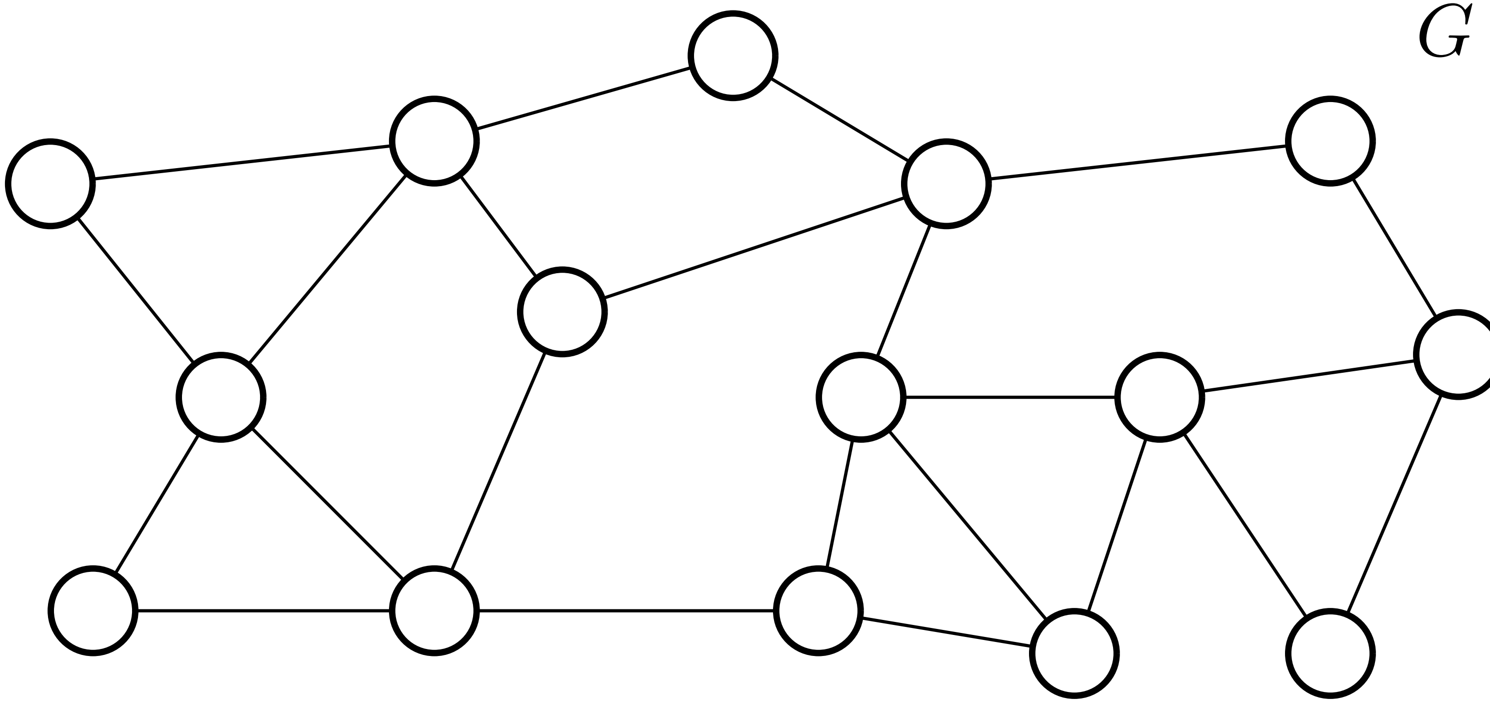
Distributed Vertex Coloring

Vertex Colorings

Vertex Coloring: Each vertex is assigned a color.

$G = (V, E)$



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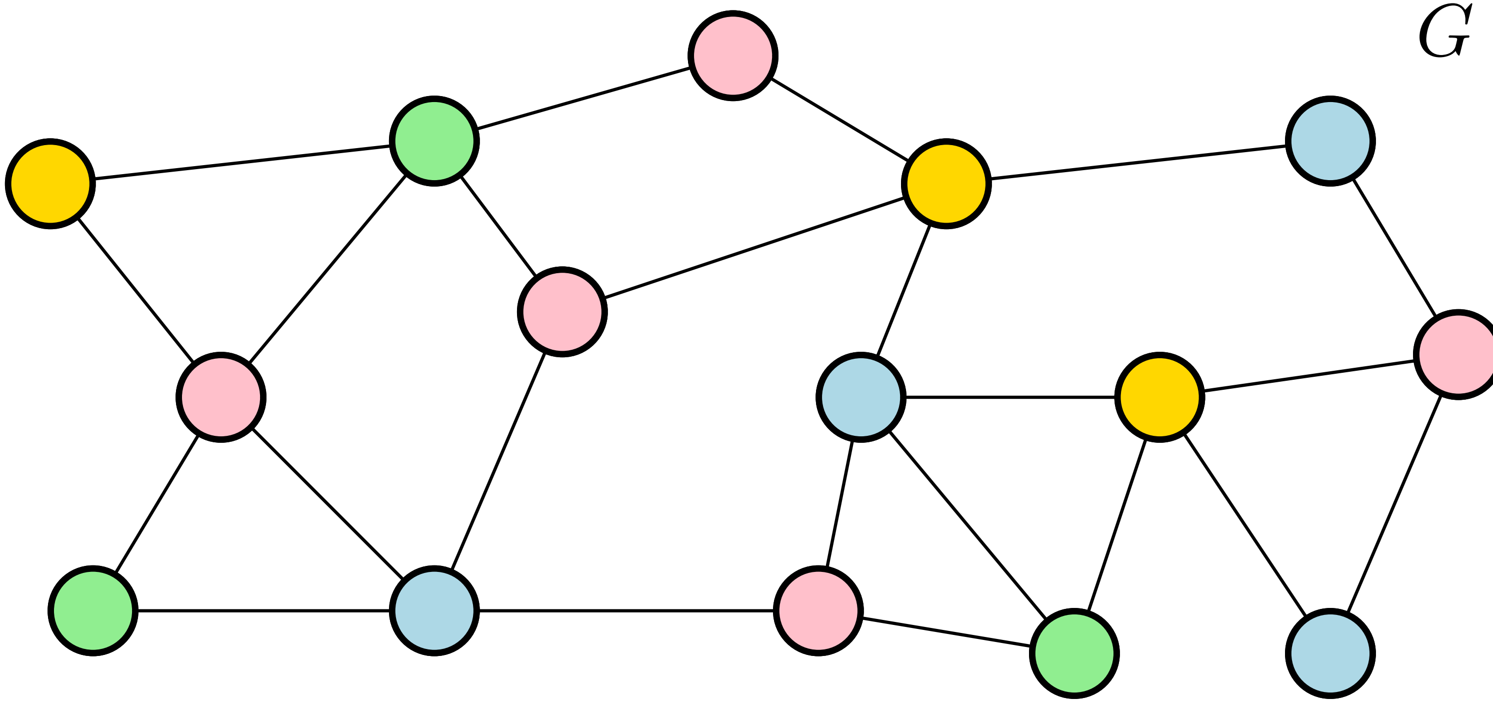

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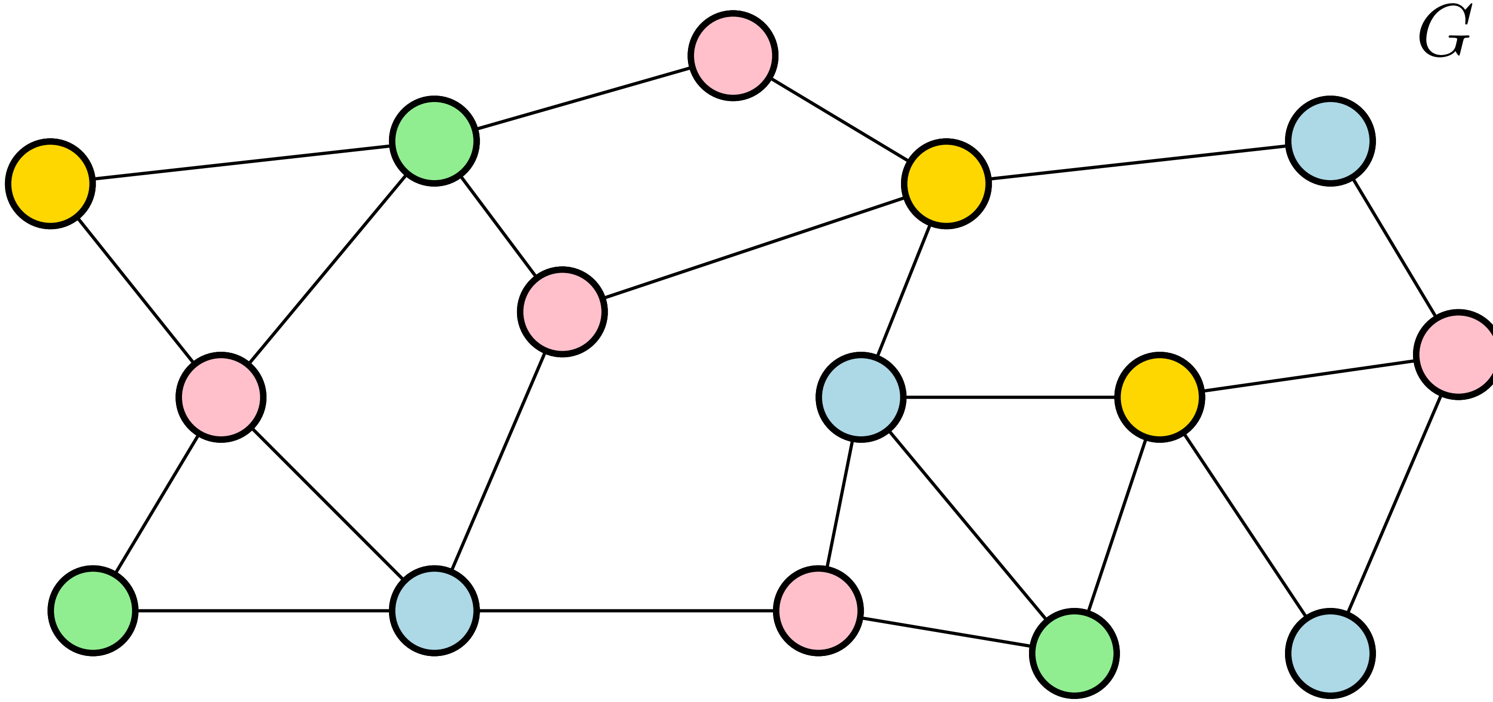

$G = (V, E)$

Valid Vertex Coloring: A vertex coloring is **valid** if no two adjacent nodes have the same color.

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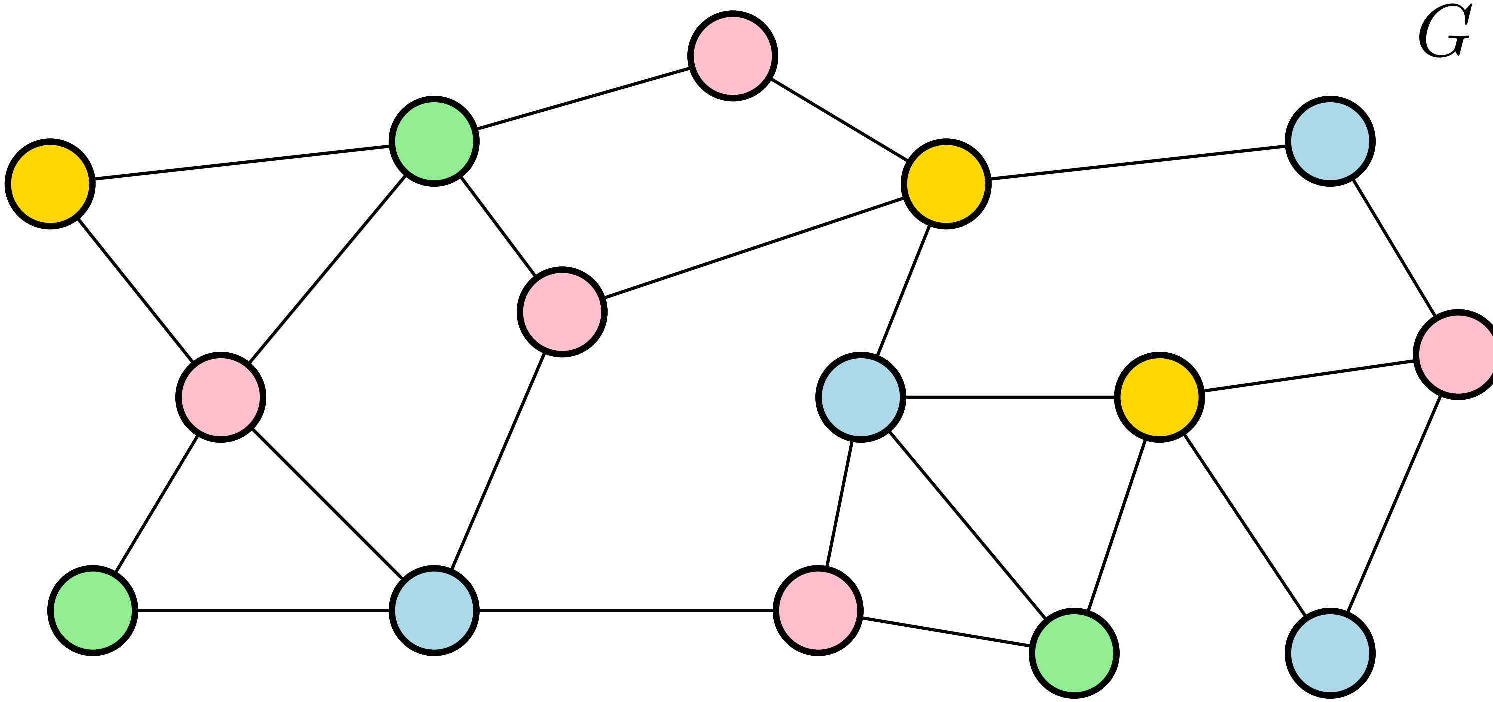

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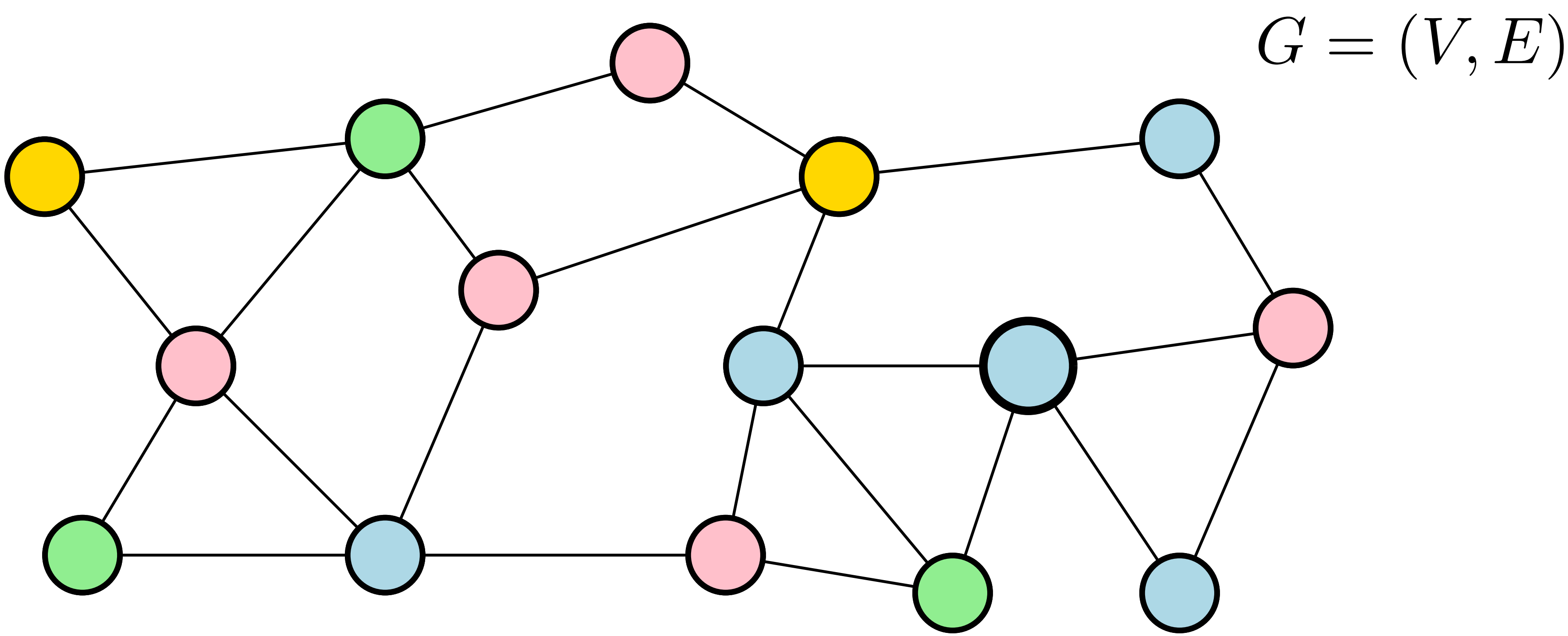

