10. Stability of Interconnected Nonlinear Systems

10.1 Preliminaries

For convenience of the reader, this section provides a quick review of the notion of *comparison functions* and their role in the well-known criterion of Lyapunov for determining stability and asymptotic stability.

Definition 10.1.1. *A continuous function* $\alpha : [0, a) \to [0, \infty)$ *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0) = 0$*. If* $a = \infty$ *and* $\lim_{r\to\infty} \alpha(r) = \infty$*, the function is said to belong to class* $\mathcal{K}_\infty$*.*

Definition 10.1.2. *A continuous function* $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ *is said to belong to class* $\mathcal{KL}$ *if, for each fixed* s*, the function*

$$\begin{array}{rccc} \alpha : & [0, a) & \to & [0, \infty) \\ & r & \mapsto & \beta(r, s) \end{array}$$

belongs to class $\mathcal{K}$ *and, for each fixed* r*, the function*

$$\begin{array}{rccc} \varphi : & [0, \infty) & \to & [0, \infty) \\ & s & \mapsto & \beta(r, s) \end{array}$$

is decreasing and $\lim_{s\to\infty} \varphi(s) = 0$*.*

Class $\mathcal{K}$ and class $\mathcal{KL}$ functions have some interesting features, that can be summarized as follows. The composition of two class $\mathcal{K}$ (respectively, class $\mathcal{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathcal{K}$ (respectively, class $\mathcal{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathcal{K}$ function, defined on $[0, a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique function, $\alpha^{-1} : [0, b) \to [0, a)$, such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \quad \text{for all } r \in [0, a) \\ \alpha(\alpha^{-1}(r)) &= r, \quad \text{for all } r \in [0, b). \end{aligned}$$

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathcal{K}$ function. If $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$. If $\beta(\cdot, \cdot)$ is a class $\mathcal{KL}$ function and $\alpha_1(\cdot), \alpha_2(\cdot)$ are class $\mathcal{K}$ functions, the function thus defined

$$\begin{array}{rccc} \gamma : & [0, a) \times [0, \infty) & \to & [0, \infty) \\ & (r, s) & \mapsto & \alpha_1(\beta(\alpha_2(r), s)) \end{array}$$

is a class $\mathcal{KL}$ function. It is also useful to know that any class $\mathcal{KL}$ function can always be estimated in terms of two other class $\mathcal{K}_\infty$ functions and of the exponential function, as indicated in the following result [1].

Lemma 10.1.1. *Assume $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function. Then, there exist two class $\mathcal{K}_\infty$ functions $\gamma(\cdot)$ and $\theta(\cdot)$ such that*

$$\beta(r,s) \le \gamma(e^{-s}\theta(r))$$

for all $(r,s) \in [0,a) \times [0,\infty)$.

Finally, another important feature of the comparison functions is the following property, which is very useful in establishing the asymptotic convergence to zero of the trajectories of a nonlinear system [2].

Lemma 10.1.2. *Consider the differential equation*

$$\dot{y} = -\alpha(y)$$

where $y \in \mathbb{R}$ and $\alpha(\cdot)$ is a locally Lipschitz class $\mathcal{K}$ function defined on $[0,a)$. For all $0 \le y^\circ < a$, this equation has a unique solution $y(t)$ satisfying $y(0) = y^\circ$, defined for all $t \ge 0$, and

$$y(t) = \varphi(y^\circ, t)$$

where $\varphi(\cdot,\cdot)$ is a class $\mathcal{KL}$ function defined on $[0,a) \times [0,\infty)$.

In what follows we denote, as usual, by $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$ and by B_ε (respectively, by $\bar{B}_\varepsilon$) the open (respectively, closed) ball of radius ε, namely

$$B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}\,, \qquad \bar{B}_\varepsilon = \{x \in \mathbb{R}^n : \|x\| \le \varepsilon\}\,.$$

Now, consider a nonlinear system

$$\dot{x} = f(x) \tag{10.1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well known criterion of Lyapunov, which, in terms of comparison function, can be expressed as follows.

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[1] For a proof of this result, see Sontag (1998).
[2] For a proof of this result, see Khalil (1996), page 656.

Theorem 10.1.3. *Let* $V : B_d \to \mathbb{R}$ *be a* C^1 *function such that, for some class* $\mathcal{K}$ *functions* $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, *defined on* $[0,d)$,

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \text{for all } \|x\| < d. \tag{10.2}$$

If

$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \text{for all } \|x\| < d, \tag{10.3}$$

the equilibrium $x = 0$ *of (10.1) is stable.*

If, for some class $\mathcal{K}$ *function* $\alpha(\cdot)$, *defined on* $[0,d)$,

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|) \qquad \text{for all } \|x\| < d, \tag{10.4}$$

the equilibrium $x = 0$ *of (10.1) is locally asymptotically stable.*

If $d = \infty$ *and, in the above inequalities,* $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ *are class* $\mathcal{K}_\infty$ *functions, the equilibrium* $x = 0$ *of (10.1) is globally asymptotically stable.*

Proof. From (10.3) it is deduced that, so long as $x(t)$ is defined, $V(x(t))$ is non increasing, i.e.

$$V(x(t)) \le V(x(0)) \, .$$

Suppose $\varepsilon < d$ and define

$$\delta = \overline{\alpha}^{-1}(\underline{\alpha}(\varepsilon)) \, .$$

Then, using (10.2), observe that $\|x(0)\| \le \delta$ implies

$$\underline{\alpha}(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \overline{\alpha}(\|x(0)\|) \le \overline{\alpha}(\delta) = \underline{\alpha}(\varepsilon) \, ,$$

i.e.

$$\|x(t)\| \le \varepsilon \, .$$

This, since $f(x)$ is locally Lipschitz, shows that $x(t)$ is defined for all $t \ge 0$ and proves the property of stability in the sense of Lyapunov.

Set $V(t) = V(x(t))$, $\theta(\cdot) = \alpha(\overline{\alpha}^{-1}(\cdot))$ and observe that, if (10.4) holds, one has

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -\theta(V(t)) \, .$$

Without loss of generality, suppose $\theta(\cdot)$ is locally Lipschitz (if this is not the case, it can be replaced in what follows by any locally Lipschitz class $\mathcal{K}$ function $\bar{\theta}(\cdot)$ such that $\theta(r) \ge \bar{\theta}(r)$). Then, by Lemma 10.1.2, the differential equation

$$\dot{y} = -\theta(y)$$

has a unique solution $y(t)$ satisfying $y(0) = V(0)$, defined for all $t \ge 0$, and

$$y(t) = \varphi(V(0), t)$$

for some class $\mathcal{KL}$ function $\varphi(\cdot, \cdot)$. By the comparison Lemma,

$$V(t) \le \varphi(V(0), t)$$

and this yields

$$\|x(t)\| \le \underline{\alpha}^{-1}(\varphi(\overline{\alpha}(\|x(0)\|), t))$$

Since the right-hand side is a class $\mathcal{KL}$ function in the arguments $\|x(0)\|$ and t, this proves local asymptotic stability. Global asymptotic stability is proven in the same way. $\triangleleft$

Remark 10.1.1. Note that the inequality on the right-hand side of (10.2) is redundant, since the existence of a function $\overline{\alpha}(x)$ for which this inequality holds is an immediate consequence of the continuity of $V(x)$ (and, of course, if $\underline{\alpha}(\cdot)$ is of class $\mathcal{K}_\infty$, so is necessarily $\overline{\alpha}(\cdot)$). However, the inequality in question turns out to be useful in establishing bounds for the trajectories, as done for instance in the proof of Theorem 10.1.3.

On the other hand, the inequality on the left-hand side of (10.2) is instrumental, together with (10.3), in establishing existence and boundedness of $x(t)$. In particular, suppose the various inequalities considered in Theorem 10.1.3 hold for $d = \infty$. The hypothesis (10.3) that $V(x(t))$ is non-increasing guarantees that $x(t)$ is defined for all $t \ge 0$ and bounded, so long as $x(0)$ is such that $\overline{\alpha}(\|x(0)\|)$ belongs to the domain of the *inverse* of the function $\underline{\alpha}(\cdot)$. If the function $\underline{\alpha}(\cdot)$ is a class $\mathcal{K}_\infty$ function, its inverse is defined on $[0, \infty)$ and therefore existence and boundedness of $x(t)$ are guaranteed for any $x(0)$. $\triangleleft$

Remark 10.1.2. Arguments similar to those used in the proof of this theorem are very useful in order to establish the *invariance*, in positive time, of certain bounded subsets of $\mathbb{R}^n$. Specifically, suppose the various inequalities considered in Theorem 10.1.3 hold for $d = \infty$ and let Ω_c denote the set of all $x \in \mathbb{R}^n$ for which $V(x) \le c$, namely

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}.$$

Note that, for any c in the image of the function $\underline{\alpha}(\cdot)$ (i.e. for any c such that $c = \underline{\alpha}(r)$ for some $0 \le r < \infty$), the inequality on the left-hand side of (10.2) yields

$$x \in \Omega_c \quad \Rightarrow \quad \|x\| \le \underline{\alpha}^{-1}(c)$$

i.e. the set Ω_c is *bounded*. This property holds for all $c > 0$ if $\underline{\alpha}(\cdot)$ is a class $\mathcal{K}_\infty$ function.

If

$$\frac{\partial V(x)}{\partial x} f(x) < 0$$

at each point x of the boundary of Ω_c, it can be concluded that, for any initial condition in the interior of Ω_c, the solution $x(t)$ of (10.1) is defined for all $t \ge 0$ and such that $x(t) \in \Omega_c$ for all $t \ge 0$. Indeed, existence and uniqueness are guaranteed by the local Lipschitz property so long as $x(t) \in \Omega_c$, because Ω_c is a compact set. The fact that $x(t)$ remains in Ω_c for all $t \ge 0$ is proved by contradiction. For, suppose that, for some trajectory $x(t)$, there is a time t_1 such that $x(t)$ is in the interior of Ω_c at all $t < t_1$ and $x(t_1)$ is on the boundary of Ω_c. Then,

$$\begin{array}{rcl} V(x(t)) & < & c \quad \text{for all } t < t_1 \\ V(x(t_1)) & = & c \end{array}$$

and this contradicts the previous inequality, which shows that the derivative of $V(x(t))$ is strictly negative at $t = t_1$. ◁

Remark 10.1.3. The existence of a class $\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$ satisfying the inequality on the left-hand side of (10.2) is equivalent to the properties that the function $V(x)$ is *positive definite* (i.e. vanishes at $x = 0$ and is positive for all nonzero x) and *proper* (i.e. *for each $c > 0$, the set Ω_c is a compact set*). The necessity of the latter property is an immediate consequence of (10.2), because $x \in \Omega_c$ implies

$$\|x\| \leq \underline{\alpha}^{-1}(V(x)) \leq \underline{\alpha}^{-1}(c)$$

which shows that Ω_c is bounded. Moreover, Ω_c is closed by definition, so it is compact. Conversely, suppose Ω_c is compact and define, for all $c \geq 0$

$$\rho(c) = \max_{x \in \Omega_c} \|x\| \, .$$

This function vanishes at $c = 0$, is positive for any nonzero c, is strictly increasing and $\lim_{c \to \infty} \rho(c) = \infty$, because $V(x)$ is continuous. Thus, $\rho(\cdot)$ is a class $\mathcal{K}_\infty$ function. Set

$$\underline{\alpha}(r) = \rho^{-1}(r) \, .$$

Now, take any x, let $c = V(x)$, and observe that indeed

$$\|x\| \leq \max_{x \in \Omega_c} \|x\| \, .$$

Then,

$$\underline{\alpha}(\|x\|) \leq \underline{\alpha}(\max_{x \in \Omega_c} \|x\|) = \underline{\alpha}(\rho(c)) = V(x)$$

i.e. this function satisfies the inequality on the left-hand side of (10.2). ◁

Remark 10.1.4. In the inequality (10.4), the function $\alpha(\cdot)$, was supposed to be of class $\mathcal{K}$. However, one can prove that, if for some $V(x)$ defined for all $x \in \mathbb{R}^n$ the inequalities (10.2) and (10.4) hold, with $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ of class $\mathcal{K}_\infty$ and $\alpha(\cdot)$ of class $\mathcal{K}$, there is another function $\widetilde{V}(x)$ which satisfies similar inequalities but with $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and $\alpha(\cdot)$ all of class $\mathcal{K}_\infty$. In other words, there is no loss of generality in assuming that $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function.

