is a zero of $T_{\rm e}(s)$. Entering the solution Π_0 of this equation into the second one yields an equation that, since $b \neq 0$, can be solved for Ψ, yielding

$$\Psi = \frac{-1}{b}[A_{10}\Pi_0 + P_1]\,.$$

In summary, it can be concluded that Francis' equations have a unique solution if and only if *none of the eigenvalues of S coincides with a zero of the transfer function $T_{\rm e}(s)$*.

Of course the condition thus found must be consistent with the condition resulting from Lemma 4.1, specialized to the present context in which $m = p = 1$. In this case, the condition (4.9) becomes

$$\det\begin{pmatrix} A-\lambda I & B \\ C_{\rm e} & 0 \end{pmatrix} \neq 0 \qquad \forall \lambda \in \sigma(S)\,.$$

Now, it is known (see section 2.1) that the roots of the polynomial

$$\det\begin{pmatrix} A-\lambda I & B \\ C_{\rm e} & 0 \end{pmatrix}$$

coincide with the zeros of the transfer function $T_{\rm e}(s)$. Thus, the condition indicated in Lemma 4.1 is the condition that none of the eigenvalues of S is a zero of $T_{\rm e}(s)$, which is precisely the condition obtained from the construction above. $\triangleleft$

4.3 The case of measurement feedback: steady state analysis

Consider now the case in which the feedback law is provided by a controller that does not have access to the full state x of the plant and the full state w of the exosystem. We assume that the controller has access to the regulated output e and, possibly, to an additional *supplementary* set of independent measured variables. In other words, we assume that the vector

$$y = Cx + Qw$$

of *measured outputs* of the plant can be split in two parts as in

$$y = \begin{pmatrix} e \\ y_\mathrm{r} \end{pmatrix} = \begin{pmatrix} C_\mathrm{e} \\ C_\mathrm{r} \end{pmatrix} x + \begin{pmatrix} Q_\mathrm{e} \\ Q_\mathrm{r} \end{pmatrix} w$$

in which e is the *regulated* output and $y_\mathrm{r} \in \mathbb{R}^{p_\mathrm{r}}$. As opposite to the case considered in the previous section, this is usually referred to as the case of "measurement feedback". In this setup the controlled plant, together with the exosystem, is modeled by equations of the form

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= Ax + Bu + Pw \\ e &= C_\mathrm{e}x + Q_\mathrm{e}w \\ y_\mathrm{r} &= C_\mathrm{r}x + Q_\mathrm{r}w\,. \end{aligned} \tag{4.15}$$

The control is provided by a generic dynamical system with input y and output u, modeled as in (4.2). Proceeding as in the first part of the proof of Proposition 4.1, it is easy to deduce the following *necessary* conditions for the solution of the problem of output regulation.

Proposition 4.2. *The problem of output regulation in the case of measurement feedback has a solution* only if

(i) the triplet $\{A,B,C\}$ is stabilizable and detectable
(ii) there exists a solution pair (Π,Ψ) of the Francis' equation (4.6).

Proof. System (4.1) controlled by (4.2) can be regarded as an autonomous linear system (see Fig. 4.2)

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{x}_\mathrm{c} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 \\ P + BD_\mathrm{c}Q & A + BD_\mathrm{c}C & BC_\mathrm{c} \\ B_\mathrm{c}Q & B_\mathrm{c}C & A_\mathrm{c} \end{pmatrix} \begin{pmatrix} w \\ x \\ x_\mathrm{c} \end{pmatrix}. \tag{4.16}$$

If a controller of the form (4.2) solves the problem of output regulation, all the eigenvalues of the matrix

$$\begin{pmatrix} A + BD_\mathrm{c}C & BC_\mathrm{c} \\ B_\mathrm{c}C & A_\mathrm{c} \end{pmatrix}$$

have negative real part. Hence, the triplet $\{A,B,C\}$ *must be stabilizable and detectable*.

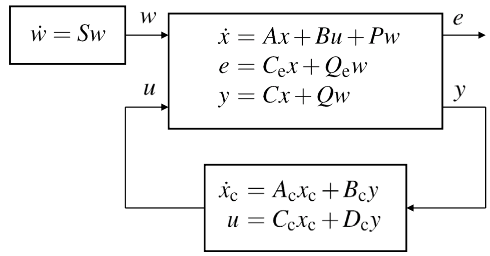

Fig. 4.2 The closed loop system, augmented with the exosystem.

Since by assumption S has all eigenvalues on the imaginary axis, system (4.16) possesses two complementary invariant subspaces: a *stable eigenspace* and a *center eigenspace*. The latter, in particular, can be expressed as

$$\mathscr{V}^{\rm c} = \mathrm{I}\begin{pmatrix} I \\ \Pi \\ \Pi_{\rm c} \end{pmatrix}$$

in which the pair $(\Pi, \Pi_{\rm c})$ is a solution of the Sylvester equation

$$\begin{pmatrix} \Pi \\ \Pi_{\rm c} \end{pmatrix} S = \begin{pmatrix} A + BD_{\rm c}C & BC_{\rm c} \\ B_{\rm c}C & A_{\rm c} \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_{\rm c} \end{pmatrix} + \begin{pmatrix} P + BD_{\rm c}Q \\ B_{\rm c}Q \end{pmatrix}. \tag{4.17}$$

Setting

$$\Psi = D_{\rm c}C\Pi + C_{\rm c}\Pi_{\rm c} + D_{\rm c}Q$$

it is observed that the pair (Π, Ψ) satisfies the first equation of (4.6).

Any trajectory of (4.16) has a unique decomposition into a component entirely contained in the stable eigenspace and a component entirely contained in the center eigenspace. The former, which asymptotically decays to 0 as $t \to \infty$, is the *transient component* of the trajectory. The latter, in which $x(t)$ and $x_{\rm c}(t)$ have, respectively, the form

$$x_{\rm ss}(t) = \Pi w(t) \qquad x_{\rm c,ss}(t) = \Pi_{\rm c} w(t)\,, \tag{4.18}$$

is the *steady-state component* of the trajectory.

If the controller (4.2) solves the problem of output regulation, the steady-state component of any trajectory must be contained in the kernel of the map $e = Q_{\rm e}w + C_{\rm e}x$ and hence the solution $(\Pi, \Pi_{\rm c})$ of the Sylvester equation (4.17) necessarily satisfies the second equation of (4.6). This shows that if the controller (4.2) solves the problem of output regulation, the equations (4.6) necessarily have a solution a pair (Π, Ψ). $\triangleleft$

In order to better understand the steady-state behavior of the closed loop system (4.16), it is convenient to split $B_{\rm c}$ and $D_{\rm c}$ consistently with the partition adopted for

y, as

$$B_c = \begin{pmatrix} B_{ce} & B_{cr} \end{pmatrix} \qquad D_c = \begin{pmatrix} D_{ce} & D_{cr} \end{pmatrix},$$

and rewrite the controller (4.2) as

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_{ce} e + B_{cr} y_r \\ u &= C_c x_c + D_{ce} e + D_{cr} y_r . \end{aligned}$$

In these notations, the closed-loop system (4.16) becomes

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} S & 0 & 0 \\ P + BD_{ce}Q_e + BD_{cr}Q_r & A + BD_{ce}C_e + BD_{cr}C_r & BC_c \\ B_{ce}Q_e + B_{cr}Q_r & B_{ce}C_e + B_{cr}C_r & A_c \end{pmatrix} \begin{pmatrix} w \\ x \\ x_c \end{pmatrix}, \tag{4.19}$$

and the Sylvester equation (4.17) becomes

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_{ce}C_e + BD_{cr}C_r & BC_c \\ B_{ce}C_e + B_{cr}C_r & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_{ce}Q_e + BD_{cr}Q_r \\ B_{ce}Q_e + B_{cr}Q_r \end{pmatrix}.$$

Bearing in mind the fact that, if the controller solves the problem of output regulation, the matrix Π must satisfy the second of (4.6), the equation above reduces to

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_{cr}C_r & BC_c \\ B_{cr}C_r & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_{cr}Q_r \\ B_{cr}Q_r \end{pmatrix},$$

and we observe that, in particular, the matrix Π_c satisfies

$$\begin{aligned} \Pi_c S &= A_c \Pi_c + B_{cr}(C_r \Pi + Q_r) \\ \Psi &= C_c \Pi_c + D_{cr}(C_r \Pi + Q_r), \end{aligned} \tag{4.20}$$

in which (Π, Ψ) is the solution pair of (4.6).

Equations (4.20), from a general viewpoint, could be regarded as a constraint on the solution (Π, Ψ) of (4.6). These equations interpret the ability, of the controller, to generate the *feedforward input* necessary to keep the regulated variable identically zero in steady state. In steady state, the state $x(t)$ of the plant evolves (see (4.18)) as

$$x_{ss}(t) = \Pi w(t),$$

the regulated output $e(t)$ is identically zero, because

$$e_{ss}(t) = (C_e \Pi + Q_e) w(t) = 0, \tag{4.21}$$

the additional measured output $y_r(t)$ evolves as

$$y_{r,ss}(t) = (C_r \Pi + Q_r) w(t) \tag{4.22}$$

and the state $x_c(t)$ of the controller evolves (see again (4.18)) as

$$x_{c,ss}(t) = \Pi_c w(t).$$

The first equation of (4.20) expresses precisely the property that $\Pi_\mathrm{c} w(t)$ is the steady-state response of the controller, when the latter is forced by a steady-state input of the form (4.21) – (4.22). The second equation of (4.20), in turn, shows that the output of the controller, in steady state, is a function of the form

$$\begin{aligned} u_\mathrm{ss}(t) &= C_\mathrm{c} x_{\mathrm{c,ss}}(t) + D_\mathrm{ce} e_\mathrm{ss}(t) + D_\mathrm{cr} y_{\mathrm{r,ss}}(t) \\ &= C_\mathrm{c} \Pi_\mathrm{c} w(t) + D_\mathrm{cr}(C_\mathrm{r}\Pi + Q_\mathrm{r}) w(t) = \Psi w(t)\,. \end{aligned}$$

The latter, as predicted by Francis' equations, is a control able to force in the controlled plant a steady-state trajectory of the form $x_\mathrm{ss}(t) = \Pi w(t)$ and consequently to keep $e_\mathrm{ss}(t)$ identically zero. The property thus described is usually referred to as the *internal model property*: any controller that solves the problem of output regulation necessarily embeds a model of the feedforward inputs needed to keep $e(t)$ identically zero.

4.4 The case of measurement feedback: construction of a controller

The possibility of constructing a controller that solves the problem in the case of measurement feedback reposes on the following preliminary result. Let

$$\psi(\lambda) = s_0 + s_1\lambda + \cdots + s_{d-1}\lambda^{d-1} + \lambda^d$$

denote the *minimal* polynomial of S and set

$$\Phi = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ -s_0 I & -s_1 I & -s_2 I & \cdots & -s_{d-1} I \end{pmatrix}, \tag{4.23}$$

in which all blocks are $p \times p$. Set also

$$G = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \end{pmatrix}^\mathrm{T} \tag{4.24}$$

in which all blocks are $p \times p$.

Note that there exists a nonsingular matrix T such that

$$T\Phi T^{-1} = \begin{pmatrix} S_0 & 0 & \cdots & 0 \\ 0 & S_0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & S_0 \end{pmatrix}, \qquad TG = \begin{pmatrix} G_0 & 0 & \cdots & 0 \\ 0 & G_0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & G_0 \end{pmatrix} \tag{4.25}$$

in which S_0 is the $d \times d$ matrix

$$S_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0 & -s_1 & -s_2 & \cdots & -s_{d-1} \end{pmatrix}. \tag{4.26}$$

and

$$G_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^{\mathrm{T}}.$$

From this, we see in particular that $\psi(\lambda)$ is also the minimal polynomial of Φ. As a consequence

$$\begin{aligned} s_0 I + s_1 S + \cdots + s_{d-1} S^{d-1} + S^d &= 0 \\ s_0 I + s_1 \Phi + \cdots + s_{d-1} \Phi^{d-1} + \Phi^d &= 0. \end{aligned} \tag{4.27}$$

With Φ and G constructed in this way, consider now the system obtained by letting the regulated output of (4.1) drive a post-processor characterized by the equations

$$\dot{\eta} = \Phi \eta + Ge. \tag{4.28}$$

Since the minimal polynomial of the matrix Φ in (4.28) coincides with the minimal polynomial of the matrix S that characterizes the exosystem, system (4.28) is usually called an *internal model* (of the exosystem). Note that, in the coordinates $\tilde{\eta} = T\eta$, with T such that (4.25) hold, system (4.28) can be seen as a bench of p identical sub-systems of the form

$$\dot{\tilde{\eta}}_i = S_0 \tilde{\eta}_i + G_0 e_i$$

each one driven by the i-th component e_i of e.

In what follows, we are going to show that the problem of output regulation can be solved by means of a controller having the following structure [4]

$$\begin{aligned} \dot{\eta} &= \Phi \eta + Ge \\ \dot{\xi} &= A_{\mathrm{s}} \xi + B_{\mathrm{s}} y + J_{\mathrm{s}} \eta \\ u &= C_{\mathrm{s}} \xi + D_{\mathrm{s}} y + H_{\mathrm{s}} \eta \end{aligned} \tag{4.29}$$

in which $A_{\mathrm{s}}, B_{\mathrm{s}}, C_{\mathrm{s}}, D_{\mathrm{s}}, J_{\mathrm{s}}, H_{\mathrm{s}}$, are suitable matrices (see Fig. 4.3). This controller consists of the post-processor (4.28) whose state η drives, along with the full measured output y, the system

$$\begin{aligned} \dot{\xi} &= A_{\mathrm{s}} \xi + B_{\mathrm{s}} y + J_{\mathrm{s}} \eta \\ u &= C_{\mathrm{s}} \xi + D_{\mathrm{s}} y + H_{\mathrm{s}} \eta, \end{aligned}$$

and can be seen as a system of the general form (4.2) if we set

[4] The arguments uses hereafter are essentially the same as those used in [1], [4] and [3].

$$x_{\mathrm{c}} = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \qquad A_{\mathrm{c}} = \begin{pmatrix} \Phi & 0 \\ J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \qquad B_{\mathrm{c}} = \begin{pmatrix} (G \quad 0) \\ B_{\mathrm{s}} \end{pmatrix}$$

$$C_{\mathrm{c}} = (H_{\mathrm{s}} \quad C_{\mathrm{s}}) \qquad D_{\mathrm{c}} = D_{\mathrm{s}} .$$

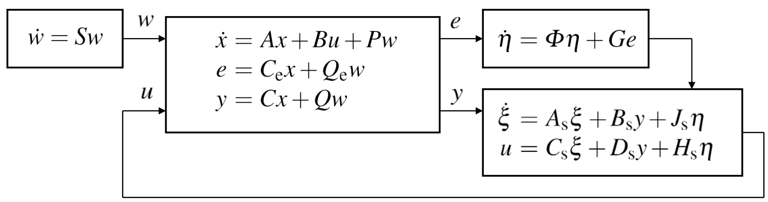

Fig. 4.3 The plant, augmented with the exosystem, controlled by (4.29).

The first step in proving that a controller of the form (4.29) can to solve the problem of output regulation consists in the analysis of the properties of stabilizability and detectability of a system – with state (x, η), input u and output y_{a} – defined as follows

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ y_{\mathrm{a}} &= \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \end{aligned} \tag{4.30}$$

that will be referred to as the *augmented* system. As a matter of fact, it turns out that the system in question has the following important property.

Lemma 4.2. *The augmented system (4.30) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C\}$ is stabilizable and detectable
(ii) the non-resonance condition

$$\operatorname{rank} \begin{pmatrix} A - \lambda I & B \\ C_{\mathrm{e}} & 0 \end{pmatrix} = n + p \qquad \forall \lambda \in \sigma(S) . \tag{4.31}$$

holds.

Proof. To check detectability we look at the linear independence of the columns of the matrix

$$\begin{pmatrix} A - \lambda I & 0 \\ GC_{\mathrm{e}} & \Phi - \lambda I \\ C & 0 \\ 0 & I \end{pmatrix}$$

for any λ having nonnegative real part. Taking linear combinations of rows (and using the fact that the rows of C_{e} are part of the rows of C), this matrix can be easily reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 \\ 0 & 0 \\ C & 0 \\ 0 & I \end{pmatrix}$$

from which it is seen that the columns are linearly independent if and only if so are those of the submatrix

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}.$$

Hence, we conclude that system (4.30) is detectable if and only if so is pair A, C.

To check stabilizability, let Φ and G be defined as above, and look at the linear independence of the rows of the matrix

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & -\lambda I & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -s_0 I & -s_1 I & -s_2 I & \cdots & -(s_{d-1}+\lambda)I & 0 \end{pmatrix}$$

for any λ having non-negative real part. Taking linear combinations of columns, this matrix is initially reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -\psi(\lambda)I & * & * & \cdots & * & 0 \end{pmatrix}$$

in which $\psi(\lambda)$ is the minimal polynomial of S. Then, taking linear combinations of rows, this matrix is reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -\psi(\lambda)I & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

After permutation of rows and columns, one finally obtains a matrix of the form

$$\begin{pmatrix} A-\lambda I & B & 0 & 0 \\ C_{\mathrm{e}} & 0 & -\psi(\lambda)I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

(where the lower-right block is an identity matrix of dimension $(d-1)p$). If λ is not an eigenvalue of S, $\psi(\lambda) \neq 0$ and the rows are independent if and only if so are those of

$$\begin{pmatrix} A-\lambda I & B \end{pmatrix}.$$

On the contrary, if λ is an eigenvalue of S, $\psi(\lambda)=0$ and the rows are independent if and only if so are those of

$$\begin{pmatrix} A-\lambda I & B \\ C_\mathrm{e} & 0 \end{pmatrix}.$$

Thus, it is concluded that system (4.30) is stabilizable if and only if so is the pair (A,B) and the *non-resonance* condition (4.31) holds . ◁

If the assumptions of this Lemma hold, the system (4.30) is stabilizable by means of a (dynamic) feedback. In other words, there exists matrices $A_\mathrm{s}, B_\mathrm{s}, C_\mathrm{s}, D_\mathrm{s}, J_\mathrm{s}, H_\mathrm{s}$, such that the closed-loop system obtained controlling (4.30) by means of a system of the form

$$\begin{aligned} \dot{\xi} &= A_\mathrm{s}\xi + B_\mathrm{s}y'_\mathrm{a} + J_\mathrm{s}y''_\mathrm{a} \\ u &= C_\mathrm{s}\xi + D_\mathrm{s}y'_\mathrm{a} + H_\mathrm{s}y''_\mathrm{a}, \end{aligned} \tag{4.32}$$

in which y'_a and y''_a are the upper and – respectively – lower block of the output y_a of (4.30), is stable. In what follows system (4.32) will be referred to as a *stabilizer*.

Remark 4.1. A simple expression of such stabilizer can be found in this way. By Lemma 4.2 the augmented system (4.30) is stabilizable. Hence, there exist matrices $L \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{R}^{m\times pd}$ such that the system

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \left(\begin{pmatrix} A & 0 \\ GC_\mathrm{e} & \Phi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} L & M \end{pmatrix} \right) \begin{pmatrix} x \\ \eta \end{pmatrix}$$

is stable. Moreover, since the pair (A,C) is detectable, there is a matrix N such $(A-NC)$ has all eigenvalues in $\mathbb{C}^-$. Let the augmented system (4.30) be controlled by

$$\begin{aligned} \dot{\xi} &= A\xi + N(y'_\mathrm{a} - C\xi) + B(L\xi + My''_\mathrm{a}) \\ u &= L\xi + My''_\mathrm{a}. \end{aligned} \tag{4.33}$$

Bearing in mind that $y'_\mathrm{a} = Cx$, that $y''_\mathrm{a} = \eta$, and using arguments identical to those used in the proof of the sufficiency in Theorem A.4, it is seen that the resulting closed-loop system is stable. ◁

With this result in mind, consider, for the solution of the problem of output regulation, a candidate controller of the form (4.29), in which the matrices $A_\mathrm{s}, B_\mathrm{s}, C_\mathrm{s}, D_\mathrm{s}, J_\mathrm{s}, H_\mathrm{s}$, are chosen in such a way that (4.32) stabilizes system (4.30). This yields a closed loop system of the form

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P+BD_\mathrm{s}Q & A+BD_\mathrm{s}C & BH_\mathrm{s} & BC_\mathrm{s} \\ GQ_\mathrm{e} & GC_\mathrm{e} & \Phi & 0 \\ B_\mathrm{s}Q & B_\mathrm{s}C & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.34}$$

If the stabilizer (4.32) stabilizes system (4.30), the matrix

$$\begin{pmatrix} A+BD_\mathrm{s}C & BH_\mathrm{s} & BC_\mathrm{s} \\ GC_\mathrm{e} & \Phi & 0 \\ B_\mathrm{s}C & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \tag{4.35}$$

has all eigenvalues with negative real part. Hence, the closed loop system possesses a *stable eigenspace* and a *center eigenspace*. The latter, in particular, can be expressed as

$$\mathscr{V}^\mathrm{c} = \mathrm{Im}\begin{pmatrix} I \\ \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} \tag{4.36}$$

in which $(\Pi_x, \Pi_\eta, \Pi_\xi)$ is a solution of the Sylvester equation

$$\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A+BD_\mathrm{s}C & BH_\mathrm{s} & BC_\mathrm{s} \\ GC_\mathrm{e} & \Phi & 0 \\ B_\mathrm{s}C & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P+BD_\mathrm{s}Q \\ GQ_\mathrm{e} \\ B_\mathrm{s}Q \end{pmatrix}. \tag{4.37}$$

In particular, it is seen that Π_η and Π_x satisfy

$$\Pi_\eta S = \Phi\Pi_\eta + G(C_\mathrm{e}\Pi_x + Q_\mathrm{e}). \tag{4.38}$$

This is a *key* property, from which it will be deduced that the proposed controller solves the problem of output regulation. In fact, the following result holds.

Lemma 4.3. *If Φ is the matrix defined in (4.23) and G is the matrix defined in (4.24), the equation (4.38)* implies

$$C_\mathrm{e}\Pi_x + Q_\mathrm{e} = 0. \tag{4.39}$$

Proof. Let Π_η be partitioned consistently with the partition of Φ, as

$$\Pi_\eta = \begin{pmatrix} \Pi_{\eta,1} \\ \Pi_{\eta,2} \\ \cdots \\ \Pi_{\eta,d} \end{pmatrix}.$$

Bearing in mind the special structure of Φ and G, equation (4.38) becomes

$$\begin{aligned} \Pi_{\eta,1}S &= \Pi_{\eta,2} \\ \Pi_{\eta,2}S &= \Pi_{\eta,3} \\ &\cdots \\ \Pi_{\eta,d-1}S &= \Pi_{\eta,d} \\ \Pi_{\eta,d}S &= -s_0\Pi_{\eta,1} - s_1\Pi_{\eta,2} - \cdots - s_{d-1}\Pi_{\eta,d} + C_\mathrm{e}\Pi_x + Q_\mathrm{e}. \end{aligned}$$

The first $d-1$ of these yield, for $i = 1, 2, \ldots d$,

$$\Pi_{\eta,i} = \Pi_{\eta,1} S^{i-1}\,,$$

which, replaced in the last one, yield in turn

$$\Pi_{\eta,1} S^d = \Pi_{\eta,1}(-s_0 I - s_1 S - \cdots - s_{d-1} S^{d-1}) + C_e \Pi_x + Q_e\,. \tag{4.40}$$

By definition, $\psi(\lambda)$ satisfies the first of (4.27). Hence, (4.40) implies (4.39). ◁

We have proven in this way that if the matrix (4.35) has all eigenvalues with negative real part, and the matrices Φ and G have the form (4.23) and (4.24), the center eigenspace of system (4.1) controlled by (4.29), whose expression is given in (4.36), is such that Π_x satisfies (4.39). Since in steady-state $x_{ss}(t) = \Pi_x w(t)$, we see that $e_{ss}(t) = C_e \Pi_x w(t) + Q_e w(t) = 0$ and conclude that the proposed controller solves the problem of output regulation. We summarize this result as follows.

Proposition 4.3. *Suppose that*

(i) the triplet $\{A,B,C\}$ is stabilizable and detectable
(ii) the non-resonance condition (4.31) holds.

Then, the problem of output regulation is solvable, in particular by means of a controller of the form (4.29), in which Φ, G have the form (4.23),(4.24) and $A_s, B_s, C_s, D_s, J_s, H_s$ are such that (4.32) stabilizes the augmented plant (4.30).

Remark 4.2. Note, that, setting

$$\Psi = C_s \Pi_\xi + D_s(C\Pi_x + Q) + H_s \Pi_\eta\,,$$

it is seen that the pair (Π_x, Ψ) is a solution of Francis's equation (4.6). ◁

4.5 Robust output regulation

We consider in this section the case in which the plant is affected by *structured uncertainties*, that is the case in which the coefficient matrices that characterize the model of the plant depend on a vector μ of uncertain parameters, as in

$$\begin{aligned} \dot w &= Sw \\ \dot x &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C_e(\mu)x + Q_e(\mu)w \\ y_r &= C_r(\mu)x + Q_r(\mu)w \end{aligned} \tag{4.41}$$

(note that S is assumed to be *independent* of μ). Coherently with the notations adopted before, we set

$$C(\mu) = \begin{pmatrix} C_e(\mu) \\ C_r(\mu) \end{pmatrix}, \qquad Q(\mu) = \begin{pmatrix} Q_e(\mu) \\ Q_r(\mu) \end{pmatrix}.$$

We show in what follows how the results discussed in the previous section can be enhanced to obtain a robust controller. First of all, observe that if a robust controller exists, this controller must solve the problem of output regulation for each of μ. Hence, for each of such values, the necessary conditions for existence of a controller determined in the earlier sections must hold: the triplet $\{A(\mu), B(\mu), C(\mu)\}$ must be stabilizable and detectable and, above all, the μ-dependent Francis equations

$$\begin{aligned} \Pi(\mu)S &= A(\mu)\Pi(\mu) + B(\mu)\Psi(\mu) + P(\mu) \\ 0 &= C_{\mathrm{e}}(\mu)\Pi(\mu) + Q_{\mathrm{e}}(\mu)\,. \end{aligned} \tag{4.42}$$

must have a solution pair $\Pi(\mu), \Psi(\mu)$ (that, in general, we expect to be μ-dependent).

The design method discussed in the previous section was based on the possibility of stabilizing system (4.30) by means of a (dynamic) feedback driven by y_{a}. The existence of such stabilizer was guaranteed by the fulfillment of the assumptions of Lemma 4.2. In the present setting, in which the parameters of the plant depend on a vector μ of uncertain parameters, one might pursue a similar approach, seeking a controller (which should be μ-independent) that stabilizes the augmented system [5]

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A(\mu) & 0 \\ GC_{\mathrm{e}}(\mu) & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B(\mu) \\ 0 \end{pmatrix} u \\ y_{\mathrm{a}} &= \begin{pmatrix} C(\mu) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.43}$$

It should be stressed, though, that a result such as that of the Lemma 4.2 cannot be invoked anymore. In fact, the (necessary) assumption that the triplet $\{A(\mu), B(\mu), C(\mu)\}$ is stabilizable and detectable for every μ no longer guarantees the existence of a *robust* stabilizer for (4.43). For example, a stabilizer having the structure (4.33) cannot be used, because the latter presumes a precise knowledge of the matrices A, B, C that characterize the model of the plant, and this is no longer the case when such matrices depend on a vector μ of uncertain parameters.

We will show later in the Chapter how (and under what assumptions) such a robust stabilizer may be found. In the preset section, we take this as an hypothesis, i.e. we *suppose* that there exists a dynamical system, modeled as in (4.32), that robustly stabilizes (4.43). To say that the plant (4.43), controlled by (4.32), is robustly stable is the same thing as to say that the matrix

$$\begin{pmatrix} A(\mu) + B(\mu)D_{\mathrm{s}}C(\mu) & B(\mu)H_{\mathrm{s}} & B(\mu)C_{\mathrm{s}} \\ GC_{\mathrm{e}}(\mu) & \Phi & 0 \\ B_{\mathrm{s}}C(\mu) & J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \tag{4.44}$$

is Hurwitz for every value of μ.

Let now the uncertain plant (4.41) be controlled by a system of the form

[5] It is worth observing that, since by assumption the matrix S is not affected by parameter uncertainties, so is its minimal polynomial $\psi(\lambda)$ and consequently so is the matrix Φ defined in (4.23).

$$
\begin{aligned}
\dot{\eta} &= \Phi\eta + Ge \\
\dot{\xi} &= A_{\mathrm{s}}\xi + B_{\mathrm{s}}y + J_{\mathrm{s}}\eta \\
u &= C_{\mathrm{s}}\xi + D_{\mathrm{s}}y + H_{\mathrm{s}}\eta \,.
\end{aligned}
\tag{4.45}
$$

The resulting closed-loop system is an autonomous linear system having the form

$$
\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P(\mu)+B(\mu)D_{\mathrm{s}}Q(\mu) & A+B(\mu)D_{\mathrm{s}}C(\mu) & B(\mu)H_{\mathrm{s}} & B(\mu)C_{\mathrm{s}} \\ GQ_{\mathrm{e}}(\mu) & GC_{\mathrm{e}}(\mu) & \Phi & 0 \\ B_{\mathrm{s}}Q(\mu) & B_{\mathrm{s}}C(\mu) & J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}.
\tag{4.46}
$$

Since the matrix (4.44) is Hurwitz for every value of μ and S has all eigenvalues on the imaginary axis, the closed loop system possesses two complementary invariant subspaces: a *stable eigenspace* and a *center eigenspace*. The latter, in particular, has the expression

$$
\mathscr{V}^{\mathrm{c}} = \mathrm{Im} \begin{pmatrix} I \\ \Pi_x(\mu) \\ \Pi_\eta(\mu) \\ \Pi_\xi(\mu) \end{pmatrix}
$$

in which $(\Pi_x(\mu), \Pi_\eta(\mu), \Pi_\xi(\mu))$ is the (unique) solution of the Sylvester equation (compare with (4.37))

$$
\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A(\mu)+B(\mu)D_{\mathrm{s}}C(\mu) & B(\mu)H_{\mathrm{s}} & B(\mu)C_{\mathrm{s}} \\ GC_{\mathrm{e}}(\mu) & \Phi & 0 \\ B_{\mathrm{s}}C(\mu) & J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix}
+ \begin{pmatrix} P(\mu)+B(\mu)D_{\mathrm{s}}Q(\mu) \\ GQ_{\mathrm{e}}(\mu) \\ B_{\mathrm{s}}Q(\mu) \end{pmatrix}.
$$

From this Sylvester equation, one deduces (compare with (4.38)) that

$$
\Pi_\eta(\mu)S = \Phi\Pi_\eta(\mu) + G(C_{\mathrm{e}}(\mu)\Pi_x(\mu) + Q_{\mathrm{e}}(\mu))\,.
$$

From this, using Lemma 4.3, it is concluded that

$$
C_{\mathrm{e}}(\mu)\Pi_x(\mu) + Q_{\mathrm{e}}(\mu) = 0\,.
$$

In steady-state $x_{\mathrm{ss}}(t) = \Pi_x(\mu)w(t)$ and, in view of equation above, we conclude that $e_{\mathrm{ss}}(t) = 0$. We summarize the discussion as follows.

Proposition 4.4. *Let Φ be a matrix of the form (4.23) and G a matrix of the form (4.24). Suppose the system (4.43) is robustly stabilized by a stabilizer of the form (4.32). Then the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.45).*

Remark 4.3. One might be puzzled by the absence of the non-resonance condition in the statement of the Proposition above. It turns out, though, that this condition is implied by the assumption of robust stabilizability of the augmented system. As a matter of fact, in this statement it is assumed that there exists a stabilizer that stabilizes the augmented plant (4.43) for every μ. As a trivial consequence, the latter is stabilizable and detectable for every μ. This being the case, it is seen from Lemma 4.2 that, if the system (4.43) is robustly stabilized by a stabilizer of the form (4.32), necessarily the triplet $\{A(\mu), B(\mu), C(\mu)\}$ is stabilizable and detectable for every μ and the non-resonance condition must hold for every μ. We stress also that the non-resonance condition, which we have seen is necessary, also implies the existence of a solution of Francis's equations (4.42). This is an immediate consequence of Lemma 4.1.

4.6 The special case in which $m = p$ and $p_{\mathrm{r}} = 0$

We discuss in this section the design of regulators in the special case of a plant in which the number of regulated outputs is equal to the number of control inputs, and no additional measurements outputs are available. Of course, the design of a regulator could be achieved by following the general construction described in the previous section, but in this special case alternative (and somewhat simpler) design procedures are available, which will be described in what follows.

Immediate consequences of the assumption $m = p$ are the fact that the nonresonance condition can be rewritten as

$$\det\begin{pmatrix} A-\lambda I & B \\ C_{\mathrm{e}} & 0 \end{pmatrix} \neq 0 \qquad \forall \lambda \in \sigma(S)\,, \tag{4.47}$$

and the fact that, if this is the case, the solution Π, Ψ of Francis's equations (4.6) is *unique*. If, in addition, $p_{\mathrm{r}} = 0$, the construction described above can be simplified and an alternative structure of the controller is possible.

If $p_{\mathrm{r}} = 0$, in fact, the structure of the controller (4.29) becomes

$$\begin{aligned} \dot\eta &= \Phi\eta + Ge \\ \dot\xi &= A_{\mathrm{s}}\xi + B_{\mathrm{s}}e + J_{\mathrm{s}}\eta \\ u &= C_{\mathrm{s}}\xi + D_{\mathrm{s}}e + H_{\mathrm{s}}\eta\,. \end{aligned} \tag{4.48}$$

Suppose that J_{s} and H_{s} are chosen as

$$\begin{aligned} J_{\mathrm{s}} &= B_{\mathrm{s}}\Gamma \\ H_{\mathrm{s}} &= D_{\mathrm{s}}\Gamma \end{aligned}$$

in which Γ is a matrix to be determined. In this case, the proposed controller can be seen as a pure *cascade connection* of a post-processing *filter* modeled by

$$\begin{aligned} \dot\eta &= \Phi\eta + Ge \\ \tilde e &= \Gamma\eta + e\,, \end{aligned} \tag{4.49}$$

whose output $\tilde{e}$ drives a stabilizer modeled by

$$\begin{aligned} \dot{\xi} &= A_s\xi + B_s\tilde{e} \\ u &= C_s\xi + D_s\tilde{e} \end{aligned} \tag{4.50}$$

as shown in Fig. 4.4.

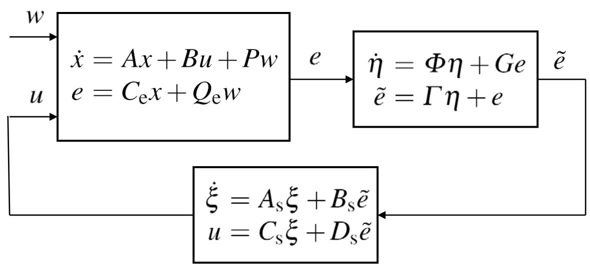

Fig. 4.4 The control consists in a *post-processing* internal model cascaded with a stabilizer.

Such special form of J_s and H_s is admissible if a system of the form (4.50) exists that stabilizes augmented plant

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_e & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ \tilde{e} &= \begin{pmatrix} C_e & \Gamma \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \end{aligned} \tag{4.51}$$

i.e. if the latter is stabilizable and detectable. In this respect, the following result is useful.

Lemma 4.4. *Let Γ be such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, the augmented system (4.51) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C_e\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.47) holds.

Proof. Stabilizability is a straightforward consequence of Lemma 4.2. Detectability holds if the columns of the matrix

$$\begin{pmatrix} A - \lambda I & 0 \\ GC_e & \Phi - \lambda I \\ C_e & \Gamma \end{pmatrix}$$

are independent for all λ having non-negative real part. Taking linear combinations of rows, we transform the latter into

$$\begin{pmatrix} A-\lambda I & 0 \\ 0 & \Phi - G\Gamma - \lambda I \\ C_\mathrm{e} & \Gamma \end{pmatrix},$$

and we observe that, if Γ is such that the matrix $\Phi - G\Gamma$ is Hurwitz, the augmented system (4.51) is detectable if and only if so is the pair (A, C_e). $\triangleleft$

Note that a matrix Γ that makes $\Phi - G\Gamma$ Hurwitz certainly exists because by construction the pair Φ, G is reachable. We can therefore conclude that, if the triplet $\{A, B, C_\mathrm{e}\}$ is stabilizable and detectable, if the non-resonance condition holds and if Γ is chosen in such that a way $\Phi - G\Gamma$ is Hurwitz (as it is always possibile), the problem of output regulation can be solved by means of a controller consisting of the cascade of (4.49) and (4.50). [6]

We summarize this in the following statement.

Proposition 4.5. *Suppose that*

(i) the triplet $\{A, B, C_\mathrm{e}\}$ is stabilizable and detectable
(ii) the non-resonance condition (4.47) holds.

Then, the problem of output regulation is solvable by means of a controller of the form

$$\begin{aligned} \dot\eta &= \Phi\eta + Ge \\ \dot\xi &= A_\mathrm{s}\xi + B_\mathrm{s}(\Gamma\eta + e) \\ u &= C_\mathrm{s}\xi + D_\mathrm{s}(\Gamma\eta + e), \end{aligned} \tag{4.52}$$

in which Φ, G have the form (4.23),(4.24), Γ is such that a $\Phi - G\Gamma$ is Hurwitz and $A_\mathrm{s}, B_\mathrm{s}$, $C_\mathrm{s}, D_\mathrm{s}$ are such that (4.50) stabilizes the augmented plant (4.51).

The controller (4.52) is, as observed, the cascade of two sub-systems. It would be nice to check whether these sub-systems could be “swapped”, i.e. whether the same result could be obtained by means of a controller consisting of a pre-processing filter of the form

$$\begin{aligned} \dot\eta &= \Phi\eta + G\tilde u \\ u &= \Gamma\eta + \tilde u \end{aligned} \tag{4.53}$$

whose input $\tilde u$ is provided by a stabilizer of the form

$$\begin{aligned} \dot\xi &= A_\mathrm{s}\xi + B_\mathrm{s}e \\ \tilde u &= C_\mathrm{s}\xi + D_\mathrm{s}e \end{aligned} \tag{4.54}$$

as shown in Fig. 4.5.

This is trivially possible if $m = 1$, because the two sub-systems in question are single-input single-output. However, as it is shown now, this is possible also if $m >$

[6] Note that the filter (4.49) is an invertible system, the inverse being given by

$$\begin{aligned} \dot\eta &= (\Phi - G\Gamma)\eta + G\tilde e \\ e &= -\Gamma\eta + \tilde e\,. \end{aligned}$$

Hence, if Γ is chosen in such that a way $\Phi - G\Gamma$ is Hurwitz, the inverse of (4.49) is a stable system.

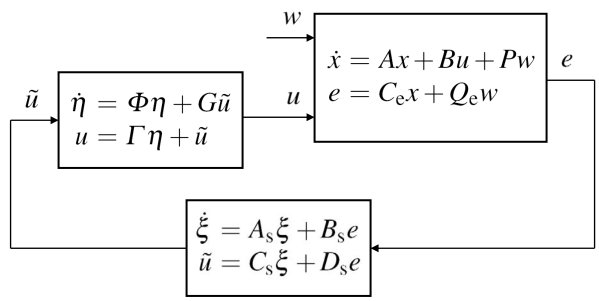

Fig. 4.5 The control consists in a stabilizer cascaded with a *pre-processing* internal model.

1. To this end, observe that controlling the plant (4.1) by means of (4.53) and (4.54) yields an overall system modeled by

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P + BD_sQ_e & A + BD_sC_e & B\Gamma & BC_s \\ GD_sQ_e & GD_sC_e & \Phi & GC_s \\ B_sQ_e & B_sC_e & 0 & A_s \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.55}$$

On the basis of our earlier discussions, it can be claimed that the problem of output regulation is solved if the matrix the matrix

$$\begin{pmatrix} A + BD_sC_e & B\Gamma & BC_s \\ GD_sC_e & \Phi & GC_s \\ B_sC_e & 0 & A_s \end{pmatrix} \tag{4.56}$$

is Hurwitz and the associated center eigenspace of (4.55) is contained in the kernel of the error map $e = C_e x + Q_e w$.

The matrix (4.56) is Hurwitz if (and only if) the stabilizer (4.54) stabilizes the augmented plant

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ G \end{pmatrix} \tilde{u} \\ e &= \begin{pmatrix} C_e & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.57}$$

In what follows the conditions under which this is possible are discussed.

Lemma 4.5. *Let Φ and G be defined as in (4.23) and (4.24). Let Γ be such that the matrix $\Phi - G\Gamma$ is Hurwitz. Then, the pair (Φ, Γ) is observable.*

Proof. Since $\Phi - G\Gamma$ is Hurwitz, the pair (Φ, Γ) is by definition detectable. But Φ has by assumption all eigenvalues with non-negative real part. In this case, detectability (of the pair (Φ, Γ)) is equivalent to observability. $\triangleleft$

Lemma 4.6. *Let Φ be defined as in (4.23). Let Γ be a $(p \times dp)$ matrix such that the pair (Φ, Γ) is observable. Then, there exists a nonsingular matrix T such that*

$$\Gamma = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} T$$

and

$$T\Phi = \Phi T\,.$$

Proof. If the pair (Φ, Γ) is observable, the square $pd \times pd$ matrix (recall that, by construction, the minimal polynomial of Φ has degree d)

$$T = \begin{pmatrix} \Gamma \\ \Gamma\Phi \\ \cdots \\ \Gamma\Phi^{d-1} \end{pmatrix}$$

is invertible. In view of the second of (4.27), a simple calculation shows that this matrix renders the two identities in the Lemma fulfilled. $\triangleleft$

Lemma 4.7. *Let Γ be such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, the augmented system (4.57) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C_\mathrm{e}\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.47) holds.

Proof. To check stabilizability, observe that the rows of the matrix

$$\begin{pmatrix} A - \lambda I & B\Gamma & B \\ 0 & \Phi - \lambda I & G \end{pmatrix}$$

which, via combination of columns, can be reduced to the matrix

$$\begin{pmatrix} A - \lambda I & 0 & B \\ 0 & \Phi - G\Gamma - \lambda I & G \end{pmatrix},$$

are linearly independent for each λ having non-negative real part if and only if the pair (A, B) is stabilizable. To check detectabililty, we need to look at the linear independence of the columns of the matrix

$$\begin{pmatrix} A - \lambda I & B\Gamma \\ 0 & \Phi - \lambda I \\ C_\mathrm{e} & 0 \end{pmatrix}. \tag{4.58}$$

By Lemma 4.5, the pair (Φ, Γ) is observable. Let T be a matrix that renders the two identities in Lemma 4.6 fulfilled, and have the matrix (4.58) replaced by the matrix

$$\begin{pmatrix} I & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & B\Gamma \\ 0 & \Phi - \lambda I \\ C_\mathrm{e} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T^{-1} \end{pmatrix} = \begin{pmatrix} A - \lambda I & B\Gamma T^{-1} \\ 0 & \Phi - \lambda I \\ C_\mathrm{e} & 0 \end{pmatrix},$$

which, in view of the forms of ΓT^{-1} and Φ can be expressed in more detail as

$$\begin{pmatrix} A-\lambda I & B & 0 & 0 & \cdots & 0 \\ 0 & -\lambda I & I & 0 & \cdots & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & -s_0 I & -s_1 I & -s_2 I & \cdots & -(s_{d-1}+\lambda)I \\ C_e & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From this, the check of detectability condition proceeds essentially as in the check of stabilizability condition in Lemma 4.2. Taking linear combinations and permutation of columns and rows, one ends up with a matrix of the form

$$\begin{pmatrix} A-\lambda I & B & 0 \\ C_e & 0 & 0 \\ 0 & -\psi(\lambda)I & 0 \\ 0 & 0 & I \end{pmatrix}$$

from which the claim follows. ◁

If the matrix (4.56) is Hurwitz, the center eigenspace of (4.55) can be expressed as

$$\mathscr{V}^c = \mathrm{Im}\begin{pmatrix} I \\ \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix}$$

in which Π_x, Π_η, Π_ξ are solutions of the Sylvester equation

$$\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A+BD_sC_e & B\Gamma & BC_s \\ GD_sC_e & \Phi & GC_s \\ B_sC_e & 0 & A_s \end{pmatrix}\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P+BD_sQ_e \\ GD_sQ_e \\ B_sQ_e \end{pmatrix}.$$

The matrices Π_x, Π_η, Π_ξ that characterize $\mathscr{V}^c$ can, in this particular setting, be easily determined. To this end, recall that, if $m = p$ and the non-resonance condition holds, the solution Π, Ψ of Francis' equations (4.6) is *unique*. As shown in the next Lemma, a special relation between the matrix Ψ and the matrices Φ and Γ that characterize the filter (4.53) can be established.

Lemma 4.8. *Let Φ be defined as in (4.23). Let Γ be a $(p \times dp)$ matrix such that the pair (Φ, Γ) is observable. Then, given any matrix Ψ, there exists a matrix Σ such that*

$$\begin{aligned} \Sigma S &= \Phi \Sigma \\ \Psi &= \Gamma \Sigma. \end{aligned} \tag{4.59}$$

Proof. Let T be a matrix such that the two identities in Lemma 4.6 hold. Change (4.59) in

$$\begin{aligned}(T\Sigma)S &= T\Phi T^{-1}(T\Sigma) = \Phi(T\Sigma)\\ \Psi &= \Gamma T^{-1}(T\Sigma) = (I \quad 0 \quad \cdots \quad 0)\,(T\Sigma)\,,\end{aligned}$$

and then check that

$$T\Sigma = \begin{pmatrix} \Psi \\ \Psi S \\ \cdot \\ \Psi S^{d-1} \end{pmatrix}$$

is a solution, thanks to the first one of (4.27). ◁

Using (4.59) and (4.6) it is easily seen that the triplet

$$\Pi_x = \Pi, \qquad \Pi_\eta = \Sigma, \qquad \Pi_\xi = 0$$

is a solution of the Sylvester equation above, as a matter of fact, the *unique* solution of that equation. Thus, in particular, the steady-state of the closed-loop system (4.55) is such that $x_{\rm ss}(t) = \Pi w(t)$ with Π obeying the second equation of (4.6). As a consequence, $e_{\rm ss}(t) = 0$ and the problem of output regulation is solved. We summarize the discussion as follows.

Proposition 4.6. *Suppose that*

(i) the triplet $\{A, B, C_{\rm e}\}$ is stabilizable and detectable
(ii) the non-resonance condition (4.47) holds.

