Web Algorithms

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Algorithmic techniques: linear programming

Characteristics

- The problem is formulated as an integer linear program (ILP)
- Integer linear program = linear program + integrality constraints
- There exists a polynomial time algorithm (ellipsoid algorithm) for solving linear problem, but ….
- Solving an integer linear program is an *NP-HARD* problem
- So what???
- The formulation as ILP makes it possible to use powerful general methods that, according to the properties of the ILP, are able to yield good approximation algorithms:
  - Rounding
  - Primal-dual

Linear programming: rounding

Characteristics

- The problem is formulated as an integer linear program (ILP)
- The linear relaxation (LP) obtained from ILP relaxing integrality constraints, that is substituting them with suitable (on integral) linear constraints
- The obtained solution (optimal for LP) is rounded to a close <u>feasible</u> <u>integral</u> solution for ILP
- The measure *m* of the obtained solution is then compared with the one of the optimal solution for LP, that is m_{LP}^*, that is a lower (MIN) or upper (MAX) bound for m^*

Min problems:

$m_{LP}^* \quad m^* \equiv m_{ILP}^* \quad m \quad r \cdot m_{LP}^* \quad r \cdot m^*$

Max problems

$r \cdot m^* \quad r \cdot m_{LP}^* \quad m \quad m^* \equiv m_{ILP}^* \quad m_{LP}^*$

Min Weighted Vertex Cover

- INPUT: graph $G=(V,E)$, an integer cost c_j associated to every $v_j \in V$

- SOLUTION: $U \subseteq V$ such that $v_j \in U$ or $v_k \in U$ $\forall \{v_j, v_k\} \in E$

- MEASURE: Total cost of U, that is

$$\sum_{v_j \in U} c_j$$

ILP:

$$\min \sum_{j=1}^{n} c_j \cdot x_j$$ ≡ objective function

$x_j + x_k \geq 1 \quad \forall \{v_j, v_k\} \in E$ ≡ constraints

$x_j \in \{0,1\} \quad \forall v_j \in V \; \forall j,\ 1 \leq j \leq n$ ≡ integral constraints

LP:

(Linear relaxation)

$$\min \sum_{j=1}^{n} c_j \cdot x_j$$

$x_j + x_k \geq 1 \quad \forall \{v_j, v_k\} \in E$

$\cancel{x_j \leq 1 \quad \forall v_j \in V}$ *(superfluous, why?)*

$x_j \geq 0 \quad \forall v_j \in V$

Algorithm Round-Vertex-Cover

Begin

Determine the ILP associated to the input instance.

Solve the linear relaxation LP of the ILP and let $<x_1^*, \ldots, x_n^*>$ be the resulting optimal solution of LP.

$\forall v_j$ let $x_j=1$ if $x_j^* \geq ½$ and $x_j=0$ if $x_j^* < ½$.

Return the cover U associated to $<x_1, \ldots, x_n>$, that is such that

$U=\{ v_j \in V \mid x_j=1\}$.

End

Theorem: Round-Vertex-Cover is a 2-approximation algorithm

Proof. It is sufficient to show that:

1. $x_1, \ldots, x_n$ is feasible for the ILP (it satisfies all the constraints), that is that *U* is a cover

1. $\frac{m}{m^*_{PL}} \le 2$ and thus also $\frac{m}{m^*} \le \frac{m}{m^*_{PL}} \le 2$

- *Let us prove fact 1)*

  By the feasibility of <$x_1^*,\ldots,x_n^*$> for LP, for every edge $\{v_j,v_k\}\in E$ it is $x_j^*+x_k^*\geq 1$, that is $x_j^*\geq 0.5$ or $x_k^*\geq 0.5$, so that $x_j=1$ or $x_k=1$, and thus $x_j+x_k\geq 1$ is satisfied in ILP.