To this end, let $\rho(\cdot)$ be the function defined by an integral of the form

$$\rho(r) = \int_0^r q(s) ds$$

in which $q(\cdot)$ is a smooth class $\mathcal{K}_\infty$ function. Indeed, also the function $\rho(\cdot)$ is a class $\mathcal{K}_\infty$ function. Define $\widetilde{V}(x)$ as

$$\widetilde{V}(x) = \rho(V(x))$$

and define

$$\tilde{\alpha}(r) = q(\underline{\alpha}(r))\alpha(r) \ .$$

By construction, the function $\widetilde{V}(x)$ is automatically C^1 and, as it easy to check, satisfies estimates of the form (10.2). Moreover, the function $\tilde{\alpha}(r)$ is a class $\mathcal{K}_\infty$ function, because so are the functions $\underline{\alpha}(\cdot)$ and $q(\cdot)$. Now, observe that

$$\frac{\partial \widetilde{V}}{\partial x} f(x) = q[V(x)] \frac{\partial V}{\partial x} f(x) \leq -q[V(x)]\alpha(\|x\|) \ .$$

But

$$-q[V(x)]\alpha(\|x\|) \leq -q(\underline{\alpha}(\|x\|))\alpha(\|x\|) = -\tilde{\alpha}(\|x\|)$$

and this completes the proof. $\triangleleft$

It is well-known that the criterion for asymptotic stability provided by the previous Theorem has a *converse*, namely, the existence of a function $V(x)$ having the properties indicated in Theorem 10.1.3 is *implied* by the property of asymptotic stability of the equilibrium $x = 0$ of (10.1). In particular, the following result holds [3].

Theorem 10.1.4. *Suppose the equilibrium $x = 0$ of (10.1) is locally asymptotically stable. Then, there exist $d > 0$, a C^1 function $V : B_d \to \mathbb{R}$, and class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (10.2) and (10.4) hold. If the equilibrium $x = 0$ of (10.1) is globally asymptotically stable, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (10.2) and (10.4) hold with $d = \infty$.*

Remark 10.1.5. Note that, combining the result of Theorem 10.1.4 with an argument used in the proof of Theorem 10.1.3, it can be deduced that, if the equilibrium $x = 0$ is globally asymptotically stable, there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that, for any x°, the solution $x(t)$ of (10.1) with initial condition $x(0) = x^\circ$ satisfies an estimate of the form

$$\|x(t)\| \leq \beta(\|x^\circ\|, t)$$

for all $t \geq 0$. Note also that, using Lemma 10.1.1, this estimate can be replaced by an estimate of the form

$$\|x(t)\| \leq \gamma(e^{-t}\theta(\|x^\circ\|))$$

in which $\gamma(\cdot)$ and $\theta(\cdot)$ are class $\mathcal{K}_\infty$ functions. Since the inverse of $\gamma(\cdot)$ is defined on $[0, \infty)$ and is a class $\mathcal{K}_\infty$ function, this shows that, if the equilibrium $x = 0$ of (10.1) is globally asymptotically stable, any trajectory $x(t)$ satisfies an estimate of the form

$$\tilde{\gamma}(\|x(t)\|) \leq e^{-t}\theta(\|x(0)\|)$$

in which $\tilde{\gamma}(\cdot)$ and $\theta(\cdot)$ are class $\mathcal{K}_\infty$ functions. $\triangleleft$

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[3] For a proof of this result, see Kurzweil (1956).

It is well-known that, for a nonlinear system, the property of asymptotic stability of the equilibrium $x = 0$ does not necessarily imply *exponential* decay to zero of $\|x(t)\|$. If the equilibrium $x = 0$ of system (10.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$ and $\lambda > 0$ such that

$$x(0) \in B_d \quad \Rightarrow \quad \|x(t)\| \leq Me^{-\lambda t}\|x(0)\| \quad \text{for all } t \geq 0$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. In what follows, a characterization of those systems possessing a globally asymptotically and locally exponentially stable equilibrium is given. This characterization, and another interesting property that is presented immediately afterwards, are of great help in addressing certain problems of asymptotic stabilization, discussed in the next Chapters.

Lemma 10.1.5. *The equilibrium $x = 0$ of nonlinear system (10.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function $V(x) : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and real numbers $\delta > 0, a > 0, b > 0$, such that*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -\alpha(\|x\|)$$

for all $x \in \mathbb{R}^n$ and

$$\underline{\alpha}(s) = as^2, \qquad \alpha(s) = bs^2$$

for all $s \in [0, \delta]$.

Proof. The equilibrium of the nonlinear system (10.1) is locally exponentially stable [4] if and only if there exists real numbers $r > 0, \underline{a} > 0, \overline{a} > 0, \underline{b} > 0$ and a smooth function $U(x) : B_r \to \mathbb{R}$ such that

$$\begin{aligned} \underline{a}\|x\|^2 \leq U(x) \leq \overline{a}\|x\|^2 \\ \frac{\partial U}{\partial x} f(x) \leq -\underline{b}\|x\|^2 \end{aligned} \tag{10.5}$$

for all $x \in B_r$.

Thus, to prove that the condition stated in the Theorem is sufficient, it is only needed to check that, for all $x \in B_\delta$ and some $\overline{a}$, $V(x) \leq \overline{a}\|x\|^2$, which is indeed the case since $V(x)$ is a smooth function vanishing at $x = 0$ together with its first derivatives.

To prove the necessity, let $U(x)$ be a function satisfying the conditions (10.5) and $V(x)$ be a function satisfying (10.2) and (10.4) with $d = \infty$. We claim that there exist real numbers $k > 0, \delta > 0, \rho > 0, c_1 > 0, c_2 > 0$ such that the set

[4] See Khalil (1996), pages 140 and 149.

$$S = \{x \in \mathbb{R}^n : c_1 \le kV(x) \le c_2\}$$

satisfies

$$\bar{B}_\delta \subset S \subset \bar{B}_\rho \subset B_r$$

and

$$kV(x) \ge U(x) \qquad \text{for all } \ x \in S \ . \tag{10.6}$$

In fact, choose any $\rho < r$, and $c_1 = \overline{a}\rho^2$, $c_2 = 2\overline{a}\rho^2$. Then, choose k so that

$$\{x \in \mathbb{R}^n : kV(x) \le c_2\} \subset \bar{B}_\rho \ .$$

Such a k indeed exists because, if $k \ge c_2/\underline{\alpha}(\rho)$

$$kV(x) \le c_2 \quad \Rightarrow \quad \|x\| \le \underline{\alpha}^{-1}(\frac{c_2}{k}) \le \rho \ .$$

In this way, $S \subset \bar{B}_\rho \subset B_r$ and

$$x \in S \quad \Rightarrow \quad kV(x) \ge c_1 = \overline{a}\rho^2 \ge U(x)$$

which proves (10.6). Finally, choose $\delta < \min_{\{kV(x)=c_1\}} \|x\|$.

Now, let $\sigma(\cdot)$ be a smooth non-decreasing function, defined on $[0,\infty)$, and such that

$$\sigma(s) = \begin{cases} 0 & \text{if } s \le c_1 \\ 1 & \text{if } c_2 \le s \ . \end{cases}$$

Its derivative, denoted $\sigma'(\cdot)$, satisfies

$$\sigma'(s) \begin{cases} = 0 & \text{if } s \le c_1 \\ \ge 0 & \text{if } c_1 < s < c_2 \\ = 0 & \text{if } c_2 \le s \ . \end{cases}$$

Set

$$\beta(x) = \sigma(kV(x))$$

and consider the function

$$W(x) = \beta(x)kV(x) + (1-\beta(x))U(x) \ ,$$

which is well-defined because, by construction, $\|x\| \ge r$ implies $1-\beta(x) = 0$. Then, for all $x \in \mathbb{R}^n$,

$$W(x) \ge \beta(x)k\underline{\alpha}(\|x\|) + (1-\beta(x))\underline{a}\|x\|^2 \ .$$

Let $0 < a \le \underline{a}$ be such that

$$as^2 < k\underline{\alpha}(s) \qquad \text{for all } \ s \in [\delta, \rho] \ .$$

Then, from the previous inequality it can be concluded that there exists a class $\mathcal{K}_\infty$ function $\underline{\tilde{\alpha}}(\cdot)$, satisfying

$$\underline{\tilde{\alpha}}(s) = \begin{cases} as^2 & \text{if } s \le \delta \\ k\underline{\alpha}(s) & \text{if } s \ge \rho\ , \end{cases}$$

such that

$$W(x) \ge \underline{\tilde{\alpha}}(\|x\|)$$

for all $x \in \mathbb{R}^n$.

Moreover

$$\frac{\partial W}{\partial x} f(x) = \beta(x) k \frac{\partial V}{\partial x} f(x) + (1-\beta(x)) \frac{\partial U}{\partial x} f(x) + (kV(x) - U(x)) \frac{\partial \beta}{\partial x} f(x).$$

Observe that

$$(kV(x) - U(x)) \frac{\partial \beta}{\partial x} f(x) = (kV(x) - U(x)) \sigma'(kV(x)) k \frac{\partial V}{\partial x} f(x) \qquad (10.7)$$

and that, by construction,

$$\sigma'(kV(x))(kV(x) - U(x)) \ge 0$$

for all $x \in \mathbb{R}^n$. Thus, the quantity (10.7) is always non-positive, and

$$\begin{aligned} \frac{\partial W}{\partial x} f(x) &\le \beta(x) k \frac{\partial V}{\partial x} f(x) + (1-\beta(x)) \frac{\partial U}{\partial x} f(x) \\ &\le -\left[\beta(x) k \alpha(\|x\|) + (1-\beta(x)) \underline{b} \|x\|^2\right] . \end{aligned}$$

Form this one can conclude, as before, that there exists a number $0 < b < \underline{b}$ and a class $\mathcal{K}_\infty$ function $\tilde{\alpha}(\cdot)$, satisfying $\tilde{\alpha}(s) = bs^2$ for all $s \in [0, \delta]$, such that

$$\frac{\partial W}{\partial x} f(x) \le -\tilde{\alpha}(\|x\|)$$

for all $x \in \mathbb{R}^n$. This completes the proof of the necessity. $\triangleleft$

The following lemma, which concludes the section, provides a useful estimate for the function $\sigma(\|x(t)\|)$ resulting from the composition of a class $\mathcal{K}$ function $\sigma(\cdot)$ with the norm of an integral curve $x(t)$ of system (10.1), under the hypothesis that the latter has a globally asymptotically and locally exponentially stable at $x = 0$.

Lemma 10.1.6. *Consider the system (10.1). Suppose the equilibrium $x = 0$ is globally asymptotically stable and locally exponentially stable. Let $\sigma(\cdot)$ be a class $\mathcal{K}$ function which is differentiable at the origin. Then, there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ and a number $\lambda > 0$ such that, for any $x^\circ \in \mathbb{R}^n$, the integral curve $x(t)$ passing through x° at time $t = 0$ is such that*

$$\sigma(\|x(t)\|) \le \alpha(\|x^\circ\|) e^{-\lambda t}$$

for all $t \ge 0$.

Proof. Set

$$F(t) = \sigma(\|x(t)\|) \ .$$

By hypothesis, $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ function $\beta(\|x^\circ\|, t)$ and this, in view of Lemma 10.1.1, implies that, for some pair of class $\mathcal{K}_\infty$ functions $\gamma(\cdot)$, $\theta(\cdot)$,

$$\|x(t)\| \le \gamma(e^{-t}\theta(\|x^\circ\|)) \ .$$

Note that $\gamma(\theta(s)) \ge s$. Consider the class $\mathcal{K}_\infty$ functions $\tilde{\sigma}(\cdot)$ and $\tilde{\gamma}(\cdot)$ defined as

$$\begin{aligned} \tilde{\sigma}(s) &= 2\max\{s, \sigma(s)\} \\ \tilde{\gamma}(s) &= \tilde{\sigma} \circ \gamma(s) \ . \end{aligned}$$

Then,

$$F(t) \le \tilde{\sigma}(\|x(t)\|) \le \tilde{\gamma}(e^{-t}\theta(\|x^\circ\|)) \ .$$

Observe also that $\tilde{\gamma}(\theta(s)) > s$ for all $s > 0$.

By hypothesis, the equilibrium $x = 0$ of the system is also locally exponentially stable. This means that there are numbers $M > 0$, $\lambda > 0$ and $d > 0$ such that, if $\|x^\circ\| \le d$, the integral curve passing through x° at time $t = 0$ satisfies

$$\|x(t)\| \le M\|x^\circ\|e^{-\lambda t}$$

for all $t \ge 0$. Moreover, the function $\sigma(\cdot)$ is by hypothesis differentiable at the origin. This implies the existence of numbers $N > 0$ and $\bar{s} > 0$ such that, if $s \in [0, \bar{s}]$, $\sigma(s) \le Ns$. Let

$$R = \min\{d, \frac{\bar{s}}{M}\} \ .$$

Then, if $\|x^\circ\| \le R$,

$$F(t) \le N\|x(t)\| \le NM\|x^\circ\|e^{-\lambda t} \tag{10.8}$$

for all $t \ge 0$.