Then, the problem of output regulation is solvable by means of a controller of the form

$$\begin{aligned}\dot\xi &= A_{\rm s}\xi + B_{\rm s}e\\ \dot\eta &= \Phi\eta + G(C_{\rm s}\xi + D_{\rm s}e)\\ u &= \Gamma\eta + (C_{\rm s}\xi + D_{\rm s}e)\,,\end{aligned} \tag{4.60}$$

in which Φ, G have the form (4.23),(4.24), Γ is such that a $\Phi - G\Gamma$ is Hurwitz and $A_{\rm s}, B_{\rm s}$, $C_{\rm s}, D_{\rm s}$ are such that

$$\begin{aligned}\dot\xi &= A_{\rm s}\xi + B_{\rm s}e\\ \tilde u &= C_{\rm s}\xi + D_{\rm s}e\,.\end{aligned} \tag{4.61}$$

stabilizes the augmented plant (4.57).

We have seen in this section that, if $m = p$ and $p_{\rm r} = 0$, the problem of output regulation can be approached, under identical assumptions, in two equivalent ways. In the first one of these, the regulated output e is *post-processed* by an *internal model* of the form (4.49) and the resulting augmented system is subsequently stabilized. In the second one of these, the control input u is *pre-processed* by an *internal model* of the form (4.53) and the resulting augmented system is stabilized (see Figs. 4.4 and 4.5).

While both modes of control yield the same result, it should be stressed that the steady state behaviors of the state variables are different. In the first mode of control, it is seen from the analysis above that in steady state

$$x_{\rm ss}(t) = \Pi_x w(t), \qquad \eta_{\rm ss}(t) = \Pi_\eta w(t), \qquad \xi_{\rm ss}(t) = \Pi_\xi w(t)\,.$$

in which, using in particular Lemma 4.3, it is observed that

$$\begin{aligned} \Pi_x &= \Pi \\ \Pi_\eta S &= \Phi \Pi_\eta \\ \Pi_\xi S &= A_s \Pi_\xi + B_s \Gamma \Pi_\eta \\ \Psi &= C_s \Pi_\xi + D_s \Gamma \Pi_\eta \end{aligned}$$

where Π, Ψ is the unique solution of Francis's equations (4.6). In the second mode of control, on the other hand, in steady state we have

$$x_{ss}(t) = \Pi_x w(t), \qquad \eta_{ss}(t) = \Pi_\eta w(t), \qquad \xi_{ss}(t) = 0\,,$$

in which

$$\begin{aligned} \Pi_x &= \Pi \\ \Pi_\eta S &= \Phi \Pi_\eta \\ \Psi &= \Gamma \Pi_\eta \end{aligned}$$

with Π, Ψ the unique solution of Francis's equations (4.6). In particular, in the second mode of control, in steady state the stabilizer (4.61) is *at rest*.

In both modes of control the *internal model* has an identical structure, that of the system

$$\begin{aligned} \dot{\eta} &= \Phi \eta + G \hat{u} \\ \hat{y} &= \Gamma \eta + \hat{u}\,, \end{aligned} \tag{4.62}$$

in which Φ is the matrix defined in (4.23). In the discussion above, the matrix G has been taken as in (4.24), and the matrix Γ was any matrix rendering $\Phi - G\Gamma$ a Hurwitz matrix. However, it is easy to check that one can reverse the roles of G and Γ. In fact, using the state transformation

$$\tilde{\eta} = T \eta$$

with T defined as in the proof of Lemma 4.6 it is seen that an equivalent realization of (4.62) is

$$\begin{aligned} \dot{\tilde{\eta}} &= \Phi \tilde{\eta} + \tilde{G} \hat{u} \\ \hat{y} &= \tilde{\Gamma} \eta + \hat{u}\,, \end{aligned} \tag{4.63}$$

in which $\tilde{G} = TG$ and

$$\tilde{\Gamma} = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} \tag{4.64}$$

(here all blocks are $p \times p$, with $p = m$ by hypothesis). Thus, one can design the internal model picking Φ as in (4.23), Γ as in (4.64) and then choosing a G that makes $\Phi - G\Gamma$ a Hurwitz matrix.

Finally, note that if we let F denote the Hurwitz matrix $F = \Phi - G\Gamma$, the internal model can be put in the form

$$\begin{aligned} \dot{\eta} &= F \eta + G \hat{y} \\ \hat{y} &= \Gamma \eta + \hat{u}\,. \end{aligned} \tag{4.65}$$

The two modes of control lend themselves to solve also a problem of *robust* regulation. From the entire discussion above, in fact, one can arrive at the following conclusion.

Proposition 4.7. *Let Φ be a matrix of the form (4.23), G a matrix of the form (4.24) and Γ a matrix such that $\Phi - G\Gamma$ is Hurwitz (alternatively: Γ a matrix of the form (4.64) and G a matrix such that $\Phi - G\Gamma$ is Hurwitz). If system (4.51) is robustly stabilized by a stabilizer of the form (4.50), the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.52). If system (4.57) is robustly stabilized by a stabilizer of the form (4.61), the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.60).*

4.7 The case of SISO systems

We consider now the case in which $m = p = 1$ and $p_{\mathrm{r}} = 0$, and the coefficient matrices that characterize the controlled plant depend on a vector μ of uncertain parameters, as in (4.41), which we will rewrite for convenience as [7]

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C(\mu)x + Q(\mu)w\,. \end{aligned} \tag{4.66}$$

Let the system be controlled by a controller of the form (4.60). We know from the previous analysis that the controller in question solves the problem of output regulation if there exists a stabilizer of the form (4.61) that stabilizes the augmented system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B(\mu) \\ G \end{pmatrix} \tilde{u} \\ e &= \begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.67}$$

Suppose the triplet in question has a well-defined relative degree r between control input $\tilde{u}$ and regulated output e, independent of μ. It is known from Chapter 2 that a single-input single-output system having well-defined relative degree and *all zeros with negative real part* can be robustly stabilized by (dynamic) output feedback. Thus, we seek assumptions ensuring that the augmented system so defined has a well-defined relative degree and all zeros with negative real part. An easy calculation shows that, if the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has relative degree r,[8] then the augmented system (4.67) still has relative degree r, between the input $\tilde{u}$ and the output e. In fact, for all $k \leq r-1$

---

[7] Note that the subscript "e" has been dropped.

[8] Here and in the following we use the abbreviation "the triplet $\{A, B, C\}$" to mean the associated system (2.1).

$$\begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ 0 & \Phi \end{pmatrix}^k = \begin{pmatrix} C(\mu)A^k(\mu) & 0 \end{pmatrix}$$

from which it is seen that the relative degree is r, with high-frequency gain

$$\begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ 0 & \Phi \end{pmatrix}^{r-1} \begin{pmatrix} B(\mu) \\ G \end{pmatrix} = C(\mu)A^{r-1}(\mu)B(\mu)\,.$$

To evaluate the zeros, we look at the roots of the polynomial

$$\det \begin{pmatrix} A(\mu)-\lambda I & B(\mu)\Gamma & B(\mu) \\ 0 & \Phi-\lambda I & G \\ C(\mu) & 0 & 0 \end{pmatrix} = 0\,.$$

The determinant is unchanged if we multiply the matrix, on the right, by a matrix of the form

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\Gamma & I \end{pmatrix}$$

no matter what Γ is. Thus, the zeros of the system are the roots of the polynomial

$$\det \begin{pmatrix} A(\mu)-\lambda I & 0 & B(\mu) \\ 0 & \Phi-G\Gamma-\lambda I & G \\ C(\mu) & 0 & 0 \end{pmatrix} = 0\,.$$

The latter clearly coincide with the roots of

$$\det \begin{pmatrix} A(\mu)-\lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix} \cdot \det(\Phi-G\Gamma-\lambda I) = 0\,.$$

Thus, the $n+d-r$ zeros of the augmented plant are given by the $n-r$ zeros of the triplet $\{A(\mu),B(\mu),C(\mu)\}$ and by the d eigenvalues of the matrix $\Phi-G\Gamma$. If, as indicated above, the matrix Γ is chosen in such a way that the matrix $\Phi-G\Gamma$ is Hurwitz (as it is always possible), we see that the zeros of the augmented system have negative real part if so are the zeros of the triplet $\{A(\mu),B(\mu),C(\mu)\}$.

We summarize the conclusion as follows.

Proposition 4.8. *Consider an uncertain system of the form (4.66). Suppose that the triplet $\{A(\mu),B(\mu),C(\mu)\}$ has a well-defined relative degree and all its $n-r$ zeros have negative real part, for every value of μ. Let Φ be a matrix of the form (4.23), G a matrix of the form (4.24) and Γ a matrix such that $\Phi-G\Gamma$ is Hurwitz. Then, the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.60), in which (4.61) is a robust stabilizer of the augmented system (4.67).*

Example 4.2. To make this result more explicit, it is useful to examine in more detail the case of a system having relative degree 1. As a byproduct, additional insight in the design procedure is gained, that proves to be useful in similar contexts in the next sections. Consider the case in which the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has relative degree 1, i.e. is such that $C(\mu)B(\mu) \neq 0$. Then, as shown in Example 4.1, system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P(\mu) & A(\mu) \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B(\mu) \end{pmatrix} u$$

$$e = \begin{pmatrix} Q(\mu) & C(\mu) \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}$$

has relative degree 1. It can be put in normal form, obtaining a system (see again Example 4.1)

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)u + P_1(\mu)w \\ e &= \xi\,, \end{aligned} \tag{4.68}$$

in which it is assumed that $b(\mu) > 0$. As shown in Example 4.1, since by hypothesis the eigenvalues of $A_{00}(\mu)$ have negative real part, the Sylvester equation [9]

$$\Pi_0(\mu)S = A_{00}(\mu)\Pi_0(\mu) + P_0(\mu) \tag{4.69}$$

has a solution $\Pi_0(\mu)$ and therefore the regulator equations (4.6) have (unique) solution

$$\Pi(\mu) = \begin{pmatrix} \Pi_0(\mu) \\ 0 \end{pmatrix}, \qquad \Psi(\mu) = \frac{-1}{b(\mu)}[A_{10}(\mu)\Pi_0(\mu) + P_1(\mu)]\,.$$

Changing z into $\tilde{z} = z - \Pi_0(\mu)w$ yields the simplified system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\ \dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[u - \Psi(\mu)w]\,. \end{aligned}$$

Let now this system be controlled by a preprocessor of the form (4.53), with Γ such that the matrix $F = \Phi - G\Gamma$ is Hurwitz and $\tilde{u}$ provided by a stabilizer (4.54). As shown in Lemma 4.8, there always exists a matrix $\Sigma(\mu)$ such that (see (4.59) in this respect)

$$\begin{aligned} \Sigma(\mu)S &= \Phi\Sigma(\mu) \\ \Psi(\mu) &= \Gamma\Sigma(\mu)\,. \end{aligned} \tag{4.70}$$

Using these identities and changing η into $\tilde{\eta} = \eta - \Sigma(\mu)w$ yields the following (augmented) system

[9] Since the parameters of the equation are μ-dependent so is expected to be its solution.

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\
\dot{\tilde{\eta}} &= \Phi\tilde{\eta} + G\tilde{u} \\
\dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[\Gamma\tilde{\eta} + \tilde{u}]\,,
\end{aligned} \tag{4.71}$$

in which, as expected, the subsystem characterized by the last three equations is *independent* of w. If such subsystem is (robustly) stabilized, then in particular $\xi \to 0$ as $t \to \infty$. Since $e = \xi$, the problem of output regulation is (robustly) solved. Now, the system in question has relative degree 1 (between input $\tilde{u}$ and output e) and its $n-1+d$ zeros all have negative real part. Hence the stabilizer (4.61) can be a pure output feedback, i.e. $\tilde{u} = -ke$, with large k. To double-check that this is the case, let $\tilde{u}$ be fixed in this way and observe that the resulting closed-loop system is a system of the form

$$\dot{x} = A(\mu)x$$

in which

$$x = \begin{pmatrix} \tilde{z} \\ \tilde{\eta} \\ \xi \end{pmatrix}, \qquad A(\mu) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ 0 & \Phi & -Gk \\ A_{10}(\mu) & b(\mu)\Gamma & a_{11}(\mu) - b(\mu)k \end{pmatrix}. \tag{4.72}$$

A similarity transformation $\bar{x} = T(\mu)x$, with

$$T(\mu) = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\frac{1}{b(\mu)}G \\ 0 & 0 & 1 \end{pmatrix} \tag{4.73}$$

changes $A(\mu)$ into the matrix (set here $F = \Phi - G\Gamma$)

$$\bar{A}(\mu) = T(\mu)A(\mu)T^{-1}(\mu) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F & \frac{1}{b(\mu)}(FG - Ga_{11}(\mu)) \\ A_{10}(\mu) & b(\mu)\Gamma & \Gamma G + a_{11}(\mu) - b(\mu)k \end{pmatrix}. \tag{4.74}$$

Since the eigenvalues of $A_{00}(\mu)$ and those of F have negative real part, the converse Lyapunov Theorem says that there exists a $(n-1+d)\times(n-1+d)$ positive definite symmetric matrix $Z(\mu)$ such that

$$Z(\mu)\begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F \end{pmatrix} + \begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F \end{pmatrix}^{\mathrm{T}} Z(\mu) < 0\,.$$

Then, it is an easy matter to show that the $(n+d)\times(n+d)$ positive definite matrix

$$P(\mu) = \begin{pmatrix} Z(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

is such that

$$P(\mu)\bar{A}(\mu) + \bar{A}^{\mathrm{T}}(\mu)P(\mu) < 0 \tag{4.75}$$

if k is large enough. [10] From this, using the direct Theorem of Lyapunov, we conclude that, if k is large enough, the matrix (4.74) has all eigenvalues with negative real part. If this is the case, the lower subsystem of (4.71) is robustly stabilized and the problem of output regulation is robustly solved. ◁

4.8 Internal model adaptation

The remarkable feature of the controller discussed in the previous section is the ability of securing asymptotic decay of the regulated output $e(t)$ in spite of parameter uncertainties.

Thus, control schemes consisting of an internal model and of a robust stabilizer efficiently address the problem of rejecting all exogenous inputs generated by the exosystem. In this sense, they generalize the classical way in which integral-control based schemes cope with constant but unknown disturbances, even in the presence of parameter uncertainties. There still is a limitation, though, in these schemes: the necessity of a precise model of the exosystem. As a matter of fact, the controller considered above contains a pair of matrices Φ, Γ whose construction (see above) requires the knowledge of the precise values of the coefficients of the minimal polynomial of S. This limitation is not sensed in a problem of set point control, where the uncertain exogenous input is constant and thus obeys a trivial, parameter independent, differential equation, but becomes immediately evident in the problem of rejecting e.g. a *sinusoidal* disturbance of unknown amplitude and phase. A robust controller is able to cope with uncertainties on amplitude and phase of the exogenous sinusoidal signal, but the *frequency* at which the internal model oscillates must exactly match the frequency of the exogenous signal: any mismatch in such frequencies results in a nonzero steady-state error.

In what follows we show how this limitation can be removed, by automatically tuning the "natural frequencies" of the robust controller. For the sake of simplicity, we limit ourselves to sketch here the main philosophy of the design method.[11]

Consider again the single-input single-output system (4.66) for which we have learned how to design a robust regulator but suppose, now, that the model of the exosystem that generates the disturbance w depends on a vector ρ of uncertain parameters, ranging on a prescribed compact set $\mathscr{Q}$, as in

$$\dot{w} = S(\rho)w\,. \tag{4.76}$$

We retain the assumption that the exosystem is neutrally stable, in which case $S(\rho)$ can only have eigenvalues on the imaginary axis (with simple multiplicity in the minimal polynomial). Therefore, uncertainty in the value of ρ is reflected in the uncertainty in the value of the imaginary part of these eigenvalues.

---

[10] See Section 2.3.

[11] The approach in this section closely follows the approach described, in a more general context, in [6].

Let
$$\psi_\rho(\lambda) = s_0(\rho) + s_1(\rho)\lambda + \cdots + s_{d-1}(\rho)\lambda^{d-1} + \lambda^d$$
denote the minimal polynomial of $S(\rho)$ and assume that the coefficients $s_0(\rho)$, $s_1(\rho)$, $\ldots, s_{d-1}(\rho)$ are continuous functions of ρ. Following the design procedure illustrated in the previous sections, consider a pair of matrices Φ_ρ, G defined as

$$\Phi_\rho = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0(\rho) & -s_1(\rho) & -s_2(\rho) & \cdots & -s_{d-1}(\rho) \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

the former of which is a continuous function of ρ.

We know from the discussion above that, if the parameter ρ were known, a controller consisting of a pre-processing internal model of the form

$$\begin{aligned} \dot\eta &= \Phi_\rho \eta + G\tilde u \\ u &= \Gamma \eta + \tilde u\,, \end{aligned} \tag{4.77}$$

with $\tilde u$ provided by a robust stabilizer of the form (4.61), would solve the problem of output regulation. We have also seen that such a stabilizer exists if Γ is *any* matrix that renders $\Phi_\rho - G\Gamma$ a Hurwitz matrix and if, in addition, all $n-r$ zeros of the triplet $\{A(\mu), B(\mu), C(\mu)\}$ have negative real part for all μ.

Note that in this context the choice of Γ is arbitrary, so long as the matrix $\Phi_\rho - G\Gamma$ is Hurwitz. In what follows, we choose this matrix as follows. Let F be a *fixed* $d \times d$ matrix

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix} \tag{4.78}$$

with a characteristic polynomial

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_{d-1}\lambda^{d-1} + \lambda^d\,,$$

having all roots with negative real part, let G be defined as above, i.e.

$$G = (0 \quad 0 \quad \cdots \quad 0 \quad 1)^{\mathrm{T}}\,, \tag{4.79}$$

and choose, for Γ, a matrix of the form

$$\Gamma_\rho = (\,(a_0 - s_0(\rho)) \quad (a_1 - s_1(\rho)) \quad \cdots \quad (a_{d-1} - s_{d-1}(\rho))\,)\,, \tag{4.80}$$

(note that we have added the subscript "ρ" to stress the dependence of such Γ on the vector ρ of possibly uncertain parameters). This choice is clearly such that

$$\Phi_\rho - G\Gamma_\rho = F$$

is a Hurwitz matrix. Hence, if ρ were known, this choice would be admissible. With this choice, the pre-processing internal model (4.77) can be rewritten as

$$\begin{aligned} \dot{\eta} &= F\eta + G(\Gamma_\rho \eta + \tilde{u}) \\ u &= \Gamma_\rho \eta + \tilde{u}\,. \end{aligned} \tag{4.81}$$

Essentially, what we have done is to "shift" the uncertain data from the matrix Φ_ρ to the vector Γ_ρ. The realization (4.81) of the internal model, though, lends itself to the implementation of some easy (and standard) adaptive control techniques.

If ρ were known, the controller (4.81), with $\tilde{u}$ provided by a robust stabilizer of the form (4.61) would be a robust controller. In case ρ is not known, one may wish to replace the vector Γ_ρ with an *estimate* $\hat{\Gamma}$, *to be tuned* by means of an appropriate adaptation law.

We illustrate how this can be achieved in the simple situation in which the system has relative degree 1. To facilitate the analysis, we assume that the controlled plant has been initially put in normal form (see Example 4.1), which in the present case will be [12]

$$\begin{aligned} \dot{w} &= S(\rho)w \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)u + P_1(\mu,\rho)w \\ e &= \xi\,. \end{aligned} \tag{4.82}$$

By assumption, the $n-1$ eigenvalues of the matrix $A_{00}(\mu)$ have negative real part for all μ.

Consider a *tunable* pre-processing internal model of the form

$$\begin{aligned} \dot{\eta} &= F\eta + G(\hat{\Gamma}\eta + \tilde{u}) \\ u &= \hat{\Gamma}\eta + \tilde{u} \end{aligned} \tag{4.83}$$

in which $\hat{\Gamma}$ is a $1 \times d$ vector to be tuned. The associated augmented system becomes

$$\begin{aligned} \dot{w} &= S(\rho)w \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)[\hat{\Gamma}\eta + \tilde{u}] + P_1(\mu,\rho)w \\ \dot{\eta} &= F\eta + G[\hat{\Gamma}\eta + \tilde{u}] \\ e &= \xi\,. \end{aligned}$$

Define an *estimation error*

$$\tilde{\Gamma} = \hat{\Gamma} - \Gamma_\rho\,,$$

and rewrite the system in question as (recall that $F + G\Gamma_\rho = \Phi_\rho$)

---

[12] It is seen from the construction in Example 4.1 that, in the normal form (4.11), the matrices P_0 and P_1 are found by means of transformations involving A,B,C,P and also S. Thus, if the former are functions of μ and the latter is a function of ρ, so are expected to be P_0 and P_1.

$$
\begin{aligned}
\dot{w} &= S(\rho)w \\
\dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\
\dot{\eta} &= \Phi_\rho \eta + G\tilde{u} + G\tilde{\Gamma}\eta \\
\dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)[\Gamma_\rho \eta + \tilde{u}] + b(\mu)\tilde{\Gamma}\eta + P_1(\mu,\rho)w \\
e &= \xi\,.
\end{aligned}
$$

We know from the analysis in Example 4.2 that, if $\tilde{\Gamma}$ were zero, the choice of a stabilizing control

$$\tilde{u} = -k\xi \tag{4.84}$$

(with $k > 0$ and large) would solve the problem of robust output regulation. Let $\tilde{u}$ be chosen in this way and consider, for the resulting closed-loop system, a change of coordinates (see again Example 4.2)

$$\tilde{z} = z - \Pi_0(\mu,\rho)w, \qquad \tilde{\eta} = \eta - \Sigma(\mu,\rho)w$$

in which $\Pi_0(\mu,\rho)$ is a solution of

$$\Pi_0(\mu,\rho)S(\rho) = A_{00}(\mu)\Pi_0(\mu,\rho) + P_0(\mu,\rho)$$

and $\Sigma(\mu,\rho)$ satisfies

$$
\begin{aligned}
\Sigma(\mu,\rho)S(\rho) &= \Phi_\rho \Sigma(\mu,\rho) \\
\Psi(\mu,\rho) &= \Gamma_\rho \Sigma(\mu,\rho)\,,
\end{aligned}
$$

in which

$$\Psi(\mu,\rho) = \frac{-1}{b(\mu)}[A_{10}(\mu)\Pi_0(\mu) + P_1(\mu,\rho)]\,.$$

This yields a system of the form

$$
\begin{aligned}
\dot{w} &= S(\rho)w \\
\dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\
\dot{\tilde{\eta}} &= \Phi_\rho \tilde{\eta} - Gk\xi + G\tilde{\Gamma}\eta \\
\dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[\Gamma_\rho \tilde{\eta} - k\xi] + b(\mu)\tilde{\Gamma}\eta
\end{aligned}
$$

(note that we *have not* modified the terms $\tilde{\Gamma}\eta$ for reasons that will become clear in a moment).

The dynamics of w is now completely decoupled, so that we can concentrate on the lower subsystem, that can be put in the form (compare with (4.72))

$$\dot{x} = A(\mu,\rho)x + B(\mu)\tilde{\Gamma}\eta \tag{4.85}$$

in which

$$x = \begin{pmatrix} \tilde{z} \\ \tilde{\eta} \\ \xi \end{pmatrix}, \quad A(\mu,\rho) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ 0 & \Phi_\rho & -Gk \\ A_{10}(\mu) & b(\mu)\Gamma_\rho & a_{11}(\mu) - b(\mu)k \end{pmatrix}, \quad B(\mu) = \begin{pmatrix} 0 \\ G \\ b(\mu) \end{pmatrix}.$$

It is known from Example 4.2 that – since the eigenvalues of $A_{00}(\mu)$ and those of $F = \Phi_\rho - G\Gamma_\rho$ have negative real part – a matrix such as $A(\mu,\rho)$ is Hurwitz, provided that k is large enough. Specifically, let x be changed in

$$\bar{x} = T(\mu)x$$

with $T(\mu)$ defined as in (4.73), which changes (4.85) in a system of the form

$$\dot{\bar{x}} = \bar{A}(\mu,\rho)\bar{x} + \bar{B}(\mu)\tilde{\Gamma}\eta\,, \tag{4.86}$$

in which $\bar{A}(\mu,\rho) = T(\mu)A(\mu,\rho)T^{-1}(\mu)$, and

$$\bar{B}(\mu) = T(\mu)B(\mu) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} b(\mu)\,.$$

It has been shown in Example 4.2 that there exists a positive definite matrix

$$P(\mu) = \begin{pmatrix} Z(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

and a number k^* such that

$$P(\mu)\bar{A}(\mu,\rho) + \bar{A}^{\mathrm{T}}(\mu,\rho)P(\mu) < 0 \tag{4.87}$$

if $k > k^*$. [13]

Consider now the positive definite quadratic form

$$U(\bar{x},\tilde{\Gamma}) = \bar{x}^{\mathrm{T}}P(\mu)\bar{x} + b(\mu)\tilde{\Gamma}\tilde{\Gamma}^{\mathrm{T}}\,,$$

and compute the derivative of this function along the trajectories of system (4.86). Letting $Q(\mu,\rho)$ denote the negative definite matrix on the left-hand side of (4.87) and observing that

$$\bar{x}^{\mathrm{T}}P(\mu)\bar{B}(\mu) = \xi b(\mu)\,,$$

this yields

$$\begin{aligned}
\dot{U} &= \bar{x}^{\mathrm{T}}[P(\mu)\bar{A}(\mu,\rho) + \bar{A}^{\mathrm{T}}(\mu,\rho)P(\mu)]\bar{x} + 2\bar{x}^{\mathrm{T}}P(\mu)\bar{B}(\mu)\tilde{\Gamma}\eta + 2b(\mu)\tilde{\Gamma}\dot{\tilde{\Gamma}}^{\mathrm{T}} \\
&= \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} + 2\xi b(\mu)\tilde{\Gamma}\eta + 2b(\mu)\tilde{\Gamma}\dot{\tilde{\Gamma}}^{\mathrm{T}} \\
&= \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} + 2b(\mu)\tilde{\Gamma}[\xi\eta + \dot{\tilde{\Gamma}}^{\mathrm{T}}]\,.
\end{aligned}$$

Observing that Γ_ρ is constant, we see that

$$\dot{\tilde{\Gamma}}^{\mathrm{T}} = \dot{\hat{\Gamma}}^{\mathrm{T}}\,.$$

[13] Recall, in this respect, that both the uncertain vectors μ and ρ range on compact sets.

The function $\dot{U}$ cannot be made negative definite, but it can be made *negative semi-definite*, by simply taking

$$\dot{\hat{\Gamma}}^{\mathrm{T}} = -\xi\eta$$

so as to obtain

$$\dot{U}(\bar{x},\tilde{\Gamma}) = \bar{x}^{\mathrm{T}} Q(\mu,\rho)\bar{x} \leq 0\,.$$

Thus, since $U(\bar{x},\tilde{\Gamma})$ is positive definite and $\dot{U}(\bar{x},\tilde{\Gamma}) \leq 0$, the trajectories of the closed-loop system are *bounded*. In addition, appealing to La Salle's invariance principle, [14] we can claim that the trajectories asymptotically converge to an invariant set contained in the locus where $\dot{U}(\bar{x},\tilde{\Gamma}) = 0$.

Clearly, from the expression above, since $Q(\mu,\rho)$ is a definite matrix, we see that

$$\dot{U}(\bar{x},\tilde{\Gamma}) = 0 \qquad \Rightarrow \qquad \bar{x} = 0\,.$$

Hence, $\lim_{t\to\infty} \bar{x}(t) = 0$, which in particular implies $\lim_{t\to\infty} \xi(t) = 0$. Therefore the problem of robust output regulation (in the presence of exosystem uncertainties) is solved.

We summarize the result as follows.

Proposition 4.9. *Consider an uncertain single-input single-output system*

$$\begin{aligned}
\dot{w} &= S(\rho)w \\
\dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\
e &= C(\mu)x + Q(\mu)w\,.
\end{aligned}$$

Suppose the system has relative degree 1 and, without loss of generality, $C(\mu)B(\mu) > 0$*. Suppose the* $n-1$ *zeros of the triplet* $\{A(\mu),B(\mu),C(\mu)\}$ *have negative real part, for every value of* μ*. Then, the problem of robust output regulation is solved by a controller of the form*

$$\begin{aligned}
\dot{\eta} &= F\eta + G(\hat{\Gamma}\eta - ke) \\
u &= \hat{\Gamma}\eta - ke
\end{aligned}$$

in which F,G *are matrices of the form (4.78) – (4.79),* $k > 0$ *is a large number and* $\hat{\Gamma}$ *is provided by the* adaptation law

$$\dot{\hat{\Gamma}}^{\mathrm{T}} = -e\eta\,.$$

The extension to systems having higher relative degree, which relies upon arguments similar to those presented in Section 2.5, is relatively straightforward, but it will not be covered here.

Example 4.3. A classical control problem arising in the steel industry is the control of the steel thickness in a rolling mill. As shown schematically in Fig. 4.6, a strip of steel of thickness H goes in on one side, and a thinner strip of steel of thickness h comes out on the other side. The exit thickness h is determined from the balance of

[14] See Theorem B.6 in Appendix B.

two forces: a force proportional to the difference between the incoming and outgoing thicknesses

$$F_H = W(H - h)$$

and a force proportional to the gap between the rolls, that can be expressed as

$$F_s = M(h - s)$$

in which s, known as "unloaded screw position", is seen as a control input. The expression of F_s presumes that the rolls are perfectly round. However, this is seldom the case. The effect of rolls that are not perfectly round (problem known as "eccentricity") can be modeled by adding a perturbation d to the gap $h - s$ in the expression of F_s, which yields

$$F_s^d = M(h - s - d)\,.$$

Rolls that are not perfectly round can be thought of as rolls of variable radius and this radius – if the rolls rotate at constant speed - is a periodically varying function of time, the period being equal to the time needed for the roll to perform a complete revolution. Thus, the term d that models the perturbation of the nominal gap $h - s$ is a periodic function of time. The period of such function is not fixed though, because it depends on the rotation speed of the rolls.

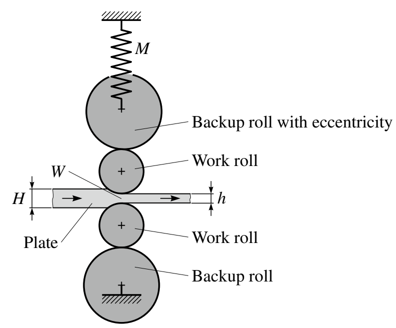

Fig. 4.6 A schematic representation of the process of thickness reduction.

Balancing the two forces F_H and F_s^d yields

$$h = \frac{1}{M + W}(Ms + WH + Md)\,.$$

The purpose of the design is to control the thickness h. This, if h_{ref} is the prescribed reference value for h, yields a tracking *error* defined as

$$e = \frac{1}{M + W}(Ms + WH + Md) - h_{\text{ref}}\,.$$

The unloaded screw position s is, in turn, proportional to the angular position of the shaft of a servomotor which, neglecting friction and mechanical losses, can be modeled as

$$\ddot{s} = bu$$

in which u is seen as a control. Setting $x_1 = s$ and $x_2 = \dot{s}$, we obtain a model of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= bu \\ e &= c_1 x_1 + q_1 h_{\text{ref}} + q_2 H + q_3 d \end{aligned}$$

in which c_1, q_1, q_2, q_3 are fixed coefficients.

In this expression, h_{ref} and H are *constant* exogenous inputs, while d is a periodic function which, for simplicity, we assume to be a *sinusoidal* function. Thus, setting $w_1 = h_{\text{ref}}$, $w_2 = H$, $w_3 = d$, the tracking error can be rewritten as

$$e = Cx + Qw$$

in which

$$C = \begin{pmatrix} c_1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} q_1 & q_2 & q_3 & 0 \end{pmatrix}$$

and $w \in \mathbb{R}^4$ satisfies

$$\dot{w} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \\ 0 & 0 & -\rho & 0 \end{pmatrix} := S_\rho w\,.$$

In normal form, the model thus found is given by

$$\begin{aligned} \dot{w} &= S_\rho w \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= c_1 bu + QS_\rho^2 w \\ e &= \xi_1\,. \end{aligned}$$

This system has relative degree 2. Thus, according to the method described in Section 2.4, we define a new variable θ as

$$\theta = \xi_2 + a_0 \xi_1 = \dot{e} + a_0 e \tag{4.88}$$

in which $a_0 > 0$, and set $z = \xi_1$ to obtain a system which, viewed as a system with input u and output θ, has relative degree 1 and one zero with negative real part

$$\begin{aligned} \dot{w} &= S_\rho w \\ \dot{z} &= -a_0 z + \theta \\ \dot{\theta} &= -a_0^2 z + a_0 \theta + c_1 bu + QS_\rho^2 w\,. \end{aligned}$$

The theory developed above can be used to design a control law, driven by the "regulated variable" θ, that will steer $\theta(t)$ to 0 as $t \to \infty$. This suffices to steer also the actual tracking error $e(t)$ to zero. In fact, in view of (4.88), it is seen that $e(t)$ is the output of a stable one-dimensional linear system

$$\dot{e} = -a_0 e + \theta$$

driven by the input θ. If $\theta(t)$ asymptotically vanishes, so does $e(t)$.

For the design of the internal model, it is observed that the minimal polynomial of S_ρ is the polynomial of degree 3

$$\psi_\rho(\lambda) = \lambda^3 + \rho^2\lambda\,,$$

which has a fixed root at $\lambda = 0$. Thus, it is natural to seek a setting in which only two parameters are adapted (those that correspond to the uncertain roots in $\pm j\rho$). This can be achieved in this way. Pick

$$F = \begin{pmatrix} 0 & H_2 \\ -G_2 & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}, \qquad \Gamma_\rho = \begin{pmatrix} 1 & \Gamma_{2,\rho} \end{pmatrix}$$

in which F_2, G_2 is the pair of matrices

$$F_2 = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with a_0 and a_1 both positive (so that F_2 is Hurwitz) and $\Gamma_{2,\rho}$ such that

$$F_2 + G_2\Gamma_{2,\rho} = \begin{pmatrix} 0 & 1 \\ -\rho^2 & 0 \end{pmatrix}.$$

Finally, let P_2 be a positive definite solution of the Lyapunov equation

$$P_2F_2 + F_2^{\mathrm{T}}P_2 = -I$$

and pick $H_2 = G_2^{\mathrm{T}}P_2$. Then, the following properties hold:[15]

(i) the matrix F is Hurwitz
(ii) the minimal polynomial of $F + G\Gamma_\rho$ is $\psi_\rho(\lambda)$
(iii) the pair $(F + G\Gamma_\rho, \Gamma_\rho)$ is observable

---

[15] To prove (i), it suffices to observe that the positive definite matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix}$$

satisfies

$$QF + F^{\mathrm{T}}Q = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix} \leq 0\,,$$

and use LaSalle's invariance principle. The proof of (ii) is achieved by direct substitution. Property (iii) is a consequence of (i) and of (ii), which says that all eigenvalues of $F + G\Gamma_\rho$ have zero real part. Property (iv) follows from Lemma 4.8.

(iv) for any $\Psi \in \mathbb{R}^{1\times 3}$ there exists a matrix $\Sigma(\rho)$ such that

$$\Sigma(\rho)S_\rho = (F + G\Gamma_\rho)\Sigma(\rho)\,, \qquad \Psi = \Gamma_\rho \Sigma(\rho)\,.$$

As shown above, if ρ were known, the controller (4.81) with $\tilde{u} = -k\theta$, namely the controller

$$\begin{aligned} \dot{\eta} &= F\eta + G[\Gamma_\rho \eta - k\theta] \\ u &= \Gamma_\rho \eta - k\theta \end{aligned}$$

would solve the problem of output regulation. Since ρ is not know, Γ_ρ has to be replaced by a vector of tunable parameters. Such vector, though, needs not to be a (1×3) vector because the first component of Γ_ρ, being equal to 1, is not uncertain. Accordingly, in the above controller, this vector is replaced by a vector of the form

$$\hat{\Gamma} = (\,1 \quad \hat{\Gamma}_2\,)$$

in which only $\hat{\Gamma}_2$ is a vector of tunable parameters.

The analysis that the suggested controller is able to solve the problem of output regulation in spite of the uncertainty about the value of ρ, if an appropriate adaptation law is chosen for $\hat{\Gamma}_2$, is identical to the one presented above, and will not repeated here. We limit ourself to conclude with the complete model of the controller which, in view of all of the above, reads as

$$\begin{aligned} \dot{\eta}_1 &= H_2\eta_2 \\ \dot{\eta}_2 &= F_2\eta_2 + G_2(\hat{\Gamma}_2\eta_2 - k(\dot{e} + a_0 e)) \\ \dot{\hat{\Gamma}}_2^{\mathrm{T}} &= -\eta_2(\dot{e} + a_0 e) \\ u &= \eta_1 + \hat{\Gamma}_2\eta_2 - k(\dot{e} + a_0 e)\,. \qquad \triangleleft \end{aligned}$$

4.9 Robust regulation via H_∞ methods

In Section 4.7, appealing to the results presented in Chapter 2, we have shown how robust regulation can be achieved in the special case $m = p = 1$, under the assumption that the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has a well-defined relative degree and all its $n - r$ zeros have negative real part, for every value of μ. In this section, we discuss how robust regulation can be achieved in a more general setting, appealing to the method for robust stabilization presented in Sections 3.5 and 3.6.[16]

For consistency with the notation used in the context of robust stabilization via H_∞ methods, we denote the controlled plant as

[16] The approach in this section essentially follows the approach of [7]. See also [5] and [8] for further reading.

$$
\begin{aligned}
\dot{w} &= Sw \\
\dot{x} &= Ax + B_1 v + B_2 u + Pw \\
z &= C_1 x + D_{11} v + D_{12} u + Q_1 w \\
y &= C_2 x + D_{21} v + Q_2 w \\
e &= C_{\mathrm{e}} x + D_{\mathrm{e}1} v + Q_{\mathrm{e}} w\,,
\end{aligned}
\tag{4.89}
$$

in which

$$
C_{\mathrm{e}} = EC_2\,, \quad D_{\mathrm{e}1} = ED_{21}\,, \quad Q_{\mathrm{e}} = EQ_2
$$

with

$$
E = \begin{pmatrix} I_p & 0 \end{pmatrix}.
$$

According to the theory presented in Sections 4.4 and 4.5, we consider a controller that has the standard structure of a *post-processing internal model*

$$
\dot{\eta} = \Phi\eta + Ge\,, \tag{4.90}
$$

in which Φ, G have the form (4.23)–(4.24), cascaded with a *robust stabilizer*.

The purpose of such stabilizer is to solve the problem of γ-suboptimal H_∞ feedback design for an *augmented* plant defined as

$$
\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{\mathrm{e}1} \end{pmatrix} v + \begin{pmatrix} B_2 \\ 0 \end{pmatrix} u \\
z &= \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + D_{11} v + D_{12} u \\
y_{\mathrm{a}} &= \begin{pmatrix} C_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} D_{21} \\ 0 \end{pmatrix} v\,.
\end{aligned}
\tag{4.91}
$$

If this is the case, in fact, on the basis of the theory of robust stabilization via H_∞ methods, one can claim that the problem of output regulation is solved, *robustly* with respect to dynamic perturbations that can be expressed as

$$
v = P(s)z\,,
$$

in which $P(s)$ is the transfer function of a stable uncertain system, satisfying $\|P\|_{H_\infty} < 1/\gamma$.

For convenience, let system (4.91) be rewritten as

$$
\begin{aligned}
\dot{x}_{\mathrm{a}} &= A_{\mathrm{a}} x_{\mathrm{a}} + B_{\mathrm{a}1} v + B_{\mathrm{a}2} u \\
z &= C_{\mathrm{a}1} x + D_{\mathrm{a}11} v + D_{\mathrm{a}12} u \\
y_{\mathrm{a}} &= C_{\mathrm{a}2} x + D_{\mathrm{a}21} v\,,
\end{aligned}
\tag{4.92}
$$

in which

$$
A_{\mathrm{a}} = \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix}, \quad B_{\mathrm{a}1} = \begin{pmatrix} B_1 \\ GD_{\mathrm{e}1} \end{pmatrix}, \quad B_{\mathrm{a}2} = \begin{pmatrix} B_2 \\ 0 \end{pmatrix},
$$

$$C_{a1} = (\, C_1 \quad 0 \,), \quad D_{a11} = D_{11}, \quad D_{a12} = D_{12},$$

$$C_{a2} = \begin{pmatrix} C_2 & 0 \\ 0 & I \end{pmatrix}, \quad D_{a21} = \begin{pmatrix} D_{21} \\ 0 \end{pmatrix}.$$

Observe, in particular, that the state x_a has dimension $n + dp$.

The necessary and sufficient conditions for the solution of a γ-suboptimal feedback design problem are those determined in Theorem 3.3. With reference to system (4.92), the conditions in question are rewritten as follows.

Theorem 4.1. *Consider a plant of modelled by equations of the form (4.92). Let* $V_{a1}, V_{a2}, Z_{a1}, Z_{a2}$ *be matrices such that*

$$\mathrm{Im}\begin{pmatrix} Z_{a1} \\ Z_{a2} \end{pmatrix} = \mathrm{Ker}\,(\, C_{a2} \quad D_{a21} \,), \qquad \mathrm{Im}\begin{pmatrix} V_{a1} \\ V_{a2} \end{pmatrix} = \mathrm{Ker}\,(\, B_{a2}^{\mathrm{T}} \quad D_{a12}^{\mathrm{T}} \,).$$

The problem of γ-suboptimal H_∞ feedback design has a solution if and only if there exist symmetric matrices S_a and R_a satisfying the following system of linear matrix inequalities

$$\begin{pmatrix} Z_{a1}^{\mathrm{T}} & Z_{a2}^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_a^{\mathrm{T}} S_a + S_a A_a & S_a B_{a1} & C_{a1}^{\mathrm{T}} \\ B_{a1}^{\mathrm{T}} S_a & -\gamma I & D_{a11}^{\mathrm{T}} \\ C_{a1} & D_{a11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_{a1} & 0 \\ Z_{a2} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.93}$$

$$\begin{pmatrix} V_{a1}^{\mathrm{T}} & V_{a2}^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_a R_a + R_a A_a^{\mathrm{T}} & R_a C_{a1}^{\mathrm{T}} & B_{a1} \\ C_{a1} R_a & -\gamma I & D_{a11} \\ B_{a1}^{\mathrm{T}} & D_{a11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_{a1} & 0 \\ V_{a2} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.94}$$

$$\begin{pmatrix} S_a & I \\ I & R_a \end{pmatrix} \geq 0. \tag{4.95}$$

In particular, there exists a solution of dimension k if and only if there exist R_a and S_a satisfying (4.93), (4.94), (4.95) and, in addition,

$$\mathrm{rank}(I - R_a S_a) \leq k. \tag{4.96}$$

In view of the special structure of the matrices that characterize (4.92), the conditions above can be somewhat simplified. Observe that the inequality (4.93) can be rewritten as

$$\begin{pmatrix} Z_{a1}^{\mathrm{T}} S_a (A_a Z_{a1} + B_{a1} Z_{a2}) + (A_a Z_{a1} + B_{a1} Z_{a2})^{\mathrm{T}} S_a Z_{a1} - \gamma Z_{a2}^{\mathrm{T}} Z_{a2} & Z_{a1}^{\mathrm{T}} C_{a1}^{\mathrm{T}} + Z_{a2}^{\mathrm{T}} D_{a11}^{\mathrm{T}} \\ C_{a1} Z_{a1} + D_{a11} Z_{a2} & -\gamma I \end{pmatrix} < 0. \tag{4.97}$$

The kernel of the matrix

$$(\, C_{a2} \quad D_{a21} \,) = \begin{pmatrix} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{pmatrix}$$

is spanned by the columns of a matrix

$$\begin{pmatrix} Z_{a1} \\ Z_{a2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} \\ Z_2 \end{pmatrix}$$

in which Z_1 and Z_2 are such that

$$\operatorname{Im} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} C_2 & D_{21} \end{pmatrix} .$$

Therefore, since $GC_eZ_1 + GD_{e1}Z_2 = GE(C_2Z_1 + D_{21}Z_2) = 0$,

$$(A_aZ_{a1} + B_{a1}Z_{a2}) = \begin{pmatrix} A & 0 \\ GC_e & \Phi \end{pmatrix} \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{e1} \end{pmatrix} Z_2 = \begin{pmatrix} AZ_1 + B_1Z_2 \\ 0 \end{pmatrix} .$$

Thus, if S_a is partitioned as

$$S_a = \begin{pmatrix} S & * \\ * & * \end{pmatrix},$$

in which S is $n \times n$, we see that

$$Z_{a1}^T S_a (A_aZ_{a1} + B_{a1}Z_{a2}) = Z_1^T S (AZ_1 + B_1Z_2) .$$

Moreover, $Z_{a2}^T Z_{a2} = Z_2^T Z_2$ and

$$C_{a1}Z_{a1} + D_{a11}Z_{a2} = C_1Z_1 + D_{11}Z_2.$$

It is therefore concluded that the inequality (4.97) reduces to the inequality

$$\begin{pmatrix} Z_1^T & Z_2^T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^TS + SA & SB_1 & C_1^T \\ B_1^TS & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.98}$$

which is *identical* to the inequality (3.76) determined in Theorem 3.3.

The inequality (4.94), in general, does not lend itself to any special simplification. We simply observe that the kernel of the matrix

$$\begin{pmatrix} B_{a2}^T & D_{a12}^T \end{pmatrix} = \begin{pmatrix} B_2^T & 0 & D_{12}^T \end{pmatrix}$$

is spanned by the columns of a matrix

$$\begin{pmatrix} V_{a1} \\ V_{a2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} V_2 & 0 \end{pmatrix} \end{pmatrix}$$

in which V_1 and V_2 are such that

$$\operatorname{Im} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} B_2^T & D_{12}^T \end{pmatrix} ,$$

and hence (4.94), in the actual plant parameters, can be rewritten as

$$\begin{pmatrix} V_1^{\mathrm{T}} & 0 & V_2^{\mathrm{T}} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} R_{\mathrm{a}} + R_{\mathrm{a}} \begin{pmatrix} A^{\mathrm{T}} & C_{\mathrm{e}}^{\mathrm{T}} G^{\mathrm{T}} \\ 0 & \Phi^{\mathrm{T}} \end{pmatrix} & R_{\mathrm{a}} \begin{pmatrix} C_1^{\mathrm{T}} \\ 0 \end{pmatrix} & \begin{pmatrix} B_1 \\ GD_{\mathrm{e1}} \end{pmatrix} \\ \begin{pmatrix} C_1 & 0 \end{pmatrix} R_{\mathrm{a}} & -\gamma I & D_{11} \\ \begin{pmatrix} B_1^{\mathrm{T}} & D_{\mathrm{e1}}^{\mathrm{T}} G^{\mathrm{T}} \end{pmatrix} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ V_2 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} < 0. \tag{4.99}$$

Thanks to the simplified version of (4.93), it is possible to show – as a Corollary of Theorem 4.1 – that the problem in question can be solved by a controller of dimension not exceeding n.