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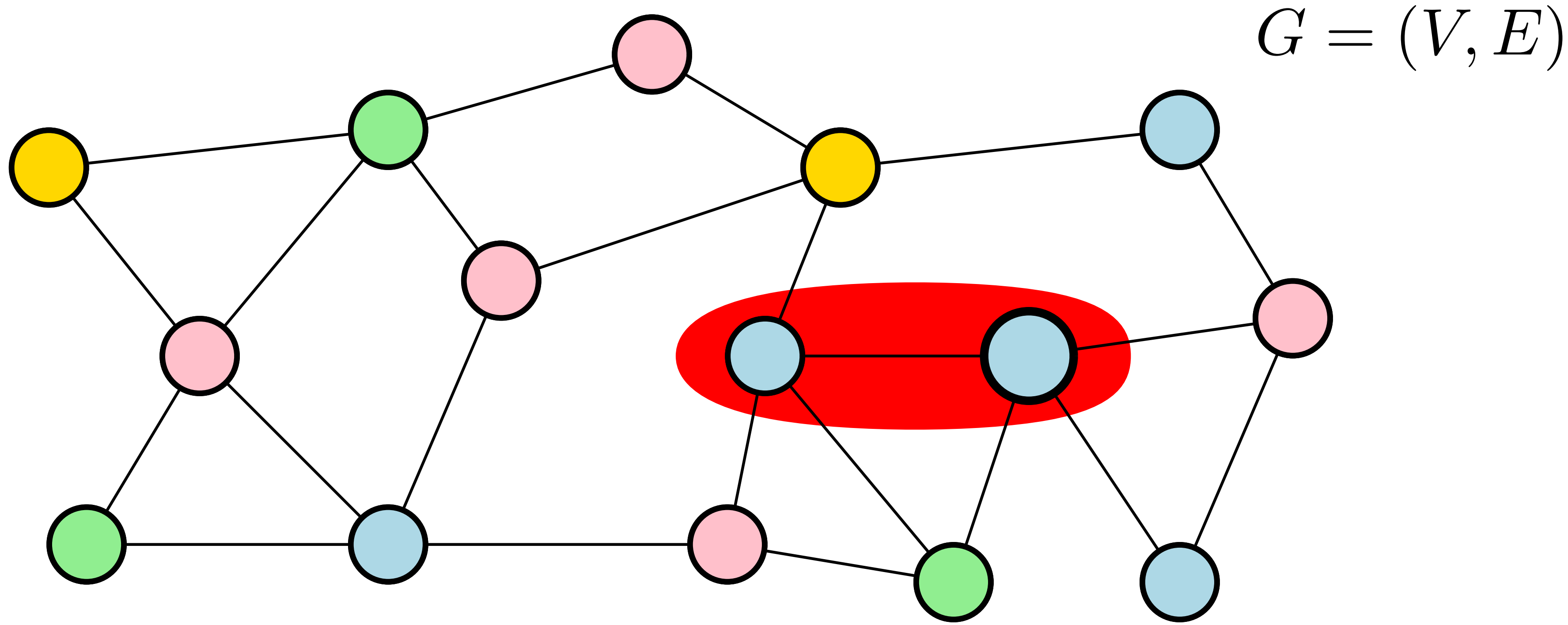

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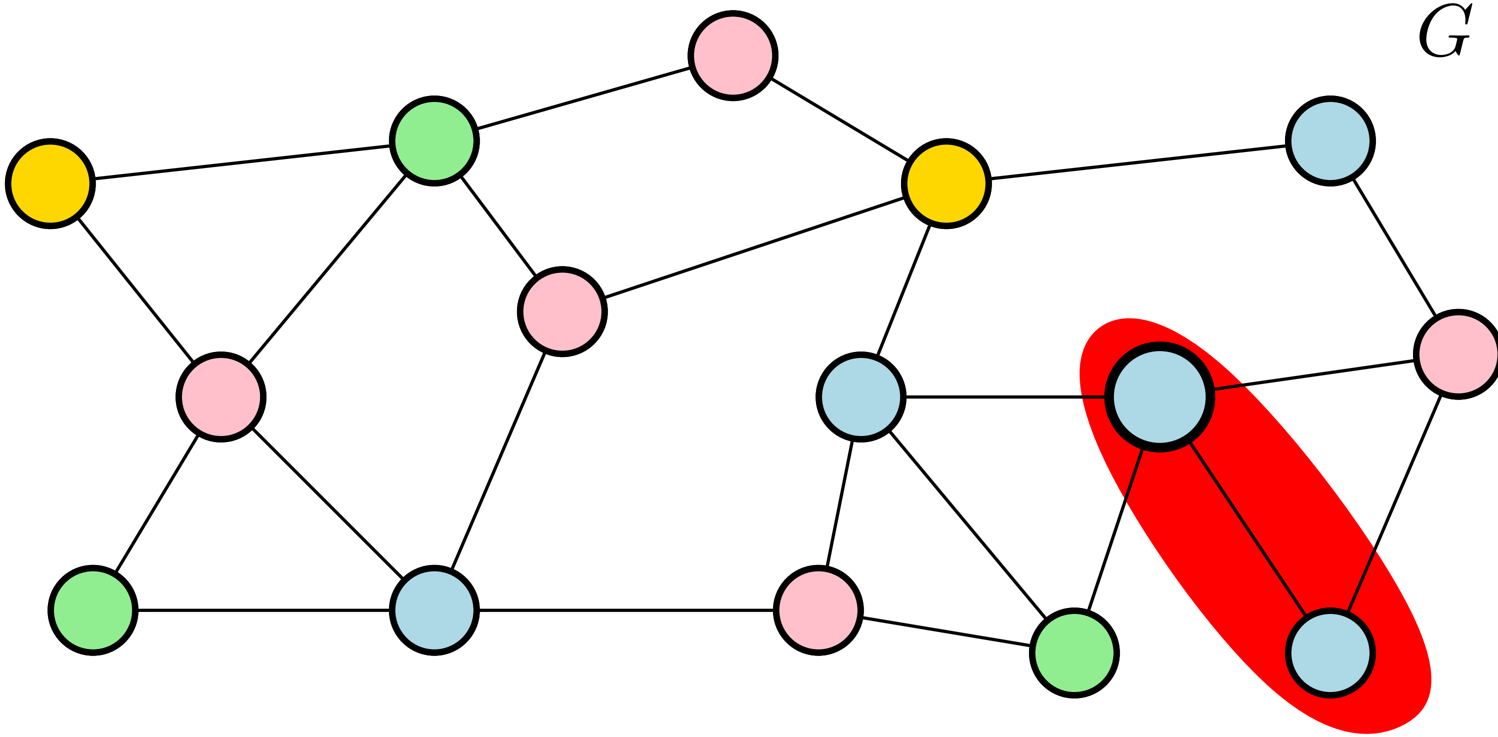

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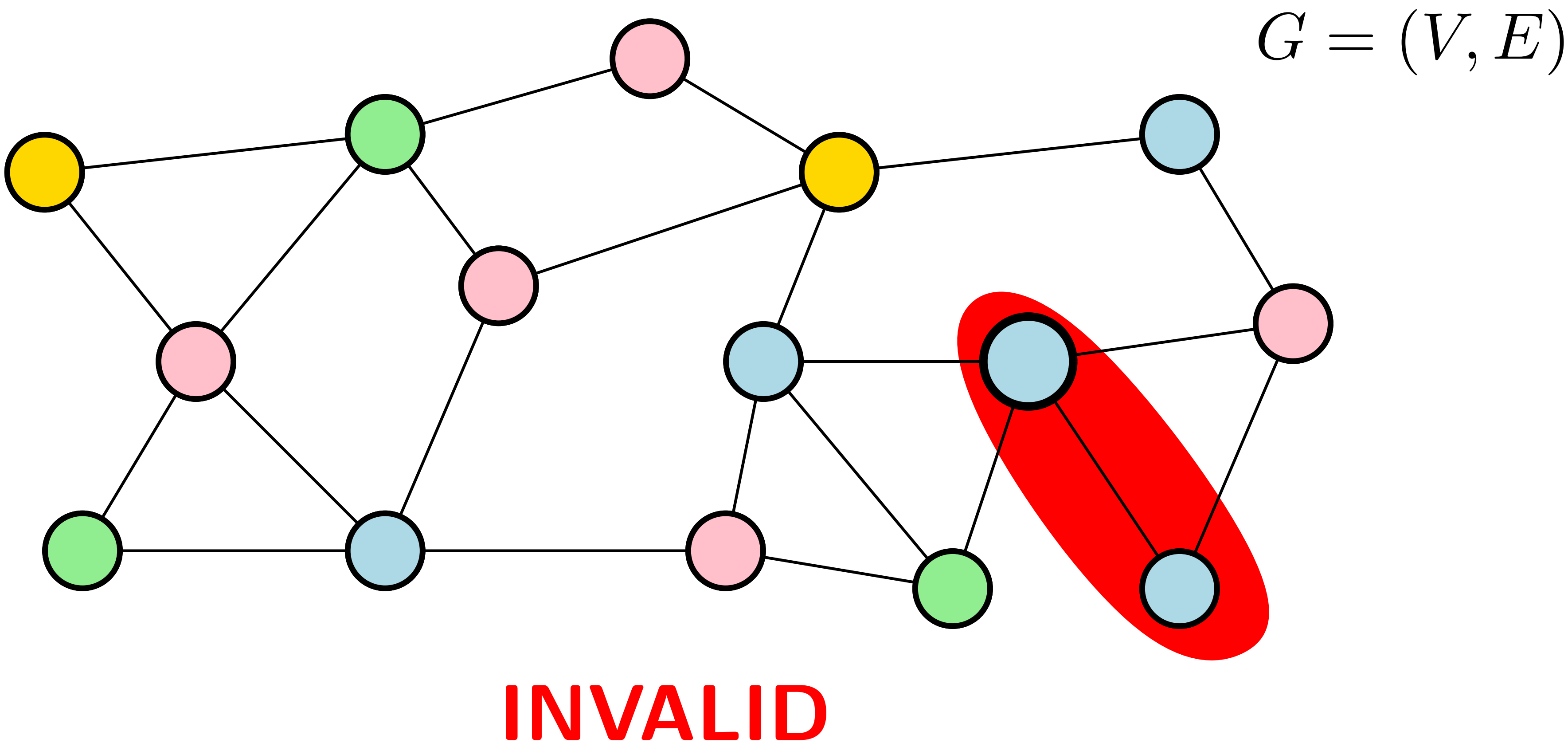

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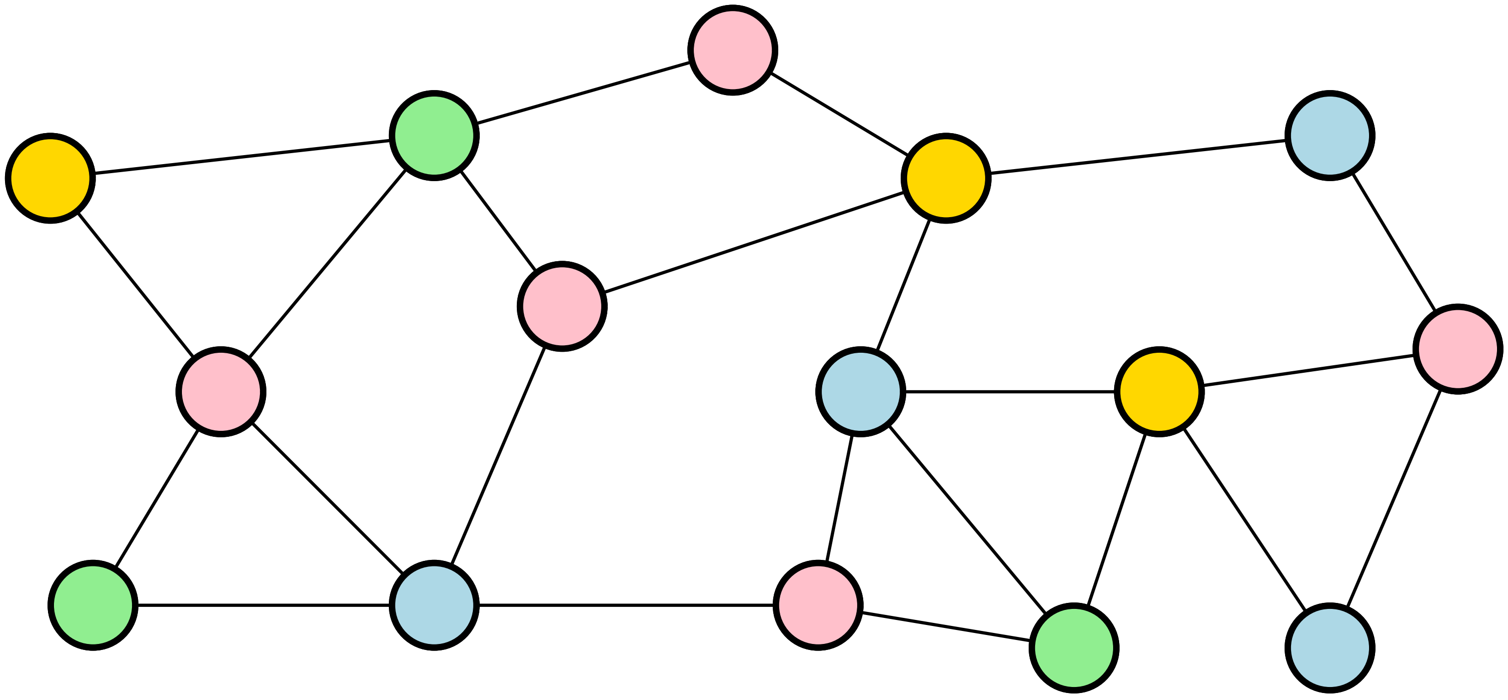
Vertex Colorings

In algorithms each color can be represented with an integer.



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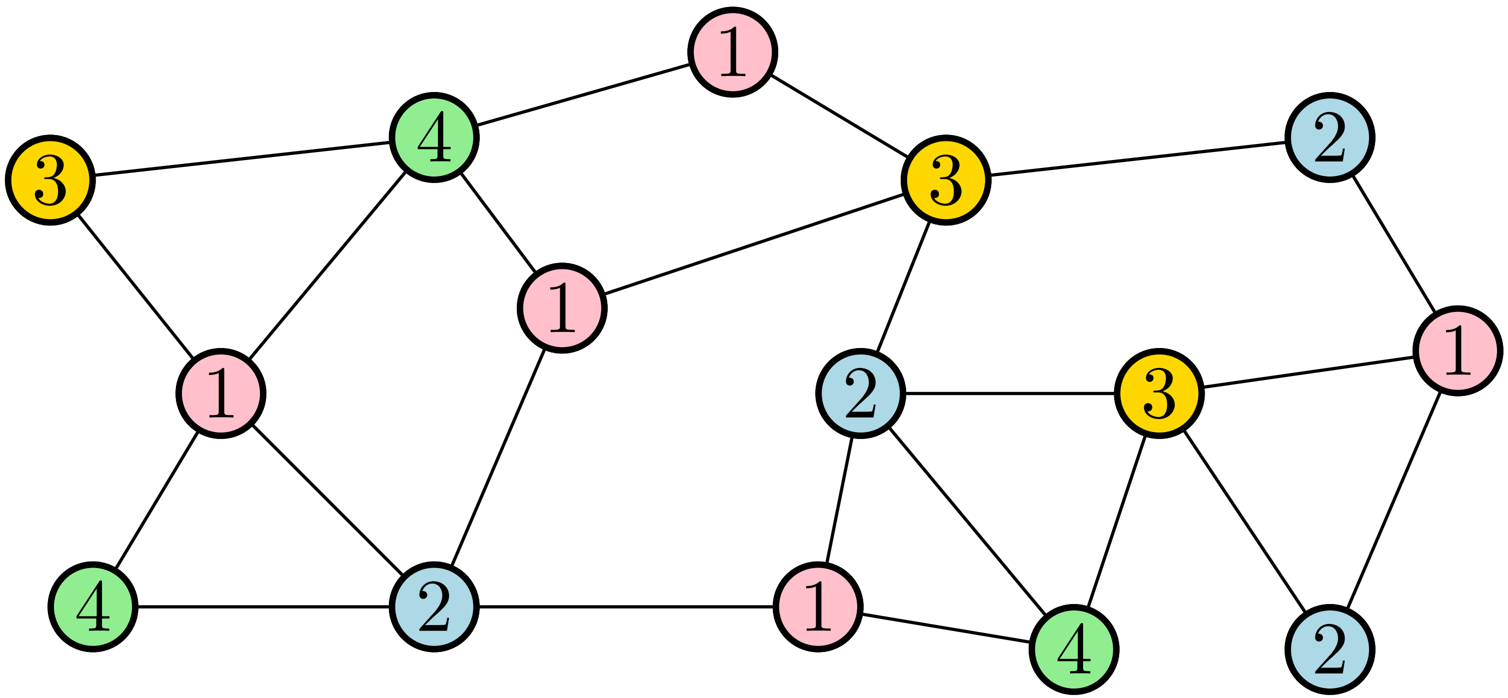

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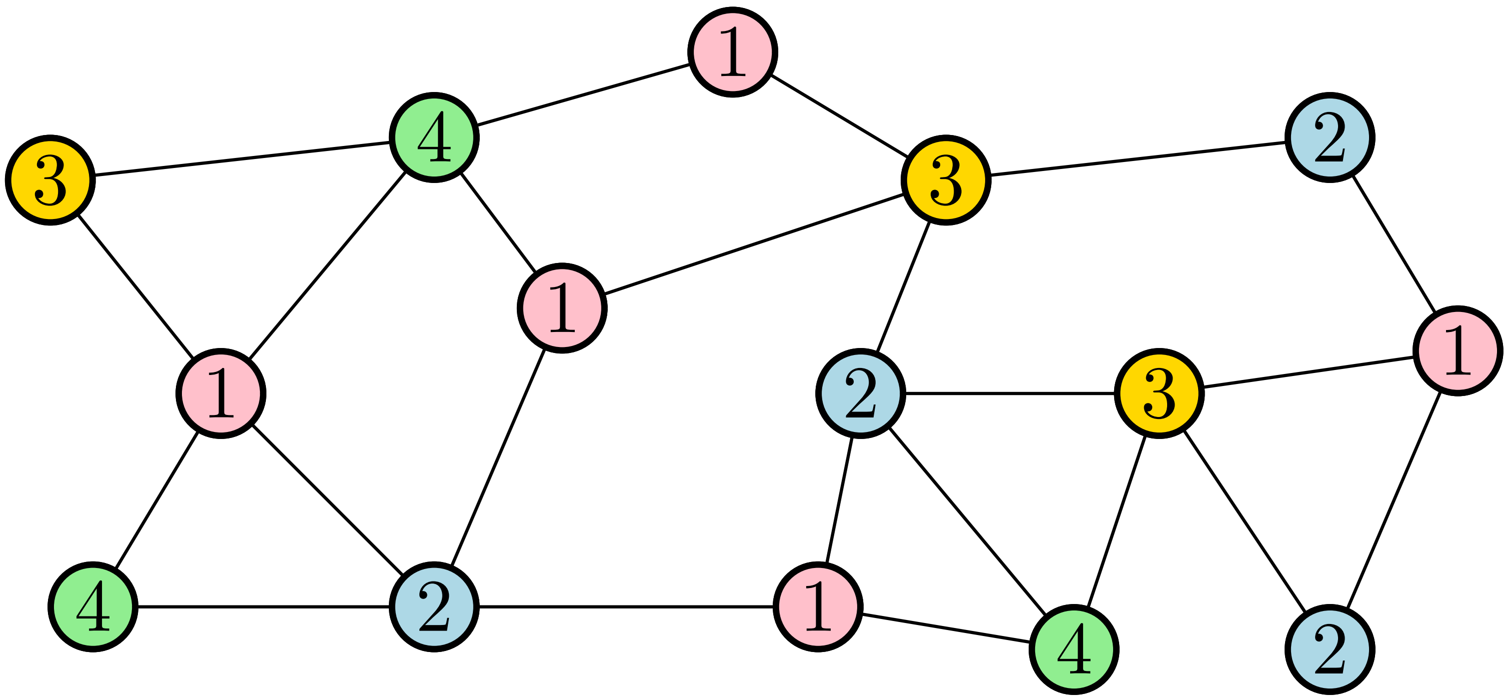

Definition: A k-**coloring** is a coloring with k colors.