Suppose, without loss of generality, $NM > 1$, define

$$A(s) = NM\tilde{\gamma}(\theta(s))\Big(\frac{\theta(s)}{\tilde{\gamma}^{-1}(R)}\Big)^{\lambda}$$

and consider the case $\|x^\circ\| > R$. Since $\tilde{\gamma}(\theta(s)) > s$ for all $s > 0$,

$$T_1 := \ln \frac{\theta(\|x^\circ\|)}{\tilde{\gamma}^{-1}(R)} > 0$$

and

$$A(\|x^\circ\|)e^{-\lambda T_1} = NM\tilde{\gamma}(\theta(\|x^\circ\|)) \ .$$

As a consequence, we have, for all $t \in [0, T_1]$,

$$\begin{aligned} F(t) &\le \tilde{\gamma}(\theta(\|x^\circ\|)) \\ &\le NM\tilde{\gamma}(\theta(\|x^\circ\|)) = A(\|x^\circ\|)e^{-\lambda T_1} \le A(\|x^\circ\|)e^{-\lambda t} \ . \end{aligned}$$

Moreover, by definition of T_1,

$$\|x(T_1)\| \leq \gamma(e^{-T_1}\theta(\|x^\circ\|)) \leq \tilde{\gamma}(e^{-T_1}\theta(\|x^\circ\|)) = R \ ,$$

and therefore, for all $t \geq T_1$,

$$\begin{array}{rcl} F(t) & \leq & NM\|x^\circ(T_1)\|e^{-\lambda(t-T_1)} \\ & \leq & NMRe^{-\lambda(t-T_1)} = NM\tilde{\gamma}(e^{-T_1}\theta(\|x^\circ\|))e^{-\lambda(t-T_1)} \\ & \leq & NM\tilde{\gamma}(\theta(\|x^\circ\|))e^{\lambda T_1}e^{-\lambda t} \leq A(\|x^\circ\|)e^{-\lambda t} \end{array}$$

This proves that, for all $\|x^\circ\| > R$,

$$F(t) \leq A(\|x^\circ\|)e^{-\lambda t}$$

for all $t \geq 0$. On the other hand, for all $\|x^\circ\| \leq R$, (10.8) holds.

Thus, let $\alpha(\cdot)$ be a class $\mathcal{K}$ function satisfying

$$\begin{array}{rcll} \alpha(s) & \geq & NMs & \text{for } \ s \in [0, R] \\ \alpha(s) & \geq & A(s) & \text{for } \ s > R \ . \end{array}$$

This proves the Lemma. $\triangleleft$

10.2 Asymptotic Stability and Small Perturbations

This section contains a result which analyzes the effect of a "small perturbation" affecting a system of the form (10.1), whose equilibrium $x = 0$ is assumed to be locally asymptotically stable. The theorem below, often referred to as the theorem of "total stability", shows that the trajectories of the perturbed system remain arbitrarily close to the equilibrium $x = 0$, although they may not necessarily asymptotically converge to it, provided that the perturbation as well as the initial state are sufficiently small.

Theorem 10.2.1. *Suppose the equilibrium $x = 0$ of (10.1) is locally asymptotically stable. Suppose $g(x, t)$ is piecewise continuous in t and satisfies a Lipschitz condition*

$$\|g(x', t) - g(x'', t)\| \leq L\|x' - x''\|$$

for all $t \geq 0$ and all x', x'' in some neighborhood U of $x = 0$. Then, given any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ (both possibly dependent on $\varepsilon > 0$) such that, if

$$\|x^\circ\| \leq \delta_1$$

$$\|g(x, t)\| \leq \delta_2 \ \ for \ \ all \ \ \|x\| \leq \varepsilon \ \ and \ \ all \ \ t \geq 0 \ ,$$

the solution $x(t)$ of the perturbed system

$$\dot{x} = f(x) + g(x, t) \tag{10.9}$$

satisfying $x(0) = x^\circ$ *is such that*

$$\|x(t)\| \le \varepsilon$$

for all $t \ge 0$.

Proof. Let $V(x)$ be a C^1 function satisfying (10.2), (10.4). Since $\dfrac{\partial V}{\partial x}$ is C^0, there exists a number $M > 0$ such that

$$\|\frac{\partial V}{\partial x}\| \le M$$

for all $x \in U$. Suppose, without loss of generality, that ε is such that $B_\varepsilon \subset U$. Fix $\varepsilon > 0$ and let $c > 0$ be such that $c \le \underline{\alpha}(\varepsilon)$. Choose δ_2 such that

$$-\alpha(\overline{\alpha}^{-1}(c)) + M\delta_2 < 0 .$$

By construction, $x \in \Omega_c$ implies

$$\|x\| \le \varepsilon .$$

In fact,

$$\underline{\alpha}(\|x\|) \le V(x) \le c$$

implies

$$\|x\| \le \underline{\alpha}^{-1}(c) \le \underline{\alpha}^{-1}(\underline{\alpha}(\varepsilon)) = \varepsilon.$$

Also, at each point x of the boundary of Ω_c,

$$\alpha(\|x\|) \ge \alpha(\overline{\alpha}^{-1}(V(x))) = \alpha(\overline{\alpha}^{-1}(c)) .$$

As a consequence, at each point x of the boundary of Ω_c,

$$\frac{\partial V}{\partial x}[f(x) + g(x,t)] \le -\alpha(\|x\|) + \|\frac{\partial V}{\partial x}\|\delta_2 \le -\alpha(\overline{\alpha}^{-1}(c)) + M\delta_2 < 0 .$$

From this, it can be concluded that, for any initial condition in the interior of Ω_c, the solution $x(t)$ of (10.9) is defined for all $t \ge 0$ and is such that $x(t) \in \Omega_c$ for all $t \ge 0$ (see Remark 10.1.2). To complete the proof it suffices to choose $\delta_1 < \overline{\alpha}^{-1}(c)$, for this guarantees that x° is in the interior of Ω_c. In fact,

$$\overline{\alpha}^{-1}(V(x^\circ)) \le \|x^\circ\| \le \delta_1 < \overline{\alpha}^{-1}(c)$$

implies $V(x^\circ) < c$. $\triangleleft$

Note that $g(x,t)$ is not assumed to be vanishing at $x = 0$. In other words, the perturbed system may not have an equilibrium at $x = 0$. Or the perturbed system may have an unstable equilibrium at $x = 0$ (as shown in the following example). Nevertheless, in both cases the theorem of total stability guarantees that, if the perturbation is small enough, the trajectories of the perturbed system remain arbitrarily close to the equilibrium $x = 0$ of the unperturbed system.

Example 10.2.1. Consider the system

$$\dot{x} = -x^3 + \gamma x \tag{10.10}$$

where $\gamma > 0$ is a small number. Indeed the "unperturbed" system $\dot{x} = -x^3$ has a globally asymptotically stable equilibrium at $x = 0$. On the other hand, the "perturbed system" has three equilibria, at $x = 0$, $x = +\sqrt{\gamma}$, $x = -\sqrt{\gamma}$. Elementary arguments, based on the principle of stability in the first approximation, show that, since γ is positive, the first one of these equilibria is unstable, while the other two are asymptotically stable. A glance at the graph of $-x^3 + \gamma x$ suggests that

$$\begin{array}{ccccccc} 0 & < & x(0) & < & +\sqrt{\gamma} & \Rightarrow & x(t) \to +\sqrt{\gamma} \\ +\sqrt{\gamma} & < & x(0) & < & +\infty & \Rightarrow & x(t) \to +\sqrt{\gamma} \\ -\sqrt{\gamma} & < & x(0) & < & 0 & \Rightarrow & x(t) \to -\sqrt{\gamma} \\ -\infty & < & x(0) & < & -\sqrt{\gamma} & \Rightarrow & x(t) \to -\sqrt{\gamma}\,. \end{array}$$

Note that, even though the equilibrium $x = 0$ of the perturbed system is unstable, the boundedness properties indicated in the Theorem of total stability hold. In particular, given any $\varepsilon > 0$, choose $\delta_1 = \varepsilon$ and $\delta_2 = \varepsilon^3$. Then, the perturbation term γx satisfies

$$|\gamma x| \le \delta_2 \quad \text{for all } |x| \le \varepsilon$$

if $\sqrt{\gamma} \le \varepsilon$. If this is the case, the two nonzero equilibria of the system lie in the interval $[-\varepsilon, +\varepsilon]$, and we see form the previous analysis that any trajectory with initial condition $|x(0)| \le \delta_1 = \varepsilon$ remains confined to the set $|x| \le \varepsilon$. $\triangleleft$

This theorem lends itself to an easy application to the study of the stability of the equilibrium of a pair of *cascade-connected* systems. More precisely, consider the composite system

$$\begin{array}{rcl} \dot{x} & = & f(x,z) \\ \dot{z} & = & g(z) \end{array} \tag{10.11}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0) = 0$, and $f(x,z)$, $g(z)$ are locally Lipschitz on a neighborhood U of $(x,z) = (0,0)$. In view of the fact that the state z of the lower subsystem acts as an input to the upper subsystem, we shall sometimes refer to the latter as to the "driven subsystem" and to the former as to the "driving subsystem". As an immediate corollary of the theorem of total stability, it can be deduced that if the equilibrium $x = 0$ of the upper subsystem, driven by $z = 0$, is locally asymptotically stable and the equilibrium $z = 0$ of the lower subsystem is stable, then the equilibrium $(x,z) = (0,0)$ of the cascade is stable.

Corollary 10.2.2. *Consider system (10.11). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is locally asymptotically stable and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is stable. Then, the equilibrium $(x,z) = (0,0)$ of (10.11) is stable.*

Proof. By hypothesis, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|z^\circ\| \leq \delta$, the integral curve $z^\circ(t)$ of $\dot{z} = g(z)$ satisfying $z^\circ(0) = z^\circ$ is such that $\|z^\circ(t)\| \leq \varepsilon$ for all $t \geq 0$. Express $f(x, z^\circ(t))$ as

$$f(x, z^\circ(t)) = f(x, 0) + g(x, t) \tag{10.12}$$

where

$$g(x, t) = f(x, z^\circ(t)) - f(x, 0)\ .$$

Since $f(x, z)$ is locally Lipschitz, there exist $\eta > 0$ and $M > 0$ such that

$$\|g(x, t)\| \leq M\|z^\circ(t)\| \leq M\varepsilon$$

for all $\|x\| \leq \eta$, all $\|z^\circ\| \leq \delta$ and $t \geq 0$. This bound for $\|g(x, t)\|$ can be rendered arbitrarily small by choosing a sufficiently small δ and the result follows from Theorem 10.2.1. $\triangleleft$

10.3 Asymptotic Stability of Cascade-Connected Systems

In this section we investigate the *asymptotic* stability of the equilibrium $(x, z) = (0, 0)$ of a pair of cascade connected subsystems of the form

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z)\ , \end{aligned} \tag{10.13}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0, 0) = 0$, $g(0) = 0$, and $f(x, z)$, $g(z)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ (see Fig. 10.1).

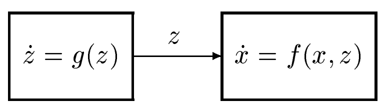

Fig. 10.1. Cascade connection.

The main result of the analysis is that local asymptotic stability of the equilibrium $x = 0$ of the upper subsystem, driven by $z = 0$, and local asymptotic stability of the equilibrium $z = 0$ of the lower subsystem always imply local asymptotic stability of the equilibrium $(x, z) = (0, 0)$ of the cascade. However, *global* asymptotic stability of the equilibrium $x = 0$ of the upper subsystem, driven by $z = 0$, and *global* asymptotic stability of the equilibrium $z = 0$ of the lower subsystem *do not* imply, in general, *global* asymptotic stability of the equilibrium $(x, z) = (0, 0)$ of the cascade. To infer global asymptotic stability of the cascade, a (strong) extra condition is needed, as shown below.

The key of the proof of these facts is the following result.

Theorem 10.3.1. *Consider system (10.13). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable. Let $\mathcal{S}$ be a set with the property that, for any $\tilde{x}^\circ \in \mathcal{S}$, the integral curve $\tilde{x}(t)$ of $\dot{\tilde{x}} = f(\tilde{x}, 0)$ satisfying $\tilde{x}(0) = \tilde{x}^\circ$ is defined for all $t \geq 0$ and is such that*

$$\lim_{t\to\infty} \tilde{x}(t) = 0 \, .$$

Pick any z° and let $z^\circ(t)$ denote the integral curve of $\dot{z} = g(z)$ satisfying $z^\circ(0) = z^\circ$. Suppose $z^\circ(t)$ is defined for all $t \geq 0$ and such that

$$\lim_{t\to\infty} z^\circ(t) = 0 \, .$$

Pick any $x^\circ \in \mathcal{S}$ and let $x^\circ(t)$ denote the integral curve of $\dot{x} = f(x, z^\circ(t))$ satisfying $x^\circ(0) = x^\circ$. Suppose $x^\circ(t)$ is defined for all $t \geq 0$, is bounded and is such that $x^\circ(t) \in \mathcal{S}$ for all $t \geq 0$. Then,

$$\lim_{t\to\infty} x^\circ(t) = 0 \, .$$

Proof. First of all, observe that, using the theorem of total stability, it is easy to prove that, since the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable, given any ε, there exist δ_1 and δ_2 such that, if $\|\bar{x}^\circ\| \leq \delta_1$ and $\|z(t)\| \leq \delta_2$ for all $t \geq 0$, the solution $\bar{x}(t)$ of

$$\dot{x} = f(x, z(t))$$

satisfying $\bar{x}(0) = \bar{x}^\circ$ is such that $\|\bar{x}(t)\| \leq \varepsilon$ for all $t \geq 0$. To this end, it suffices to split $f(x, z(t))$ as in (10.12) and use arguments similar to those used in the proof of Corollary 10.2.2. As a consequence, the theorem will be proven if we can show that, for any ε, it is possible to find a time T such that $\|x^\circ(T)\| \leq \delta_1$, and $\|z^\circ(t)\| \leq \delta_2$ for all $t \geq T$.