Corollary 4.1. *Consider the problem of γ-suboptimal H_∞ feedback design for the augmented plant (4.91). Suppose there exists positive definite symmetric matrices S and R_{a} satisfying the system of linear matrix equations (4.98), (4.99) and*

$$\begin{pmatrix} S & \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \begin{pmatrix} I_n \\ 0 \end{pmatrix} & R_{\mathrm{a}} \end{pmatrix} > 0. \tag{4.100}$$

Then the problem can be solved, by a controller of dimension not execeeding n.

Proof. As a consequence of (4.100), the matrix R_{a} is positive definite and hence nonsingular. Let R_{a} be partitioned as

$$R_{\mathrm{a}} = \begin{pmatrix} R_{\mathrm{a11}} & R_{\mathrm{a12}} \\ R_{\mathrm{a12}}^{\mathrm{T}} & R_{\mathrm{a22}} \end{pmatrix}$$

in which R_{a11} is $n \times n$, and let R_{a}^{-1} be partitioned as

$$R_{\mathrm{a}}^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix}$$

in which Y_{11} is $n \times n$. As shown above, a $(n+dp) \times (n+dp)$ matrix S_{a} of the form

$$S_{\mathrm{a}} = \begin{pmatrix} S & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix} \tag{4.101}$$

is a solution of (4.93). It is possible to show that condition (4.100) implies (4.95). In fact, with

$$T = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ -Y_{12} & 0 & I & 0 \\ -Y_{22} & 0 & 0 & I \end{pmatrix}$$

one obtains

$$T^{\mathrm{T}} \begin{pmatrix} S_{\mathrm{a}} & I \\ I & R_{\mathrm{a}} \end{pmatrix} T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S & I & 0 \\ 0 & I & R_{\mathrm{a11}} & R_{\mathrm{a12}} \\ 0 & 0 & R_{\mathrm{a12}}^{\mathrm{T}} & R_{\mathrm{a22}} \end{pmatrix}.$$

By (4.100), the matrix on the right is positive semidefinite and this implies (4.95). Thus, (4.98), (4.99) and (4.100) altogether imply (4.93), (4.94) and (4.95).

Finally, observe that

$$R_a S_a = R_a \begin{pmatrix} S & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & I \end{pmatrix}$$

and hence the last dp columns of $(I - R_a S_a)$ are zero. As a consequence

$$\operatorname{rank}(I - S_a R_a) \leq n\,,$$

from which it is concluded that the problem can be solved by a controller dimension not exceeding n. $\triangleleft$

Remark 4.4. It is worth stressing the fact that the dimension of the robust stabilizer does not exceed the dimension n of the controlled plant, and this despite of the fact that the robust stabilizer is designed for an augmented plant of dimension $n + dp$. This is essentially due to the structure of the augmented plant and in particular to the fact that the component η of the state of such augmented plant is not affected by the exogenous input v and is directly available for feedback. $\triangleleft$

If the hypotheses of this Corollary are fulfilled, there exists a controller

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c y_a' + J_s y_a'' \\ u &= C_c \xi + D_c y_a' + H_s y_a''\,, \end{aligned}$$

in which y_a' and y_a'' are the upper and – respectively – lower blocks of the output y_a of (4.91), that solves the problem of γ-suboptimal H_∞ feedback design for the augmented plant (4.91). The matrices $A_c, B_c, C_c, D_c, J_s, H_s$ can be found solving an appropriate linear matrix inequality. [17] With these matrices, one can build a controller of the form

$$\begin{aligned} \dot{\eta} &= \Phi \eta + G e \\ \dot{\xi} &= A_c \xi + B_c y + J_s \eta \\ u &= C_c \xi + D_c y + H_s \eta \end{aligned}$$

that solves the problem of output regulation for the perturbed plant (4.89), robustly with respect to dynamic perturbations that can be expressed as $v = P(s)z$, in which $P(s)$ is the transfer function of a stable uncertain system, satisfying $\|P\|_{H_\infty} < 1/\gamma$.

References

1. B. Francis, O.A. Sebakhy, W.M. Wonham, Synthesis of multivariable regulators: the internal model principle, *Appl. Math. Optimiz.*, 1, pp. 64-86, 1974.
2. B.A Francis, The linear multivariable regulator problem, *SIAM J. Control Optimiz.*, 14, pp. 486–505, 1976.

[17] See Section 3.6.

3. E.J. Davison, The robust control of a servomechanism problem for linear time-invariant multivariable systems, *IEEE Trans. Autom. Control*, 21, pp. 25–34, 1976.
4. B.A Francis, M.W. Wonham, The internal model principle of control theory, *Automatica*, 12, pp. 457–465, 1977.
5. J. Abedor, K. Nagpal, P.P. Khargonekar, K. Poolla, Robust regulation in the presence of norm-bounded uncertainty, *IEEE Trans. Automat. Control*, 40, pp. 147-153, 1995.
6. A. Serrani, A. Isidori, L. Marconi, Semiglobal nonlinear output regulation with adaptive internal model, *IEEE Trans. Automatic Control*, AC-46, 1178–1194, 2001.
7. H. Köroglu, C. W. Scherer, An LMI approach to H_∞ synthesis subject to almost asymptotic regulation constraints. *Systems & Control Letters*, 57, pp. 300-308, 2008
8. H. Köroglu, C. W. Scherer, Scheduled control for robust attenuation of non-stationary sinusoidal disturbances with measurable frequencies, *Automatica*, 47, pp. 504-514, 2011.

Appendix B
Stability and Asymptotic Behavior of Nonlinear Systems

B.1 The theorems of Lyapunov for nonlinear systems

We assume in what follows that the reader is familiar with basic concepts concerning the stability of an equilibrium in a nonlinear system. In this section we provide a sketchy summary of some fundamental results, mainly to the purpose of introducing notations and results that are currently used throughout the book. [1]

Comparison functions. A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathscr{K}_\infty$. A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathscr{K}\mathscr{L}$ if, for each fixed t, the function

$$\begin{aligned} \alpha : [0,a) &\to [0,\infty) \\ r &\mapsto \beta(r,t) \end{aligned}$$

belongs to class $\mathscr{K}$ and, for each fixed r, the function

$$\begin{aligned} \varphi : [0,\infty) &\to [0,\infty) \\ t &\mapsto \beta(r,t) \end{aligned}$$

is decreasing and $\lim_{t\to\infty} \varphi(t) = 0$.

The composition of two class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathscr{K}$ function, defined on $[0,a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique function, $\alpha^{-1} : [0,b) \to [0,a)$, such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \text{ for all } r \in [0,a) \\ \alpha(\alpha^{-1}(r)) &= r, \text{ for all } r \in [0,b). \end{aligned}$$

[1] For further reading, see [2], [4],[6], [5].

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathscr{K}$ function. If $\alpha(\cdot)$ is a class $\mathscr{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$. If $\beta(\cdot,\cdot)$ is a class $\mathscr{K}\mathscr{L}$ function and $\alpha_1(\cdot), \alpha_2(\cdot)$ are class $\mathscr{K}$ functions, the function thus defined

$$\begin{aligned}\gamma\colon [0,a)\times[0,\infty) &\to [0,\infty)\\ (r,t) \qquad &\mapsto \alpha_1(\beta(\alpha_2(r),t))\end{aligned}$$

is a class $\mathscr{K}\mathscr{L}$ function.

The Theorems of Lyapunov. Consider an autonomous nonlinear system

$$\dot{x} = f(x) \tag{B.1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well known criterion of Lyapunov, which, using comparison functions, can be expressed as follows. Let B_d denote the open ball of radius d in $\mathbb{R}^n$, i.e.

$$B_d = \{x \in \mathbb{R}^n : \|x\| < d\}\,.$$

Theorem B.1. [Direct Theorem] *Let $V : B_d \to \mathbb{R}$ be a C^1 function such that, for some class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, defined on $[0,d)$,*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \text{for all } x \in B_d. \tag{B.2}$$

If

$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \text{for all } x \in B_d, \tag{B.3}$$

the equilibrium $x = 0$ of (B.1) is stable.

If, for some class $\mathscr{K}$ function $\alpha(\cdot)$, defined on $[0,d)$,

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|) \qquad \text{for all } x \in B_d, \tag{B.4}$$

the equilibrium $x = 0$ of (B.1) is locally asymptotically stable.

If $d = \infty$ and, in the above inequalities, $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathscr{K}_\infty$ functions, the equilibrium $x = 0$ of (B.1) is globally asymptotically stable.

Remark B.1. The usefulness of the comparison functions, in the statement of the Theorem, is motivated by the following simple arguments. Suppose (B.3) holds. Then, so long as $x(t) \in B_d$, $V(x(t))$ is non-increasing, i.e. $V(x(t)) \le V(x(0))$. Pick $\varepsilon < d$ and define $\delta = \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)$. Then, using (B.2), it is seen that, if $\|x(0)\| < \delta$,

$$\underline{\alpha}(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \overline{\alpha}(\|x(0)\|) \le \overline{\alpha}(\delta) = \underline{\alpha}(\varepsilon)$$

which implies $\|x(t)\| \le \varepsilon$. This shows that $x(t)$ exists for all t and the equilibrium $x = 0$ is stable.

Suppose now that (B.4) holds. Define $\gamma(r) = \alpha(\overline{\alpha}^{-1}(r))$, which is a class $\mathscr{K}$ function. Using the estimate on the right of (B.2), it is seen that $\alpha(\|x\|) \ge \gamma(V(x))$

and hence

$$\frac{\partial V}{\partial x} f(x) \leq -\gamma(V(x)) .$$

Since $V(x(t))$ is a continuous function of t, non-increasing and non-negative for each t, there exists a number $V^* \geq 0$ such that $\lim_{t\to\infty} V(x(t)) = V^*$. Suppose V^* is strictly positive. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\gamma(V(x(t)) \leq -\gamma(V^*) < 0 .$$

Integration with respect to time yields

$$V(x(t)) \leq V(x(0)) - \gamma(V^*)t$$

for all t. This cannot be the case, because for large t the right-hand side is negative, while the left-hand side is non-negative. From this it follows that $V^* = 0$ and therefore, using the fact that $V(x)$ vanishes only at $x = 0$, it is concluded that $\lim_{t\to\infty} x(t) = 0$. Note also that identical arguments hold for the analysis of the asymptotic properties of a time-dependent system

$$\dot{x} = f(x,t)$$

so long $f(0,t) = 0$ for all $t \geq 0$ and $V(x)$ is independent of t. $\triangleleft$

Sometimes, in the design of feedback laws, while it is difficult to obtain a system whose equilibrium $x = 0$ is globally asymptotically stable, it is relatively more easy to obtain a system in which trajectories are bounded (maybe for a specific set of initial conditions) and have suitable decay properties. Instrumental, in such context, is the notion of *sublevel set* of a Lyapunov function $V(x)$ which, for a fixed non-negative real number c, is defined as

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}.$$

The function $V(x)$, which is *positive definite* (i.e. is positive for all nonzero x and zero at $x = 0$) is said to be *proper* if, for each $c \in \mathbb{R}$, the sublevel set Ω_c is a *compact* set. Now, it is easy to check that the function $V(x)$ is proper if and only the inequality on the left-hand side of (B.2) holds for all $x \in \mathbb{R}^n$, with a function $\underline{\alpha}(\cdot)$ which is of class $\mathscr{K}_\infty$. Note also that, if $V(x)$ is proper, for any $c > 0$ it is possible to find a numbers $c_1 > 0$ and $c_2 > 0$ such that

$$B_{c_1} \subset \Omega_c \subset B_{c_2} .$$

An typical example of how sublevel sets can be used to analyze boundedness and decay of trajectories is the following one. Let r_1 and r_2 be two positive numbers, with $r_2 > r_1$. Suppose $V(x)$ is a function satisfying (B.2), with $\underline{\alpha}(\cdot)$ a class $\mathscr{K}_\infty$ function. Pick any pair of positive numbers c_1, c_2, such that

$$\Omega_{c_1} \subset B_{r_1} \subset B_{r_2} \subset \Omega_{c_2} .$$

and let $S_{c_1}^{c_2}$ denote the "annular" compact set

$$S_{c_1}^{c_2} = \{x \in \mathbb{R}^n : c_1 \le V(x) \le c_2\}\,.$$

Suppose that, for some $a > 0$,

$$\frac{\partial V}{\partial x} f(x) \le -a \qquad \text{for all } x \in S_{c_1}^{c_2}\,.$$

Then, for each initial condition $x(0) \in B_{r_2}$, the trajectory $x(t)$ of (B.1) is defined for all t and there exists a finite time T such that $x(t) \in B_{r_1}$ for all $t \ge T$. In fact, take any $x(0) \in B_{r_2} \setminus \Omega_{c_1}$. Such $x(0)$ is in $S_{c_1}^{c_2}$. So long as $x(t) \in S_{c_1}^{c_2}$, the function $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t) \le -a$$

and hence

$$V(x(t)) \le V(x(0)) - at \le c_2 - at\,.$$

Thus, at a time $T \le (c_2 - c_1)/a$, $x(T)$ is on the boundary of the set Ω_{c_1}. On the boundary of Ω_{c_1} the derivative of $V(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_{c_1} and remains there for all $t \ge T$.

It is well-known that the criterion for asymptotic stability provided by the previous Theorem has a *converse*, namely, the existence of a function $V(x)$ having the properties indicated in Theorem B.1 is *implied* by the property of asymptotic stability of the equilibrium $x = 0$ of (B.1). In particular, the following result holds.

Theorem B.2. [Converse Theorem] *Suppose the equilibrium $x = 0$ of (B.1) is locally asymptotically stable. Then, there exist $d > 0$, a C^1 function $V : B_d \to \mathbb{R}$, and class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold. If the equilibrium $x = 0$ of (B.1) is globally asymptotically stable, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold with $d = \infty$.*

It is well-known that, for a nonlinear system, the property of asymptotic stability of the equilibrium $x = 0$ does not necessarily imply *exponential* decay to zero of $\|x(t)\|$. If the equilibrium $x = 0$ of system (B.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$ and $\lambda > 0$ such that

$$x(0) \in B_d \quad \Rightarrow \quad \|x(t)\| \le M e^{-\lambda t} \|x(0)\| \quad \text{for all } t \ge 0$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. In this context, the following criterion is useful.

Lemma B.1. *The equilibrium $x = 0$ of nonlinear system (B.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function $V(x) : \mathbb{R}^n \to \mathbb{R}$, three class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and three real numbers $\delta > 0, \underline{a} > 0, a > 0$, such that*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|)$$

for all $x \in \mathbb{R}^n$ and

$$\underline{\alpha}(s) = \underline{a}s^2, \qquad \alpha(s) = as^2$$

for all $s \in B_\delta$.

B.2 Input-to-state stability and the theorems of Sontag

In the analysis of *forced* nonlinear systems, the property of *input-to-state stability*, introduced and thoroughly studied by E.D. Sontag, plays a role of paramount importance.[2] Consider a forced nonlinear system

$$\dot{x} = f(x,u) \tag{B.5}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0,\infty) \to \mathbb{R}^m$ of (B.5) can be any piecewise continuous bounded function. The space of all such functions is endowed with the so-called supremum norm $\|u(\cdot)\|_\infty$, which is defined as

$$\|u(\cdot)\|_\infty = \sup_{t \ge 0} \|u(t)\|\,.$$

Definition B.1. System (B.5) is said to be input-to-state stable if there exist a class $\mathscr{K}\mathscr{L}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$, called a *gain function*, such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \beta(\|x(0)\|,t) + \gamma(\|u(\cdot)\|_\infty) \tag{B.6}$$

for all $t \ge 0$.

Since, for any pair $\beta > 0, \gamma > 0$, $\max\{\beta,\gamma\} \le \beta + \gamma \le \max\{2\beta, 2\gamma\}$, an alternative way to say that a system is input-to-state stable is to say that there exists a class $\mathscr{K}\mathscr{L}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \max\{\beta(\|x(0)\|,t), \gamma(\|u(\cdot)\|_\infty)\} \tag{B.7}$$

for all $t \ge 0$. Note also that, letting $\|u(\cdot)\|_{[0,t]}$ denote the supremum norm of the restriction of $u(\cdot)$ to the interval $[0,t]$, namely

[2] The concept of input-to-state stability, its properties and applications have been introduced in the sequence of papers [10, 11, 14]. []A summary of the most relevant aspect of the theory can also be found in [5, p. 17-31]

$$\|u(\cdot)\|_{[0,t]} = \sup_{s\in[0,t]} \|u(s)\|\,,$$

the bound (B.6) can be also expressed in the alternative form [3]

$$\|x(t)\| \le \beta(\|x(0)\|,t) + \gamma(\|u(\cdot)\|_{[0,t]})\,, \tag{B.8}$$

and the bound (B.7) in the alternative form

$$\|x(t)\| \le \max\{\beta(\|x(0)\|,t), \gamma(\|u(\cdot)\|_{[0,t]})\}\,,$$

both holding for all $t \ge 0$.

The property of input-to-state stability can be given a characterization which extends the well known criterion of Lyapunov for asymptotic stability. The key tool for such characterization is the notion of *ISS-Lyapunov function*.

Definition B.2. A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (B.5) if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \tag{B.9}$$

and

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) \quad \text{for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfying } \|x\| \ge \chi(\|u\|). \tag{B.10}$$

An equivalent form in which the notion of an ISS-Lyapunov function can be described is the following one.

Lemma B.2. *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (B.5) if and only if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\sigma(\cdot)$ such that (B.9) holds and*

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) + \sigma(\|u\|) \qquad \textit{for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m\,. \tag{B.11}$$

The existence of an ISS-Lyapunov function turns out to be a necessary and sufficient condition for input-to-state stability.

Theorem B.3. *System (B.5) is input-to-state stable if and only if there exists an ISS-Lyapunov function. In particular, if such function exists, then an estimate of the form (B.6) holds with $\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)))$.*

The following elementary examples describe how the property of input-to-state stability can be checked and an estimate of the gain function can be evaluated.

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[3] In fact, since $\|u(\cdot)\|_{[0,t]} \le \|u(\cdot)\|_\infty$ and $\gamma(\cdot)$ is increasing, (B.8) implies (B.6). On the other hand, since $x(t)$ depends only on the restriction of $u(\cdot)$ to the interval $[0,t]$, one could in (B.6) replace $u(\cdot)$ with an input $\bar{u}(\cdot)$ defined as $\bar{u}(s) = u(s)$ for $0 \le s \le t$ and $\bar{u}(s) = 0$ for $s > t$, in which case $\|\bar{u}(\cdot)\|_\infty = \|u(\cdot)\|_{[0,t]}$, and observe that (B.6) implies (B.8).

Example B.1. A stable linear system

$$\dot{x} = Ax + Bu$$

is input-to-state stable, with a linear gain function. In fact, let P denote the unique positive definite solution of the Lyapunov equation $PA + A^{\mathrm{T}}P = -I$ and observe that $V(x) = x^{\mathrm{T}}Px$ satisfies

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\|x\|^2 + c\|x\|\,\|u\|$$

for some $c > 0$. Pick $0 < \varepsilon < 1$ and set $\ell = c/(1-\varepsilon)$. Then, it is easy to see that

$$\|x\| \geq \ell\|u\| \qquad \Rightarrow \qquad \frac{\partial V}{\partial x}(Ax + Bu) \leq -\varepsilon\|x\|^2 \,.$$

The system is input-to-state, with $\chi(r) = \ell r$. Since $\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$, we obtain following the estimate for the (linear) gain function

$$\gamma(r) = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\ell r \,. \qquad \triangleleft$$

Example B.2. Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k + bx^p u \,,$$

in which $k \in \mathbb{N}$ is *odd*, $a > 0$ and $p \in \mathbb{N}$ is such that $p < k$. Pick $V(x) = \frac{1}{2}x^2$ and note that, since $k+1$ is even,

$$\frac{\partial V}{\partial x}(-ax^k + bx^p u) \leq -a|x|^{k+1} + |b||x|^{p+1}|u|$$

Pick $0 < \varepsilon < a$ and define (recall that $k > p$)

$$\chi(r) = \left(\frac{|b|r}{a-\varepsilon}\right)^{\frac{1}{k-p}} .$$

Then, it is easy to see that

$$\|x\| \geq \chi(\|u\|) \qquad \Rightarrow \qquad \frac{\partial V}{\partial x}(-ax^k + bx^p u) \leq -\varepsilon|x|^{k+1} \,.$$

The system is input-to-state stable, with $\gamma(r) = \chi(r)$.

Note that the condition $k > p$ is essential. In fact, the following system, in which $k = p = 1$,

$$\dot{x} = -x + xu$$

is not input-to-state stable. Under the bounded (constant) input $u(t) = 2$ the state $x(t)$ evolves as a solution of $\dot{x} = x$ and hence diverges to infinity. $\triangleleft$

The notion of input-to-state stability lends itself to a number of alternative (equivalent) characterizations, among which one of the most useful can be expressed as follows.

Theorem B.4. *System (B.5) is input-to-state stable if and only if there exists class* $\mathscr{K}$ *functions* $\gamma_0(\cdot)$ *and* $\gamma(\cdot)$ *such that, for any bounded input and any* $x(0) \in \mathbb{R}^n$*, the response* $x(t)$ *satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(\|x(0)\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \gamma(\limsup_{t\to\infty} \|u(t)\|)\ . \end{aligned}$$

B.3 Cascade-connected systems

In this section we investigate the *asymptotic* stability of the equilibrium $(z, \xi) = (0,0)$ of a pair of cascade connected subsystems of the form

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi)\ , \end{aligned} \tag{B.12}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0) = 0$, and $f(z,\xi)$, $g(\xi)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ (see Fig. B.1).

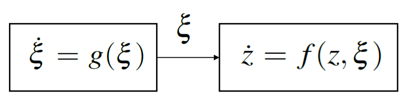

Fig. B.1 A cascade-connection of systems.

Since similar cascade connections occur quite often in the analysis (and feedback design) of nonlinear systems, it is important to understand under what conditions the stability properties of the two components subsystems determine the stability of the cascade. If both systems were linear systems, the cascade would be a system modeled as

$$\begin{aligned} \dot{z} &= Fz + G\xi \\ \dot{\xi} &= A\xi\ , \end{aligned}$$

and it is trivially seen that if both F and G have all eigenvalues in $\mathbb{C}^-$, the cascade is an asymptotically stable system. The nonlinear counterpart of such property, though, requires some extra care.

The simplest scenario, in this respect, is one in which one is interested in seeking only local stability. In this case, the following result holds. [4]

Lemma B.3. *Suppose the equilibrium $z = 0$ of*

$$\dot{z} = f(z, 0) \tag{B.13}$$

is locally asymptotically stable and the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is stable. Then the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is stable. If the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is locally asymptotically stable, then the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is locally asymptotically stable.

It must be stressed, though, that in this Lemma only the property of *local* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ is considered. In fact, by means of a simple counterexample, it can be shown that the *global* asymptotic stability of $z = 0$ as an equilibrium of (B.13) and the *global* asymptotic stability of $\xi = 0$ as an equilibrium of $\dot{\xi} = g(\xi)$ *do not* imply, in general, *global* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ of the cascade. As a matter of fact, the cascade connection of two such systems may even have finite escape times. To infer global asymptotic stability of the cascade, a (strong) extra condition is needed, as shown below.

Example B.3. Consider the case in which

$$\begin{aligned} f(z, \xi) &= -z + z^2 \xi \\ g(\xi) &= -\xi\,. \end{aligned}$$

Clearly $z = 0$ is a globally asymptotically equilibrium of $\dot{z} = f(z, 0)$ and $\xi = 0$ is a globally asymptotically equilibrium of $\dot{\xi} = g(\xi)$. However, this system has finite escape times. To show that this is the case, consider the differential equation

$$\dot{\tilde{z}} = -\tilde{z} + \tilde{z}^2 \tag{B.14}$$

with initial condition $\tilde{z}(0) = z_0$. Its solution is

$$\tilde{z}(t) = \frac{-z_0}{z_0 - 1 - z_0 \exp(-t)} \exp(-t)\,.$$

Suppose $z_0 > 1$. Then, the $\tilde{z}(t)$ escapes to infinity in finite time. In particular, the maximal (positive) time interval on which $\tilde{z}(t)$ is defined is the interval $[0, t_{\max}(z_0))$ with

$$t_{\max}(z_0) = \ln\Big(\frac{z_0}{z_0 - 1}\Big)\,.$$

Now, return to system (B.12), with initial condition (z_0, ξ_0) and let ξ_0 be such that

$$\xi(t) = \exp(-t)\xi_0 \ge 1 \qquad \text{for all } t \in [0, t_{\max}(z_0))\,.$$

[4] More details and proofs of the results stated in this section can be found in [5, p. 11-17 and 31-36].

Clearly, on the time interval $[0, t_{\max}(z_0))$, we have

$$\dot{z} = -z + z^2\xi \geq -z + z^2 .$$

By comparison with (B.14), it follows that

$$z(t) \geq \tilde{z}(t) .$$

Hence $z(t)$ escapes to infinity, at a time $t^* \leq t_{\max}(z_0)$. The lesson learned from this example is that, even if $\xi(t)$ exponentially decreases to 0, this may not suffice to prevent finite escape time in the upper system. The state $z(t)$ escapes to infinity at a time in which the effect of $\xi(t)$ on the upper equation is still not negligible. $\triangleleft$.

The following results provide the extra condition needed to ensure global asymptotic stability in the cascade.

Lemma B.4. *Suppose the equilibrium $z = 0$ of (B.13) is asymptotically stable, and let S be a subset of the domain of attraction of such equilibrium. Consider the system*

$$\dot{z} = f(z, \xi(t)) . \tag{B.15}$$

in which $\xi(t)$ is a continuous function, defined for all $t \geq 0$ and suppose that $\lim_{t\to\infty} \xi(t) = 0$. Pick $z_0 \in S$, and suppose that the integral curve $z(t)$ of (B.15) satisfying $z(0) = z_0$ is defined for all $t \geq 0$, bounded, and such that $z(t) \in S$ for all $t \geq 0$. Then $\lim_{t\to\infty} z(t) = 0$.

This last result implies, in conjunction with Lemma B.3, that if the equilibrium $z = 0$ of (B.13) is globally asymptotically stable, if the equilibrium $\xi = 0$ of the lower subsystem of (B.12) is globally asymptotically stable, and all trajectories of the composite system (B.12) *are bounded*, the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is globally asymptotically stable.

To be in a position to use this result in practice, one needs to determine conditions under which the boundedness property holds. This is indeed the case if the upper subsystem of the cascade, viewed as a system with state z and input ξ, is input-to-state stable. In view of this, it can be claimed that if the upper subsystem of the cascade is input-to-state stable and the lower subsystem is globally asymptotically stable (at the equilibrium $\xi = 0$), the cascade is globally asymptotically stable (at the equilibrium $(z, \xi) = (0, 0)$).

As a matter fact, a more general result holds, which is stated as follows.

Theorem B.5. *Suppose that system*

$$\dot{z} = f(z, \xi) , \tag{B.16}$$

viewed as a system with input ξ and state z is input-to-state stable and that system

$$\dot{\xi} = g(\xi, u) , \tag{B.17}$$

viewed as a system with input u and state ξ is input-to-state stable as well. Then, system

$$\begin{aligned}\dot{z} &= f(z,\xi)\\ \dot{\xi} &= g(\xi,u)\end{aligned}$$

is input-to-state stable.

Example B.4. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\xi_1\xi_2\\ \dot{x}_2 &= -x_2 + \xi_1^2 - x_1\xi_1\xi_2\\ \dot{\xi}_1 &= -\xi_1^3 + \xi_1 u_1\\ \dot{\xi}_2 &= -\xi_2 + u_2\,.\end{aligned}$$

The subsystem consisting of the two top equations, seen as a system with state $x = (x_1, x_2)$ and input $\xi = (\xi_1, \xi_2)$ is input-to-state stable. In fact, let this system be written as

$$\dot{x} = f(x,\xi)$$

and consider the candidate ISS-Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

for which we have

$$\frac{\partial V}{\partial x} f(x,\xi) = -(x_1^2 + x_2^2) + x_2\xi_1^2 \le -x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{2}\xi_1^4 \le -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|\xi\|^4\,.$$

Thus, the function $V(x)$ satisfies the condition indicated in Lemma B.2, with

$$\alpha(r) = \frac{1}{2}r^2\,, \qquad \sigma(r) = \frac{1}{2}r^4\,.$$

The subsystem consisting of the two bottom equations is composed of two separate subsystems, both of which are input-to-state stable, as seen in examples B.2 and B.1. Thus, the overall system is input-to-state stable. $\triangleleft$

B.4 Limit sets

Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{B.18}$$

with $x \in \mathbb{R}^n$. It is well known that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for all $x_0 \in \mathbb{R}^n$ the solution of (B.18) with initial condition $x(0) = x_0$, denoted by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique.

Definition B.3. Let $x_0 \in \mathbb{R}^n$ be fixed. Suppose that $x(t,x_0)$ is defined for all $t \geq 0$. A point x is said to be an ω-limit *point* of the motion $x(t,x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k,x_0) = x\,.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is the union of all ω-limit points of the motion $x(t,x_0)$.

It is obvious from this definition that an ω-limit point *is not* necessarily a limit of $x(t,x_0)$ as $t \to \infty$, because the solution in question may not admit any limit as $t \to \infty$ (see for instance fig. B.2). [5]

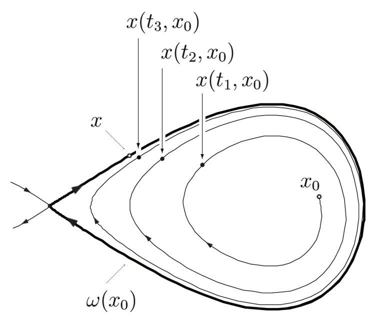

Fig. B.2 The ω-limit set of the point x_0.

However, it is known that, if the motion $x(t,x_0)$ is *bounded*, then $x(t,x_0)$ asymptotically approaches the set $\omega(x_0)$, as specified in the Lemma that follows.[6] In this respect, recall that a set $S \subset \mathbb{R}^n$ is said to be *invariant* under (B.18) if for all initial conditions $x_0 \in S$ the solution $x(t,x_0)$ of (B.18) exists for all $t \in (-\infty,+\infty)$ and $x(t,x_0) \in S$ for all such t. [7]. Moreover, the *distance* of a point $x \in \mathbb{R}$ from a set $S \subset \mathbb{R}^n$, denoted $\mathrm{dist}(x,S)$, is the non-negative real number defined as

$$\mathrm{dist}(x,S) = \inf_{z\in S} \|x-z\|\,.$$

[5] Figures B.2, B.3, B.4 are reprinted from *Annual Reviews in Control*, Vol. 32, A.Isidori and C.I.Byrnes, Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection, Pages 1-16, Copyright (2008), with permission from Elsevier.

[6] See [1, page 198].

[7] We recall, for the sake of completeness, that a set S is said to be *positively invariant*, or *invariant in positive time* (respectively, *negatively invariant* or *invariant in negative time*) if for all initial conditions $x_0 \in X$, the solution $x(t,x_0)$ exists for all $t \geq 0$ and $x(t,x_0) \in X$ for all $t \geq 0$ (respectively exists for all $t \leq 0$ and $x(t,x_0) \in X$ for all $t \leq 0$). Thus, a set is invariant if it is both positively invariant and negatively invariant.

Lemma B.5. *Suppose there is a number M such that $\|x(t,x_0)\| \le M$ for all $t \ge 0$. Then, $\omega(x_0)$ is a nonempty connected compact set, invariant under (B.18). Moreover,*

$$\lim_{t\to\infty} \mathrm{dist}(x(t,x_0),\omega(x_0)) = 0\,.$$

Example B.5. Consider the classical (stable) Van der Pol oscillator, written in state-space form as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \mu(1-x_1^2)x_2\end{aligned} \tag{B.19}$$

in which, as it is well known, the damping term $\mu(1-x_1^2)y$ can be seen as a model of a nonlinear resistor, negative for small x_1 and positive for large x_1 (see [6]). From the phase portrait of this system (depicted in fig. B.3 for $\mu = 1$) it is seen that all motions except the trivial motion occurring for $x_0 = 0$ are bounded in positive time and approach, as $t \to \infty$, the limit cycle $\mathscr{L}$. As consequence, $\omega(x_0) = \mathscr{L}$ for any $x_0 \neq 0$, while $\omega(0) = \{0\}$. ◁

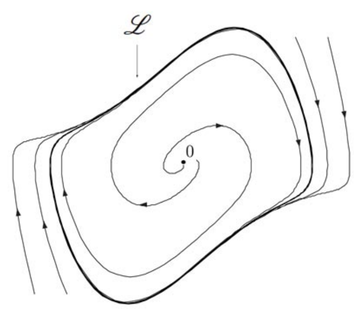

Fig. B.3 The phase portrait of the Van der Pol oscillator.

An important useful application of the notion of ω-limit set of a point is found in the proof of the following result, commonly known as LaSalle's invariance principle.[8]

Theorem B.6. *Consider system (B.18). Suppose there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \textit{for all } x \in \mathbb{R}^n,$$

for some pair of class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and such that

$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \textit{for all } x \in \mathbb{R}^n\,. \tag{B.20}$$

[8] See e.g. [6]

Let $\mathscr{E}$ denote the set

$$\mathscr{E} = \{x \in \mathbb{R}^n : \frac{\partial V}{\partial x} f(x) = 0\}. \tag{B.21}$$

Then, for each x_0, the integral curve $x(t,x_0)$ of (B.18) passing through x_0 at time $t = 0$ is bounded, and

$$\omega(x_0) \subset \mathscr{E}.$$

Proof. A direct consequence of (B.20) is that, for any x_0, the motion $x(t,x_0)$ is bounded in positive time. In fact, this property yields $V(x(t,x_0)) \le V(x_0)$ for all $t \ge 0$ and this in turn implies (see Remark B.1)

$$\|x(t,x_0)\| \le \underline{\alpha}^{-1}(\overline{\alpha}(\|x_0\|)).$$

Thus, the limit set $\omega(x_0)$ is nonempty, compact and invariant. The non-negative-valued function $V(x(t,x_0))$ is non-increasing for $t \ge 0$. Thus, there is a number $V_0 \ge 0$, possibly dependent on x_0, such that

$$\lim_{t\to\infty} V(x(t,x_0)) = V_0.$$

By definition of limit set, for each point $x \in \omega(x_0)$, there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that $\lim_{k\to\infty} x(t_k,x_0) = x$. Thus, since $V(x)$ is continuous,

$$V(x) = \lim_{k\to\infty} V(x(t_k,x_0)) = V_0.$$

In other words, the function $V(x)$ takes the same value V_0 at any point $x \in \omega(x_0)$. Now, pick any initial condition $\bar{x}_0 \in \omega(x_0)$. Since the latter is invariant, we have $x(t,\bar{x}_0) \in \omega(x_0)$ for all $t \in \mathbb{R}$. Thus, along this particular motion, $V(x(t,\bar{x}_0)) = V_0$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} V(x(t,\bar{x}_0)) = \frac{\partial V}{\partial x} f(x)\Big|_{x=x(t,\bar{x}_0)}.$$

This, implies

$$x(t,\bar{x}_0) \in \mathscr{E}, \qquad \text{for all } t \in \mathbb{R}$$

and, since $\bar{x}_0$ is any point in $\omega(x_0)$, proves the Theorem. ◁

This Theorem is often used to determine the asymptotic properties of the integral curves of (B.18). In fact, in view of Lemma B.5, it is seen that if a function $V(x)$ can be found such that (B.20) holds, any trajectory of (B.18) is bounded and converges, asymptotically, to an invariant set that is entirely contained in the set $\mathscr{E}$ defined by (B.21). In particular, if system (B.18) has an equilibrium at $x = 0$ and it can be determined that, in the set $\mathscr{E}$, the only possible invariant set is the point $x = 0$, then the equilibrium in question is globally asymptotically stable.

Example B.6. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1(1 + x_1 x_2),\end{aligned}$$

pick $V(x) = x_1^2 + x_2^2$, and observe that

$$\frac{\partial V}{\partial x} f(x) = -2(x_1 x_2)^2 \,.$$

The function on the right-hand side is not negative definite, but it is negative semi-definite, i.e. satisfies (B.20). Thus, trajectories converge to bounded sets that are invariant and contained in the set

$$\mathscr{E} = \{x \in \mathbb{R}^2 : x_1 x_2 = 0\} \,.$$

Now, it is easy to see that no invariant set may exist, other than the equilibrium, entirely contained in the set $\mathscr{E}$. In fact, if a trajectory of the system is contained in $\mathscr{E}$ for all $t \in \mathbb{R}$, this trajectory must be a solution of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \,. \end{aligned}$$

This system, an harmonic oscillator, has only one trajectory entirely contained in $\mathscr{E}$, the trivial trajectory $x(t) = 0$. Thus, the equilibrium point $x = 0$ is the only possible invariant set contained in $\mathscr{E}$ and therefore this equilibrium is globally asymptotically stable. $\triangleleft$

Returning to the analysis of the properties of limit sets, let $B \subset \mathbb{R}^n$ be a fixed bounded set and suppose that *all* motions with initial condition $x_0 \in B$ are bounded in positive time. Since any motion $x(t, x_0)$ asymptotically approaches the limit set $\omega(x_0)$ as $t \to \infty$, it seems reasonable to look at the set

$$\Omega = \bigcup_{x_0 \in B} \omega(x_0) \,, \tag{B.22}$$

as to a "target" set that is asymptotically approached by motions of (B.18) with initial conditions in B. However, while it is true that the distance of $x(t, x_0)$ from the set (B.22) tends to 0 as $t \to \infty$ for any x_0, the convergence to such set may fail be *uniform* in x_0, even if the set B is compact. In this respect, recall that – by definition – the distance of $x(t, x_0)$ from a set S tends to 0 as $t \to \infty$ if for every ε there exists T such that

$$\mathrm{dist}(x(t, x_0), S) \le \varepsilon \,, \qquad \text{for all } t \ge T \,. \tag{B.23}$$

The number T in this expression depends on ε but also on x_0. [9] The distance of $x(t, x_0)$ from S is said to tend to 0, as $t \to \infty$, *uniformly* in x_0 on B, if for every ε there exists T, which depends on ε *but not* on x_0, such that (B.23) holds for all $x_0 \in B$.

Example B.7. Consider again the example B.5, in which the set Ω defined by (B.22) consists of the union of the equilibrium point $\{0\}$ and of the limit cycle $\mathscr{L}$ and let

[9] In fact it is likely that, the more is x_0 distant from S, the longer one has to wait until $x(t, x_0)$ becomes ε-distant from S.

B be a compact set satisfying $B \supset \mathscr{L}$. All $x_0 \in B$ are such that $\mathrm{dist}(x(t,x_0),\Omega) \to 0$ as $t \to \infty$. However, the convergence is not uniform in x_0. In fact, observe that, if $x_0 \neq 0$ is inside $\mathscr{L}$, the motion $x(t,x_0)$ is bounded in negative time and remains inside $\mathscr{L}$ for all $t \leq 0$ (as a matter of fact, it converges to 0 as $t \to -\infty$). Pick any $x_1 \neq 0$ inside $\mathscr{L}$ such that $\mathrm{dist}(x_1,\mathscr{L}) > \varepsilon$ and let T_1 be the minimal time needed to have $\mathrm{dist}(x(t,x_1),\mathscr{L}) \leq \varepsilon$ for all $t \geq T_1$. Let $T_0 > 0$ be fixed and define $x_0 = x(-T_0,x_1)$. If T_0 is large, x_0 is close to 0, and the minimal time T needed to have $\mathrm{dist}(x(t,x_0),\Omega) \leq \varepsilon$ for all $t \geq T$ is $T = T_0 + T_1$. Since the time T_0 can be taken arbitrarily large, it follows that the time $T > 0$ needed to have $\mathrm{dist}(x(t,x_0),\Omega) \leq \varepsilon$ for all $t \geq T$ can be made arbitrarily large, even if x_0 is taken within a compact set. $\triangleleft$

Uniform convergence to the target set is important for various reason. On one side, for practical purposes it is important to have a fixed bound on the time needed to get within an ε-distance of that set. On another side, uniform convergence plays a relevant role in the existence Lyapunov functions, an indispensable tool in analysis and design of feedback systems. While convergence to the set (B.22) is not guaranteed to be uniform, there is a larger set – though – for which such property holds.

Definition B.4. Let B be a bounded subset of $\mathbb{R}^n$ and suppose $x(t,x_0)$ is defined for all $t \geq 0$ and all $x_0 \in B$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k,t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k,x_k) = x\,.$$

It is clear from the definition that, if B consists of only one single point x_0, all x_k's in the definition above are necessarily equal to x_0 and the definition in question returns the definition of ω-limit set of a point. It is also clear that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, with all the x_k's being taken equal to x_0. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B)\,.$$

However, the converse inclusion is not true in general.

Example B.8. Consider again the system in the example B.5, and let B be a compact set satisfying $B \supset \mathscr{L}$. We know that $\{0\}$ and $\mathscr{L}$, being ω-limit sets of points of B, are in $\omega(B)$. But it is also easy to see that any other point inside $\mathscr{L}$ is a point of $\omega(B)$. In fact, let $\bar{x}$ be any of such points and pick any sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$. Since $x(t,\bar{x})$ remains inside $\mathscr{L}$ (and hence in B) for all negative values of t, it is seen that $x_k := x(-t_k,\bar{x})$ is a point in B for all k. The sequence $\{x_k,t_k\}$ is such that $x(t_k,x_k) = \bar{x}$ and therefore the property required for $\bar{x}$ to be in $\omega(B)$ is trivially satisfied. This shows that $\omega(B)$ includes not just $\{0\}$ and $\mathscr{L}$, but also all points of the open region surrounded by $\mathscr{L}$. $\triangleleft$

The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma B.5, can be summarized as follows.[10]

Lemma B.6. *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t,x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Moreover, the distance of $x(t,x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

Thus, as it is the case for the ω-limit set of a point, the ω-limit set *of a bounded set B* is compact and invariant. Being invariant, the set $\omega(B)$ is filled with motions which exist for all $t \in (-\infty, +\infty)$ and all such motions, since this set is compact, are bounded in positive and in negative time. Moreover, this set is uniformly approached by motions with initial conditions $x_0 \in B$. We conclude the section with another property, that will be used later to define the concept of *steady-state behavior* of a system. [11]

Lemma B.7. *If B is a compact set invariant for (B.18), then $\omega(B) = B$.*

B.5 Limit sets and stability

It is well known that, in a nonlinear system, an equilibrium point which attracts all motions with initial conditions in some open neighborhood of this point is not necessarily stable in the sense of Lyapunov. A classical example showing that convergence to an equilibrium does not imply stability is provided by the following 2-dimensional system. [12]

Example B.9. Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(x_1,x_2) \\ g(x_1,x_2) \end{pmatrix} \tag{B.24}$$

in which $f(0,0) = g(0,0) = 0$ and

$$\begin{pmatrix} f(x_1,x_2) \\ g(x_1,x_2) \end{pmatrix} = \frac{1}{(x_1^2+x_2^2)(1+(x_1^2+x_2^2)^2)} \begin{pmatrix} x_1^2(x_2-x_1)+x_2^5 \\ x_2^2(x_2-2x_1) \end{pmatrix}$$

for $(x_1,x_2) \neq (0,0)$. The phase portrait of this system is the one depicted in fig. B.4. This system has only one equilibrium at $(x,y) = (0,0)$ and any initial condition $(x_1(0),x_2(0))$ in the plane produces a motion that asymptotically tends to this point. However, it is not possible to find, for every $\varepsilon > 0$, a number $\delta > 0$ such that every initial condition in a disc of radius δ produces a motion which remains in a disc of radius ε for all $t \geq 0$. $\triangleleft$

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[10] For a proof see, e.g. , [3], [7] and [8].

[11] For a proof, see [17].

[12] See [9] and [2, p. 191-194].

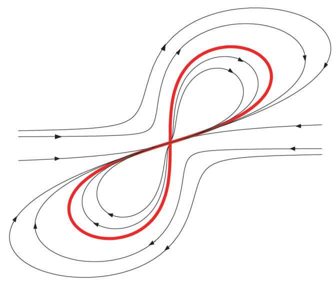

Fig. B.4 The phase portrait of system (B.24).

It is also known – though – that if the convergence to the equilibrium is *uniform*, then the equilibrium in question *is stable*, in the sense of Lyapunov. This property is a consequence of the fact that $x(t,x_0)$ depends continuously on x_0 (see for example [2, p. 181]).

We have seen before that bounded motions of (B.18) with initial conditions in a bounded set B asymptotically approach the compact invariant set $\omega(B)$. Thus, the question naturally arises to determine whether or not this set is also stable in the sense of Lyapunov. In this respect, we recall that the notion of *asymptotic stability of a closed invariant set* $\mathscr{A}$ is defined as follows. The set $\mathscr{A}$ is asymptotically stable if:

(i) for every $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\mathrm{dist}(x_0,\mathscr{A}) \le \delta \qquad \Rightarrow \qquad \mathrm{dist}(x(t,x_0),\mathscr{A}) \le \varepsilon \quad \text{for all } t \ge 0.$$

(ii) there exists a number $d > 0$ such that

$$\mathrm{dist}(x_0,\mathscr{A}) \le d \qquad \Rightarrow \qquad \lim_{t\to\infty} \mathrm{dist}(x(t,x_0),\mathscr{A}) = 0\,.$$

It is not difficult to show (see [12] or [15]) that if the set $\mathscr{A}$ is also bounded and hence compact, and the convergence in (ii) is *uniform* in x_0, then property (ii) implies property (i). This yields the following important property of the set $\omega(B)$.

Lemma B.8. *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t,x_0)\| \le M$ for all $t \ge 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Suppose also that $\omega(B)$ is contained in the interior of B. Then, $\omega(B)$ is asymptotically stable, with a domain of attraction that contains B.*

B.6 The steady state behavior of a nonlinear system

We use the concepts introduced in the previous section to define a notion of *steady state* for a nonlinear system.