Vertex Colorings

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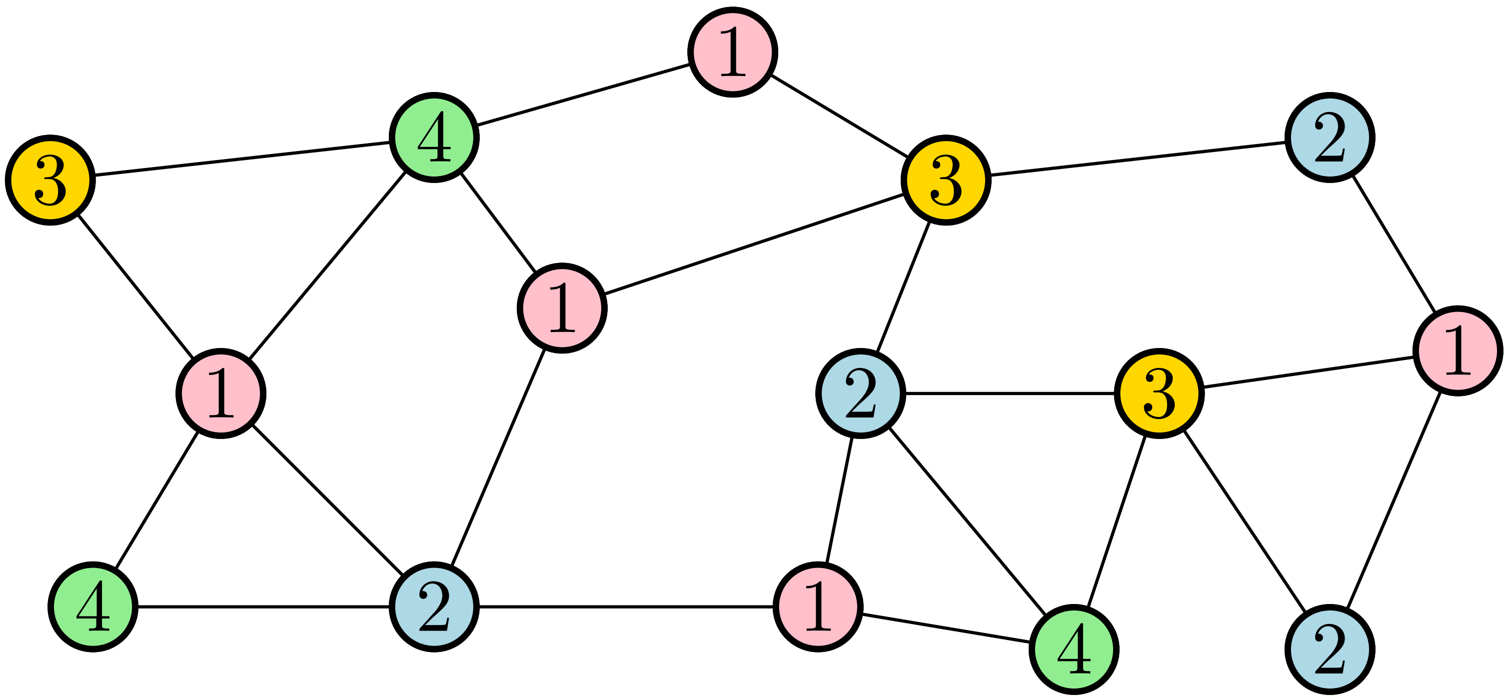
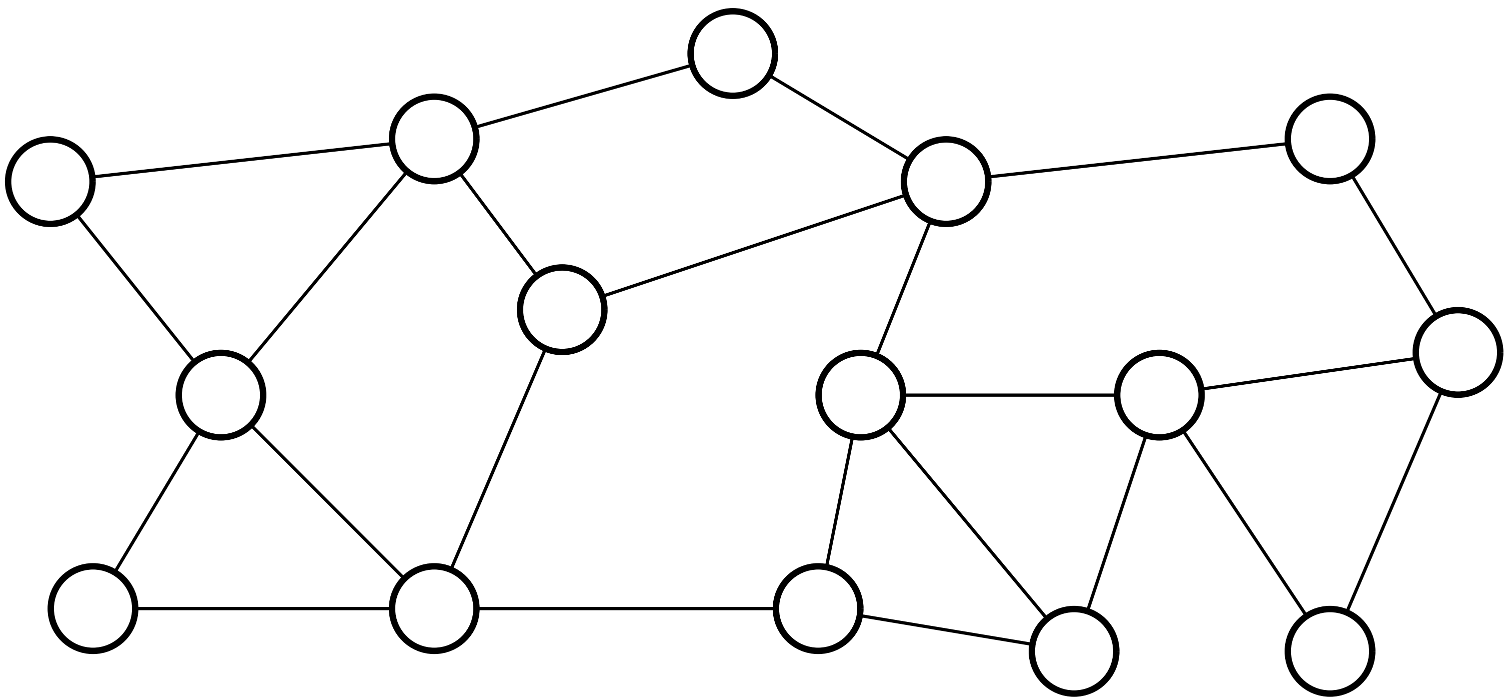


4-coloring

Definition: A k-**coloring** is a coloring with k colors.

Chromatic Number

Definition: Given a graph G, the **chromatic number** $\varphi(G)$ of G is the smallest integer k such that there exists a valid k-coloring of G.



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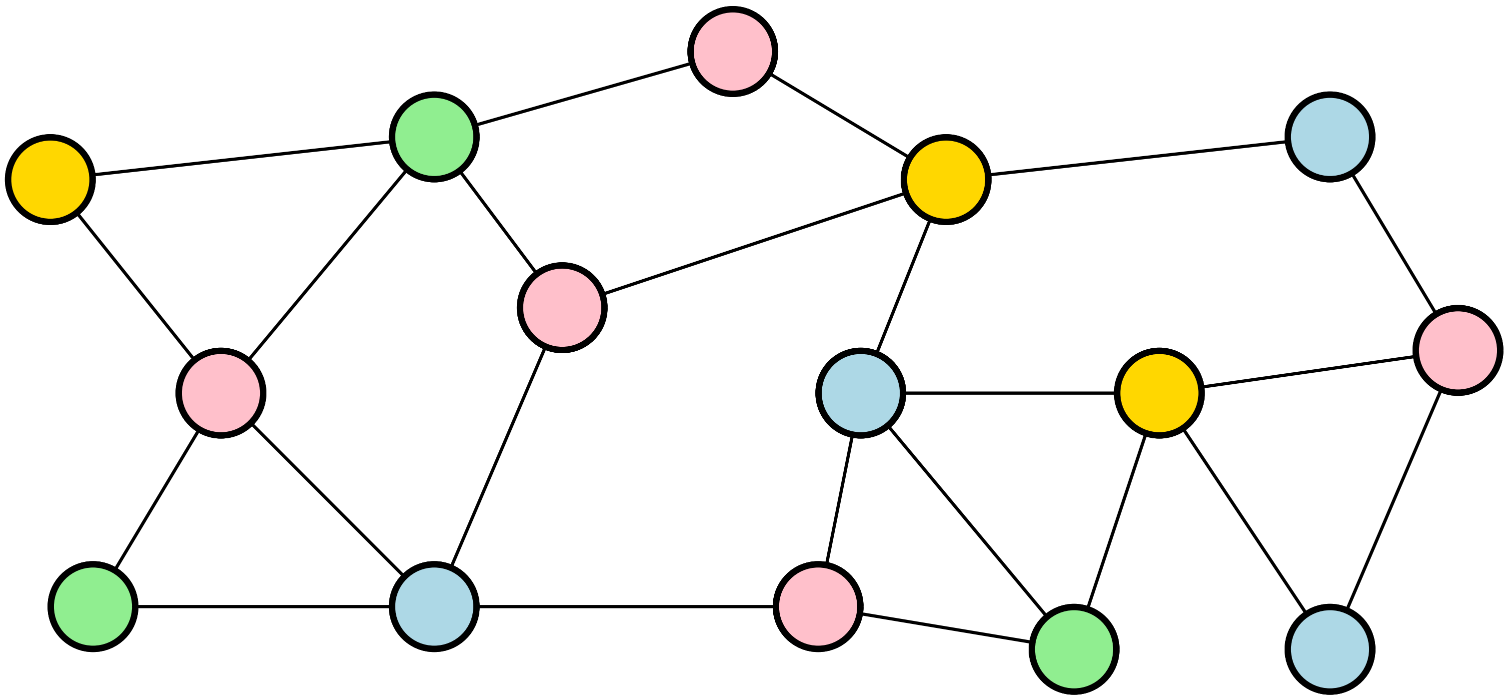

$\varphi(G) \leq 4$

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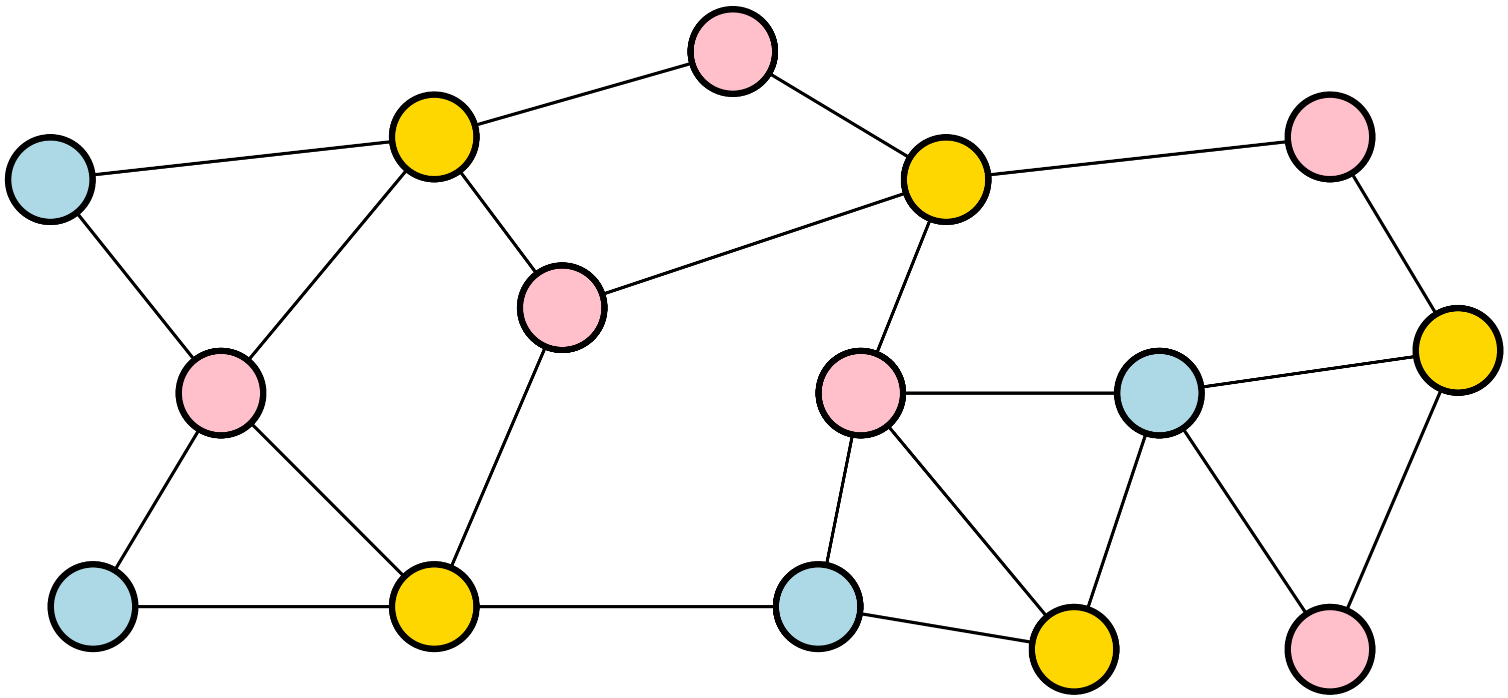

$\varphi(G) \leq 3$

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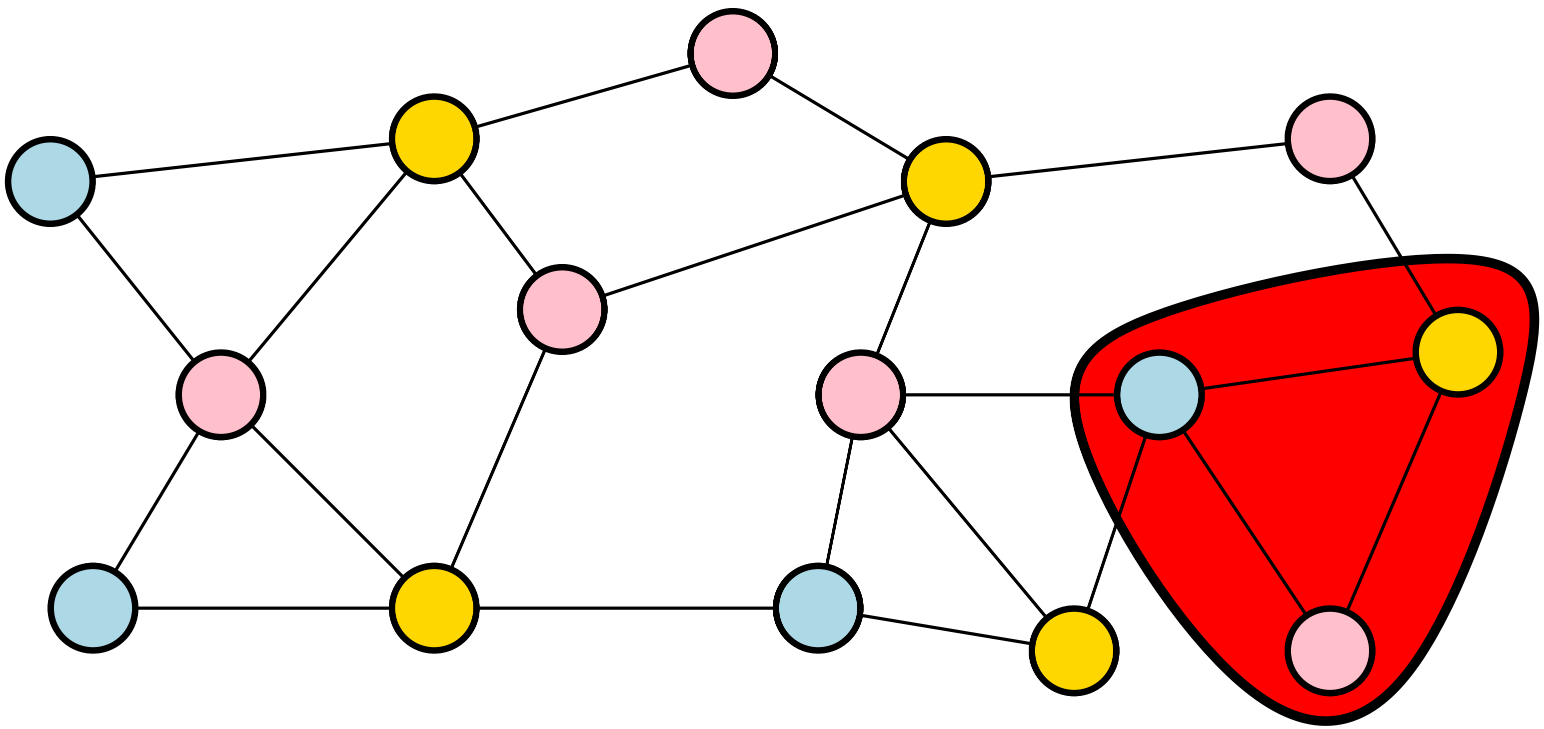

