Web Algorithms – Sponsored Search

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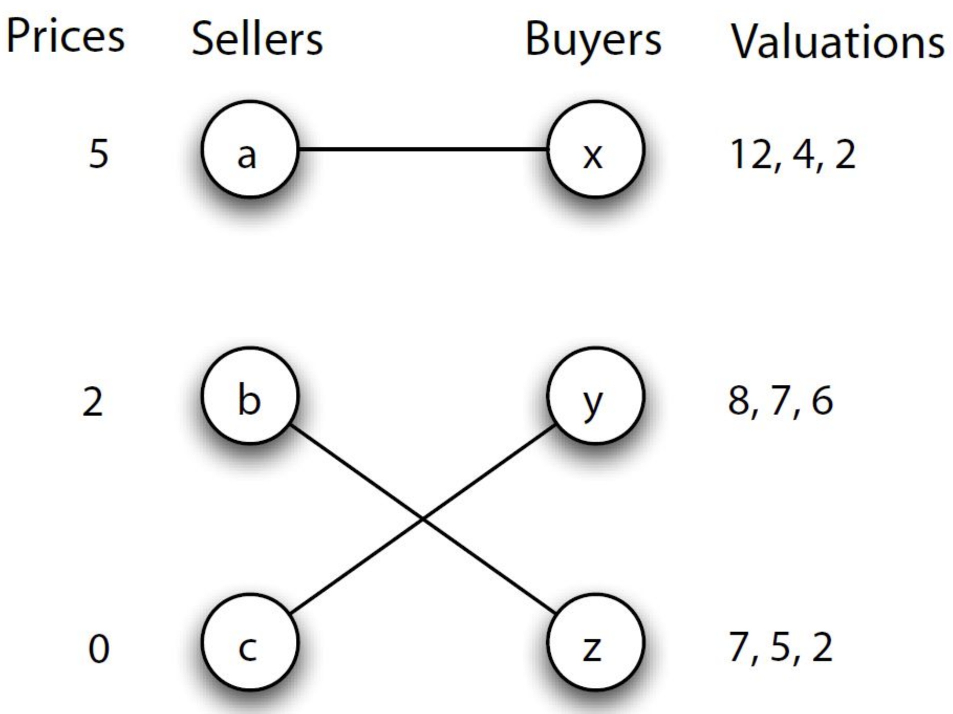
Matching Markets (Second Part)

2^{nd} Scenario: Houses Sales

- Set *S* of Sellers (of houses) and *B* of buyers
- Individual decisions of buyers based on prices and own valuations
- More standard picture of a market



- Each seller *i* puts his house up for a price $p_i \geq 0$

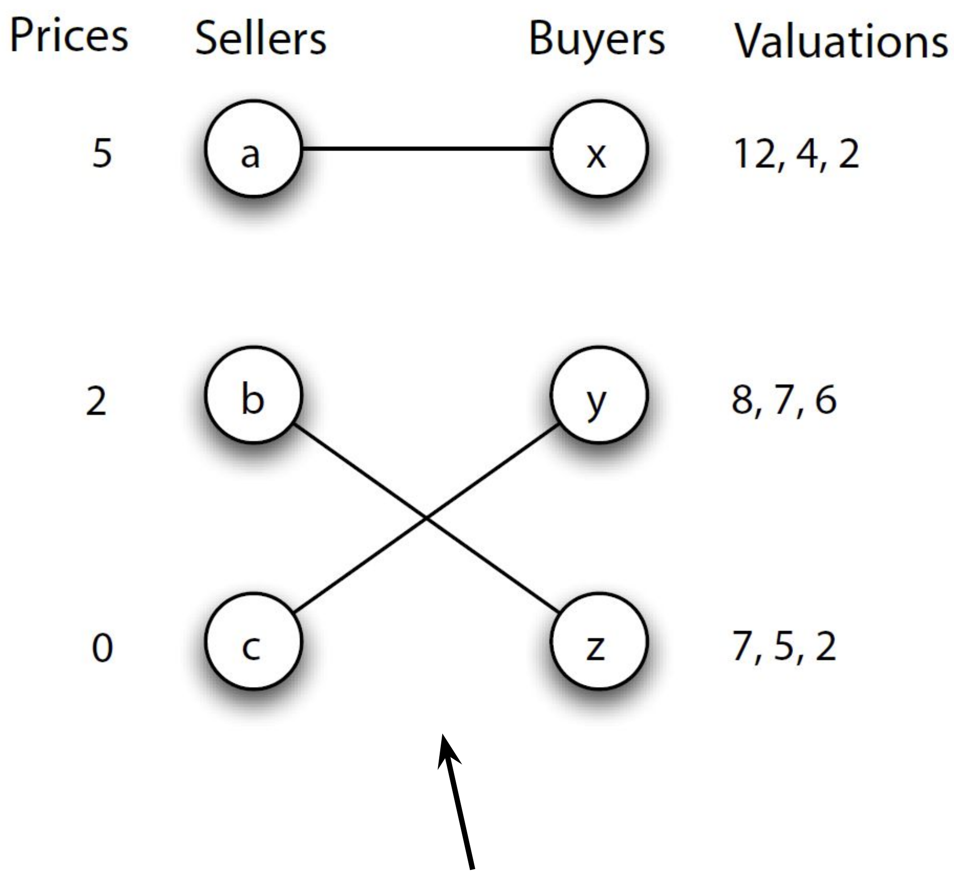
- Buyer *j* payoff for seller *i*: $v_{i,j} - p_i$