- *Let us prove fact 2)*

$$m=\sum_{j=1}^{n} c_j\cdot x_j \leq \sum_{j=1}^{n} c_j\cdot 2\cdot x_j^* = 2\cdot \overbrace{\sum_{j=1}^{n} c_j\cdot x_j^*}^{m_{LP}^*} = 2\cdot m_{LP}^*$$

By the rounding:
$x_j\leq 2\cdot x_j^*$

that is

$$\frac{m}{m^*}\leq\frac{m}{m_{LP}^*}\leq 2.$$

□

Min Weighted Set Cover

- INPUT: Universe $U=\{o_1,\ldots,o_n\}$ of n objects, family $\hat{S}=\{S_1,\ldots,S_h\}$ of h subsets of U, integer cost c_j associated to every $S_j\in\hat{S}$
- SOLUTION: Cover if U, that is subfamily $\hat{C}\subseteq\hat{S}$ such that

$$\bigcup_{S_j \in C} S_j = U$$

- MEASURE: Total cost of the cover, that is $\sum_{S_j \in \hat{C}} c_j$

f = max frequency of an object in the subsets of $\hat{S}$, that is every object occurs in at most f subsets.

Min Weighted Set Cover

Given a set of elements $\{1, 2, \ldots, n\}$ (called the universe) and a collection S of m sets whose union equals the universe, the set cover problem is to identify the smallest sub-collection of S whose union equals the universe. For example, consider the universe $U = \{1, 2, 3, 4, 5\}$ and the collection of sets $S = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$. Clearly the union of S is U. However, we can cover all of the elements with the following, smaller number of sets: $\{\{1, 2, 3\}, \{4, 5\}\}$.

ILP:

$$\min \sum_{j=1}^{h} c_j \cdot x_j$$ ≡ objective function

$$\sum_{S_j \mid o_i \in S_j} x_j \geq 1 \qquad \forall o_i \in U$$ ≡ constraints

$$x_j \in \{0,1\} \qquad \forall S_j \in \hat{S}$$ ≡ integral constraints

LP:

$$\min \sum_{j=1}^{h} c_j \cdot x_j$$

$$\sum_{S_j | o_i \in S_j} x_j \geq 1 \qquad \forall o_i \in U$$

$$\cancel{x_j \leq 1 \qquad \forall S_j \in \hat{S}}$$

(superfluous, why?)

$$x_j \geq 0 \qquad \forall S_j \in \hat{S}$$

Algorithm Round-Set-Cover

Begin

Determine the ILP associated to the input instance.

Solve the linear relaxation LP of the ILP and let $<x_1^*, \ldots, x_n^*>$ be the resulting optimal solution of LP.

$\forall\, S_j$ let $x_j=1$ if $x_j^* \geq 1/f$ and $x_j=0$ if $x_j^* < 1/f$.

Return the resulting cover, i.e. $\hat{C}=\{\, S_j \in \hat{S} \mid x_j=1 \,\}$.

End

Theorem: Round-Set-Cover is f-approximating ($f \geq 1$)

Proof. It is sufficient to show that:

1. $x_1, \ldots, x_n$ is feasible for the ILP

1. $\frac{m}{m^*_{LP}} \leq f$ (and thus also $\frac{m}{m^*} \leq \frac{m}{m^*_{LP}} \leq f$)

- *Let us prove claim 1)*

  By the feasibility of $<x_1^*,\ldots,x_n^*>$ for LP, $\forall o_i \in U$

$$\sum_{S_j | o_i \in S_j} x_j^* \geq 1$$

  and since the summation has at most f terms, there must exist S_j containing o_i s.t. $x_j^* \geq 1/f$, that is such that $x_j=1$, and thus

$$\sum_{S_j | o_i \in S_j} x_j \geq 1.$$

- *Let us prove claim 2)*

m_{LP}^*

that is

$$m = \sum_{j=1}^{h} c_j \cdot x_j \leq \sum_{j=1}^{h} c_j \cdot f \cdot x_j^* = f \cdot \underbrace{\sum_{j=1}^{f} c_j \cdot x_j^*}_{m_{LP}^*} = f \cdot m_{LP}^*$$

By the rounding:
$x_j \leq f \cdot x_j^*$

$$\frac{m}{m^*} \leq \frac{m}{m_{LP}^*} \leq f.$$

□