Chapter 4
Regulation and Tracking in Linear Systems

Abstract In this Chapter, we will study in some generality the problem of designing a feedback law to the purpose of making a controlled plant stable, and securing exact asymptotic tracking of external commands (respectively, exact asymptotic rejection of external disturbances) which belong to a fixed family of functions. The problem in question can be seen as a (broad) generalization of the classical set-point control problem in the elementary theory of servomechanisms.

4.1 The problem of asymptotic tracking and disturbance rejection

In the last part of the previous Chapter, we have considered a plant modeled as in (3.48), and – regarding y as an output available for *measurement* and u as an input available for *control* – we have studied the problem of finding a controller of the form (3.49) yielding a *stable* closed loop system having a transfer function – between input v and output z – whose H_∞ norm does not exceed a given number γ. The problem in question was motivated by the interest in solving a problem of robust stability, but it has an independent interest *per se*. In fact, regarding v as a vector of external *disturbances*, affecting the behavior of the controlled plant, and z as set of variables of special interest, the problem considered in section 3.6 can be regarded as the problem of finding a controller that – while keeping the closed loop stable – enforces a prescribed *attenuation* of the effect of the disturbance v on the variable of interest z, the attenuation being expressed in terms of the H_∞ norm (or of the $\mathscr{L}_2$ gain, if desired) of the resulting system.

In the present Chapter, we continue to study a problem of this kind, i.e. the control of a plant affected by external disturbances, from which certain variables of interest have to be "protected", but with some differences. Specifically, while in section 3.6 we have considered the case in which the influence of the disturbances on the variables of interest had to be attenuated by a given factor, we consider in this section the case in which the influence of the disturbances on the variables of interest should ultimately vanish. This is indeed a much stronger requirement and it is unlikely that it might be enforced in general. It can be enforced, though, if the disturbances happen to belong to a specific (well-defined) family of signals. This gives rise to a specific setup, known as *problem of asymptotic disturbance rejection and/or tracking*, or more commonly *problem of output regulation*, that will be explained in more detail hereafter.

For consistency with the notations currently used in the literature dealing with the specific problem addressed in this Chapter, the controlled plant (compare with (3.48)) is assumed to be modeled by equations written in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w \\ y &= Cx + Qw\,. \end{aligned} \tag{4.1}$$

The first equation of this system describes a *plant* with *state* $x \in \mathbb{R}^n$ and *control input* $u \in \mathbb{R}^m$, subject to a set of *exogenous input* variables $w \in \mathbb{R}^{n_w}$ which includes *disturbances* (to be rejected) and/or *references* (to be tracked). The second equation defines a set of *regulated* (or *error*) variables $e \in \mathbb{R}^p$, which are expressed as a linear combination of the plant state x and of the exogenous input w.[1] The third equation defines a set of *measured* variables $y \in \mathbb{R}^q$, which are assumed to be available for feedback, and are also expressed as a linear combination of the plant state x and of the exogenous input w.

The control action to (4.1) is to be provided by a feedback *controller* which processes the measured information y and generates the appropriate control input. In general, the controller in question is a system modeled by equations of the form

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}y \\ u &= C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y\,, \end{aligned} \tag{4.2}$$

with state $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, which yields a closed-loop system modeled by equations of the form

$$\begin{aligned} \dot{x} &= (A + BD_{\mathrm{c}}C)x + BC_{\mathrm{c}}x_{\mathrm{c}} + (P + BD_{\mathrm{c}}Q)w \\ \dot{x}_{\mathrm{c}} &= B_{\mathrm{c}}Cx + A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}Qw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w\,. \end{aligned} \tag{4.3}$$

This system is seen as a system with *input* w and *output* e. The purpose of the control is to make sure that the closed loop system be *asymptotically stable* and that the regulated variable e, viewed as a function of time, *asymptotically decays to zero* as time tends to ∞, for every possible initial state and for every possible *exogenous input* in a prescribed family of functions of time. This requirement is also known as property of *output regulation*. For the sake of mathematical simplicity, and also because in this way a large number of relevant practical situations can be covered, it

[1] If some components of w are (external) commands that certain variables of interest are required to track, then some of the components of e can be seen as *tracking errors*, that is differences between the actual values of those variables of interest and their expected reference values. Overall, the components of e can simply be seen as variables on which the effect of w is expected to vanish asymptotically.

is assumed that the family of the exogenous inputs $w(\cdot)$ which affect the plant, and for which asymptotic decay of the regulated variable is to be achieved, is the family of all functions of time which are solution of a homogeneous linear differential equation

$$\dot{w} = Sw \tag{4.4}$$

with state $w \in \mathbb{R}^{n_w}$, for all possible initial conditions $w(0)$. This system, which is viewed as a mathematical model of a "generator" for all possible exogenous input functions, is called the *exosystem*. In this Chapter, we will discuss this design problem at various levels of generality.

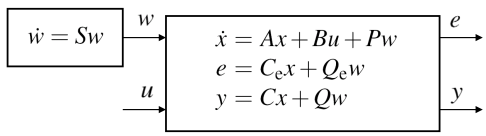

Fig. 4.1 The controlled plant and the exosystem

Note that, without loss of generality, in the analysis of this problem it can be assumed that all the eigenvalues of S have non-negative real part. In fact, if there were eigenvalues having negative real part, system (4.4) could be split (after a similarity transformation) as

$$\begin{aligned} \dot{w}_s &= S_s w_s \\ \dot{w}_u &= S_u w_u \end{aligned}$$

with S_s having *all* eigenvalues in $\mathbb{C}^-$ and S_u having *no* eigenvalue in $\mathbb{C}^-$. The input $w_s(t)$ generated by the upper subsystem is asymptotically vanishing. Thus, once system (4.3) has been rendered stable, the influence of $w_s(t)$ on the regulated variable $e(t)$ asymptotically vanishes as well. In other words, the only exogenous inputs that matter in the problem under consideration are those generated by the lower subsystem.

This being said, it must be observed – though – that if some of the eigenvalues of S have *positive* real part, there will be initial conditions from which the exosystem (4.4) generates signals that grow unbounded as time increases. Thus, also the associated response of the state of (4.3) can grow unbounded as time increases and this is not a reasonable setting. In view of all of this, in the analysis which follows we will proceed under the assumption that all eigenvalues of S have zero real part (and are simple roots of its minimal polynomial), even though all results that will be presented are also valid in the more general setting in which S has eigenvalues with positive real part (and/or has eigenvalues with zero real part which are multiple roots of its minimal polynomial).

4.2 The case of full information and Francis' equations

For expository reasons we consider first the (non realistic) case in which the full state x of the plant and the full state w of the exosystem are available for measurement, i.e. the case in which the measured variable y in (4.1) is $y = \mathrm{col}(x, w)$. This is called the case of "full information". We also assume that all system parameters are known exactly.

In this setup, we consider the special version of the controller (4.2) in which $u = D_c y$, that we rewrite for convenience as

$$u = Fx + Lw\,. \tag{4.5}$$

Proposition 4.1. *The problem of output regulation in the case of full information has a solution if and only if*

(i) the matrix pair (A, B) is stabilizable
(ii) there exists a solution pair (Π, Ψ) of the linear matrix equations

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C_e \Pi + Q_e\,. \end{aligned} \tag{4.6}$$

Proof. [Necessity] System (4.1) controlled by (4.5) can be regarded as an autonomous linear system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P + BL & A + BF \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}. \tag{4.7}$$

Suppose the pair of matrices F, L is such the controller (4.5) solves the problem of output regulation. Then, the matrix $A + BF$ has all eigenvalues with negative real part and, hence, the matrix pair (A, B) *must be stabilizable*.

The eigenvalues of system (4.7) are those of $(A + BF)$ and those of S. The former are all in $\mathbb{C}^-$ while none of the latter, in view of the standing assumption described at the end of the previous section, is in $\mathbb{C}^-$. Hence this system possesses a *stable* eigenspace and a *center* eigenspace. The former can be described as

$$\mathscr{V}^s = \mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix},$$

while the latter, which is complementary to the stable eigenspace, can be described as

$$\mathscr{V}^c = \mathrm{Im}\begin{pmatrix} I \\ \Pi \end{pmatrix},$$

in which Π is a solution of the Sylvester equation

$$\Pi S = (A + BF)\Pi + (P + BL)\,. \tag{4.8}$$

Setting

$$\Psi = L + F\Pi$$

we deduce that the pair Π, Ψ satisfies the first equation of (4.6).

Any trajectory of (4.7) has a unique decomposition into a component entirely contained in the stable eigenspace and a component entirely contained in the center eigenspace. The former, which asymptotically decays to 0 as $t \to \infty$, is the *transient component* of the trajectory. The latter, on the contrary, is persistent: it is the *steady-state component* of the trajectory. We denote it as

$$x_{\mathrm{ss}}(t) = \Pi w(t) .$$

As a consequence, the steady-state component of the regulated output $e(t)$ is

$$e_{\mathrm{ss}}(t) = (C_{\mathrm{e}}\Pi + Q_{\mathrm{e}})w(t) .$$

If the controller (4.5) solves the problem of output regulation, we must have $e_{\mathrm{ss}}(t) = 0$ and this can only occur if Π satisfies $C_{\mathrm{e}}\Pi + Q_{\mathrm{e}} = 0$. In terms of state trajectories, this is equivalent to say that the steady-state component of any state trajectory *must be contained in the kernel of the map* $e = Q_{\mathrm{e}}w + C_{\mathrm{e}}x$. We have shown in this way that the solution Π of the Sylvester equation (4.8) necessarily satisfies the second equation of (4.6). In summary, if the pair of matrices F, L is such that the controller (4.5) solves the problem of output regulation, the equations (4.6) *must have a solution* pair (Π, Ψ).

[Sufficiency] Suppose that (A, B) is stabilizable and pick F such that the eigenvalues of $A + BF$ have negative real part. Suppose the equations (4.6) have a solution (Π, Ψ) and pick L as

$$L = \Psi - F\Pi ,$$

that is, consider a control law of the form

$$u = \Psi w + F(x - \Pi w) .$$

The corresponding closed loop system is

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P + B(\Psi - F\Pi) & A + BF \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} .$$

Consider the change of variables

$$\tilde{x} = x - \Pi w$$

and, bearing in mind the first equation of (4.6) and the choice of L, observe that, in the new coordinates,

$$\dot{\tilde{x}} = (A + BF)\tilde{x} .$$

Hence $\lim_{t\to\infty} \tilde{x}(t) = 0$, because the eigenvalues of $A + BF$ have negative real part.

In the new coordinates,

$$e(t) = C_e\tilde{x}(t) + (C_e\Pi + Q_e)w(t) = C_e\tilde{x}(t)\,.$$

Therefore, using the second equation of (4.6), we conclude that

$$\lim_{t\to\infty} e(t) = 0$$

and we see that the proposed law solves the problem. ◁

Equations (4.6) are called the *linear regulator equations* or, also, the *Francis's equations*.[2] The following result is useful to determine the existence of solutions.[3]

Lemma 4.1. *The Francis' equation (4.6) have a solution for any (P, Q_e) if and only if*

$$\text{rank}\begin{pmatrix} A-\lambda I & B \\ C_e & 0 \end{pmatrix} = \#\ \text{rows} \qquad \forall\lambda \in \sigma(S)\,. \tag{4.9}$$

If this is the case and the matrix on the left-hand side of (4.9) is square (i.e. the control input u and the regulated output e have the same number of components), the solution (Π,Ψ) is unique.

The condition (4.9) is usually referred to as the *non-resonance condition*. Note that the condition is necessary and sufficient if the existence of a solution *for any* (P, Q_e) is sought. Otherwise, it is simply a sufficient condition. Note also that such condition requires $m \ge p$, i.e. that the number of components of the control input u be larger than or equal to the number of components of the regulated output.

Example 4.1. As an example of what the non-resonance condition (4.9) means and why it plays a crucial role in the solution of (4.6), consider the case in which $\dim(e) = \dim(u) = 1$. Observe that the first two equations of (4.1), together with (4.4), can be written in the form of a composite system with input u and output e as

$$\begin{aligned}\begin{pmatrix}\dot w\\ \dot x\end{pmatrix} &= \begin{pmatrix} S & 0 \\ P & A\end{pmatrix}\begin{pmatrix} w\\ x\end{pmatrix} + \begin{pmatrix}0\\ B\end{pmatrix}u \\ e &= \begin{pmatrix} Q_e & C_e\end{pmatrix}\begin{pmatrix} w\\ x\end{pmatrix}.\end{aligned} \tag{4.10}$$

Bearing in mind the conditions (2.2) used to identify the value of the relative degree of a system, let r be such that $C_eA^kB = 0$ for all $k < r-1$ and $C_eA^{r-1}B \neq 0$. A simple calculation shows that, for any $k \ge 0$,

$$\begin{pmatrix} S & 0 \\ P & A\end{pmatrix}^k \begin{pmatrix}0\\ B\end{pmatrix} = \begin{pmatrix}0\\ A^kB\end{pmatrix}.$$

Therefore

[2] See [2].

[3] See Appendix A.2

$$\begin{pmatrix} Q_e & C_e \end{pmatrix} \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^k \begin{pmatrix} 0 \\ B \end{pmatrix} = 0$$

for $k = 1, \dots, r-2$ and

$$\begin{pmatrix} Q_e & C_e \end{pmatrix} \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^{r-1} \begin{pmatrix} 0 \\ B \end{pmatrix} = C_e A^{r-1} B.$$

Thus, system (4.10), viewed as system with input u and output e has relative degree r and can be brought to (strict) normal form, by means of a suitable change of variables. In order to identify such change of variables, let

$$\xi_1 = Q_e w + C_e x$$

and define recursively $\xi_2, \dots, \xi_r$ so that

$$\xi_{i+1} = \dot{\xi}_i$$

for $i = 1, \dots, r-1$, as expected. It is easy to see that ξ_i can be given an expression of the form

$$\xi_i = R_i w + C_e A^{i-1} x.$$

in which R_i is a suitable matrix. This is indeed the case for $i = 1$ if we set $R_1 = Q_e$. Assuming that it is the case for a generic i, it is immediate to check that

$$\begin{aligned} \xi_{i+1} &= \dot{\xi}_i = R_i \dot{w} + C_e A^{i-1} \dot{x} \\ &= R_i S w + C_e A^{i-1}(Ax + Bu + Pw) = (R_i S + C_e A^{i-1} P) w + C_e A^{i-1} x, \end{aligned}$$

which has the required form with $R_{i+1} = (R_i S + C_e A^{i-1} P)$.

Then, set (compare with (2.4))

$$T_1 = \begin{pmatrix} C_e \\ C_e A \\ \cdots \\ C_e A^{r-1} \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 \\ R_2 \\ \cdots \\ R_r \end{pmatrix},$$

let T_0 be a matrix, satisfying $T_0 B = 0$, such that (compare with (2.5))

$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix}$$

is nonsingular and define the new variables as

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0 x \\ R w + T_1 x \end{pmatrix}$$

(the variable w is left unchanged).

