Pricing problem

- INPUT:
  - *m* identical and indivisible *items;*
  - *n buyers*, where each buyer has a valuation function $v_i : \{1,2,\ldots,m\} \mapsto \Re^+$
- OUTPUT:
  - A <u>feasible and envy-free</u> outcome consisting of an allocation vector $X=<x_1,\ldots, x_n>$ (where x_i is the number of items we sell or assign to buyers *i*) and an item price $p\geq 0$ (*p* is the price for a single item).
- GOAL:
  - Maximizing the revenue that is $p\sum_{i=1}^{n} x_i$
- Notice that the input size is *nm*.
- Does such problem admit a polynomial time algorithm that finds a feasible and envy-free outcome that maximizes the revenue?
- It is possible to show that the problem is NP-Hard. Indeed it is even worst, i.e., it is possible to show that, under common complexity assumptions, the problem cannot be approximated in polynomial time within a factor $O(\log^{\varepsilon} n)$, for some small $\varepsilon>0$. (However we do not see the proof in this course).
- We are going to see a polynomial time approximation algorithm.

A useful Definition

Definition: *An outcome is <u>nearly-feasible</u> if it satisfies only individual rationality (i.e., it does not satisfy supply constraint).*

Example

- m=5, n=4
- buyers' valuations:

|items Buyer|	1	2	3	4	5
Buyer 1	3	4	5	6	8
Buyer 2	2	1	7	7	7
Buyer 3	3	4	5	8	8
Buyer 4	3	5	5	5	3

Example (2)

- Let us consider the following outcome:
- $X=<x_1,x_2, x_3,x_4>= <1,1,4,2>; \quad p=2$
- The outcome is *nearly-feasible*! In fact…
  - Individual rationality:
  - Buyer 1: $u_1(x_1,p)=v_1(x_1) - px_1 \longrightarrow u_1(1,2)=v_1(1) - 2*1= 3\text{-}2=1 \geq 0$ *OK!*
  - Buyer 2: $u_2(x_2,p)=v_2(x_2) - px_2 \longrightarrow u_2(1,2)=v_2(1) - 2*1= 2\text{-}2=0 \geq 0$ *OK!*
  - Buyer 3: $u_3(x_3,p)=v_3(x_3) - px_3 \longrightarrow u_3(4,2)=v_3(4) - 2*4= 8\text{-}8=0 \geq 0$ *OK!*
  - Buyer 4: $u_4(x_4,p)=v_4(x_4) - px_4 \longrightarrow u_4(2,2)=v_4(2) - 2*2= 5\text{-}4=1 \geq 0$ *OK!*
  - Supply constraint: $\sum_{i=1}^{n} x_i = 1+1+4+2 = 8 > 5$ NOT SATISFIED!

That is, the outcome is *nearly-feasible*, since it satisfies the individual rationality and does not satisfy the supply constraint.

Notice that we cannot sell more than *m* items to one buyer.

A useful Lemma

Lemma A: *Given a nearly-feasible and envy-free outcome (X,p).*
There exists a feasible and envy-free outcome (X',p') such that p'≥p and

$$\sum_{i=1}^{n} x'_i \geq \frac{m}{2}$$

That is in the outcome (X',p') we assign or sell at least m/2 items at a price at least p.

Proof:

- Suppose there exists a buyer that receives $y \geq m/2$ items in X (with y≤m). Let us consider the buyer *i* having the highest valuation for *y* items. Then we return the outcome where we sell *y* items at buyer *i* at item price $p' = v_i(y)/y \geq p$. Notice that this is possible since the *outcome (X,p)* satisfies the individual rationality. Moreover, the new outcome is envy-free because the buyer *i* has utility 0 and all the other buyers have utility at most 0 for receiving *y* items at price *p'*. Thus, in this case the lemma holds.
- Therefore for now on we consider that all the buyers receive less than *m/2* items in *X*.
- Let n_j be the number of buyers who receive *j* items in *X*.
- We give a constructive proof where two more cases are considered as follows.

A useful Lemma (2)

<u>(Case 1)</u> First, suppose there exists a $j \in \{1,\ldots,m\}$ such that $jn_j \geq m/2$.