Definition B.5. Consider system (B.18) with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose that X is positively invariant. The motions of this system are said to be *ultimately bounded* if there is a bounded subset $B \subset X$ with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \geq T$ and all $x_0 \in X_0$.

Motions with initial conditions in a set B having the property indicated in the previous definition are indeed bounded and hence it makes sense to consider the limit set $\omega(B)$, which – according to Lemma B.6 – is nonempty and has all the properties indicated in that Lemma. What it is more interesting, though, is that – while a set B having the property indicated in the previous definition is clearly not unique – the set $\omega(B)$ is a unique well-defined set.

Lemma B.9. [13] *Let the motions of (B.18) be ultimately bounded and let B' be any other bounded subset of X with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B'$ for all $t \geq T$ and all $x_0 \in X_0$. Then, $\omega(B) = \omega(B')$.*

It is seen from this that, in any system whose motions are ultimately bounded, all motions asymptotically converge to a well-defined compact invariant set, which is filled with trajectories that are bounded in positive and negative time. This motivates the following definition.

Definition B.6. Suppose the motions of system (B.18), with initial conditions in a closed and positively invariant set X, are ultimately bounded. A *steady state motion* is any motion with initial condition in $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady state locus* of (B.18) and the restriction of (B.18) to $\omega(B)$ is the *steady state behavior* of (B.18). $\triangleleft$

This definition characterizes the steady state *behavior* of a nonlinear *autonomous* system, such as system (B.18). It can be used to characterized the steady state *response* of a *forced* nonlinear system

$$\dot{z} = f(z, u) \tag{B.25}$$

so long as the input u can be seen as the output of an *autonomous* "input generator"

$$\begin{aligned} \dot{w} &= s(w) \\ u &= q(w) \,. \end{aligned} \tag{B.26}$$

[13] See [17] for a proof.

In this way, the concept of steady state response (to specific classes of inputs) can be extended to nonlinear systems.

The idea of seeing the steady state response of a forced system as a particular response of an augmented autonomous system has been already exploited in section A.5, in the analysis of the steady state response of a stable linear system to harmonic inputs. In the present setting, the results of such analysis can be recast as follows. Let (B.25) be a stable linear system, written as

$$\dot{z} = Az + Bu\,, \tag{B.27}$$

in which $z \in \mathbb{R}^n$, and let (B.26) be the "input generator" defined in (A.26). The composition of (B.25) and (B.26) is the autonomous linear system (compare with (A.28))

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{B.28}$$

Pick a set W_c defined as

$$W_c = \{w \in \mathbb{R}^2 : \|w\| \le c\}$$

in which c is a fixed number, and consider the set $X = W_c \times \mathbb{R}^n$. The set X is a closed set, positively invariant for the motion of (B.28). Moreover, since the lower subsystem of (B.28) is a linear asymptotically stable system driven by a bounded input, the motions of system (B.28), with initial conditions taken in X, are ultimately bounded. In fact, let Π be the solution of the Sylvester equation (A.29) and recall that the difference $z(t) - \Pi w(t)$ tends to zero as $t \to \infty$. Then, any bounded set B of the form

$$B = \{(w,z) \in W_c \times \mathbb{R}^n : \|z - \Pi w\| \le d\}$$

in which d is any positive number, has the property requested in the definition of ultimate boundedness. It is easy to check that

$$\omega(B) = \{(w,z) \in W_c \times \mathbb{R}^n : z = \Pi w\}\,,$$

that is, $\omega(B)$ is the graph of the restriction of the linear map $x = \Pi w$ to the set W_c. The set $\omega(B)$ is invariant for (B.28), and the restriction of (B.28) to the set $\omega(B)$ characterizes the steady state response of (B.27) to harmonic inputs of fixed angular frequency ω, and amplitude not exceeding c.

A totally similar result holds if the input generator is a nonlinear system of the form (B.26), whose initial conditions are chosen in a *compact invariant* set W. The fact that W is invariant for the dynamics of (B.26) implies, as a consequence of Lemma B.8, that the steady state locus of (B.26) is the set W itself, i.e. that the input generator is "in steady state". [14] The composition of (B.26) and (B.27) yields an augmented system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= Az + Bq(w)\,, \end{aligned} \tag{B.29}$$

[14] Note that the set W_c considered in the previous example had exactly this property.

in which $(w,z) \in X := W \times \mathbb{R}^n$. Note that, since W is invariant for (B.26), the set X is invariant for (B.29).

Since the inputs generated by (B.26) are bounded and the lower subsystem of (B.29) is input-to-state stable, the motions of system (B.29), with initial conditions taken in X, are ultimately bounded. In fact, since W is compact and invariant, there exists a number U such that $\|q(w(t))\| \leq U$ for all $t \in \mathbb{R}$ and all $w(0) \in W$. Therefore, standard arguments can be invoked to deduce the existence of positive numbers K, λ and M such that

$$\|z(t)\| \leq Ke^{-\lambda t}\|z(0)\| + MU$$

for all $t \geq 0$. From this, it is immediate to check that any bounded set B of the form

$$B = \{(w,z) \in W \times \mathbb{R}^n : \|z\| \leq (1+d)MU\}$$

in which d is any positive number, has the property requested in the definition of ultimate boundedness. This being the case, it can be shown that the steady state locus of (B.29) is the graph of the (nonlinear) map [15]

$$\begin{aligned} \pi : W &\to \mathbb{R}^n \\ w &\mapsto \pi(w) = \int_{\infty}^{0} e^{-A\tau} Bq(\bar{w}(\tau, w))d\tau \,, \end{aligned} \tag{B.30}$$

i.e. that

$$\omega(B) = \{(w,z) \in W \times \mathbb{R}^n : z = \pi(w)\}\,,$$

To check that this is the case, observe first of all that – since $q(\bar{w}(t,w))$ is by hypothesis a bounded function of t and all eigenvalues of A have negative real part – the integral on the right-hand side of (B.30) is finite for every $w \in W$. Then, observe that the graph of the map $z = \pi(w)$ is invariant for (B.29). In fact, pick an initial state for (B.29) on the graph of this map, i.e. a pair (w_0, z_0) satisfying $z_0 = \pi(w_0)$ and compute the solution $z(t)$ of the lower equation of (B.29), via the classical variation of constants formula, to obtain

$$\begin{aligned} z(t) &= e^{At}\int_{-\infty}^{0} e^{-A\tau} Bq(\bar{w}(\tau, w_0))d\tau + \int_{0}^{t} e^{A(t-\tau)} Bq(\bar{w}(\tau, w_0))d\tau \\ &= \int_{\infty}^{0} e^{-A\theta} Bq(\bar{w}(\theta + t, w_0))d\theta = \int_{\infty}^{0} e^{-A\theta} Bq(\bar{w}(\theta, w(t, w_0)))d\theta\,. \end{aligned}$$

This shows that $z(t) = \pi(w(t))$ and proves the invariance of the graph of $\pi(\cdot)$ for (B.29). Since the graph of $\pi(\cdot)$ is a compact set invariant for (B.29), this set is necessarily a subset of the steady state locus of (B.29). Finally, observe that, since the eigenvalues of A have negative real part, all motions of (B.29) whose initial conditions are not on the graph of $\pi(\cdot)$ are unbounded in negative time and therefore cannot be contained in the steady state locus, which by definition is a bounded in-

[15] In the following formula, $\bar{w}(t,w)$ denotes the integral curve of $\dot{w} = s(w)$ passing through w at time $t = 0$. Note that, as a consequence of the fact that W is closed and invariant, $\bar{w}(t,w)$ is defined for all $(t,w) \in \mathbb{R} \times W$.

variant set. Thus, the only points in the steady state locus are precisely the points of the graph of $\pi(\cdot)$.

This result shows that the steady state response of a stable linear system to an input generated by a nonlinear system of the form (B.26), with initial conditions $w(0)$ taken in a compact invariant set W, can be expressed in the form

$$z_{\mathrm{ss}}(t) = \pi(w(t))$$

in which $\pi(\cdot)$ is the map defined in (B.30).

Remark B.2. Note that the motions of the autonomous input generator (B.26) are not necessarily periodic motions, as it was the case for the input generator (A.26). For instance, the system in question could be a stable Van der Pol oscillator, with W defined as the set of all points inside and on the boundary of the limit cycle. In this case, it is possible to think of the steady state response of (B.27) not just as of the (single) periodic input obtained when the initial condition of (B.26) is taken on the limit cycle, but also as of all (non periodic) inputs obtained when the initial condition is taken in the interior of W. $\triangleleft$

Consider now the case of a general nonlinear system of the form (B.25), in which $z \in \mathbb{R}^n$, with input u supplied by a nonlinear input generator of the form (B.26). Suppose that system (B.25) is input-to-state stable and that the initial conditions of the input generator are taken in compact invariant set W. It is easy to see that the motions of the augmented system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, q(w))\,, \end{aligned} \tag{B.31}$$

with initial conditions in the set $X = W \times \mathbb{R}^n$, are ultimately bounded. In fact, since W is a compact set, there exists a number $U > 0$ such that

$$\|u(\cdot)\|_\infty = \|q(w(\cdot))\|_\infty \le U$$

for all $w(0) \in W$. Since (B.25) is input-to-state stable, there exist a class $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that

$$\|z(t)\| \le \beta(\|z(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \le \beta(\|z(0)\|, t) + \gamma(U)$$

for all $t \ge 0$. Since $\beta(\cdot,\cdot)$ is a class $\mathscr{KL}$ function, for any compact set Z and any number $d > 0$, there exists a time T such that $\beta(\|z(0)\|, t) \le d\gamma(U)$ for all $z(0) \in Z$ and all $t \ge T$. Thus, it follows that the set

$$B = \{(w,z) \in W \times \mathbb{R}^n : \|z\| \le (1+d)\gamma(U)\}$$

has the property requested in the definition of ultimate boundedness.

Since the motions of the augmented system (B.31) are ultimately bounded, its steady state locus $\omega(B)$ is well-defined. As a matter of fact, it is possible to prove that also in this case the set in question is the *graph of a map* defined on W.

Lemma B.10. *Consider a system of the form (B.31) with $(w,z) \in W \times \mathbb{R}^n$. Suppose its motions are ultimately bounded. If W is a compact set invariant for $\dot{w} = s(w)$, the steady state locus of (B.31) is the graph of a (possibly set-valued) map defined on W.*

Proof. Since W is compact and invariant for (B.26), $\omega(W) = W$. As a consequence, for all $\bar{w} \in W$ there is a sequence $\{w_k, t_k\}$ with w_k in W for all k such that $\bar{w} = \lim_{k\to\infty} w(t_k, w_k)$. Set $x = \mathrm{col}(w,z)$ and let $x(t,x_0)$ denote the integral curve of (B.31) passing through x_0 at time $t = 0$. Pick any point $z_0 \in \mathbb{R}^n$ and let $x_k = \mathrm{col}(w_k, z_0)$. All such x_k's are in a compact set. Hence, by definition of ultimate boundedness, there is a bounded set B and a integer $k^* > 0$ such that $x(t_{k^*+\ell}, x_k) \in B$ for all $\ell \geq 0$ and all k. Set $\bar{x}_\ell = x(t_{k^*}, x_\ell)$ and $\tau_\ell = t_{k^*+\ell} - t_{k^*}$, for $\ell \geq 0$, and observe that, by construction, $x(\tau_\ell, \bar{x}_\ell) = x(t_{k^*+\ell}, x_\ell)$, which shows that all $x(\tau_\ell, \bar{x}_\ell)$'s are in B, a bounded set. Hence, there exists a subsequence $\{x(\tau_h, \bar{x}_h)\}$ converging to a point $\hat{x} = \mathrm{col}(\hat{w}, \hat{z})$, which is a point of $\omega(B)$ because all $\bar{x}_h$'s are in B. Since system (B.31) is upper triangular, necessarily $\hat{w} = \bar{w}$. This shows that, for any point $\bar{w} \in W$, there is at least one point $\hat{z} \in \mathbb{R}^n$ such that $(\bar{w}, \hat{z}) \in \omega(B)$. $\triangleleft$

It should be stressed that the map whose graph characterizes the steady state locus of (B.31) may fail to be single-valued and, also, may fail to be continuously differentiable, as shown in the examples below.

Example B.10. Consider the system

$$\dot{z} = -z^3 + zu, \tag{B.32}$$

which is input-to-state stable, with input u provided by the input generator

$$\begin{aligned} \dot{w} &= 0 \\ u &= w \end{aligned}$$

for which we take $W = \{w \in \mathbb{R} : |w| \leq 1\}$. Thus, $u(t) = w(t) = w(0) := w_0$. If $w_0 \leq 0$, system (B.32) has a globally asymptotically stable equilibrium at $z = 0$. If $w_0 > 0$, system (B.32) has one unstable equilibrium at $z = 0$ and two locally asymptotically stable equilibria at $z = \pm\sqrt{w_0}$. For every fixed $w_0 > 0$, trajectories of (B.32) with initial conditions satisfying $|z_0| > \sqrt{w_0}$ asymptotically converge to either one of the two asymptotically stable equilibria, while the compact set

$$\{(w,z); w = w_0, |z| \leq \sqrt{w_0}\}$$

is invariant. As a consequence, the steady state locus of the augmented system

$$\begin{aligned} \dot{z} &= -z^3 + zw \\ \dot{w} &= 0 \end{aligned}$$

is the graph of the set-valued map

$$\pi \,:\, w \in W \quad \mapsto \quad \pi(w) \subset \mathbb{R}$$

defined as

$$\begin{aligned} -1 \le w \le 0 \quad &\Rightarrow \quad \pi(w) = \{0\} \\ 0 < w \le 1 \quad &\Rightarrow \quad \pi(w) = \{z \in \mathbb{R} : |z| \le \sqrt{w}\}. \qquad \triangleleft \end{aligned}$$

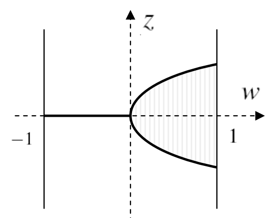

Fig. B.5 The steady state locus of system (B.32).

Example B.11. Consider the system

$$\dot{z} = -z^3 + u,$$

which is input to state stable, with input u provided by the harmonic oscillator

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 \\ u &= w_1 \end{aligned}$$

for which we take $W = \{w \in \mathbb{R}^2 : \|w\| \le 1\}$. It can be shown [16] that, for each $w(0) \in W$, there is one and only one value $z(0) \in \mathbb{R}$ from which the motion of the resulting augmented system (B.31) is bounded both in positive and negative time. The set of all such pairs identifies a single-valued map $\pi : W \to \mathbb{R}$, whose graph characterizes the steady state locus of the system. The map in question is continuously differentiable at any nonzero w, but it is only continuous at $w = 0$. $\triangleleft$

If the map whose graph characterizes the steady state locus of (B.31) *is single-valued*, the steady state response of an input-to-state stable system of the form (B.25) to an input generated by a system of the form (B.26) can be expressed as

$$z_{ss}(t) = \pi(w(t)),$$

in which $\pi(\cdot)$ is a map defined on W. In general, it is not easy to give explicit expressions of this map (such as the one considered earlier in (B.30)). However, if $\pi(\cdot)$ is continuosly differentiable, a very expressive implicit characterization is possible. In fact, recall that the steady state locus of (B.31) is by definition an invariant set,

[16] See [16].

i.e. $z(t) = \pi(w(t))$ for all $t \in \mathbb{R}$ along any trajectory of (B.31) with initial condition satisfying $z(0) = \pi(w(0))$. Along all such trajectories,

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = f(z(t), q(w(t))) = f(\pi(w(t)), q(w(t))\,.$$

If $\pi(w)$ is continuously differentiable, then

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\partial \pi}{\partial w}\Big|_{w=w(t)} \frac{\mathrm{d}w(t)}{\mathrm{d}t} = \frac{\partial \pi}{\partial w}\Big|_{w=w(t)} s(w(t))$$

and hence it is seen that $\pi(w)$ satisfies the partial differential equation

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), q(w)) \qquad \text{for all } w \in W. \tag{B.33}$$

References

1. G.D. Birkhoff , *Dynamical Systems*, American Mathematical Society, 1927.
2. W. Hahn, *Stability of Motion*, Springer Verlag, 1967.
3. N.P. Bhatia , G.P. Szego, *Stability Theory of Dynamical Systems*, Springer Verlag, Berlin, 1970.
4. T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Springer-Verlag, New York, 1975.
5. A. Isidori, *Nonlinear Control Systems II*, Springer-Verlag, New York, 1999.
6. H. Khalil, *Nonlinear Systems* (3rd ed.), Prentice Hall, 2002.
7. J.K. Hale, L.T. Magalhães , W.M. Oliva, *Dynamics in Infinite Dimensions*, Springer Verlag(New York, NY), 2002.
8. G.R. Sell, Y. You, *Dynamics of Evolutionary Equations*, Springer Verlag(New York, NY), 2002.
9. R. E. Vinograd, The inadequacy of the method of characteristic exponents for the study of nonlinear differential equations, *Mat. Sbornik*, **41**, pp. 431–438, 1957 (in Russian).
10. E.D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Trans. on Autom. Control*, **34**, pp. 435–443, 1989.
11. E.D. Sontag, Further facts about input to state stabilization, *IEEE Trans. on Autom. Control*, **35**, pp. 473–476, 1990.
12. E. D. Sontag, Y. Wang, On the characterizations of input-to-state stability with respect to compact sets, *IEEE Conf. on Decision and Control*, pp. 226–231, 1995.
13. Y. Lin, E.D. Sontag, Y. Wang, A smooth converse Lyapunov theorem for robust stability, *SIAM J. Contr. Optimiz.*, **34**, 124–160, 1996.
14. E.D. Sontag and Yang Y., New characterizations of input-to-state stability, *IEEE Trans. on Autom. Control*, **41**, pp. 1283–1294, 1996.
15. F. Celani, *Omega-limit sets of nonlinear systems that are semiglobally practically stabilized*, Doctoral Dissertation, Washington University in St. Louis, 2003.
16. C.I. Byrnes, D.S. Gilliam, A. Isidori, J. Ramsey, On the steady-state behavior of forced nonlinear systems, in *New Trends in Nonlinear Dynamics and Control, and their Applications*, W.Kang, M. Xiao and C. Borges Eds., Springer Verlag (London), pp. 119–144, 2003.
17. A. Isidori, C.I. Byrnes, Steady-state behaviors in nonlinear systems, with an application to robust disturbance rejection, *Annual Reviews in Control*, **32**, pp. 1–16, 2008.

NORMAL FORMS AND FEEDBACK STABILIZATION

1 Relative degree and local normal forms

The purpose of this Section is to show how single-input single-output nonlinear systems can be locally given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which several important properties can be elucidated.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally described in the following way. The single-input single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

is said to have *relative degree* r at a point x° if [1]

(i) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r-1$

(ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

Note that there may be points where a relative degree cannot be defined. This occurs, in fact, when the first function of the sequence

$$L_g h(x), L_g L_f h(x), \ldots, L_g L_f^k h(x), \ldots$$

which is not identically zero (in a neighborhood of x°) has a zero exactly at the point $x = x^\circ$. However, the set of points where a relative degree can be defined is clearly an open and dense subset of the set U where the system (1) is defined.

Remark. In order to compare the notion thus introduced with a familiar concept, let us calculate the relative degree of a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,. \end{aligned}$$

In this case, since $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$, it easily seen that

$$L_f^k h(x) = CA^k x$$

and therefore

$$L_g L_f^k h(x) = CA^k B\,.$$

[1] Let λ be real-valued function and f an n-vector-valued vector, both defined on a subset U of $\mathbb{R}^n$. The function $L_f\lambda$ is the real-valued function defined as

$$L_f\lambda(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x) := \frac{\partial \lambda}{\partial x} f(x)\,.$$

This function is sometimes called *derivative of λ along f*.

Thus, the integer r is characterized by the conditions

$$\begin{aligned} CA^k B &= 0 \qquad \text{for all } k < r-1 \\ CA^{r-1} B &\neq 0 \, . \end{aligned}$$

It is well-known that the integer satisfying these conditions is exactly equal to the *difference* between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function

$$T(s) = C(sI - A)^{-1} B$$

of the system. $\triangleleft$

We illustrate now a simple interpretation of the notion of relative degree, which is not restricted to the assumption of linearity considered in the previous Remark. Assume the system at some time t° is in the state $x(t^\circ) = x^\circ$ and suppose we wish to calculate the value of the output $y(t)$ and of its derivatives with respect to time $y^{(k)}(t)$, for $k = 1, 2, \ldots$, at $t = t^\circ$. We obtain

$$\begin{aligned} y(t^\circ) &= h(x(t^\circ)) = h(x^\circ) \\ y^{(1)}(t) &= \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &= L_f h(x(t)) + L_g h(x(t))u(t) \, . \end{aligned}$$

If the relative degree r is larger than 1, for all t such that $x(t)$ is near x°, i.e. for all t near t°, we have $L_g h(x(t)) = 0$ and therefore

$$y^{(1)}(t) = L_f h(x(t)) \, .$$

This yields

$$\begin{aligned} y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{dx}{dt} &= \frac{\partial L_f h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &= L_f^2 h(x(t)) + L_g L_f h(x(t))u(t) \, . \end{aligned}$$

Again, if the relative degree is larger than 2, for all t near t° we have $L_g L_f h(x(t)) = 0$ and

$$y^{(2)}(t) = L_f^2 h(x(t)) \, .$$

Continuing in this way, we get

$$\begin{aligned} y^{(k)}(t) &= L_f^k h(x(t)) \qquad \text{for all } k < r \text{ and all } t \text{ near } t^\circ \\ y^{(r)}(t^\circ) &= L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ) u(t^\circ) \, . \end{aligned}$$

Thus, the relative degree r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t^\circ$ in order to have the value $u(t^\circ)$ of the input explicitly appearing.

Note also that if

$$L_g L_f^k h(x) = 0 \qquad \text{for all } x \text{ in a neighborhood of } x^\circ \text{ and all } k \geq 0$$

(in which case no relative degree can be defined at any point around x°) then the output of the system is not affected by the input, for all t near t°. As a matter of fact, if this is the case, the previous calculations show that the Taylor series expansion of $y(t)$ at the point $t = t^\circ$ has the form

$$y(t) = \sum_{k=0}^{\infty} L_f^k h(x^\circ) \frac{(t - t^\circ)^k}{k!}$$

i.e. that $y(t)$ is a function depending only on the initial state and not on the input.

These calculations suggest that the functions $h(x)$, $L_f h(x), \ldots, L_f^{r-1} h(x)$ must have a special importance. As a matter of fact, it is possible to show that they can be used in order to define, at least partially, a local coordinates transformation around x°(recall that x° is a point where $L_g L_f^{r-1} h(x^\circ) \neq 0$). This fact is based on the following property.

Lemma 1 *The row vectors* [2]

$$dh(x^\circ), dL_f h(x^\circ), \ldots, dL_f^{r-1} h(x^\circ)$$

are linearly independent.

Lemma 1 shows that necessarily $r \leq n$ and that the r functions $h(x), L_f h(x), \ldots, L_f^{r-1} h(x)$ qualify as a partial set of new coordinate functions around the point x°. As we shall see in a moment, the choice of these new coordinates entails a particularly simple structure for the equations describing the system. However, before doing this, it is convenient to summarize the results discussed so far in a formal statement, that also illustrates a way in which the set of new coordinates can be completed in case the relative degree r is strictly less than n.

Proposition 1 *Suppose the system has relative degree r at x°. Then $r \leq n$. If r is strictly less than n, it is always possible to find $n - r$ more functions $\psi_1(x), \ldots, \psi_{r-r}(x)$ such that the mapping*

$$\Phi(x) = \begin{pmatrix} \psi_1(x) \\ \cdots \\ \psi_{n-r}(x) \\ h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

has a jacobian matrix which is nonsingular at x° and therefore qualifies as a local coordinates transformation in a neighborhood of x°. The value at x° of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\psi_1(x), \ldots, \psi_{n-r}(x)$ in such a way that

$$L_g \psi_i(x) = 0 \qquad \text{for all } 1 \leq i \leq n - r \text{ and all } x \text{ around } x^\circ.$$

[2]Let λ be a real-valued function defined on a subset U of $\mathbb{R}^n$. Its *differential*, denote $d\lambda(x)$ is the row vector

$$d\lambda(x) = \begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \cdots & \frac{\partial \lambda}{\partial x_n} \end{pmatrix} := \frac{\partial \lambda}{\partial x}.$$

The description of the system in the new coordinates is found very easily. Set

$$z = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \cdots \\ \psi_{n-r}(x) \end{pmatrix}, \quad \xi = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

and

$$\tilde{x} = (z, \xi) \,.$$

Looking at the calculations already carried out at the beginning, we obtain for $\xi_1, \ldots, \xi_r$

$$\begin{aligned} \frac{d\xi_1}{dt} &= \frac{\partial h}{\partial x}\frac{dx}{dt} = L_f h(x(t)) = \xi_2(t) \\ &\cdots \\ \frac{d\xi_{r-1}}{dt} &= \frac{\partial (L_f^{r-2}h)}{\partial x}\frac{dx}{dt} = L_f^{r-1}h(x(t)) = \xi_r(t) \,. \end{aligned}$$

For ξ_r we obtain

$$\frac{d\xi_r}{dt} = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t).$$

On the right-hand side of this equation we must now replace $x(t)$ with its expression as a function of $\tilde{x}(t)$, which will be written as $x(t) = \Phi^{-1}(z(t), \xi(t))$. Thus, setting

$$\begin{aligned} q(z,\xi) &= L_f^r h(\Phi^{-1}(z,\xi)) \\ b(z,\xi) &= L_g L_f^{r-1} h(\Phi^{-1}(z,\xi)) \end{aligned}$$

the equation in question can be rewritten as

$$\frac{d\xi_r}{dt} = q(z(t),\xi(t)) + b(z(t),\xi(t))u(t) \,.$$

Note that at the point $(z^\circ, \xi^\circ) = \Phi(x^\circ)$, $b(z^\circ, \xi^\circ) \neq 0$ by definition. Thus, the coefficient $a(z,\xi)$ is nonzero for all (z,ξ) in a neighborhood of (z°, ξ°).

As far as the other new coordinates are concerned, we cannot expect any special structure for the corresponding equations, if nothing else has been specified. However, if $\psi_1(x), \ldots, \psi_{n-r}(x)$ have been chosen in such a way that $L_g \psi_i(x) = 0$, then

$$\frac{dz_i}{dt} = \frac{\partial \psi_i}{\partial x}(f(x(t)) + g(x(t))u(t)) = L_f\psi_i(x(t)) + L_g\psi_i(x(t))u(t) = L_f\psi_i(x(t)).$$

Setting

$$f_0(z,\xi) = \begin{pmatrix} L_f\psi_1(\Phi^{-1}(z,\xi)) \\ \cdots \\ L_f\psi_{n-r}(\Phi^{-1}(z,\xi)) \end{pmatrix}$$

the latter can be rewritten as

$$\frac{dz}{dt} = f_0(z(t),\xi(t)) \,.$$

Thus, in summary, the state-space description of the system in the new coordinates will be as follows

$$\begin{aligned} \dot{z} &= f_0(z,\xi) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z,\xi) + b(z,\xi)u \;. \end{aligned} \tag{2}$$

In addition to these equations one has to specify how the output of the system is related to the new state variables. But, being $y = h(x)$, it is immediately seen that

$$y = \xi_1 \;. \tag{3}$$

The equations thus defined are said to be in *normal form.* They are useful in understanding how certain control problems can be solved. The equations in question can be given a compact expression if we introduce three matrices $\hat{A} \in \mathbb{R}^r \times \mathbb{R}^r$, $\hat{B} \in \mathbb{R}^r \times \mathbb{R}$ and $\hat{C} \in \mathbb{R} \times \mathbb{R}^r$ defined as follows

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{C} = (1 \quad 0 \quad 0 \quad \cdots \quad 0) \;.$$

In fact, it is readily seen that, with the aid of these notations, the equations (2) and (2) can be re-written as

$$\begin{aligned} \dot{z} &= f_0(z,\xi) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)u] \\ y &= \hat{C}\xi \;. \end{aligned} \tag{4}$$

2 Global Normal Forms

We address, in this section, the problem of deriving the global version of the coordinates transformation and normal form introduced in section 1. Consider a single-input single-output system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{5}$$

in which $f(x)$ and $g(x)$ are smooth vector fields, and $h(x)$ is a smooth function, defined on $\mathbb{R}^n$. Assume, as usual, that $f(0) = 0$ and $h(0) = 0$. This system is said to have *uniform* relative degree r if it has relative degree r at each $x^\circ \in \mathbb{R}^n$.

If system (5) has uniform relative degree r, the r differentials

$$dh(x), dL_f h(x), \ldots, dL_f^{r-1} h(x)$$

are linearly independent at each $x \in \mathbb{R}^n$ and therefore the set

$$Z^* = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \ldots = L_f^{r-1} h(x) = 0\}$$

(which is nonempty in view of the hypothesis that $f(0) = 0$ and $h(0) = 0$) is a smooth embedded submanifold of $\mathbb{R}^n$, of dimension $n - r$. In particular, each connected component of Z^* is a maximal integral manifold of the (nonsingular and involutive) distribution

$$\Delta^* = (\mathrm{span}\{dh, dL_f h, \ldots, dL_f^{r-1} h\})^\perp \, .$$

The submanifold Z^* is the point of departure for the construction a globally defined version of the coordinates transformation considered in section 1.

Proposition 2 *Suppose (5) has uniform relative degree r. Set*

$$\alpha(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \qquad \beta(x) = \frac{1}{L_g L_f^{r-1} h(x)}$$

and consider the (globally defined) vector fields

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \qquad \tilde{g}(x) = g(x)\beta(x) \, .$$

Suppose the vector fields

$$\tau_i = (-1)^{i-1} ad_{\tilde{f}}^{i-1} \tilde{g}(x), \qquad 1 \le i \le r \tag{6}$$

are complete.

Then Z^* is connected. Moreover, the smooth mapping

$$\begin{array}{rccl} \Phi & : & Z^* \times \mathbb{R}^r & \to & \mathbb{R}^n \\ & & (z, (\xi_1, \ldots, \xi_r)) & \mapsto & \Phi_{\xi_r}^{\tau_1} \circ \Phi_{\xi_{r-1}}^{\tau_2} \circ \cdots \circ \Phi_{\xi_1}^{\tau_r}(z) \, , \end{array} \tag{7}$$

in which – as usual – $\Phi_t^\tau(x)$ denotes the flow of the vector field τ, has a globally defined smooth inverse

$$(z, (\xi_1, \ldots, \xi_r)) = \Phi^{-1}(x) \tag{8}$$

in which

$$\begin{array}{rcl} z & = & \Phi_{-h(x)}^{\tau_r} \circ \cdots \circ \Phi_{-L_{\tilde{f}}^{r-2} h(x)}^{\tau_2} \circ \Phi_{-L_{\tilde{f}}^{r-1} h(x)}^{\tau_1}(x) \\ \xi_i & = & L_{\tilde{f}}^{i-1} h(x) \qquad 1 \le i \le r \, . \end{array}$$

The globally defined diffeomorphism (8) changes system (5) into a system described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & f_0(z, \xi_1, \ldots, \xi_r) \\ \dot{\xi}_1 & = & \xi_2 \\ & & \cdots \\ \dot{\xi}_{r-1} & = & \xi_r \\ \dot{\xi}_r & = & q(z, \xi_1, \ldots, \xi_r) + b(z, \xi_1, \ldots, \xi_r) u \\ y & = & \xi_1 \end{array} \tag{9}$$

where

$$\begin{aligned} q(z, \xi_1, \ldots, \xi_r) &= L_f^r h \circ \Phi(z, (\xi_1, \ldots, \xi_r)) \\ b(z, \xi_1, \ldots, \xi_r) &= L_g L_f^{r-1} h \circ \Phi(z, (\xi_1, \ldots, \xi_r)) \,. \end{aligned}$$

If, and only if, the vector fields (6) are such that [3]

$$[\tau_i, \tau_j] = 0 \qquad \text{for all } \ 1 \le i, j \le r \,,$$

then the globally defined diffeomorphism (8) changes system (5) into a system described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z, \xi_1, \ldots, \xi_r) + b(z, \xi_1, \ldots, \xi_r) u \\ y &= \xi_1 \,. \end{aligned} \tag{10}$$

Note also that, if $r < n$, the submanifold Z^* is the largest (with respect to inclusion) smooth submanifold of $h^{-1}(0)$ with the property that, at each $x \in Z^*$, there is $u^*(x)$ such that $f^*(x) = f(x) + g(x)u^*(x)$ is tangent to Z^*. Actually, for each $x \in Z^*$ there is only one $u^*(x)$ rendering this condition satisfied, namely,

$$u^*(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \,.$$

The submanifold Z^* is an *invariant* manifold of the (autonomous) system

$$\dot{x} = f^*(x) \tag{11}$$

and the restriction of this system to Z^* can be identified with $(n-r)$-dimensional system

$$\dot{z} = f_0(z, 0, \ldots, 0)$$

of Z^*.

3 The Zero Dynamics

In this section we introduce and discuss an important concept, that in many instances plays a role exactly similar to that of the "zeros" of the transfer function in a linear system. We have already seen that the relative degree r of a linear system can be interpreted as the difference between the number of poles and the number of zeros in the transfer function. In particular,

[3]Let f and g be vector fields on $\mathbb{R}^n$. Their *Lie bracket*, denoted $[f, g]$, is the vector field of $\mathbb{R}^n$ defined as

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \,.$$

any linear system in which r is strictly less than n has zeros in its transfer function. On the contrary, if $r = n$ the transfer function has no zeros; thus, a nonlinear system having relative degree $r = n$ in some sense analogue to a linear systems without zeros. We shall see in this section that this kind of analogy can be pushed much further.

Consider a nonlinear system with r strictly less than n and look at its normal form. Recall that, if x° is such that $f(x^\circ) = 0$ and $h(x^\circ) = 0$, then necessarily the set ξ of the last r new coordinates is 0 at x°. Note also that it is always possible to choose arbitrarily the value at x° of the first $n-r$ new coordinates, thus in particular being 0 at x°. Therefore, without loss of generality, one can assume that $\xi = 0$ and $z = 0$ at x°. Thus, if x° was an equilibrium for the system in the original coordinates, its corresponding point $(z, \xi) = (0, 0)$ is an equilibrium for the system in the new coordinates and from this we deduce that

$$\begin{aligned} q(0,0) &= 0 \\ f_0(0,0) &= 0 \,. \end{aligned}$$

Suppose now we want to analyze the following problem, called the *Problem of Zeroing the Output*. Find, if any, pairs consisting of an initial state x° and of an input function $u^\circ(\cdot)$, defined for all t in a neighborhood of $t = 0$, such that the corresponding output $y(t)$ of the system is identically zero for all t in a neighborhood of $t = 0$. Of course, we are interested in finding *all* such pairs (x°, u°) and not simply in the trivial pair $x^\circ = 0, u^\circ = 0$ (corresponding to the situation in which the system is initially at rest and no input is applied). We perform this analysis on the normal form of the system.

Recalling that in the normal form

$$y(t) = \xi_1(t) \,,$$

we see that the constraint $y(t) = 0$ for all t implies

$$\xi_1(t) = \xi_2(t) = \ldots = \xi_r(t) = 0 \,,$$

that is $\xi(t) = 0$ for all t.

Thus, we see that when the output of the system is identically zero its state is constrained to evolve in such a way that also $\xi(t)$ is identically zero. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = q(z(t), 0) + b(z(t), 0)u(t)$$

(recall that $b(z(t), 0) \neq 0$ if $z(t)$ is close to 0). As far as the variable $z(t)$ is concerned, it is clear that, being $\xi(t)$ identically zero, its behavior is governed by the differential equation

$$\dot{z}(t) = f_0(z(t), 0) \,. \tag{12}$$

From this analysis we deduce the following facts. If the output $y(t)$ has to be zero, then necessarily the initial state of the system must be set to a value such that $\xi(0) = 0$, whereas $z(0) = z^\circ$ can be chosen arbitrarily. According to the value of z°, the input must be set as

$$u(t) = -\frac{q(z(t), 0)}{b(z(t), 0)}$$

where $z(t)$ denotes the solution of the differential equation

$$\dot{z}(t) = f_0(z(t), 0) \qquad \text{with initial condition } z(0) = z^\circ.$$

Note also that for each set of initial data $\xi = 0$ and $z = z^\circ$ the input thus defined is the *unique* input capable to keep $y(t)$ identically zero for all times.

The dynamics of (12) correspond to the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. These dynamics, which are rather important in many of our developments, are called the *zero dynamics* of the system.

Remark. In the case of a linear system, functions $f_0(z, \xi)$ and $q(z, \xi)$ are linear functions, and $b(z, \xi)$ is a constant. The normal form (in the notation (4) can be written as

$$\begin{aligned} \dot{z} &= Fz + G\xi \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[Hz + K\xi + bu] \\ y &= \hat{C}\xi\,. \end{aligned}$$

Using

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

the system can be rewritten as

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu \\ y &= C\tilde{x} \end{aligned}$$

in which

$$A = \begin{pmatrix} F & G \\ \hat{B}H & \bar{A} + \hat{B}K \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 \\ \hat{B}b \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & \hat{C} \end{pmatrix}$$

A simple calculation shows that

$$\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = b \det(sI - F)\,.$$

On the other hand, it is known that the transfer function $T(s) = C(sI - A)^{-1}B$ of the system can be expressed as

$$T(s) = \frac{\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix}}{\det(sI - A)}\,.$$

Hence,

$$T(s) = b\,\frac{\det(sI - F)}{\det(sI - A)}\,,$$

from which it is concluded that in a (controllable and observable) linear system, the zeros of the transfer function $T(s)$ coincide with the eigenvalues of F. In other words, in a linear system the zero dynamics are linear dynamics with eigenvalues coinciding with the zeros of the transfer function of the system. $\triangleleft$

As we will see in the sequel of this Chapter, the asymptotic properties of the system identified by the upper equation in (9) play a role of paramount importance in the design of stabilizing feedback laws. In this context, the properties expressed by the following two definitions are considered.

Definition. Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal. The system is *globally minimum phase* if the equilibriun $z = 0$ of

$$\dot{z} = f_0(z, 0)$$

is globally asymptotically stable.

According to the well-known criterion of Lyapunov, a system is minimum phase if there exists a positive definite and proper smooth real-valued function $V(z)$ satisfying [4]

$$\frac{\partial V}{\partial z} f(z, 0) \leq -\alpha(|z|) \qquad \text{for all } z \in \mathbb{R}^n\,,$$

in which $\alpha(|z|)$ is a class $\mathcal{K}_\infty$ function.

Definition. Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal. The system is *strongly minimum phase* if system

$$\dot{z} = f_0(z, \xi)\,, \tag{13}$$

viewed as a system with input ξ and state z, is input-to-state stable.

By definition – then – a system is strongly minimum phase if there exist a class $\mathcal{K}L$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{L}$ function $\gamma_z(\cdot)$ such that the following property holds: the response $z(t)$ of (13)from the initial state $z(0) = z_0$ to a piecewise-continuous bounded input $\xi_0(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies, for any z_0 and for any such $\xi_0(\cdot)$,

$$|z(t)| \leq \max\{\beta(|z_0|, t), \gamma_z(\|\xi_0(\cdot)\|_\infty)\} \qquad \text{for all } t \geq 0.$$

The function $\gamma_z(\cdot)$ is called a *gain function* of (13).

According to the well-known criterion of Sontag, a system is strongly minimum phase if there exists a positive definite and proper smooth real-valued function $V(z)$ and a class $\mathcal{K}$ function $\chi(\cdot)$ satisfying

$$\frac{\partial V}{\partial z} f(z, 0) \leq -\alpha(|z|) \qquad \text{for all } z \in \mathbb{R}^n\,, \qquad \text{for all } (z, \xi) \text{ such that } |z| \geq \chi(|\xi|)\,,$$

in which $\alpha(|z|)$ is a class $\mathcal{K}_\infty$ function.

[4]To say that a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is proper and positive definite is equivalent to say that there exist class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)\,, \qquad \text{for all } z \in \mathbb{R}^n\,.$$

4 Stabilization via full-state feedback

A typical setting in which normal forms are useful is the derivation of systematic method for stabilization *in the large* of certain classes of nonlinear system, even in the presence of parameter uncertainties. We begin this analysis with the observation that, if the system is *strongly minumum phase*, it is quite easy to design a globally stabilizing *state feedback* law. To be precise, consider again a system in normal form (4), which we assume to be globally defined, and assume that the system is strongly minimum phase, i.e. assume that $f_0(0,0) = 0$ and that

$$\dot{z} = f_0(z, \xi)\,,$$

viewed as a system with input ξ and state z, is input-to-state stable. Assume also that and that the coefficient $b(z, \xi)$ satisfies

$$b(z, \xi) \geq b_0 > 0 \qquad \text{for all } (x, \xi)$$

for some b_0. Consider the feedback law

$$u = \frac{1}{b(z, \xi)}\,(-q(z, \xi) - \hat{K}\xi)\,, \tag{14}$$

in which $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is a vector of design parameters. Under this feedback law, the system becomes

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi} &= (\hat{A} + \hat{B}\hat{K})\xi\,. \end{aligned} \tag{15}$$

Since the pair $(\hat{A}, \hat{B})$ is controllable, it is possible to pick $\hat{K}$ so that the matrix $(\hat{A}+\hat{B}\hat{K})$ is a Hurwitz matrix. If this is the case, system (15) appears as a cascade-connection in which a globally asymptotically stable system (the lower sub-system) drives an input-to-stable system (the upper-system). By known properties, such cascade-connection is globally asymptotically stable. In other words, if $\hat{K}$ is chosen in this way, the feedback law (14) *globally asymptotically* stabilizes the equilibrium $(z, \xi) = (0, 0)$ of the closed-loop system.

The feedback law (14) is expressed in the (z, ξ) coordinates that characterize the normal form (4). To express it in the original coordinates that characterize the model (1), it suffices to bear in mind that

$$b(z, \xi) = L_g L_f^{r-1} h(x)\,, \qquad q(z, \xi) = L_f^r h(x)$$

observe that

$$\hat{K}\xi = \sum_{i=1}^{r} \hat{k}_i L_f^i h(x)$$

in which $\hat{k}_1, \ldots, \hat{k}_r$ are the entries of the row vector $\hat{K}$. Thus, can conclude what follows.

Lemma 2 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal. Suppose the system is strongly minimum phase. If $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is any vector such that $\sigma(\hat{A} + \hat{B}\hat{K}) \in \mathbb{C}^-$, the* state feedback *law*

$$u(x) = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) - \sum_{i=1}^{r} \hat{k}_i L_f^i h(x)\right), \tag{16}$$

globally asymptotically stabilizes the equilibrium $x = 0$.

This feedback strategy, although very intuitive and elementary, is not useful in a practical context because it relies upon exact cancelation of certain nonlinear function and, as such, possibly *non-robust.* Uncertainties in $q(z, \xi)$ and $b(z, \xi)$ would make this strategy unapplicable. Moreover, it relies upon the assumption that the system is *strongly* minimum phase, which is a somewhat stronger assumption. Finally, the implementation of such control law requires the availability, for feedback purposes, of the *full* state (z, ξ) of the system, a condition that might be hard to ensure. Thus, motivated by these considerations, we readdress the problem in what follows, by seeking feedback law depending on fewer measurements (hopefully only on the measurement output y), requiring less a stringent assumption (the property of being simply *minimum-phase*, other than being strongly minimum-phase) and possibly robust with respect to model uncertainties. Of course, in return, some price has to be paid.

We conduct this analysis in the subsequent sections. For the time being we conclude by showing how, in the context of full state feedback, the assumption that the system is *strongly* minimum-phase can be weakened. This is possible, to some extent, if the normal form of the system has the special structure (10). To this end, we use again a feedback of the form (16) but in which now $\hat{K}$ has the following structure

$$\hat{K} = \begin{pmatrix} -a_0 k^r & -a_1 k^{r-1} & \cdots & -a_{r-2} k^2 & -a_{r-1} k \end{pmatrix}$$

where $k > 0$ is a design parameter and the a_i's are such that

$$d(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1 \lambda + a_0$$

is a Hurwitz polynomial. In fact, it is possible to show that, with a feedback law of this kind, it is possible to asymptotically stabilize the equilibrium $x = 0$ of the closed-loop system, with a domain of attraction that includes a fixed (but otherwise arbitrary) compact set.

Lemma 3 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal form having the special structure (10). Suppose the zero dynamics are globally asymptotically stable. Let the control be provided by the state feedback law*

$$u_k(x) = \frac{1}{L_g L_f^{r-1} h(x)} \Big(-L_f^r h(x) - \sum_{i=1}^{r} k^{r+1-i} a_{i-1} L_f^i h(x) \Big). \tag{17}$$

Then, for every choice of a compact set $\mathcal{C}$ there is a number k^* such that, for all $k \geq k^*$ the feedback law (17) asymptotically stabilizes the equilibrium $x = 0$ of the resulting closed-loop system, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$. If, in addition, the zero dynamics are also locally exponentially stable, then the equilibrium $x = 0$ of the resulting closed-loop system is also locally expoentially stable.

Proof. Let the system (5) and the feedback law (17) be expressed in normal form. By assumption, there is a positive definite and proper smooth function $V_0 : \mathbb{R}^{n-r}$ satisfying

$$\begin{aligned} \underline{\alpha}_0(|z|) \leq V_0(z) \leq \overline{\alpha}_0(|z|) \\ \frac{\partial V_0}{\partial z} f_0(z, 0) \leq -\alpha(|z|) \end{aligned} \qquad \text{for all } z \in \mathbb{R}^{n-r}$$

in which $\underline{\alpha}_0(\cdot)$, $\overline{\alpha}_0(\cdot)$ and $\alpha_0(\cdot)$ are class $\mathcal{K}_\infty$ functions. Moreover, by construction, the matrix

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{r-2} & -a_{r-1} \end{pmatrix}.$$

is a Hurwitz matrix and therefore there exists a positive definite symmetric matrix P solution of the Lyapunov equation

$$A^{\mathrm{T}}P + PA = -I\,.$$

Change now the ξ_i's in new variables ζ_i defined as

$$\zeta_i = \frac{1}{k^{i-1}}\xi_i, \qquad 1 \le i \le r$$

The closed loop system, after this transformation of coordinates, is described by equations of the form (recall that the normal form of (5) has the special structure (10)

$$\begin{array}{rcl} \dot{z} & = & f(z, \zeta_1) \\ \dot{\zeta} & = & kA\zeta \end{array} \tag{18}$$

in which

$$\zeta = \mathrm{col}(\zeta_1, \zeta_2, \ldots, \zeta_r)\,.$$

5 Stabilization via partial-state feedback

5.1 Systems having relative degree 1

We discuss, in this section, the case of systems having *relative degree 1*, i.e. we consider systems modeled by equations of the form

$$\begin{array}{rcl} \dot{z} & = & f(z, \xi) \\ \dot{\xi} & = & q(z, \xi) + b(z, \xi)u \\ y & = & \xi_1 \end{array} \tag{19}$$

in which $z \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$. All functions are smooth functions of their arguments. [5]

About this system we assume that

$$\begin{array}{rcl} f(0,0) & = & 0 \\ q(0,0) & = & 0 \end{array}$$

[5] Note that, in order to simplify the notation, we have dropped the subscript "0" from the symbol f_0 that characterizes the right-hand-side of the first equation in the normal form. This is legitimate because we assume that the system is already given in normal form and hence no conflict of notation may occur with the symbol "f" used earlier in (1).

and that the coefficient $b(z, \xi)$ satisfies

$$b(z, \xi) \geq b_0 > 0 \qquad \text{for all } (x, \xi)$$

for some b_0. We also assume that the system is *minimum-phase*, i.e. the equilibrium $z = 0$ of

$$\dot{z} = f_0(z, 0)$$

is globally asymptotically stable.