This being the case, it is readily seen that the system, in the new coordinates, reads as

$$\begin{aligned}
\dot w &= Sw\\
\dot z &= T_0Ax + T_0Pw\\
\dot\xi_1 &= \xi_2\\
&\cdots\\
\dot\xi_{r-1} &= \xi_r\\
\dot\xi_r &= C_eA^rx + bu + (R_rS + C_eA^{r-1}P)w\\
e &= \xi_1\,,
\end{aligned}$$

in which

$$b = C_eA^{r-1}B \neq 0\,.$$

To complete the transformation, it remains to replace x by its expression in terms of z, ξ, w. Proceeding as in section 3.1, let M_0 and M_1 be partitions of the inverse of T and observe that

$$x = M_0z + M_1\xi - M_1Rw\,.$$

Then, it can be concluded that the (strict) normal form of such system has the following expression

$$\begin{aligned}
\dot w &= Sw\\
\dot z &= A_{00}z + A_{01}\xi + P_0w\\
\dot\xi_1 &= \xi_2\\
&\cdots\\
\dot\xi_{r-1} &= \xi_r\\
\dot\xi_r &= A_{10}z + A_{11}\xi + bu + P_1w\\
e &= \xi_1\,,
\end{aligned}\tag{4.11}$$

in which the four matrices $A_{00}, A_{01}, A_{10}, A_{11}$ are *precisely* the matrices that characterize the (strict) normal form of a system defined as

$$\begin{aligned}
\dot x &= Ax + Bu\\
e &= C_ex\,,
\end{aligned}$$

and P_0, P_1 are suitable matrices. Hence, as shown in Section 2.1, the matrix A_{00} is a matrix whose eigenvalues coincide with the zeros of the transfer function

$$T_e(s) = C_e(sI - A)^{-1}B\,. \tag{4.12}$$

The normal form (4.11) can be used to determine a solution of Francis' equations. To this end, let Π be partitioned as

$$\Pi = \begin{pmatrix}\Pi_0\\ \Pi_1\end{pmatrix}$$

consistently with the partition of $\tilde x$, and let $\pi_{1,i}$ denote the i-th row of Π_1.

Then, it is immediate to check that Francis's equations are rewritten as

$$
\begin{aligned}
\Pi_0 S &= A_{00}\Pi_0 + A_{01}\Pi_1 + P_0\\
\pi_{1,1}S &= \pi_{1,2}\\
&\cdots\\
\pi_{1,r-1}S &= \pi_{1,r}\\
\pi_{1,r}S &= A_{10}\Pi_0 + A_{11}\Pi_1 + b\Psi + P_1\\
0 &= \pi_{1,1}\,.
\end{aligned}
\tag{4.13}
$$

The last equation, along with the second, third and r-th, yield $\pi_{1,i} = 0$ for all i, i.e.

$$\Pi_1 = 0\,.$$

As a consequence, the remaining equations reduce to

$$
\begin{aligned}
\Pi_0 S &= A_{00}\Pi_0 + P_0\\
0 &= A_{10}\Pi_0 + b\Psi + P_1\,.
\end{aligned}
\tag{4.14}
$$

The first of these equations is a Sylvester equation in Π_0, that has a unique solution if and only if the none of the eigenvalues of S is an eigenvalue of the matrix A_{00}. Since the eigenvalues of A_{00} are the zeros of (4.12), this equation has a unique solution if and only if none of the eigenvalues of S is a zero of $T_\mathrm{e}(s)$. Entering the solution Π_0 of this equation into the second one yields an equation that, since $b \neq 0$, can be solved for Ψ, yielding

$$\Psi = \frac{-1}{b}[A_{10}\Pi_0 + P_1]\,.$$

In summary, it can be concluded that Francis' equations have a unique solution if and only if *none of the eigenvalues of S coincides with a zero of the transfer function $T_\mathrm{e}(s)$*.

Of course the condition thus found must be consistent with the condition resulting from Lemma 4.1, specialized to the present context in which $m = p = 1$. In this case, the condition (4.9) becomes

$$\det\begin{pmatrix} A - \lambda I & B\\ C_\mathrm{e} & 0\end{pmatrix} \neq 0 \qquad \forall\lambda \in \sigma(S)\,.$$

Now, it is known (see section 2.1) that the roots of the polynomial

$$\det\begin{pmatrix} A - \lambda I & B\\ C_\mathrm{e} & 0\end{pmatrix}$$

coincide with the zeros of the transfer function $T_\mathrm{e}(s)$. Thus, the condition indicated in Lemma 4.1 is the condition that none of the eigenvalues of S is a zero of $T_\mathrm{e}(s)$, which is precisely the condition obtained from the construction above. $\triangleleft$

4.3 The case of measurement feedback: steady state analysis

Consider now the case in which the feedback law is provided by a controller that does not have access to the full state x of the plant and the full state w of the exosystem. We assume that the controller has access to the regulated output e and, possibly, to an additional *supplementary* set of independent measured variables. In other words, we assume that the vector

$$y = Cx + Qw$$

of *measured outputs* of the plant can be split in two parts as in

$$y = \begin{pmatrix} e \\ y_{\mathrm{r}} \end{pmatrix} = \begin{pmatrix} C_{\mathrm{e}} \\ C_{\mathrm{r}} \end{pmatrix} x + \begin{pmatrix} Q_{\mathrm{e}} \\ Q_{\mathrm{r}} \end{pmatrix} w$$

in which e is the *regulated* output and $y_{\mathrm{r}} \in \mathbb{R}^{p_{\mathrm{r}}}$. As opposite to the case considered in the previous section, this is usually referred to as the case of "measurement feedback". In this setup the controlled plant, together with the exosystem, is modeled by equations of the form

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= Ax + Bu + Pw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w \\ y_{\mathrm{r}} &= C_{\mathrm{r}}x + Q_{\mathrm{r}}w\,. \end{aligned} \tag{4.15}$$

The control is provided by a generic dynamical system with input y and output u, modeled as in (4.2). Proceeding as in the first part of the proof of Proposition 4.1, it is easy to deduce the following *necessary* conditions for the solution of the problem of output regulation.

Proposition 4.2. *The problem of output regulation in the case of measurement feedback has a solution* only if

(i) the triplet $\{A, B, C\}$ is stabilizable and detectable
(ii) there exists a solution pair (Π, Ψ) of the Francis' equation (4.6).

Proof. System (4.1) controlled by (4.2) can be regarded as an autonomous linear system (see Fig. 4.2)

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{x}_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 \\ P + BD_{\mathrm{c}}Q & A + BD_{\mathrm{c}}C & BC_{\mathrm{c}} \\ B_{\mathrm{c}}Q & B_{\mathrm{c}}C & A_{\mathrm{c}} \end{pmatrix} \begin{pmatrix} w \\ x \\ x_{\mathrm{c}} \end{pmatrix}. \tag{4.16}$$

If a controller of the form (4.2) solves the problem of output regulation, all the eigenvalues of the matrix

$$\begin{pmatrix} A + BD_{\mathrm{c}}C & BC_{\mathrm{c}} \\ B_{\mathrm{c}}C & A_{\mathrm{c}} \end{pmatrix}$$

have negative real part. Hence, the triplet $\{A, B, C\}$ *must be stabilizable and detectable*.

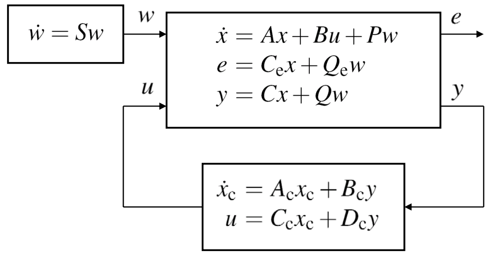

Fig. 4.2 The closed loop system, augmented with the exosystem.

Since by assumption S has all eigenvalues on the imaginary axis, system (4.16) possesses two complementary invariant subspaces: a *stable eigenspace* and a *center eigenspace*. The latter, in particular, can be expressed as

$$\mathscr{V}^c = \mathrm{I}\begin{pmatrix} I \\ \Pi \\ \Pi_c \end{pmatrix}$$

in which the pair (Π, Π_c) is a solution of the Sylvester equation

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_c C & BC_c \\ B_c C & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_c Q \\ B_c Q \end{pmatrix}. \tag{4.17}$$

Setting

$$\Psi = D_c C\Pi + C_c \Pi_c + D_c Q$$

it is observed that the pair (Π, Ψ) satisfies the first equation of (4.6).

Any trajectory of (4.16) has a unique decomposition into a component entirely contained in the stable eigenspace and a component entirely contained in the center eigenspace. The former, which asymptotically decays to 0 as $t \to \infty$, is the *transient component* of the trajectory. The latter, in which $x(t)$ and $x_c(t)$ have, respectively, the form

$$x_{ss}(t) = \Pi w(t) \qquad x_{c,ss}(t) = \Pi_c w(t), \tag{4.18}$$

is the *steady-state component* of the trajectory.

If the controller (4.2) solves the problem of output regulation, the steady-state component of any trajectory must be contained in the kernel of the map $e = Q_e w + C_e x$ and hence the solution (Π, Π_c) of the Sylvester equation (4.17) necessarily satisfies the second equation of (4.6). This shows that if the controller (4.2) solves the problem of output regulation, the equations (4.6) necessarily have a solution a pair (Π, Ψ). $\triangleleft$

In order to better understand the steady-state behavior of the closed loop system (4.16), it is convenient to split B_c and D_c consistently with the partition adopted for

y, as

$$B_{\rm c} = (B_{\rm ce} \quad B_{\rm cr}) \qquad D_{\rm c} = (D_{\rm ce} \quad D_{\rm cr})\,,$$

and rewrite the controller (4.2) as

$$\begin{aligned}\dot{x}_{\rm c} &= A_{\rm c}x_{\rm c} + B_{\rm ce}e + B_{\rm cr}y_{\rm r}\\ u &= C_{\rm c}x_{\rm c} + D_{\rm ce}e + D_{\rm cr}y_{\rm r}\,.\end{aligned}$$

In these notations, the closed-loop system (4.16) becomes

$$\begin{pmatrix}\dot{w}\\ \dot{x}\\ \dot{x}_{\rm c}\end{pmatrix} = \begin{pmatrix} S & 0 & 0\\ P + BD_{\rm ce}Q_{\rm e} + BD_{\rm cr}Q_{\rm r} & A + BD_{\rm ce}C_{\rm e} + BD_{\rm cr}C_{\rm r} & BC_{\rm c}\\ B_{\rm ce}Q_{\rm e} + B_{\rm cr}Q_{\rm r} & B_{\rm ce}C_{\rm e} + B_{\rm cr}C_{\rm r} & A_{\rm c}\end{pmatrix}\begin{pmatrix} w\\ x\\ x_{\rm c}\end{pmatrix}, \tag{4.19}$$

and the Sylvester equation (4.17) becomes

$$\begin{pmatrix}\Pi\\ \Pi_{\rm c}\end{pmatrix}S = \begin{pmatrix} A + BD_{\rm ce}C_{\rm e} + BD_{\rm cr}C_{\rm r} & BC_{\rm c}\\ B_{\rm ce}C_{\rm e} + B_{\rm cr}C_{\rm r} & A_{\rm c}\end{pmatrix}\begin{pmatrix}\Pi\\ \Pi_{\rm c}\end{pmatrix} + \begin{pmatrix} P + BD_{\rm ce}Q_{\rm e} + BD_{\rm cr}Q_{\rm r}\\ B_{\rm ce}Q_{\rm e} + B_{\rm cr}Q_{\rm r}\end{pmatrix}.$$

Bearing in mind the fact that, if the controller solves the problem of output regulation, the matrix Π must satisfy the second of (4.6), the equation above reduces to

$$\begin{pmatrix}\Pi\\ \Pi_{\rm c}\end{pmatrix}S = \begin{pmatrix} A + BD_{\rm cr}C_{\rm r} & BC_{\rm c}\\ B_{\rm cr}C_{\rm r} & A_{\rm c}\end{pmatrix}\begin{pmatrix}\Pi\\ \Pi_{\rm c}\end{pmatrix} + \begin{pmatrix} P + BD_{\rm cr}Q_{\rm r}\\ B_{\rm cr}Q_{\rm r}\end{pmatrix},$$

and we observe that, in particular, the matrix $\Pi_{\rm c}$ satisfies

$$\begin{aligned}\Pi_{\rm c}S &= A_{\rm c}\Pi_{\rm c} + B_{\rm cr}(C_{\rm r}\Pi + Q_{\rm r})\\ \Psi &= C_{\rm c}\Pi_{\rm c} + D_{\rm cr}(C_{\rm r}\Pi + Q_{\rm r})\,,\end{aligned} \tag{4.20}$$

in which (Π, Ψ) is the solution pair of (4.6).

Equations (4.20), from a general viewpoint, could be regarded as a constraint on the solution (Π, Ψ) of (4.6). These equations interpret the ability, of the controller, to generate the *feedforward input* necessary to keep the regulated variable identically zero in steady state. In steady state, the state $x(t)$ of the plant evolves (see (4.18)) as

$$x_{\rm ss}(t) = \Pi w(t)\,,$$

the regulated output $e(t)$ is identically zero, because

$$e_{\rm ss}(t) = (C_{\rm e}\Pi + Q_{\rm e})w(t) = 0\,, \tag{4.21}$$

the additional measured output $y_{\rm r}(t)$ evolves as

$$y_{\rm r,ss}(t) = (C_{\rm r}\Pi + Q_{\rm r})w(t) \tag{4.22}$$

and the state $x_{\rm c}(t)$ of the controller evolves (see again (4.18)) as

$$x_{\rm c,ss}(t) = \Pi_{\rm c}w(t)\,.$$

The first equation of (4.20) expresses precisely the property that $\Pi_c w(t)$ is the steady-state response of the controller, when the latter is forced by a steady-state input of the form (4.21) – (4.22). The second equation of (4.20), in turn, shows that the output of the controller, in steady state, is a function of the form

$$\begin{aligned} u_{ss}(t) &= C_c x_{c,ss}(t) + D_{ce} e_{ss}(t) + D_{cr} y_{r,ss}(t) \\ &= C_c \Pi_c w(t) + D_{cr}(C_r \Pi + Q_r) w(t) = \Psi w(t)\,. \end{aligned}$$

The latter, as predicted by Francis' equations, is a control able to force in the controlled plant a steady-state trajectory of the form $x_{ss}(t) = \Pi w(t)$ and consequently to keep $e_{ss}(t)$ identically zero. The property thus described is usually referred to as the *internal model property*: any controller that solves the problem of output regulation necessarily embeds a model of the feedforward inputs needed to keep $e(t)$ identically zero.

4.4 The case of measurement feedback: construction of a controller

The possibility of constructing a controller that solves the problem in the case of measurement feedback reposes on the following preliminary result. Let

$$\psi(\lambda) = s_0 + s_1\lambda + \cdots + s_{d-1}\lambda^{d-1} + \lambda^d$$

denote the *minimal* polynomial of S and set

$$\Phi = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ -s_0 I & -s_1 I & -s_2 I & \cdots & -s_{d-1} I \end{pmatrix}, \qquad (4.23)$$

in which all blocks are $p \times p$. Set also

$$G = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \end{pmatrix}^{\mathrm{T}} \qquad (4.24)$$

in which all blocks are $p \times p$.