Now, pick $T_1 > 0$, set

$$z^\circ_{T_1}(t) = \begin{cases} z^\circ(t) & \text{if } t \leq T_1 \\ 0 & \text{if } t > T_1 . \end{cases}$$

and let $x^\circ_{T_1}(t)$ denote the integral curve of

$$\dot{x} = f(x, z^\circ_{T_1}(t))$$

satisfying $x^\circ_{T_1}(0) = x^\circ$. Indeed, $x^\circ_{T_1}(t) = x^\circ(t)$ for all $0 \leq t \leq T_1$. Moreover, for $t > T_1$, $x^\circ_{T_1}(t)$ is a solution of

$$\dot{x} = f(x, 0)$$

and hence tends to 0 as $t \to \infty$, because $x^\circ_{T_1}(T_1) \in \mathcal{S}$. In particular, there exist T_2 such that

$$\|x^\circ_{T_1}(T_1 + T_2)\| \leq \frac{\delta_1}{2} \, . \tag{10.14}$$

The time T_2 thus characterized may depend on $x^\circ_{T_1}(T_1)$ and hence on T_1 but, since by hypothesis $x^\circ(t)$ is bounded, using a compactness argument one can conclude that there exists a T_2 depending only on x° for which the inequality (10.14) holds for any T_1.

Set

$$T = T_1 + T_2 \ .$$

We will show now that, if T_1 is large enough

(a) $\|z^\circ(t)\| \le \delta_2$ for all $t \ge T$, and

(b) $\|x^\circ(T) - x^\circ_{T_1}(T)\| \le \dfrac{\delta_1}{2}$,

and this will conclude the proof.

Property (a) is an immediate consequence of the fact that $z^\circ(t)$ converges to 0 as $t \to \infty$. To prove (b), we proceed as follows. Observe that $x^\circ(t)$ and $x^\circ_{T_1}(t)$ are integral curves of

$$\dot{x} = f(x, z^\circ(t))$$

and, respectively,

$$\dot{x} = f(x, 0)$$

satisfying $x^\circ(T_1) = x^\circ_{T_1}(T_1)$. Thus, for $t \ge T_1$,

$$x^\circ(t) = x^\circ(T_1) + \int_{T_1}^{t} f(x^\circ(s), z^\circ(s))ds$$

and

$$x^\circ_{T_1}(t) = x^\circ(T_1) + \int_{T_1}^{t} f(x^\circ_{T_1}(s), 0)ds \ .$$

Since $x^\circ(s), x^\circ_{T_1}(s), z^\circ(s)$ are defined for all $s \ge T_1$ and bounded and $f(x,z)$ is locally Lipschitz, there exist $L > 0$ and $M > 0$ such that

$$\|f(x^\circ(s), z^\circ(s)) - f(x^\circ_{T_1}(s), 0)\| \le L\|x^\circ(s) - x^\circ_{T_1}(s)\| + M\|z^\circ(s)\|$$

for all $s \ge T_1$. Thus,

$$\|x^\circ(t) - x^\circ_{T_1}(t)\| \le L \int_{T_1}^{t} \|x^\circ(s) - x^\circ_{T_1}(s)\|ds + M \int_{T_1}^{t} \|z^\circ(s)\|ds \ .$$

Since $z^\circ(s)$ converges to 0 as $t \to \infty$, given any $\delta > 0$, there is T_1 such that $\|z^\circ(s)\| \le \delta$ for all $t \ge T_1$. Thus, for all $t \ge T_1$ we can write

$$\|x^\circ(t) - x^\circ_{T_1}(t)\| \le L \int_{T_1}^{t} \|x^\circ(s) - x^\circ_{T_1}(s)\|ds + M\delta(t - T_1) \ .$$

Gronwall-Bellman's lemma yields

$$\|x^\circ(t) - x^\circ_{T_1}(t)\| \le \frac{M\delta}{L}(e^{L(t-T_1)} - 1)$$

and, thus,

$$\|x^\circ(T) - x^\circ_{T_1}(T)\| \leq \frac{M\delta}{L}(e^{LT_2} - 1) .$$

In the right-hand side, M, L, T_2 are fixed numbers, but δ can be rendered arbitrarily small by choosing an appropriately large T_1. This proves (b) and completes the proof of the theorem. $\triangleleft$

From this theorem it is easy to deduce the following corollaries, which express the result outlined at the beginning of the section.

Corollary 10.3.2. *Consider the system (10.13). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is locally asymptotically stable. Then, the equilibrium $(x, z) = (0, 0)$ of (10.13) is locally asymptotically stable.*

Proof. Indeed, for some sufficiently small $\delta > 0$ one can choose $\mathcal{S} = B_\delta$. By Corollary 10.2.2, the equilibrium $(x, z) = (0, 0)$ is stable. Thus, there exist δ_1 and δ_2 such that, if x° and z° satisfy

$$\|x^\circ\| \leq \delta_1, \qquad \|z^\circ\| \leq \delta_2 ,$$

then $x^\circ(t)$ belongs to B_δ. From this, the result follows. $\triangleleft$

Corollary 10.3.3. *Consider the system (10.13). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is globally asymptotically stable and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is globally asymptotically stable. Suppose the integral curves of the composite system are defined for all $t \geq 0$ and bounded. Then, the equilibrium $(x, z) = (0, 0)$ of (10.13) is globally asymptotically stable.*

Proof. In this case, $\mathcal{S} = \mathbb{R}^n$ and the hypothesis needed to show, using Theorem 10.3.1, that for any initial condition (x°, z°) the trajectory $(x^\circ(t), z^\circ(t))$ converges to zero as $t \to \infty$ is simply the boundedness of $x^\circ(t)$. $\triangleleft$

10.4 Input-to-State Stability

In the previous section, we have discussed the asymptotic behavior of the response $x(t)$ of a system of the form

$$\dot{x} = f(x, z)$$

to an input $z(t)$ which was, in turn, the response of the autonomous nonlinear system

$$\dot{z} = g(z) .$$

Since the latter was assumed to be locally (respectively, globally) asymptotically stable, the input $z(t)$ driving the first system was a function of time which, for sufficiently small $z(0)$ (respectively, for any $z(0)$), asymptotically decayed to 0 as $t \to \infty$.

In this section we extend this type of analysis, to the case in which the input driving the first system is simply a *bounded* function of time. Of course, we cannot expect anymore that the state $x(t)$ decays to 0 as $t \to \infty$; rather, we are interested in the case in which $x(t)$ remains bounded, and the bound on the state can be expressed as a (possibly nonlinear) function of the bound on the input. In the special case in which the input tends to 0 (in particular, when the input is identically zero), we still expect that $x(t)$ converges to 0 as time tends to ∞, as it happened in the earlier situation. These requirements altogether lead to the notion of *input-to-state stability*, which was introduced by E. Sontag [5] and can be formally characterized as follows.

Consider a nonlinear system

$$\dot{x} = f(x, u) \tag{10.15}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0, \infty) \to \mathbb{R}^m$ of (10.15) can be any piecewise continuous bounded function. The set of all such functions, endowed with the supremum norm

$$\|u(\cdot)\|_\infty = \sup_{t \geq 0} \|u(t)\|$$

is denoted by L_∞^m.

Definition 10.4.1. *System (10.15) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, called a gain function, such that, for any input $u(\cdot) \in L_\infty^m$ and any $x^\circ \in \mathbb{R}^n$, the response $x(t)$ of (10.15) in the initial state $x(0) = x^\circ$ satisfies*

$$\|x(t)\| \leq \beta(\|x^\circ\|, t) + \gamma(\|u(\cdot)\|_\infty) \tag{10.16}$$

for all $t \geq 0$.

Remark 10.4.1. Since, for any pair $\beta > 0, \gamma > 0$, $\max\{\beta, \gamma\} \leq \beta + \gamma \leq \max\{2\beta, 2\gamma\}$, it is seen that an alternative way to say that a system is input-to-state stable is that there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L_\infty^m$ and any $x^\circ \in \mathbb{R}^n$, the response $x(t)$ of (10.15) in the initial state $x(0) = x^\circ$ satisfies

$$\|x(t)\| \leq \max\{\beta(\|x^\circ\|, t), \gamma(\|u(\cdot)\|_\infty)\} \tag{10.17}$$

for all $t \geq 0$. Both estimates (10.16) and (10.17) will be indifferently used in the sequel. $\triangleleft$

[5] See Sontag (1989).

In what follows, it will be shown that the property, for a given system, of being input-to-state stable, can be given a characterization which extends the well known criterion of Lyapunov for asymptotic stability. The key tool for this analysis is the notion of *ISS-Lyapunov function*, defined as follows.

Definition 10.4.2. *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (10.15) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\chi(\cdot)$ such that*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \quad \textit{for all } x \in \mathbb{R}^n \tag{10.18}$$

and

$$\|x\| \ge \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) \quad \textit{for all } x \in \mathbb{R}^n. \tag{10.19}$$

The reason why a function characterized in this way can play an important role in establishing the property of input-to-state stability can be explained as follows. First of all, observe that if $V(x)$ is an ISS-Lyapunov function for (10.15), evaluation of (10.19) at $u = 0$ yields

$$\frac{\partial V}{\partial x} f(x,0) \le -\alpha(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \ ,$$

and therefore $V(x)$ is a Lyapunov function in the usual sense of the term for the autonomous system $\dot{x} = f(x,0)$. Thus, as seen in Section 10.1, the response of (10.15) with $u(t) = 0$ for all $t \ge 0$, and arbitrary initial state $x(0) = x^\circ$, satisfies an estimate of the form

$$\|x(t)\| \le \beta(\|x^\circ\|, t)$$

for all $t \ge 0$, where $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function, as expected from the definition of input-to-state stability.

Consider now the case of a nonzero input $u(\cdot) \in L_\infty^m$, let

$$M = \|u(\cdot)\|_\infty,$$

and set

$$c = \overline{\alpha}(\chi(M)) \ .$$

Then, from the inequality on the right-hand side of (10.18), it is deduced that the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}$$

is such that

$$B_{\chi(M)} \subset \Omega_c \ .$$

As a consequence, $\|x\| \ge \chi(M)$ at each x on the boundary of Ω_c and therefore, at any $t \ge 0$ such that $x(t)$ is on the boundary of Ω_c, $\|x(t)\| \ge \chi(\|u(t)\|)$. Suppose now that (10.19) holds. Then,

$$\frac{\partial V(x)}{\partial x} f(x(t), u(t)) < 0$$

at any $t \geq 0$ such that $x(t)$ is on the boundary of Ω_c, and (see Section 10.1), it can be concluded that for any initial condition $x'(0)$ in the interior of Ω_c, the solution $x'(t)$ of (10.15) is defined for all $t \geq 0$ and $x'(t) \in \Omega_c$ for all $t \geq 0$.

In particular, for all $t \geq 0$, $x'(t)$ satisfies

$$\|x'(t)\| \leq \underline{\alpha}^{-1}(V(x'(t))) \leq \underline{\alpha}^{-1}(c) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(M))) \ .$$

Setting

$$\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r) \ ,$$

we see that, for all $t \geq 0$, $x'(t)$ satisfies

$$\|x'(t)\| \leq \gamma(\|u(\cdot)\|_\infty) \ , \tag{10.20}$$

or, what is the same,

$$\|x'(\cdot)\|_\infty \leq \gamma(\|u(\cdot)\|_\infty) \ .$$

Let now $x(t)$ be the solution of (10.15), satisfying $x(0) = x^\circ$. If $x^\circ \in \Omega_c$, the previous arguments show that $\|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty)$ for all $t \geq 0$ and this proves that the estimate (10.16) holds. If not, i.e. if $V(x^\circ) > c$, observe that, so long as $V(x(t)) > c$, $\|x(t)\| > \chi(\|u(t)\|)$ and this yields

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = \frac{\partial V}{\partial x} f(x(t), u(t)) \leq -\alpha(\|x(t)\|) < 0 \ .$$

Thus, so long as $V(x(t)) > c$, the function $V(x(t))$ is decreasing, and this shows in particular that $x(t)$ is bounded (in fact, $\|x(t)\| \leq \underline{\alpha}^{-1}(V(x(t))) \leq \underline{\alpha}^{-1}(V(x(0)))$). Moreover, it is possible to show that, for some finite time t_0,

$$\begin{aligned} & V(x(t)) > c, \quad \text{for all} \quad 0 \leq t < t_0 \\ & V(t_0) = c \ . \end{aligned}$$

By contradiction, suppose this is not the case. Then $V(x(t)) > c$ for all $t \geq 0$, and therefore,

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leq -\alpha(\overline{\alpha}^{-1}(V(x(t)))) \tag{10.21}$$

for all $t \geq 0$. In other words, the function $V(t) = V(x(t))$ is such that

$$\dot{V}(t) \leq -\alpha(\overline{\alpha}^{-1}(V(t))$$

for all $t \geq 0$ and this shows (see proof of Theorem 10.1.3) that, for some class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$,

$$\|x(t)\| \leq \beta(\|x^\circ\|, t) \tag{10.22}$$

for all $t \geq 0$. In particular, $x(t)$ tends to 0 as $t \to \infty$ and thus $V(x(t))$ tends to 0 as $t \to \infty$, which contradicts the assumption that $V(x(t))$ was bounded from below by $c > 0$ for all $t \geq 0$.