The system will be controlled the very simple feedback law

$$u = -ky$$

in which $k > 0$, which yields a closed loop system

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= q(z, \xi) - b(z, \xi)k\xi\,. \end{aligned} \tag{20}$$

For convenience, we set

$$x = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

and rewrite the latter as

$$\dot{x} = F_k(x)$$

in which

$$F_k(x) = \begin{pmatrix} f(z, \xi) \\ q(z, \xi) - b(z, \xi)k\xi \end{pmatrix}.$$

Proposition 3 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree 1 and possesses a globally defined normal form. Suppose the system is globally minimum phase. Let the control be provided by the output feedback $u = ky$. Then, for every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a number k^* such that, for all $k \geq k^*$ there is a finite time T such that all trajectories of the closed-loop system with initial condition $x(0) \in \mathcal{C}$ remain bounded and satisfy $|x(t)| \leq \varepsilon$ for all $t \geq T$.*

Proof. Consider, for this system, the candidate Lyapunov function

$$W(x) = V(z) + \frac{1}{2}\xi^2$$

which is positive definite and proper. For any real number $a \geq 0$, let

$$\Omega_a = \{x \in \mathbb{R}^n : W(x) \leq a\}$$

denote the sublevel set consisting of all points of $\mathbb{R}^n$ at which the value of $W(x)$ is less than or equal to a and let

$$B_a = \{x \in \mathbb{R}^n : |x| \leq a\}$$

denotes the closed ball consisting of all points of $\mathbb{R}^n$ whose norm does not exceed a. Since $W(x)$ is proper, the set Ω_a is a compact set for any a. Pick any arbitrarily large number

R, any arbitrarily small number r. Since $W(x)$ is positive definite and proper, there exist numbers numbers $0 < d < c$ such that

$$\Omega_d \subset B_r \subset B_R \subset \Omega_c \,.$$

Consider also the compact "annular" region

$$S_d^c = \{x \in \mathbb{R}^n : d \leq V(x) \leq c\} \,.$$

Our goal is to show that – if the gain coefficient k is large enough – the function

$$\dot{W}(x) := \frac{\partial W}{\partial x} F_k(x)$$

is negative at each point of S_d^c. Observe, in this respect, that

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi) k \xi^2 \,.$$

To this end, we proceed as follows. Consider the compact set

$$Z = S_d^c \bigcap \{x \in \mathbb{R}^n : \xi = 0\} \,.$$

At each point of Z

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,0) \leq -\alpha(|z|)$$

Since $z \neq 0$ at each point of Z, there is a number $a > 0$ such that

$$\dot{W}(x) \leq -a \qquad \forall x \in Z \,.$$

Hence, by continuity, there is an open set Z_ε containing Z such

$$\dot{W}(x) \leq -a/2 \qquad \forall x \in Z_\varepsilon \,. \tag{21}$$

Consider now the set

$$\tilde{S} = \{x \in S_d^c : x \notin Z_\varepsilon\} \,.$$

which is a compact set, let

$$M = \max_{x \in \tilde{S}} \{ \frac{\partial V}{\partial x} f_0(z,\xi) + \xi q(z,\xi) \}$$

$$m = \min_{x \in \tilde{S}} \{ b(z,\xi) \xi^2 \}$$

and observe that $m > 0$ because $b(z,\xi) \geq b_0 > 0$ and ξ cannot vanish at any point of $\tilde{S}$. Thus, since $k > 0$, we obtain

$$\dot{W}(x) \leq M - km \qquad \forall x \in \tilde{S} \,.$$

Let k_1 be such that $M - k_1 m = -a/2$. Then, if $k \geq k_1$,

$$\dot{W}(x) \leq -a/2 \qquad \forall x \in \tilde{S} \,. \tag{22}$$

This, together with (21) shows that

$$k \geq k_1 \qquad \Rightarrow \qquad \dot{W}(x) \leq -a/2 \qquad \forall x \in S_d^c\,,$$

as requested. This being the case, we see that for any trajectory with initial condition in S_d^c, so long as $z(t) \in S_d^c$ we have

$$W(x(t)) \leq W(x(0)) - (a/2)t\,.$$

As a consequence, the trajectory in finite time enters the set Ω_d, and remains there (because $\dot{W}(x)$ is negative on the boundary of this set). $\triangleleft$

This property is commonly known as property of *semi-global practical* stabilizability, of the equilibrium $(z, \xi) = (0, 0)$ of (19).

To obtain *asymptotic* stability, either a nonlinear control law $u = -\kappa(y)$ is needed or, if one insists in using a linear law $u = -ky$, extra assumptions are necessary. Note that in the former case, the closed-loop system becomes (compare with (20))

$$\begin{aligned}\dot{z} &= f(z, \xi)\\ \dot{\xi} &= q(z, \xi) - b(z, \xi)\kappa(\xi)\,.\end{aligned}$$

which will be rewritten as

$$\dot{x} = F_{\kappa(\cdot)}(x)$$

in which

$$F_{\kappa(\cdot)}(x) = \begin{pmatrix} f(z, \xi) \\ q(z, \xi) - b(z, \xi)\kappa(\xi) \end{pmatrix}\,.$$

The following results hold.

Proposition 4 *Consider a system of the form (5), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree 1 and possesses a globally defined normal form. Suppose the zero dynamics are globally asymptotically stable. Then, for every choice of a compact set $\mathcal{C}$ there exists a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that the equilibrium $x = 0$ is asymptotically stabilized by the feedback law $u = -\kappa(y)$, with a domain of attraction $\mathcal{A}$ that includes the set $\mathcal{C}$. If, in addition, the zero dynamics are also locally exponentially stable, then for each compact set $\mathcal{C}$ there is a number k^* such that, for all $k \geq k^*$, the same result holds with $\kappa(y) = -ky$.*

Proof. To prove the second part, we observe that to say that the equilibrium $z = 0$ of $\dot{z} = f(z, 0)$ is locally exponentially stable is to say that the eigenvalues of the matrix

$$F := \frac{\partial f}{\partial z}(0, 0) \tag{23}$$

have negative real part. Linear arguments can be invoked to prove the existence of a number k_2 such that, if $k \geq k_2$, the equilibrium $x = 0$ of (19) is locally asymptotically stable. It is also possible to show that there is a number r' such that, for all $k \geq k_2$, the closed ball $B_{r'}$

is always contained in the domain of attraction of $x = 0$. To check that this is the case, let P be the positive definite solution of

$$PF_0 + F_0^{\mathrm{T}} P = -2I\,,$$

which exists because all eigenvalues of F_0 have negative real part and consider, for the system the candidate quadratic Lyapunov function

$$U(x) = z^{\mathrm{T}} \tilde{P} z + \frac{1}{2}\xi^2$$

yielding

$$\dot{U}(x) = 2z^{\mathrm{T}} P f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi)k\xi^2\,.$$

Expand $f_0(z,\xi)$ as

$$f_0(z,\xi) = F_0 z + g(z) + [f_0(z,\xi) - f_0(z,0)]$$

in which

$$\lim_{z\to 0} \frac{|g(z)|}{|z|} = 0\,.$$

Then, there is a number δ such that,

$$|z| \le \delta \qquad \Rightarrow \qquad |g(z)| \le \frac{1}{2|P|}|z|$$

and this yields

$$2z^{\mathrm{T}} P[F_0 z + g(z)] = -2|z|^2 + 2z^{\mathrm{T}} P g(z) \le -|z|^2 \qquad \forall z \in B_\delta\,.$$

Since the function $[f_0(z,\xi) - f_0(z,0)]$ is a continuously differentiable function that vanish at $\xi = 0$, there is a number M_1 such that

$$|2z^{\mathrm{T}} P[f_0(z,\xi) - f_0(z,0)]| \le M_1|z||\xi| \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \le \delta\,.$$

Likewise, since $q(z,\xi)$ is a continuously differentiable function that vanish at $(z,\xi) = (0,0)$, there are numbers N_1 and N_2 such that

$$|\xi q(z,\xi)| \le N_1|z||\xi| + N_2|\xi|^2 \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \le \delta\,.$$

Finally, since $b(z,\xi)$ is positive and nowhere zero, there is a number b_0 such that (recall that $k > 0$)

$$-kb(z,\xi) \le -kb_0 \qquad \text{for all } (z,\xi) \text{ such that } z \in B_\delta \text{ and } |\xi| \le \delta\,.$$

Putting all these inequalities together, one finds that, for all for all (z,ξ) such that $z \in B_\delta$ and $|\xi| \le \delta$

$$\dot{U}(x) \le -|z|^2 + (M_1 + N_1)|z||\xi| - (kb_0 - N_2)|\xi|^2\,.$$

It is easy to check that if k is such that

$$(2kb_0 - 2N_2 - 1) \ge (M_1 + N_1)^2\,,$$

$$\dot{U}(x) \le -\frac{1}{2}|x|^2\,.$$

This shows that there is a number k_2 such that, if $k \ge k_2$, the function $\dot{U}(x)$ is negative definite for all x satisfying $z \in B_\delta$ and $|\xi| \le \delta$. Pick now any (nontrivial) sublevel set $\tilde{\Omega}_c$ of $U(x)$ entirely contained in the set of all x satisfying $z \in B_\delta$ and $|\xi| \le \delta$ and let r' be such that $B_{r'} \subset \tilde{\Omega}_c$. Then, the argument above shows that, for all $k \ge k_2$, the equilibrium $x = 0$ is asymptotically stable with a domain of attraction that contains $B_{r'}$, which is a set independent of the choice of k.

Pick now $r \le r'$, pick any $R > 0$ and use the result of the Proposition above. There is a number k_1 such that, if $k \ge k_1$ all trajectories with initial condition in B_R in finite time enter the region B_r, and hence enter the region of attraction of $x = 0$. It can be concluded, setting $k^* = \max\{k_1, k_2\}$, that for all $k \ge k^*$ the equilibrium $x = 0$ of (20) is asymptotically stable, with a domain of attraction that contains B_R.

5.2 Systems having relative degree $r > 1$

Consider now the general case of a system having relative degree $r > 1$, which in normal form is expressed as

$$\begin{array}{rcl}
\dot{z} &=& f_0(z, \xi_1, \ldots, \xi_{r-1}, \xi_r)\\
\dot{\xi}_1 &=& \xi_2\\
&& \cdots\\
\dot{\xi}_{r-1} &=& \xi_r\\
\dot{\xi}_r &=& q(z, \xi_1, \ldots, \xi_{r-1}, \xi_r) + b(z, \xi_1, \ldots, \xi_{r-1}, \xi_r)u
\end{array}\,.$$

For this system, choose a control of the form

$$u = -k(\xi_r + d_{r-2}\xi_{r-1} + \cdots + d_1\xi_2 + d_0\xi_1)\,.$$

To analyze the corresponding closed-loop system, we change the variable ξ_r into a variable ζ defined as

$$\theta = d_0\xi_1 + d_1\xi_2 + \cdots + d_{r-2}\xi_{r-1} + \xi_r\,.$$

With this change of coordinates, the closed-loop system can be rewritten in the form

$$\begin{array}{rcl}
\dot{z} &=& f_0(z, \xi_1, \ldots, \xi_{r-1}, -\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta)\\
\dot{\xi}_1 &=& \xi_2\\
&& \cdots\\
\dot{\xi}_{r-2} &=& \xi_{r-1}\\
\dot{\xi}_{r-1} &=& -\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta\\
\dot{\theta} &=& d_0\xi_2 + d_1\xi_3 + \cdots + d_{r-2}(-\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta) + q(z, \xi_1, \ldots, \xi_{r-1}, -\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta)\\
&& \quad + b(z, \xi_1, \ldots, \xi_{r-1}, -\sum_{i=1}^{r-1} d_{i-1}\xi_i + \theta)(-k\theta)
\end{array}\,.$$

This system has a structure which is identical to that of system (20). In fact, if we set

$$\bar{z} = \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}) \in \mathbb{R}^{n-1}$$

and define $\bar{f}_0(\bar{z},\theta)$, $\bar{q}(\bar{z},\theta)$, $\bar{b}(\bar{z},\theta)$ as

$$\bar{f}_0(\bar{z},\theta) = \begin{pmatrix} f_0(z,\xi_1,\ldots,\xi_{r-1},-\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta) \\ \xi_2 \\ \cdots \\ \xi_{r-1} \\ -\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta \end{pmatrix}$$

$$\begin{aligned} \bar{q}(\bar{z},\theta) &= d_0\xi_2+d_1\xi_3+\cdots+d_{r-2}(-\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta)+q(z,\xi_1,\ldots,\xi_{r-1},-\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta) \\ \bar{b}(\bar{z},\theta) &= b(z,\xi_1,\ldots,\xi_{r-1},-\textstyle\sum_{i=1}^{r-1} d_{i-1}\xi_i+\theta)\,, \end{aligned}$$

the system cam be rewritten in the form

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}_0(\bar{z},\theta) \\ \dot{\theta} &= \bar{q}(\bar{z},\theta)-\bar{b}(\bar{z},\theta)k\theta\,, \end{aligned}$$

identical to that of (20).

Thus, identical stability results can be obtained, if $\bar{f}_0(\bar{z},\theta)$, $\bar{q}(\bar{z},\theta)$, $\bar{b}(\bar{z},\theta)$ satisfy conditions corresponding to those assumed on $f_0(z,\xi)$, $q(z,\xi)$, $b(z,\xi)$.

From this viewpoint, it is trivial to check that if the functions $f_0(z,\xi)$, $q(z,\xi)$, and $b(z,\xi)$ satisfy

$$\begin{aligned} f_0(0,0) &= 0 \\ q(0,0) &= 0 \\ b(z,\xi) &\geq b_0>0 \qquad \text{for all } (x,\xi) \end{aligned}$$

then also

$$\begin{aligned} \bar{f}_0(0,0) &= 0 \\ \bar{q}(0,0) &= 0 \\ \bar{b}(\bar{z},\theta) &\geq b_0>0 \qquad \text{for all } (\bar{z},\theta)\,. \end{aligned}$$

Thus, in order to be able to use the stabilization results developed in the previous subsection, it remains to check the input-to-state stability property of the system

$$\dot{\bar{z}} = \bar{f}_0(\bar{z},\theta)\,.$$

This system can be interpreted as a cascade interconnection

$$\begin{aligned} \dot{z} &= f_0(z,C\zeta+D\theta) \\ \dot{\zeta} &= A\theta+B\theta \end{aligned} \tag{24}$$

in which we have used ζ to denote the vector

$$\zeta = \mathrm{col}(\xi_1,\xi_2,\ldots,\xi_{r-1})$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{r-2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ -d_0 & -d_1 & \cdots & -d_{r-3} & -d_{r-2} \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}.$$

If the d_i's are such that the matrix A is Hurwitz, the lower subsystem is input-to-state stable. If the original system is strongly minimum phase, the upper subsystem, viewed as a system with input $C\zeta + D\theta$ and state z is input-to-state stable. Thus, system (24), being the cascade of two input-to-state stable systems, is input-to-state stable.

6 Examples and counterexamples

Finite escape time in a cascade

Practical but not asymptotic stability

NONLINEAR OBSERVERS AND SEPARATION PRINCIPLE

1 The Observability Canonical Form

In this Chapter we discuss the design of observers for nonlinear systems modelled by equations of the form

$$\begin{array}{rcl} \dot{x} & = & f(x,u) \\ y & = & h(x,u) \end{array} \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}$. In particular, we focus our discussion on the design of the so-called *high-gain observers*, which have been thoroughly investigated by Gauthier and Kupca in [**?**].

A key requirement for the existence of observers of this kind is the existence of a global diffeomorphism

$$\begin{array}{rcccl} \Phi & : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & & x & \mapsto & z \end{array}$$

carrying system (1) into a system of the form

$$\begin{array}{rcl} \dot{z}_1 & = & \tilde{f}_1(z_1, z_2, u) \\ \dot{z}_2 & = & \tilde{f}_2(z_1, z_2, z_3, u) \\ & & \cdots \\ \dot{z}_{n-1} & = & \tilde{f}_{n-1}(z_1, z_2, \ldots, z_n, u) \\ \dot{z}_n & = & \tilde{f}_n(z_1, z_2, \ldots, z_n, u) \\ y & = & \tilde{h}(z_1, u) \end{array} \tag{2}$$

in which the $\tilde{h}(z_1, u)$ and $f_i(z_1, z_2, \ldots, z_{i+1}, u)$ satisfy

$$\frac{\partial \tilde{h}}{\partial z_1} \neq 0\,, \quad \text{and} \quad \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \neq 0\,, \quad \text{for all} \quad i = 1, \ldots, n-1 \tag{3}$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$. This form will be referred to as the *Gauthier-Kupca's observability canonical form.*

For the sake of completeness, it is useful to review how the existence of canonical forms of this kind can be checked and the canonical form itself can be constructed. We begin with the description of a set of *necessary* conditions for the existence of this canonical form.

Consider again system (1), suppose that $f(0,0) = 0$, $h(0,0) = 0$, and define – recursively – a sequence of real-valued functions $\varphi_i(x,u)$ as follows

$$\varphi_1(x,u) := h(x,u)\,, \qquad \varphi_i(x,u) := \frac{\partial \varphi_{i-1}}{\partial x} f(x,u)\,.$$

for $i = 1, \ldots, n$. From these functions, define a sequence of i-vector-valued functions $\Phi_i(x, u)$ as follows

$$\Phi_i(x,u) = \begin{pmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{pmatrix}$$

for $i = 1, \ldots, n$. Finally, with each of the $\Phi_i(x, u)$'s, associate the subspace

$$K_i(x,u) = \ker\left[\frac{\partial \Phi_i}{\partial x}\right]_{(x,u)}.$$

Note that the map

$$D_i(u) \quad : \quad x \quad \mapsto \quad K_i(x,u)$$

identifies a *distribution* on $\mathbb{R}^n$. The collection of all these distributions has been called, by Gauthier and Kupca, the *canonical flag* of (1). The notation chosen stresses the fact that the map in question, in general, depends on the parameter $u \in \mathbb{R}^m$. The canonical flag is said to be *uniform* if:

(i) for all $i = 1, \ldots, n$, for all $u \in \mathbb{R}^m$ and for all $x \in \mathbb{R}^n$

$$\dim K_i(x,u) = n - i.$$

(ii) for all $i = 1, \ldots, n$ and for all $x \in \mathbb{R}^n$

$$K_i(x,u) = \text{independent of } u.$$

In other words, condition (i) says that the distribution $D_i(u)$ has constant dimension $n - i$. This condition is referred to as the "regularity" condition. Condition (ii) says that, for each x, the subspace $K_i(x, u)$ is always the same, regardless of what $u \in \mathbb{R}^m$ is. This condition is referred to as the"u-independency" condition.

Proposition 1 *System (1) is globally diffeomorphic to a system in Gauthier-Kupca's observability canonical form only if its canonical flag is uniform.*

Proof. (Sketch of) Suppose a system is already in observability canonical form and compute its canonical flag. A simple calculation shows that the each of the functions $\varphi_i(x, u)$ are functions of the form

$$\tilde{\varphi}_i(z_1, \ldots, z_i, u),$$

and

$$\frac{\partial \tilde{\varphi}_i}{\partial z_i} \neq 0, \qquad \text{for all } z_1, \ldots, z_i, u.$$

Thus, for each $i = 1, \ldots, n$

$$K_i(z,u) = \operatorname{span}\begin{pmatrix} 0 \\ I_{n-i} \end{pmatrix}.$$

This shows that the canonical flag of a system in observability canonical form is uniform. This property is not altered by a diffeomorphism and hence the condition in question is a necessary condition for an observability canonical form to exist. $\triangleleft$

Remark. These necessary condition thus identified is also sufficient for the existence of a *local* diffeomorphism carrying system (1) into a system in observability canonical form (as proven in [**?**], Chapter 3, Theorem 2.1). $\triangleleft$

We describe now a set of *sufficient* conditions for a system to be globally diffeomorphic to a system in Gauthier-Kupca's observability canonical form.

Proposition 2 *Consider the nonlinear system (1) and define a map*

$$\begin{array}{rccl} \Phi & : & \mathbb{R}^n & \rightarrow & \mathbb{R}^n \\ & & x & \mapsto & z = \Phi(x) \end{array}$$

as

$$\Phi(x) = \begin{pmatrix} \varphi_1(x,0) \\ \varphi_2(x,0) \\ \cdots \\ \varphi_n(x,0) \end{pmatrix} .$$

Suppose that:

(i) *the canonical flag of (1) is uniform,*

(i) $\Phi(x)$ *is a global diffeomorphism.*

Then, system (1) is globally diffeomorphic, via $\Phi(x)$*, to a system in Gauthier-Kupca's observability canonical form.*

Proof. By assumption,

$$\ker\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{(x,u)}$$

has constant dimension $n-i$ and it is independent of u. Now denote by $T(z)$ the inverse of the diffeomorphism $\Phi(x)$. Since $\Phi(x) = \Phi_n(x,0)$, $T(z)$ is a globally defined mapping which satisfies

$$\Phi_n(T(z),0) = z\,.$$

Then

$$\Big[\frac{\partial \Phi_n}{\partial x}\Big]_{\substack{x=T(z)\\ u=0}} \frac{\partial T}{\partial z} = I$$

for all $z \in \mathbb{R}^n$. This implies, for all $j > i$,

$$\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{\substack{x=T(z)\\ u=0}} \frac{\partial T}{\partial z_j} = 0\,,$$

or, what is the same,

$$\frac{\partial T}{\partial z_j} \in \ker\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{\substack{x=T(z)\\ u=0}}\,, \qquad \forall z \in \mathbb{R}^n$$

But this, because the $K_i(x,u)$'s are independent of u, implies

$$\frac{\partial T}{\partial z_j} \in \ker\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{\substack{x=T(z)\\ u=u}}\,, \qquad \forall j > i\,, \forall z \in \mathbb{R}^n\,, \forall u \in \mathbb{R}^m\,. \tag{4}$$

Suppose the map $z = \Phi(x)$ is used to change coordinates in (1) and consider the system in the new coordinates

$$\begin{array}{rcl} \dot{z} &=& \tilde{f}(z,u) \\ y &=& \tilde{h}(z,u) \end{array}$$

where

$$\tilde{h}(z,u) = h(T(z),u)\,, \qquad \tilde{f}(z,u) = \left[\frac{\partial \Phi_n}{\partial x}\right]_{\substack{x=T(z)\\u=0}} f(T(z),u)\,.$$

Define

$$\tilde{\varphi}_1(z,u) := \tilde{h}(z,u)\,, \qquad \tilde{\varphi}_i(z,u) := \frac{\partial \tilde{\varphi}_{i-1}}{\partial z}\tilde{f}(z,u) \qquad \text{for } i = 1,\ldots,n.$$

and note that

$$\tilde{\varphi}_1(z,u) = \varphi_1(T(z),u)\,, \qquad \tilde{\varphi}_i(z,u) = \varphi_i(T(z),u)\,,$$

which implies

$$\tilde{\Phi}_i(z,u) := \begin{pmatrix} \tilde{\varphi}_1(z,u) \\ \vdots \\ \tilde{\varphi}_i(z,u) \end{pmatrix} = \Phi_i(T(z),u)\,.$$

Using (4) for $i = 1$, we obtain

$$\frac{\partial \tilde{\varphi}_1(z,u)}{\partial z_j} = \left[\frac{\partial \varphi_1}{\partial x}\right]_{\substack{x=T(z)\\u=u}} \frac{\partial T}{\partial z_j} = 0$$

for all $j > 1$ which means that $\tilde{h}(z,u) = \tilde{\varphi}_1(z,u)$ only depends on z_1. Note also that

$$\frac{\partial \tilde{\varphi}_1}{\partial z} = \begin{pmatrix} \frac{\partial \tilde{\varphi}_1}{\partial z_1} & 0 & \cdots & 0 \end{pmatrix}$$

and hence, by the uniformity assumption,

$$\frac{\partial \tilde{h}}{\partial z_1} = \frac{\partial \tilde{\varphi}_1}{\partial z} \neq 0\,.$$

This being the case,

$$\tilde{\varphi}_2(z,u) = \frac{\partial \tilde{h}}{\partial z_1}\tilde{f}_1(z,u)\,.$$

Use now (4) for $i = 2$, to obtain (in particular)

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z_j} = \left[\frac{\partial \varphi_2}{\partial x}\right]_{\substack{x=T(z)\\u=u}} \frac{\partial T}{\partial z_j} = 0$$

for all $j > 2$ which means that $\tilde{\varphi}_2(z,u)$ only depends on z_1, z_2. Looking at the form of $\tilde{\varphi}_2(z,u)$, we see that

$$\frac{\partial \tilde{f}_1}{\partial z_j} = 0$$

for all $j > 2$ which means that $\tilde{f}_1(z,u)$ only depends on z_1, z_2. Moreover, an easy check shows that

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z} = \begin{pmatrix} * & \frac{\partial \tilde{h}}{\partial z_1}\frac{\partial \tilde{f}_1}{\partial z_2} & 0 & \cdots & 0 \end{pmatrix}$$

and hence

$$\frac{\partial \tilde{f}_1}{\partial z_2} \neq 0$$

because otherwise the uniformity assumption would be contradicted. Continuing in the same way, the result follows. $\triangleleft$

2 The case of input-affine systems

Consider and input-affine system, namely described by equations of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \tag{5}$$

It is easy to check that the functions $\varphi_i(x,u)$ have the following expressions

$$\begin{array}{rcl} \varphi_1(x,u) & = & h(x) \\ \varphi_2(x,u) & = & L_f h(x) + L_g h(x) u \\ \varphi_3(x,u) & = & L_f^2 h(x) + [L_g L_f h(x) + L_f L_g h(x)]u + L_g^2 h(x) u^2 \\ \varphi_4(x,u) & = & L_f^3 h(x) + [L_g L_f^2 h(x) + L_f L_g L_f h(x) + L_f^2 L_g h(x)]u + \\ & & +[L_g^2 L_f h(x) + L_g L_f L_g h(x) + L_f L_g^2 h(x)]u^2 + L_g^3 h(x) u^3 \\ \varphi_5(x,u) & = & \cdots \end{array}$$

Hence

$$\Phi_n(x,0) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{pmatrix} := \Phi(x).$$

If the canonical flag of (5) is uniform and if $\Phi_n(x,0)$ is a global diffeomorphism, the system is transformable into "uniform observability" canonical form. The form in question is

$$\begin{array}{rcl} \dot{z} & = & \tilde{f}(z) + \tilde{g}(z)u \\ y & = & \tilde{h}(z) \end{array}$$

in which

$$\tilde{f}(z) = \Big[\frac{\partial \Phi(x)}{\partial x} f(x)\Big]_{x=\Phi^{-1}(z)}, \qquad \tilde{g}(z) = \Big[\frac{\partial \Phi(x)}{\partial x} g(x)\Big]_{x=\Phi^{-1}(z)}$$
$$\tilde{h}(z) = h(\Phi^{-1}(z))$$

By construction,

$$\tilde{h}(z) = z_1$$

Moreover, looking at the special structure of $\Phi(x)$, it is easy to check that $\tilde{f}(z)$ has the following form

$$\tilde{f}(z) = \begin{pmatrix} z_2 \\ z_3 \\ \cdots \\ z_n \\ \tilde{f}_n(z_1, \ldots, z_n) \end{pmatrix}$$

Finally, since we know that the i-th entry of

$$\tilde{f}(z) + \tilde{g}(z)u$$

can only depend on $z_1, z_2, \dots, z_{i+1}$, we deduce that $\tilde{g}(z)$ must necessarily be of the form

$$\tilde{g}(z) = \begin{pmatrix} \tilde{g}_(z_1, z_2) \\ \tilde{g}_2(z_1, z_2, z_3) \\ \cdots \\ \tilde{g}_{n-1}(z_1, z_2, \dots, z_n) \\ \tilde{g}_n(z_1, z_2, \dots, z_n) \end{pmatrix}$$

However, it is also possible to show that g_i actually is independent of z_{i+1}. We check this for $i = 1$ (the other checks are similar). Observe that

$$\begin{array}{rcl} \tilde{\varphi}_1(z,u) & = & z_1 \\ \tilde{\varphi}_2(z,u) & = & z_2 + \tilde{g}_1(z_1, z_2)u \end{array}$$

Computing Jacobians we obtain

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1(z,u)}{\partial z} \\ \dfrac{\partial \tilde{\varphi}_2(z,u)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \dfrac{\partial \tilde{g}_1}{\partial z_1}u & (1 + \dfrac{\partial \tilde{g}_1}{\partial z_2}u) & 0 & \cdots & 0 \end{pmatrix}.$$

If, at some point (z_1, z_2)

$$\frac{\partial \tilde{g}_1}{\partial z_2} \neq 0$$

it is possible to find u such that

$$1 + \frac{\partial \tilde{g}_1}{\partial z_2}u = 0$$

in which case

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1(z,u)}{\partial z} \\ \dfrac{\partial \tilde{\varphi}_2(z,u)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

At this value of (z,u) the subspace $K_2(z,u)$ has dimension $n-1$ and not $n-2$ as prescribed. Hence the uniformity conditions are violated. We conclude that

$$\frac{\partial \tilde{g}_1}{\partial z_2} = 0$$

for all (z_1, z_2), i.e. that $\tilde{g}_1$ is independent of z_2.

The conclusion is that the "uniform observability" canonical form of an input-affine system has the following structure

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 + \tilde{g}_1(z_1)u \\ \dot{z}_2 & = & z_3 + \tilde{g}_2(z_1, z_2)u \\ & & \cdots \\ \dot{z}_{n-1} & = & z_n + \tilde{g}_{n-1}(z_1, z_2, \dots, z_{n-1})u \\ \dot{z}_n & = & \tilde{f}_n(z_1, \dots, z_n) + \tilde{g}_n(z_1, z_2, \dots, z_n)u \\ y & = & z_1 \end{array}$$

3 High-gain Nonlinear Observers

In this section, we describe how to design a *global* asymptotic state observer for a system in Gauthier-Kupca's observability canonical form. Letting $\mathbf{z}_i$ denote the vector

$$\mathbf{z}_i = \mathrm{col}(z_1, \dots, z_i)$$

the canonical form in question can be rewritten in more concise form as

$$\begin{array}{rcl} \dot{z}_1 & = & f_1(\mathbf{z}_1, z_2, u) \\ \dot{z}_2 & = & f_2(\mathbf{z}_2, z_3, u) \\ & & \cdots \\ \dot{z}_{n-1} & = & f_{n-1}(\mathbf{z}_{n-1}, z_n, u) \\ \dot{z}_n & = & f_n(\mathbf{z}_n, u) \\ y & = & h(z_1, u) \,. \end{array} \tag{6}$$

The construction described below reposes on the follwing two additional technical assumptions:

(i) each of the maps $f_i(\mathbf{z}_i, z_{i+1}, u)$, for $i = 1, \dots, n$, is globally Lipschitz with respect to $\mathbf{z}_i$, uniformly in z_{i+1} and u,

(ii) there exist two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \le \left| \frac{\partial \tilde{h}}{\partial z_1} \right| \le \beta \,, \quad \text{and} \quad \alpha \le \left| \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \right| \le \beta \,, \quad \text{for all} \quad i = 1, \dots, n-1$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$.

Remark. Note that the assumption in question is automatically satisfied if it is known – a priori – that $z(t)$ remains in a compact set Z. $\triangleleft$

The observer for (6) consist in a *copy* of the dynamics of (6) corrected by an *innovation* term proportional to the difference between the output of (6) and the output of the copy. More precisely, the observer in question is a system of the form

$$\begin{array}{rcl} \dot{\hat{z}}_1 & = & f_1(\hat{\mathbf{z}}_1, \hat{z}_2, u) + \kappa c_{n-1}(y - h(\hat{z}_1, u)) \\ \dot{\hat{z}}_2 & = & f_2(\hat{\mathbf{z}}_2, \hat{z}_3, u) + \kappa^2 c_{n-2}(y - h(\hat{z}_1, u)) \\ & & \cdots \\ \dot{\hat{z}}_{n-1} & = & f_{n-1}(\hat{\mathbf{z}}_{n-1}, \hat{z}_n, u) + \kappa^{n-1} c_1(y - h(\hat{z}_1, u)) \\ \dot{\hat{z}}_n & = & f_n(\hat{\mathbf{z}}_n, u) + \kappa^n c_0(y - h(\hat{z}_1, u)) \,, \end{array} \tag{7}$$

in which κ and $c_{n-1}, c_{n-2}, \dots, c_0$ are design parameters.

Letting the "observation error" be defined as

$$e_i = \hat{z}_i - z_i, \qquad i = 1, 2, \dots, n \,,$$

an estimate of this error can be obtained as follows. Observe that, using the mean value theorem, one can write

$$\begin{aligned} f_i(\hat{\mathbf{z}}_i, \hat{z}_{i+1}, u) - f_i(\mathbf{z}_i, z_{i+1}, u) &= f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) \\ &\quad + f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), z_{i+1}(t), u(t)) \\ &= f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) \\ &\quad + \frac{\partial f_i}{\partial z_{i+1}}(\mathbf{z}_i(t), \delta_i(t), u(t))\, e_{i+1}(t) \end{aligned}$$

in which $\delta_i(t)$ is a number in the interval $[\hat{z}_{i+1}(t), z_{i+1}(t)]$. Note also that

$$h(\hat{z}_1, u) - y = h(\hat{z}_1(t), u(t)) - h(z_1(t), u(t)) = \frac{\partial h}{\partial z_1}(\delta_0(t), u(t))\, e_1$$

in which $\delta_0(t)$ is a number in the interval $[\hat{z}_1(t), z_1(t)]$. Setting

$$g_{i+1}(t) = \frac{\partial f_i}{\partial z_{i+1}}(\mathbf{z}_i(t), \delta_i(t), u(t))$$

and

$$g_1(t) = \frac{\partial h}{\partial z_1}(\delta_0(t), u(t))$$

the relation above yields

$$\dot{e}_i = f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t)) + g_{i+1}(t) e_{i+1} - \kappa^i c_{n-i} g_1(t) e_1 \,.$$

The equations thus found can be organized in matrix form as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} -\kappa c_{n-1} g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -\kappa^2 c_{n-2} g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -\kappa^{n-1} c_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -\kappa^n c_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_{n-1} \\ e_n \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ \cdots \\ F_{n-1} \\ F_n \end{pmatrix} \tag{8}$$

in which

$$F_i = f_i(\hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\mathbf{z}_i(t), \hat{z}_{i+1}(t), u(t))$$

for $i = 1, 2, \ldots, n-1$ and

$$F_n = f_n(\hat{\mathbf{z}}_n(t), u(t)) - f_i(\mathbf{z}_i(t), u(t)) \,.$$

Consider now a "rescaled" observation error defined as

$$\tilde{e}_i = \frac{1}{\kappa^i}\, e_i, \qquad i = 1, 2, \ldots, n.$$

Simple calculations show that

$$\begin{pmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \\ \cdots \\ \dot{\tilde{e}}_{n-1} \\ \dot{\tilde{e}}_n \end{pmatrix} = \kappa \begin{pmatrix} -c_{n-1} g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -c_{n-2} g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -c_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \cdots \\ \tilde{e}_{n-1} \\ \tilde{e}_n \end{pmatrix} + \begin{pmatrix} \kappa^{-1} F_1 \\ \kappa^{-2} F_2 \\ \cdots \\ \kappa^{-n+1} F_{n-1} \\ \kappa^{-n} F_n \end{pmatrix}. \tag{9}$$

The right-hand side of this equation consists of a term which is linear in the vector

$$\tilde{e} = \mathrm{col}(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n)$$

and of a nonlinear term. The nonlinear term, though, can be bounded by a quantity which is linear in $\|\tilde{e}\|$. In fact, observe that, because of Assumption (i), there is a number L such that

$$\|F_i\| \leq L\|\hat{\mathbf{z}}_i(t) - \mathbf{z}_i(t)\|\,.$$

Clearly,

$$\|\hat{\mathbf{z}}_i - \mathbf{z}_i\| = \sqrt{e_1^2 + e_2^2 + \ldots + e_i^2} = \sqrt{\kappa\tilde{e}_1^2 + \kappa^2\tilde{e}_2^2 + \ldots + \kappa^i\tilde{e}_i^2}$$

from which we see that, if $\kappa > 1$,

$$\|\kappa^{-i}F_i\| \leq L\sqrt{\tilde{e}_1^2 + \tilde{e}_2^2 + \ldots + \tilde{e}_i^2} \leq L\,\|\tilde{e}\|\,. \tag{10}$$

As far as the properties of the linear part are concerned, the following useful Lemma can be invoked (see [**?**], Chapter 6, Lemma 2.1).

Lemma 1 *Consider a matrix of the form*

$$A(t) = \begin{pmatrix} 0 & g_2(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_n(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} c_{n-1} \\ c_{n-2} \\ \\ \cdots \\ c_1 \\ c_0 \end{pmatrix} \begin{pmatrix} g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and suppose there exists two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \leq g_i(t) \leq \beta \qquad \textit{for all } t \geq 0 \textit{ and } i = 1, 2, \ldots, n\,. \tag{11}$$

Then, there is a set of real numbers $c_0, c_1, \ldots, c_{n-1}$, a real number $\lambda > 0$ and a symmetric positive definite $n \times n$ matrix S, with λ and S only depending on α and β, such that

$$A(t)S + SA^{\mathrm{T}}(t) \leq -\lambda I\,. \tag{12}$$

Note that the hypothesis that the $g_i(t)$'s satisfy (11) is in the present case fulfilled as a straightforward consequence of Assumption (ii). With this result in mind consider, for system (9), a candidate Lyapunov function

$$V(\tilde{e}) = \tilde{e}^{\mathrm{T}}S\tilde{e}\,.$$

Then, we have

$$\dot{V}(\tilde{e}(t)) = \kappa\tilde{e}^{\mathrm{T}}(t)[A(t)S + SA^{\mathrm{T}}(t)]\tilde{e}(t) + 2\tilde{e}^{\mathrm{T}}(t)S\tilde{F}(t)$$

having denoted by $\tilde{F}(t)$ the vector

$$\tilde{F}(t) = \mathrm{col}(\kappa^{-1}F_1, \kappa^{-2}F_2, \ldots, \kappa^{-n}F_n)\,.$$

Using the estimates (10) and (12), it is seen that

$$\dot V(\tilde e(t)) \le -\kappa\lambda \|\tilde e\|^2 + 2\|S\|\, L\sqrt{n}\, \|\tilde e\|^2 \,.$$

Set

$$\kappa^* = \frac{2\|S\|\, L\sqrt{n}}{\lambda}\,.$$

Then, if $\kappa > \kappa^*$, the estimate

$$\dot V(\tilde e(t)) \le -cV(\tilde e(t))$$

holds, for some $c > 0$. As a consequence, by standard results, it follows that

$$\lim_{t\to\infty} \tilde e(t) = 0\,.$$

In summary, the observer (7) asymptotically tracks the state of system (6) if the coefficients $c_0, c_1, \ldots, c_{n-1}$ are such that the property indicated in Lemma 1 holds (which is always possible by virtue of Assumption (ii)) and if the number κ is sufficiently large. This is why the observer in question is called a "high-gain" observer.

4 The Gains of the Nonlinear Observer

In this section, we prove Lemma 1. The result is obviously true in case $n = 1$. In this case, in fact, the matrix $A(t)$ reduces to the scalar quantity

$$A(t) = -c_0 g_1(t)\,.$$

Taking $S = 1$ we need to prove that

$$-2c_0 g_1(t) \le -\lambda$$

for some $\lambda > 0$, which is indeed possible if the sign[1] of c_0 is the same as that of $g_1(t)$.

For $n > 2$ we proceed by induction. The induction hypothesis is that, having defined $A_0(t)$ as

$$A_0(t) = \begin{pmatrix} 0 & g_3(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_4(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_n(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} d_{n-2} \\ d_{n-3} \\ \ldots \\ d_1 \\ d_0 \end{pmatrix} \begin{pmatrix} g_2(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

there is a choice of $d_{n-2}, \ldots, d_0$ and a matrix S_0, which depend only on α and β, such that

$$S_0 A_0(t) + A_0^{\mathrm{T}}(t) S_0 \le -\lambda_0 I$$

for some $\lambda_0 > 0$.

[1]The $g_i(t)$'s are continuous functions that never vanish. Thus, each of them has a well-defined sign.

This being the case, set

$$F_0(t) = \begin{pmatrix} 0 & g_3(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_4(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & g_n(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad C_0 = -\begin{pmatrix} d_{n-2} \\ d_{n-3} \\ \\ \cdots \\ d_1 \\ d_0 \end{pmatrix}$$

$$K = \begin{pmatrix} c_{n-2} \\ c_{n-3} \\ \\ \cdots \\ c_1 \\ c_0 \end{pmatrix} \qquad H_0(t) = (\, g_2(t) \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0\,)$$

and note that

$$A_0(t) = F_0(t) + C_0 H_0(t)$$

and

$$A(t) = \begin{pmatrix} -c_{n-1} g_1(t) & H_0(t) \\ -K g_1(t) & F_0(t) \end{pmatrix}$$

Change variables as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_0 & I_{n-1} \end{pmatrix} z := Tz\,,$$

to obtain

$$\tilde{A}(t) = TA(t)T^{-1} = \begin{pmatrix} -c_{n-1} g_1(t) - H_0(t) C_0 & H_0(t) \\ -(K + C_0 c_{n-1}) g_1(t) - A_0(t) C_0 & A_0(t) \end{pmatrix}$$

In these new coordinates, we look for a matrix $\tilde{S}$ of the form

$$\tilde{S} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & S_0 \end{pmatrix}$$

and we prove that the quadratic form

$$Q(x) = x^{\mathrm{T}} (\tilde{S} \tilde{A}(t) + \tilde{A}(t)^{\mathrm{T}} \tilde{S}) x$$

satisfies the required inequality. Simple calculations show that

$$\begin{aligned} Q(x) \quad = \quad & (-c_{n-1} g_1(t) - H_0(t) C_0) x_1^2 + H_0(t) x_1 x_2 \\ & + 2 x_2^{\mathrm{T}} S_0 [-(K + C_0 c_{n-1}) g_1(t) - A_0(t) C_0] x_1 + 2 x_2^{\mathrm{T}} S_0 A_0(t) x_2 \end{aligned}$$

Choose

$$K = -c_{n-1} C_0$$

and observe that (recall that $\|H_0(t)\| = |g_2(t)|$)

$$\begin{aligned} Q(x) &\leq (-c_{n-1}g_1(t) - H_0(t)C_0)x_1^2 \\ &\quad + (|g_2(t)| + 2\|S_0\|\,\|A_0(t)C_0\|)|x_1|\,\|x_2\| - \lambda_0\|x_2\|^2 \\ &:= (-c_{n-1}g_1(t) - H_0(t)C_0)x_1^2 + \delta(t)|x_1|\,\|x_2\| - \lambda_0\|x_2\|^2\,, \end{aligned}$$

in which we have set

$$\delta(t) = |g_2(t)| + 2\|S_0\|\,\|A_0(t)C_0\|\,.$$

For any $\epsilon > 0$,

$$|x_1|\,\|x_2\| \leq \frac{\epsilon}{2}\|x_2\|^2 + \frac{1}{2\epsilon}|x_1|^2$$

and this yields

$$Q(x) \leq (-c_{n-1}g_1(t) - H_0(t)C_0 + \frac{\delta(t)}{2\epsilon})|x_1|^2 + (-\lambda_0 + \delta(t)\frac{\epsilon}{2})\|x_2\|^2\,.$$

The function $\delta(t)$ is bounded from above, by a number that only depends on α and β (in fact, $|g_2(t)|$ and the entries of $A_0(t)$ are bounded by β, while S_0 and C_0, determined at the earlier stage of the algorithm, only depend on α and β). There exists a (sufficiently small) number ϵ, that only depends on α and β, such that

$$(-\lambda_0 + \delta(t)\frac{\epsilon}{2}) < -\frac{\lambda_0}{2}.$$

Since $g_1(t)$ is bounded from below (and has a fixed sign) and $|H_0(t)C_0|$ is bounded from above (by a number that only depends on α and β), there exists a (sufficiently large, in magnitude) number c_{n-1} (having the same sign as $g_1(t)$) such that

$$(-c_{n-1}g_1(t) - H_0(t)C_0 + \frac{\delta(t)}{2\epsilon}) < -\frac{\lambda_0}{2}.$$

In this way, we obtain

$$Q(x) \leq -\frac{\lambda_0}{2}\|x\|^2\,.$$

Reverting to the original coordinates proves the Lemma. In fact, it suffices to set

$$S = T^{\mathrm{T}}\tilde{S}T$$

to obtain

$$z^{\mathrm{T}}[A^{\mathrm{T}}(t)S + SA(t)]z = x^{\mathrm{T}}[\tilde{A}^{\mathrm{T}}(t)\tilde{S} + \tilde{S}\tilde{A}(t)]x \leq -\frac{\lambda_0}{2}\|x\|^2 = -\frac{\lambda_0}{2}\|Tz\|^2 \leq -\frac{\lambda_0}{2}\gamma\|z\|^2 := -\lambda\|z\|^2$$

in which $\gamma = \lambda_{min}(T^{\mathrm{T}}T)$.