Note that there exists a nonsingular matrix T such that

$$T\Phi T^{-1} = \begin{pmatrix} S_0 & 0 & \cdots & 0 \\ 0 & S_0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & S_0 \end{pmatrix}, \qquad TG = \begin{pmatrix} G_0 & 0 & \cdots & 0 \\ 0 & G_0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & G_0 \end{pmatrix} \qquad (4.25)$$

in which S_0 is the $d \times d$ matrix

$$S_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0 & -s_1 & -s_2 & \cdots & -s_{d-1} \end{pmatrix}. \tag{4.26}$$

and

$$G_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^{\mathrm{T}}.$$

From this, we see in particular that $\psi(\lambda)$ is also the minimal polynomial of Φ. As a consequence

$$\begin{aligned} s_0 I + s_1 S + \cdots + s_{d-1} S^{d-1} + S^d = 0 \\ s_0 I + s_1 \Phi + \cdots + s_{d-1} \Phi^{d-1} + \Phi^d = 0. \end{aligned} \tag{4.27}$$

With Φ and G constructed in this way, consider now the system obtained by letting the regulated output of (4.1) drive a post-processor characterized by the equations

$$\dot{\eta} = \Phi \eta + G e. \tag{4.28}$$

Since the minimal polynomial of the matrix Φ in (4.28) coincides with the minimal polynomial of the matrix S that characterizes the exosystem, system (4.28) is usually called an *internal model* (of the exosystem). Note that, in the coordinates $\tilde{\eta} = T\eta$, with T such that (4.25) hold, system (4.28) can be seen as a bench of p identical sub-systems of the form

$$\dot{\tilde{\eta}}_i = S_0 \tilde{\eta}_i + G_0 e_i$$

each one driven by the i-th component e_i of e.

In what follows, we are going to show that the problem of output regulation can be solved by means of a controller having the following structure [4]

$$\begin{aligned} \dot{\eta} &= \Phi \eta + G e \\ \dot{\xi} &= A_s \xi + B_s y + J_s \eta \\ u &= C_s \xi + D_s y + H_s \eta \end{aligned} \tag{4.29}$$

in which $A_s, B_s, C_s, D_s, J_s, H_s$, are suitable matrices (see Fig. 4.3). This controller consists of the post-processor (4.28) whose state η drives, along with the full measured output y, the system

$$\begin{aligned} \dot{\xi} &= A_s \xi + B_s y + J_s \eta \\ u &= C_s \xi + D_s y + H_s \eta, \end{aligned}$$

and can be seen as a system of the general form (4.2) if we set

[4] The arguments uses hereafter are essentially the same as those used in [1], [4] and [3].

$$x_{\rm c} = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \qquad A_{\rm c} = \begin{pmatrix} \Phi & 0 \\ J_{\rm s} & A_{\rm s} \end{pmatrix} \qquad B_{\rm c} = \begin{pmatrix} (G \quad 0) \\ B_{\rm s} \end{pmatrix}$$

$$C_{\rm c} = (H_{\rm s} \quad C_{\rm s}) \qquad D_{\rm c} = D_{\rm s}\,.$$

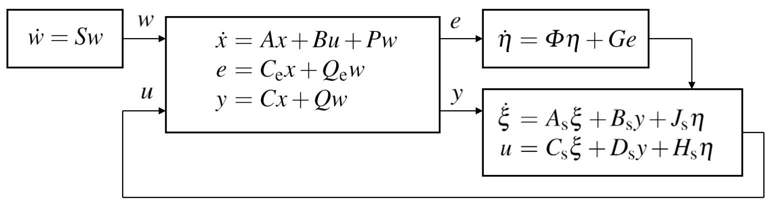

Fig. 4.3 The plant, augmented with the exosystem, controlled by (4.29).

The first step in proving that a controller of the form (4.29) can to solve the problem of output regulation consists in the analysis of the properties of stabilizability and detectability of a system – with state (x,η), input u and output $y_{\rm a}$ – defined as follows

$$\begin{aligned} \begin{pmatrix} \dot x \\ \dot\eta \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_{\rm e} & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ y_{\rm a} &= \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \end{aligned} \tag{4.30}$$

that will be referred to as the *augmented* system. As a matter of fact, it turns out that the system in question has the following important property.

Lemma 4.2. *The augmented system (4.30) is stabilizable and detectable if and only if*

(i) the triplet $\{A,B,C\}$ is stabilizable and detectable
(ii) the non-resonance condition

$$\text{rank} \begin{pmatrix} A-\lambda I & B \\ C_{\rm e} & 0 \end{pmatrix} = n+p \qquad \forall \lambda \in \sigma(S)\,. \tag{4.31}$$

holds.

Proof. To check detectability we look at the linear independence of the columns of the matrix

$$\begin{pmatrix} A-\lambda I & 0 \\ GC_{\rm e} & \Phi - \lambda I \\ C & 0 \\ 0 & I \end{pmatrix}$$

for any λ having nonnegative real part. Taking linear combinations of rows (and using the fact that the rows of $C_{\rm e}$ are part of the rows of C), this matrix can be easily reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 \\ 0 & 0 \\ C & 0 \\ 0 & I \end{pmatrix}$$

from which it is seen that the columns are linearly independent if and only if so are those of the submatrix

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}.$$

Hence, we conclude that system (4.30) is detectable if and only if so is pair A, C.

To check stabilizability, let Φ and G be defined as above, and look at the linear independence of the rows of the matrix

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & -\lambda I & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -s_0 I & -s_1 I & -s_2 I & \cdots & -(s_{d-1}+\lambda)I & 0 \end{pmatrix}$$

for any λ having non-negative real part. Taking linear combinations of columns, this matrix is initially reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -\psi(\lambda)I & * & * & \cdots & * & 0 \end{pmatrix}$$

in which $\psi(\lambda)$ is the minimal polynomial of S. Then, taking linear combinations of rows, this matrix is reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -\psi(\lambda)I & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

After permutation of rows and columns, one finally obtains a matrix of the form

$$\begin{pmatrix} A-\lambda I & B & 0 & 0 \\ C_{\mathrm{e}} & 0 & -\psi(\lambda)I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

(where the lower-right block is an identity matrix of dimension $(d-1)p$). If λ is not an eigenvalue of S, $\psi(\lambda) \neq 0$ and the rows are independent if and only if so are those of

$$\begin{pmatrix} A-\lambda I & B \end{pmatrix}.$$

On the contrary, if λ is an eigenvalue of S, $\psi(\lambda) = 0$ and the rows are independent if and only if so are those of

$$\begin{pmatrix} A-\lambda I & B \\ C_\mathrm{e} & 0 \end{pmatrix}.$$

Thus, it is concluded that system (4.30) is stabilizable if and only if so is the pair (A,B) and the *non-resonance* condition (4.31) holds . ◁

If the assumptions of this Lemma hold, the system (4.30) is stabilizable by means of a (dynamic) feedback. In other words, there exists matrices $A_\mathrm{s}, B_\mathrm{s}, C_\mathrm{s}, D_\mathrm{s}, J_\mathrm{s}, H_\mathrm{s}$, such that the closed-loop system obtained controlling (4.30) by means of a system of the form

$$\begin{aligned} \dot{\xi} &= A_\mathrm{s}\xi + B_\mathrm{s}y'_\mathrm{a} + J_\mathrm{s}y''_\mathrm{a} \\ u &= C_\mathrm{s}\xi + D_\mathrm{s}y'_\mathrm{a} + H_\mathrm{s}y''_\mathrm{a}, \end{aligned} \tag{4.32}$$

in which y'_a and y''_a are the upper and – respectively – lower block of the output y_a of (4.30), is stable. In what follows system (4.32) will be referred to as a *stabilizer*.

Remark 4.1. A simple expression of such stabilizer can be found in this way. By Lemma 4.2 the augmented system (4.30) is stabilizable. Hence, there exist matrices $L \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{R}^{m\times pd}$ such that the system

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \left(\begin{pmatrix} A & 0 \\ GC_\mathrm{e} & \Phi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} L & M \end{pmatrix} \right) \begin{pmatrix} x \\ \eta \end{pmatrix}$$

is stable. Moreover, since the pair (A,C) is detectable, there is a matrix N such $(A-NC)$ has all eigenvalues in $\mathbb{C}^-$. Let the augmented system (4.30) be controlled by

$$\begin{aligned} \dot{\xi} &= A\xi + N(y'_\mathrm{a} - C\xi) + B(L\xi + My''_\mathrm{a}) \\ u &= L\xi + My''_\mathrm{a}. \end{aligned} \tag{4.33}$$

Bearing in mind that $y'_\mathrm{a} = Cx$, that $y''_\mathrm{a} = \eta$, and using arguments identical to those used in the proof of the sufficiency in Theorem A.4, it is seen that the resulting closed-loop system is stable. ◁

With this result in mind, consider, for the solution of the problem of output regulation, a candidate controller of the form (4.29), in which the matrices $A_\mathrm{s}, B_\mathrm{s}, C_\mathrm{s}, D_\mathrm{s}, J_\mathrm{s}, H_\mathrm{s}$, are chosen in such a way that (4.32) stabilizes system (4.30). This yields a closed loop system of the form

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P+BD_\mathrm{s}Q & A+BD_\mathrm{s}C & BH_\mathrm{s} & BC_\mathrm{s} \\ GQ_\mathrm{e} & GC_\mathrm{e} & \Phi & 0 \\ B_\mathrm{s}Q & B_\mathrm{s}C & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.34}$$

If the stabilizer (4.32) stabilizes system (4.30), the matrix

$$\begin{pmatrix} A+BD_\mathrm{s}C & BH_\mathrm{s} & BC_\mathrm{s} \\ GC_\mathrm{e} & \Phi & 0 \\ B_\mathrm{s}C & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \tag{4.35}$$

has all eigenvalues with negative real part. Hence, the closed loop system possesses a *stable eigenspace* and a *center eigenspace*. The latter, in particular, can be expressed as

$$\mathscr{V}^\mathrm{c} = \mathrm{Im} \begin{pmatrix} I \\ \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} \tag{4.36}$$

in which $(\Pi_x, \Pi_\eta, \Pi_\xi)$ is a solution of the Sylvester equation

$$\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A+BD_\mathrm{s}C & BH_\mathrm{s} & BC_\mathrm{s} \\ GC_\mathrm{e} & \Phi & 0 \\ B_\mathrm{s}C & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P+BD_\mathrm{s}Q \\ GQ_\mathrm{e} \\ B_\mathrm{s}Q \end{pmatrix}. \tag{4.37}$$

In particular, it is seen that Π_η and Π_x satisfy

$$\Pi_\eta S = \Phi \Pi_\eta + G(C_\mathrm{e}\Pi_x + Q_\mathrm{e}). \tag{4.38}$$

This is a *key* property, from which it will be deduced that the proposed controller solves the problem of output regulation. In fact, the following result holds.

Lemma 4.3. *If Φ is the matrix defined in (4.23) and G is the matrix defined in (4.24), the equation (4.38)* implies

$$C_\mathrm{e}\Pi_x + Q_\mathrm{e} = 0. \tag{4.39}$$

Proof. Let Π_η be partitioned consistently with the partition of Φ, as

$$\Pi_\eta = \begin{pmatrix} \Pi_{\eta,1} \\ \Pi_{\eta,2} \\ \cdots \\ \Pi_{\eta,d} \end{pmatrix}.$$

Bearing in mind the special structure of Φ and G, equation (4.38) becomes

$$\begin{aligned} \Pi_{\eta,1}S &= \Pi_{\eta,2} \\ \Pi_{\eta,2}S &= \Pi_{\eta,3} \\ &\cdots \\ \Pi_{\eta,d-1}S &= \Pi_{\eta,d} \\ \Pi_{\eta,d}S &= -s_0\Pi_{\eta,1} - s_1\Pi_{\eta,2} - \cdots - s_{d-1}\Pi_{\eta,d} + C_\mathrm{e}\Pi_x + Q_\mathrm{e}. \end{aligned}$$

The first $d-1$ of these yield, for $i = 1, 2, \ldots d$,

$$\Pi_{\eta,i} = \Pi_{\eta,1} S^{i-1} \,,$$

which, replaced in the last one, yield in turn

$$\Pi_{\eta,1} S^d = \Pi_{\eta,1}(-s_0 I - s_1 S - \cdots - s_{d-1} S^{d-1}) + C_\mathrm{e}\Pi_x + Q_\mathrm{e} \,. \tag{4.40}$$

By definition, $\psi(\lambda)$ satisfies the first of (4.27). Hence, (4.40) implies (4.39). ◁

We have proven in this way that if the matrix (4.35) has all eigenvalues with negative real part, and the matrices Φ and G have the form (4.23) and (4.24), the center eigenspace of system (4.1) controlled by (4.29), whose expression is given in (4.36), is such that Π_x satisfies (4.39). Since in steady-state $x_\mathrm{ss}(t) = \Pi_x w(t)$, we see that $e_\mathrm{ss}(t) = C_\mathrm{e}\Pi_x w(t) + Q_\mathrm{e} w(t) = 0$ and conclude that the proposed controller solves the problem of output regulation. We summarize this result as follows.

Proposition 4.3. *Suppose that*

(i) the triplet $\{A,B,C\}$ is stabilizable and detectable
(ii) the non-resonance condition (4.31) holds.

Then, the problem of output regulation is solvable, in particular by means of a controller of the form (4.29), in which Φ, G have the form (4.23),(4.24) and $A_\mathrm{s}, B_\mathrm{s}, C_\mathrm{s}, D_\mathrm{s}, J_\mathrm{s}, H_\mathrm{s}$ are such that (4.32) stabilizes the augmented plant (4.30).

Remark 4.2. Note, that, setting

$$\Psi = C_\mathrm{s}\Pi_\xi + D_\mathrm{s}(C\Pi_x + Q) + H_\mathrm{s}\Pi_\eta \,,$$

it is seen that the pair (Π_x, Ψ) is a solution of Francis's equation (4.6). ◁

4.5 Robust output regulation

We consider in this section the case in which the plant is affected by *structured uncertainties*, that is the case in which the coefficient matrices that characterize the model of the plant depend on a vector μ of uncertain parameters, as in

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C_\mathrm{e}(\mu)x + Q_\mathrm{e}(\mu)w \\ y_\mathrm{r} &= C_\mathrm{r}(\mu)x + Q_\mathrm{r}(\mu)w \end{aligned} \tag{4.41}$$

(note that S is assumed to be *independent* of μ). Coherently with the notations adopted before, we set

$$C(\mu) = \begin{pmatrix} C_\mathrm{e}(\mu) \\ C_\mathrm{r}(\mu) \end{pmatrix}, \qquad Q(\mu) = \begin{pmatrix} Q_\mathrm{e}(\mu) \\ Q_\mathrm{r}(\mu) \end{pmatrix}.$$

We show in what follows how the results discussed in the previous section can be enhanced to obtain a robust controller. First of all, observe that if a robust controller exists, this controller must solve the problem of output regulation for each of μ. Hence, for each of such values, the necessary conditions for existence of a controller determined in the earlier sections must hold: the triplet $\{A(\mu), B(\mu), C(\mu)\}$ must be stabilizable and detectable and, above all, the μ-dependent Francis equations

$$\begin{aligned} \Pi(\mu)S &= A(\mu)\Pi(\mu) + B(\mu)\Psi(\mu) + P(\mu) \\ 0 &= C_e(\mu)\Pi(\mu) + Q_e(\mu)\,. \end{aligned} \tag{4.42}$$

must have a solution pair $\Pi(\mu), \Psi(\mu)$ (that, in general, we expect to be μ-dependent).