The inequality (10.21) holds for all $t \in [0, t_0)$. Then, an estimate of the form (10.22) holds for all $t \in [0, t_0)$. Since for $t \geq t_0$ we have

$$\|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty)$$

we conclude that

$$\|x(t)\| \leq \max\{\beta(\|x^\circ\|, t), \gamma(\|u(\cdot)\|_\infty)\}$$

for all $t \geq 0$, and this completes the proof of the fact that if $V(x)$ is an ISS-Lyapunov function for (10.15), then this system is input-to-state stable.

Remark 10.4.2. The previous argument shows also how the function $\gamma(\cdot)$ which appears in the estimate (10.17) can be computed from the functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, and $\chi(\cdot)$ which characterize (10.18) and (10.19). As a matter of fact, $\gamma(\cdot)$ can be given the expression

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r) \,. \triangleleft$$

The arguments above show that the existence of an ISS-Lyapunov function is a *sufficient condition* for input-to-state stability. As in the case of asymptotic stability, this result has also a *converse*, namely the existence of a function $V(x)$ having the properties indicated in (10.18) and (10.19) is *implied* by the property of input-to-state stability [6]

Theorem 10.4.1. *System (10.15) is input-to-state stable if and only if there exists an ISS-Lyapunov function.*

There is an alternative way to check whether or not a function $V(x)$ is an ISS-Lyapunov function for a given system, which is useful in many circumstances.

Lemma 10.4.2. *Consider system (10.15). A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (10.15) if and only if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that (10.18) holds and*

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \qquad \textit{for all } x \in \mathbb{R}^n \textit{ and all } u \in \mathbb{R}^m \,. \quad (10.23)$$

Proof. Suppose (10.23) holds and define

$$\chi(r) = \alpha^{-1}(k\sigma(r)) \,,$$

with $k > 1$. Then, $\|x\| \geq \chi(\|u\|)$ implies

[6] For a proof of this result, see Sontag and Wang (1995).

$$\frac{1}{k}\alpha(\|x\|) \geq \sigma(\|u\|)$$

which in turn implies

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \leq -\frac{k-1}{k}\alpha(\|x\|) \ .$$

This shows that a relation of the form (10.19) holds.

Observe now that, if (10.19) holds and $\|x\| \geq \chi(\|u\|)$, then (10.23) holds for any $\sigma(\cdot)$. To complete the proof, define

$$\phi(r) = \max_{\|u\|=r, \|x\| \leq \chi(r)} \{\frac{\partial V}{\partial x} f(x,u) + \alpha(\chi(\|u\|)\} \ .$$

Then, for $\|x\| \leq \chi(\|u\|)$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|) + \phi(\|u\|) \ .$$

Define

$$\sigma(r) = \max\{0, \phi(r)\} \ ,$$

The function $\sigma(\cdot)$ is continuous, nonnegative, and $\sigma(0) = 0$. If $\sigma(\cdot)$ is not a class $\mathcal{K}$ function, majorize it by a class $\mathcal{K}$ function, to have property (10.23) fulfilled. $\triangleleft$

Remark 10.4.3. From the proof of the previous Lemma, it is possible to deduce how the gain function $\gamma(\cdot)$ which appears in the estimate (10.17) can be computed from the functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ and $\sigma(\cdot)$ which characterize (10.18) and (10.23). As a matter of fact, recall that if $V(x)$ is a (positive definite and proper) function satisfying (10.19), an estimate of the form (10.17) holds with a gain function $\gamma(\cdot)$ given by

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r) \ .$$

On the other hand, the proof of the previous Lemma shows that if $V(x)$ is a (positive definite and proper) function satisfying (10.23), then (10.19) holds for $\chi(r) = \alpha^{-1}(k\sigma(r))$ and $k > 1$. Thus, an estimate of the form (10.17) holds with a gain function $\gamma(\cdot)$ given by

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha^{-1} \circ k\sigma(r)$$

where k is any number satisfying $k > 1$. $\triangleleft$

The concepts introduced above are illustrated in the following simple examples.

Example 10.4.1. Consider a linear system

$$\dot{x} = Ax + Bu$$

and suppose that all the eigenvalues of the matrix A have negative real part. Let $P > 0$ denote the unique solution of the Lyapunov equation $PA + A^{\mathrm{T}}P = -I$, observe that the function $V(x) = x^{\mathrm{T}}Px$ satisfies

$$\underline{a}\|x\|^2 \leq V(x) \leq \overline{a}\|x\|^2$$

for suitable $\underline{a} > 0$ and $\overline{a} > 0$, and that

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\|x\|^2 + 2\|x\|\|P\|\|B\|\|u\| .$$

Pick any $0 < \varepsilon < 1$ and set

$$c = \frac{2}{1-\varepsilon}\|P\|\|B\| , \qquad \chi(r) = cr .$$

Then

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}(Ax + Bu) \leq -\varepsilon\|x\|^2 .$$

Thus, the system is input-to-state stable, with a gain function

$$\gamma(r) = (\overline{a}/\underline{a})cr$$

which is a *linear* function. $\triangleleft$

Example 10.4.2. Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k + bx^p\varphi(u) ,$$

in which $k \in \mathbb{I}$ is *odd*, $p \in \mathbb{I}$ satisfies

$$p < k , \tag{10.24}$$

$a > 0$, and $\varphi(\cdot)$ is a C^1 function satisfying $\varphi(0) = 0$.

Indeed, since $a > 0$ and k is odd, the system is globally asymptotically stable. Choose a candidate ISS-Lyapunov function as $V(x) = \frac{1}{2}x^2$, which yields

$$\dot{V} = \frac{\partial V}{\partial x}f(x,u) = -ax^{k+1} + bx^{p+1}\varphi(u) .$$

Let $\theta(\cdot)$ be a class $\mathcal{K}$ function satisfying

$$|\varphi(u)| \leq \theta(|u|)$$

for all $u \in \mathbb{R}$, and observe that, since $k+1$ is even

$$\dot{V} \leq -a|x|^{k+1} + |b||x|^{p+1}\theta(|u|) .$$

Set

$$\nu = k - p$$

and obtain

$$\dot{V} \leq |x|^{p+1} \Big(-a|x|^\nu + |b|\theta(|u|) \Big) .$$

Thus, using the class $\mathcal{K}_\infty$ function

$$\alpha(r) = \varepsilon r^{k+1} ,$$

in which $\varepsilon > 0$, it is deduced that

$$\dot{V} \leq -\alpha(|x|)$$

if

$$-a|x|^\nu + |b|\theta(|u|) \leq -\varepsilon|x|^\nu ,$$

i.e. if

$$(a - \varepsilon)|x|^\nu \geq |b|\theta(|u|) .$$

Taking, without loss of generality, $\varepsilon < a$, it is concluded that condition (10.19) holds for the class $\mathcal{K}$ function

$$\chi(r) = \Big(\frac{|b|\theta(r)}{a - \varepsilon} \Big)^{\frac{1}{\nu}} .$$

Thus, the system is input-to-state stable, with a gain function $\gamma(\cdot)$ which is bounded by this function $\chi(\cdot)$. ◁

Example 10.4.3. Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k \mathrm{sgn}(x) + bx^p \varphi(u) ,$$

in which $k \in \mathbb{I}$ is *even*, $p \in \mathbb{I}$ satisfies

$$p < k ,$$

$a > 0$, and $\varphi(\cdot)$ is a C^1 function satisfying $\varphi(0) = 0$. Note that the function $x^k \mathrm{sgn}(x)$ is a C^1 function, for any even integer $k > 0$.

Choose the same candidate ISS-Lyapunov function as in the previous example, $V(x) = \frac{1}{2}x^2$, to obtain

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) = -ax^{k+1}\mathrm{sgn}(x) + bx^{p+1}\varphi(u) .$$

This, since $k + 1$ is odd, yields the same inequality found in the previous example, namely

$$\dot{V} \leq -a|x|^{k+1} + |b||x|^{p+1}\theta(|u|) .$$

from which identical conclusions follow. ◁

Example 10.4.4. An important feature of the two previous examples, which made it possible to prove that both systems were input-to-state stable, was the inequality (10.24). In fact, if this inequality does not hold, the system may fail to be input-to-state stable. This can be seen, for instance, in the simple example

$$\dot{x} = -x + xu \ .$$

In fact, suppose $u(t) = 2$ for all $t \geq 0$. The state response of the system, to this input, from the initial state $x(0) = x^\circ$ coincides with that of the autonomous system

$$\dot{x} = x \ ,$$

i.e. $x(t) = e^t x^\circ$, which shows that the bound (10.16) cannot hold. ◁

Example 10.4.5. Let $n = 2$, $m = 1$ and consider the system

$$\begin{aligned} \dot{z} &= -z^3 + zy \\ \dot{y} &= az^2 - y + u \ . \end{aligned}$$

where a is a real parameter. To check whether or not this system might be input-to-state stable, for some value of a, choose the candidate ISS-Lyapunov function

$$V(z,y) = \frac{1}{2}(z^2 + y^2)$$

to obtain

$$\dot{V} = \frac{\partial V}{\partial x} f(x,u) = -z^4 + (1+a)z^2 y - y^2 + yu \ .$$

Recall that

$$z^2 y \leq \frac{1}{2} z^4 + \frac{1}{2} y^2$$

and that, for any number $\delta > 0$,

$$yu \leq \frac{\delta}{2} y^2 + \frac{1}{2\delta} u^2 \ .$$

Thus, it is seen that

$$\dot{V} \leq (-1 + \frac{|1+a|}{2}) z^4 + (-1 + \frac{|1+a|}{2} + \frac{\delta}{2}) y^2 + \frac{1}{2\delta} u^2 \ .$$

From this, it is easy to conclude that the system is input-to-state stable if $|a| < 1$. In fact, if this is the case, in the previous inequality the coefficient

$$-1 + \frac{|1+a|}{2}$$

of z^4 is negative, and it is possible to find $\delta > 0$ such that also the coefficient

$$-1 + \frac{|1+a|}{2} + \frac{\delta}{2}$$

of y^2 is negative. In other words, there exist numbers $d_1 > 0$, $d_2 > 0$ such that

$$\dot{V} \le -(d_1 z^4 + d_2 y^2) + \frac{1}{2\delta}u^2 \ .$$

Consider now the function

$$W(z,y) = d_1 z^4 + d_2 y^2 \ .$$

This function is positive definite and also proper because (see Remark 10.1.3), for any $c > 0$, the closed set

$$\Omega_c = \{(z,y) \in \mathbb{R}^2 : W(z,y) \le c\}$$

is bounded. In fact

$$W(z,y) \le c \quad \Rightarrow \quad |z| \le \left(\frac{c}{d_1}\right)^{\frac{1}{4}}, |y| \le \left(\frac{c}{d_2}\right)^{\frac{1}{2}} .$$

Therefore, there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$\alpha(\|x\|) \le W(z,y)$$

for all $x \in \mathbb{R}^2$. As a consequence,

$$\dot{V} \le -\alpha(\|x\|) + \frac{1}{2\delta}|u|^2$$

and it is concluded that an inequality of the form (10.23) holds, for

$$\sigma(r) = \frac{1}{2\delta}r^2 \ .$$

This shows that, if $|a| < 1$, the system is input-to-state stable. $\triangleleft$

As shown, for instance, in the third of these examples, in a system of the form (10.15), the global asymptotic stability of the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ *does not* necessarily imply input-to state stability. Nevertheless, it is always possible *to render* such a system input-to-state stable by means of a *feedback transformation* of the form

$$u = \beta(x)v$$

where $\beta(x)$ is an $m \times m$ matrix of smooth functions of x, defined and invertible for all $x \in \mathbb{R}^n$. Note that feedback transformations of this kind, which actually correspond to x-dependent changes of coordinates in the space of input values, have been encountered several times before in the design of nonlinear control systems.

Theorem 10.4.3. *Consider system (10.15). Suppose the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is globally asymptotically stable. Then, there exists an $m \times m$ matrix $\beta(x)$ of smooth functions of x, which is defined for all $x \in \mathbb{R}^n$ and is nonsingular for all x, such that*

$$\dot{x} = f(x, \beta(x)u) \tag{10.25}$$

is input-to-state stable.