5 A Nonlinear Separation Principle

In this section, we show how the theory of nonlinear observers discussed earlier to the purpose of achieving asymptotic stability via dynamic output feedback. Consider a nonlinear system in observability canonical form (2), which we rewrite in compact form as

$$\begin{aligned} \dot{z} &= f(z,u) \\ y &= h(z,u)\,, \end{aligned} \tag{13}$$

with $f(0,0) = 0$ and $h(0,0) = 0$ and suppose there exists a feedback law $u = \alpha(z)$, with $\alpha(0) = 0$, such that the equilibrium $z = 0$ of

$$\dot{z} = f(z, \alpha(z))$$

is globally asymptotically stable. For convenience, let the latter system be rewritten as

$$\dot{z} = F(z)\,. \tag{14}$$

Note that, by the inverse Lyapunov theorem, there exists a smooth function $W(z)$, satisfying

$$\underline{\alpha}(|z|) \le W(z) \le \overline{\alpha}(|z|) \qquad \text{for all } z, \tag{15}$$

and

$$\frac{\partial W}{\partial z} F(z) \le -\alpha(|z|) \qquad \text{for all } z, \tag{16}$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$.

Assume, for the time being, that the hypotheses (i) and (ii) of Section 3 hold (we shall see later how these can be removed) and consider an observer of the form (7), which we rewrite in compact form as

$$\dot{\hat{z}} = f(\hat{z}, u) + G(y - h(\hat{z}, u))\,. \tag{17}$$

An obvious choice to achieve asymptotic stability, suggested by the analogy with linear systems, would be to replace z by its estimate $\hat{z}$ in the map $\alpha(z)$. However, this simple choice may prove to be dangerous, for the following reason. Bearing in mind the analysis carried out in Section 3, define

$$D_\kappa = \mathrm{diag}(\kappa, \kappa^2, \ldots, \kappa^n)$$

and observe that

$$D_\kappa \tilde{e} = z - \hat{z}\,.$$

Thus, feeding the system (13) with a control $u = \alpha(\hat{z})$ would result in a system

$$\dot{z} = f(z, \alpha(z - D_\kappa \tilde{e}))\,.$$

which contains the possibly large parameter κ. This, since the system is nonlinear, may cause finite escape times. To avoid this, as a precautionary measure, it is appropriate to "saturate" the control, by choosing instead a law of the form

$$u = \sigma_L(\alpha(\hat{z}))) \tag{18}$$

in which $\sigma(r)$ is any function that coincides with r when $|r| \le L$, is strictly increasing and satisfies $|\sigma(r) \le 2L$ for all $r \in \mathbb{R}$. The consequence of this is that global asymptotic stability is no longer assured. However, as it it will be shown, *semiglobal stability* can still be obtained.

Consider now the aggregate of (13), of (17) and of (18), namely

$$\begin{aligned} \dot{z} &= f(z, \sigma_L(\alpha(\hat{z}))) \\ \dot{\hat{z}} &= f(\hat{z}, \sigma_L(\alpha(\hat{z}))) + G(h(z, \sigma_L(\alpha(\hat{z}))) - h(\hat{z}, \sigma_L(\alpha(\hat{z}))) \,. \end{aligned} \tag{19}$$

Replacing $\hat{z}$ by its expression in terms of $\tilde{e}$, we obtain for the first equation a system that can be written as

$$\begin{aligned} \dot{z} &= f(z, \sigma_L(\alpha(z - D_\kappa \tilde{e}))) \\ &= f(z, \alpha(z)) - f(z, \alpha(z)) + f(z, \sigma_L(\alpha(z - D_\kappa \tilde{e}))) \\ &= F(z) + H(z, \tilde{e}) \end{aligned}$$

in which

$$H(z, \tilde{e}) = f(z, \sigma_L(\alpha(z - D_\kappa \tilde{e}))) - f(z, \alpha(z)) \,.$$

In what follows, we shall show a *semiglobal* stabilizability property, namely the property that, for every compact set $\mathcal{A}$ of initial conditions in the state space, there is a choice of design parameters such that the equilibrium $(z, \hat{z}) = (0, 0)$ of the closed loop system is asymptotically stable, with a domain of attraction that contains $\mathcal{A}$. For simplicity let the set in question be of the form $B_R \times B_R$ in which B_R is a closed ball of radius R in $\mathbb{R}^n$.

The analysis proceeds as follows. Choose a number c such that

$$\Omega_c = \{z : W(z) \le c\} \supset B_R \,,$$

and then choose the parameter L in the definition of $\sigma_L(\cdot)$ as

$$L = \max_{z \in \Omega_{c+1}} \alpha(z) + 1 \,.$$

With this choice, it is easy to realize that there is a number M_0 such that

$$|H(z, \tilde{e})| \le M_0, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } \tilde{e} \in \mathbb{R}^n$$

and a pair of numbers M_1, δ such that

$$|H(z, \tilde{e})| \le M_1 |D_\kappa \tilde{e}|, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } |D_\kappa \tilde{e}| \le \delta \,.$$

These numbers are independent of κ and only depend on number R that characterizes the radius of the ball B_R in which $z(0)$ is taken.

Let $z(0) \in B_R \subset \Omega_c$. Regardless of what $\tilde{e}(t)$ is, so long as $z(t) \in \Omega_{c+1}$, we have

$$\dot{W}(z(t)) = \frac{\partial W}{\partial z}[F(z) + H(z, \tilde{e})] \le -\alpha(|z|) + |\frac{\partial W}{\partial z}| M_0$$

Setting

$$M_2 = \max_{z \in \Omega_{c+1}} |\frac{\partial W}{\partial z}|$$

and

$$M = M_2 M_0$$

we obtain

$$\dot{W}(z(t)) \leq Mt$$

which in turn yields

$$W(t) - W(0) \leq Mt$$

and we deduce that $z(t)$ remains in Ω_{c+1} at least until time $T = 1/M$. This time may be very small but, because of the presence of the saturation function $\sigma_L(\cdot)$, it is independent of κ. It rather only depends on the number R that characterizes the radius of the ball B_R in which $z(0)$ is taken.

Recall now that the variable $\tilde{e}$ satisfies the estimate established in Section 3. Letting $V(\tilde{e})$ denote the quadratic form $V(\tilde{e}) = \tilde{e}^{\mathrm{T}} S \tilde{e}$, we know that

$$\dot{V}(\tilde{e}(t)) \leq -cV(\tilde{e})$$

in which

$$c := \kappa\lambda - 2\|S\| L\sqrt{n}$$

is a number that can be made arbitrarily large by increasing κ (recall that λ and $\|S\|$ only depend of the bounds α and β in assumption (ii) and L on the Lipschitz constant in assumption (i)). From this inequality, bearing in mind the fact that

$$a_1|\tilde{e}|^2 \leq V(\tilde{e}) \leq a_2|\tilde{e}|^2$$

in which a_1, a_2 are numbers depending on S and hence only on α, β, we obtain

$$|\tilde{e}(t)| \leq \sqrt{\frac{a_2}{a_1}}\, e^{-\frac{c}{2}t}\, |\tilde{e}(0)|$$

which is valid actually for all t, so long $z(t)$ exists.

To be able to use the estimates thus obtained, we need to bear in mind that, in the dynamics of z, $\tilde{e}$ is multiplied by D_κ. If, without loss of generality, $\kappa > 1$, we have

$$|D_\kappa \tilde{e}| \leq \kappa^n |\tilde{e}|, \qquad |\tilde{e}(0)| \leq |e(0)| = |z(0) - \hat{z}(0)| \leq 2R\,.$$

Hence we see that

$$|D_\kappa \tilde{e}(t)| \leq \kappa^n \sqrt{\frac{a_2}{a_1}}\, e^{-\frac{c}{2}t}\, 2R\,.$$

For any fixed time, the right-hand side is a polynomial function of κ multiplied by an exponentially decaying function of κ. Thus, bearing in mind the definitions given before of of time T, we see that there for any choice of ε, there is a κ^* such that, for all $\kappa \geq \kappa^*$,

$$|D_\kappa \tilde{e}(t)| \leq \varepsilon\,, \qquad \text{for all } t \geq T.$$

Consider now again the inequality

$$\dot{W}(z(t)) = \frac{\partial W}{\partial z}[F(z) + H(z,\tilde{e})] \leq -\alpha(|z|) + |\frac{\partial W}{\partial z}||H(z,\tilde{e})|\,.$$

We know from the argument above that, if κ is large enough, we can make $|D_\kappa \tilde{e}(t)| \le \delta$ for all $t \ge T$ and hence, so long as $z(t) \in \Omega_{c+1}$, we have

$$\dot{W}(z(t)) \le -\alpha(|z(t)|) + M_2 M_1 \delta \,.$$

Pick now any number $d \ll c$ and consider the "annular" compact set

$$S_d^{c+1} = \{z : d \le W(z) \le c+1\} \,.$$

Let r be

$$r = \min_{z \in S_d^{c+1}} |z|$$

By construction

$$\alpha(|z|) \ge \alpha(r) \qquad \text{for all } z \in S_d^{c+1}.$$

If δ is small enough

$$M_2 M_1 \delta \le \frac{1}{2}\alpha(r) \,,$$

and hence

$$\dot{W}(z(t)) \le -\frac{1}{2}\alpha(r) \,,$$

so long as $z(t) \in S_d^{c+1}$. By standard arguments, this proves that any trajectory which starts in B_R in finite time (which only depends on R) enters the set Ω_d and remain in this set thereafter.

The argument so far have shown that, given any arbitrary number $d \ll c$, there exist a number κ^* such that, if $\kappa \ge \kappa^*$, all trajectories starting in $B_R \times B_R$ are such $z(t)$ enters in finite time the set Ω_d and $\tilde{e}(t)$ decays to zero. To conclude the proof of convergence to the equilibrium, it suffices to argue as follows. We have shown that trajectries with initial conditions in $B_R \times B_R$ are bounded. Thus $\omega(B_R \times B_R)$ exists and, since $\tilde{e}(t)$ decays asymptotically to 0, the set in question is a subset of the set in which $\tilde{e} = 0$. The restriction of the entire system to the set in which $\tilde{e} = 0$ is nothing else than system

$$\dot{z} = F(z)$$

in which $z = 0$ is by assumption a globally asymptotically stable equilibrium. Hence, we conclude that

$$\omega(B_R \times B_R) = \{(0,0)\}$$

and this completes the proof.

It remains to discuss the role of the assumptions (i) and (ii). Having proven that the trajectories of the system starting in $B_R \times B_R$ remain in a bounded region, it suffices to look for numbers α and β and a Lipschitz constant L making assumptions (i) and (ii) valid *only on this bounded region* and these numbers indeed exist (as a simple consequence of the existence of the observability canonical form). Specifically, let c and L be fixed as before and consider a system

$$\begin{array}{rcl} \dot{z} &=& f_c(z,u) \\ y &=& h_c(z,u) \end{array} \qquad (20)$$

with $f_c(z,u)$ and $h_c(z,u)$ satisfying

$$
\begin{array}{rcll}
f_c(z,u) & = & f(z,u), & \forall (z,u) : z \in \Omega_{c+1}, |u| \le 2L \\
h_c(z,u) & = & h(z,u), & \forall (z,u) : z \in \Omega_{c+1}, |u| \le 2L\,.
\end{array}
$$

Also, let $\alpha_c(z)$ be a function satisfying $\alpha_c(z) = \alpha(z)$ for all $z \in \Omega_{c+1}$ and such that the equilibrium $z = 0$ of

$$
\dot{z} = f_c(z, \alpha_c(z))
$$

is globally asymptotically stable.

Since $f_c(z,u)$ and $h_c(z,u)$ are arbitrary outside a compact set, assumptions (i) and (ii) can be fulfilled [for details, see J.P. Gauthier, I. Kupka: *Deterministic Observation Theory and Applications*, Cambridge University Press, 2001]. As a consequence, the previous result show how to find a controller for (20) steering all trajectories of the closed loop system

$$
\begin{array}{rcl}
\dot{z} & = & f_c(z, \sigma_L(\alpha_c(\hat{z}))) \\
\dot{\hat{z}} & = & f_c(\hat{z}, \sigma_L(\alpha_c(\hat{z}))) + G(h_c(z, \sigma_L(\alpha_c(\hat{z}))) - h_c(\hat{z}, \sigma_L(\alpha_c(\hat{z})))
\end{array}
\tag{21}
$$

with initial conditions in $B_R \times B_R$ to the equilibrium $(z, \hat{z}) = (0,0)$. This controller generates an input always satisfying $|u| \le 2L$ and induces in (20) a trajectory which always satisfies $z(t) \in \Omega_{c+1}$. Since (20) and the original plant agree on this set, the controller constructed in this way achieves the same result if used for the original plant.

6 Exercises

Exercise 1. Consider the system

$$
\begin{array}{rcl}
\dot{x}_1 & = & x_3 + (x_1 + x_1^3 + x_2)^3 u \\
\dot{x}_2 & = & x_1 - 3x_1^2 x_3 + (x_2 + x_1^3)u - 3x_1^2(x_1 + x_1^3 + x_2)^3 u \\
\dot{x}_3 & = & (x_2 + x_1^3)^2 \\
y & = & x_2 + x_1^3
\end{array}
$$

We want to see whether or not it is transformable into a system in “uniform observability” canonical form. We compute the $\varphi_i(x,u)$ and obtain

$$
\begin{array}{rcl}
\varphi_1(x,u) & = & x_2 + x_1^3 \\
\varphi_2(x,u) & = & x_1 + (x_2 + x_1^3)u \\
\varphi_3(x,u) & = & x_3 + (x_1 + x_1^3 + x_2)^3 u + (x_1 - 3x_1^2 x_3 + (x_2 + x_1^3)u - 3x_1^2(x_1 + x_1^3 + x_2)^3 u)u \\
& & \quad + 3x_1^2(x_3 + (x_1 + x_1^3 + x_2)^3 u)u
\end{array}
$$

Computing the Jacobians, we obtain

$$
\begin{pmatrix}
\dfrac{\partial \varphi_1(x,u)}{\partial x} \\
\dfrac{\partial \varphi_2(x,u)}{\partial x} \\
\dfrac{\partial \varphi_3(x,u)}{\partial x}
\end{pmatrix}
=
\begin{pmatrix}
3x_1^2 & 1 & 0 \\
1 + 3x_1^2 u & u & 0 \\
* & * & 1
\end{pmatrix}
$$

from which we see that

$$K_1(x,u) = \text{span}\begin{pmatrix} 1 & 0 \\ -3x_1^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad K_2(x,u) = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad K_3(x,u) = \{0\}\,.$$

Hence, the "uniformity conditions" hold:

- $K_1(x,u)$ has constant dimension 2 and is independent of u.
- $K_2(x,u)$ has constant dimension 1 and is independent of u.
- $K_3(x,u)$ has constant dimension 0 and (trivially) is independent of u.

Moreover

$$z = \Phi_n(x,0) = \begin{pmatrix} x_2 + x_1^3 \\ x_1 \\ x_3 \end{pmatrix}$$

is a global diffemorphism. Its inverse is

$$x = \begin{pmatrix} z_2 \\ z_1 - z_2^3 \\ z_3 \end{pmatrix}.$$

In the new coordinates, the system reads as

$$\begin{aligned} \dot z_1 &= z_2 + z_1 u \\ \dot z_2 &= z_3 + (z_2 + z_1)^3 u \\ \dot z_3 &= z_1^2 \\ y &= z_1\,. \end{aligned}$$

Exercise 2. Consider the system

$$\begin{aligned} \dot x_1 &= x_2 + x_1 x_2 u \\ \dot x_2 &= x_3 \\ \dot x_3 &= (x_2 + x_1^3)^2 + u \\ y &= x_1 \end{aligned}$$

This system is not transformable into a system in "uniform observability" canonical form. In fact, we have

$$\begin{aligned} \varphi_1(x,u) &= x_1 \\ \varphi_2(x,u) &= x_2 + x_1 x_2 u \end{aligned}$$

and

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_2 u & 1 + x_1 u & 0 \end{pmatrix}.$$

Hence

$$K_1(x,u) = \text{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$K_2(x,u) = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } 1 + x_1 u \neq 0$$

$$K_2(x,u) = \text{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } 1 + x_1 u = 0$$

and the uniformity conditions are not fulfilled. The transformation is not possible.

Exercise 3. Consider the system

$$\begin{aligned} \dot{z} &= z - z_3 + \xi_1 \\ \dot{\xi}_1 &= \xi_1 + \xi_2 \\ \dot{\xi}_2 &= z + \xi_1^2 + u \\ y &= \xi_1 \end{aligned}$$

This system is a relative degree 2 system in normal form, but its zero dynamics

$$\dot{z} = z - z_3$$

is unstable. Hence, stabilization methods based on high gain feedback from ξ_1, ξ_2 (or estimates of $y, y^{(1)}$) cannot be used.

However, the system is (semiglobally) stabilizable by dynamic output feedback. In fact, the system is stabilizable by *state feedback*, and is transformable into "uniform observability" canonical form. Thus, it can be semiglobally stabilized by dynamic output feedback via the separation principle.

Construction of a stabilizing feedback $u(z, \xi_1, \xi_2)$. The z dynamics can be stabilized by the virtual control $\xi_1^*(z) = -2z$. Then, back-step. Change ξ_1 into $\eta_1 = \xi_1 - \xi_1^*(z) = \xi_1 - 2z$, to obtain

$$\begin{aligned} \dot{z} &= -z - z_3 + \eta_1 \\ \dot{\eta}_1 &= 4z + 2z^3 - \eta_1 + \xi_2 \\ \dot{\xi}_2 &= z + (\eta_1 + 2z)^2 + u \end{aligned}$$

Change again ξ_2 into $\eta_2 = 4z + 2z^3 - \eta_1 + \xi_2$ to obtain a system of the form

$$\begin{aligned} \dot{z} &= -z - z_3 + \eta_1 \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= a(z, \eta_1, \eta_2) + u \end{aligned}$$

Since we are interested only in semiglobal stability, we can obtain this results by means of a simple feedback law of the form

$$u = -\gamma(\eta_2 + k a_0 \eta_1)$$

and k, γ positive and sufficiently large (note that, in the original coordinates, this feedback depends on *all* the state variables z, ξ_1, ξ_2). If we were interested in global stability, we would proceed differently (by standard back-stepping).

Construction of the observer. Note that

$$
\begin{array}{rcl}
\varphi_1(x,u) & = & \xi_1 \\
\varphi_2(x,u) & = & \xi_1 + \xi_2 \\
\varphi_3(x,u) & = & \xi_1 + \xi_2 + z + \xi_1^2 + u
\end{array}
$$

and therefore

$$
\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \\ \dfrac{\partial \varphi_3(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1+2\xi_1 & 1 \end{pmatrix}
$$

Thus, the uniformity conditions are fulfilled and the change of coordinates is

$$
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \Phi_n(x,0) = \begin{pmatrix} \xi_1 \\ \xi_1 + \xi_2 \\ \xi_1 + \xi_1^2 + \xi_2 + z \end{pmatrix}
$$

whose inverse is given by

$$
\begin{pmatrix} z \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} z_3 - z_1 - z_1^2 - z_2 \\ z_1 \\ z_2 - z_1 \end{pmatrix}.
$$

Compute now the canonical form and construct the observer.

ROBUST STABILIZATION BY DYNAMIC OUTPUT FEEDBACK

1 A robust "observer"

Consider now a system having relative degree $r > 1$, written in normal form as

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)u] \\ y &= \hat{C}\xi\,, \end{aligned} \tag{1}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^r$ and $\hat{A}, \hat{B}, \hat{C}$ are matrices defined as in (XXX). About this system we assume that

$$\begin{aligned} f(0,0) &= 0 \\ q(0,0) &= 0 \end{aligned} \tag{2}$$

and that the coefficient $b(z,\xi)$ satisfies

$$0 < b_{\min} \le b(z,\xi) \le b_{\max} \qquad \text{for all } (z,\xi) \tag{3}$$

for some $b_{\min}, b_{\max}$. We assume also that the system is strongly minimum phase, i.e. that

$$\dot{z} = f(z,\xi)\,,$$

viewed as a system with input ξ and state z, is input-to-state stable.

We know that the feedback law

$$u = \frac{1}{b(z,\xi)}\,(-q(z,\xi) + K\xi)\,, \tag{4}$$

if K is such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix, globally asymptotically stabilizes the equilibrium $(z,\xi) = (0,0)$ of the resulting closed-loop system. However, the implementation of this law requires accurate knowledge of $b(z,\xi)$ and $q(z,\xi)$ and availability of the full state (z,ξ). We see in what follows how a suitable "asymptotic proxy" of this law can be designed, which does not suffer such limitations.

The idea is to use the measured output y to drive an appropriate dynamical system to the purpose of estimating the components of ξ as well as to overcome the necessity of knowing the functions $b(z,\xi)$ and $q(z,\xi)$. To this end, let $\psi(\xi,\sigma)$ be the function defined as

$$\psi(\xi,\sigma) = b_0^{-1}[K\xi - \sigma]\,,$$

in which $\xi \in \mathbb{R}^r$, $\sigma \in \mathbb{R}$, b_0 is a design parameter and K a vector with the properties indicated above (i.e. such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix), and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth "saturation" function, characterized as follows: $g(s) = s$ if $|s| \le L$, $g(s)$ is odd and monotonically increasing, with $0 < g'(s) \le 1$, and $\lim_{s\to\infty} g(s) = L(1+c)$ with $0 < c \ll 1$. The "saturation level" L, which in order to simplify the notation is not explicitly indicated the in the symbol used to denote the function in question, is a design parameter that will be determined later.

System (1) will be controlled by a control law of the form

$$u = g(\psi(\hat{\xi}, \sigma)) \tag{5}$$

in which $\hat{\xi} \in \mathbb{R}^r$ and σ are states of the dynamical system [1]

$$\begin{array}{rcl}
\dot{\hat{\xi}}_1 & = & \hat{\xi}_2 + \kappa\alpha_1(y - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 & = & \hat{\xi}_3 + \kappa^2\alpha_2(y - \hat{\xi}_1) \\
 & & \cdots \\
\dot{\hat{\xi}}_{r-1} & = & \hat{\xi}_r + \kappa^{r-1}\alpha_{r-1}(y - \hat{\xi}_1) \\
\dot{\hat{\xi}}_r & = & \sigma + b_0 g(\psi(\hat{\xi}, \sigma)) + \kappa^r\alpha_r(y - \hat{\xi}_1) \\
\dot{\sigma} & = & \kappa^{r+1}\alpha_{r+1}(y - \hat{\xi}_1)\,.
\end{array} \tag{6}$$

The coefficients κ and $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are design parameters.

The dynamical system thus defined has the typical structure of an "observer". In the analysis of the asymptotic properties of the resulting closed-loop system, it is convenient to replace $\hat{\xi}_1, \ldots, \hat{\xi}_r, \sigma$ by means of (scaled) "error" variables, defined as follows

$$\begin{array}{rcl}
e_1 & = & \kappa^r(\xi_1 - \hat{\xi}_1) \\
e_2 & = & \kappa^{r-1}(\xi_2 - \hat{\xi}_2) \\
 & & \cdots \\
e_r & = & \kappa(\xi_r - \hat{\xi}_r) \\
e_{r+1} & = & q(z, \xi) + [b(z, \xi) - b_0]g(\psi(\xi, \sigma)) - \sigma\,.
\end{array} \tag{7}$$

The first r of these relations can be trivially inverted, to recover each $\hat{\xi}_i$, as function of e_i and ξ_i. To recover σ form the latter, we need to choose b_0 appropriately. To this end, bearing in mind the expression of $\psi(\xi, \sigma)$, observe that the relation in question is equivalent to the following one

$$\frac{K\xi - q(z, \xi) + e_{r+1}}{b(z, \xi)} = \frac{b_0}{b(z, \xi)}\Big[\Big(\frac{b(z, \xi) - b_0}{b_0}\Big)g(\psi(\xi, \sigma)) + \psi(\xi, \sigma)\Big]. \tag{8}$$

If we set

$$\psi^*(z, \xi, e_{r+1}) = \frac{K\xi - q(z, \xi) + e_{r+1}}{b(z, \xi)}$$

and we define a function $F : \mathbb{R} \to \mathbb{R}$ as

$$F(s) = \frac{b_0}{b(z, \xi)}\Big[\Big(\frac{b(z, \xi) - b_0}{b_0}\Big)g(s) + s\Big] \tag{9}$$

the relation (8) can be simply rewritten as

$$\psi^* = F(\psi)\,.$$

[1] In the expression that follows we use $\hat{\xi}$ to denote the vector $\mathrm{col}(\hat{\xi}_1, \hat{\xi}_2, \ldots \hat{\xi}_r)$.

Since $b(z,\xi)$, by assumption, is bounded as in (3), it is always possible to pick b_0 so as to make

$$\left|\frac{b(z,\xi)-b_0}{b_0}\right| \leq \delta_0 < 1 \tag{10}$$

for some δ_0. Thus, since $0 < g'(s) \leq 1$ by hypothesis, if b_0 is chosen in this way, $F'(s)$ is strictly positive, i.e. $F(s)$ is a strictly increasing (odd) function. Moreover, since $\lim_{s\to\infty} g(s) = L(1+c)$, it is seen that $\lim_{s\to\infty} F(s) = \infty$, and consequently $F(\mathbb{R}) = \mathbb{R}$. In summary, $F(s)$ is globally invertible. It is also worth noting that, so long as $|s| \leq L$, the function $F(s)$ is an identity, i.e. $F(s) = s$.

Hence, if b_0 is chosen to satisfy (10), we have

$$\psi = F^{-1}(\psi^*)$$

and this – bearing in mind the expressions of ψ and ψ^* – shows that σ can always be recovered, from the last of (7), as a smooth function of z, ξ, e_{r+1}. This makes the change (7) a diffeomorphism.

We return now to the change of variables (7). The expressions that will be derived involve functions of $z, \xi, \hat{\xi}, \sigma$. In order to simplify the notations we find it prop1, whenever appropriate, to replace the pair of variables (z,ξ) with

$$x = \mathrm{col}(z,\xi)$$

to let e denote the vector

$$e = \mathrm{col}(e_1, e_2, \ldots, e_{r+1})\,,$$

and to bear in mind that $\hat{\xi}$ and σ are functions of (x,e).

It is readily seen that $e_1, \ldots, e_{r-1}$ satisfy the following identities

$$\begin{aligned}
\dot{e}_1 &= \kappa(e_2 - \alpha_1 e_1)\\
\dot{e}_2 &= \kappa(e_3 - \alpha_2 e_1)\\
&\cdots\\
\dot{e}_{r-1} &= \kappa(e_r - \alpha_{r-1} e_1)\,.
\end{aligned}$$

Moreover

$$\begin{aligned}
\dot{e}_r &= \kappa[q(z,\xi) + b(z,\xi)g(\psi(\hat{\xi},\sigma)) - \sigma - b_0 g(\psi(\hat{\xi},\sigma)) - \kappa^r\alpha_r(y - \hat{\xi}_1)]\\
&= \kappa(e_{r+1} - \alpha_r e_1 + [b(z,\xi) - b_0][g(\psi(\hat{\xi},\sigma)) - g(\psi(\xi,\sigma))]\\
&:= \kappa(e_{r+1} - \alpha_r e_1) + \Delta_1\,,
\end{aligned}$$

in which we have set

$$\Delta_1(x,e) = \kappa[b(z,\xi) - b_0][g(\psi(\hat{\xi},\sigma) - g(\psi(\xi,\sigma)]\,.$$

In this respect, it is worth observing that $\Delta_1(x,e)$ is a smooth function that vanishes when $e_1 = e_2 = \cdots = e_r = 0$, because in that case $\hat{\xi} = \xi$, and is globally bounded independently of κ (provided that $\kappa > 1$). In fact, $[b(z,\xi) - b_0]$ is globally bounded by assumption. Moreover $g(\cdot)$ is globally Lipschitz and hence from some $\hat{L}$ we have

$$\begin{aligned}
|g(\psi(\hat{\xi},\sigma)) - g(\psi(\xi,\sigma))| &\leq \hat{L}|\psi(\hat{\xi},\sigma) - \psi(\xi,\sigma)| \leq b_0^{-1}|K||\hat{\xi} - \xi|\\
&\leq b_0^{-1}|K|\left(\sum_{i=1}^{r}(\kappa^{-r-1+i}e_i)^2\right)^{1/2} \leq b_0^{-1}|K|\kappa^{-1}\left(\sum_{i=1}^{r}(e_i)^2\right)^{1/2},
\end{aligned}$$

in the last of which we have used the fact that $\kappa > 1$. In summary, we have

$$|\Delta_1(x, e)| \le \delta_1 |e|, \tag{11}$$

for some $\delta_1 > 0$.

Finally, we obtain for the time derivative of e_{r+1} an expression of the form

$$\dot{e}_{r+1} = -\kappa\alpha_{r+1}e_1 + \dot{q} + \dot{b}g(\psi(\xi, \sigma)) + [b - b_0]g'(\psi(\xi, \sigma))b_0^{-1}[K\dot{\xi} - \dot{\sigma}]$$

in which, for convenience, we have omitted the indication of some arguments. This expression can be further elaborated as follows. Bearing in mind the dynamics of σ, we can write

$$[b(z, \xi) - b_0]g'(\psi(\xi, \sigma))b_0^{-1}\dot{\sigma} = [b(z, \xi) - b_0]g'(\psi(\xi, \sigma))b_0^{-1}\kappa\alpha_{r+1}e_1 := \Delta_0\kappa\alpha_{r+1}e_1$$

in which

$$\Delta_0(x, e) = [b(z, \xi) - b_0]g'(\psi(\xi, \sigma))b_0^{-1} \,,$$

Hence, we can write

$$\dot{e}_{r+1} = -\kappa\alpha_{r+1}(1 + \Delta_0)e_1 + \Delta_2$$

in which

$$\Delta_2(x, e) = \dot{q} + \dot{b}g(\psi) + [b - b_0]g'(\psi)b_0^{-1}K\dot{\xi} \,.$$

The functions $\Delta_0(x, e)$ and $\Delta_2(x, e)$ introduced here have the following properties. Regarding $\Delta_0(x, e)$, recall that $g'(\cdot)$ is by assumption bounded by 1 and that b_0 is chosen so as to satisfy (10). Thus, we may conclude that

$$|\Delta_0(x, e)| \le \delta_0 < 1 \,. \tag{12}$$

Regarding $\Delta_2(x, e)$, a closer look reveals that

$$\begin{aligned} \Delta_2(x, e) \;=\; & \tfrac{\partial q}{\partial z} f(z, \xi) + \tfrac{\partial q}{\partial \xi}[\hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)g(\psi(\hat{\xi}, \sigma))]] \\ & + \tfrac{\partial b}{\partial z} f(z, \xi)g(\psi(\hat{\xi}, \sigma)) + \tfrac{\partial b}{\partial \xi}[\hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)g(\psi(\hat{\xi}, \sigma))]]g(\psi(\hat{\xi}, \sigma)) \\ & + [b(z, \xi) - b_0]b_0^{-1}g'(\psi(\hat{\xi}, \sigma))K[\hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)g(\psi(\hat{\xi}, \sigma))]] \,. \end{aligned}$$

Since $g(\cdot)$ and $g'(\cdot)$ are bounded functions, it is observed that so long as x remains in a bounded region, the function $\Delta_2(x, e)$ remains bounded regardless of the value of e, with a bound which is independent of κ.

Putting the dynamics of the e_i's all together yields a system of the form

$$\dot{e} = \kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x, e)]e + \mathbf{B}_1\Delta_1(x, e) + \mathbf{B}_2\Delta_2(x, e) \tag{13}$$

in which

$$\mathbf{A} = \begin{pmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ -\alpha_r & \cdot & \cdot & \cdots & 1 \\ -\alpha_{r+1} & \cdot & \cdot & \cdots & 0 \end{pmatrix} \qquad \mathbf{B} = \mathbf{B}_2 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{B}_1 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{C} = (\, \alpha_{r+1} \quad 0 \quad \cdots \quad 0 \quad 0 \,)$$

and $\Delta_0, \Delta_1, \Delta_2$ are the functions indicated above.

2 Convergence analysis

We return now to the equations describing the (controlled) plant (1). Adding and subtracting $\hat{B}K\xi$, the last equation can be rewritten as

$$\dot{\xi} = (\hat{A} + \hat{B}K)\xi + \hat{B}\Delta_3(x,e)$$

in which

$$\Delta_3(x,e) = q(z,\xi) + b(z,\xi)g(\frac{K\hat{\xi} - \sigma}{b_0}) - K\xi\,,$$

where, consistently with the notation used in the previous section, as arguments of Δ_3 we have used x, e instead of $z, \xi, \hat{\xi}, \sigma$,

Accordingly, the entire controlled plant (1) can be written as

$$\dot{x} = \mathbf{F}(x) + \mathbf{G}\Delta_3(x,e) \tag{14}$$

in which

$$\mathbf{F}(x) = \begin{pmatrix} f(z,\xi) \\ (\hat{A} + \hat{B}K)\xi \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 0 \\ \hat{B} \end{pmatrix},$$

highlighting the structure of a globally asymptotically stable system affected by a perturbation.

We proceed now with the asymptotic analysis of the entire closed-loop system, which is a system described by equations of the form

$$\begin{aligned} \dot{x} &= \mathbf{F}(x) + \mathbf{G}\Delta_3(x,e) \\ \dot{e} &= \kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]e + \mathbf{B}_1\Delta_1(x,e) + \mathbf{B}_2\Delta_2(x,e)\,. \end{aligned} \tag{15}$$

Inspection of the various expressions of $\Delta_1(x,e), \Delta_2(x,e), \Delta_3(x,e)$ reveals that $(x,e) = (0,0)$ is an equilibrium point. In what follows, we will see how the free design parameters can be adjusted in order to obtain asymptotic stability of this equilibrium, with a region of attraction that contains a fixed (but otherwise arbitrary large) compact set $\mathcal{C}$. Note that the set in question must be assigned in terms of the *original* state variables, namely the state x of the controlled plant (1) and the states $\hat{\xi}$ and σ that characterize the dynamic controller (5)–(6). In this respect, it is observed that in the equations (15), resulting from the change of variables (7), the vector e replaces $(\hat{\xi}, \sigma)$. Thus, looking at the change (7), it should be borne in mind that the (compact) set to which the initial values of x and e belong is influenced by the choice of κ. We will return on this issue later on.

In what follows, occasionally, we use the notation $\hat{\xi}_{\text{ext}}$ to indicate the vector

$$\hat{\xi}_{\text{ext}} = \text{col}(\hat{\xi}, \sigma)$$

that characterizes the state of the dynamic controller (5)–(6). With $\mathcal{C}$ being the fixed compact set of initial conditions of the system, pick a number $R > 0$ such that

$$(x, \hat{\xi}_{\text{ext}}) \in \mathcal{C} \qquad \Rightarrow \qquad \begin{array}{c} x \in B_R \\ \hat{\xi}_{\text{ext}} \in B_R \end{array}$$

in which B_R denotes a ball of radius R in $\mathbb{R}^n$

$$B_R = \{x \in \mathbb{R}^n : |x| \leq R\}\,.$$

The first thing is to fix the "saturation" level L that characterizes the function $g(\cdot)$. To this end we begin with the observation that, since the "unperturbed" system

$$\dot{x} = \mathbf{F}(x)$$

is by assumption globally asymptotically stable, there exists a positive definite and proper smooth function $V(x)$ satisfying

$$\frac{\partial V}{\partial x}\mathbf{F}(x) \leq -\alpha(|x|)$$

for some class $\mathcal{L}_\infty$ function $\alpha(\cdot)$.

Pick a number $c > 0$ such that

$$\Omega_c = \{x : V(x) \leq c\} \supset B_R.$$

As we will see, among other things, one of the purposes of the design is to make sure that, for all times $t \geq 0$, the vector $x(t)$ remains in Ω_{c+1}. With this goal in mind, we pick for L the value

$$L = \max_{x\in\Omega_{c+1}} \Big[\frac{K\xi - q(z,\xi)}{b(z,\xi)}\Big] + 1\,.$$

We begin by analyzing what happens to $x(t)$ for small values of $t \geq 0$. Looking at the expression of the perturbation term $\Delta_3(x, e)$ in the upper system of (15) it is seen that, since $g(\cdot)$ is bounded, there exist a number δ_3, such that

$$|\Delta_3(x,e)| \leq \delta_3 \qquad \text{for all } x \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^{r+1}\,.$$

This number δ_3 is independent of the choice of κ and depends only on the choice of the number R and hence, indirectly, only on the choice of the set $\mathcal{C}$ in which the initial conditions are taken.

Let $x(0) \in B_R \subset \Omega_c$. Regardless of what $e(t)$ is, so long as $x(t) \in \Omega_{c+1}$, we have

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}\Delta_3(x,e)] \leq -\alpha(|x|) + \Big|\frac{\partial W}{\partial x}\Big|\delta_3$$

Setting

$$M = \max_{x\in\Omega_{c+1}} \Big|\frac{\partial W}{\partial x}\Big|$$

we obtain

$$\dot{V}(x(t)) \leq (M\delta_3)t$$

which in turn yields

$$V(t) - V(0) \leq (M\delta_3)t\,.$$

Thus, since $V(0) \leq c$, we see that $x(t)$ remains in Ω_{c+1} at least until time $T_0 = 1/M\delta_3$. This time may be very small but, because of the presence of the saturation function $g(\cdot)$, it is independent of κ. It rather only depends only on the choice of the number R and hence, indirectly, only on the choice of the set $\mathcal{C}$ in which the initial conditions are taken.

We proceed now to the analysis of what happens to $e(t)$. To this end, the following results are relevant.

Lemma 1 *There exist a choice of the coefficients $\alpha_1, \ldots, \alpha_{r+1}$, a positive definite and symmetric $(r+1) \times (r+1)$ matrix P and a number $\lambda > 0$ such that*

$$P[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)] + [\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]^{\mathrm{T}} P \leq -\lambda I\,. \tag{16}$$

Proof. Observe that the system

$$\dot{e} = [\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]e$$

can be seen as resulting from the interconnection of

$$\begin{array}{rcl} \dot{e} & = & \mathbf{A}e + \mathbf{B}u \\ y & = & \mathbf{C}e \end{array} \tag{17}$$

with

$$u = -\Delta_0(x,e)y\,. \tag{18}$$

Looking at the expressions of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ it is seen that the transfer function $T(s)$ of (17) is

$$T(s) = \frac{\alpha_{r+1}}{s^{r+1} + \alpha_1 s^r + \cdots \alpha_r s + \alpha_{r+1}}\,.$$

If the α_i's are such that the poles of $T(s)$ are all real and with negative real part, $\|T(\cdot)\|_\infty = 1$. As a consequence

$$\|T(\cdot)\|_\infty < \delta_0^{-1}$$

in which $\delta_0 < 1$ is any number for which (10) holds. By the Bounded Real Lemma, there is a positive definite and symmetric matrix P and a number $\lambda > 0$ such that

$$2e^{\mathrm{T}} P[\mathbf{A}e + \mathbf{B}u] \leq -\lambda|e|^2 + (\delta_0^{-1})^2|u|^2 - |y|^2$$

for all e and u. Using (18) and bearing in mind (12), which implies $|u|^2 \leq \delta_0^2|y|^2$, we obtain

$$2e^{\mathrm{T}} P[\mathbf{A}e - \mathbf{B}\mathbf{C}\Delta_0(x,e)e] \leq -\lambda|e|^2 + (\delta_0^{-1})^2\delta_0^2|y|^2 - |y|^2 \leq -\lambda|e|^2$$

which proves the Lemma. $\triangleleft$

Lemma 2 *Let the α_i's be chosen so as to make (16) satisfied. Suppose $x(t) \in \Omega_{c+1}$ for all $t \in [0, T_{\max})$ and suppose that $|\hat{\xi}_{\mathrm{ext}}(0)| \leq R$. Then, for every $T < T_{\max}$ and every $\epsilon > 0$, there is a κ^* such that, for all $\kappa \geq \kappa^*$,*

$$|e(t)| \leq 2\epsilon \qquad \text{for all } t \in [T, T_{\max}).$$

Proof. Set $U(e) = e^{\mathrm{T}} P e$, and observe that

$$\begin{array}{rcl} \dot{U}(e(t)) & = & 2e^{\mathrm{T}} P[\kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x,e)]e + \mathbf{B}_1\Delta_1 + \mathbf{B}_2\Delta_2] \leq -\kappa\lambda|e|^2 + 2|e||P||\Delta_1| + 2|e||P||\Delta_2| \\ & \leq & -(\kappa\lambda - 2\delta_1|P|)|e|^2 + 2|e||\Delta_2| \leq -(\kappa\lambda - 2\delta_1|P| - \mu)|e|^2 + \frac{1}{\mu}|\Delta_2| \end{array}$$

where we have taken advantage of the bound (11) for Δ_1 and used Young's inequality.

Recalling the expression $\Delta_2(x,e)$, observe that, since $g(\cdot)$ is bounded, there exist a number $\delta_2 > 0$, such that

$$|\Delta_2(x,e)| \le \delta_2 \qquad \text{for all } x \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^{r+1}\,.$$

This number δ_2 is independent of the choice of κ and depends only on the choice of the number R and hence, indirectly, only on the choice of the set $\mathcal{C}$ in which the initial conditions are taken.

Now, let $\epsilon > 0$ be fixed and pick μ so that

$$\frac{\delta_2}{\mu} \le \epsilon^2\,.$$

Using the standard estimates

$$a_1|e|^2 \le U(e) \le a_2|e|^2$$

and setting, for convenience

$$\alpha = \frac{\kappa\lambda - 2\delta_1|P| - \mu}{a_2}$$

(in which we take a large κ, so that $\alpha > 0$), we end-up with the inequality

$$\dot U(e(t)) \le -\alpha U(e(t)) + \epsilon^2\,,$$

which is guaranteed to hold so long as $x(t) \in \Omega_{c+1}$. Standard arguments show that

$$|e(t)| \le Ae^{-\alpha t}|e(0)| + \frac{\epsilon}{\sqrt{\alpha a_1}} \qquad \text{where } A = \Big(\frac{a_2}{a_1}\Big)^{\frac{1}{2}}\,.$$

In this expression, we have now to take into account an estimate of $|e(0)|$. Looking at (7) and assuming $\kappa > 1$, it is seen that so long as $|x(0)| \le R$ and $|\hat\xi_{\rm ext}(0)| \le R$, we have

$$|e(0)| \le \kappa^r R'$$

in which R' is a number only depending on R and not on κ. Entering this in the previous inequality, and picking κ so that $\alpha a_1 > 1$, we obtain an estimate of the form

$$|e(t)| \le A'e^{-\kappa t}\kappa^r + \epsilon\,, \qquad \text{where } A' = AR'e^{\frac{2\delta_1|P|+\mu}{a_2}}\,,$$

which, we stress, is valid so long as $x(t) \in \Omega_{c+1}$. Pick $0 < T < T_{\max}$ and let κ^* be such that

$$A'e^{-\kappa^* T}(\kappa^*)^r = \epsilon\,,$$

as it is always possible. Then, we see that, for all $\kappa \ge \kappa^*$,

$$|e(T)| \le 2\epsilon\,.$$

Moreover, since $x(t) \in \Omega_{c+1}$ for $t \in [T, T_{\max})$, we have

$$|e(t)| \le A'e^{-\kappa^* T}(\kappa^*)^r e^{-\kappa^*(t-T)} + \epsilon = \epsilon e^{-\kappa^*(t-T)} + \epsilon \le 2\epsilon \qquad \text{for all } t \in [T, T_{\max})\,.$$

This concludes the proof. $\triangleleft$

We return now to the examine the motion of $x(t)$, which we know is in Ω_{c+1} for $t \le T_0$. Suppose that, for some $T_{\max} > T_0$,

$$x(t) \in \Omega_{c+1} \qquad \text{for all } t \in [T_0, T_{\max}).$$

We know from Lemma 2 that, for any choice of ϵ, there is a value of κ^* such that, if $\kappa \ge \kappa^*$, $|e(t)| \le 2\epsilon$ for all $t \in [T_0, T_{\max})$. If $4\epsilon < b_{\min}$, we see in particular that for all $t \in [T_0, T_{\max})$

$$\Big|\frac{e_{r+1}}{b(z,\xi)}\Big| < \frac{1}{2}\,,$$

which, in view of the choice of L, implies

$$\Big|\frac{K\xi - q(z,\xi) + e_{r+1}}{b(z,\xi)}\Big| \le L - \frac{1}{2}\,.$$

From this, using (8) and taking advantage of the fact that function $F(s)$ defined in (9) is strictly increasing, with $F(s) = s$ when $|s| \le L$, we deduce that

$$|\psi(\xi,\sigma)| \le L - \frac{1}{2}\,.$$

Recall now

$$\hat{\xi} = \xi - D(\kappa)e$$

in which $D(\kappa)$ is $r \times (r+1)$ a matrix in which $d_{ii}(\kappa) = \kappa^{-r-1+i}$, for $i = 1, \ldots, r$, while all other entries are 0. Note that $|D(\kappa)| \le 1$, if $\kappa > 1$. Thus

$$\psi(\hat{\xi},\sigma) = \psi(\xi,\sigma) - b_0^{-1}KD(\kappa)e$$

and

$$|\psi(\hat{\xi},\sigma)| \le |\psi(\xi,\sigma)| + b_0^{-1}|K||e|\,.$$

If $4|K|\epsilon < b_0$, we have

$$|\psi(\hat{\xi},\sigma)| \le L\,,$$

and consequently

$$g(\psi(\hat{\xi},\sigma)) = \psi(\hat{\xi},\sigma)\,.$$

This being the case, we have for all $t \in [T_0, T_{\max})$

$$\begin{array}{rcl}\Delta_3(x,e) & = & q(z,\xi) + b(z,\xi)g(\psi(\hat{\xi},\sigma)) - K\xi = q(z,\xi) + b(z,\xi)\psi(\hat{\xi},\sigma) - K\xi \\ & = & q(z,\xi) + b(z,\xi)[\psi(\xi,\sigma) - b_0^{-1}KD(\kappa)e] - K\xi \\ & = & -e_{r+1} - b(z,\xi)b_0^{-1}KD(k)e\end{array}$$

where we have used again the property that the function (9) is an identity if $|s| \le L$, which yields

$$\psi(\xi,\sigma) = F(\psi(\xi,\sigma)) = \frac{K\xi - q(z,\xi) + e_{r+1}}{b(x,\xi)}\,.$$

As a consequence of this we can conclude that

$$|\Delta_3(x,e)| \leq (1 + b_{\max} b_0^{-1}|K|)2\epsilon \qquad \text{for all } t \in [T_0, T_{\max}).$$

From all of the above it is seen that, for all $t \in [T_0, T_{\max})$,

$$\frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}\Delta_3(x,e)] \leq -\alpha(|x|) + M(1 + b_{\max} b_0^{-1}|K|)2\epsilon\,.$$

Pick now any number $d \ll c$ and consider the "annular" compact set

$$S_d^{c+1} = \{z : d \leq V(x) \leq c+1\}\,.$$

Let ρ be

$$\rho = \min_{x \in S_d^{c+1}} |x|$$

By construction

$$\alpha(|x|) \geq \alpha(\rho) \qquad \text{for all } x \in S_d^{c+1}.$$

If ϵ is such

$$(1 + b_{\max} b_0^{-1}|K|)2\epsilon \leq \frac{1}{2}\,\alpha(\rho)\,,$$

there follows that

$$\dot V(x(t)) \leq -\frac{1}{2}\,\alpha(\rho)\,,$$

so long as $x(t) \in S_d^{c+1}$. This, in turn, implies

$$V(x(t)) \leq V(x(T_0)) - \frac{1}{2}\,\alpha(\rho) \leq c + 1 - \frac{1}{2}\,\alpha(\rho)$$

so long as $x(t) \in S_d^{c+1}$. Clearly, there is a time $T_1 > T_0$ such that $x(t) \in \Omega_{c+1}$ for all $t \in [T_0, T_1]$ and $V(x(T_1)) = d$. Since $\dot V(x(t))$ is negative on the boundary of Ω_d, it is concluded that $x(t) \in \Omega_d$ for all $t \geq T_1$ and $T_{\max} = \infty$.