The design method discussed in the previous section was based on the possibility of stabilizing system (4.30) by means of a (dynamic) feedback driven by y_a. The existence of such stabilizer was guaranteed by the fulfillment of the assumptions of Lemma 4.2. In the present setting, in which the parameters of the plant depend on a vector μ of uncertain parameters, one might pursue a similar approach, seeking a controller (which should be μ-independent) that stabilizes the augmented system [5]

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A(\mu) & 0 \\ GC_e(\mu) & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B(\mu) \\ 0 \end{pmatrix} u \\ y_a &= \begin{pmatrix} C(\mu) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.43}$$

It should be stressed, though, that a result such as that of the Lemma 4.2 cannot be invoked anymore. In fact, the (necessary) assumption that the triplet $\{A(\mu), B(\mu), C(\mu)\}$ is stabilizable and detectable for every μ no longer guarantees the existence of a *robust* stabilizer for (4.43). For example, a stabilizer having the structure (4.33) cannot be used, because the latter presumes a precise knowledge of the matrices A, B, C that characterize the model of the plant, and this is no longer the case when such matrices depend on a vector μ of uncertain parameters.

We will show later in the Chapter how (and under what assumptions) such a robust stabilizer may be found. In the preset section, we take this as an hypothesis, i.e. we *suppose* that there exists a dynamical system, modeled as in (4.32), that robustly stabilizes (4.43). To say that the plant (4.43), controlled by (4.32), is robustly stable is the same thing as to say that the matrix

$$\begin{pmatrix} A(\mu) + B(\mu)D_s C(\mu) & B(\mu)H_s & B(\mu)C_s \\ GC_e(\mu) & \Phi & 0 \\ B_s C(\mu) & J_s & A_s \end{pmatrix} \tag{4.44}$$

is Hurwitz for every value of μ.

Let now the uncertain plant (4.41) be controlled by a system of the form

[5] It is worth observing that, since by assumption the matrix S is not affected by parameter uncertainties, so is its minimal polynomial $\psi(\lambda)$ and consequently so is the matrix Φ defined in (4.23).

$$\begin{aligned}\dot{\eta} &= \Phi\eta + Ge\\ \dot{\xi} &= A_s\xi + B_s y + J_s\eta\\ u &= C_s\xi + D_s y + H_s\eta\,.\end{aligned} \tag{4.45}$$

The resulting closed-loop system is an autonomous linear system having the form

$$\begin{pmatrix}\dot{w}\\ \dot{x}\\ \dot{\eta}\\ \dot{\xi}\end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0\\ P(\mu)+B(\mu)D_sQ(\mu) & A+B(\mu)D_sC(\mu) & B(\mu)H_s & B(\mu)C_s\\ GQ_e(\mu) & GC_e(\mu) & \Phi & 0\\ B_sQ(\mu) & B_sC(\mu) & J_s & A_s\end{pmatrix}\begin{pmatrix}w\\ x\\ \eta\\ \xi\end{pmatrix}. \tag{4.46}$$

Since the matrix (4.44) is Hurwitz for every value of μ and S has all eigenvalues on the imaginary axis, the closed loop system possesses two complementary invariant subspaces: a *stable eigenspace* and a *center eigenspace*. The latter, in particular, has the expression

$$\mathscr{V}^c = \operatorname{Im}\begin{pmatrix} I\\ \Pi_x(\mu)\\ \Pi_\eta(\mu)\\ \Pi_\xi(\mu)\end{pmatrix}$$

in which $(\Pi_x(\mu), \Pi_\eta(\mu), \Pi_\xi(\mu))$ is the (unique) solution of the Sylvester equation (compare with (4.37))

$$\begin{pmatrix}\Pi_x\\ \Pi_\eta\\ \Pi_\xi\end{pmatrix} S = \begin{pmatrix} A(\mu)+B(\mu)D_sC(\mu) & B(\mu)H_s & B(\mu)C_s\\ GC_e(\mu) & \Phi & 0\\ B_sC(\mu) & J_s & A_s\end{pmatrix}\begin{pmatrix}\Pi_x\\ \Pi_\eta\\ \Pi_\xi\end{pmatrix} + \begin{pmatrix} P(\mu)+B(\mu)D_sQ(\mu)\\ GQ_e(\mu)\\ B_sQ(\mu)\end{pmatrix}.$$

From this Sylvester equation, one deduces (compare with (4.38)) that

$$\Pi_\eta(\mu)S = \Phi\Pi_\eta(\mu) + G(C_e(\mu)\Pi_x(\mu) + Q_e(\mu))\,.$$

From this, using Lemma 4.3, it is concluded that

$$C_e(\mu)\Pi_x(\mu) + Q_e(\mu) = 0\,.$$

In steady-state $x_{ss}(t) = \Pi_x(\mu)w(t)$ and, in view of equation above, we conclude that $e_{ss}(t) = 0$. We summarize the discussion as follows.

Proposition 4.4. *Let Φ be a matrix of the form (4.23) and G a matrix of the form (4.24). Suppose the system (4.43) is robustly stabilized by a stabilizer of the form (4.32). Then the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.45).*

Remark 4.3. One might be puzzled by the absence of the non-resonance condition in the statement of the Proposition above. It turns out, though, that this condition is implied by the assumption of robust stabilizability of the augmented system. As a matter of fact, in this statement it is assumed that there exists a stabilizer that stabilizes the augmented plant (4.43) for every μ. As a trivial consequence, the latter is stabilizable and detectable for every μ. This being the case, it is seen from Lemma 4.2 that, if the system (4.43) is robustly stabilized by a stabilizer of the form (4.32), necessarily the triplet $\{A(\mu), B(\mu), C(\mu)\}$ is stabilizable and detectable for every μ and the non-resonance condition must hold for every μ. We stress also that the non-resonance condition, which we have seen is necessary, also implies the existence of a solution of Francis's equations (4.42). This is an immediate consequence of Lemma 4.1.

4.6 The special case in which $m = p$ and $p_{\rm r} = 0$

We discuss in this section the design of regulators in the special case of a plant in which the number of regulated outputs is equal to the number of control inputs, and no additional measurements outputs are available. Of course, the design of a regulator could be achieved by following the general construction described in the previous section, but in this special case alternative (and somewhat simpler) design procedures are available, which will be described in what follows.

Immediate consequences of the assumption $m = p$ are the fact that the nonresonance condition can be rewritten as

$$\det \begin{pmatrix} A - \lambda I & B \\ C_{\rm e} & 0 \end{pmatrix} \neq 0 \qquad \forall \lambda \in \sigma(S), \tag{4.47}$$

and the fact that, if this is the case, the solution Π, Ψ of Francis's equations (4.6) is *unique*. If, in addition, $p_{\rm r} = 0$, the construction described above can be simplified and an alternative structure of the controller is possible.

If $p_{\rm r} = 0$, in fact, the structure of the controller (4.29) becomes

$$\begin{aligned} \dot{\eta} &= \Phi \eta + G e \\ \dot{\xi} &= A_{\rm s} \xi + B_{\rm s} e + J_{\rm s} \eta \\ u &= C_{\rm s} \xi + D_{\rm s} e + H_{\rm s} \eta \,. \end{aligned} \tag{4.48}$$

Suppose that $J_{\rm s}$ and $H_{\rm s}$ are chosen as

$$\begin{aligned} J_{\rm s} &= B_{\rm s} \Gamma \\ H_{\rm s} &= D_{\rm s} \Gamma \end{aligned}$$

in which Γ is a matrix to be determined. In this case, the proposed controller can be seen as a pure *cascade connection* of a post-processing *filter* modeled by

$$\begin{aligned} \dot{\eta} &= \Phi \eta + G e \\ \tilde{e} &= \Gamma \eta + e \,, \end{aligned} \tag{4.49}$$

whose output $\tilde{e}$ drives a stabilizer modeled by

$$\begin{aligned} \dot{\xi} &= A_s \xi + B_s \tilde{e} \\ u &= C_s \xi + D_s \tilde{e} \end{aligned} \tag{4.50}$$

as shown in Fig. 4.4.

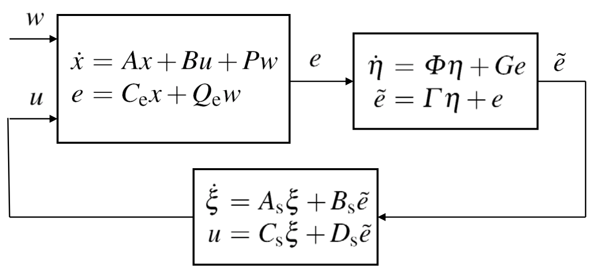

Fig. 4.4 The control consists in a *post-processing* internal model cascaded with a stabilizer.

Such special form of J_s and H_s is admissible if a system of the form (4.50) exists that stabilizes augmented plant

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_e & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ \tilde{e} &= \begin{pmatrix} C_e & \Gamma \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \end{aligned} \tag{4.51}$$

i.e. if the latter is stabilizable and detectable. In this respect, the following result is useful.

Lemma 4.4. *Let Γ be such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, the augmented system (4.51) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C_e\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.47) holds.

Proof. Stabilizability is a straightforward consequence of Lemma 4.2. Detectability holds if the columns of the matrix

$$\begin{pmatrix} A - \lambda I & 0 \\ GC_e & \Phi - \lambda I \\ C_e & \Gamma \end{pmatrix}$$

are independent for all λ having non-negative real part. Taking linear combinations of rows, we transform the latter into

$$\begin{pmatrix} A-\lambda I & 0 \\ 0 & \Phi - G\Gamma - \lambda I \\ C_{\rm e} & \Gamma \end{pmatrix},$$

and we observe that, if Γ is such that the matrix $\Phi - G\Gamma$ is Hurwitz, the augmented system (4.51) is detectable if and only if so is the pair $(A, C_{\rm e})$. $\triangleleft$

Note that a matrix Γ that makes $\Phi - G\Gamma$ Hurwitz certainly exists because by construction the pair Φ, G is reachable. We can therefore conclude that, if the triplet $\{A, B, C_{\rm e}\}$ is stabilizable and detectable, if the non-resonance condition holds and if Γ is chosen in such that a way $\Phi - G\Gamma$ is Hurwitz (as it is always possibile), the problem of output regulation can be solved by means of a controller consisting of the cascade of (4.49) and (4.50). [6]

We summarize this in the following statement.

Proposition 4.5. *Suppose that*

(i) the triplet $\{A, B, C_{\rm e}\}$ is stabilizable and detectable
(ii) the non-resonance condition (4.47) holds.

Then, the problem of output regulation is solvable by means of a controller of the form

$$\begin{aligned} \dot{\eta} &= \Phi\eta + Ge \\ \dot{\xi} &= A_{\rm s}\xi + B_{\rm s}(\Gamma\eta + e) \\ u &= C_{\rm s}\xi + D_{\rm s}(\Gamma\eta + e), \end{aligned} \tag{4.52}$$

in which Φ, G have the form (4.23),(4.24), Γ is such that a $\Phi - G\Gamma$ is Hurwitz and $A_{\rm s}, B_{\rm s}$, $C_{\rm s}, D_{\rm s}$ are such that (4.50) stabilizes the augmented plant (4.51).

The controller (4.52) is, as observed, the cascade of two sub-systems. It would be nice to check whether these sub-systems could be "swapped", i.e. whether the same result could be obtained by means of a controller consisting of a pre-processing filter of the form

$$\begin{aligned} \dot{\eta} &= \Phi\eta + G\tilde{u} \\ u &= \Gamma\eta + \tilde{u} \end{aligned} \tag{4.53}$$

whose input $\tilde{u}$ is provided by a stabilizer of the form

$$\begin{aligned} \dot{\xi} &= A_{\rm s}\xi + B_{\rm s}e \\ \tilde{u} &= C_{\rm s}\xi + D_{\rm s}e \end{aligned} \tag{4.54}$$

as shown in Fig. 4.5.

This is trivially possible if $m = 1$, because the two sub-systems in question are single-input single-output. However, as it is shown now, this is possible also if $m >$

[6] Note that the filter (4.49) is an invertible system, the inverse being given by

$$\begin{aligned} \dot{\eta} &= (\Phi - G\Gamma)\eta + G\tilde{e} \\ e &= -\Gamma\eta + \tilde{e}. \end{aligned}$$

Hence, if Γ is chosen in such that a way $\Phi - G\Gamma$ is Hurwitz, the inverse of (4.49) is a stable system.

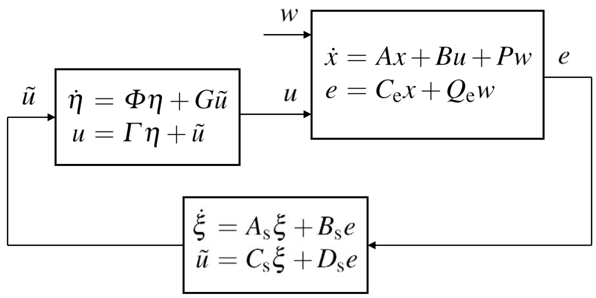

Fig. 4.5 The control consists in a stabilizer cascaded with a *pre-processing* internal model.

1. To this end, observe that controlling the plant (4.1) by means of (4.53) and (4.54) yields an overall system modeled by

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P + BD_{\mathrm{s}}Q_{\mathrm{e}} & A + BD_{\mathrm{s}}C_{\mathrm{e}} & B\Gamma & BC_{\mathrm{s}} \\ GD_{\mathrm{s}}Q_{\mathrm{e}} & GD_{\mathrm{s}}C_{\mathrm{e}} & \Phi & GC_{\mathrm{s}} \\ B_{\mathrm{s}}Q_{\mathrm{e}} & B_{\mathrm{s}}C_{\mathrm{e}} & 0 & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.55}$$

On the basis of our earlier discussions, it can be claimed that the problem of output regulation is solved if the matrix the matrix

$$\begin{pmatrix} A + BD_{\mathrm{s}}C_{\mathrm{e}} & B\Gamma & BC_{\mathrm{s}} \\ GD_{\mathrm{s}}C_{\mathrm{e}} & \Phi & GC_{\mathrm{s}} \\ B_{\mathrm{s}}C_{\mathrm{e}} & 0 & A_{\mathrm{s}} \end{pmatrix} \tag{4.56}$$

is Hurwitz and the associated center eigenspace of (4.55) is contained in the kernel of the error map $e = C_{\mathrm{e}}x + Q_{\mathrm{e}}w$.