Proof. By the converse Lyapunov theorem, there exists a C^1 positive definite and proper function $V(x)$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$\frac{\partial V}{\partial x} f(x,0) < -\alpha(\|x\|) \quad \text{for all nonzero } x \in \mathbb{R}^n .$$

We will prove the theorem by showing that, for some $\beta(x)$ having the properties indicated above and for some class $\mathcal{K}$ function $\chi(\cdot)$,

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, \beta(x)u) \leq -\frac{1}{2}\alpha(\|x\|) \quad \text{for all } x \in \mathbb{R}^n ,$$

thus showing that $V(x)$ is an ISS-Lyapunov function for (10.25).

To this end, let $g(\cdot)$ be any smooth function $g : [0,\infty) \to [0,\infty)$, which is positive for all $s \geq 0$ and identically equal to 1 on the interval $[0,1]$, and set

$$\beta(x) = g(\|x\|) I \; .$$

The matrix $\beta(x)$ thus defined is a matrix of smooth functions of x, invertible for all $x \in \mathbb{R}^n$.

Define

$$\delta(s,r) = \max_{\|x\|=s, \|v\|=r} \{ \frac{\partial V}{\partial x} f(x,v) + \frac{1}{2}\alpha(s) \} \; ,$$

and observe that, for any choice of $g(\cdot)$,

$$\frac{\partial V}{\partial x} f(x, \beta(x)u) + \frac{1}{2}\alpha(\|x\|) \leq \delta(\|x\|, g(\|x\|)\|u\|) \; , \tag{10.26}$$

for all x and u.

The function $\delta(s,r)$, a continuous function defined for all $(s,r) \in [0,\infty) \times [0,\infty)$, by hypothesis is such that $\delta(s,0) < 0$ for every $s > 0$. Thus, using continuity arguments, it is possible to prove the existence of a continuous function $\rho(s)$, defined for all $s \geq 0$, with $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$, such that

$$r \leq \rho(s) \quad \Rightarrow \quad \delta(s,r) < 0$$

for all $s > 0$. In view of this and of (10.26), the result is proven if one can show the existence of a function $\chi(\cdot)$ and complete the definition of $g(\cdot)$ so that

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad g(\|x\|)\|u\| \leq \rho(\|x\|) \; . \tag{10.27}$$

Let $\theta(\cdot)$ be any class $\mathcal{K}_\infty$ function which satisfies

$$\begin{array}{ll} \theta(s) < \rho(s) & \text{for } s \leq 2 \\ \theta(s) < s & \text{for } s \geq 2, \end{array}$$

and set

$$\chi(s) = \theta^{-1}(s) \ .$$

Moreover, let $g(\cdot)$ be such that

$$\begin{array}{ll} g(s) \leq 1 & \text{for all } s \\ g(s) < \dfrac{\rho(s)}{s} & \text{for } s \geq 2 \ . \end{array}$$

By construction, the functions $\theta(\cdot)$ and $g(\cdot)$ are such that

$$g(s)\theta(s) \leq \rho(s) \quad \text{for all } s > 0 \ .$$

Observe now that

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad \theta(\|x\|) \geq \|u\| \ .$$

Therefore,

$$\|x\| \geq \chi(\|u\|) \quad \Rightarrow \quad g(\|x\|)\|u\| \leq g(\|x\|)\theta(\|x\|) \leq \rho(\|x\|)$$

which shows that (10.27) holds and this completes the proof. ◁

The notion of input-to-state stability lends itself to a number of alternative (equivalent) characterizations, among which the most relevant one can be derived as follows. Recall (see Remark 10.4.1) that a system is input-to-state stable if the response $x(\cdot)$ to an input $u(\cdot) \in L_\infty^m$ satisfies an estimate of the form (10.17). Observe that, for any $t \geq 0$,

$$\beta(\|x^\circ\|, t) \leq \beta(\|x^\circ\|, 0)$$

and $\beta(\cdot, 0)$ is a class $\mathcal{K}$ function. Then, using (10.17), it is seen that, in an input-to-state stable system, the response $x(\cdot)$ to any input $u(\cdot) \in L_\infty^m$ is bounded and, in particular,

$$\|x(\cdot)\|_\infty \leq \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\} \tag{10.28}$$

for some class $\mathcal{K}$ function $\gamma_0(\cdot)$ (while $\gamma(\cdot)$ is the same class $\mathcal{K}$ function for which the estimate (10.17) holds). Moreover, since

$$\lim_{t\to\infty} \beta(\|x^\circ\|, t) = 0 \ ,$$

in an input-to-state stable system the response $x(t)$ to any input $u(\cdot) \in L_\infty^m$ satisfies

$$\limsup_{t\to\infty} \|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty) \tag{10.29}$$

(where $\gamma(\cdot)$ is again the same class $\mathcal{K}$ function for which the estimate (10.17) holds).

The estimate (10.29) can be given an alternative characterization, which only involves the behavior of $\|u(t)\|$ for large t. In fact, the following result holds.

Lemma 10.4.4. *Property (10.29) is equivalent to the property*

$$\limsup_{t\to\infty} \|x(t)\| \le \gamma(\limsup_{t\to\infty} \|u(t)\|) \ . \tag{10.30}$$

Proof. Since

$$\gamma(\limsup_{t\to\infty} \|u(t)\|) \le \gamma(\|u(\cdot)\|_\infty) \ ,$$

then (10.30) implies (10.29), actually with the same function $\gamma(\cdot)$. Conversely, suppose (10.29) holds. Pick any $x^\circ \in \mathbb{R}^n$, any $u(\cdot) \in L_\infty^m$ and $\varepsilon > 0$. Let

$$r = \limsup_{t\to\infty} \|u(t)\|$$

and let $h > 0$ be such that

$$\gamma(r+h) - \gamma(r) < \varepsilon \ .$$

By definition of r, there exists $T > 0$ such that $\|u(t)\| \le r + h$ for all $t \ge T$.

Let $\tilde{x}(t)$ denote the response of system (10.15) from the initial state $\tilde{x}^\circ = x(T)$ and input $\tilde{u}(\cdot)$ defined as

$$\tilde{u}(t) = u(t+T) \ .$$

Clearly, $\tilde{x}(t) = x(t+T)$, where $x(\cdot)$ is the response form the initial state x° and input $u(\cdot)$. By definition of T,

$$\|\tilde{u}(t)\| \le r + h \qquad \text{for all} \ \ t \ge 0$$

i.e. $\|\tilde{u}(\cdot)\|_\infty \le r + h$. Then (10.29) implies

$$\limsup_{t\to\infty} \|x(t)\| = \limsup_{t\to\infty} \|\tilde{x}(t)\| \le \gamma(\|\tilde{u}(\cdot)\|_\infty) \le \gamma(r+h) < \gamma(r) + \varepsilon \ .$$

Letting $\varepsilon \to 0$ yields (10.30), with the same $\gamma(\cdot)$ as in (10.29). $\triangleleft$

Remark 10.4.4. From the proof of this Lemma it is deduced that, if (10.17) holds, then also (10.28) and (10.30) hold, with the *same* class $\mathcal{K}$ function $\gamma(\cdot)$. Thus, in particular, if it is known that $V(x)$ is an ISS-Lyapunov function for system (10.15), from Remark 10.4.2 it follows that the response $x(\cdot)$ to any input $u(\cdot) \in L_\infty^m$ is such that the estimates (10.28) and (10.30) hold, with $\gamma(\cdot) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(\cdot)$. $\triangleleft$

The arguments above show that if the estimate (10.17) holds, then necessarily also (10.28) and (10.30) hold, for the same class $\mathcal{K}$ function $\gamma(\cdot)$ and the class $\mathcal{K}$ function $\gamma_0(\cdot) = \beta(\cdot, 0)$. However, it turns out that the properties expressed by these two inequalities are not just implications, but *equivalent characterizations*, of the property of input-to-state stability [7].

Theorem 10.4.5. *System (10.15) is input-to-state stable if and only if there exists class $\mathcal{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L_\infty^m$ and any $x^\circ \in \mathbb{R}^n$, the response $x(t)$ in the initial state $x(0) = x^\circ$ satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \gamma(\limsup_{t\to\infty} \|u(t)\|) \ . \end{aligned}$$

Note that, if (10.17) holds, then – as shown before – also (10.28) and (10.30) hold, with the *same* class $\mathcal{K}$ function $\gamma(\cdot)$. Theorem 10.4.5 above says that the fulfillment of estimates of the form (10.28) and (10.30) implies input-to-state stability, i.e the fulfillment of an estimate of the form (10.17). However, is important to stress that this may occur for a possibly *different* function $\gamma(\cdot)$. This can be seen, for instance, in the following example.

Example 10.4.6. Consider the system

$$\dot{x} = -\frac{1}{1+u^2} x \ ,$$

for which

$$x(t) = x(0) \exp(-\int_0^t \frac{1}{1+u^2(\tau)} d\tau) \ .$$

Clearly,

$$|x(t)| \leq |x(0)| \qquad \text{and} \qquad \lim_{t\to\infty} x(t) = 0$$

for any $u(\cdot) \in L_\infty$ and any $x(0)$, so that (10.28) and (10.30) hold for $\gamma_0(s) = s$ and $\gamma(s) = 0$. However, an estimate of the form (10.17) with $\gamma(\cdot) = 0$ cannot hold, because the rate of converge of $|x(t)|$ to zero decreases as $\|u(\cdot)\|_\infty$ increases.

It is easy to see that $V(x) = x^4$ is an ISS-Lyapunov function for this system. In fact, observe that, for any $\varepsilon > 0$,

$$|x| \geq \varepsilon |u| \qquad \Rightarrow \qquad \frac{4\varepsilon^2 x^4}{\varepsilon^2 + x^2} \leq \frac{4x^4}{1+u^2} \ .$$

Thus, the two class $\mathcal{K}_\infty$ functions

$$\chi(r) = \varepsilon r \qquad \alpha(r) = \frac{4\varepsilon^2 r^4}{\varepsilon^2 + r^2}$$

[7] For a proof of this result, see Sontag and Wang (1996).

are such that

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}(-\frac{1}{1+u^2}x) = -\frac{4x^4}{1+u^2} \leq -\alpha(|x|) \ .$$

Thus, an estimate of the form (10.17) holds, in which $\gamma(\cdot)$ can be given the expression $\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r)$. Since one can pick $\underline{\alpha}(r) = \overline{\alpha}(r) = r^4$, this yields

$$\gamma(r) = \varepsilon r \ ,$$

where ε is any small (but nonzero) positive number. ◃

10.5 Input-to-State Stability of Cascade-Connected Systems

In this section we discuss a problem similar to that considered in Section 10.3, this time referred to the property of input-to-state stability. More precisely, we shall consider the cascade connection of two subsystems, each one being input-to-state stable, and we will prove that the composed system is still input-to-state stable. As a preliminary step in the analysis, we first discuss a property of input-to-state stable systems, which will be used later in the proof of the main result of this section, but that also has its own independent interest.

Recall that, according to the results discussed in the previous section and, in particular, to Lemma 10.4.2, a system

$$\dot{x} = f(x, u)$$

is input-to-state stable if there exists a continuously differentiable function $V(x)$ satisfying (10.18), a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \tag{10.31}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

The main significance of the latter inequality is essentially the following one. Consider a bounded input $u(\cdot)$, let $\varepsilon > 0$ be any small number, set

$$d = \alpha^{-1}(\sigma(\|u(\cdot)\|_\infty) + \varepsilon)$$

and

$$c(d) = \max_{\|x\| \leq d} V(x) \ .$$

By definition,

$$\Omega_{c(d)} \supset \bar{B}_d \ .$$

If (10.31) holds, the derivative of $V(x(t))$ is strictly negative at each $x(t)$ such that $\|x(t)\| \geq d$ (thus, in particular, on the boundary of $\Omega_{c(d)}$). In fact, at all such points,

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leq -\alpha(\|x(t)\|) + \sigma(\|u(t)\|) \leq -\alpha(d) + \sigma(\|u(\cdot)\|_\infty)) = -\varepsilon\ .$$

This shows that, under this input, for any trajectory $x(t)$ there is a time t_0 such that $V(x(t)) \leq c(d)$ for all $t \geq t_0$. In other words, the composed class $\mathcal{K}$ function $\alpha^{-1}(\sigma(\cdot))$ characterizes how the bound $\|u(\cdot)\|_\infty$ on the input function determines the (finite) number $c(d)$ by which $V(x(t))$ is guaranteed to be bounded for suitably large times.