In summary, we have shown that, by tuning the design parameter κ, all trajectories of the closed-loop system with initial conditions in $\mathcal{C}$ remain bounded and enter, in finite time, an arbitrarily small compact set. As a matter of fact, the following result has been just proven.

Proposition 1 *Consider system (1), controlled by (5)–(6. Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Suppose (1) is strongly minimum phase. Let $\hat K$ such that $\hat A + \hat B \hat K$ is Hurwitz. For every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$ there is a finite time T such that all trajectories of the closed-loop system with initial conditions $(x(0), \hat\xi_{\rm ext}(0)) \in \mathcal{C}$ remain bounded and satisfy $|(x(t)| \leq \varepsilon$ and $|\hat\xi_{\rm ext}(t)| \leq \varepsilon$ for all $t \geq T$.*

If asymptotic stability is sought, it suffices to strengthen the assumption of strong minimum phase by requiring that the equilibrium $z = 0$ of $\dot z = f(z, 0)$ is also locally exponentially stable.

Proposition 2 *Consider system (1), controlled by (5)–(6). Suppose that (2) hold and that $b(x, \xi)$ is bounded as in (3). Suppose (1) is strongly minimum phase. Let $\hat{K}$ such that $\hat{A}+\hat{B}\hat{K}$ is Hurwitz. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(x, \hat{\xi}_{\text{ext}}) = (0, 0)$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.*

Remark. In both these results, to streamline the analysis, we have assumed that the controlled system is strongly minimum phase, but this assumption can be weakened, to some extent. In fact, in both results, due to the necessity of using the "rough" observer (6), it has been possible to prove convergence (to a small neighborhood of the origin or to the origin) only from a *compact set* of initial conditions. In other words, we have not proven global convergence, but only convergence with a guaranteed region of attraction. This being the case, it should be observed that identical results hold if the assumption that the system is strongly minimum phase is replaced by the assumption that the system possesses a globally defined normal form in which the first equation of (1) has the special structure

$$\dot{z} = f(z, \xi_1)$$

and is only minimum phase (rather than being strongly minimum phase). The proof, in which $\hat{K}$ has to be chosen of the special form

$$\hat{K} = \begin{pmatrix} -a_0 k^r & -a_1 k^{r-1} & \cdots & -a_{r-2} k^2 & -a_{r-1} k \end{pmatrix}$$

with $k > 0$ a design parameter to be tuned in accordance with choice of the set $\mathcal{C}$, is not conceptually different from the one described above and is not covered here. $\triangleleft$

3 Extensions

The design procedure described in the previous sections applies to systems that are strongly minimum phase (or, under a special assumption regarding the structure of the normal form, minimum phase). We discuss in this section how these methods might be used – to some extent – to handle also systems that are not minimum phase.

Consider again a plant modeled by equations of the form (1), in which we assume $q(0, 0) = 0$ and $b(z, \xi)$ bounded as in (3), and let its dynamics be extended by that of a system

$$\dot{\varrho} = L(\varrho, \xi_1, \ldots, \xi_{r-1}) + Mu\,,$$

with state $\varrho \in \mathbb{R}^\nu$. Setting

$$x_{\text{e}} = \text{col}(z, \xi, \varrho)$$

the resulting *extended* system defined in this way can be seen as a system of the form

$$\dot{x}_{\text{e}} = f_{\text{e}}(x_{\text{e}}) + g_{\text{e}}(x_{\text{e}})u \tag{19}$$

in which

$$f_{\text{e}}(x_{\text{e}}) = \begin{pmatrix} f(z, \xi) \\ \hat{A}\xi + \hat{B}q(z, \xi) \\ L(\varrho, \xi_1, \ldots, \xi_{r-1}) \end{pmatrix}, \qquad g_{\text{e}}(x_{\text{e}}) = \begin{pmatrix} 0 \\ \hat{B}b(z, \xi) \\ M \end{pmatrix}. \tag{20}$$

Let now $h_e(x_e)$ be a function defined as

$$h_e(x_e) = \xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1})\,, \tag{21}$$

in which we assume that

$$\frac{\partial N}{\partial \varrho} M = 0\,. \tag{22}$$

If this property holds, the derivative of $N(\varrho, \xi_1, \ldots, \xi_{r-1})$ along the trajectories of the extended system is a function of ϱ and ξ only, which we will write as

$$\dot{N}(\varrho, \xi) := \frac{\partial N}{\partial \varrho} L(\varrho) + \sum_{i=1}^{r-1} \frac{\partial N}{\partial \xi_i} \xi_{i+1}\,. \tag{23}$$

In what follows we assume also that this function $\dot{N}(\varrho, \xi)$ is globally Lipschitz in the argument ξ, uniformly in ϱ, i.e. that

$$|\dot{N}(\varrho, \hat{\xi}) - \dot{N}(\varrho, \xi)| \le L_N |\hat{\xi} - \xi|\,, \tag{24}$$

for some fixed $L_N > 0$,

Using (22) it is seen that

$$L_{g_e} h_e(x_e) = b(z, \xi)$$

and therefore system (19)–(20) with output (21) has relative degree $r = 1$. Since $b(z, \xi)$ is bounded as in (3), the vector field

$$\tilde{g}_e(x_e) = \frac{1}{L_{g_e} h_e(x_e)} g_e(x_e)$$

is complete. As a consequence, system (19)–(20) with output (21) possesses a globally defined normal form.

The stabilization techniques described in the previous sections are applicable if the dynamically extended system introduced in this way is strongly minimum phase. In view of this we assume the following.

Assumption 1 *There exist an integer ν and a triplet*

$$L(\varrho, \xi_1, \ldots, \xi_{r-1}),\ M,\ N(\varrho, \xi_1, \ldots, \xi_{r-1})$$

in which $L(\cdot)$ and $N(\cdot)$ are smooth functions satisfying (22)–(24) and in which $L(0, 0, \ldots, 0) = 0$, $N(0, 0, \ldots, 0) = 0$, *such that system (19)–(20) with output (21) is strongly minimum phase.*

We will return later on the interpretation of this Assumption. For the time being, we observe that if this Assumption holds and if the full state x_e of this dynamically extended system is available for measurement, global stabilization can be achieved by means of a feedback law of the form

$$u = u(x_e) = \frac{-L_{f_e} h_e(x_e) + k h_e(x_e)}{L_{g_e} h_e(x_e)}$$

in which k is any negative number. In fact, this is precisely the equivalent, in the current context, of the feedback law XXX).

The control $u(x_\mathrm{e})$ can easily be expressed in the (z, ξ, ϱ) coordinates. To this end, it suffices to take into account the definition (21) of $h_\mathrm{e}(x_\mathrm{e})$ and to observe that (recall (23))

$$L_{f_\mathrm{e}} h_\mathrm{e}(x_\mathrm{e}) = q(w, \xi) - \dot{N}(\varrho, \xi)\,.$$

In this way, we arrive at the following preliminarily conclusion: if Assumption 1 holds, a *dynamic* control law of the form

$$\begin{array}{rcl} \dot{\varrho} &=& L(\varrho, \xi_1, \ldots, \xi_{r-1}) + Mu \\ u &=& [b(z,\xi)]^{-1}[-\, q(z,\xi) + \dot{N}(\varrho,\xi) + k(\xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1}))] \end{array} \tag{25}$$

(in which k is a negative number) globally asymptotically stabilizes the equilibrium $(z, \xi, \varrho) = (0, 0, 0)$ of the closed loop system (1)–(25).

If only the output $y = \xi_1$ of system (1) is available for feedback, this law cannot be directly implemented, because $\xi_2, \ldots, \xi_r$ and $q(z, \xi)$ and $b(z, \xi)$ are not directly available. But these quantities are precisely the same quantities that needed to be estimated for the implementation of the feedback law (4) in the context of the design problem considered in the previous sections. Thus, it is expected that the very same procedure used earlier to estimate such quantities can be successfully used also in the present context. With this in mind, let $\psi_\mathrm{e}(\xi, \sigma, \varrho)$ be the function defined as

$$\psi_\mathrm{e}(\xi, \sigma, \varrho) = b_0^{-1}[\dot{N}(\varrho, \xi) + k(\xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1})) - \sigma] \tag{26}$$

in which b_0 is a design parameter, $\dot{N}(\varrho, \xi)$ is the function defined above, k is a negative number and $\sigma \in \mathbb{R}$. Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a smooth "saturation" function characterized by the properties: $g(s) = s$ if $|s| \le L$, $g(s)$ is odd and monotonically increasing, with $0 < g'(s) \le 1$, and $\lim_{s\to\infty} g(s) = L(1 + c)$ with $0 < c \ll 1$.

Plant (1) will be controlled by a law of the form

$$\begin{array}{rcl} \dot{\varrho} &=& L(\varrho, \hat{\xi}_1, \ldots, \hat{\xi}_{r-1}) + Mu \\ u &=& g(\psi_\mathrm{e}(\hat{\xi}, \sigma, \varrho)) \end{array} \tag{27}$$

in which $\hat{\xi} \in \mathbb{R}^r$ and $\sigma \in \mathbb{R}$ are states of the dynamical system

$$\begin{array}{rcl} \dot{\hat{\xi}}_1 &=& \hat{\xi}_2 + \kappa\alpha_1(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 &=& \hat{\xi}_3 + \kappa^2\alpha_2(y - \hat{\xi}_1) \\ && \cdots \\ \dot{\hat{\xi}}_{r-1} &=& \hat{\xi}_r + \kappa^{r-1}\alpha_{r-1}(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_r &=& \sigma + b_0 g(\psi_\mathrm{e}(\hat{\xi}, \sigma, \varrho)) + \kappa^r\alpha_r(y - \hat{\xi}_1) \\ \dot{\sigma} &=& \kappa^{r+1}\alpha_{r+1}(y - \hat{\xi}_1)\,. \end{array} \tag{28}$$

Note that the first $r - 1$ and the $(r + 1)$-th equations in this set are exactly the same as in (6), while the r-th one differs from the r-th of (6) in the replacement of $\psi(\hat{\xi}, \sigma)$ by $\psi_\mathrm{e}(\hat{\xi}, \sigma, \varrho)$.

With the "observer" (28), associate "error" variables $e_1, \ldots, e_r, e_{r+1}$, the first r of which are defined exactly as in (7), while e_{r+1} has the same structure as the last one of (7), with $\psi(\xi, \sigma)$ now replaced by the function $\psi_{\mathrm{e}}(\xi, \sigma, \varrho)$ defined by (26), that is

$$e_{r+1} = q(z, \xi) + [b(z, \xi) - b_0] g(\psi_{\mathrm{e}}(\xi, \sigma, \varrho)) - \sigma \,.$$

The new variables defined in this way characterize a global diffeomorphism. In particular, the possibility of recovering σ from e_{r+1} relies upon arguments identical to those used to prove this property in the previous context. In fact, setting

$$\psi_{\mathrm{e}}^*(z, \xi, \varrho, e_{r+1}) = \frac{\dot{N}(\varrho, \xi) + k(\xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1})) - q(z, \xi) + e_{r+1}}{b(z, \xi)}$$

and using the function $F(s)$ defined as in (9), it is seen that

$$\psi_{\mathrm{e}}^* = F(\psi_{\mathrm{e}})$$

and therefore, assuming that b_0 is chosen to fulfill (10), one can recover σ via

$$\psi_{\mathrm{e}} = F^{-1}(\psi_{\mathrm{e}}^*) \,.$$

In these new coordinates, we have

$$\dot{e}_i = \kappa(e_{i+1} - \alpha_i e_1) \qquad \text{for } i = 1, \ldots, r-1$$

and

$$\dot{e}_r = \kappa(e_{r+1} - \alpha_r e_1) + \Delta_1(x_{\mathrm{e}}, e)$$

in which

$$\Delta_1(x_{\mathrm{e}}, e) = \kappa[b(z, \xi) - b_0][g(\psi_{\mathrm{e}}(\hat{\xi}, \sigma, \varrho) - g(\psi_{\mathrm{e}}(\xi, \sigma, \varrho)] \,.$$

As in the previous section, bearing in mind the definition of $\psi_{\mathrm{e}}(\xi, \sigma, \varrho)$, using property (24), and picking $\kappa > 1$, it can be proven that

$$|\Delta_1(x_{\mathrm{e}}, e) \leq \delta_1 |e|$$

for some $\delta_1 > 0$.

Finally, we obtain for the time derivative of e_{r+1} an expression of the form

$$\dot{e}_{r+1} = -\kappa \alpha_{r+1} e_1 + \dot{q} + \dot{b} g(\psi_{\mathrm{e}}(\xi, \sigma, \varrho)) + [b - b_0] g'(\psi_{\mathrm{e}}(\xi, \sigma, \varrho)) b_0^{-1} [\ddot{N}(\varrho, \xi) + k(\dot{\xi}_r - \dot{N}(\varrho, \xi)) - \dot{\sigma}]$$

in which, for convenience, we have omitted the indication of some arguments. Bearing in mind the expression of $\dot{\sigma}$, the latter can be rewritten as

$$\dot{e}_{r+1} = -\kappa \alpha_{r+1}(1 + \Delta_0) e_1 + \Delta_2$$

in which

$$\Delta_0(x_{\mathrm{e}}, e) = [b(z, \xi) - b_0] g'(\psi_{\mathrm{e}}(\xi, \sigma, \varrho)) b_0^{-1} \,.$$

$$\Delta_2(x_{\mathrm{e}}, e) = \dot{q} + \dot{b} g(\psi_{\mathrm{e}}) + [b - b_0] g'(\psi_{\mathrm{e}}) b_0^{-1} [\ddot{N}(\varrho, \xi) + k \dot{\xi}_r] \,.$$

The function $\Delta_0(x_e, e)$ is bounded as in (12), i.e.

$$|\Delta_0(x_e, e)| \le \delta_0 < 1\,,$$

if b_0 is chosen so as to satisfy (10). Regarding $\Delta_2(x_e, e)$, observe in particular that

$$\begin{array}{rcl} \ddot{N}(\varrho,\xi) + k(\dot{\xi}_r - \dot{N}(\varrho,\xi)) & = & \frac{\partial \dot{N}}{\partial \varrho}[L(\varrho) + Mg(\psi_e(\hat{\xi},\sigma,\varrho))] \\ & & + \frac{\partial \dot{N}}{\partial \xi}[\hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)g(\psi_e(\hat{\xi},\sigma,\varrho))] \\ & & + k[q(z,\xi) + b(z,\xi)g(\psi_e(\hat{\xi},\sigma,\varrho)) - \dot{N}(\varrho,\xi)]\,. \end{array}$$

With this in mind, arguments identical to those used in the previous section can be invoked to claim that, since $g(\cdot)$ and $g'(\cdot)$ are bounded functions, so long as x_e remains in a bounded region, the function $\Delta_2(x_e, e)$ remains bounded regardless of the value of e, with a bound which is independent of κ.

Putting the dynamics of the e_i's all together yields a system of the form

$$\dot{e} = \kappa[\mathbf{A} - \mathbf{BC}\Delta_0(x_e, e)]e + \mathbf{B}_1\Delta_1(x_e, e) + \mathbf{B}_2\Delta_2(x_e, e) \tag{29}$$

in which $\mathbf{A}, \mathbf{B}, \mathbf{B}_2, \mathbf{B}_2, \mathbf{C}$ have exactly the same form as in the previous section, and $\Delta_0, \Delta_1, \Delta_2$ are the functions indicated above.

Passing to the equations that describe the controlled plant, note that

$$\dot{x}_e = f_e(x_e) + g_e(x_e)g(\psi_e(\hat{\xi},\sigma,\varrho)) = f_e(x_e) + g_e(x_e)u_e(x_e) + g_e(x_e)[g(\psi_e(\hat{\xi},\sigma,\varrho)) - u_e(x_e)]$$

which we will write as

$$\dot{x}_e = \mathbf{F}(x_e) + g_e(x_e)\Delta_3(x_e, e)$$

where $\mathbf{F}(x_e) = f_e(x_e) + g_e(x_e)u_e(x_e)$ and

$$\Delta_3(x_e, e) = g(\psi_e(\hat{\xi},\sigma,\varrho)) - [b(z,\xi)]^{-1}[-\,q(z,\xi) + \dot{N}(\varrho,\xi) + k(\xi_r - N(\varrho,\xi_1,\ldots,\xi_{r-1}))] \tag{30}$$

highlighting the structure of a globally asymptotically stable system affected by a perturbation.

It is realized at this point that the controlled system has a *structure identical* to that of (15), i.e.

$$\begin{array}{rcl} \dot{x}_e & = & \mathbf{F}(x_e) + g_e(x_e)\Delta_3(x_e, e) \\ \dot{e} & = & \kappa[\mathbf{A} - \mathbf{BC}\Delta_0(x_e, e)]e + \mathbf{B}_1\Delta_1(x_e, e) + \mathbf{B}_2\Delta_2(x_e, e) \end{array}$$

and that *identical hypotheses* hold.[2] Therefore, the proposed controller - if Assumption 1 holds - yields identical asymptotic properties.

Proposition 3 *Consider system (1), controlled by (27)–(28). Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Let $L(\varrho,\xi_1,\ldots,\xi_{r-1}), M, N(\varrho,\xi_1,\ldots,\xi_{r-1})$ be such that Assumption 1 holds. For every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a choice of the design parameters b_0, L and $\alpha_1,\ldots,\alpha_{r+1}$ and a number κ^* such that, for all $\kappa \ge \kappa^*$ there is a finite time T such that all trajectories of the closed-loop system with initial conditions $(x_e(0), \hat{\xi}_{\rm ext}(0)) \in \mathcal{C}$ remain bounded and satisfy $|(x_e(t)| \le \varepsilon$ and $|\hat{\xi}_{\rm ext}(t)| \le \varepsilon$ for all $t \ge T$.*

[2] Note that, while in (15) the vector field that multiplies Δ_3 is a constant vector, in the present context it is an x_e-dependent vector. However, $g_e(x_e)$ is globally bounded and this suffices to use arguments identical to those used earlier to establish the desired asymptotic properties.

If asymptotic stability is sought, it is convenient to strengthen Assumption 1 as follows.

Assumption 2 *There exist an integer ν and a triplet*

$$L(\varrho, \xi_1, \ldots, \xi_{r-1}),\ M,\ N(\varrho, \xi_1, \ldots, \xi_{r-1})$$

in which $L(\cdot)$ and $N(\cdot)$ are smooth functions satisfying (22)–(24) and in which $L(0,0,\ldots,0) = 0$, $N(0,0,\ldots,0) = 0$, such that system (19)–(20) with output (21) is strongly minimum phase and the zero dynamics is locally exponentially stable.

Proposition 4 *Consider system (1), controlled by (27)–(28). Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Let $L(\varrho, \xi_1, \ldots, \xi_{r-1}), M, N(\varrho, \xi_1, \ldots, \xi_{r-1})$ be such that Assumption 2 holds. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(x_{\rm e}, \hat{\xi}_{\rm ext})$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.*

4 The controlled zero dynamics

The convergence results described Propositions YYY and YYY have been derived under the assumption that the controlled system is strongly minimum phase. Under this assumption, the proof the results in question consists in a straightforward use "off the shelf" the results presented earlier in section XXX, and for this reason has not been repeated. However, it happens that - since system (19)–(20) with output (21) has relative degree 1 - the same conclusions hold under the weaker assumption that this system is just minimum phase. In this case, though, the proof of the results requires a modification of the arguments presented earlier, as it will be explained in what follows.

To this end it is convenient, first of all, to bring the system in question, namely system (19)–(20) with output (21), in normal form. Bearing in mind the result established earlier on the existence of globally defined normal forms and the fact that the system in question has relative degree 1, it is known that the transformation is obtained via a global diffeomorphism

$$\tilde{x}_{\rm e} = \Phi(x_{\rm e})$$

defined as

$$\tilde{x}_{\rm e} = \begin{pmatrix} z_{\rm e} \\ \theta \end{pmatrix} = \begin{pmatrix} \Phi^{\tilde{g}_{\rm e}}_{-h_{\rm e}(x_{\rm e})}(x_{\rm e}) \\ h_{\rm e}(x_{\rm e}) \end{pmatrix}$$

in which $\Phi^{\tilde{g}_{\rm e}}_t(x_{\rm e})$ denotes the flow of the vector field $\tilde{g}_{\rm e}$.

A simple calculation shows that

$$z_{\rm e} = \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}, \chi)$$

in which

$$\chi = \varrho + M \int_0^{N(\varrho,\xi_1,\ldots,\xi_{r-1})-\xi_r} \frac{1}{b(z, \xi_1, \ldots, \xi_{r-1}, \xi_r + s)} ds$$

and, of course,

$$\theta = \xi_r - N(\varrho, \xi_1, \ldots, \xi_{r-1})\,.$$

Note also that the new variable χ can be also expressed as

$$\chi = \varrho + M \int_{\xi_r}^{N(\varrho,\xi_1,\ldots,\xi_{r-1})} \frac{1}{b(z,\xi_1,\ldots,\xi_{r-1},s)} ds := T(z,\xi,\varrho)$$

in which $T(z,\xi,\varrho)$ is an identity at $\xi_r = N(\varrho,\xi_1,\ldots,\xi_{r-1})$, i.e. when $\theta = 0$. In the new coordinates, the system is described by equations of the form

$$\begin{array}{rcl} \dot{z}_{\mathrm{e}} &=& \tilde{f}(z_{\mathrm{e}},\theta) \\ \dot{\theta} &=& \tilde{q}(z_{\mathrm{e}},\theta) + \tilde{b}(z_{\mathrm{e}},\theta)u\,. \end{array} \tag{31}$$

To this system we impose the same control as before, namely the control consisting of (27)–(28). In this respect, note that, by definition, we have

$$\tilde{q}(z_{\mathrm{e}},\theta) + \tilde{b}(z_{\mathrm{e}},\theta)u = q(z,\xi) - \dot{N}(\varrho,\xi) + b(z,\xi)u$$

and therefore if u is chosen as

$$u = g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho))$$

we have

$$\begin{array}{rcl} \dot{\theta} &=& \tilde{q}(z_{\mathrm{e}},\theta) + \tilde{b}(z_{\mathrm{e}},\theta)g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho)) = q(z,\xi) - \dot{N}(\varrho,\xi) + b(z,\xi)g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho)) \\ &=& k\theta + \tilde{\Delta}_3(x_{\mathrm{e}},e) \end{array}$$

where, in analogy to what it was before with (30), we have denoted by $\tilde{\Delta}_3(x_{\mathrm{e}},e)$ the "perturbation"

$$\tilde{\Delta}_3(x_{\mathrm{e}},e) = b(z,\xi)g(\psi_{\mathrm{e}}(\hat{\xi},\sigma,\varrho)) + q(z,\xi) - \dot{N}(\varrho,\xi) - k(\xi_r - N(\varrho,\xi_1,\ldots,\xi_{r-1}))\,.$$

In this respect, note that

$$\tilde{\Delta}_3(x_{\mathrm{e}},e) = b(z,\xi)\Delta_3(x_{\mathrm{e}},e)\,.$$

In summary, the controlled closed loop system, in the new coordinates, is modeled as

$$\begin{array}{rcl} \dot{z}_{\mathrm{e}} &=& \tilde{f}(z_{\mathrm{e}},\theta) \\ \dot{\theta} &=& k\theta + \tilde{\Delta}_3(x_{\mathrm{e}},e) \\ \dot{e} &=& \kappa[\mathbf{A} - \mathbf{B}\mathbf{C}\Delta_0(x_{\mathrm{e}},e)]e + \mathbf{B}_1\Delta_1(x_{\mathrm{e}},e) + \mathbf{B}_2\Delta_2(x_{\mathrm{e}},e)\,. \end{array} \tag{32}$$

With a forgivable abuse of notation, in the right-hand sides, we have left x_{e} as argument of the various Δ_i's, while – for consistency – we should have replaced it by $x_{\mathrm{e}} = \Phi^{-1}(\tilde{x}_{\mathrm{e}})$. This does not affect the analysis, though, because what matters is only the possibility of establish bounds that are independent of κ, which is in fact the case.

At this point, it is an easy matter to arrive at the following conclusion.

Assumption 3 *There exist an integer ν and a triplet*

$$L(\varrho,\xi_1,\ldots,\xi_{r-1}),\ M,\ N(\varrho,\xi_1,\ldots,\xi_{r-1})$$

in which $L(\cdot)$ and $N(\cdot)$ are smooth functions satisfying (22)–(24) and in which $L(0,0,\ldots,0) = 0$, $N(0,0,\ldots,0) = 0$, such that the zero dynamics of system (19)–(20) with output (21) are globally asymptotically and locally exponentially stable.

Proposition 5 *Consider system (1), controlled by (27)–(28). Suppose that (2) hold and that $b(x,\xi)$ is bounded as in (3). Let $L(\varrho,\xi_1,\dots,\xi_{r-1}), M, N(\varrho,\xi_1,\dots,\xi_{r-1})$ be such that Assumption 3 holds. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters k, b_0, L and $\alpha_1,\dots,\alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(x_\mathrm{e},\hat{\xi}_\mathrm{ext})$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.*

Proof. By assumption, the zero dynamics of system (19)–(20) with output (21) are globally asymptotically and locally exponentially stable. Looking at the normal form (31) of this system, we have that the equilibrium $z_\mathrm{e}=0$ of

$$\dot{z}_\mathrm{e} = \tilde{f}(z_\mathrm{e},0)$$

is globally asymptotically and locally exponentially stable. Thus, there exists a smooth function $W(z_\mathrm{e})$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and $\alpha(\cdot)$, that are quadratic in a neighborhood of the origin, such that

$$\begin{aligned}\underline{\alpha}(|z_\mathrm{e}|) \leq W(z_\mathrm{e}) &\leq \overline{\alpha}(|z_\mathrm{e}|)\\ \frac{\partial W}{\partial z_\mathrm{e}}\tilde{f}(z_\mathrm{e},0) &\leq -\alpha(|z_\mathrm{e}|)\,.\end{aligned}$$

Observe now that the first two equations of (32) can be written as (compare with (14))

$$\dot{\tilde{x}}_\mathrm{e} = \mathbf{F}(\tilde{x}_\mathrm{e}) + \mathbf{G}\Delta_3(x_\mathrm{e},e) \tag{33}$$

in which

$$\mathbf{F}(\tilde{x}_\mathrm{e}) = \begin{pmatrix}\tilde{f}(z_\mathrm{e},\theta)\\ k\theta\end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix}0\\1\end{pmatrix},$$

and consider the (positive definite and proper) function

$$V(\tilde{x}_\mathrm{e}) = W(z_\mathrm{e}) + \theta^2\,.$$

With $\mathcal{C}$ being the fixed compact set of initial conditions of the system, pick a number $R>0$ such that

$$(x_\mathrm{e},\hat{\xi}_\mathrm{ext}) \in \mathcal{C} \qquad \Rightarrow \qquad \tilde{x}_\mathrm{e} \in B_R$$

and pick a number c such that

$$\Omega_c = \{\tilde{x}_\mathrm{e} \in \mathbb{R}^{n+\nu} : V(\tilde{x}_\mathrm{e}) \leq c\} \supset B_R\,.$$

Observe now that, along the trajectories of (33),

$$\frac{\partial V}{\partial z_\mathrm{e}}\mathbf{F}(\tilde{x}_\mathrm{e}) = \frac{\partial W}{\partial z_\mathrm{e}}\tilde{f}(z_\mathrm{e},0) + \frac{\partial W}{\partial z_\mathrm{e}}[\tilde{f}(z_\mathrm{e},\theta) - \tilde{f}(z_\mathrm{e},0)] - 2|k|\theta^2\,.$$

In this respect, it should be noted that $\frac{\partial W}{\partial z_\mathrm{e}}$ is a smooth function vanishing a $z_\mathrm{e}=0$, while $[\tilde{f}(z_\mathrm{e},\theta) - \tilde{f}(z_\mathrm{e},0)]$ is a smooth function vanishing at $\theta=0$. Thus, there is a number $K_1>0$ such that, for all $\tilde{x}_\mathrm{e} \in \Omega_{c+1}$, we have

$$\left|\frac{\partial W}{\partial z_\mathrm{e}}[\tilde{f}(z_\mathrm{e},\theta) - \tilde{f}(z_\mathrm{e},0)]\right| \leq K_1|z_\mathrm{e}||\theta|\,.$$

Let

$$K_2 = \max_{\tilde{x}_e \in \Omega_{c+1}} |z_e| \,.$$

Since the function $\alpha(s)$ is quadratic in a neighborhood of the origin, it is always possible to find a number a such that

$$\alpha(s) \geq a s^2 \qquad \text{for all } 0 \leq s \leq K_2 \,.$$

Hence, we can conclude that

$$\dot{V}(\tilde{x}_e) \leq -a|z_e|^2 + K_1|z_e||\theta| - 2|k|\theta^2 \qquad \text{for all } \tilde{x}_e \in \Omega_{c+1} \,.$$

From this, using standard arguments, we can conclude that, if k is sufficiently large,

$$\frac{\partial V}{\partial z_e}\mathbf{F}(\tilde{x}_e) \leq -d|\tilde{x}_e|^2 \qquad \text{for all } \tilde{x}_e \in \Omega_{c+1} \,,$$

for some $d > 0$.

A property of this kind is precisely the property that was used in the previous discussions and hence we can conclude that, if k is chosen in the way indicated, the remaining design parameters can be chosen so as to obtain the desired convergence result. $\triangleleft$

We conclude the section with some observation regarding the zero dynamics of (19)–(20) with output (21), whose asymptotic properties are used in the proof of this result. As it is well-known, to identify these dynamics it is not really necessary to use the diffeomorphism that brings the system to its normal form (31). In fact, it suffices to determine the unique u that forces $h_e(x_e)$ to be identically zero and to replace it into the dynamics of ϱ, along with the constraint resulting from $h_e(x_e) = 0$. To simplify the notation, let $\xi^*(\varrho, \xi_1, \ldots, \xi_{r-1})$ denote the r-dimensional function of ϱ and $\xi_1, \ldots, \xi_{r-1}$ defined as

$$\xi^*(\varrho, \xi_1, \ldots, \xi_{r-1}) = \begin{pmatrix} \xi_1 \\ \cdots \\ \xi_{r-1} \\ N(\varrho, \xi_1, \ldots, \xi_{r-1}) \end{pmatrix}$$

and note that the unique u forcing $\theta = \xi_{r-1} - N(\varrho, \xi_1, \ldots, \xi_{r-1})$ to be zero is the solution of

$$0 = q(z, \xi^*) + b(z, \xi^*)u - \dot{N}(\varrho, \xi^*) \,.$$

With this in mind, it is readily seen that the zero dynamics of the system can be expressed as

$$\begin{aligned} \dot{z} &= f(z, \xi^*) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= N(\varrho, \xi_1, \ldots, \xi_{r-1}) \\ \dot{\varrho} &= L(\varrho, \xi_1, \ldots, \xi_{r-1}) + M[b(z, \xi^*)]^{-1}[\dot{N}(\varrho, \xi^*) - q(z, \xi^*)] \,. \end{aligned} \tag{34}$$

The system thus obtained can be given the following interpretation. Let $b_0(\xi))$ be a fixed function of ξ and define

$$\overline{L}(\varrho, \xi_1, \ldots, \xi_{r-1}) = L(\varrho, \xi_1, \ldots, \xi_{r-1}) + [b_0(\xi^*)]^{-1} M \dot{N}(\varrho, \xi^*) \,,$$

$$\Delta_b(z,\xi) = \frac{b_0(\xi) - b(z,\xi)}{b_0(\xi)b(z,\xi)}\,.$$

Then, the last equation that characterizes the system can be written as

$$\dot\varrho = \overline{L}(\varrho,\xi_1,\ldots,\xi_{r-1}) + M\Big(\Delta_b(z,\xi^*)\dot N(\varrho,\xi^*) - \frac{q(z,\xi^*)}{b(z,\xi^*)}\Big)\,.$$

In this way, system (34) can be seen as the interconnection of a system, with state $z,\xi_1,\ldots,\xi_{r-}$, inputs $u_\mathrm{a}, v_\mathrm{a}$ and outputs $\xi_1,\ldots,\xi_{r-1},y_\mathrm{a}$, defined as

$$\begin{array}{rcl}\dot z &=& f(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})\\ \dot\xi_1 &=& \xi_2\\ && \cdots\\ \dot\xi_{r-1} &=& u_\mathrm{a}\\ y_\mathrm{a} &=& \dfrac{q(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})}{b(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})} - \Delta_b(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})v_\mathrm{a}\end{array} \tag{35}$$

and a system with state ϱ, defined as

$$\begin{array}{rcl}\dot\varrho &=& \overline{L}(\varrho,\xi_1,\ldots,\xi_{r-1}) - My_\mathrm{a}\\ u_\mathrm{a} &=& N(\varrho,\xi_1,\ldots,\xi_{r-1})\\ v_\mathrm{a} &=& \dot N(\varrho,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})\,.\end{array} \tag{36}$$

In this interpretation, the Assumption XXX above can be simply interpreted as the assumption that system (36) globally asymptotically stabilizes system (35). Note, in particular, that if $b(z,\xi)$ is independent of z, one could pick $b_0(\xi) = b(z,\xi)$ resulting in

$$\Delta_b(z,\xi) = 0\,.$$

In this special case, system (35) reduces to

$$\begin{array}{rcl}\dot z &=& f(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})\\ \dot\xi_1 &=& \xi_2\\ && \cdots\\ \dot\xi_{r-1} &=& u_\mathrm{a}\\ y_\mathrm{a} &=& \dfrac{q(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})}{b(z,\xi_1,\ldots,\xi_{r-1},u_\mathrm{a})}\end{array}$$

and system (36) reduces to

$$\begin{array}{rcl}\dot\varrho &=& \overline{L}(\varrho,\xi_1,\ldots,\xi_{r-1}) - My_\mathrm{a}\\ u_\mathrm{a} &=& N(\varrho,\xi_1,\ldots,\xi_{r-1})\,.\end{array}$$

4 A Unified Approach to Problems of Asymptotic Tracking and Disturbance Rejection

A.Isidori

4.1 Introduction

In this Chapter, we address the problem of *output regulation* (sometimes also known as *generalized tracking* problem, or *generalized servomechanism* problem) for nonlinear systems. Formally, the problem of output regulation is cast in the following terms. The controlled plant is a finite-dimensional, time-invariant, nonlinear system modelled by equations of the form

$$\begin{aligned} \dot{x} &= F(w,x,u) \\ e &= H(w,x) \\ y &= K(w,x) \end{aligned} \tag{4.1}$$

in which $x \in \mathbb{R}^n$ is a vector of state variables, $u \in \mathbb{R}$ is the input used for *control* purposes, $w \in \mathbb{R}^s$ is a vector of inputs which cannot be controlled and include *exogenous* commands, exogenous disturbances and model uncertainties, $e \in \mathbb{R}$ is a vector of *regulated* outputs which include tracking errors and any other variable that needs to be steered to 0, $y \in \mathbb{R}^p$ is a vector of outputs that are available for *measurement.* The exogenous input $w(t)$ is assumed to be a (undefined) member of the family of all solutions of a fixed ordinary differential equation of the form

$$\dot{w} = s(w) \tag{4.2}$$

obtained when the initial condition $w(0)$ is allowed to vary on a prescribed set W. This system is usually referred to as the *exosystem.* The initial state of (4.1) and of (4.2) are assumed to range over fixed compact sets X and W, the latter being invariant under the dynamics of (4.2). The problem of output regulation is to design a controller

$$\begin{aligned} \dot{\xi} &= \varphi(\xi,y) \\ u &= \gamma(\xi,y) \end{aligned}$$

with initial state in a compact set Ξ, yielding a closed-loop system in which

- the positive orbit of $W \times X \times \Xi$ is bounded,
- $\lim\limits_{t\to\infty} e(t) = 0$, uniformly in the initial condition (on $W \times X \times \Xi$).

We observe that, in the general setup presented above, the vector w of exogenous inputs may well include (constant) uncertain parameters, which are hence assumed to range on a given compact set. Thus, if a controller solves the problem at issue, the goal of asymptotic regulation is achieved robustly with respect to (constant) parameter uncertainties.

The theory of output regulation of nonlinear systems, which uses a combination of geometry and nonlinear dynamical systems theory, was initiated by pioneering works of [10, 9] who showed how to design a controller that provides a local solution near an equilibrium point, in the presence of exogenous signals which were produced by a neutrally stable system. Since these early contributions, the theory has experienced a tremendous growth, culminating in the recent development of design methods able to handle issues of global convergence (as in [3]), the case of parametric uncertainties affecting the autonomous (linear) system which generates the exogenous signals (such as in [15]), the case of nonlinear exogenous systems (such as in [2]), or a combination thereof (as in [13]). A thorough presentation of several recent advances in this area can also be found in the recent books [12, 8, 14].

4.2 The plant and the basic assumptions

In what follows, we consider a nonlinear system having relative degree r between control input $u \in \mathbb{R}$ and regulated output $e \in \mathbb{R}$ described in normal form as

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w,z,\xi,\zeta) \\ \dot{\xi} & = & A\xi + B\zeta \\ \dot{\zeta} & = & q(w,z,\xi,\zeta) + b(w,z,\xi,\zeta)u \\ e & = & C\xi \end{array} \tag{4.3}$$

in which $z \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{r-1}$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

As indicated above, the functions characterizing the model (4.3) are assumed to be smooth functions of their arguments. In addition, we assume the existence of a pair of real numbers (b_0, b_1) such that

$$0 < b_0 \le b(w,z,\xi,\zeta) \le b_1 \,. \tag{4.4}$$

Note that

$$\xi = \mathrm{col}(e, e^{(1)}, \ldots, e^{(r-2)}), \qquad \zeta = e^{(r-1)} \,.$$

Motivated by well-known standard design procedures (see e.g. [6]), we assume throughout that the entire *partial state* $(\xi_1, \ldots, \xi_{r-1}, \zeta)$ is available for measurement, i.e.

$$y = \mathrm{col}(\xi_1, \ldots, \xi_{r-1}, \zeta) \,.$$

The states w and z are, on the contrary, not available for measurement.

The point of departure for the solution of the problem is, as usual, the assumption of the existence of a (smooth) manifold which can be rendered invariant by feedback and on which the regulated output vanishes (see [10]). In the case of system (4.3), this amounts to the assumption of the existence of a smooth map $\pi : W \to \mathbb{R}^m$ satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(w, \pi(w), 0, 0) \qquad \forall w \in W \,. \tag{4.5}$$

This being the case, it is readily seen that the set

$$\mathcal{S}^* = \{(w,z,\xi,\zeta) : w \in W, z = \pi(w), \xi = 0, \zeta = 0\}$$

is rendered invariant by the control

$$u^*(w) = -\frac{q(w,\pi(w),0,0)}{b(w,\pi(w),0,0)} \tag{4.6}$$

and, indeed, the regulated variable $e = \xi_1$ vanishes on this set. The input $u^*(w)$ is the input which forces e to remain identically zero.

The second step in the solution of the problem usually consists in making assumptions that make it possible to build an "internal" model for the control $u^*(w)$. In a series of recent papers, it was shown how these assumptions could be progressively weakened, moving from the so-called assumption of "immersion into a linear observable system" (as in [9]) to "immersion into a nonlinear uniformly observable system" (as in [2]) to the recent results of [13], in which it was shown that no assumption is in fact needed for the construction of an internal model if only continuous (thus possibly not locally Lipschitz) controllers are acceptable. Motivated by this, we assume, in what follows, the existence of $d \in \mathbb{N}$, a map $F : \mathbb{R}^d \to \mathbb{R}^d$, a $d \times 1$ column vector G_0, a map $\gamma : \mathbb{R}^d \to \mathbb{R}$ and a map $\tau : W \to \mathbb{R}^d$ satisfying

$$\begin{array}{rcll} \dfrac{\partial \tau}{\partial w} s(w) & = & F(\tau(w)) + G_0 \gamma(\tau(w)) & \forall w \in W \\ u^*(w) & = & \gamma(\tau(w)) & \forall w \in W\,. \end{array} \tag{4.7}$$

Coherently with the assumptions on (4.3), $F(\cdot)$, $\gamma(\cdot)$ and $\tau(\cdot)$ are assumed to be smooth maps.

Remark 4.1 Under the (mild) assumption that

$$L_s^d u^*(w) = \phi(u^*(w), L_s u^*(w), \ldots, L_s^{d-1} u^*(w))\,, \tag{4.8}$$

for some smooth $\phi(x_1, \ldots, x_d)$ and all $w \in W$, a smooth internal model can be obtained taking

$$F(x) = \begin{pmatrix} x_2 \\ \cdots \\ x_d \\ \phi(x_1, \ldots, x_d) \end{pmatrix} - G_0 x_1 \qquad \gamma(x) = x_1\,, \tag{4.9}$$

in which case (4.7) hold with

$$\tau(w) = \mathrm{col}(u^*(w), \ldots, L_s^{d-1} u^*(w))\,.$$

This includes the (classical) case of linear internal models. Recent advances in the theory of nonlinear observers (see e.g. [13]) show that, if d is large enough, and $F(x) = F_0 x$ with F_0 Hurwitz and (F_0, G_0) controllable, a C^1 map $\tau(\cdot)$ and a C^0 map $\gamma(\cdot)$ which do fulfill (4.7) always exist. $\lhd$

4.3 The control

Consider, for the plant (4.3), a controller of the form

$$\begin{array}{rcl} u & = & \gamma(\eta) + \beta \dot{N}(\varphi) + v \\ \dot{\varphi} & = & L(\varphi) - M v \\ \dot{\eta} & = & F(\eta) + G_0[\gamma(\eta) + v] \\ v & = & -k[\zeta - H\xi - N(\varphi)] \end{array} \tag{4.10}$$

which is a dynamic controller, with internal state (φ, η), "driven" only by the measured variables (ξ, ζ). Assume (without loss of generality) that

$$\frac{\partial N}{\partial \varphi} M = 0 \tag{4.11}$$

in which case

$$\dot{N}(\varphi) = \frac{\partial N}{\partial \varphi} L(\varphi)\,.$$

Changing ζ into

$$\theta = \zeta - H\xi - N(\varphi)$$

yields a closed-loop system of the form

$$\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, \xi, H\xi + N(\varphi) + \theta) \\
\dot{\xi} &= A\xi + B[H\xi + N(\varphi) + \theta] \\
\dot{\varphi} &= L(\varphi) - Mv \\
\dot{\eta} &= F(\eta) + G_0[\gamma(\eta) + v] \\
\dot{\theta} &= Q(w, z, \xi, H\xi + N(\varphi) + \theta) + b(w, z, \xi, H\xi + N(\varphi) + \theta)[\gamma(\eta) + v] \\
&\quad + \Delta(w, z, \xi, H\xi + N(\varphi) + \theta)\dot{N}(\varphi)\,,
\end{aligned}$$

in which we have set

$$\begin{aligned}
Q(w, z, \xi, \zeta) &= q(w, z, \xi, \zeta) - H(A\xi + B\zeta) \\
\Delta(w, z, \xi, \zeta) &= b(w, z, \xi, \zeta)\beta - 1\,,
\end{aligned}$$

with control

$$v = -k\theta\,.$$

Remark 4.2 Note that, in case the coefficient $b(w, z, \xi, \zeta)$ only depends on the measured variables (ξ, ζ), one can choose $\beta = 1/b$, obtaining in this way $\Delta(w, z, \xi, \zeta) = 0$. $\triangleleft$

This system can be regarded as a system with input v and output θ, having relative degree 1, with input v to be chosen as $v = -k\theta$, that is as a negative output feedback. To facilitate the analysis, we bring this system in normal form. Since $b(w, z, \xi, \zeta)$ is bounded as in (4.4), by the method of the characteristics one can obtain a globally defined change of coordinates

$$X \,:\, \eta \;\mapsto\; x = X(w, z, \xi, \varphi, \eta, \theta)$$

in which X satisfies

$$\frac{\partial X}{\partial \eta} G_0 + \frac{\partial X}{\partial \theta} b(w, z, \xi, H\xi + N(\varphi) + \theta) = 0\,.$$

At $\theta = 0$, the map X is the identity map, namely $X(w, z, \xi, \varphi, \eta, 0) = \eta$ which in turn implies

$$\left[\frac{\partial X}{\partial \eta}\right]_{\theta=0} = I\,.$$

Actually, it is not difficult to find a closed form for X, which turns out to be

$$X(w, z, \xi, \varphi, \eta, \theta) = \eta - G_0 \int_0^\theta \frac{1}{b(w, z, \xi, H\xi + N(\varphi) + t)}\, dt\,.$$

From this, using our earlier assumption (4.11), it is readily seen that

$$\frac{\partial X}{\partial \varphi} M = 0\,.$$

Likewise, by the method of the characteristics one can obtain a globally defined change of coordinates

$$K \,:\, \varphi \;\mapsto\; \chi = K(w, z, \xi, \varphi, \theta)$$

in which K satisfies

$$\frac{\partial K}{\partial \varphi} M - \frac{\partial K}{\partial \theta} b(w, z, \xi, H\xi + N(\varphi) + \theta) = 0\,.$$

At $\theta = 0$, the map K is the identity map, namely $K(w, z, \xi, \varphi, 0) = \varphi$ which in turn implies

$$\left[\frac{\partial K}{\partial \varphi}\right]_{\theta=0} = I\,.$$

The inverses of K and X define a pair of maps

$$\begin{array}{rcl} \varphi & = & \hat{K}(w, z, \xi, \chi, \theta) \\ \eta & = & \hat{X}(w, z, \xi, \chi, x, \theta) \end{array}$$

which, at $\theta = 0$, are identities in χ and – respectively – in x, that is

$$\hat{K}(w, z, \xi, \chi, 0) = \chi\,, \qquad \hat{X}(w, z, \xi, \chi, x, 0) = x\,.$$

Changing coordinates in this way yields a system of the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w, z, \xi, H\xi + N(\hat{K}) + \theta) \\ \dot{\xi} & = & A\xi + B[H\xi + N(\hat{K}) + \theta] \\ \dot{\chi} & = & \dfrac{\partial K}{\partial \varphi}\Big[L(\hat{K}) + M\Big(\dfrac{Q}{b}(w, z, \xi, \theta + N(\hat{K}) + H\xi) + \dfrac{\Delta}{b}\,\dot{N}(\hat{K}) + \gamma(\hat{X})\Big)\Big] + R_\chi \\ \dot{x} & = & F(\hat{X}) - G_0\Big(\dfrac{Q}{b}(w, z, \xi, \theta + N(\hat{K}) + H\xi) + \dfrac{\Delta}{b}\,\dot{N}(\hat{K})\Big) + R_x \\ \dot{\theta} & = & Q(w, z, \xi, H\xi + N(\hat{K}) + \theta) + b(w, z, \xi, H\xi + N(\hat{K}) + \theta)[\gamma(\hat{X}) + v] \\ & & \quad + \Delta(w, z, \xi, H\xi + N(\hat{K}) + \theta)\dot{N}(\hat{K})\,, \end{array} \tag{4.12}$$

in which, for readability, we have omitted the indication of the arguments of $\hat{K}$, $\hat{X}$ and Δ/b, and we have set

$$R_\chi = \frac{\partial K}{\partial w}s(w) + \frac{\partial K}{\partial z}f(w, z, \xi, H\xi + N(\hat{K}) + \theta) + \frac{\partial K}{\partial \xi}(A\xi + B[H\xi + N(\varphi) + \theta])$$

$$R_x = \frac{\partial X}{\partial w}s(w) + \frac{\partial X}{\partial z}f(w, z, \xi, H\xi + N(\hat{K}) + \theta) + \frac{\partial X}{\partial \xi}(A\xi + B[H\xi + N(\varphi) + \theta]) + \frac{\partial X}{\partial \varphi}L(\hat{K})\,.$$

Note that at $\theta = 0$ both these terms vanish, because at $\theta = 0$ the map K is simply an identity in φ and the map X is simply an identity in η.