$\varphi(G) \leq 3$ $\varphi(G) \geq 3$

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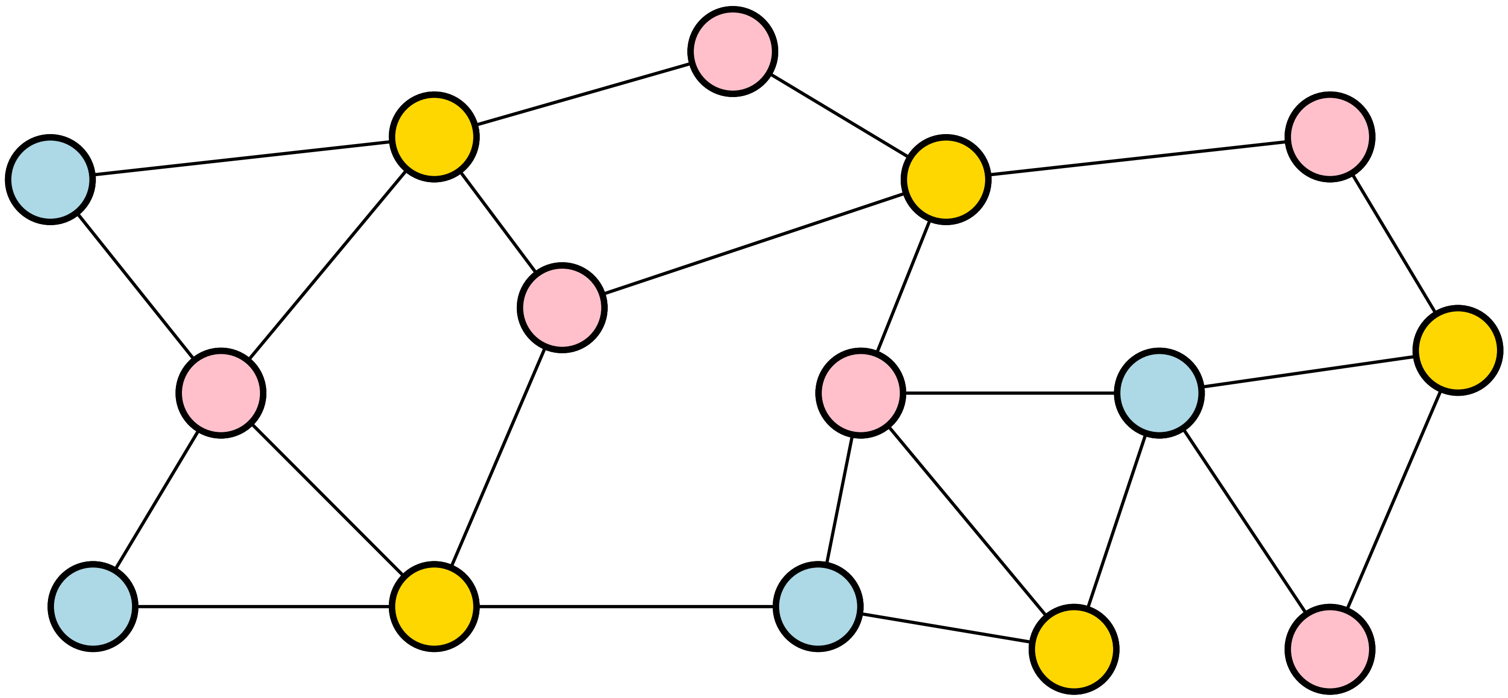

$$\varphi(G) = 3$$

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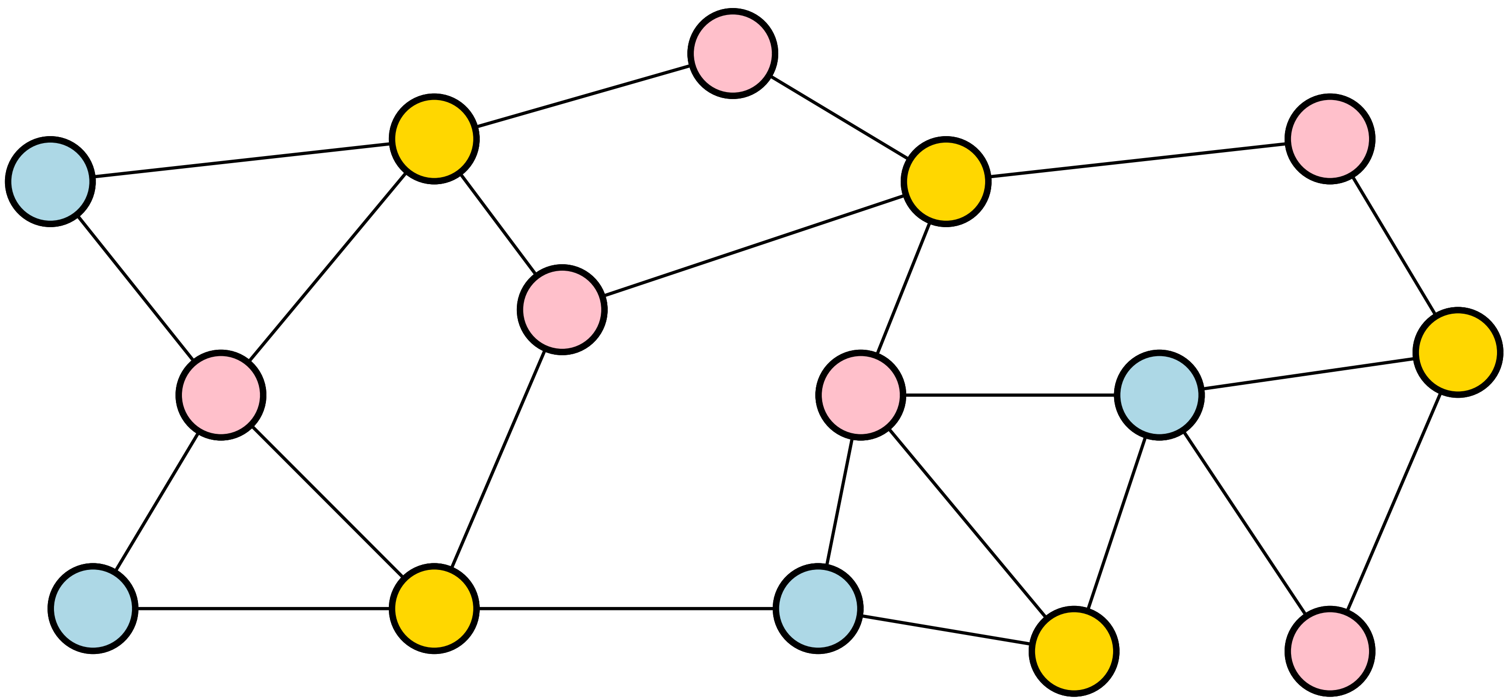

If P $\neq$ NP, the chromatic number of a graph cannot be approximated within a factor of $n^{1-\epsilon}$, for any constant $\epsilon > 0$.

(Where $n = |V|$)

A $(\Delta + 1)$-coloring

Given $v \in V$, let $\delta(v)$ be the degree of v in G.

Let $\Delta = \max_{v \in V} \delta(v)$ be the **maximum degree** of G.

Claim: Any graph G admits a $(\Delta + 1)$-coloring, i.e.,

$$\varphi(G) \leq \Delta + 1$$

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Claim: Any graph G admits a $(\Delta + 1)$-coloring, i.e.,

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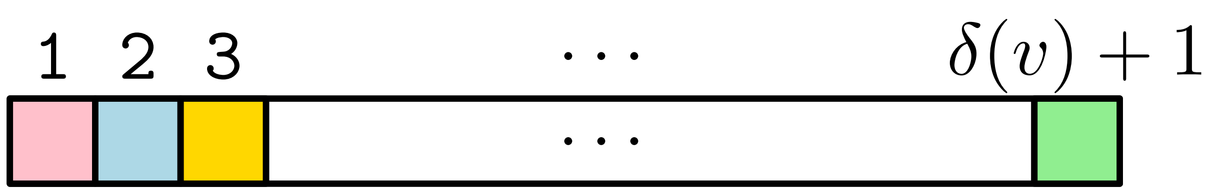
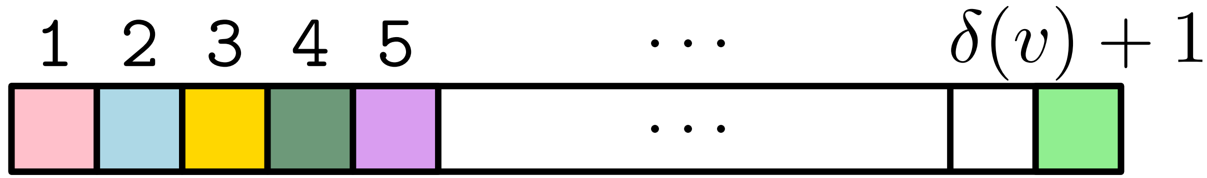
Proof: we will show an algorithm that computes a $\Delta + 1$ coloring.

Each node v keeps a *palette* of $\delta(v) + 1$ avaliable colors: $1, 2, \ldots, \delta(v) + 1$.



Computing a $(\Delta + 1)$-coloring

v's palette:



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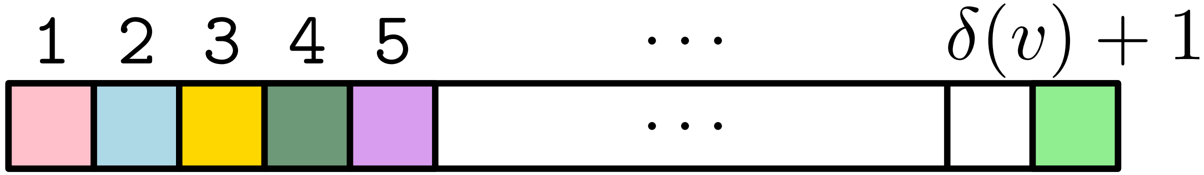
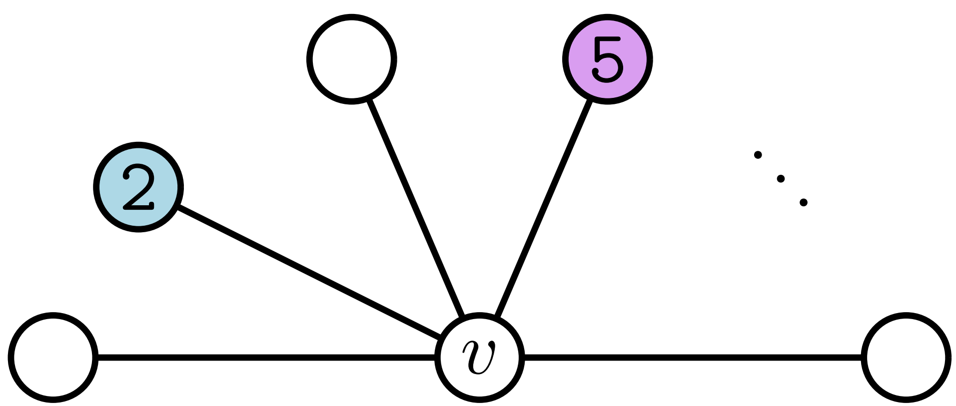


If r neighbors of v have already been colored:



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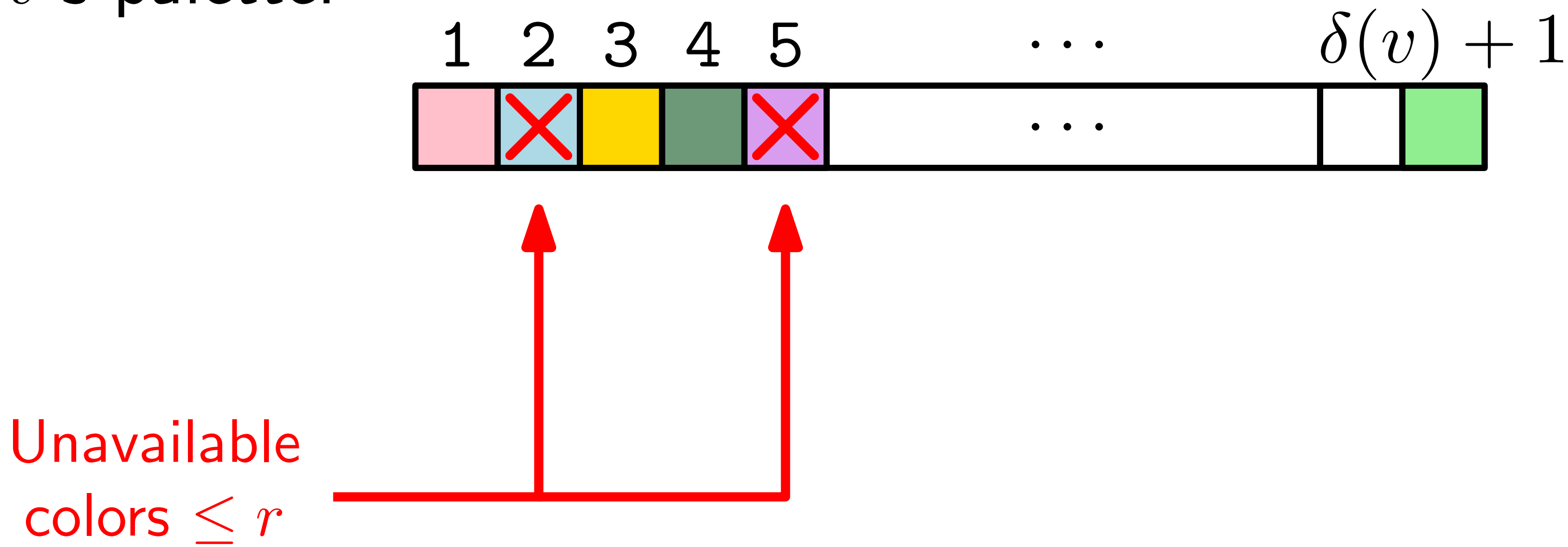

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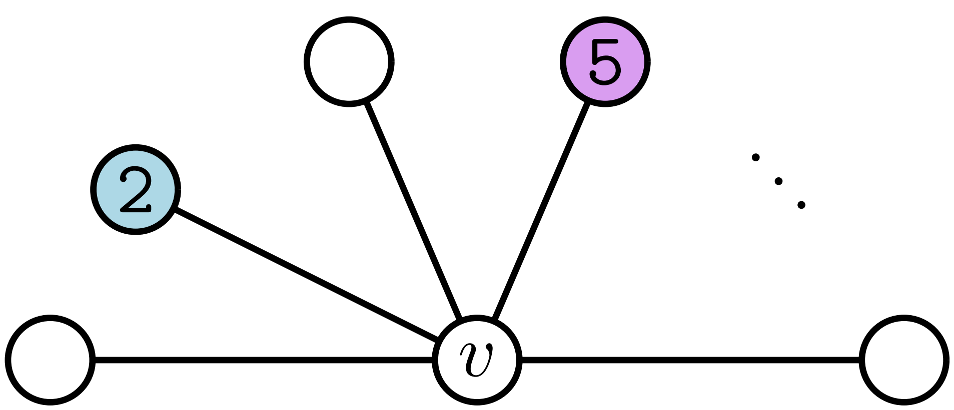

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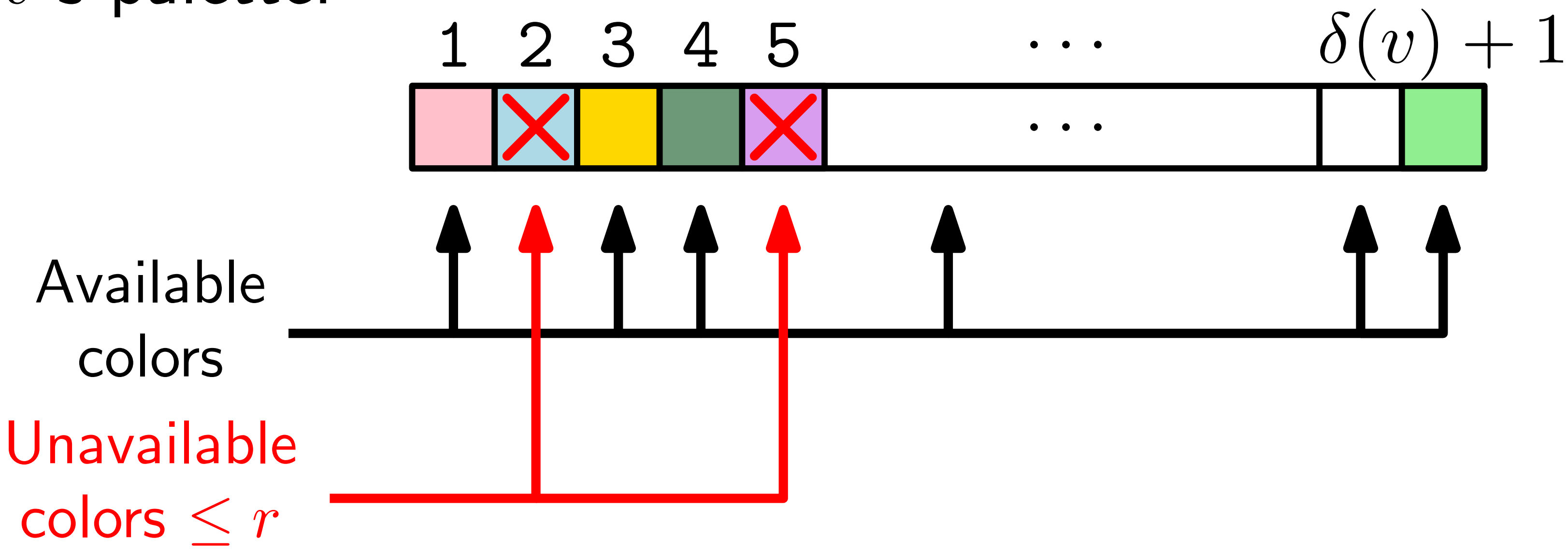

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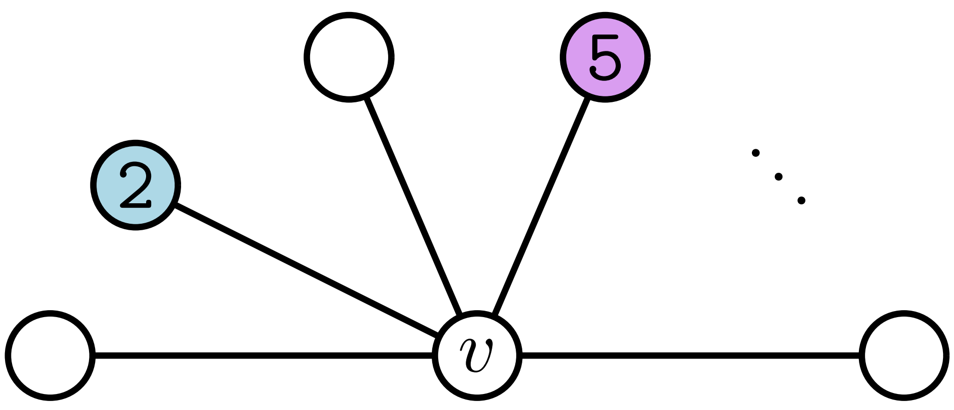