- In this case, the idea is constructing an outcome where only bundles (i.e., subset of items) of size j are sold at price p per item.
- By the property of envy-freeness, if we just sell the bundle of j items, in an envy-free outcome, all buyers with positive (i.e., greater than 0) utility $u_i(j,p)$ would demand (want) j items (i.e., if any of such buyers i with utility greater than 0 does not get j items, but some buyer i' gets j items, then i would envy i'). In addition, buyers with $u_i(j,p) = 0$ are indifferent between receiving j items and nothing. Finally, buyers with negative $u_i(j,p)$ demand nothing (they do not want to buy j items).
- Now let m' be the number of items demanded by buyers with positive $u_i(j,p)$ (recall that we suppose to sell only bundle of size j).
- If $m' \leq m$, we sell j items to all buyers with positive $u_i(j,p)$ and as many buyers with $u_i(j,p) = 0$ as possible (that is until m). In this way, we get a feasible and envy-free outcome that extracts a revenue of at least $m/2*p$ (In fact, recall that we are in the (Case 1) where $jn_j \geq m/2$, which means that it is always possible to sell at least $m/2$ items in an envy-free way, where only bundle of size j are sold).

A useful Lemma (3)

- If $m' > m$, we increase the price from p to p' which is the minimum price such that we have enough items to satisfy the demands of buyers with positive $u_i(j,p')$.
- We can get p' in the following way: among all the buyers with positive utility for j items at price p, consider the buyer i with minimum utility. We increase the price so that the utility of buyer i becomes 0. If the new m' is such that $m' \leq m$, then we stop, otherwise we keep to increase the price in the same way. Let us call p' the price when we stop this procedure (i.e., when $m' \leq m$).

- At price p', we are able to sell at least m/2 items due to the following reason:
  - Since $j < m/2$, by increasing price we lose at most $m/2$ items (notice that we could have more than one buyer getting utility zero when we increase the price, however we can sell items also to buyers with utility zero in an envy-free solution).
  - It suggests that we sell at least $m/2$ items at the first time we have enough items to satisfy the demand of all buyers (i.e., when $m' \leq m$).
- Therefore, at price $p' \geq p$ we get a feasible and envy-free outcome that extracts a revenue of at least $m/2*p$ by selling j items to all buyers with positive $u_i(j,p')$ and as many buyers with $u_i(j,p') = 0$ as possible.

A useful Lemma (4)

(Case 2) Now let us consider the remaining case where for all $j \in \{1,\ldots,m\}$ it holds that $jn_j < m/2$.

- We repeat the following **process** until there are enough items to satisfy the overall demand of all buyers or Case 1 is reached (and then we apply the rules of Case 1).
- **The process**: Arbitrarily pick a bundle of size *j* that is sold in *X*, remove *j* from the bundles sold and then re-compute the best bundles (i.e., the bundle that each buyer prefers most, where each buyer prefers the bundle that maximizes his utility) for all the buyers given the fact that bundles of size *j* are not available, that is, we only consider the bundles sold in *X* without *j*. Here we consider that a buyer prefers a non-empty bundle giving him zero utility with respect to buying no item. Moreover, we assume that the buyer prefers the bundle of maximum size among all the ones giving him the maximum utility. Notice that the resulting outcome is envy-free. Moreover, since we are in Case 2, it implies that at most a revenue of $m/2*p$ is lost when we remove bundle *j*.
- We now show that by applying the above **process** we are able to always sell at least *m/2* items at item price at least *p*.
- As the process does not decrease the price at any point of time, we only need to focus on the number of items it sells.
- Assume that the process ends with selling less than *m/2* items. In such case, considering the previous round, it must be the case that there exists a bundle such that the total demand for it is more than *m/2* items. Indeed it is the reason why the demand drops dramatically after we removing that bundle. However, we reach a contradiction since in this case there exists a $j \in \{1,\ldots,m\}$ such that $jn_j \geq m/2$ and then the process would end in the previous round by applying rules of Case 1.
- It concludes that by this **process** we will obtain a feasible outcome that extracts a revenue of at least $m/2*p$. □

Property of Lemma A

Corollary 1: *Given a nearly-feasible and envy-free outcome (X,p).*

It is possible to compute in <u>*polynomial time*</u> *a feasible and envy-free outcome (X',p') such that p'≥p and* $\sum_{i=1}^{n} x'_i \geq \frac{m}{2}$

Proof:

- It follows from the fact that the constructive proof of Lemma A can be performed in polynomial time.