The matrix (4.56) is Hurwitz if (and only if) the stabilizer (4.54) stabilizes the augmented plant

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ G \end{pmatrix} \tilde{u} \\ e &= \begin{pmatrix} C_{\mathrm{e}} & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.57}$$

In what follows the conditions under which this is possible are discussed.

Lemma 4.5. *Let Φ and G be defined as in (4.23) and (4.24). Let Γ be such that the matrix $\Phi - G\Gamma$ is Hurwitz. Then, the pair (Φ, Γ) is observable.*

Proof. Since $\Phi - G\Gamma$ is Hurwitz, the pair (Φ, Γ) is by definition detectable. But Φ has by assumption all eigenvalues with non-negative real part. In this case, detectability (of the pair (Φ, Γ)) is equivalent to observability. ◁

Lemma 4.6. *Let Φ be defined as in (4.23). Let Γ be a $(p \times dp)$ matrix such that the pair (Φ, Γ) is observable. Then, there exists a nonsingular matrix T such that*

$$\Gamma = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} T$$

and

$$T\Phi = \Phi T\,.$$

Proof. If the pair (Φ, Γ) is observable, the square $pd \times pd$ matrix (recall that, by construction, the minimal polynomial of Φ has degree d)

$$T = \begin{pmatrix} \Gamma \\ \Gamma\Phi \\ \cdots \\ \Gamma\Phi^{d-1} \end{pmatrix}$$

is invertible. In view of the second of (4.27), a simple calculation shows that this matrix renders the two identities in the Lemma fulfilled. $\triangleleft$

Lemma 4.7. *Let Γ be such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, the augmented system (4.57) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C_\mathrm{e}\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.47) holds.

Proof. To check stabilizability, observe that the rows of the matrix

$$\begin{pmatrix} A - \lambda I & B\Gamma & B \\ 0 & \Phi - \lambda I & G \end{pmatrix}$$

which, via combination of columns, can be reduced to the matrix

$$\begin{pmatrix} A - \lambda I & 0 & B \\ 0 & \Phi - G\Gamma - \lambda I & G \end{pmatrix},$$

are linearly independent for each λ having non-negative real part if and only if the pair (A, B) is stabilizable. To check detectabililty, we need to look at the linear independence of the columns of the matrix

$$\begin{pmatrix} A - \lambda I & B\Gamma \\ 0 & \Phi - \lambda I \\ C_\mathrm{e} & 0 \end{pmatrix}. \tag{4.58}$$

By Lemma 4.5, the pair (Φ, Γ) is observable. Let T be a matrix that renders the two identities in Lemma 4.6 fulfilled, and have the matrix (4.58) replaced by the matrix

$$\begin{pmatrix} I & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & B\Gamma \\ 0 & \Phi - \lambda I \\ C_\mathrm{e} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T^{-1} \end{pmatrix} = \begin{pmatrix} A - \lambda I & B\Gamma T^{-1} \\ 0 & \Phi - \lambda I \\ C_\mathrm{e} & 0 \end{pmatrix},$$

which, in view of the forms of ΓT^{-1} and Φ can be expressed in more detail as

$$\begin{pmatrix} A-\lambda I & B & 0 & 0 & \cdots & 0 \\ 0 & -\lambda I & I & 0 & \cdots & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & -s_0 I & -s_1 I & -s_2 I & \cdots & -(s_{d-1}+\lambda)I \\ C_{\mathrm{e}} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From this, the check of detectability condition proceeds essentially as in the check of stabilizability condition in Lemma 4.2. Taking linear combinations and permutation of columns and rows, one ends up with a matrix of the form

$$\begin{pmatrix} A-\lambda I & B & 0 \\ C_{\mathrm{e}} & 0 & 0 \\ 0 & -\psi(\lambda) I & 0 \\ 0 & 0 & I \end{pmatrix}$$

from which the claim follows. $\triangleleft$

If the matrix (4.56) is Hurwitz, the center eigenspace of (4.55) can be expressed as

$$\mathscr{V}^{\mathrm{c}} = \mathrm{Im}\begin{pmatrix} I \\ \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix}$$

in which Π_x, Π_η, Π_ξ are solutions of the Sylvester equation

$$\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A+BD_{\mathrm{s}}C_{\mathrm{e}} & B\Gamma & BC_{\mathrm{s}} \\ GD_{\mathrm{s}}C_{\mathrm{e}} & \Phi & GC_{\mathrm{s}} \\ B_{\mathrm{s}}C_{\mathrm{e}} & 0 & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P+BD_{\mathrm{s}}Q_{\mathrm{e}} \\ GD_{\mathrm{s}}Q_{\mathrm{e}} \\ B_{\mathrm{s}}Q_{\mathrm{e}} \end{pmatrix}.$$

The matrices Π_x, Π_η, Π_ξ that characterize $\mathscr{V}^{\mathrm{c}}$ can, in this particular setting, be easily determined. To this end, recall that, if $m = p$ and the non-resonance condition holds, the solution Π, Ψ of Francis' equations (4.6) is *unique*. As shown in the next Lemma, a special relation between the matrix Ψ and the matrices Φ and Γ that characterize the filter (4.53) can be established.

Lemma 4.8. *Let Φ be defined as in (4.23). Let Γ be a $(p \times dp)$ matrix such that the pair (Φ, Γ) is observable. Then, given any matrix Ψ, there exists a matrix Σ such that*

$$\begin{aligned} \Sigma S &= \Phi \Sigma \\ \Psi &= \Gamma \Sigma. \end{aligned} \tag{4.59}$$

Proof. Let T be a matrix such that the two identities in Lemma 4.6 hold. Change (4.59) in

$$\begin{aligned}(T\Sigma)S &= T\Phi T^{-1}(T\Sigma) = \Phi(T\Sigma)\\ \Psi &= \Gamma T^{-1}(T\Sigma) = \begin{pmatrix} I & 0 & \cdots & 0\end{pmatrix}(T\Sigma),\end{aligned}$$

and then check that

$$T\Sigma = \begin{pmatrix} \Psi \\ \Psi S \\ \cdot \\ \Psi S^{d-1} \end{pmatrix}$$

is a solution, thanks to the first one of (4.27). ◁

Using (4.59) and (4.6) it is easily seen that the triplet

$$\Pi_x = \Pi, \qquad \Pi_\eta = \Sigma, \qquad \Pi_\xi = 0$$

is a solution of the Sylvester equation above, as a matter of fact, the *unique* solution of that equation. Thus, in particular, the steady-state of the closed-loop system (4.55) is such that $x_{\rm ss}(t) = \Pi w(t)$ with Π obeying the second equation of (4.6). As a consequence, $e_{\rm ss}(t) = 0$ and the problem of output regulation is solved. We summarize the discussion as follows.

Proposition 4.6. *Suppose that*

(i) the triplet $\{A, B, C_{\rm e}\}$ is stabilizable and detectable
(ii) the non-resonance condition (4.47) holds.

Then, the problem of output regulation is solvable by means of a controller of the form

$$\begin{aligned}\dot{\xi} &= A_{\rm s}\xi + B_{\rm s}e\\ \dot{\eta} &= \Phi\eta + G(C_{\rm s}\xi + D_{\rm s}e)\\ u &= \Gamma\eta + (C_{\rm s}\xi + D_{\rm s}e),\end{aligned} \tag{4.60}$$

in which Φ, G have the form (4.23),(4.24), Γ is such that a $\Phi - G\Gamma$ is Hurwitz and $A_{\rm s}, B_{\rm s}$, $C_{\rm s}, D_{\rm s}$ are such that

$$\begin{aligned}\dot{\xi} &= A_{\rm s}\xi + B_{\rm s}e\\ \tilde{u} &= C_{\rm s}\xi + D_{\rm s}e\,.\end{aligned} \tag{4.61}$$

stabilizes the augmented plant (4.57).

We have seen in this section that, if $m = p$ and $p_{\rm r} = 0$, the problem of output regulation can be approached, under identical assumptions, in two equivalent ways. In the first one of these, the regulated output e is *post-processed* by an *internal model* of the form (4.49) and the resulting augmented system is subsequently stabilized. In the second one of these, the control input u is *pre-processed* by an *internal model* of the form (4.53) and the resulting augmented system is stabilized (see Figs. 4.4 and 4.5).

While both modes of control yield the same result, it should be stressed that the steady state behaviors of the state variables are different. In the first mode of control, it is seen from the analysis above that in steady state

$$x_{\rm ss}(t) = \Pi_x w(t), \qquad \eta_{\rm ss}(t) = \Pi_\eta w(t), \qquad \xi_{\rm ss}(t) = \Pi_\xi w(t)\,.$$

in which, using in particular Lemma 4.3, it is observed that

$$\begin{aligned} \Pi_x &= \Pi \\ \Pi_\eta S &= \Phi \Pi_\eta \\ \Pi_\xi S &= A_s \Pi_\xi + B_s \Gamma \Pi_\eta \\ \Psi &= C_s \Pi_\xi + D_s \Gamma \Pi_\eta \end{aligned}$$

where Π, Ψ is the unique solution of Francis's equations (4.6). In the second mode of control, on the other hand, in steady state we have

$$x_{ss}(t) = \Pi_x w(t), \qquad \eta_{ss}(t) = \Pi_\eta w(t), \qquad \xi_{ss}(t) = 0\,,$$

in which

$$\begin{aligned} \Pi_x &= \Pi \\ \Pi_\eta S &= \Phi \Pi_\eta \\ \Psi &= \Gamma \Pi_\eta \end{aligned}$$

with Π, Ψ the unique solution of Francis's equations (4.6). In particular, in the second mode of control, in steady state the stabilizer (4.61) is *at rest*.

In both modes of control the *internal model* has an identical structure, that of the system

$$\begin{aligned} \dot\eta &= \Phi\eta + G\hat u \\ \hat y &= \Gamma\eta + \hat u\,, \end{aligned} \tag{4.62}$$

in which Φ is the matrix defined in (4.23). In the discussion above, the matrix G has been taken as in (4.24), and the matrix Γ was any matrix rendering $\Phi - G\Gamma$ a Hurwitz matrix. However, it is easy to check that one can reverse the roles of G and Γ. In fact, using the state transformation

$$\tilde\eta = T\eta$$

with T defined as in the proof of Lemma 4.6 it is seen that an equivalent realization of (4.62) is

$$\begin{aligned} \dot{\tilde\eta} &= \Phi\tilde\eta + \tilde G\hat u \\ \hat y &= \tilde\Gamma\eta + \hat u\,, \end{aligned} \tag{4.63}$$

in which $\tilde G = TG$ and

$$\tilde\Gamma = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} \tag{4.64}$$

(here all blocks are $p \times p$, with $p = m$ by hypothesis). Thus, one can design the internal model picking Φ as in (4.23), Γ as in (4.64) and then choosing a G that makes $\Phi - G\Gamma$ a Hurwitz matrix.

Finally, note that if we let F denote the Hurwitz matrix $F = \Phi - G\Gamma$, the internal model can be put in the form

$$\begin{aligned} \dot\eta &= F\eta + G\hat y \\ \hat y &= \Gamma\eta + \hat u\,. \end{aligned} \tag{4.65}$$

The two modes of control lend themselves to solve also a problem of *robust* regulation. From the entire discussion above, in fact, one can arrive at the following conclusion.

Proposition 4.7. *Let Φ be a matrix of the form (4.23), G a matrix of the form (4.24) and Γ a matrix such that $\Phi - G\Gamma$ is Hurwitz (alternatively: Γ a matrix of the form (4.64) and G a matrix such that $\Phi - G\Gamma$ is Hurwitz). If system (4.51) is robustly stabilized by a stabilizer of the form (4.50), the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.52). If system (4.57) is robustly stabilized by a stabilizer of the form (4.61), the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.60).*

4.7 The case of SISO systems

We consider now the case in which $m = p = 1$ and $p_{\mathrm{r}} = 0$, and the coefficient matrices that characterize the controlled plant depend on a vector μ of uncertain parameters, as in (4.41), which we will rewrite for convenience as [7]

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C(\mu)x + Q(\mu)w\,. \end{aligned} \tag{4.66}$$

Let the system be controlled by a controller of the form (4.60). We know from the previous analysis that the controller in question solves the problem of output regulation if there exists a stabilizer of the form (4.61) that stabilizes the augmented system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B(\mu) \\ G \end{pmatrix} \tilde{u} \\ e &= \begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.67}$$

Suppose the triplet in question has a well-defined relative degree r between control input $\tilde{u}$ and regulated output e, independent of μ. It is known from Chapter 2 that a single-input single-output system having well-defined relative degree and *all zeros with negative real part* can be robustly stabilized by (dynamic) output feedback. Thus, we seek assumptions ensuring that the augmented system so defined has a well-defined relative degree and all zeros with negative real part. An easy calculation shows that, if the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has relative degree r,[8] then the augmented system (4.67) still has relative degree r, between the input $\tilde{u}$ and the output e. In fact, for all $k \leq r - 1$

---

[7] Note that the subscript "e" has been dropped.

[8] Here and in the following we use the abbreviation "the triplet $\{A, B, C\}$" to mean the associated system (2.1).

$$\begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ 0 & \Phi \end{pmatrix}^k = \begin{pmatrix} C(\mu)A^k(\mu) & 0 \end{pmatrix}$$

from which it is seen that the relative degree is r, with high-frequency gain

$$\begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ 0 & \Phi \end{pmatrix}^{r-1} \begin{pmatrix} B(\mu) \\ G \end{pmatrix} = C(\mu)A^{r-1}(\mu)B(\mu)\,.$$

To evaluate the zeros, we look at the roots of the polynomial

$$\det \begin{pmatrix} A(\mu)-\lambda I & B(\mu)\Gamma & B(\mu) \\ 0 & \Phi-\lambda I & G \\ C(\mu) & 0 & 0 \end{pmatrix} = 0\,.$$

The determinant is unchanged if we multiply the matrix, on the right, by a matrix of the form

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\Gamma & I \end{pmatrix}$$

no matter what Γ is. Thus, the zeros of the system are the roots of the polynomial

$$\det \begin{pmatrix} A(\mu)-\lambda I & 0 & B(\mu) \\ 0 & \Phi-G\Gamma-\lambda I & G \\ C(\mu) & 0 & 0 \end{pmatrix} = 0\,.$$

The latter clearly coincide with the roots of

$$\det \begin{pmatrix} A(\mu)-\lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix} \cdot \det(\Phi-G\Gamma-\lambda I) = 0\,.$$

Thus, the $n+d-r$ zeros of the augmented plant are given by the $n-r$ zeros of the triplet $\{A(\mu),B(\mu),C(\mu)\}$ and by the d eigenvalues of the matrix $\Phi-G\Gamma$. If, as indicated above, the matrix Γ is chosen in such a way that the matrix $\Phi-G\Gamma$ is Hurwitz (as it is always possible), we see that the zeros of the augmented system have negative real part if so are the zeros of the triplet $\{A(\mu),B(\mu),C(\mu)\}$.

We summarize the conclusion as follows.