Computing a $(\Delta + 1)$-coloring

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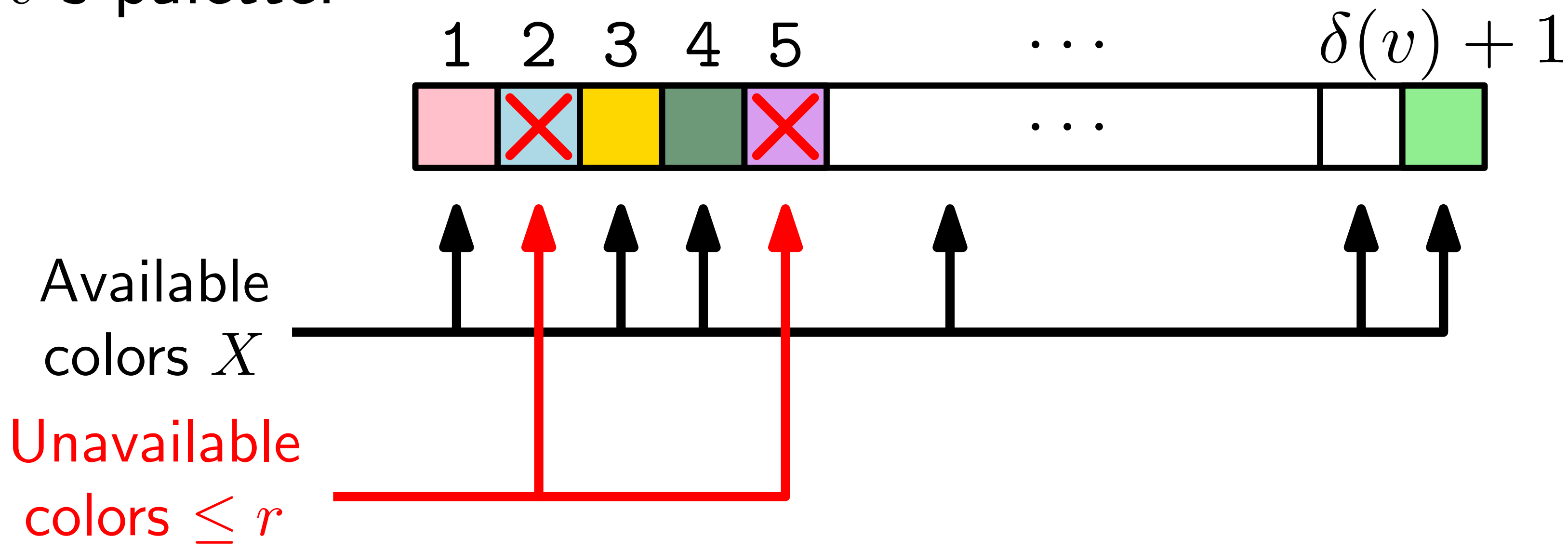
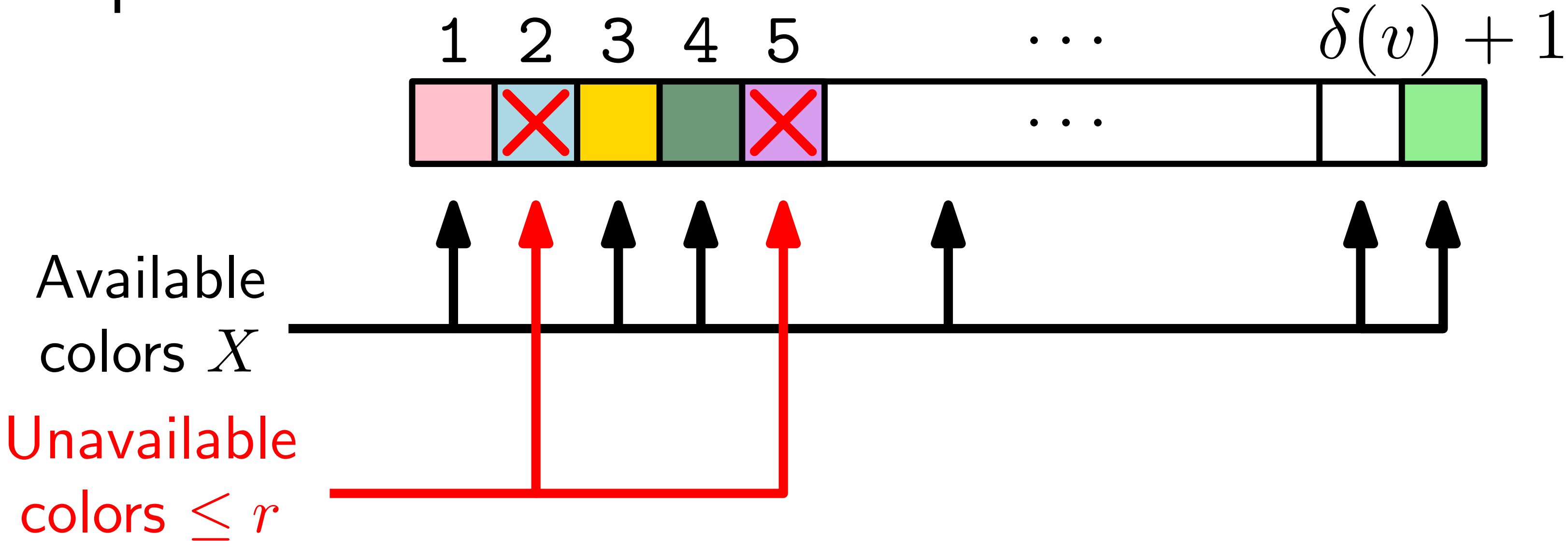


Number of available colors in v's palette:

$$X = \delta(v) + 1 - r$$

Computing a $(\Delta + 1)$-coloring

v's palette:



Number of available colors in v's palette:

$$X = \delta(v) + 1 - r \geq \delta(v) + 1 - \delta(v)$$

Computing a $(\Delta + 1)$-coloring

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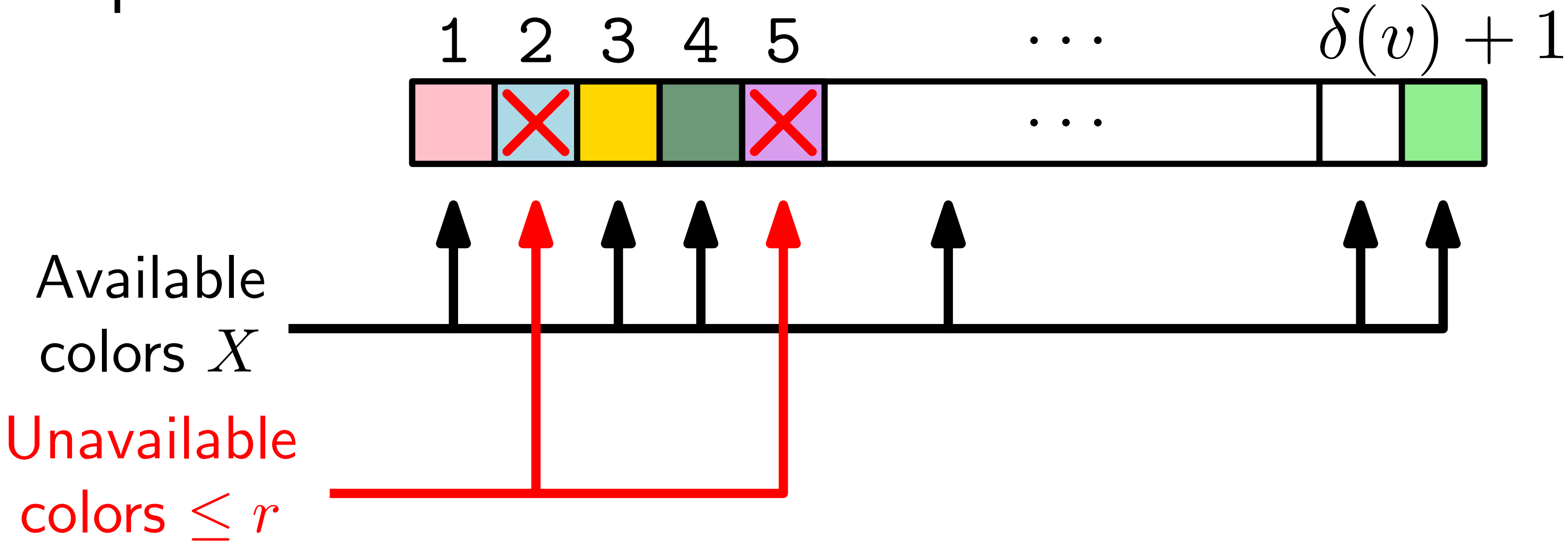


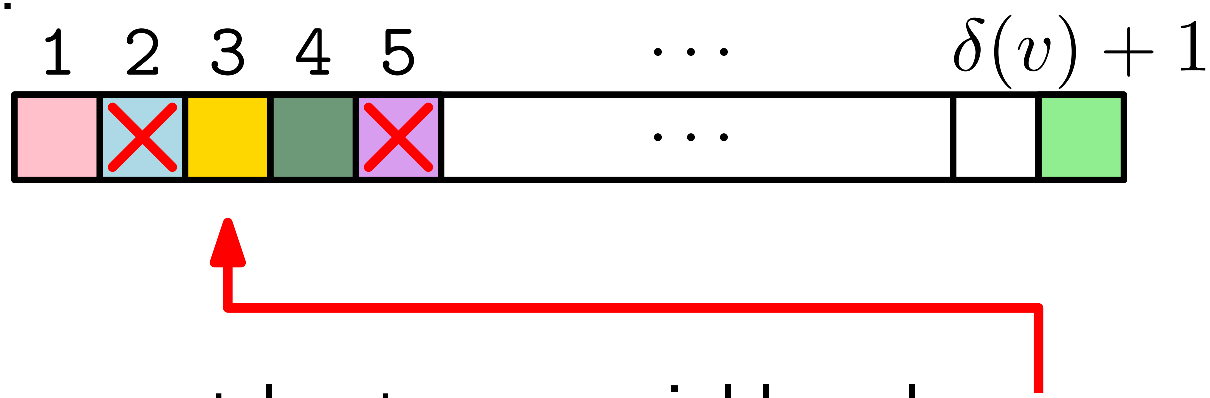
Number of available colors in v's palette:

$$X = \delta(v) + 1 - r \geq \delta(v) + 1 - \delta(v) = 1$$

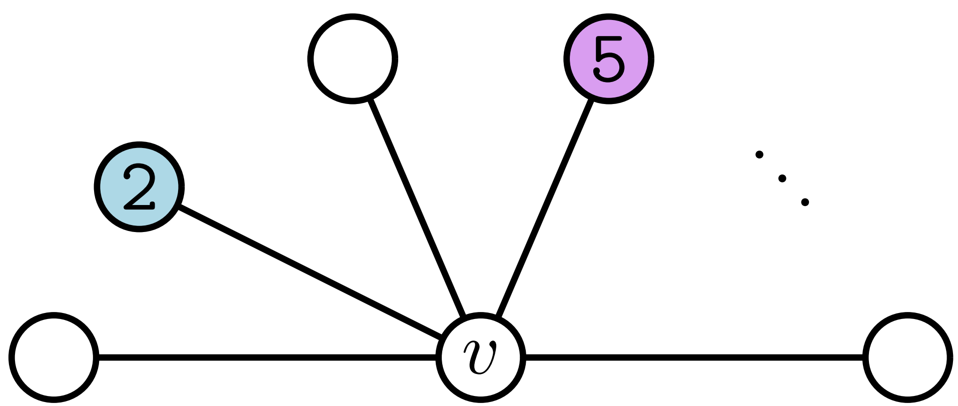
There is always at least one available color.

Computing a $(\Delta + 1)$-coloring

v's palette:



There is always at least one avaiable color c.



Computing a $(\Delta + 1)$-coloring

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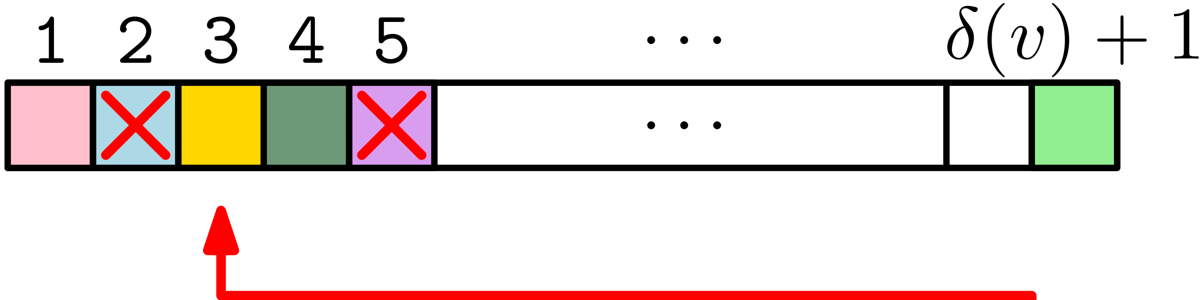

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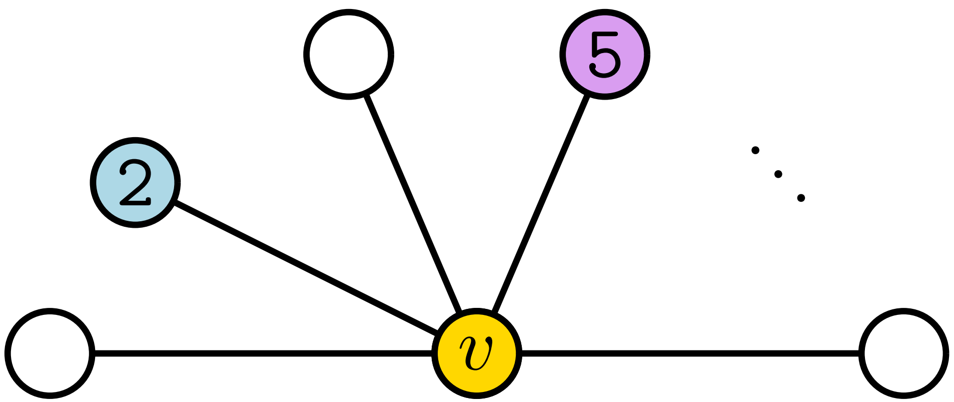


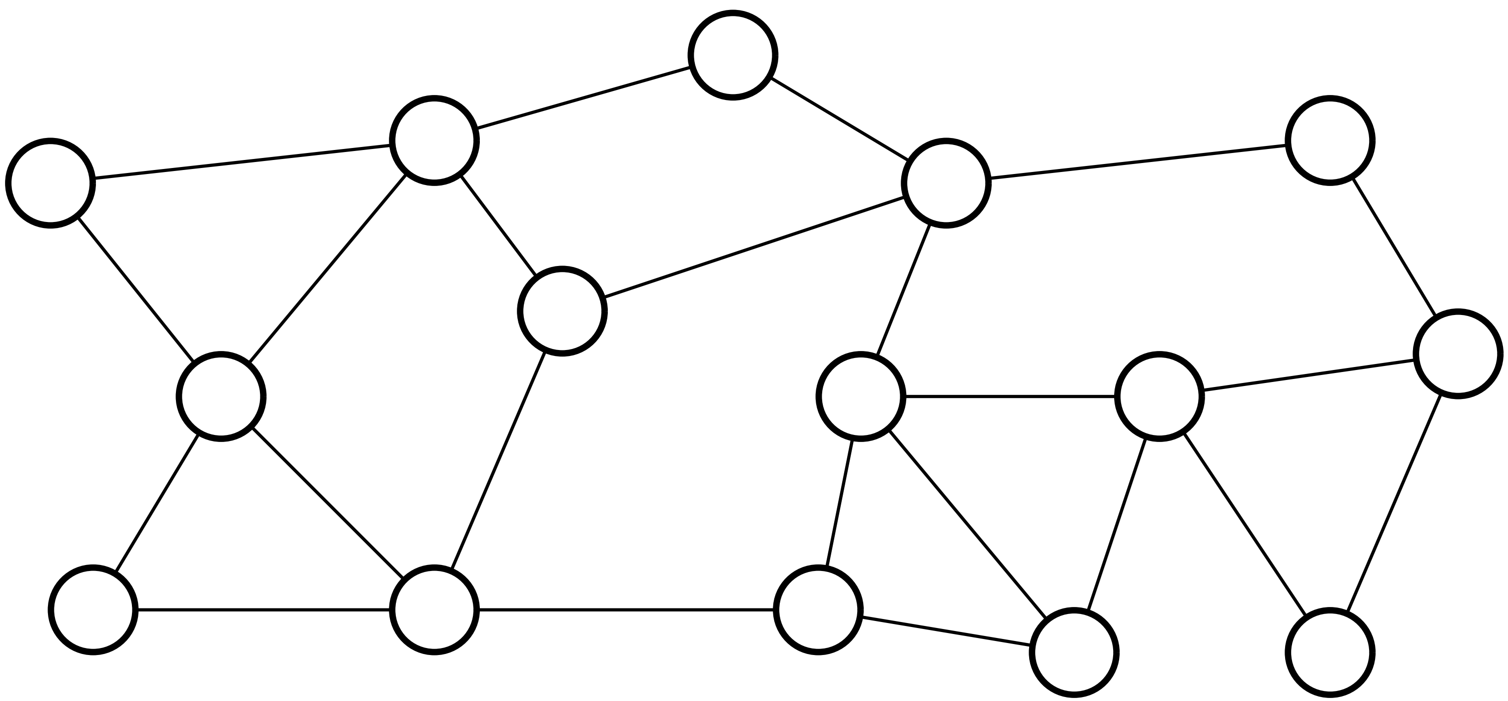
Vertex v can pick color c for itself!