 (her valuation for her house minus the amount of money she has to pay)

- Preferred sellers buyer *j*:
  - Sellers that maximize her payoff
  - If negative payoff for every seller, no preferred seller (buyer does not transact)

- Preferred-seller graph: edges between buyers and their preferred sellers

- Assumption: all $v_{i,j}$ and p_i are integers ≥ 0

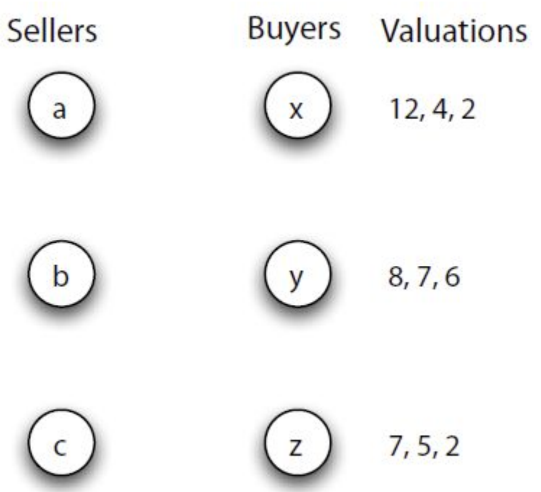


Preferred graph

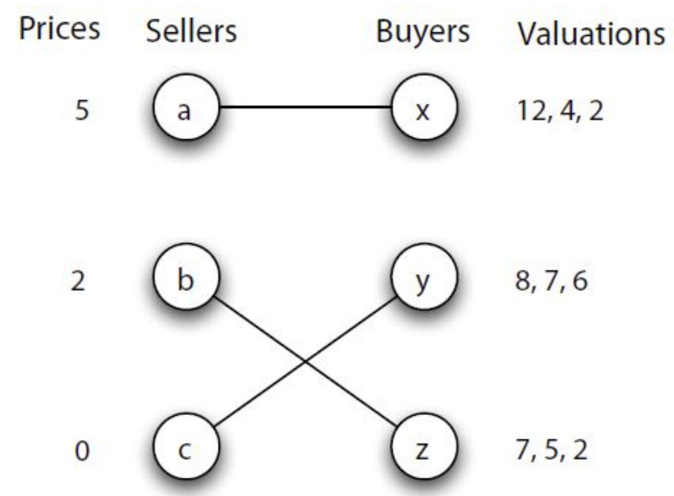
Payoffs of each buyer for each house:

	a	b	c
X	**7**	**2**	**2**
Y	**3**	**5**	**6**
Z	**2**	**3**	**2**

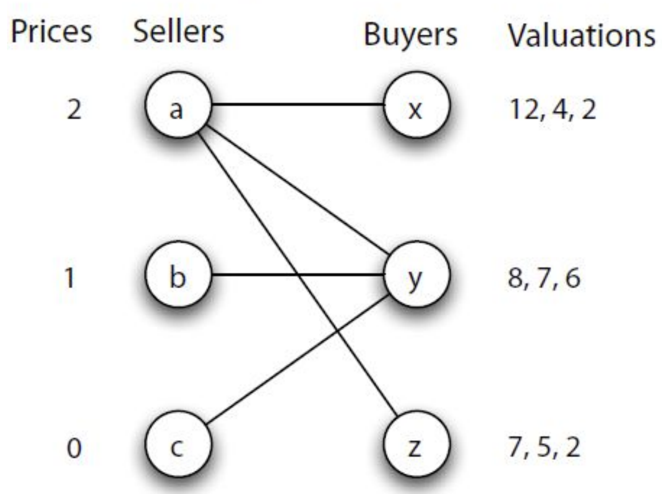
Pricing Examples



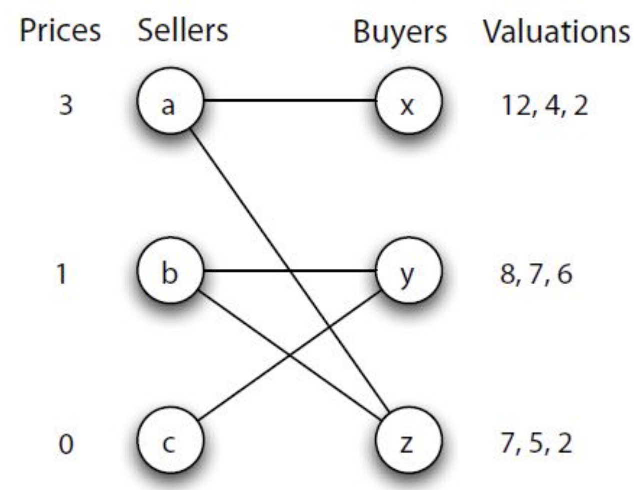
(a) *Buyer Valuations*



(b) *Market-Clearing Prices*



(c) *Prices that Don't Clear the Market*



(d) *Market-Clearing Prices (Tie-Breaking Required)*

- Nice property of Example (b) in previous slide: making the most convenient choice, each buyer ends up with a different house

- In case of ties (more than one preferred seller for some buyers) some coordination is required (Example (d))

- However, it is still possible to assign to each buyer one of her most convenient houses

- In other words, the preferred-seller graph has a perfect matching

- Such a pricing is called market-clearing: it resolves conflicts

Question 1: do market-clearing prices exist for any possible set of buyers valuations?

Question 2: what is the quality of the assignments induced by market-clearing prices (in term of total valuation of buyers)

Existence Market-Clearing Prices

Theorem. Market-clearing prices exist for any possible set of valuations.

Proof.
We prove the claim by providing a suitable procedure ending up with market clearing prices.

Let us first discuss informally the main involved ideas

Idea 1 (raising of prices)

- If a set of prices P is not market-clearing, then there exists a constricted set of buyers C

- That is, the set $N(C)$ of the neighbors of buyers in C in the preferred-sellers graph (i.e. all the preferred sellers of buyers in C) is such that $|N(C)|<|C|$

- Then $N(C)$ is in high demand and usually markets react raising prices of $N(C)$

- The procedure then raises of *1* the prices of $N(C)$, trying to dissuade some buyer in C and eliminate the constricted set

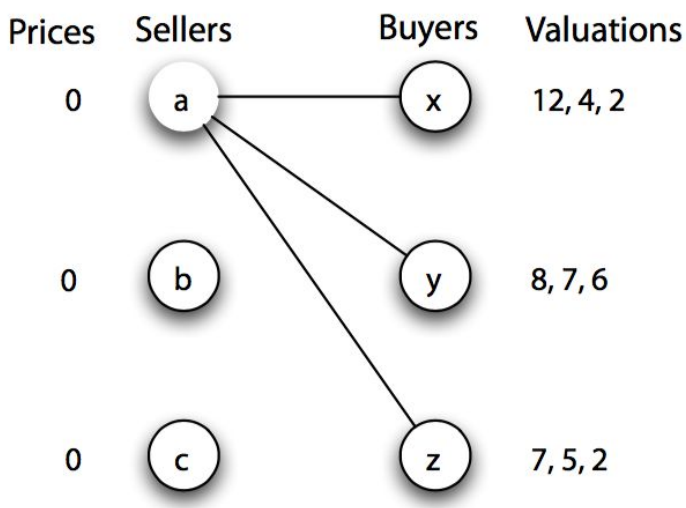
Idea 2 (reduction argument)

- Before the beginning of each round, all prices are eventually decreased by a fixed amount δ (i.e. $p'_i = p_i - \delta \ \forall\ i$) in such a way that the smallest price is equal to *0*
- For each buyer this just increases all the payoffs $(v_{i,j} - p_i)$ due to *P* of δ, that is the new set of prices *P'* does not modify the order of the payoffs