The system obtained in this way can be seen as feedback interconnection of a system with input θ and state (w, z, ξ, χ, x) and of a system with input (w, z, ξ, χ, x) and state θ. In fact, setting

$$\mathbf{p} = \mathrm{col}\{w, z, \xi, \chi, x\}$$

the system cam be viewed as a system of the form

$$\begin{array}{rcl} \dot{\mathbf{p}} & = & \mathbf{F}(\mathbf{p}) + \mathbf{G}(\mathbf{p}, \theta)\theta \\ \dot{\theta} & = & \mathbf{H}(\mathbf{p}) + \mathbf{H}(\mathbf{p}, \theta)\theta - \mathbf{b}(\mathbf{p}, \theta)v \end{array} \tag{4.13}$$

with control to be chosen as

$$v = -k\theta\,. \tag{4.14}$$

The advantage of seeing system (4.12) in this form is that we can appeal to the following result (see e.g. [13]).

Theorem 4.1 *Consider a system of the form (4.13) with v as in (4.14). The functions $\mathbf{F}(\mathbf{p})$ and $\mathbf{H}(\mathbf{p})$ are locally Lipschitz and the functions $\mathbf{G}(\mathbf{p},\theta)$ and $\mathbf{H}(\mathbf{p},\theta)$ are continuous. Let $\mathbf{P}$ be an arbitrary fixed compact set. Suppose that $\mathbf{b}(\mathbf{p},\theta) > 0$ for all $(\mathbf{p},\theta)$. Suppose there exists a set $\mathcal{A}$ which is locally exponentially stable for*

$$\dot{\mathbf{p}} = \mathbf{F}(\mathbf{p}),$$

with a domain of attraction that contains the set $\mathbf{P}$. Suppose also that

$$\mathbf{H}(\mathbf{p}) = 0, \qquad \forall \mathbf{p} \in \mathcal{A}.$$

Then, for any choice of a compact set Θ, there is a number k^* such that, for all $k > k^*$, the set $\mathcal{A}\times\{0\}$ is locally exponentially stable, with a domain of attraction that contains $\mathbf{P}\times\Theta$.

If the assumption of this Theorem are fulfilled and, in addition, the regulated variable $e = \xi_1$ vanishes $\mathcal{A}$, we conclude that the proposed controller is able to solve the problem of output regulation.

4.4 The structure of the core subsystem

All of the above suggests the use of the degrees of freedom in the choice of the parameters of the controller in order to impose appropriate asymptotic properties on the subsystem obtained by setting $\theta = 0$ in (4.12). The latter, in view of the fact that at $\theta = 0$

$$\hat{K} = \chi\,, \quad \hat{X} = x\,, \quad \frac{\partial K}{\partial \varphi} = 0\,, \quad R_\chi = 0\,, \quad R_x = 0\,,$$

reads as

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f(w,z,\xi,H\xi + N(\chi))\\
\dot{\xi} &= A\xi + B(H\xi + N(\chi))\\
\dot{\chi} &= L(\chi) + M\Big(\frac{Q}{b}(w,z,\xi,H\xi + N(\chi)) + \frac{\Delta}{b}\dot{N}(\chi) + \gamma(x)\Big)\\
\dot{x} &= F(x) - G_0\Big(\frac{Q}{b}(w,z,\xi,H\xi + N(\chi)) + \frac{\Delta}{b}\dot{N}(\chi)\Big)
\end{aligned} \tag{4.15}$$

Theorem 4.1 above identifies an auxiliary problem which, if solved, makes it possible to use the controller (4.10) for the solution of the problem of output regulation for the original plant: shape the internal model $\{F(x), G_0, \gamma(x)\}$ and find, if possible, a triplet $\{L(\chi), M, N(\chi)\}$ in such a way that system (4.15) possesses a compact invariant set $\mathcal{A}$ which is locally exponentially stable and attracts all admissible initial conditions, and that both ξ_1 and the map

$$Q(w,z,\xi,H\xi + N(\chi)) + b(w,z,\xi,H\xi + N(\chi))\gamma(x) + \Delta(w,z,\xi,H\xi + N(\chi))\dot{N}(\chi)\,. \tag{4.16}$$

vanish on this set.

Recall now that, by assumption, there exists $\pi(w)$ and $\tau(w)$ satisfying (4.5) and (4.7). Hence, it is readily seen that if $L(0) = 0$ and $N(0) = 0$, the set

$$\mathcal{A} = \{(w,z,\xi,\chi,x) : w \in W, z = \pi(w), \xi = 0, \chi = 0, x = \tau(w)\}$$

is a compact invariant set of (4.15). Moreover, by construction, the map (4.16) vanishes on this set. Trivially, also ξ_1 vanishes on this set. Thus, it is concluded if the set $\mathcal{A}$ can be made local exponentially stable, with a domain of attraction that contains the compact set of all admissible initial conditions, the proposed controller, with large k solves the problem of output regulation.

System (4.15) is not terribly difficult to handle. As a matter of fact, it can be regarded as interconnection of three much simpler subsystems. To see this, set

$$\begin{array}{rcl} z_{\mathrm{a}} & = & z - \pi(w) \\ \tilde{x} & = & x - \tau(w) \end{array}$$

and define

$$f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta) = f(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta) - f(w, \pi(w), 0, 0)$$

$$h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta) = \frac{Q}{b}(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta) - \frac{Q}{b}(w, \pi(w), 0, 0)$$

and

$$\Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta) = \frac{\Delta}{b}(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta) = \beta - \frac{1}{b(w, z_{\mathrm{a}} + \pi(w), \xi, \zeta)} .$$

In the new coordinates thus introduced, the invariant manifold $\mathcal{A}$ is simply the set

$$\mathcal{A} = \{(w, z_{\mathrm{a}}, \xi, \chi, \tilde{x}) : w \in W, (z_{\mathrm{a}}, \xi, \chi, \tilde{x}) = (0, 0, 0, 0)\} .$$

Bearing in mind (4.5), (4.7) and (4.6), it is readily seen that

$$\dot{z}_{\mathrm{a}} = f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi))$$

and

$$\frac{Q}{b}(w, z, \xi, H\xi + N(\chi)) = h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) - \gamma(\tau(w)) .$$

In view of this, using again (4.7), the core subsystem (4.15) can be seen as a system with input $\bar{u}$ and output $\bar{y}$ defined as

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z}_{\mathrm{a}} & = & f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) \\ \dot{\xi} & = & A\xi + B(H\xi + N(\chi)) \\ \dot{\chi} & = & L(\chi) + M[h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) + \Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi))\dot{N}(\chi) + \bar{u}] \\ \dot{\tilde{x}} & = & F(\tilde{x} + \tau(w)) - F(\tau(w)) - G_0[h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi)) + \Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + N(\chi))\dot{N}(\chi)] \\ \bar{y} & = & \gamma(\tilde{x} + \tau(w)) - \gamma(\tau(w)) \end{array} \tag{4.17}$$

subject to unitary output feedback

$$\bar{u} = \bar{y} .$$

System (4.17), in turn, can be seen as the cascade of an “inner loop” consisting of a subsystem, which we call the “auxiliary plant”, modelled by equations of the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z}_{\mathrm{a}} & = & f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + u_{\mathrm{a}}) \\ \dot{\xi} & = & (A + BH)\xi + Bu_{\mathrm{a}} \\ y_{\mathrm{a}} & = & h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + u_{\mathrm{a}}) + \Delta_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi + u_{\mathrm{a}})v_{\mathrm{a}} , \end{array} \tag{4.18}$$

controlled by

$$\begin{array}{rcl} \dot{\chi} & = & L(\chi) + M[y_{\mathrm{a}} + \bar{u}] \\ u_{\mathrm{a}} & = & N(\chi) \\ v_{\mathrm{a}} & = & \dot{N}(\chi) , \end{array} \tag{4.19}$$

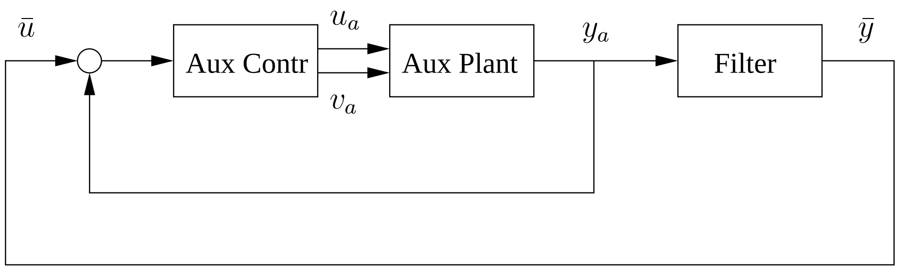

Figure 4.1: The feedback structure of system (4.15)

cascaded with a system, which we call a "weighting filter", modelled by equations of the form

$$\begin{aligned} \dot{\tilde{x}} &= F(\tilde{x}+\tau(w)) - F(\tau(w)) - G_0 y_{\mathrm{a}} \\ \bar{y} &= \gamma(\tilde{x}+\tau(w)) - \gamma(\tau(w)) \,. \end{aligned} \tag{4.20}$$

All of this is depicted in Fig.F 4.1.

With this interpretation in mind, the idea is now to use the small-gain Theorem to enforce the desired asymptotic properties on system (4.15). To this purpose, set

$$\mathbf{x} = \mathrm{col}(z_{\mathrm{a}}, \xi, \chi, \tilde{x})$$

and observe that system (4.17) is a system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\mathbf{x}} &= \mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\bar{u} \\ \bar{y} &= \mathbf{C}(w,\mathbf{x}) \end{aligned} \tag{4.21}$$

in which

$$\mathbf{A}(w,\mathbf{0}) = \mathbf{0} \qquad \mathbf{C}(w,\mathbf{0}) = \mathbf{0}\,.$$

Let $W \times \mathbf{X}$ be the desired (compact) set of initial conditions for $(w, \mathbf{x})$ in system (4.15). Suppose there exists a smooth positive definite and proper function $\mathbf{V}(\mathbf{x})$, with quadratic bounds for small $|\mathbf{x}|$, namely satisfying

$$a_1|\mathbf{x}|^2 \le \mathbf{V}(\mathbf{x}) \le a_2|\mathbf{x}|^2, \qquad \forall |\mathbf{x}| \le d$$

for some $a_1 > 0, a_2 > 0, d > 0$, such that

$$\begin{gathered} \frac{\partial \mathbf{V}}{\partial \mathbf{x}}[\mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\bar{u}] - \gamma^2\bar{u}^2 + \delta^2[\mathbf{C}(w,\mathbf{x})]^2 \le -a|\mathbf{x}|^2 \\ \forall (w,\mathbf{x},\bar{u}) \in W \times \mathbf{V}^{-1}([0,c]) \times R \end{gathered} \tag{4.22}$$

for some $a > 0$, with $c > 0$ such that

$$\mathbf{X} \in \mathbf{V}^{-1}([0,c])\,.$$

If $\gamma \le \delta$ in the inequality above, trivially

$$\frac{\partial \mathbf{V}}{\partial \mathbf{x}}[\mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\mathbf{C}(w,\mathbf{x})] \le -a|\mathbf{x}|^2 \qquad \forall (w,\mathbf{x}) \in W \times \mathbf{V}^{-1}([0,c])\,.$$

Since system

$$\dot{\mathbf{x}} = \mathbf{A}(w,\mathbf{x}) + \mathbf{B}(w,\mathbf{x})\mathbf{C}(w,\mathbf{x})$$

is precisely system (4.15), we conclude that if the inequality above holds for $\gamma \le \delta$, the set $\mathcal{A} = W \times \mathbf{0}$ is a locally exponentially stable invariant set of (4.15), with a domain of attraction that contains the set $W \times \mathbf{X}$ of all admissible initial conditions, as sought.

We have in this way shown that, if it is possible to use the degrees of freedom in the parameters of the controller so as to enforce a *dissipation inequality* of the form (4.22) for some $\gamma \le \delta$, system (4.15) has the desired asymptotic properties and the proposed controller solves the problem of output regulation. In the next sections we will see how this can be achieved.

4.5 The asymptotic properties of the core subsystem

We concentrate now on the issue of enforcing the dissipation inequality (4.22) for some $\gamma \le \delta$ on system (4.17). The latter is the cascade of two subsystems: the "inner loop", consisting of (4.18) and (4.19), and the "filter" (4.20). An obvious prerequisite of (4.22) is the local exponential stability, with appropriate regions of attraction, of both subsystems of this cascade. Stability of the filter is not an issue, as we will show below.

4.5.1 Stability of the filter

For convenience, assume that $F(x)$ and $\gamma(x)$ have the form (4.9), and consider initially the simpler case in which the function $\phi(\cdot)$ in (4.8) is a linear function. In this case, $F(x)$ is a linear function, which we write as

$$F(x) = F_0 x = (\Phi - G_0 H)x$$

where

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0 & -s_1 & 0 & \cdots & -s_{d-1} \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

The pair (Φ, H) is an observable pair and therefore one can always find a vector G_0 assigning to F_0 any prescribed set of eigenvalues. Note also that, if the spectra of F_0 and Φ are disjoint, any such G_0 makes the pair (F_0, G_0) a controllable pair.

The equations (4.20) describing the filter reduce to linear equations

$$\begin{aligned} \dot{\tilde{x}} &= F_0 \tilde{x} - G_0 y_{\mathrm{a}} \\ \bar{y} &= H\tilde{x} \end{aligned} \tag{4.23}$$

in which F_0 is a Hurwitz matrix and (F_0, G_0) is controllable. The degrees of freedom in this structure are the entries of G_0, which are subject only to the condition that F_0 is Hurwitz. Alternatively, one may consider changing system (4.23), via state-space isomorphism, into a system

$$\begin{aligned} \dot{\hat{x}} &= F\hat{x} - G y_{\mathrm{a}} \\ \bar{y} &= \Psi\hat{x} \end{aligned} \tag{4.24}$$

in which

$$F = TF_0T^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & 0 & \cdots & -c_{d-1} \end{pmatrix}, \qquad G = TG_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix},$$

and

$$\Psi = HT^{-1} = \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 & \cdots & \psi_{d-1} \end{pmatrix} .$$

In this case, the degrees of freedom are the entries $c_0, \ldots, c_{d-1}$ of F, subject to the condition that the latter is Hurwitz. Note that the matrix Ψ necessarily satisfies

$$T\Phi T^{-1} = F + G\Psi$$

and hence is the unique matrix which assigns to $F + G\Psi$ the eigenvalues of Φ.

If the function $\phi(\cdot)$ in (4.8) is nonlinear, one can proceed as follows. Let $\phi_c(\cdot)$ be any function which agrees with $\phi(\cdot)$ on $\tau(W)$ and is globally Lipschitz (in which case a condition identical to (4.8) continues to hold with $\phi_c(\cdot)$ replacing $\phi(\cdot)$). Then, proceeding as in [2] (see also [7]), choose

$$G_0 = D_\kappa \bar{G}$$

in which

$$
\begin{aligned}
D_\kappa &= \operatorname{diag}(\kappa, \kappa^2, \ldots, \kappa^d) \\
\bar{G} &= \operatorname{col}(c_{d-1}, c_{d-2}, \ldots, c_0)
\end{aligned}
$$

with $c_{d-1}, \ldots, c_0$ coefficients of a Hurwitz polynomial

$$p(\lambda) = c_0 + c_1\lambda + \cdots + c_{d-1}\lambda^{d-1} + \lambda^d \,.$$

Set $\bar{x} = D_\kappa^{-1}\tilde{x}$ and observe that

$$\dot{\bar{x}} = \kappa \bar{A}\bar{x} - \bar{B}\Delta(\bar{x}, \tau(w), \kappa) - \bar{G}y_{\mathrm{a}}$$

in which $\bar{A}$ is a Hurwitz matrix (only determined by the choice of $c_{d-1}, \ldots, c_0$),

$$\bar{B} = \operatorname{col}(0, 0, \ldots, 0, 1)\,,$$

and

$$\Delta(\bar{x}, \tau(w), \kappa) = \frac{1}{\kappa^d}[\phi_c(D_k\bar{x} + \tau(w)) - \phi_c(\tau(w))]$$

Since $\phi_c(\cdot)$ is globally Lipschitz, with a Lipschitz constant L, for any $\kappa > 1$ we have

$$|\Delta(\bar{x}, \tau(w), \kappa)| \le \frac{1}{\kappa^d} L|D_\kappa \bar{x}| \le L|\bar{x}|\,.$$

Let $\bar{P}$ be a positive definite solution of the Lyapunov equation $\bar{P}\bar{A} + \bar{A}^{\mathrm{T}}\bar{P} = -I$ and set $\bar{V}(\bar{x}) = \bar{x}^{\mathrm{T}}\bar{P}\bar{x}$. Then,

$$\dot{\bar{V}} \le -\kappa|\bar{x}|^2 + 2|\bar{P}|\,L\,|\bar{x}|^2 + 2|\bar{P}\bar{G}|\,|\bar{x}|\,|y_{\mathrm{a}}| \le -(\kappa - 2|\bar{P}|L - |\bar{P}\bar{G}|^2)|\bar{x}|^2 + |y_{\mathrm{a}}|^2\,.$$

From this, it is seen that, if κ is large enough,

$$\dot{\bar{V}} \le -d|\bar{x}|^2 + |y_{\mathrm{a}}|^2 \tag{4.25}$$

for some $d < 0$, i.e. the filter is globally input-to-state stable, actually with a linear gain function.

4.5.2 Design for minimum-phase plants

The simplest situation in which the design paradigm outlined above can be successfully implemented is the case in which the controlled plant satisfies the following assumptions:

- the function $f(w, z, \xi, \zeta)$ in (4.3) does not depend on $\xi_2, \xi_3, \ldots, \xi_{r-1}, \zeta$, in which case this function will be rewritten as $f(w, z, C\xi)$ and, accordingly, the function $f_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, \zeta)$ will be rewritten as $f_{\mathrm{a}}(w, z_{\mathrm{a}}, C\xi)$,

- there exists a smooth positive definite and proper function $V(z_{\mathrm{a}})$, with quadratic bounds for small $|z_{\mathrm{a}}|$, satisfying
$$\frac{\partial V}{\partial z_{\mathrm{a}}} f_{\mathrm{a}}(w, z_{\mathrm{a}}, 0) \le -\alpha(|z_{\mathrm{a}}|)$$
some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ which is quadratic for small values of the argument.

A plant which satisfies this assumption is said to be globally asymptotically and locally exponentially *minimum phase* (see e.g. [1]).

With this in mind, set $M = 0$, $N = 0$ and let $\dot{\chi} = L(\chi)$ any arbitrary globally stable system. In this case system (4.17) reduces to

$$\begin{array}{rcl}\dot{w} & = & s(w)\\ \dot{z_{\mathrm{a}}} & = & f_{\mathrm{a}}(w, z_{\mathrm{a}}, C\xi)\\ \dot{\xi} & = & (A+BH)\xi\\ \dot{\chi} & = & L(\chi)\\ \dot{\tilde{x}} & = & F(\tilde{x}+\tau(w)) - F(\tau(w)) - G_0 h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi)\\ \bar{y} & = & \gamma(\tilde{x}+\tau(w)) - \gamma(\tau(w))\end{array} \tag{4.26}$$

In this system, the input $\bar{u}$ is no longer present. Thus, it suffices to fulfill the dissipation inequality (4.22) for $\gamma = \delta = 0$. Neglecting the dynamics of χ, which is totally decoupled, and letting $\mathbf{x} = (z_{\mathrm{a}}, \xi, \tilde{x})$, we seek a positive definite and proper function $\mathbf{V}(\mathbf{x})$, with quadratic bounds for small $|\mathbf{x}|$, satisfying

$$\dot{\mathbf{V}}(w, \mathbf{x}) \leq -a|\mathbf{x}|^2 \qquad \forall (w, \mathbf{x}) \in W \times \mathbf{V}^{-1}([0, c])\,, \tag{4.27}$$

where $c > 0$ is such that $\mathbf{X} \in \mathbf{V}^{-1}([0, c])$. If the latter holds, the proposed controller solves the problem of output regulation.

The only free parameters left in the design of the controller are the coefficients of the vector H. Let $\mathbf{Z}_{\mathrm{a}} \times \mathbf{\Xi}$ be a fixed compact set of initial conditions for (z_{a}, ξ). It is well known that (see e.g. [11]), under the assumptions above, it is always possible to find a vector H and a positive definite matrix P such that the positive definite and proper function

$$U(z_{\mathrm{a}}, \xi) = V(z_{\mathrm{a}}) + \xi^{\mathrm{T}} P \xi,$$

satisfies

$$\frac{\partial U}{\partial z_{\mathrm{a}}} f_{\mathrm{a}}(w, z_{\mathrm{a}}, C\xi) + \frac{\partial U}{\partial \xi}(A+BH)\xi \leq -a'(|z_{\mathrm{a}}|^2 + |\xi|^2)\,, \qquad \forall (w, z_{\mathrm{a}}, \xi) \in W \times U^{-1}([0, c+1])$$

for some $a > 0$, where c is such that

$$\mathbf{Z}_{\mathrm{a}} \times \mathbf{\Xi} \subset U^{-1}([0, c])\,.$$

From this, using (4.25), the estimate

$$|y_{\mathrm{a}}|^2 = |h_{\mathrm{a}}(w, z_{\mathrm{a}}, \xi, H\xi)|^2 \leq a''(|z_{\mathrm{a}}|^2 + |\xi|^2), \qquad \forall (w, z_{\mathrm{a}}, \xi) \in W \times U^{-1}([0, c+1])$$

and standard arguments, it is easily seen that there exists a number $\lambda > 0$ such that the function

$$\mathbf{V}(z_{\mathrm{a}}, \xi, \tilde{x}) = U(z_{\mathrm{a}}, \xi) + \lambda \bar{V}(D_\kappa \tilde{x})$$

satisfies an inequality of the form (4.27).

4.5.3 Taking advantage of the degrees of freedom in the filter

In the case of minimum-phase plants, the only property expected, in the design of the filter, is the property of global input-to-state stability and this, as shown above, can always be achieved. If the plant is not minimum phase, a more elaborate analysis is needed, which reposes – among other things – on the exploitation of the degrees of freedom left in the design of the filter. In fact, if the plant is non-minimum phase, a nontrivial controller (4.19) is needed to stabilize the auxiliary plant (4.18), and – because of this – system (4.15) is not anymore the simple cascade-connection of two stable subsystems. Rather, system (4.15) has now the feedback structure depicted in Fig. 4.1: system (4.17) controlled by $\bar{u} = \bar{y}$. In this setting, it is therefore reasonable to exploit the degrees of freedom left in the design of the filter to the purpose of lowering as much as possible the value of the gain of the system (4.17) between input $\bar{u}$ and output $\bar{y}$.

To better understand what the problem is about, it is useful to consider –as a preliminary example– the case in which the controlled plant is a linear system, and the exogenous inputs (to be followed and/or rejected) are sinusoidal functions of time. Accordingly, the auxiliary plant (4.18) is a linear system, modelled as

$$\begin{aligned} \dot{\bar{z}}_{\mathrm{a}} &= A_{\mathrm{a}}\bar{z}_{\mathrm{a}} + B_{\mathrm{a}}u_{\mathrm{a}} \\ y_{\mathrm{a}} &= C_{\mathrm{a}}\bar{z}_{\mathrm{a}} + D_{\mathrm{a}}u_{\mathrm{a}} + \Delta_{\mathrm{a}}v_{\mathrm{a}}\,, \end{aligned} \tag{4.28}$$

in which we have set $\bar{z}_{\mathrm{a}} = \mathrm{col}(z_{\mathrm{a}}, \xi)$. The matrices $A_{\mathrm{a}}, B_{\mathrm{a}}, C_{\mathrm{a}}, D_{\mathrm{a}}, \Delta_{\mathrm{a}}$ are, possibly, continuous functions of a vector μ of uncertain parameters, ranging on a compact set. Also the filter (4.20) is a linear system, as in (4.24), with parameters chosen in such a way that $F + G\Psi$ has purely imaginary eigenvalues at $\pm\, i\,\Omega_k$, $k = 1, \ldots, p$ (the Ω_i's being the frequencies of the exogenous inputs).

If a linear controller

$$\begin{aligned} \dot{\chi} &= L\chi + M[\bar{u} + y_{\mathrm{a}}] \\ u_{\mathrm{a}} &= N\chi \\ v_{\mathrm{a}} &= NL\chi \end{aligned} \tag{4.29}$$

is chosen, the composition of (4.28) and (4.29) is a linear system with input $\bar{u}$ and output y_{a}, characterized by a transfer function $T(s)$. The filter, on the other hand, is a linear system with input y_{a} and output $\bar{y}$, characterized by a transfer function $\Phi(s)$. Thus, in this case, system (4.17) is a linear system with transfer function $W(s) = \Phi(s)T(s)$. In view of the discussion above, the goal is to choose the parameters of the filter and the controller (4.29) in such a way that the composition of (4.28) and (4.29) is a *stable* linear system and

$$\|\Phi(s)T(s)\|_\infty < 1\,. \tag{4.30}$$

As observed above, the vector Ψ in (4.24) is necessarily the unique vector which assigns to $F + \Psi G$ the characteristic polynomial $q(s) = \prod_{k=1}^{p}(s^2 + \Omega_k^2)$. Since the characteristic polynomial of $F + \Psi G$ is precisely the numerator polynomial of the transfer function of

$$\begin{aligned} \dot{x} &= Fx - Gu \\ y &= \Psi x + u \end{aligned}$$

we see that

$$\frac{\prod_{k=1}^{p}(s^2 + \Omega_k^2)}{\det(sI - F)} = 1 - \Psi(sI - F)^{-1}G = 1 + \Phi(s)\,.$$

Therefore, necessarily, $\Phi(\pm\, i\,\Omega_k) = -1$ regardless of how F is chosen. This may seem disappointing, because it implies $\|\Phi(s)\|_\infty \geq 1$. In particular, this shows that (4.30) cannot be approached by trying to enforce the conservative estimate

$$\|\Phi(s)T(s)\|_\infty \leq \|\Phi(s)\|_\infty \|T(s)\|_\infty < 1\,,$$

because it is very unlikely that $\|T(s)\|_\infty$ can be made less than 1. Consider, in this respect, the case in which A_{a} has a pole at the origin and $\Delta_{\mathrm{a}} = 0$, and observe that necessarily $T(0) = 1$ (yielding $\|T(s)\|_\infty \geq 1$) no matter how the controller is chosen.

Despite of the fact that the magnitude of $\Phi(\pm i\Omega_k)$ is necessarily 1, a clever choice of F may still be sought to the purpose of lowering the magnitude of $\Phi(i\omega)$ at frequencies other than Ω_k, in view of the fact that – after all – it is the H_∞ norm of the *product* $\Phi(s)T(s)$ that matters in the basic condition (4.30).

It is immediate to see that, if the characteristic polynomial of F is chosen as

$$d_0(s) = \prod_{k=1}^{p}(s+\Omega_k)^2 \,, \tag{4.31}$$

the transfer function $\Phi(s)$ of the filter (4.24) has a zero at the origin and, consequently the first entry ψ_0 of Ψ is 0. The fact that $\Phi(s)$ is zero at $s=0$ may alleviate the task of fulfilling the inequality (4.30). In fact, since $\psi_0 = 0$, the first component $\hat{x}_1$ of the state $\hat{x}$ of the filter (4.24) can be replaced

$$\bar{x}_1 = -c_0\hat{x} - y_{\rm a} \,,$$

yielding equations of the form

$$\begin{array}{rcl} \dot{\bar{x}} & = & \bar{F}\bar{x} - \bar{G}\dot{y}_{\rm a} \\ \bar{y} & = & \Psi\bar{x} \,, \end{array} \tag{4.32}$$

in which the output map has remained the same. Having done this change, system (4.17) can be seen as the cascade of a linear system having transfer function $sT(s)$ and of a (modified) filter having transfer function $Q(s) = -\Psi(sI - \bar{F})^{-1}\bar{G}$. Accordingly, the bound (4.30) becomes

$$\|\Upsilon(s)sT(s)\|_\infty < 1 \,.$$

The advantage of this different viewpoint is that now the (still conservative) estimate

$$\|\Upsilon(s)sT(s)\|_\infty \leq \|\Upsilon(s)\|_\infty \|sT(s)\|_\infty < 1 \,,$$

can be more easily handled. As a matter of fact, it is possible to show (see [5]) that

$$\|\Upsilon(s)\|_\infty \leq \sum_{k=1}^{p} \frac{2}{\Omega_k} \,, \tag{4.33}$$

and hence the basic design goal is achieved if

$$\|sT(s)\|_\infty \leq \Big(\sum_{k=1}^{p} \frac{2}{\Omega_k}\Big)^{-1} . \tag{4.34}$$

We discuss in the next section cases in which the inequality (4.34) can be enforced.

4.5.4 Examples of design for non-minimum-phase plants

As a first example of a non-minimum phase plant to which the design procedure outlined in the previous section can be successfully applied consider again the case in which the controlled plant is a linear system, and the exogenous inputs are sinusoidal functions of time. In this case, the auxiliary plant is an $n_{\rm a}$-dimensional linear system, as in (4.28), whose coefficient matrices are continuous functions of a vector μ of uncertain parameters, ranging on a *compact* set. Suppose that the matrix $A_{\rm a}$ has a fixed number $m \geq 1$ of eigenvalues at the origin, while the remaining $n_{\rm a} - m$ eigenvalues have negative real part. This case can be handled as follows. Set $u_{\rm a} = \chi_0$ with χ_0 generated by a system

$$\dot{\chi}_0 \;\; = \;\; -\chi_0 + u'_{\rm a} \tag{4.35}$$

in which case, trivially, $v_{\rm a} = \dot{u}_{\rm a} = -\chi_0 + u'_{\rm a}$. Composing with (4.28) yields a system with input $u'_{\rm a}$ and output $y_{\rm a}$ modelled by

$$\begin{array}{rcl} \dot{z}_{\rm a} & = & A_{\rm a}z_{\rm a} + B_{\rm a}\chi_0 \\ \dot{\chi}_0 & = & -\chi_0 + u'_{\rm a} \\ y_{\rm a} & = & C_{\rm a}z_{\rm a} + [D_{\rm a} - \Delta_{\rm a}]\chi_0 + \Delta_{\rm a}u'_{\rm a} \,. \end{array} \tag{4.36}$$

This is a $(n_a + 1)$-dimensional linear system, having m eigenvalues at the origin and $n_a - m + 1$ eigenvalues with negative real part, for every value of μ. Recall that

$$\Delta_a = \beta - 1/b$$

in which b is a positive coefficient, possibly (continuously) dependent on μ, bounded from below and from above. Indeed, β can be chosen in such a way that $\Delta_a > 0$ for all values of μ. In this case (4.36) has relative degree zero between input u'_a and output y_a, and a high-frequency gain Δ_a which is positive for every μ. As a result, the system has $n_a + 1$ zeros for every value of μ which range, in view of the continuity hypotheses, on some compact set. Let $P'_a(s)$ denote the transfer function of (4.36).

Suppose now that the control u' to (4.36) is provided by another controller, written as

$$\begin{array}{rcl} \dot{\chi}_1 & = & L_1\chi_1 + N_1[\bar{u} + y_a] \\ u' & = & N_1\chi_1 \,. \end{array} \tag{4.37}$$

This results in a system, with input $\bar{u}$ and output y_a, characterized by a transfer function of the form

$$T(s) = \frac{R(s)P'_a(s)}{1 - R(s)P'_a(s)} \tag{4.38}$$

in which

$$R(s) = N_1(sI - L_1)^{-1}M_1 \,.$$

If we succeed in choosing L_1, M_1, N_1 in such a way that the H_∞ norm of $sT(s)$ satisfies the bound (4.34), the problem is solved, with (4.29) being the cascade of (4.35) and (4.37), namely

$$\begin{array}{rcl} \dot{\chi}_1 & = & L_1\chi_1 + N_1[\bar{u} + y_a] \\ \dot{\chi}_0 & = & -\chi_0 + N_1\chi_1 \\ u_a & = & \chi_0 \\ v_a & = & -\chi_0 + N_1\chi_1 \,. \end{array}$$

Enforcing (4.34), in this case, is not difficult at all. In fact, this depends on the following Lemma.

Lemma 4.1 *Let $N_0(s)$ and $D_0(s)$ be polynomial of fixed degrees, whose coefficients are continuous functions of a vector μ of uncertain parameters, ranging on a compact set. Let $D_0(s)$ be monic and Hurwitz for every value of μ and let*

$$\deg[D_0] \geq \deg[N_0] \,.$$

Set, for $k = 1, \ldots,$

$$\begin{array}{rcl} N_1(s) & = & gN_0(s) \\ N_k(s) & = & (s + z_{k-1})N_{k-1}(s) \\ D_1(s) & = & sD_0(s) \\ D_k(s) & = & sD_{k-1}(s) \end{array}$$

and

$$P_k(s) = D_k(s) + N_k(s) \qquad T_k(s) = \frac{N_k(s)}{D_k(s) + N_k(s)} \,.$$

Let $\gamma > 0$ be fixed. For any choice of $a > 1$ there is a choice of real numbers $g, z_1, z_2, \ldots, z_k$ such that $P_{k+1}(s)$ is Hurwitz, $T_{k+1}(0) = 1$ and

$$\|sT_{k+1}(s)\|_\infty \leq a^k\gamma \,.$$

The proof of this Lemma requires elementary arguments and is not included here. It can be found in [5]. In the context above, the Lemma is used as follows. Let $N_a(s)$ and $s^m D_a(s)$ be the numerator and, respectively, monic denominator of $P'_a(s)$ (recall, to this end, that $P'_a(s)$ has m poles at the origin). Set

$$N_0(s) = N_a(s), \qquad D_0(s) = (s+1)^m D_a(s) \,.$$

The polynomials thus defined meet the assumptions of the Lemma. Thus, using the result of the Lemma for $k = m$, it is immediately seen that, for any $\gamma > 0$ and $a > 0$ there is a choice of parameters $g, z_1, \ldots, z_{m-1}$, such that, choosing the transfer function of (4.37) as

$$R(s) = -g\frac{(s+z_{m-1})\ldots(s+z_1)}{(s+1)^m},$$

one obtains

$$\|sT(s)\|_\infty = \|s\,\frac{R(s)P'_{\rm a}(s)}{1-R(s)P'_{\rm a}(s)}\|_\infty = \|sT_m(s)\|_\infty \le a^{m-1}\gamma\,.$$

Since a and γ are arbitrary, this shows that the inequality (4.34) can always be enforced.

The peculiar feature of this example was the possibility of finding a controller which does stabilize the auxiliary plant (which is unstable) while, at the same time, keeping an arbitrarily low gain between $\bar{u}$ and the output $\dot{y}_{\rm a}$. It is precisely in this respect that the idea of replacing $y_{\rm a}$ by $\dot{y}_{\rm a}$ has helped. This feature may still be exploited for certain classes of nonlinear systems. In [4], for instance, it was shown that the same design result can be achieved in the special case of a nonlinear auxiliary plant in feed-forward form

$$\begin{array}{rcl} \dot{z}_{\rm a1} &=& p(z_{\rm a1}, z_{\rm a2}, w)z_{\rm a2} \\ \dot{z}_{\rm a2} &=& u_{\rm a} \\ y_{\rm a} &=& z_{\rm a1} \end{array} \tag{4.39}$$

in which $p(z_{\rm a1}, z_{\rm a2}, w)$ is a continuous function bounded as

$$0 < p_0 \le p(z_{\rm a1}, z_{\rm a2}, w) \le p_1\,.$$

As a second example, we consider a case in which $r = 1$ and the zero dynamics of the controlled plant (4.3) are unstable. In order to keep the example elementary, a number of strong simplifying assumptions are made. To begin with, we assume that $\Delta_{\rm a} = 0$ and that $f_{\rm a}(\cdot)$ and $h_{\rm a}(\cdot)$ are affine in $u_{\rm a}$, in which case the auxiliary plant (4.18) can be rewritten in the form

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z}_{\rm a} &=& A(w, z_{\rm a}) + B(w, z_{\rm a})u_{\rm a} \\ y_{\rm a} &=& C(w, z_{\rm a}) + D(w, z_{\rm a})u_{\rm a}\,. \end{array} \tag{4.40}$$

Setting, as before, $u_{\rm a} = \chi_0$ with χ_0 generated by (4.35), yields a composite system

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z}_{\rm a} &=& A(w, z_{\rm a}) + B(w, z_{\rm a})\chi_0 \\ \dot{\chi} &=& -\chi_0 + u'_{\rm a} \\ y_{\rm a} &=& C(w, z_{\rm a}) + D(w, z_{\rm a})\chi_0\,. \end{array}$$

If $D(w, z_{\rm a})$ is nowhere zero, this system has uniform relative degree 1 between input $u'_{\rm a}$ and output $y_{\rm a}$ and can be rewritten in normal form as

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z}_{\rm a} &=& \hat{A}(w, z_{\rm a}) + \hat{B}(w, z_{\rm a})\xi \\ \dot{\xi} &=& \hat{C}(w, z_{\rm a}) + \hat{D}(w, z_{\rm a})\xi + D(w, z_{\rm a})u'_{\rm a} \\ y_{\rm a} &=& \xi\,, \end{array}$$

in which $\hat{A}(w, z_{\rm a}), \hat{B}(w, z_{\rm a}), \hat{C}(w, z_{\rm a}), \hat{D}(w, z_{\rm a})$ are appropriate functions. Next, we seek conditions under which the simplest possible control $u'_{\rm a}$, namely

$$u'_{\rm a} = K[\xi + \bar{u}]$$

yields the required asymptotic properties (if K is appropriately chosen). The easiest case in which this occurs is when there exists a function $V(z_{\rm a})$ satisfying, for some choice of positive numbers a_1, a_2, a_3,

$$a_1|z_{\rm a}|^2 \le V(z_{\rm a}) \le a_2|z_{\rm a}|^2, \qquad |\frac{\partial V}{\partial z_{\rm a}}| \le a_3|z_{\rm a}|, \qquad \forall z_{\rm a} \in \mathbb{R}^m$$

such that, for some positive a_4,

$$\frac{\partial V}{\partial z_{\rm a}}\hat{A}(w, z_{\rm a}) \le -a_4|z_{\rm a}|^2, \qquad \forall (w, z)_{\rm a}) \in W \times \mathbb{R}^m \,.$$

If this is the case and, in addition, there are positive numbers $b_0, c_0, d_0, \delta_0, \delta_1$ such that

$$|\hat{B}(w, z_{\rm a})| \le b_0, \quad |\hat{C}(w, z_{\rm a})| \le c_0|z_{\rm a}|, \quad |\hat{D}(w, z_{\rm a})| \le d_0, \quad \delta_0 \le |D(w, z_{\rm a})| \le \delta_1,$$
$$\forall (w, z)_{\rm a}) \in W \times \mathbb{R}^m \,,$$

standard arguments show that the closed loop system

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z}_{\rm a} &=& \hat{A}(w, z_{\rm a}) + \hat{B}(w, z_{\rm a})\xi \\ \dot{\xi} &=& \hat{C}(w, z_{\rm a}) + \hat{D}(w, z_{\rm a})\xi + D(w, z_{\rm a})K[\xi + \bar{u}] \,, \end{array}$$

if $D(w, z_{\rm a})K < 0$ and $|K|$ is large enough, can be forced to satisfy a dissipation inequality of the form

$$\frac{\partial V}{\partial z_{\rm a}}\dot{z}_{\rm a} + \frac{\partial \xi^2}{\partial \xi}\dot{\xi} - K^2\bar{u}^2 \le -a(|z_{\rm a}|^2 + \xi^2) \,,$$

for some $a > 0$. As a consequence, the same system, with output

$$\dot{y}_{\rm a} = \hat{C}(w, z_{\rm a}) + \hat{D}(w, z_{\rm a})\xi + D(w, z_{\rm a})K[\xi + \bar{u}]$$

satisfies a dissipation inequality of the form

$$\frac{\partial V}{\partial z_{\rm a}}\dot{z}_{\rm a} + \frac{\partial \xi^2}{\partial \xi}\dot{\xi} - K^2\bar{u}^2 + \delta^2[\dot{y}_{\rm a}]^2 \le -\frac{a}{2}(|z_{\rm a}|^2 + \xi^2) \,,$$

for some small $\delta > 0$. This being the case, from the previous arguments it follows that if the frequencies of the exogenous input satisfy

$$\frac{K}{\delta} \le \Big(\sum_{k=1}^{p} \frac{2}{\Omega_k}\Big)^{-1}$$

the proposed controller solves the problem of output regulation.

Remark 4.3 It is worth stressing that, while we have implicitly assumed that the zero dynamics of the auxiliary plant (4.40) are globally asymptotically stable, the zero dynamics of the original plant (4.3) may well be unstable. In fact, while the zero dynamics of the original plant in the setting of the previous example are those of

$$\dot{z}_{\rm a} = A(w, z_{\rm a}) \,,$$

the zero dynamics of the auxiliary plant are those of

$$\dot{z}_{\rm a} = A(w, z_{\rm a}) - \frac{B(w, z_{\rm a})C(w, z_{\rm a})}{D(w, z_{\rm a})} \,.$$

It is precisely in the case of a plant having unstable zero dynamics that this (more elaborate) design procedure may help. $\lhd$

Remark 4.4 We have shown that the method is applicable if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value determined by the gain needed to solve an auxiliary stabilization problem. In other words, the minimal gain needed to stabilize the unstable zero dynamics of the original plant determines a lower limit on the frequencies of the exogenous inputs for which the desired steady state *performance* can be achieved. $\lhd$

Bibliography

[1] C.I. Byrnes, A. Isidori, Asymptotic stabilization of minimum phase nonlinear systems, *IEEE Trans. Automatic Control*, **AC-36**: 1122–1137, 1991.

[2] C.I. Byrnes, A. Isidori, Nonlinear Internal Models for Output Regulation, *IEEE Trans. Automatic Control*, **AC-49**: 2244–2247, 2004 .

[3] Z. Chen and J. Huang, A general formulation and solvability of the global robust output regulation problem, *IEEE Trans. Automatic Control*, **AC-50**: 448–462, 2005.

[4] F. Delli Priscoli, L. Marconi, A. Isidori, A dissipativity-based approach to output regulation of non-minimum-phase systems, *Systems & Control Lett.*, to appear.

[5] F. Delli Priscoli, L. Marconi, A. Isidori, A Method for Robust Regulation of Non-mimimum-phase Linear Systems, ROCOND 09.

[6] F. Esfandiari, H. Khalil, Output feedback stabilization of fully linearizable systems. International J. of Control **56**: 1007–1037, 1992.

[7] J.P. Gauthier, I. Kupka, *Deterministic Observation Theory and Applications*, Cambridge Univeraity Press, 2001.

[8] J. Huang, *Nonlinear Output Regulation: Theory and Applications.* SIAM, Philadephia (2004)

[9] J. Huang and C.F. Lin. On a robust nonlinear multivariable servomechanism problem. *IEEE Trans. Automatic Control*, **AC-39**: 1510–1513, 1994.

[10] A. Isidori and C.I. Byrnes. Output regulation of nonlinear systems. *IEEE Trans. Automatic Control*, **AC-25**: 131–140, 1990.

[11] A. Isidori, *Nonlinear Control Systems: volume II*, Springer Verlag, London (1999).

[12] A. Isidori, L. Marconi, A. Serrani, *Robust Autonomous Guidance: and Internal-Model Approach*, Springer Verlag, London (2003).

[13] L. Marconi, L. Praly, A. Isidori A, Output Stabilization via Nonlinear Luenberger Observers, SIAM J. Control and Optimization, **45**: 2277–2298, 2006.

[14] A. Pavlov, N. van de Wouw, H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: a Convergent Dynamics Approach.* Birkhauser, Boston (2006)

[15] A. Serrani, A. Isidori, L. Marconi, Semiglobal nonlinear output regulation with adaptive internal model, *IEEE Trans. Automatic Control*, **AC-46**: 1178–1194, 2001.