In view of this, it appears that only the composition of the two functions $\alpha^{-1}(\cdot)$ and $\sigma(\cdot)$ matters in establishing a correspondence between a bound on the input and a bound on the state, and that, of course, infinitely many other such pairs might yield the same result. Motivated by this observation, we will examine below the problem of constructing families of pairs of functions $\alpha(\cdot)$ and $\sigma(\cdot)$ rendering an inequality of the form (10.31) true for a given input-to-state system. For convenience, we will say that a pair $\{\alpha(\cdot), \sigma(\cdot)\}$, in which the former is a class $\mathcal{K}_\infty$ function and the latter a class $\mathcal{K}$ function, is an *ISS-pair* for system (10.15) if, for some C^1 function $V(x)$ satisfying estimates of the form (10.18), the inequality (10.31) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Lemma 10.5.1. *Assume $\{\alpha(\cdot), \sigma(\cdot)\}$ is an ISS-pair for (10.15).*

(i) Let $\tilde{\sigma}(\cdot)$ be a class $\mathcal{K}$ function such that $\sigma(r) = \mathcal{O}[\tilde{\sigma}(r)]$ as $r \to \infty$. Then, there exists a class $\mathcal{K}_\infty$ function $\tilde{\alpha}(\cdot)$ such that $\{\tilde{\alpha}(\cdot), \tilde{\sigma}(\cdot)\}$ is an ISS-pair.

(ii) Let $\tilde{\alpha}(\cdot)$ be a class $\mathcal{K}_\infty$ function such that $\tilde{\alpha}(r) = \mathcal{O}[\alpha(r)]$ as $r \to 0^+$. Then, there exists a class $\mathcal{K}$ function $\tilde{\sigma}(\cdot)$ such that $\{\tilde{\alpha}(\cdot), \tilde{\sigma}(\cdot)\}$ is an ISS-pair.

Proof. For both parts of the theorem, the proof will be conducted by considering a candidate ISS-Lyapunov function $W(x)$ of the form

$$W(x) = \rho(V(x))$$

where $\rho(\cdot)$ is a class $\mathcal{K}_\infty$ function defined by an integral of the form

$$\rho(s) = \int_0^s q(t)dt$$

in which $q(\cdot)$ is a smooth function $[0, \infty) \to [0, \infty)$, which is non-decreasing and such that $q(s) > 0$ for $s > 0$ (the class of all such functions is often denoted as $\mathcal{SN}$). For this function, we wish to obtain an inequality of the form (10.31). To this end, observe that

$$\frac{\partial W}{\partial x} f(x,u) = q[V(x)] \frac{\partial V}{\partial x} f(x,u) \leq q[V(x)][-\alpha(\|x\|) + \sigma(\|u\|)]\ . \tag{10.32}$$

Set $\theta(s) = \overline{\alpha}(\alpha^{-1}(2\sigma(s)))$. Then, it is easy to see that the right-hand side of (10.32) is bounded by

$$-\frac{1}{2} q[V(x)]\alpha(\|x\|) + q[\theta(\|u\|)]\sigma(\|u\|)\ . \tag{10.33}$$

In fact, this is indeed the case (no matter what $\theta(\cdot)$ is) if $\alpha(\|x\|) \geq 2\sigma(\|u\|)$. If, $\alpha(\|x\|) \leq 2\sigma(\|u\|)$, observe that $V(x) \leq \overline{\alpha}(\|x\|) \leq \theta(\|u\|)$, in which case the right-hand side of (10.32) is bounded by the quantity $-q[V(x)]\alpha(\|x\|) + q[\theta(\|u\|)]\sigma(\|u\|)$.

The quantity (10.33) can in turn be bounded by

$$-\frac{1}{2}q[\underline{\alpha}(\|x\|)]\alpha(\|x\|) + q[\theta(\|u\|)]\sigma(\|u\|) ,$$

and, therefore, part (i) is proven if one can show that it is possible to find $q(\cdot)$ and $\tilde{\alpha}(\cdot)$ such that

$$\begin{aligned} q[\underline{\alpha}(s)]\alpha(s) &\geq 2\tilde{\alpha}(s) \\ q[\theta(r)]\sigma(r) &\leq \tilde{\sigma}(r) . \end{aligned} \tag{10.34}$$

Assume, without loss of generality, that $\sigma(\cdot)$ is class $\mathcal{K}_\infty$ (if this is not the case, majorize it by a class $\mathcal{K}_\infty$ function), so that also $\theta(\cdot)$ is class $\mathcal{K}_\infty$ and define

$$\beta(r) = \sigma(\theta^{-1}(r)), \qquad \tilde{\beta}(r) = \tilde{\sigma}(\theta^{-1}(r)) .$$

Both these functions are class $\mathcal{K}_\infty$ and, since $\sigma(r) = \mathcal{O}[\tilde{\sigma}(r)]$ as $r \to \infty$, also $\beta(r) = \mathcal{O}[\tilde{\beta}(r)]$ as $r \to \infty$. Using this property, it is easy to see that there exists a class $\mathcal{SN}$ function $q(\cdot)$ such that

$$q(r)\beta(r) \leq \tilde{\beta}(r)$$

for all $r \in [0, \infty)$. Thus,

$$q[\theta(r)]\sigma(r) \leq \tilde{\sigma}(r) . \tag{10.35}$$

Define

$$\tilde{\alpha}(s) = \frac{1}{2}q[\underline{\alpha}(s)]\alpha(s) . \tag{10.36}$$

This is a class $\mathcal{K}_\infty$ function, because so is $\alpha(\cdot)$ and $q(\cdot)$ is of class $\mathcal{SN}$. Indeed, (10.35) and (10.36) prove (10.34) and this completes the proof of (i).

To prove part (ii), we need to find $q(\cdot)$ and $\tilde{\sigma}(\cdot)$ such that (10.34) holds. To this end, define

$$\beta(r) = \frac{1}{2}\alpha(\theta^{-1}(s)), \qquad \tilde{\beta}(r) = \tilde{\alpha}(\theta^{-1}(s)) .$$

These functions are such that $\tilde{\beta}(r) = \mathcal{O}[\beta(r)]$ as $r \to 0^+$. Using this property, it is easy to see that there exists a class $\mathcal{SN}$ function $q(\cdot)$ such that

$$\tilde{\beta}(s) \leq q(s)\beta(s)$$

for all $s \in [0, \infty)$. Thus

$$-\frac{1}{2}q[\theta(s)]\alpha(s) \leq -\tilde{\alpha}(s) . \tag{10.37}$$

Define

$$\tilde{\sigma}(r) = q[\theta(r)]\sigma(s) , \tag{10.38}$$

which is a class $\mathcal{K}_\infty$ function. Indeed, (10.37) and (10.38) prove (10.34) and this completes the proof of (ii). $\triangleleft$

As an application of this result, we find the proof of the property that the cascade connection of two input-to-state stable systems is input-to-state stable. More precisely, consider a system of the form

$$\begin{aligned} \dot{x} &= f(x,z) \\ \dot{z} &= g(z,u)\ , \end{aligned} \tag{10.39}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0,0) = 0$, and $f(x,z)$, $g(z,u)$ are locally Lipschitz (see Fig. 10.2).

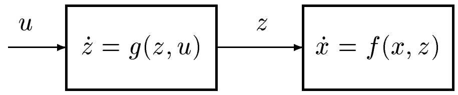

Fig. 10.2. Cascade connection with input.

Theorem 10.5.2. *Suppose that system*

$$\dot{x} = f(x,z)\ , \tag{10.40}$$

viewed as a system with input z and state x is input-to-state stable and that system

$$\dot{z} = g(z,u)\ , \tag{10.41}$$

viewed as a system with input u and state z is input-to-state stable as well. Then, system (10.39) is input-to-state stable.

Proof. By hypothesis, there exist an ISS-pair $\{\alpha(\cdot), \sigma(\cdot)\}$ for system (10.40) and an ISS-pair $\{\beta(\cdot), \zeta(\cdot)\}$ for system (10.41). Define a function $\tilde{\beta}(\cdot)$ in the following way

$$\tilde{\beta}(s) = \begin{cases} \beta(s) & \text{for small } s \\ \sigma(s) & \text{for large } s. \end{cases}$$

Then, by Lemma 10.5.1, part (ii), there exists $\tilde{\zeta}(\cdot)$ such that $\{\tilde{\beta}(\cdot), \tilde{\zeta}(\cdot)\}$ is an ISS-pair for system (10.41). Also, define a function $\tilde{\sigma}(\cdot)$ as

$$\tilde{\sigma}(s) = \frac{1}{2}\tilde{\beta}(s)\ .$$

Then, by Lemma 10.5.1, part (i), there exists $\tilde{\alpha}(\cdot)$ such that $\{\tilde{\alpha}(\cdot), \frac{1}{2}\tilde{\beta}(\cdot)\}$ is an ISS-pair for system (10.40).

As a consequence, there exist positive definite and proper functions $V(x)$ and $U(z)$, such that

$$\begin{aligned}\frac{\partial V}{\partial x} f(x,z) &\leq -\tilde{\alpha}(\|x\|) + \tfrac{1}{2}\tilde{\beta}(\|z\|) \\ \frac{\partial U}{\partial z} g(z,u) &\leq -\tilde{\beta}(\|z\|) + \tilde{\zeta}(\|u\|) \, .\end{aligned}$$

The composite function $W(x,z) = V(x) + U(z)$ satisfies

$$\frac{\partial W}{\partial x} f(x,z) + \frac{\partial W}{\partial z} g(z,u) \leq -\tilde{\alpha}(\|x\|) - \frac{1}{2}\tilde{\beta}(\|z\|) + \tilde{\zeta}(\|u\|) \, .$$

and therefore $W(x,z)$ is an ISS-Lyapunov function for the composite system (10.39). ◁

As an immediate corollary of this theorem, it is possible to derive a result analogous to that established with Corollary 10.3.3.

Corollary 10.5.3. *Consider system (10.13). Suppose that system*

$$\dot{x} = f(x,z) \, ,$$

viewed as a system with input z and state x is input-to-state stable and that the equilibrium $z = 0$ of

$$\dot{z} = g(z) \, ,$$

is globally asymptotically stable. Then, the equilibrium $(x,z) = (0,0)$ of system (10.13) is globally asymptotically stable.

Proof. The driving system is trivially input-to-state stable and so is the cascade (10.13), by Theorem 10.5.2. The latter, which has no input, is therefore globally asymptotically stable. ◁

Another application of this theorem is the following one. Suppose a system

$$\dot{x} = f(x,u)$$

has an ISS-pair $\{\alpha(\cdot), \sigma(\cdot)\}$ in which, for some integer $q > 0$,

$$\sigma(r) = \mathcal{O}[r^q] \qquad \text{as } r \to \infty.$$

Then, for any $K > 0$, the system in question has an ISS-pair $\{\tilde{\alpha}(\cdot), \tilde{\sigma}(\cdot)\}$ in which $\tilde{\sigma}(r) = Kr^q$. Accordingly, for some positive definite and proper C^1 function $V(x)$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -\tilde{\alpha}(\|x\|) + K\|u\|^q \, . \tag{10.42}$$

Suppose now the input function $u : [0,\infty) \to \mathbb{R}^m$ is a piecewise continuous function satisfying

$$\lim_{T\to\infty} \int_0^T \|u(t)\|^q dt < \infty \, .$$

The set of all such functions, endowed with the norm

$$\|u(\cdot)\|_q = \left(\int_0^\infty \|u(t)\|^q dt\right)^{\frac{1}{q}} ,$$

is denoted by L_q^m.

Integration of (10.42) on the interval $[0,t]$ yields

$$V(x(t)) - V(x^\circ) \le K \int_0^t \|u(s)\|^q ds \le K \int_0^\infty \|u(s)\|^q ds$$

and therefore

$$V(x(t)) \le V(x^\circ) + K\Big[\|u(\cdot)\|_q\Big]^q ,$$

which in turn, using the estimates (10.18) yields

$$\|x(t)\| \le \underline{\alpha}^{-1}\Big(\overline{\alpha}(\|x^\circ\|) + K\Big[\|u(\cdot)\|_q\Big]^q\Big) .$$

This shows that the response $x(t)$ to any input $u(\cdot) \in L_q^m$, in any initial state x°, exists for all $t \ge 0$ and is bounded, by a quantity which depends on the norm of the initial state and on the norm of the input function.

10.6 The "Small-Gain" Theorem for Input-to-State Stable Systems

In this section we investigate the stability property of *feedback interconnected* nonlinear systems, and we will see that the property of input-to-state stability lends itself to a simple characterization of an important *sufficient condition* under which the feedback interconnection of two globally asymptotically stable systems remains globally asymptotically stable.

Consider the following interconnected system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2, u) , \end{aligned} \tag{10.43}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$ and $f_1(0,0) = 0$, $f_2(0,0,0) = 0$ (see Fig. 10.3). We suppose that the first subsystem, viewed as a system with internal state x_1 and input x_2 is input-to-state stable. Likewise, we suppose the second subsystem, viewed as a system with internal state x_2 and inputs x_1 and u is input-to-state stable.