Proposition 4.8. *Consider an uncertain system of the form (4.66). Suppose that the triplet $\{A(\mu),B(\mu),C(\mu)\}$ has a well-defined relative degree and all its $n-r$ zeros have negative real part, for every value of μ. Let Φ be a matrix of the form (4.23), G a matrix of the form (4.24) and Γ a matrix such that $\Phi-G\Gamma$ is Hurwitz. Then, the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.60), in which (4.61) is a robust stabilizer of the augmented system (4.67).*

Example 4.2. To make this result more explicit, it is useful to examine in more detail the case of a system having relative degree 1. As a byproduct, additional insight in the design procedure is gained, that proves to be useful in similar contexts in the next sections. Consider the case in which the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has relative degree 1, i.e. is such that $C(\mu)B(\mu) \neq 0$. Then, as shown in Example 4.1, system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P(\mu) & A(\mu) \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B(\mu) \end{pmatrix} u$$

$$e = \begin{pmatrix} Q(\mu) & C(\mu) \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}$$

has relative degree 1. It can be put in normal form, obtaining a system (see again Example 4.1)

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)u + P_1(\mu)w \\ e &= \xi\,, \end{aligned} \tag{4.68}$$

in which it is assumed that $b(\mu) > 0$. As shown in Example 4.1, since by hypothesis the eigenvalues of $A_{00}(\mu)$ have negative real part, the Sylvester equation [9]

$$\Pi_0(\mu)S = A_{00}(\mu)\Pi_0(\mu) + P_0(\mu) \tag{4.69}$$

has a solution $\Pi_0(\mu)$ and therefore the regulator equations (4.6) have (unique) solution

$$\Pi(\mu) = \begin{pmatrix} \Pi_0(\mu) \\ 0 \end{pmatrix}, \qquad \Psi(\mu) = \frac{-1}{b(\mu)}[A_{10}(\mu)\Pi_0(\mu) + P_1(\mu)]\,.$$

Changing z into $\tilde{z} = z - \Pi_0(\mu)w$ yields the simplified system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\ \dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[u - \Psi(\mu)w]\,. \end{aligned}$$

Let now this system be controlled by a preprocessor of the form (4.53), with Γ such that the matrix $F = \Phi - G\Gamma$ is Hurwitz and $\tilde{u}$ provided by a stabilizer (4.54). As shown in Lemma 4.8, there always exists a matrix $\Sigma(\mu)$ such that (see (4.59) in this respect)

$$\begin{aligned} \Sigma(\mu)S &= \Phi\Sigma(\mu) \\ \Psi(\mu) &= \Gamma\Sigma(\mu)\,. \end{aligned} \tag{4.70}$$

Using these identities and changing η into $\tilde{\eta} = \eta - \Sigma(\mu)w$ yields the following (augmented) system

[9] Since the parameters of the equation are μ-dependent so is expected to be its solution.

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\
\dot{\tilde{\eta}} &= \Phi\tilde{\eta} + G\tilde{u} \\
\dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[\Gamma\tilde{\eta} + \tilde{u}]\,,
\end{aligned} \tag{4.71}$$

in which, as expected, the subsystem characterized by the last three equations is *independent* of w. If such subsystem is (robustly) stabilized, then in particular $\xi \to 0$ as $t \to \infty$. Since $e = \xi$, the problem of output regulation is (robustly) solved. Now, the system in question has relative degree 1 (between input $\tilde{u}$ and output e) and its $n-1+d$ zeros all have negative real part. Hence the stabilizer (4.61) can be a pure output feedback, i.e. $\tilde{u} = -ke$, with large k. To double-check that this is the case, let $\tilde{u}$ be fixed in this way and observe that the resulting closed-loop system is a system of the form

$$\dot{x} = A(\mu)x$$

in which

$$x = \begin{pmatrix} \tilde{z} \\ \tilde{\eta} \\ \xi \end{pmatrix}, \qquad A(\mu) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ 0 & \Phi & -Gk \\ A_{10}(\mu) & b(\mu)\Gamma & a_{11}(\mu) - b(\mu)k \end{pmatrix}. \tag{4.72}$$

A similarity transformation $\bar{x} = T(\mu)x$, with

$$T(\mu) = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\frac{1}{b(\mu)}G \\ 0 & 0 & 1 \end{pmatrix} \tag{4.73}$$

changes $A(\mu)$ into the matrix (set here $F = \Phi - G\Gamma$)

$$\bar{A}(\mu) = T(\mu)A(\mu)T^{-1}(\mu) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F & \frac{1}{b(\mu)}(FG - Ga_{11}(\mu)) \\ A_{10}(\mu) & b(\mu)\Gamma & \Gamma G + a_{11}(\mu) - b(\mu)k \end{pmatrix}. \tag{4.74}$$

Since the eigenvalues of $A_{00}(\mu)$ and those of F have negative real part, the converse Lyapunov Theorem says that there exists a $(n-1+d)\times(n-1+d)$ positive definite symmetric matrix $Z(\mu)$ such that

$$Z(\mu)\begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F \end{pmatrix} + \begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F \end{pmatrix}^{\mathrm{T}} Z(\mu) < 0\,.$$

Then, it is an easy matter to show that the $(n+d)\times(n+d)$ positive definite matrix

$$P(\mu) = \begin{pmatrix} Z(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

is such that

$$P(\mu)\bar{A}(\mu) + \bar{A}^{\mathrm{T}}(\mu)P(\mu) < 0 \tag{4.75}$$

if k is large enough. [10] From this, using the direct Theorem of Lyapunov, we conclude that, if k is large enough, the matrix (4.74) has all eigenvalues with negative real part. If this is the case, the lower subsystem of (4.71) is robustly stabilized and the problem of output regulation is robustly solved. $\triangleleft$

4.8 Internal model adaptation

The remarkable feature of the controller discussed in the previous section is the ability of securing asymptotic decay of the regulated output $e(t)$ in spite of parameter uncertainties.

Thus, control schemes consisting of an internal model and of a robust stabilizer efficiently address the problem of rejecting all exogenous inputs generated by the exosystem. In this sense, they generalize the classical way in which integral-control based schemes cope with constant but unknown disturbances, even in the presence of parameter uncertainties. There still is a limitation, though, in these schemes: the necessity of a precise model of the exosystem. As a matter of fact, the controller considered above contains a pair of matrices Φ, Γ whose construction (see above) requires the knowledge of the precise values of the coefficients of the minimal polynomial of S. This limitation is not sensed in a problem of set point control, where the uncertain exogenous input is constant and thus obeys a trivial, parameter independent, differential equation, but becomes immediately evident in the problem of rejecting e.g. a *sinusoidal* disturbance of unknown amplitude and phase. A robust controller is able to cope with uncertainties on amplitude and phase of the exogenous sinusoidal signal, but the *frequency* at which the internal model oscillates must exactly match the frequency of the exogenous signal: any mismatch in such frequencies results in a nonzero steady-state error.

In what follows we show how this limitation can be removed, by automatically tuning the "natural frequencies" of the robust controller. For the sake of simplicity, we limit ourselves to sketch here the main philosophy of the design method.[11]

Consider again the single-input single-output system (4.66) for which we have learned how to design a robust regulator but suppose, now, that the model of the exosystem that generates the disturbance w depends on a vector ρ of uncertain parameters, ranging on a prescribed compact set $\mathscr{Q}$, as in

$$\dot{w} = S(\rho)w\,. \tag{4.76}$$

We retain the assumption that the exosystem is neutrally stable, in which case $S(\rho)$ can only have eigenvalues on the imaginary axis (with simple multiplicity in the minimal polynomial). Therefore, uncertainty in the value of ρ is reflected in the uncertainty in the value of the imaginary part of these eigenvalues.

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[10] See Section 2.3.

[11] The approach in this section closely follows the approach described, in a more general context, in [6].

Let

$$\psi_\rho(\lambda) = s_0(\rho) + s_1(\rho)\lambda + \cdots + s_{d-1}(\rho)\lambda^{d-1} + \lambda^d$$

denote the minimal polynomial of $S(\rho)$ and assume that the coefficients $s_0(\rho)$, $s_1(\rho), \ldots, s_{d-1}(\rho)$ are continuous functions of ρ. Following the design procedure illustrated in the previous sections, consider a pair of matrices Φ_ρ, G defined as

$$\Phi_\rho = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0(\rho) & -s_1(\rho) & -s_2(\rho) & \cdots & -s_{d-1}(\rho) \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

the former of which is a continuous function of ρ.

We know from the discussion above that, if the parameter ρ were known, a controller consisting of a pre-processing internal model of the form

$$\begin{aligned} \dot\eta &= \Phi_\rho \eta + G\tilde u \\ u &= \Gamma \eta + \tilde u\,, \end{aligned} \tag{4.77}$$

with $\tilde u$ provided by a robust stabilizer of the form (4.61), would solve the problem of output regulation. We have also seen that such a stabilizer exists if Γ is *any* matrix that renders $\Phi_\rho - G\Gamma$ a Hurwitz matrix and if, in addition, all $n-r$ zeros of the triplet $\{A(\mu), B(\mu), C(\mu)\}$ have negative real part for all μ.

Note that in this context the choice of Γ is arbitrary, so long as the matrix $\Phi_\rho - G\Gamma$ is Hurwitz. In what follows, we choose this matrix as follows. Let F be a *fixed* $d \times d$ matrix

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix} \tag{4.78}$$

with a characteristic polynomial

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_{d-1}\lambda^{d-1} + \lambda^d\,,$$

having all roots with negative real part, let G be defined as above, i.e.

$$G = (0 \quad 0 \quad \cdots \quad 0 \quad 1)^{\mathrm{T}}\,, \tag{4.79}$$

and choose, for Γ, a matrix of the form

$$\Gamma_\rho = (\,(a_0 - s_0(\rho)) \quad (a_1 - s_1(\rho)) \quad \cdots \quad (a_{d-1} - s_{d-1}(\rho))\,)\,, \tag{4.80}$$

(note that we have added the subscript "ρ" to stress the dependence of such Γ on the vector ρ of possibly uncertain parameters). This choice is clearly such that

$$\Phi_\rho - G\Gamma_\rho = F$$

is a Hurwitz matrix. Hence, if ρ were known, this choice would be admissible. With this choice, the pre-processing internal model (4.77) can be rewritten as

$$\begin{aligned} \dot{\eta} &= F\eta + G(\Gamma_\rho \eta + \tilde{u}) \\ u &= \Gamma_\rho \eta + \tilde{u} . \end{aligned} \tag{4.81}$$

Essentially, what we have done is to "shift" the uncertain data from the matrix Φ_ρ to the vector Γ_ρ. The realization (4.81) of the internal model, though, lends itself to the implementation of some easy (and standard) adaptive control techniques.

If ρ were known, the controller (4.81), with $\tilde{u}$ provided by a robust stabilizer of the form (4.61) would be a robust controller. In case ρ is not known, one may wish to replace the vector Γ_ρ with an *estimate* $\hat{\Gamma}$, *to be tuned* by means of an appropriate adaptation law.

We illustrate how this can be achieved in the simple situation in which the system has relative degree 1. To facilitate the analysis, we assume that the controlled plant has been initially put in normal form (see Example 4.1), which in the present case will be [12]

$$\begin{aligned} \dot{w} &= S(\rho)w \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)u + P_1(\mu,\rho)w \\ e &= \xi . \end{aligned} \tag{4.82}$$

By assumption, the $n-1$ eigenvalues of the matrix $A_{00}(\mu)$ have negative real part for all μ.

Consider a *tunable* pre-processing internal model of the form

$$\begin{aligned} \dot{\eta} &= F\eta + G(\hat{\Gamma}\eta + \tilde{u}) \\ u &= \hat{\Gamma}\eta + \tilde{u} \end{aligned} \tag{4.83}$$

in which $\hat{\Gamma}$ is a $1 \times d$ vector to be tuned. The associated augmented system becomes

$$\begin{aligned} \dot{w} &= S(\rho)w \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)[\hat{\Gamma}\eta + \tilde{u}] + P_1(\mu,\rho)w \\ \dot{\eta} &= F\eta + G[\hat{\Gamma}\eta + \tilde{u}] \\ e &= \xi . \end{aligned}$$

Define an *estimation error*

$$\tilde{\Gamma} = \hat{\Gamma} - \Gamma_\rho ,$$

and rewrite the system in question as (recall that $F + G\Gamma_\rho = \Phi_\rho$)

[12] It is seen from the construction in Example 4.1 that, in the normal form (4.11), the matrices P_0 and P_1 are found by means of transformations involving A,B,C,P and also S. Thus, if the former are functions of μ and the latter is a function of ρ, so are expected to be P_0 and P_1.

$$\begin{aligned}
\dot{w} &= S(\rho)w \\
\dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\
\dot{\eta} &= \Phi_\rho \eta + G\tilde{u} + G\tilde{\Gamma}\eta \\
\dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)[\Gamma_\rho \eta + \tilde{u}] + b(\mu)\tilde{\Gamma}\eta + P_1(\mu,\rho)w \\
e &= \xi\,.
\end{aligned}$$

We know from the analysis in Example 4.2 that, if $\tilde{\Gamma}$ were zero, the choice of a stabilizing control

$$\tilde{u} = -k\xi \tag{4.84}$$

(with $k > 0$ and large) would solve the problem of robust output regulation. Let $\tilde{u}$ be chosen in this way and consider, for the resulting closed-loop system, a change of coordinates (see again Example 4.2)

$$\tilde{z} = z - \Pi_0(\mu,\rho)w, \qquad \tilde{\eta} = \eta - \Sigma(\mu,\rho)w$$

in which $\Pi_0(\mu,\rho)$ is a solution of

$$\Pi_0(\mu,\rho)S(\rho) = A_{00}(\mu)\Pi_0(\mu,\rho) + P_0(\mu,\rho)$$

and $\Sigma(\mu,\rho)$ satisfies

$$\begin{aligned}
\Sigma(\mu,\rho)S(\rho) &= \Phi_\rho \Sigma(\mu,\rho) \\
\Psi(\mu,\rho) &= \Gamma_\rho \Sigma(\mu,\rho)\,,
\end{aligned}$$

in which

$$\Psi(\mu,\rho) = \frac{-1}{b(\mu)}[A_{10}(\mu)\Pi_0(\mu) + P_1(\mu,\rho)]\,.$$

This yields a system of the form

$$\begin{aligned}
\dot{w} &= S(\rho)w \\
\dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\
\dot{\tilde{\eta}} &= \Phi_\rho \tilde{\eta} - Gk\xi + G\tilde{\Gamma}\eta \\
\dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[\Gamma_\rho \tilde{\eta} - k\xi] + b(\mu)\tilde{\Gamma}\eta
\end{aligned}$$

(note that we *have not* modified the terms $\tilde{\Gamma}\eta$ for reasons that will become clear in a moment).