Sequential $(\Delta + 1)$-coloring algorithm

- For each node v: create a palette of $\delta(v) + 1$ colors.
- Mark all colors (in all palettes) as available.
- While $\exists$ uncolored node v:
  - Let c be any available color from v's palette
    (recall that such a color always exists)
  - Color v with color c
  - For every neighbor u of v:
    - Mark c as unavailable in u's palette

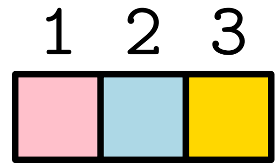
$\square$

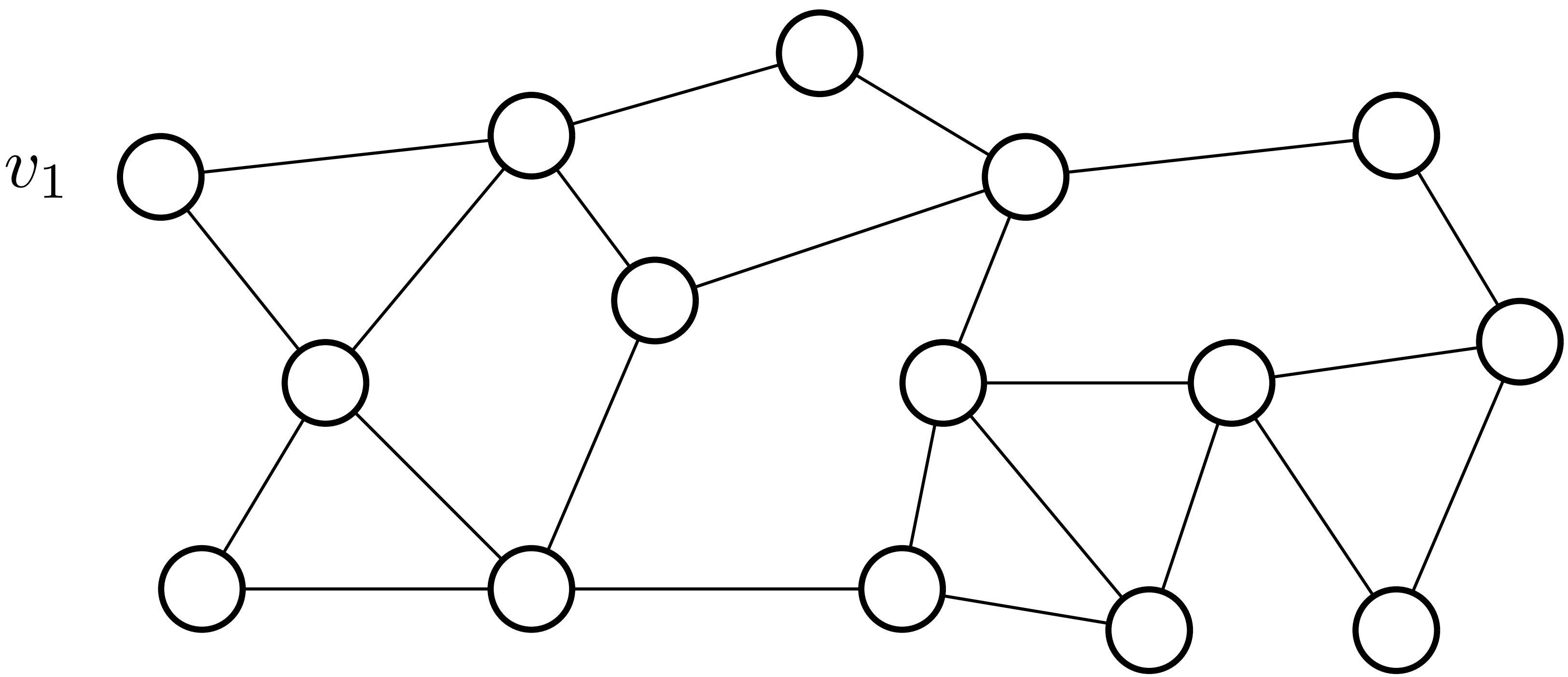
A sample execution



A sample execution

v_1's palette

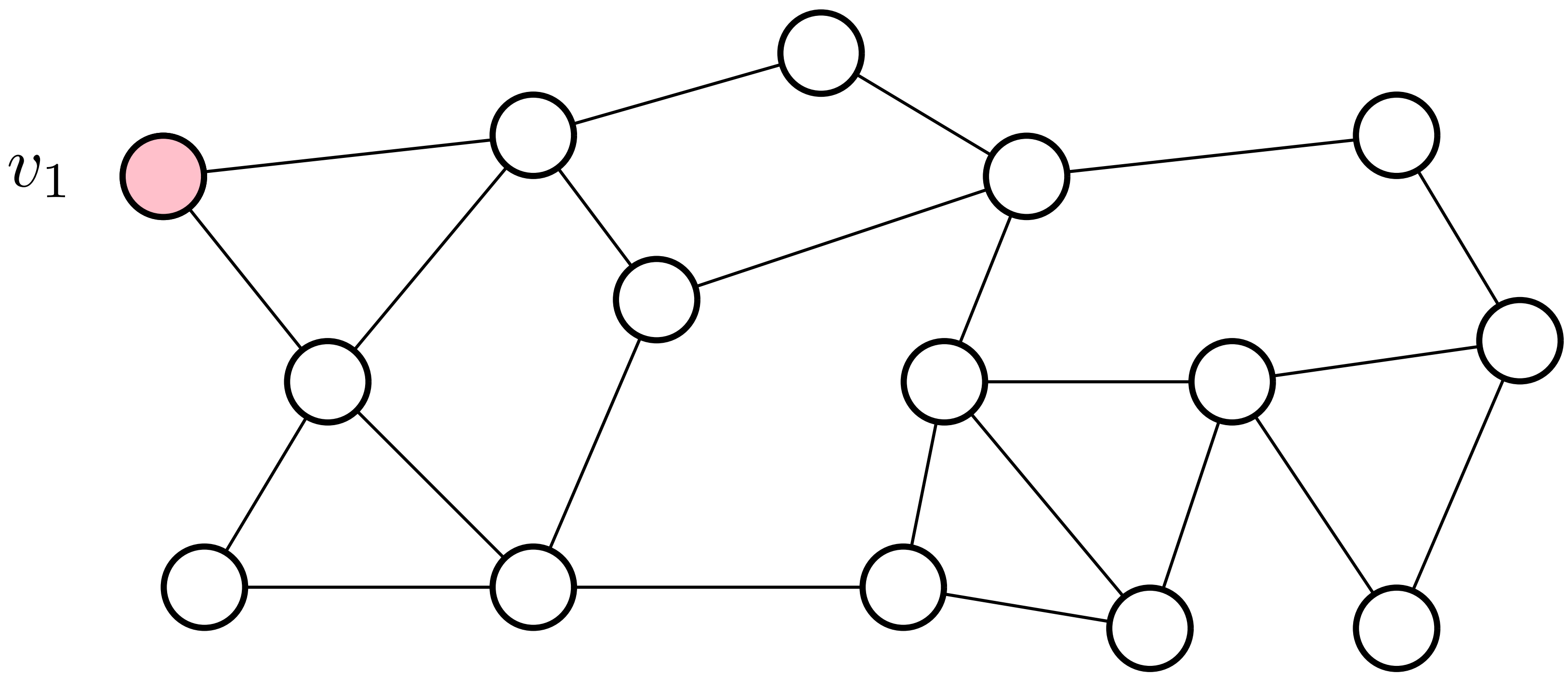




A sample execution

v_1's palette

1	2	3



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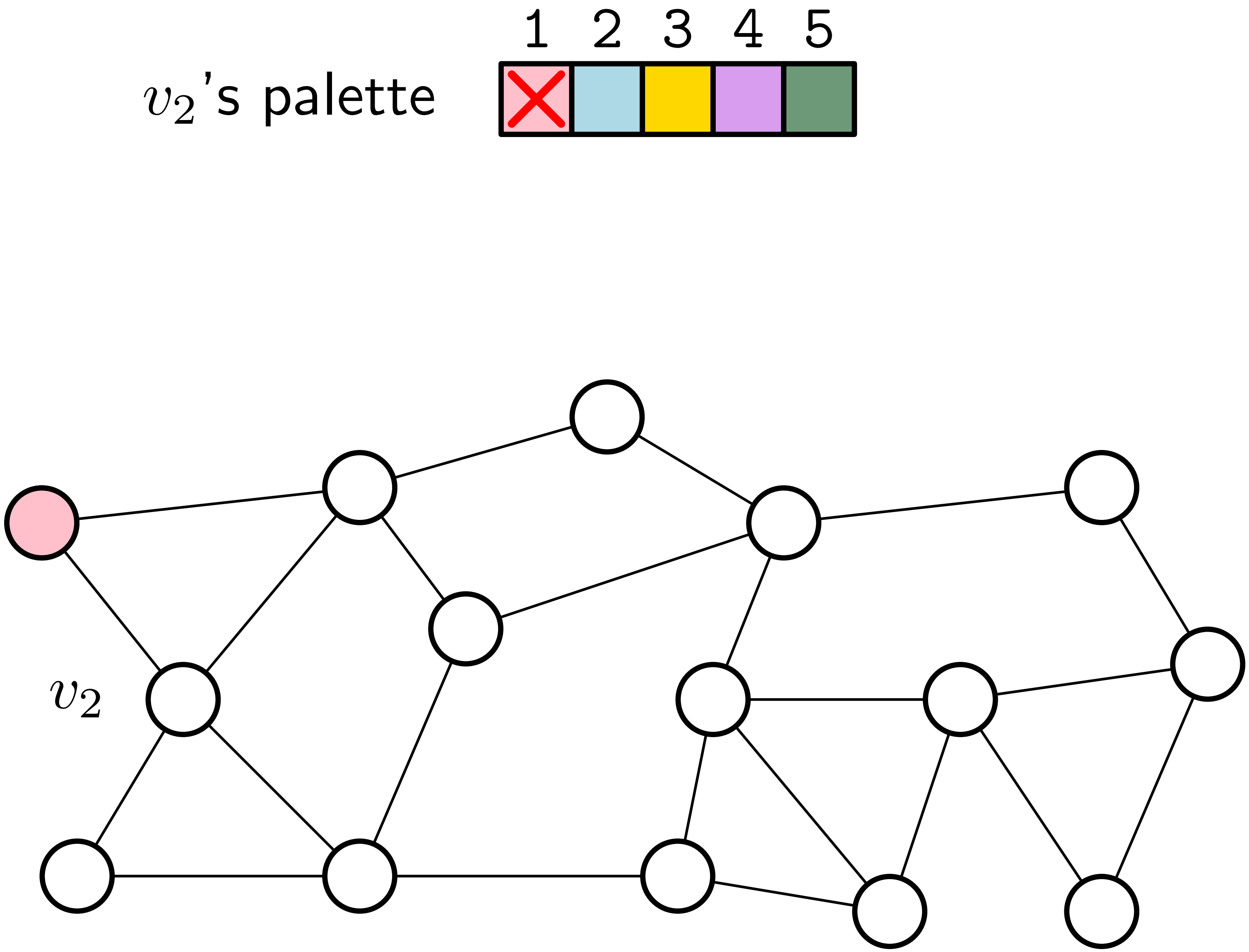

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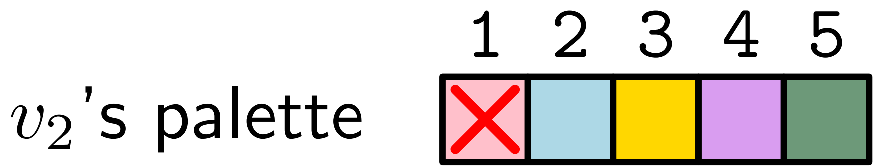
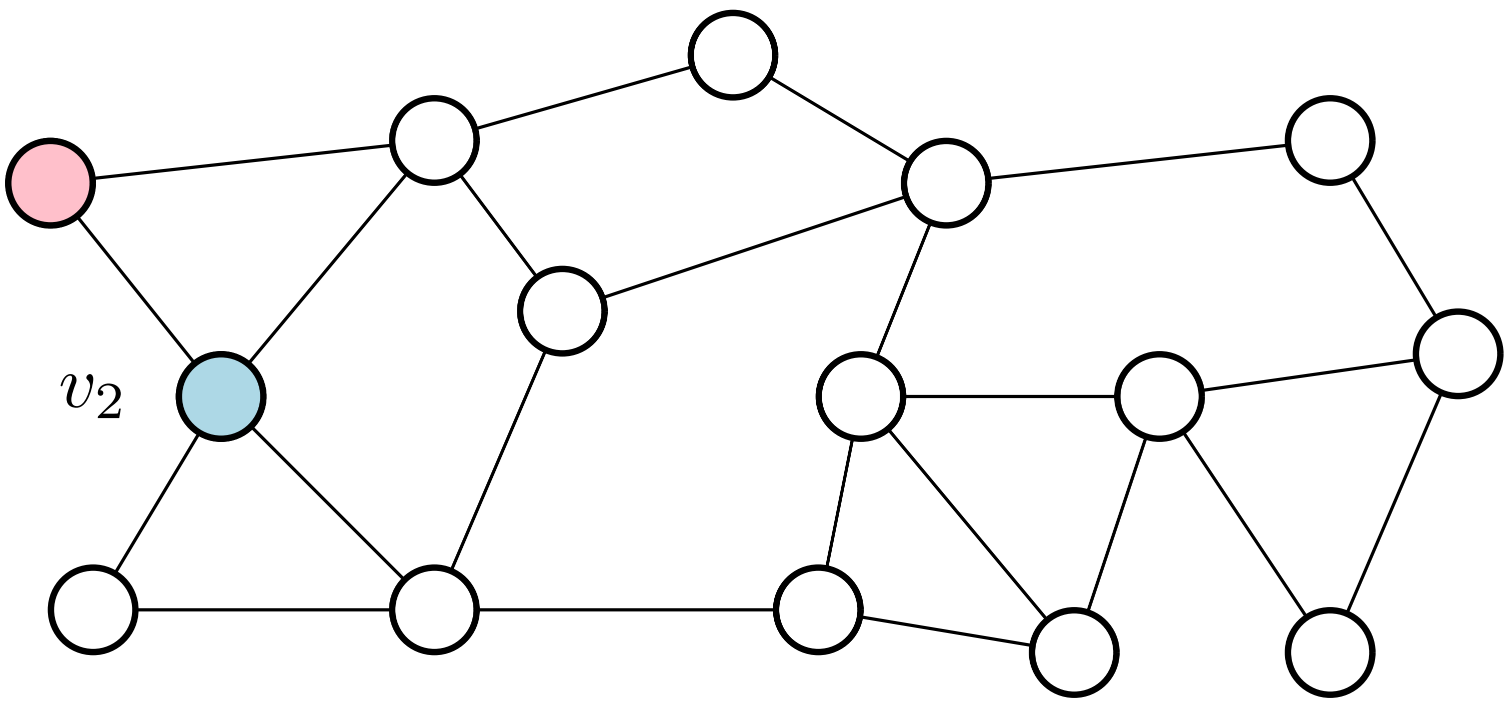
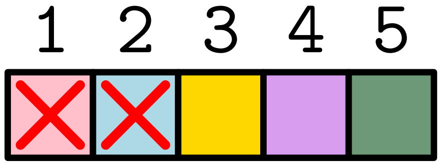
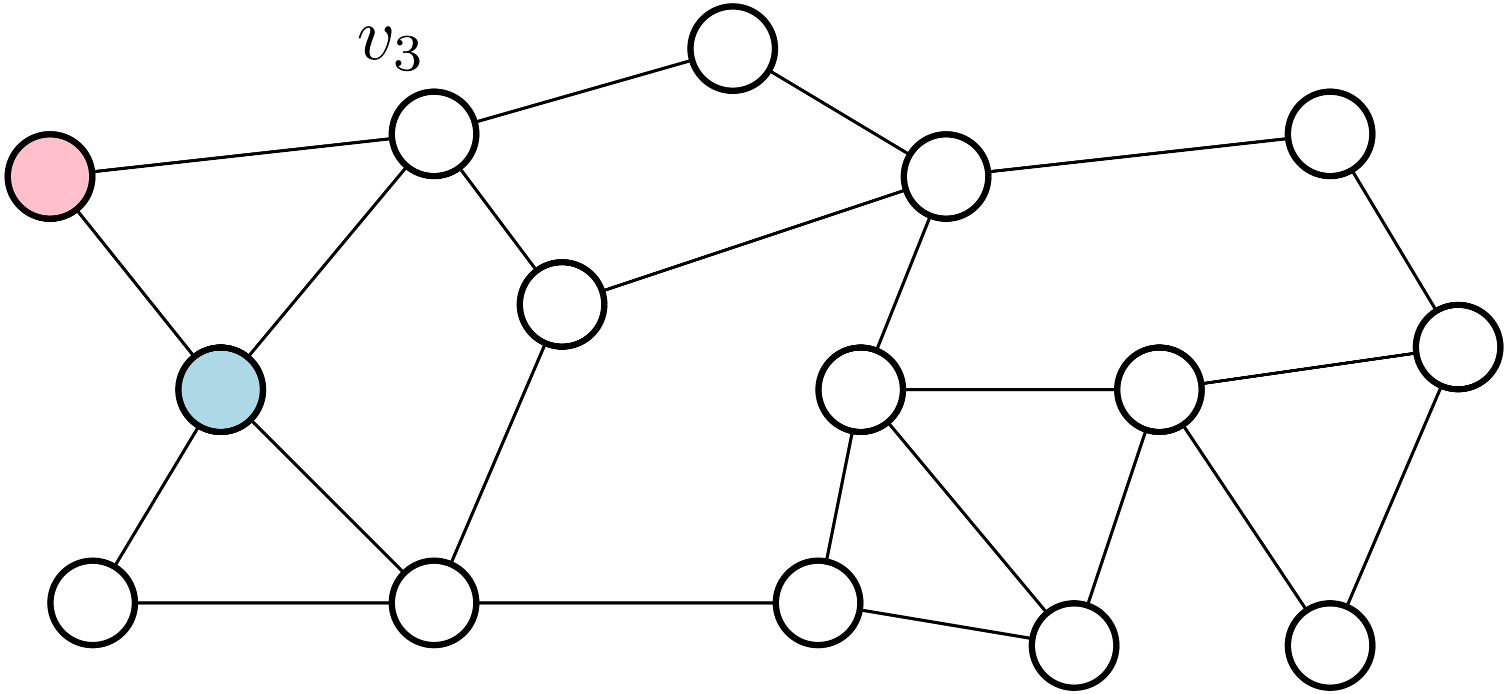