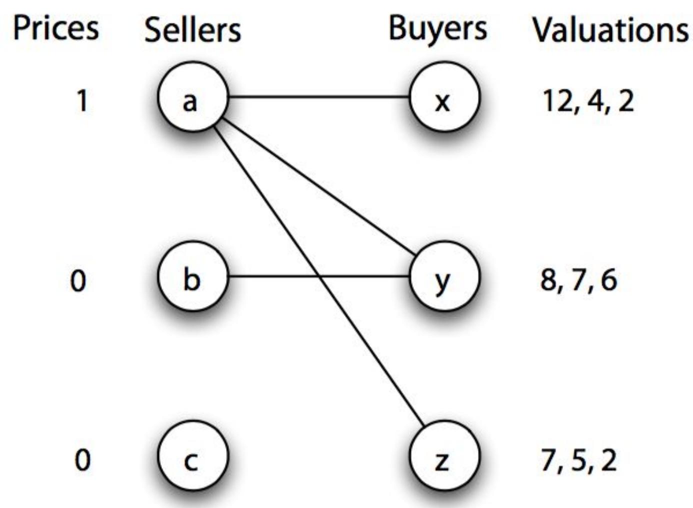
  Example. If the payoffs of a given buyer are in the order 8, 7, 5 and 2, decreasing all the prices by 1 the new payoffs are 9 8, 6 and 3, respectively
- But guarantees that every buyer has payoff at least 0 for at least one seller (the one with price 0), and thus at least one preferred seller, at the beginning of each round and at the end of the procedure

Procedure (for market clearing prices)

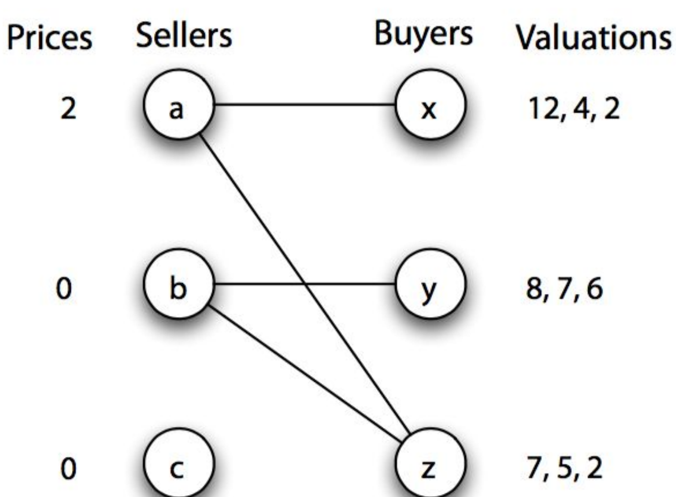
1. Set $p_i=0 \ \forall \ I$
2. Construct the preferred-seller graph G
3. If perfect matching in G STOP: current prices are market-clearing
4. If not find a constricted set of buyers C and their neighbors $N(C)$
5. Raise the price of each seller in $N(C)$ by 1
6. Reduce prices decreasing all of them by the same amount δ=1 so that the lowest price becomes 0
7. Repeat from step 2.



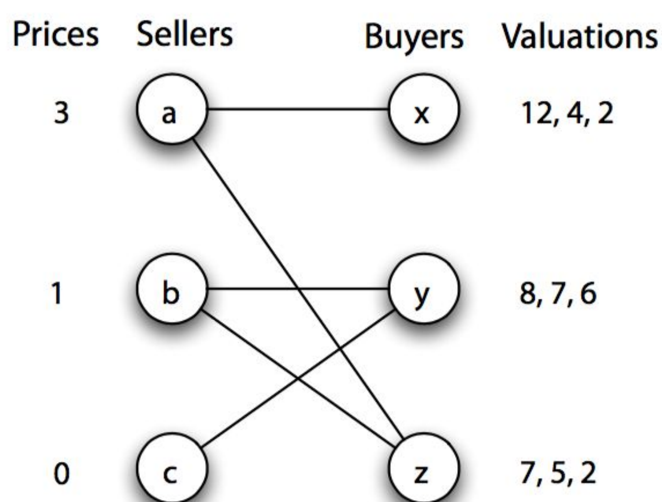
(a) *Start of first round*



(b) *Start of second round*



(c) *Start of third round*



(d) *Start of fourth round*

Remarks on the example

- In the second round there are different constricted sets
- It just means that there can be multiple options for how to run the procedure
- These options in general lead to different market-clearing prices
- Thus, market-clearing prices are not unique

- Let us now complete the proof of the theorem
- Observe first that at the end of the execution, by the reduction argument the lowest price is 0
- Hence every buyer has payoff at least *0* for at least one seller (the one with price *0*)
- So every buyer has at least one preferred seller and there are not constricted sets: there is a perfect matching!
- Hence the final set of prices is market-clearing