The dynamics of w is now completely decoupled, so that we can concentrate on the lower subsystem, that can be put in the form (compare with (4.72))

$$\dot{x} = A(\mu,\rho)x + B(\mu)\tilde{\Gamma}\eta \tag{4.85}$$

in which

$$x = \begin{pmatrix} \tilde{z} \\ \tilde{\eta} \\ \xi \end{pmatrix}, \quad A(\mu,\rho) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ 0 & \Phi_\rho & -Gk \\ A_{10}(\mu) & b(\mu)\Gamma_\rho & a_{11}(\mu) - b(\mu)k \end{pmatrix}, \quad B(\mu) = \begin{pmatrix} 0 \\ G \\ b(\mu) \end{pmatrix}.$$

It is known from Example 4.2 that – since the eigenvalues of $A_{00}(\mu)$ and those of $F = \Phi_\rho - G\Gamma_\rho$ have negative real part – a matrix such as $A(\mu,\rho)$ is Hurwitz, provided that k is large enough. Specifically, let x be changed in

$$\bar{x} = T(\mu)x$$

with $T(\mu)$ defined as in (4.73), which changes (4.85) in a system of the form

$$\dot{\bar{x}} = \bar{A}(\mu,\rho)\bar{x} + \bar{B}(\mu)\tilde{\Gamma}\eta\,, \tag{4.86}$$

in which $\bar{A}(\mu,\rho) = T(\mu)A(\mu,\rho)T^{-1}(\mu)$, and

$$\bar{B}(\mu) = T(\mu)B(\mu) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} b(\mu)\,.$$

It has been shown in Example 4.2 that there exists a positive definite matrix

$$P(\mu) = \begin{pmatrix} Z(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

and a number k^* such that

$$P(\mu)\bar{A}(\mu,\rho) + \bar{A}^{\mathrm{T}}(\mu,\rho)P(\mu) < 0 \tag{4.87}$$

if $k > k^*$. [13]

Consider now the positive definite quadratic form

$$U(\bar{x},\tilde{\Gamma}) = \bar{x}^{\mathrm{T}}P(\mu)\bar{x} + b(\mu)\tilde{\Gamma}\tilde{\Gamma}^{\mathrm{T}}\,,$$

and compute the derivative of this function along the trajectories of system (4.86). Letting $Q(\mu,\rho)$ denote the negative definite matrix on the left-hand side of (4.87) and observing that

$$\bar{x}^{\mathrm{T}}P(\mu)\bar{B}(\mu) = \xi b(\mu)\,,$$

this yields

$$\begin{aligned}
\dot{U} &= \bar{x}^{\mathrm{T}}[P(\mu)\bar{A}(\mu,\rho) + \bar{A}^{\mathrm{T}}(\mu,\rho)P(\mu)]\bar{x} + 2\bar{x}^{\mathrm{T}}P(\mu)\bar{B}(\mu)\tilde{\Gamma}\eta + 2b(\mu)\tilde{\Gamma}\dot{\tilde{\Gamma}}^{\mathrm{T}} \\
&= \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} + 2\xi b(\mu)\tilde{\Gamma}\eta + 2b(\mu)\tilde{\Gamma}\dot{\tilde{\Gamma}}^{\mathrm{T}} \\
&= \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} + 2b(\mu)\tilde{\Gamma}[\xi\eta + \dot{\tilde{\Gamma}}^{\mathrm{T}}]\,.
\end{aligned}$$

Observing that Γ_ρ is constant, we see that

$$\dot{\tilde{\Gamma}}^{\mathrm{T}} = \dot{\hat{\Gamma}}^{\mathrm{T}}\,.$$

[13] Recall, in this respect, that both the uncertain vectors μ and ρ range on compact sets.

The function $\dot{U}$ cannot be made negative definite, but it can be made *negative semi-definite*, by simply taking

$$\dot{\hat{\Gamma}}^{\mathrm{T}} = -\xi\eta$$

so as to obtain

$$\dot{U}(\bar{x},\tilde{\Gamma}) = \bar{x}^{\mathrm{T}} Q(\mu,\rho)\bar{x} \le 0\,.$$

Thus, since $U(\bar{x},\tilde{\Gamma})$ is positive definite and $\dot{U}(\bar{x},\tilde{\Gamma}) \le 0$, the trajectories of the closed-loop system are *bounded*. In addition, appealing to La Salle's invariance principle, [14] we can claim that the trajectories asymptotically converge to an invariant set contained in the locus where $\dot{U}(\bar{x},\tilde{\Gamma}) = 0$.

Clearly, from the expression above, since $Q(\mu,\rho)$ is a definite matrix, we see that

$$\dot{U}(\bar{x},\tilde{\Gamma}) = 0 \qquad \Rightarrow \qquad \bar{x} = 0\,.$$

Hence, $\lim_{t\to\infty}\bar{x}(t) = 0$, which in particular implies $\lim_{t\to\infty}\xi(t) = 0$. Therefore the problem of robust output regulation (in the presence of exosystem uncertainties) is solved.

We summarize the result as follows.

Proposition 4.9. *Consider an uncertain single-input single-output system*

$$\begin{aligned} \dot{w} &= S(\rho)w \\ \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C(\mu)x + Q(\mu)w\,. \end{aligned}$$

Suppose the system has relative degree 1 and, without loss of generality, $C(\mu)B(\mu) > 0$*. Suppose the* $n-1$ *zeros of the triplet* $\{A(\mu),B(\mu),C(\mu)\}$ *have negative real part, for every value of* μ*. Then, the problem of robust output regulation is solved by a controller of the form*

$$\begin{aligned} \dot{\eta} &= F\eta + G(\hat{\Gamma}\eta - ke) \\ u &= \hat{\Gamma}\eta - ke \end{aligned}$$

in which F, G *are matrices of the form (4.78) – (4.79),* $k > 0$ *is a large number and* $\hat{\Gamma}$ *is provided by the* adaptation law

$$\dot{\hat{\Gamma}}^{\mathrm{T}} = -e\eta\,.$$

The extension to systems having higher relative degree, which relies upon arguments similar to those presented in Section 2.5, is relatively straightforward, but it will not be covered here.

Example 4.3. A classical control problem arising in the steel industry is the control of the steel thickness in a rolling mill. As shown schematically in Fig. 4.6, a strip of steel of thickness H goes in on one side, and a thinner strip of steel of thickness h comes out on the other side. The exit thickness h is determined from the balance of

[14] See Theorem B.6 in Appendix B.

two forces: a force proportional to the difference between the incoming and outgoing thicknesses
$$F_H = W(H-h)$$
and a force proportional to the gap between the rolls, that can be expressed as
$$F_s = M(h-s)$$
in which s, known as "unloaded screw position", is seen as a control input. The expression of F_s presumes that the rolls are perfectly round. However, this is seldom the case. The effect of rolls that are not perfectly round (problem known as "eccentricity") can be modeled by adding a perturbation d to the gap $h-s$ in the expression of F_s, which yields
$$F_s^d = M(h-s-d)\,.$$
Rolls that are not perfectly round can be thought of as rolls of variable radius and this radius – if the rolls rotate at constant speed - is a periodically varying function of time, the period being equal to the time needed for the roll to perform a complete revolution. Thus, the term d that models the perturbation of the nominal gap $h-s$ is a periodic function of time. The period of such function is not fixed though, because it depends on the rotation speed of the rolls.

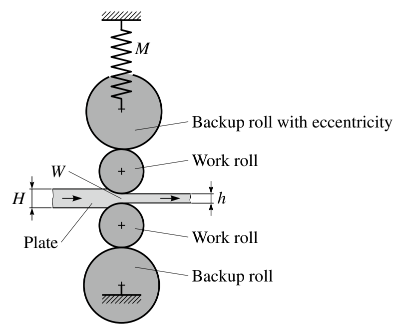

Fig. 4.6 A schematic representation of the process of thickness reduction.

Balancing the two forces F_H and F_s^d yields
$$h = \frac{1}{M+W}(Ms + WH + Md)\,.$$
The purpose of the design is to control the thickness h. This, if h_{ref} is the prescribed reference value for h, yields a tracking *error* defined as
$$e = \frac{1}{M+W}(Ms + WH + Md) - h_{\text{ref}}\,.$$

The unloaded screw position s is, in turn, proportional to the angular position of the shaft of a servomotor which, neglecting friction and mechanical losses, can be modeled as

$$\ddot{s} = bu$$

in which u is seen as a control. Setting $x_1 = s$ and $x_2 = \dot{s}$, we obtain a model of the form

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= bu\\ e &= c_1x_1 + q_1h_{\text{ref}} + q_2H + q_3d\end{aligned}$$

in which c_1, q_1, q_2, q_3 are fixed coefficients.

In this expression, h_{ref} and H are *constant* exogenous inputs, while d is a periodic function which, for simplicity, we assume to be a *sinusoidal* function. Thus, setting $w_1 = h_{\text{ref}}$, $w_2 = H$, $w_3 = d$, the tracking error can be rewritten as

$$e = Cx + Qw$$

in which

$$C = \begin{pmatrix} c_1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} q_1 & q_2 & q_3 & 0 \end{pmatrix}$$

and $w \in \mathbb{R}^4$ satisfies

$$\dot{w} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \rho\\ 0 & 0 & -\rho & 0 \end{pmatrix} := S_\rho w\,.$$

In normal form, the model thus found is given by

$$\begin{aligned}\dot{w} &= S_\rho w\\ \dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= c_1bu + QS_\rho^2 w\\ e &= \xi_1\,.\end{aligned}$$

This system has relative degree 2. Thus, according to the method described in Section 2.4, we define a new variable θ as

$$\theta = \xi_2 + a_0\xi_1 = \dot{e} + a_0e \tag{4.88}$$

in which $a_0 > 0$, and set $z = \xi_1$ to obtain a system which, viewed as a system with input u and output θ, has relative degree 1 and one zero with negative real part

$$\begin{aligned}\dot{w} &= S_\rho w\\ \dot{z} &= -a_0z + \theta\\ \dot{\theta} &= -a_0^2z + a_0\theta + c_1bu + QS_\rho^2 w\,.\end{aligned}$$

The theory developed above can be used to design a control law, driven by the "regulated variable" θ, that will steer $\theta(t)$ to 0 as $t \to \infty$. This suffices to steer also the actual tracking error $e(t)$ to zero. In fact, in view of (4.88), it is seen that $e(t)$ is the output of a stable one-dimensional linear system

$$\dot{e} = -a_0 e + \theta$$

driven by the input θ. If $\theta(t)$ asymptotically vanishes, so does $e(t)$.

For the design of the internal model, it is observed that the minimal polynomial of S_ρ is the polynomial of degree 3

$$\psi_\rho(\lambda) = \lambda^3 + \rho^2\lambda\,,$$

which has a fixed root at $\lambda = 0$. Thus, it is natural to seek a setting in which only two parameters are adapted (those that correspond to the uncertain roots in $\pm j\rho$). This can be achieved in this way. Pick

$$F = \begin{pmatrix} 0 & H_2 \\ -G_2 & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}, \qquad \Gamma_\rho = \begin{pmatrix} 1 & \Gamma_{2,\rho} \end{pmatrix}$$

in which F_2, G_2 is the pair of matrices

$$F_2 = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with a_0 and a_1 both positive (so that F_2 is Hurwitz) and $\Gamma_{2,\rho}$ such that

$$F_2 + G_2\Gamma_{2,\rho} = \begin{pmatrix} 0 & 1 \\ -\rho^2 & 0 \end{pmatrix}.$$

Finally, let P_2 be a positive definite solution of the Lyapunov equation

$$P_2F_2 + F_2^{\mathrm{T}}P_2 = -I$$

and pick $H_2 = G_2^{\mathrm{T}}P_2$. Then, the following properties hold:[15]

(i) the matrix F is Hurwitz
(ii) the minimal polynomial of $F + G\Gamma_\rho$ is $\psi_\rho(\lambda)$
(iii) the pair $(F + G\Gamma_\rho, \Gamma_\rho)$ is observable

---

[15] To prove (i), it suffices to observe that the positive definite matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix}$$

satisfies

$$QF + F^{\mathrm{T}}Q = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix} \le 0\,,$$

and use LaSalle's invariance principle. The proof of (ii) is achieved by direct substitution. Property (iii) is a consequence of (i) and of (ii), which says that all eigenvalues of $F + G\Gamma_\rho$ have zero real part. Property (iv) follows from Lemma 4.8.

(iv) for any $\Psi \in \mathbb{R}^{1\times 3}$ there exists a matrix $\Sigma(\rho)$ such that

$$\Sigma(\rho)S_\rho = (F + G\Gamma_\rho)\Sigma(\rho)\,, \qquad \Psi = \Gamma_\rho\Sigma(\rho)\,.$$

As shown above, if ρ were known, the controller (4.81) with $\tilde{u} = -k\theta$, namely the controller

$$\begin{aligned}\dot{\eta} &= F\eta + G[\Gamma_\rho\eta - k\theta]\\ u &= \Gamma_\rho\eta - k\theta\end{aligned}$$

would solve the problem of output regulation. Since ρ is not know, Γ_ρ has to be replaced by a vector of tunable parameters. Such vector, though, needs not to be a (1×3) vector because the first component of Γ_ρ, being equal to 1, is not uncertain. Accordingly, in the above controller, this vector is replaced by a vector of the form

$$\hat{\Gamma} = (\,1 \quad \hat{\Gamma}_2\,)$$

in which only $\hat{\Gamma}_2$ is a vector of tunable parameters.

The analysis that the suggested controller is able to solve the problem of output regulation in spite of the uncertainty about the value of ρ, if an appropriate adaptation law is chosen for $\hat{\Gamma}_2$, is identical to the one presented above, and will not repeated here. We limit ourself to conclude with the complete model of the controller which, in view of all of the above, reads as

$$\begin{aligned}\dot{\eta}_1 &= H_2\eta_2\\ \dot{\eta}_2 &= F_2\eta_2 + G_2(\hat{\Gamma}_2\eta_2 - k(\dot{e} + a_0e))\\ \dot{\hat{\Gamma}}_2^{\mathrm{T}} &= -\eta_2(\dot{e} + a_0e)\\ u &= \eta_1 + \hat{\Gamma}_2\eta_2 - k(\dot{e} + a_0e)\,. \qquad \triangleleft\end{aligned}$$

4.9 Robust regulation via H_∞ methods

In Section 4.7, appealing to the results presented in Chapter 2, we have shown how robust regulation can be achieved in the special case $m = p = 1$, under the assumption that the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has a well-defined relative degree and all its $n - r$ zeros have negative real part, for every value of μ. In this section, we discuss how robust regulation can be achieved in a more general setting, appealing to the method for robust stabilization presented in Sections 3.5 and 3.6.[16]

For consistency with the notation used in the context of robust stabilization via H_∞ methods, we denote the controlled plant as

[16] The approach in this section essentially follows the approach of [7]. See also [5] and [8] for further reading.

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{x} &= Ax + B_1 v + B_2 u + Pw \\
z &= C_1 x + D_{11} v + D_{12} u + Q_1 w \\
y &= C_2 x + D_{21} v + Q_2 w \\
e &= C_{\mathrm{e}} x + D_{\mathrm{e}1} v + Q_{\mathrm{e}} w\,,
\end{aligned} \tag{4.89}$$

in which

$$C_{\mathrm{e}} = EC_2\,, \quad D_{\mathrm{e}1} = ED_{21}\,, \quad Q_{\mathrm{e}} = EQ_2$$

with

$$E = \begin{pmatrix} I_p & 0 \end{pmatrix}.$$

According to the theory presented in Sections 4.4 and 4.5, we consider a controller that has the standard structure of a *post-processing internal model*

$$\dot{\eta} = \Phi\eta + Ge\,, \tag{4.90}$$

in which Φ, G have the form (4.23)–(4.24), cascaded with a *robust stabilizer*.