A sample execution

v_3's palette





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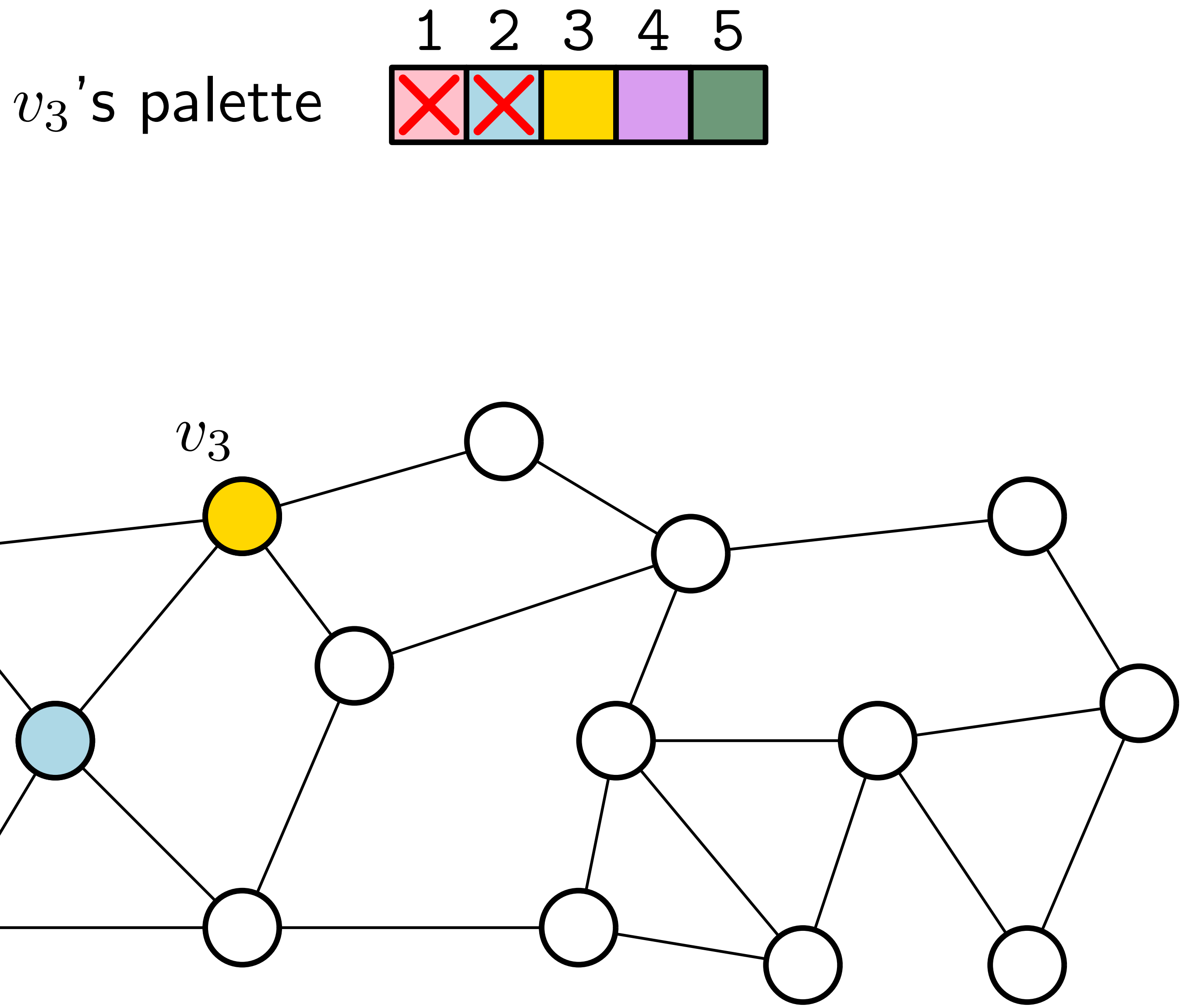

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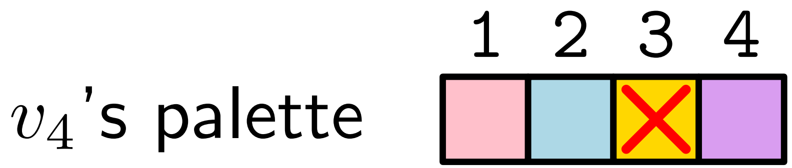
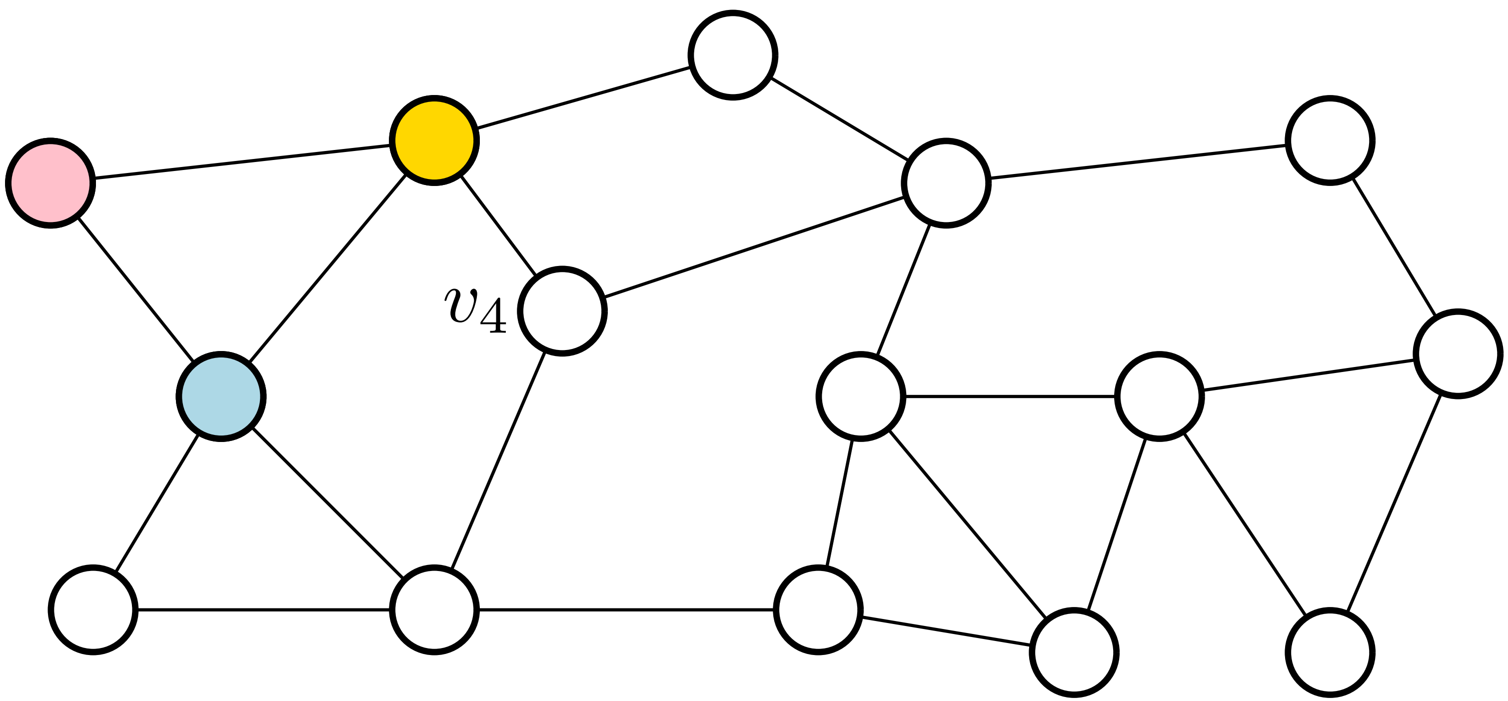
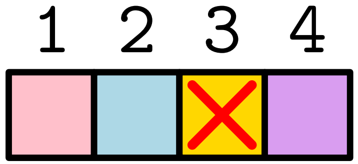
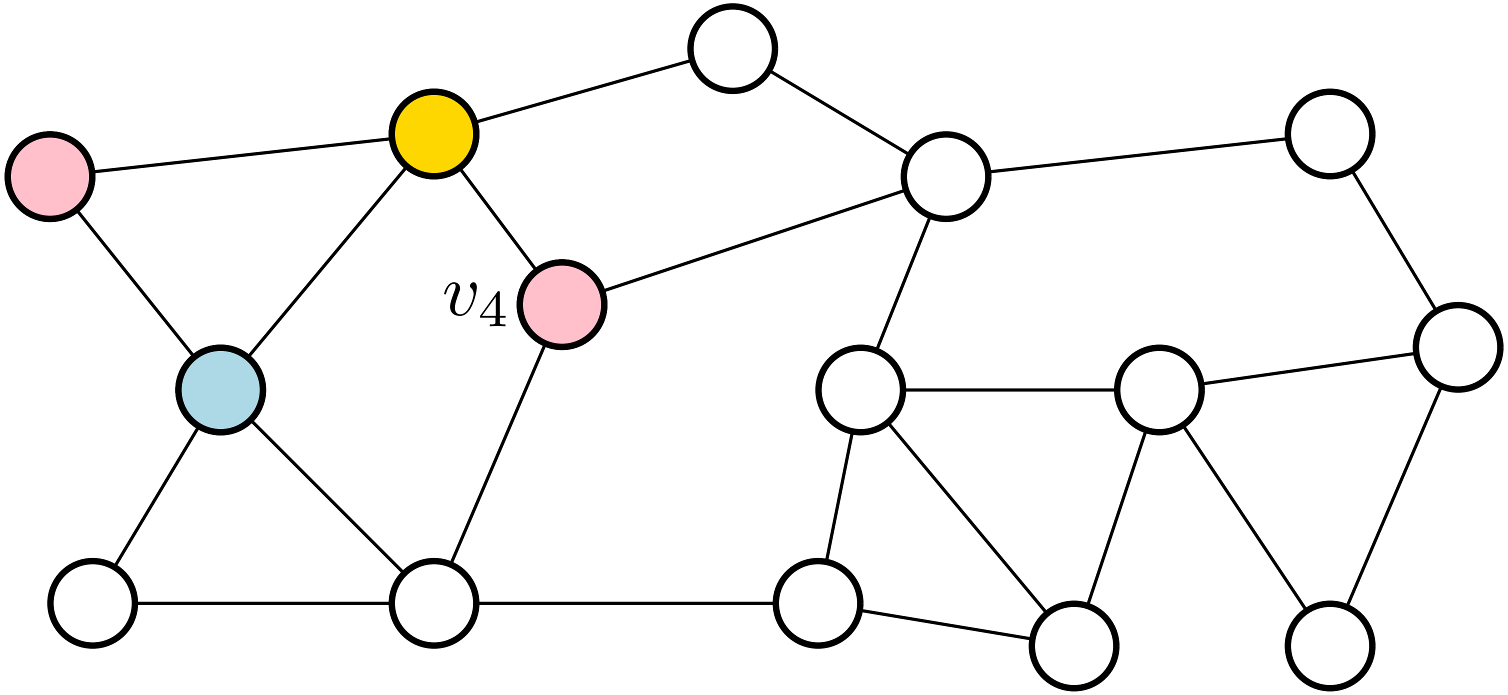




A sample execution

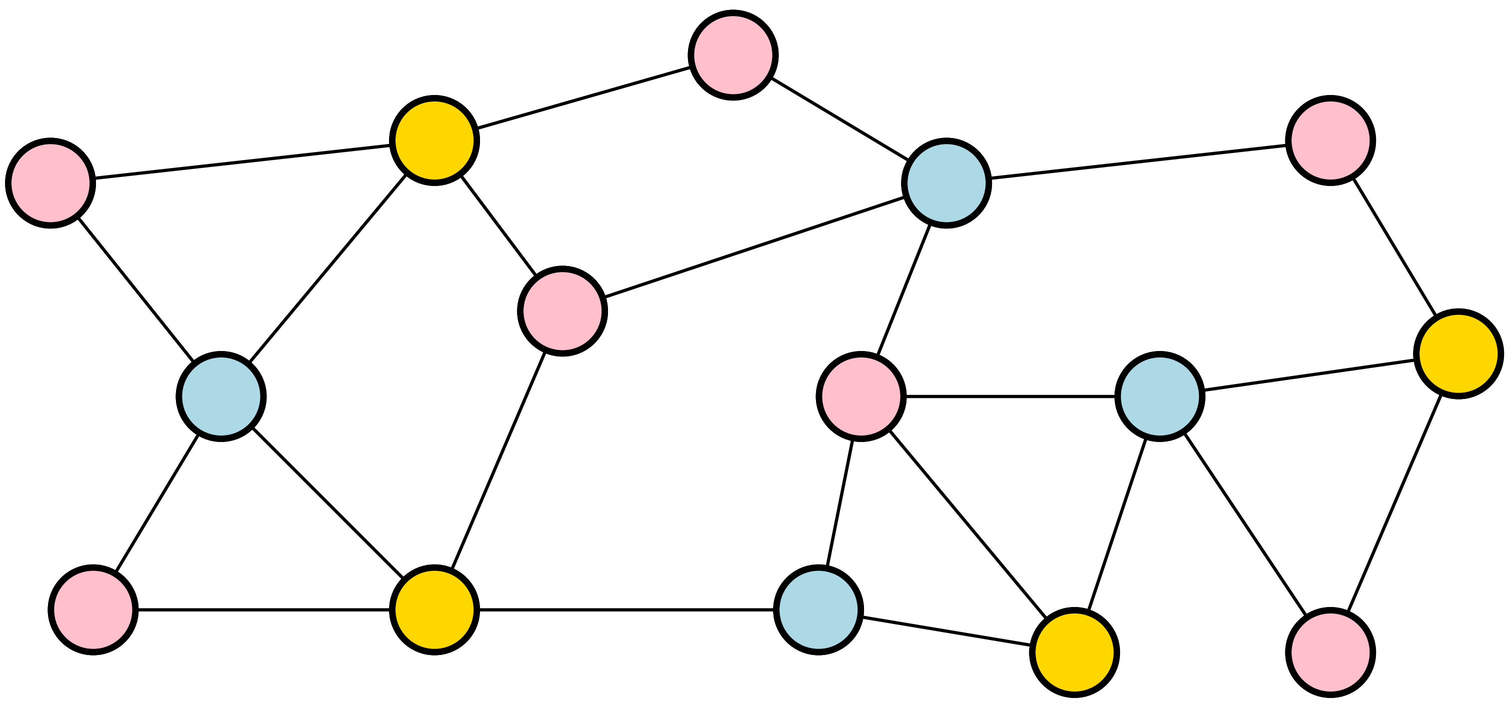
v_4's palette





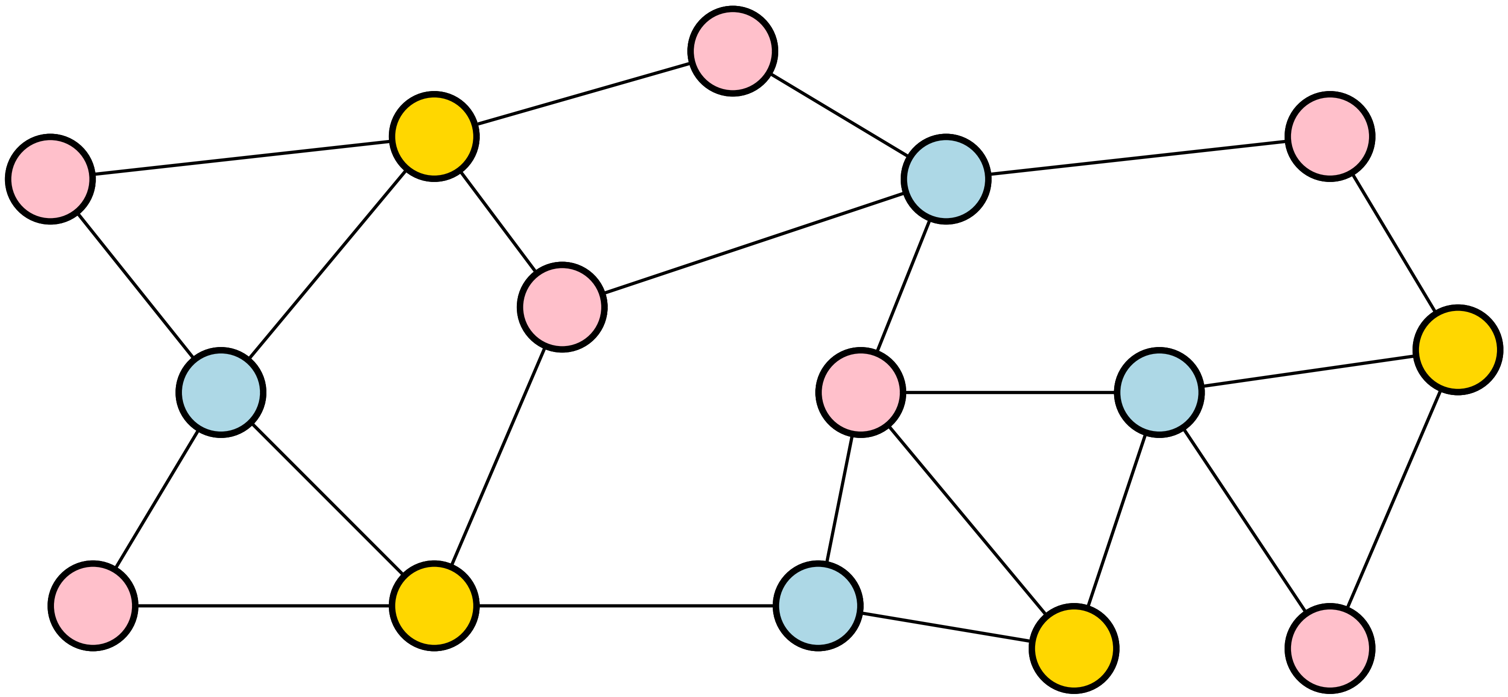
A sample execution

Termination



Coloring and Independent Sets

In any valid coloring, the nodes of the same color form an independent set.