In view of the results discussed at the end of Section 10.4, the hypothesis of input-to-state stability of the first subsystem is equivalent to the existence of two class $\mathcal{K}$ functions $\gamma_{01}(\cdot)$, $\gamma_1(\cdot)$ such that the response $x_1(\cdot)$ to any input $x_2(\cdot) \in L_\infty^{n_2}$ satisfies

$$\|x_1(t)\| \le \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2(\cdot)\|_\infty)\} \tag{10.44}$$

for all $t \ge 0$, and

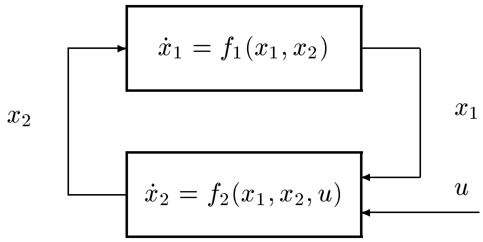

Fig. 10.3. Feedback connection with input.

$$\limsup_{t\to\infty} \|x_1(t)\| \leq \gamma_1(\limsup_{t\to\infty} \|x_2(t)\|) \ . \tag{10.45}$$

Likewise the hypothesis of input-to-state stability of the second subsystem is equivalent to the existence of three class $\mathcal{K}$ functions $\gamma_{02}(\cdot)$, $\gamma_2(\cdot)$, $\gamma_u(\cdot)$ such that the response $x_2(\cdot)$ to any input $x_1(\cdot) \in L_\infty^{n_1}$, $u(\cdot) \in L_\infty^m$ satisfies

$$\|x_2(t)\| \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\} \tag{10.46}$$

for all $t \geq 0$, and

$$\limsup_{t\to\infty} \|x_2(t)\| \leq \max\{\gamma_2(\limsup_{t\to\infty} \|x_1(t)\|), \gamma_u(\limsup_{t\to\infty} \|u(t)\|)\} \ . \tag{10.47}$$

In what follows we shall prove that if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a *simple contraction*, i.e. if

$$\gamma_1(\gamma_2(r)) < r \quad \text{for all } r > 0, \tag{10.48}$$

the composite system is input-to-state stable. This result is usually referred to as the *small-gain* theorem.

Theorem 10.6.1. *If the condition (10.48) holds, system (10.43), viewed as a system with state $x = (x_1, x_2)$ and input u, is input-to-state stable. In particular, the class $\mathcal{K}$ functions*

$$\begin{aligned} \gamma_0(r) &= \max\{2\gamma_{01}(r), 2\gamma_{02}(r), 2\gamma_1 \circ \gamma_{02}(r), 2\gamma_2 \circ \gamma_{01}(r)\} \\ \gamma(r) &= \max\{2\gamma_1 \circ \gamma_u(r), 2\gamma_u(r)\} \end{aligned}$$

are such that response $x(t)$ to any input $u(\cdot) \in L_\infty^m$ is bounded and

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(\|x^\circ\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\leq \gamma(\limsup_{t\to\infty} \|u(t)\|) \ . \end{aligned}$$

Proof. Pick $x_1^\circ \in \mathbb{R}^{n_1}$, $x_2^\circ \in \mathbb{R}^{n_2}$, $u(\cdot) \in L_\infty^m$. We show first that the corresponding trajectories $x_1(t)$ and $x_2(t)$ exist for all $t \geq 0$ and are bounded. For, suppose this is not the case. Then, for every number $R > 0$, there exists a time $T > 0$, such that the trajectories are defined on $[0, T]$, and

$$\text{either} \quad \|x_1(T)\| > R \quad \text{or} \quad \|x_2(T)\| > R\,. \tag{10.49}$$

Choose R such that

$$\begin{aligned} R &> \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \\ R &> \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\}\,, \end{aligned} \tag{10.50}$$

and let T be such that (10.49) hold. Define, for $i = 1, 2$,

$$\begin{aligned} x_i^T(t) &= x_i(t) \quad \text{if} \quad t \in [0, T] \\ &= 0 \qquad\ \ \text{if} \quad t > T\,. \end{aligned}$$

and let $\tilde{x}_1(t)$ denote the response of the top subsystem of (10.43), in the initial state x_1°, to the input $x_2^T(\cdot)$. Since the latter is bounded on $[0, \infty)$, we have

$$\|\tilde{x}_1(t)\| \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2^T(\cdot)\|_\infty)\}$$

for all $t \geq 0$.

Since, by causality,

$$\tilde{x}_1(t) = x_1(t) \qquad \text{for all} \ \ t \in [0, T]$$

we deduce that

$$\|x_1^T(\cdot)\|_\infty = \max_{t \in [0,T]} \|x_1(t)\| \leq \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2^T(\cdot)\|_\infty)\}\,. \tag{10.51}$$

Similarly, let $\tilde{x}_2(t)$ denote the response of the bottom subsystem of (10.43), in the initial state x_2°, to the input $x_1^T(\cdot)$, $u(\cdot)$. Since the latter are bounded on $[0, \infty)$, we have

$$\|\tilde{x}_2(t)\| \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1^T(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}$$

for all $t \geq 0$. This, in turn, since

$$\tilde{x}_2(t) = x_2(t) \qquad \text{for all} \ \ t \in [0, T]$$

yields

$$\|x_2^T(\cdot)\|_\infty = \max_{t \in [0,T]} \|x_2(t)\| \leq \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1^T(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}\,. \tag{10.52}$$

Observe now that, if $a \leq \max\{b, c, \theta(a)\}$ and $\theta(a) < a$, then necessarily $\max\{b, c, \theta(a)\} = \max\{b, c\}$. Thus, replacing the estimate (10.52) into (10.51) and using the hypothesis that $\gamma_1 \circ \gamma_2(r) < r$ yields

$$\|x_1^T(\cdot)\|_\infty \le \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} .$$

Observe also that if $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then also $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction. In fact, let $\gamma_1^{-1}(\cdot)$ denote the inverse of the function $\gamma_1(\cdot)$, which is defined on an interval of the form $[0, r_1^*)$ where

$$r_1^* = \lim_{r\to\infty} \gamma_1(r) .$$

If $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then

$$\gamma_2(r) < \gamma_1^{-1}(r) \ \text{ for all } \ 0 < r < r_1^* ,$$

and this shows that

$$\gamma_2(\gamma_1(r)) < r \ \text{ for all } \ r > 0 ,$$

i.e. $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction.

Therefore, an argument identical to the one used before shows that

$$\|x_2^T(\cdot)\|_\infty \le \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\} .$$

In particular, using (10.50), we have

$$\begin{array}{rcl} \|x_1(T)\| & \le & \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \quad < \quad R \\ \|x_2(T)\| & \le & \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\} \qquad\quad < \quad R , \end{array}$$

which contradicts (10.49).

Having shown that the trajectories are defined for all $t \ge 0$ and bounded, (10.44) and (10.46) yield

$$\begin{array}{rcl} \|x_1(\cdot)\|_\infty & \le & \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1(\|x_2(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_\infty & \le & \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2(\|x_1(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\} , \end{array}$$

combining which, and using the property that $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, one obtains

$$\begin{array}{rcl} \|x_1(\cdot)\|_\infty & \le & \max\{\gamma_{01}(\|x_1^\circ\|), \gamma_1 \circ \gamma_{02}(\|x_2^\circ\|), \gamma_1 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_\infty & \le & \max\{\gamma_{02}(\|x_2^\circ\|), \gamma_2 \circ \gamma_{01}(\|x_1^\circ\|), \gamma_u(\|u(\cdot)\|_\infty)\} . \end{array}$$

In a similar way, combining now (10.45) and (10.47) (in which all the limits are finite since $x_1(\cdot)$ and $x_2(\cdot)$ are bounded) and using the property that $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, one obtains

$$\begin{array}{rcl} \limsup\limits_{t\to\infty} \|x_1(t)\| & \le & \gamma_1 \circ \gamma_u(\limsup\limits_{t\to\infty} \|u(t)\|) \\ \limsup\limits_{t\to\infty} \|x_2(t)\| & \le & \gamma_u(\limsup\limits_{t\to\infty} \|u(t)\|) . \end{array} \tag{10.53}$$

From this, observing that

$$\|x(\cdot)\|_\infty \le \max\{2\|x_1(\cdot)\|_\infty, 2\|x_2(\cdot)\|_\infty\}$$

and

$$\limsup_{t\to\infty} \|x(t)\| \le \max\{2 \limsup_{t\to\infty} \|x_1(t)\|, 2 \limsup_{t\to\infty} \|x_2(t)\|\} ,$$

the result follows. ◁

The condition (10.48), i.e. the condition that the composed function $\gamma_1 \circ \gamma_2(\cdot)$ is a contraction, is usually referred to as the *small gain condition*. Of course, it can be written in different alternative ways depending on how the functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are estimated. For instance, if it is known that $V_1(x_1)$ is an ISS-Lyapunov functions for the upper subsystem of (10.43), i.e. a function such

$$\underline{\alpha}_1(\|x_1\|) \le V_1(x_1) \le \overline{\alpha}_1(\|x_1\|)$$

$$\|x_1\| \ge \chi_1(\|x_2\|) \quad \Rightarrow \quad \frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) \le -\alpha(\|x_1\|)$$

then (see Remark 10.4.4) estimates of the form (10.44) and (10.45) hold with

$$\gamma_1(r) = \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1(r) \; .$$

Likewise, if $V_2(x_2)$ is a function such that

$$\underline{\alpha}_2(\|x_2\|) \le V_2(x_2) \le \overline{\alpha}_2(\|x_2\|)$$

$$\|x_2\| \ge \max\{\chi_2(\|x_1\|), \chi_u(\|u\|)\} \quad \Rightarrow \quad \frac{\partial V_2}{\partial x_2} f_2(x_1, x_2, u) \le -\alpha(\|x_2\|) \; ,$$

then, by means of arguments similar to those used in Section 10.4, it is easy to deduce that estimates of the form (10.46) and (10.47) hold with

$$\begin{aligned} \gamma_2(r) &= \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r) \\ \gamma_u(r) &= \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_u(r) \; . \end{aligned}$$

In this case the small-gain condition can be written in the form

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r) < r \; .$$

We conclude the section with a simple example.

Example 10.6.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 \\ \dot{x}_2 &= a x_1^2 - x_2 + u \; . \end{aligned} \tag{10.54}$$

in which a is a real parameter, viewed as a system with state $(x_1, x_2) \in \mathbb{R}^2$ and input u. We regard this as interconnection of two one-dimensional systems, described by the upper and, respectively, lower equation, and we use the result of Theorem 10.6.1 to determine whether or not there are values of a such that the system is input-to-state stable.

For the upper subsystem

$$\dot{x}_1 = -x_1^3 + x_1 x_2$$

viewed as a system with state x_1 and input x_2, we consider the ISS-Lyapunov function $V(x_1) = \frac{1}{2} x_1^2$ and obtain

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1(x_1, x_2) \le -|x_1|^4 + |x_1|^2 |x_2| \ .$$

Choose any $0 < \varepsilon < 1$ and observe that

$$(1-\varepsilon)|x_1|^2 \ge |x_2|$$

implies

$$\dot{V} \le -\alpha(|x_1|)$$

for the class $\mathcal{K}_\infty$ function $\alpha(r) = \varepsilon r^4$. Thus, the inequality (10.19) holds with

$$\chi(r) = \frac{\sqrt{r}}{\sqrt{1-\varepsilon}} \ .$$

Since we can take

$$\underline{\alpha}(r) = \overline{\alpha}(r) = \frac{1}{2} r^2$$

it is deduced that the upper subsystem of (10.54) is input-to-state stable and estimates of the form (10.44) and (10.45) hold, with

$$\gamma_1(r) = \frac{\sqrt{r}}{\sqrt{1-\varepsilon}} \ .$$

For the lower subsystem of (10.54)

$$\dot{x}_2 = a x_1^2 - x_2 + u$$

viewed as a system with state x_2 and input (x_1, u), consider again a quadratic ISS-Lyapunov function $V(x_2) = \frac{1}{2} x_2^2$ and obtain

$$\dot{V} = \frac{\partial V}{\partial x_2} f_2(x_1, x_2, u) \le |x_2|(|a||x_1|^2 - |x_2| + |u|) \ .$$

Choose again any $0 < \varepsilon < 1$ and observe that

$$(1-\varepsilon)|x_2| \ge |a||x_1|^2 + |u|$$

yields

$$\dot{V} \le -\alpha(|x_2|)$$

for the class $\mathcal{K}_\infty$ function $\alpha(r) = \varepsilon r^2$. Thus, the class $\mathcal{K}$ functions

$$\chi_2(r) = \frac{2|a|r^2}{1-\varepsilon}, \qquad \chi_u(r) = \frac{2r}{1-\varepsilon}$$

are such that

$$|x_2| \ge \max\{\chi_2(|x_1|), \chi_u(|u|)\} \quad \Rightarrow \quad \dot{V} \le -\alpha(|x_2|) \ .$$

As a consequence, estimates of the form (10.46) and (10.47) hold, with

$$\gamma_2(r) = \frac{2|a|r^2}{1-\varepsilon}, \qquad \chi_u(r) = \frac{2r}{1-\varepsilon} \ .$$

Checking the small gain condition on these estimates yields

$$\gamma_2(\gamma_1(r)) = \frac{2|a|r}{(1-\varepsilon)^2} < r \ ,$$

which can be fulfilled, for all $r > 0$, if $|a| < 1/2$. $\triangleleft$