The purpose of such stabilizer is to solve the problem of γ-suboptimal H_∞ feedback design for an *augmented* plant defined as

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{\mathrm{e}1} \end{pmatrix} v + \begin{pmatrix} B_2 \\ 0 \end{pmatrix} u \\
z &= \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + D_{11} v + D_{12} u \\
y_{\mathrm{a}} &= \begin{pmatrix} C_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} D_{21} \\ 0 \end{pmatrix} v\,.
\end{aligned} \tag{4.91}$$

If this is the case, in fact, on the basis of the theory of robust stabilization via H_∞ methods, one can claim that the problem of output regulation is solved, *robustly* with respect to dynamic perturbations that can be expressed as

$$v = P(s)z\,,$$

in which $P(s)$ is the transfer function of a stable uncertain system, satisfying $\|P\|_{H_\infty} < 1/\gamma$.

For convenience, let system (4.91) be rewritten as

$$\begin{aligned}
\dot{x}_{\mathrm{a}} &= A_{\mathrm{a}} x_{\mathrm{a}} + B_{\mathrm{a}1} v + B_{\mathrm{a}2} u \\
z &= C_{\mathrm{a}1} x + D_{\mathrm{a}11} v + D_{\mathrm{a}12} u \\
y_{\mathrm{a}} &= C_{\mathrm{a}2} x + D_{\mathrm{a}21} v\,,
\end{aligned} \tag{4.92}$$

in which

$$A_{\mathrm{a}} = \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix}, \quad B_{\mathrm{a}1} = \begin{pmatrix} B_1 \\ GD_{\mathrm{e}1} \end{pmatrix}, \quad B_{\mathrm{a}2} = \begin{pmatrix} B_2 \\ 0 \end{pmatrix},$$

$$C_{\mathrm{a}1} = \begin{pmatrix} C_1 & 0 \end{pmatrix}, \quad D_{\mathrm{a}11} = D_{11}, \quad D_{\mathrm{a}12} = D_{12},$$
$$C_{\mathrm{a}2} = \begin{pmatrix} C_2 & 0 \\ 0 & I \end{pmatrix}, \quad D_{\mathrm{a}21} = \begin{pmatrix} D_{21} \\ 0 \end{pmatrix}.$$

Observe, in particular, that the state x_{a} has dimension $n+dp$.

The necessary and sufficient conditions for the solution of a γ-suboptimal feedback design problem are those determined in Theorem 3.3. With reference to system (4.92), the conditions in question are rewritten as follows.

Theorem 4.1. *Consider a plant of modelled by equations of the form (4.92). Let* $V_{\mathrm{a}1}, V_{\mathrm{a}2}, Z_{\mathrm{a}1}, Z_{\mathrm{a}2}$ *be matrices such that*

$$\mathrm{Im}\begin{pmatrix} Z_{\mathrm{a}1} \\ Z_{\mathrm{a}2} \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} C_{\mathrm{a}2} & D_{\mathrm{a}21} \end{pmatrix}, \qquad \mathrm{Im}\begin{pmatrix} V_{\mathrm{a}1} \\ V_{\mathrm{a}2} \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} B_{\mathrm{a}2}^{\mathrm{T}} & D_{\mathrm{a}12}^{\mathrm{T}} \end{pmatrix}.$$

The problem of γ-suboptimal H_∞ feedback design has a solution if and only if there exist symmetric matrices S_{a} and R_{a} satisfying the following system of linear matrix inequalities

$$\begin{pmatrix} Z_{\mathrm{a}1}^{\mathrm{T}} & Z_{\mathrm{a}2}^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_{\mathrm{a}}^{\mathrm{T}} S_{\mathrm{a}} + S_{\mathrm{a}} A_{\mathrm{a}} & S_{\mathrm{a}} B_{\mathrm{a}1} & C_{\mathrm{a}1}^{\mathrm{T}} \\ B_{\mathrm{a}1}^{\mathrm{T}} S_{\mathrm{a}} & -\gamma I & D_{\mathrm{a}11}^{\mathrm{T}} \\ C_{\mathrm{a}1} & D_{\mathrm{a}11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_{\mathrm{a}1} & 0 \\ Z_{\mathrm{a}2} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.93}$$

$$\begin{pmatrix} V_{\mathrm{a}1}^{\mathrm{T}} & V_{\mathrm{a}2}^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_{\mathrm{a}} R_{\mathrm{a}} + R_{\mathrm{a}} A_{\mathrm{a}}^{\mathrm{T}} & R_{\mathrm{a}} C_{\mathrm{a}1}^{\mathrm{T}} & B_{\mathrm{a}1} \\ C_{\mathrm{a}1} R_{\mathrm{a}} & -\gamma I & D_{\mathrm{a}11} \\ B_{\mathrm{a}1}^{\mathrm{T}} & D_{\mathrm{a}11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_{\mathrm{a}1} & 0 \\ V_{\mathrm{a}2} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.94}$$

$$\begin{pmatrix} S_{\mathrm{a}} & I \\ I & R_{\mathrm{a}} \end{pmatrix} \geq 0. \tag{4.95}$$

In particular, there exists a solution of dimension k if and only if there exist R_{a} and S_{a} satisfying (4.93), (4.94), (4.95) and, in addition,

$$\mathrm{rank}(I - R_{\mathrm{a}} S_{\mathrm{a}}) \leq k. \tag{4.96}$$

In view of the special structure of the matrices that characterize (4.92), the conditions above can be somewhat simplified. Observe that the inequality (4.93) can be rewritten as

$$\begin{pmatrix} Z_{\mathrm{a}1}^{\mathrm{T}} S_{\mathrm{a}}(A_{\mathrm{a}} Z_{\mathrm{a}1} + B_{\mathrm{a}1} Z_{\mathrm{a}2}) + (A_{\mathrm{a}} Z_{\mathrm{a}1} + B_{\mathrm{a}1} Z_{\mathrm{a}2})^{\mathrm{T}} S_{\mathrm{a}} Z_{\mathrm{a}1} - \gamma Z_{\mathrm{a}2}^{\mathrm{T}} Z_{\mathrm{a}2} & Z_{\mathrm{a}1}^{\mathrm{T}} C_{\mathrm{a}1}^{\mathrm{T}} + Z_{\mathrm{a}2}^{\mathrm{T}} D_{\mathrm{a}11}^{\mathrm{T}} \\ C_{\mathrm{a}1} Z_{\mathrm{a}1} + D_{\mathrm{a}11} Z_{\mathrm{a}2} & -\gamma I \end{pmatrix} < 0. \tag{4.97}$$

The kernel of the matrix

$$\begin{pmatrix} C_{\mathrm{a}2} & D_{\mathrm{a}21} \end{pmatrix} = \begin{pmatrix} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{pmatrix}$$

is spanned by the columns of a matrix

$$\begin{pmatrix} Z_{a1} \\ Z_{a2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} \\ Z_2 \end{pmatrix}$$

in which Z_1 and Z_2 are such that

$$\mathrm{Im} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} C_2 & D_{21} \end{pmatrix} .$$

Therefore, since $GC_e Z_1 + GD_{e1} Z_2 = GE(C_2 Z_1 + D_{21} Z_2) = 0$,

$$(A_a Z_{a1} + B_{a1} Z_{a2}) = \begin{pmatrix} A & 0 \\ GC_e & \Phi \end{pmatrix} \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{e1} \end{pmatrix} Z_2 = \begin{pmatrix} AZ_1 + B_1 Z_2 \\ 0 \end{pmatrix} .$$

Thus, if S_a is partitioned as

$$S_a = \begin{pmatrix} S & * \\ * & * \end{pmatrix} ,$$

in which S is $n \times n$, we see that

$$Z_{a1}^T S_a (A_a Z_{a1} + B_{a1} Z_{a2}) = Z_1^T S (AZ_1 + B_1 Z_2) .$$

Moreover, $Z_{a2}^T Z_{a2} = Z_2^T Z_2$ and

$$C_{a1} Z_{a1} + D_{a11} Z_{a2} = C_1 Z_1 + D_{11} Z_2 .$$

It is therefore concluded that the inequality (4.97) reduces to the inequality

$$\begin{pmatrix} Z_1^T & Z_2^T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^T S + SA & SB_1 & C_1^T \\ B_1^T S & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.98}$$

which is *identical* to the inequality (3.76) determined in Theorem 3.3.

The inequality (4.94), in general, does not lend itself to any special simplification. We simply observe that the kernel of the matrix

$$\begin{pmatrix} B_{a2}^T & D_{a12}^T \end{pmatrix} = \begin{pmatrix} B_2^T & 0 & D_{12}^T \end{pmatrix}$$

is spanned by the columns of a matrix

$$\begin{pmatrix} V_{a1} \\ V_{a2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} V_2 & 0 \end{pmatrix} \end{pmatrix}$$

in which V_1 and V_2 are such that

$$\mathrm{Im} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} B_2^T & D_{12}^T \end{pmatrix} ,$$

and hence (4.94), in the actual plant parameters, can be rewritten as

$$\begin{pmatrix} V_1^{\mathrm{T}} & 0 & V_2^{\mathrm{T}} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \left(\begin{array}{ccc} \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} R_{\mathrm{a}} + R_{\mathrm{a}} \begin{pmatrix} A^{\mathrm{T}} & C_{\mathrm{e}}^{\mathrm{T}} G^{\mathrm{T}} \\ 0 & \Phi^{\mathrm{T}} \end{pmatrix} & R_{\mathrm{a}} \begin{pmatrix} C_1^{\mathrm{T}} \\ 0 \end{pmatrix} & \begin{pmatrix} B_1 \\ GD_{\mathrm{e}1} \end{pmatrix} \\ \begin{pmatrix} C_1 & 0 \end{pmatrix} R_{\mathrm{a}} & -\gamma I & D_{11} \\ \begin{pmatrix} B_1^{\mathrm{T}} & D_{\mathrm{e}1}^{\mathrm{T}} G^{\mathrm{T}} \end{pmatrix} & D_{11}^{\mathrm{T}} & -\gamma I \end{array} \right) \begin{pmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ V_2 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} < 0 \,. \tag{4.99}$$

Thanks to the simplified version of (4.93), it is possible to show – as a Corollary of Theorem 4.1 – that the problem in question can be solved by a controller of dimension not exceeding n.

Corollary 4.1. *Consider the problem of γ-suboptimal H_∞ feedback design for the augmented plant (4.91). Suppose there exists positive definite symmetric matrices S and R_{a} satisfying the system of linear matrix equations (4.98), (4.99) and*

$$\begin{pmatrix} S & \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \begin{pmatrix} I_n \\ 0 \end{pmatrix} & R_{\mathrm{a}} \end{pmatrix} > 0 \,. \tag{4.100}$$

Then the problem can be solved, by a controller of dimension not execeeding n.

Proof. As a consequence of (4.100), the matrix R_{a} is positive definite and hence nonsingular. Let R_{a} be partitioned as

$$R_{\mathrm{a}} = \begin{pmatrix} R_{\mathrm{a}11} & R_{\mathrm{a}12} \\ R_{\mathrm{a}12}^{\mathrm{T}} & R_{\mathrm{a}22} \end{pmatrix}$$

in which $R_{\mathrm{a}11}$ is $n \times n$, and let R_{a}^{-1} be partitioned as

$$R_{\mathrm{a}}^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix}$$

in which Y_{11} is $n \times n$. As shown above, a $(n+dp) \times (n+dp)$ matrix S_{a} of the form

$$S_{\mathrm{a}} = \begin{pmatrix} S & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix} \tag{4.101}$$

is a solution of (4.93). It is possible to show that condition (4.100) implies (4.95). In fact, with

$$T = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ -Y_{12} & 0 & I & 0 \\ -Y_{22} & 0 & 0 & I \end{pmatrix}$$

one obtains

$$T^{\mathrm{T}} \begin{pmatrix} S_{\mathrm{a}} & I \\ I & R_{\mathrm{a}} \end{pmatrix} T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S & I & 0 \\ 0 & I & R_{\mathrm{a}11} & R_{\mathrm{a}12} \\ 0 & 0 & R_{\mathrm{a}12}^{\mathrm{T}} & R_{\mathrm{a}22} \end{pmatrix} .$$

By (4.100), the matrix on the right is positive semidefinite and this implies (4.95). Thus, (4.98), (4.99) and (4.100) altogether imply (4.93), (4.94) and (4.95).

Finally, observe that

$$R_\mathrm{a}S_\mathrm{a} = R_\mathrm{a}\begin{pmatrix} S & Y_{12} \\ Y_{12}^\mathrm{T} & Y_{22} \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & I \end{pmatrix}$$

and hence the last dp columns of $(I - R_\mathrm{a}S_\mathrm{a})$ are zero. As a consequence

$$\mathrm{rank}(I - S_\mathrm{a}R_\mathrm{a}) \leq n\,,$$

from which it is concluded that the problem can be solved by a controller dimension not exceeding n. ◁

Remark 4.4. It is worth stressing the fact that the dimension of the robust stabilizer does not exceed the dimension n of the controlled plant, and this despite of the fact that the robust stabilizer is designed for an augmented plant of dimension $n + dp$. This is essentially due to the structure of the augmented plant and in particular to the fact that the component η of the state of such augmented plant is not affected by the exogenous input v and is directly available for feedback. ◁

If the hypotheses of this Corollary are fulfilled, there exists a controller

$$\begin{aligned} \dot{\xi} &= A_\mathrm{c}\xi + B_\mathrm{c}y'_\mathrm{a} + J_\mathrm{s}y''_\mathrm{a} \\ u &= C_\mathrm{c}\xi + D_\mathrm{c}y'_\mathrm{a} + H_\mathrm{s}y''_\mathrm{a}\,, \end{aligned}$$

in which y'_a and y''_a are the upper and – respectively – lower blocks of the output y_a of (4.91), that solves the problem of γ-suboptimal H_∞ feedback design for the augmented plant (4.91). The matrices $A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c}, D_\mathrm{c}, J_\mathrm{s}, H_\mathrm{s}$ can be found solving an appropriate linear matrix inequality. [17] With these matrices, one can build a controller of the form

$$\begin{aligned} \dot{\eta} &= \Phi\eta + Ge \\ \dot{\xi} &= A_\mathrm{c}\xi + B_\mathrm{c}y + J_\mathrm{s}\eta \\ u &= C_\mathrm{c}\xi + D_\mathrm{c}y + H_\mathrm{s}\eta \end{aligned}$$

that solves the problem of output regulation for the perturbed plant (4.89), robustly with respect to dynamic perturbations that can be expressed as $v = P(s)z$, in which $P(s)$ is the transfer function of a stable uncertain system, satisfying $\|P\|_{H_\infty} < 1/\gamma$.

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[17] See Section 3.6.

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