- But does the procedure terminate?
- In fact, raising prices helps to eliminate the identified constricted set, but can create new constricted sets
- In order to complete the proof of the theorem we now show that the procedure terminates, that is it does not loop forever
- We prove this by means of a potential function Φ defined on the current set of prices P

- Namely, denoted as P_0 the initial set of prices (all *0*) and by P_k the prices at the beginning of round *k*, we prove that
  - $\Phi(P_0) \geq 0$ (the initial potential function is at least *0)*
  - $\Phi(P_k) \geq 0 \;\; \forall\, k$ (it never goes below 0)
  - $\Phi(P_{k+1}) < \Phi(P_k)$ (it strictly decreases at each round)
- Hence the procedure cannot loop forever
- Let's prove the potential function argument

- Given a pricing P, let $i(j,P)$ or simply $i(j)$ one preferred seller of buyer j according to P

- Potential buyer j = max payoff of j, that is $v_{i(j),j} - p_{i(j)}$

- Potential seller i = price seller i, that is p_i

- Potential pricing P = sum potential of buyers and sellers

$$\Phi(P) = (\textstyle\sum_{j \in B} v_{i(j),j} - p_{i(j)}) + (\textstyle\sum_{i \in S} p_i)$$

- Let us now show that Φ is always ≥ 0 and strictly decreases at every round

- Consider any round k with initial pricing P_k

- Since as already observed the minimum price in P_k is equal to 0 and thus the max payoff of each buyer is ≥ 0

$$\Phi(P_k) = (\sum_{j \in B} v_{i(j),j} - p_{i(j)}) + (\sum_{i \in S} p_i) \geq 0$$

- Let us now show that $\Phi(P_{k+1}) \leq \Phi(P_k)-1$

- The only steps of the procedure affecting Φ are the ones that modify prices, that is steps 5. and 6.

- During step 5., if C is the corresponding constricted set,
  - seller prices in $N(C)$ are raised of 1, hence in total they increase Φ of $|N(C)|$
  - the payoffs of buyers in C are decreased of 1, hence in total they decrease Φ of $|C|$
  - Therefore, since by def. of constricted set $|N(C)|<|C|$, i.e. $|N(C)|\leq|C|-1$, Φ is decreased at least of 1

- During step 6.
  - all the sellers prices decrease of the same factor δ, hence in total they decrease Φ of $\delta|S|$
  - all the buyers payoffs are increased of δ, hence in total they increase Φ of $\delta|B|=\delta|S|$
  - hence Φ is not changed by the step

- Summarizing, $\Phi(P_{k+1}) = \Phi(P_k) + |N(C)| - |C| \leq \Phi(P_k) -1$ □

Optimality Market-Clearing Prices

- Question: given the introduction of prices, how should we evaluate a given assignment, that is the corresponding matching *M* of buyers and sellers

- Let us assume that sellers gain their prices, that is seller i has payoff p_i

- Moreover let us denote as *i(j)* the seller matched to buyer j by *M*

- Two basic criteria follow …

- Social Welfare: global happiness of all the participants

- In other words, the sum of all the sellers and buyers payoffs

$$SW(M) = \left(\sum_{j \in B} v_{i(j),j} - p_{i(j)}\right) + \left(\sum_{i \in S} p_i\right) = \sum_{j \in B} v_{i(j),j}$$

$= \text{Total valuation of } M$!!!

- Thus the Social Welfare corresponds exactly to the previously introduced Total valuation of *M*

- Theorem. For any set of market-clearing prices, any perfect matching M in the resulting preferred-seller graph G has the maximum social welfare of any assignment of sellers to buyers.

Namely, $SW(M) = \sum_{j \in B} v_{i(j),j}$ is maximized

Proof.

Observe first that, since *M* matches every buyer to her preferred seller, that is a seller maximizing her payoff, *M* maximizes

Total payoff of (buyers in) *M* $= \sum_{j\in B} v_{i(j),j} - p_{i(j)}$

$= \sum_{j\in B} v_{i(j),j} - \sum_{i\in S} p_i$

= SW(*M*) – Sum of prices

Since Sum of prices doesn't depend on *M* and *M* maximizes Total payoff, then *M* maximizes also *SW(M)*.

□

- Revenue: total revenue of sellers

- That is, $REV(M)=\sum_{i\in S} p_i$

- Awful, as the procedure might return all prices equal *0*

  Example: Case in which all the buyers have the same valuation for all the sellers

Theorem. There exists an $\Omega(1/\log n)$-approximation