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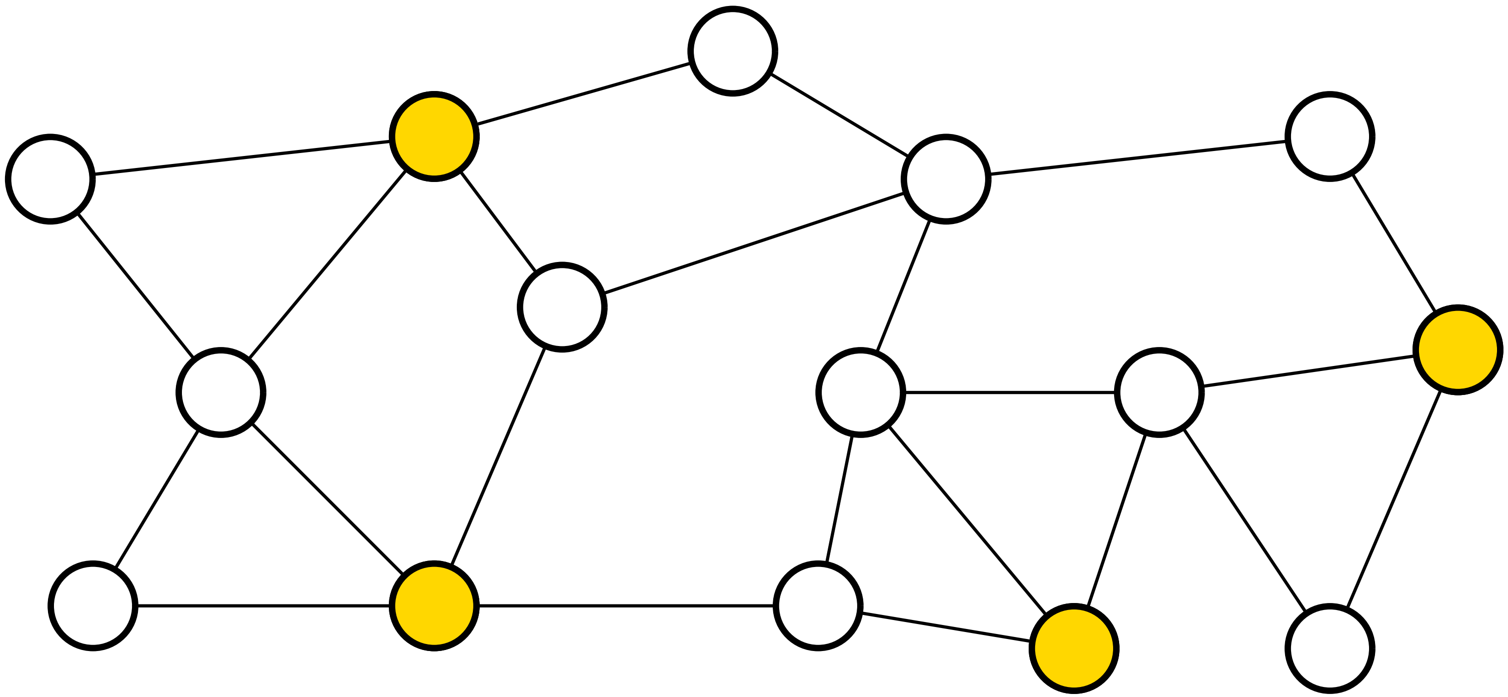

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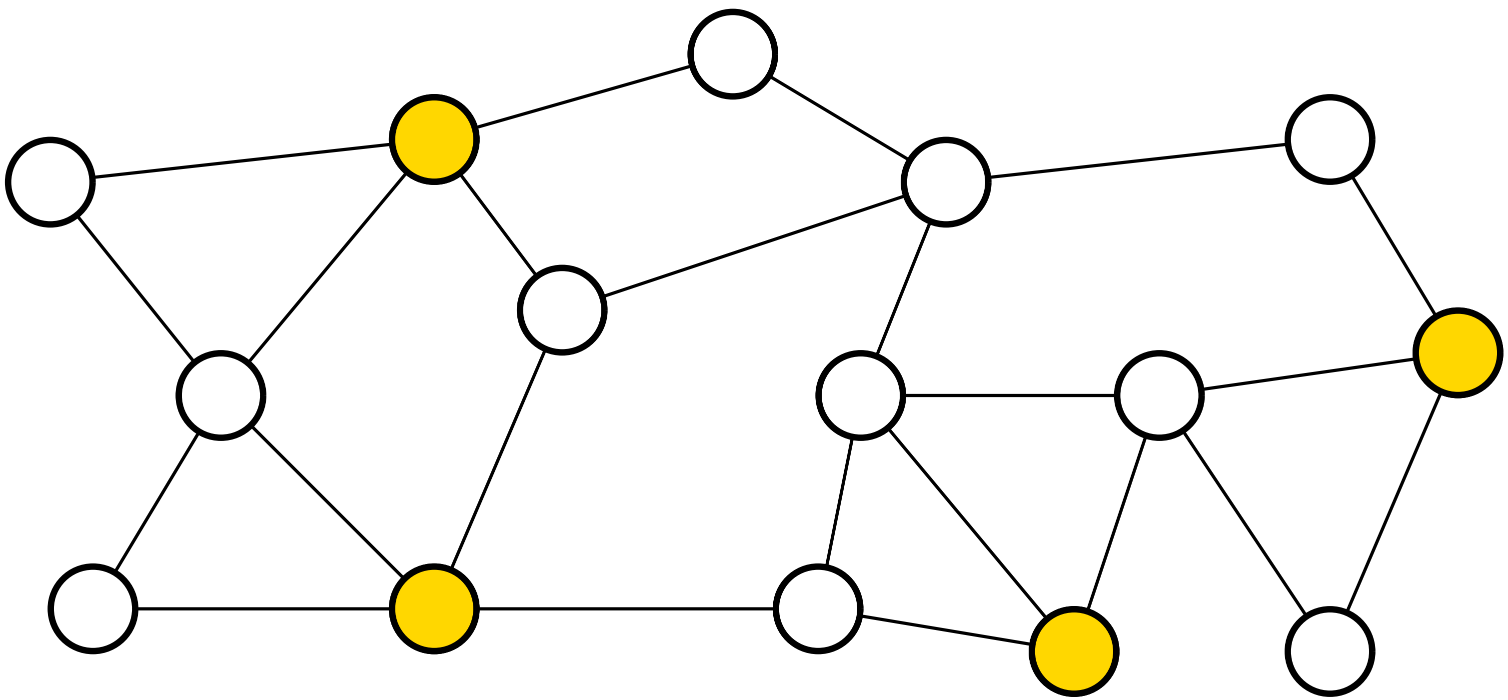

Such an independent set is not necessarily maximal

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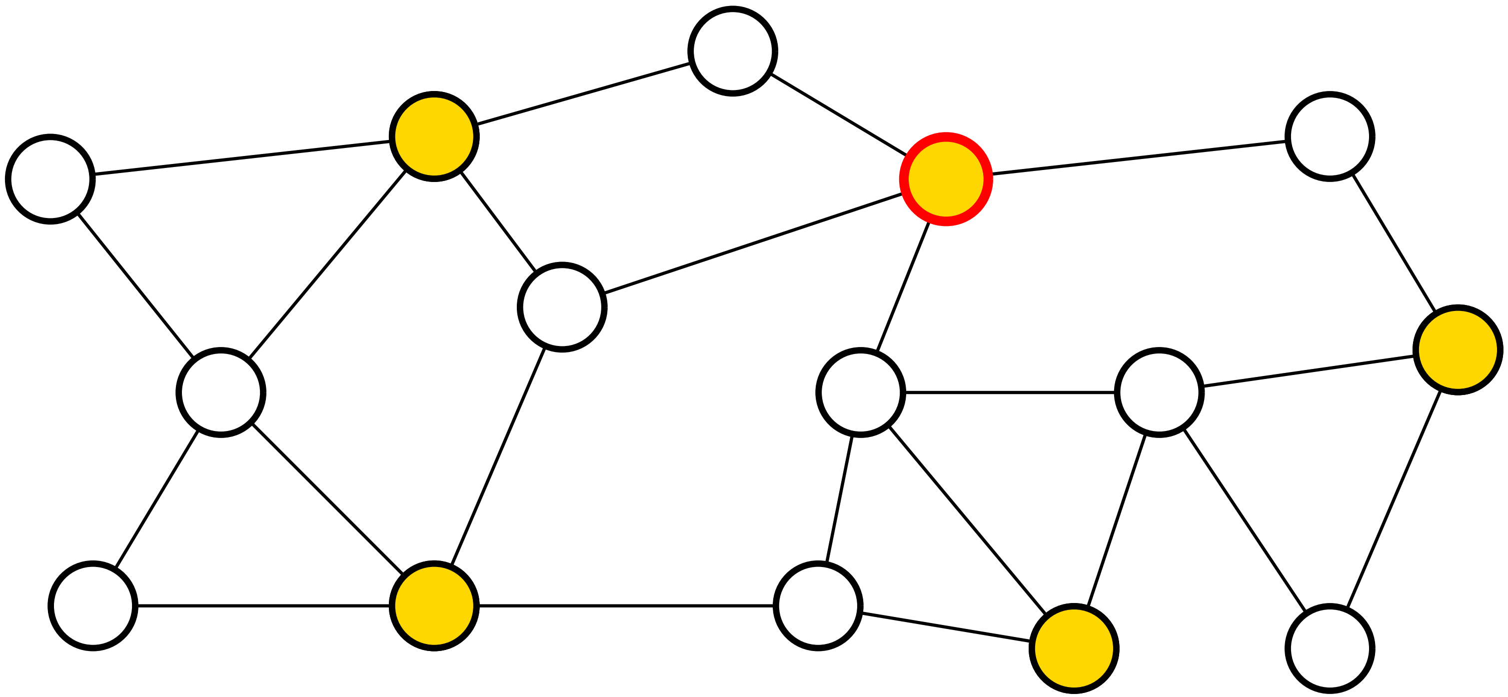


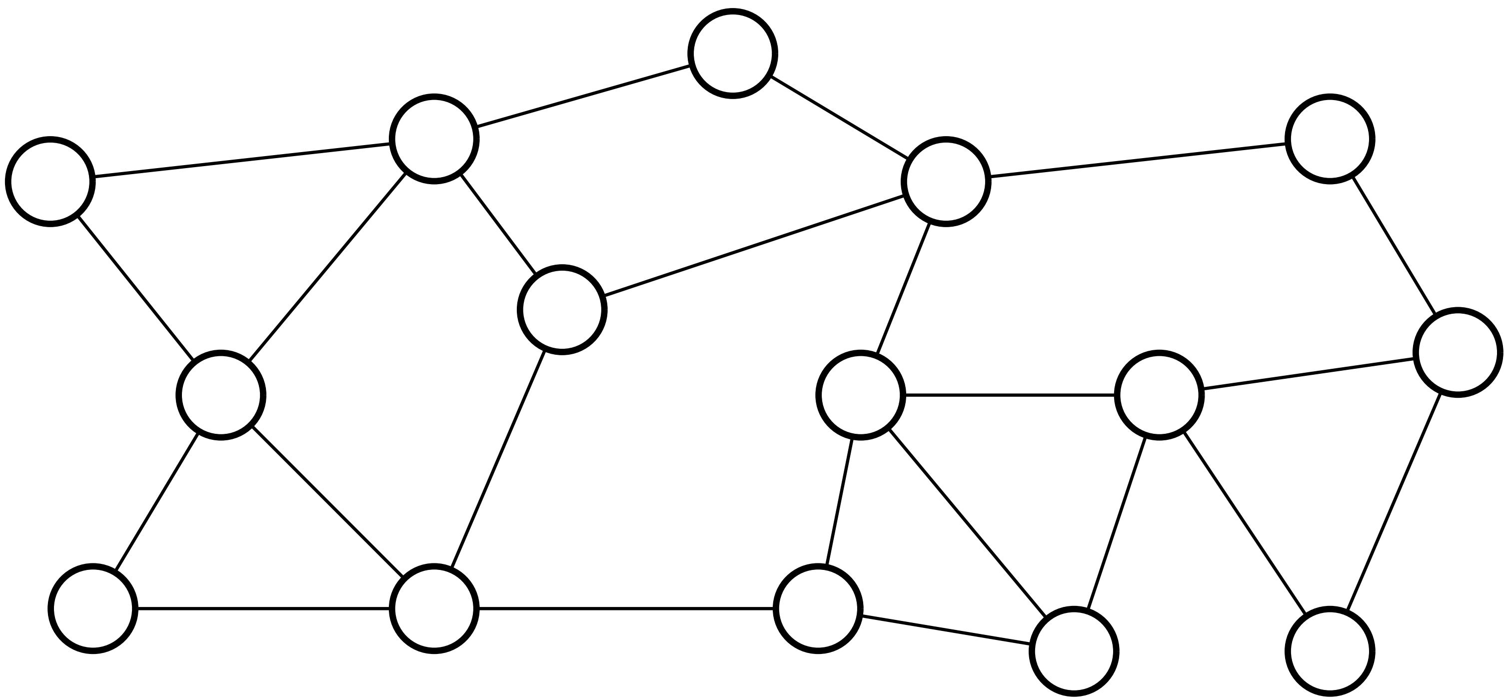
Such an independent set is not necessarily maximal

Coloring via Maximal Independent Sets

- $c \leftarrow 1$
- While $\exists$ uncolored nodes in G:
  - Find a MIS $\mathcal{I}$ of the subraph of G induced by the *uncolored* nodes.
  - Assign color c to all nodes in $\mathcal{I}$
  - $c \leftarrow c + 1$

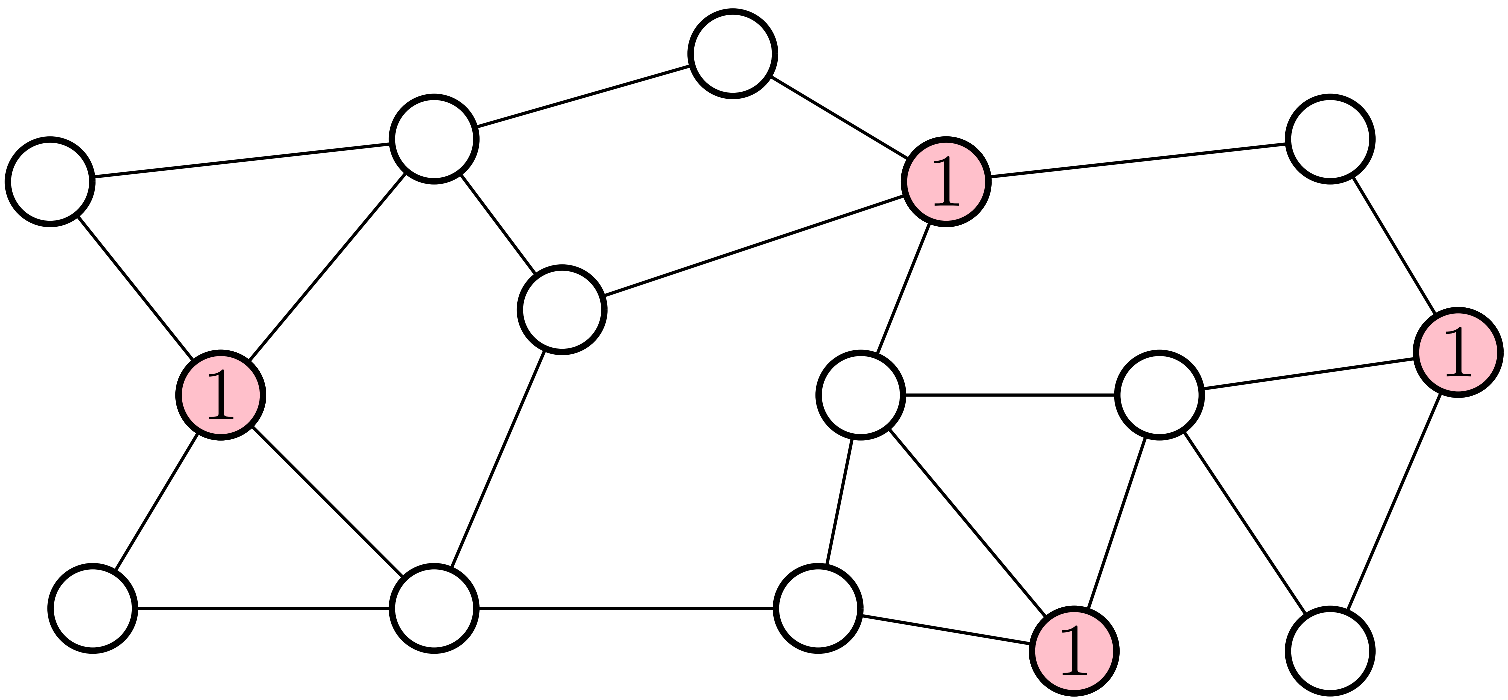
A sample execution

Initially all nodes are uncolored



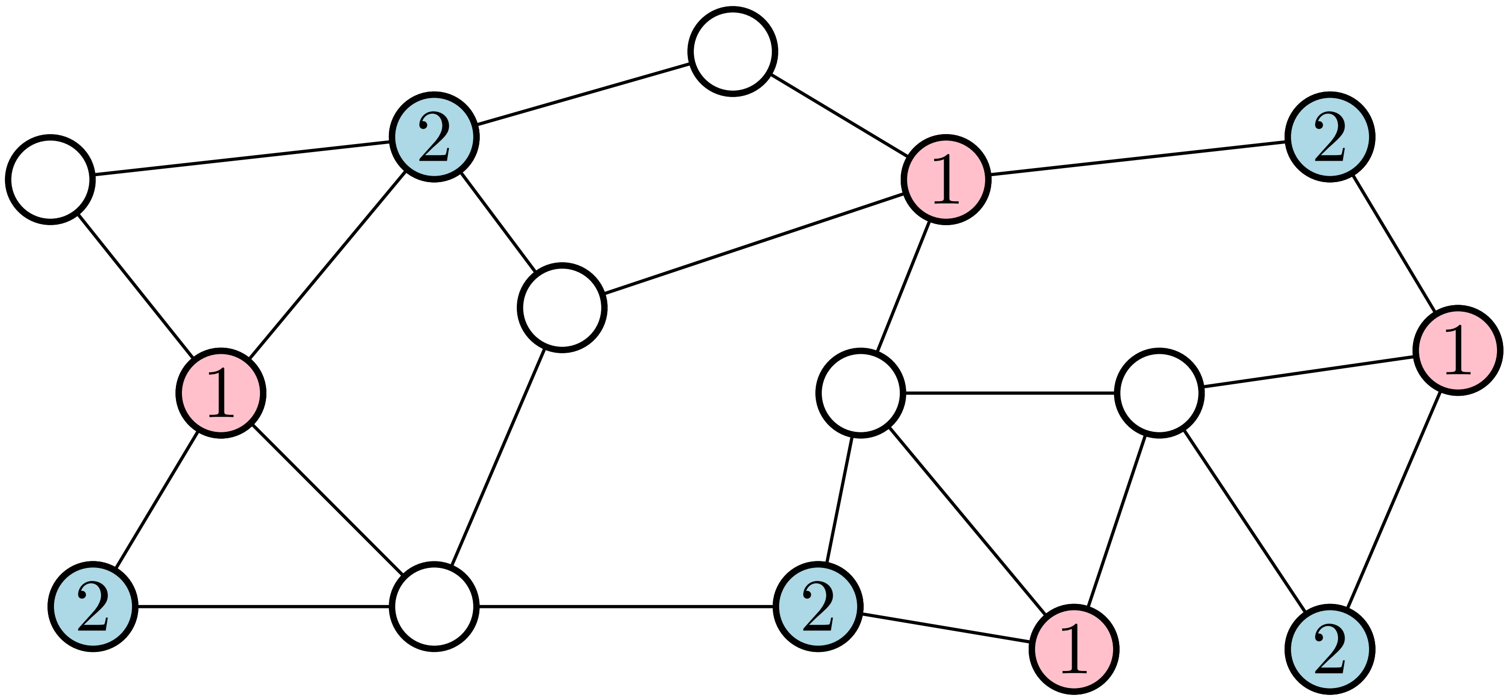
A sample execution

Iteration 1: Find a MIS $\mathcal{I}$ of the uncolored nodes and assign color 1 to the nodes in $\mathcal{I}$



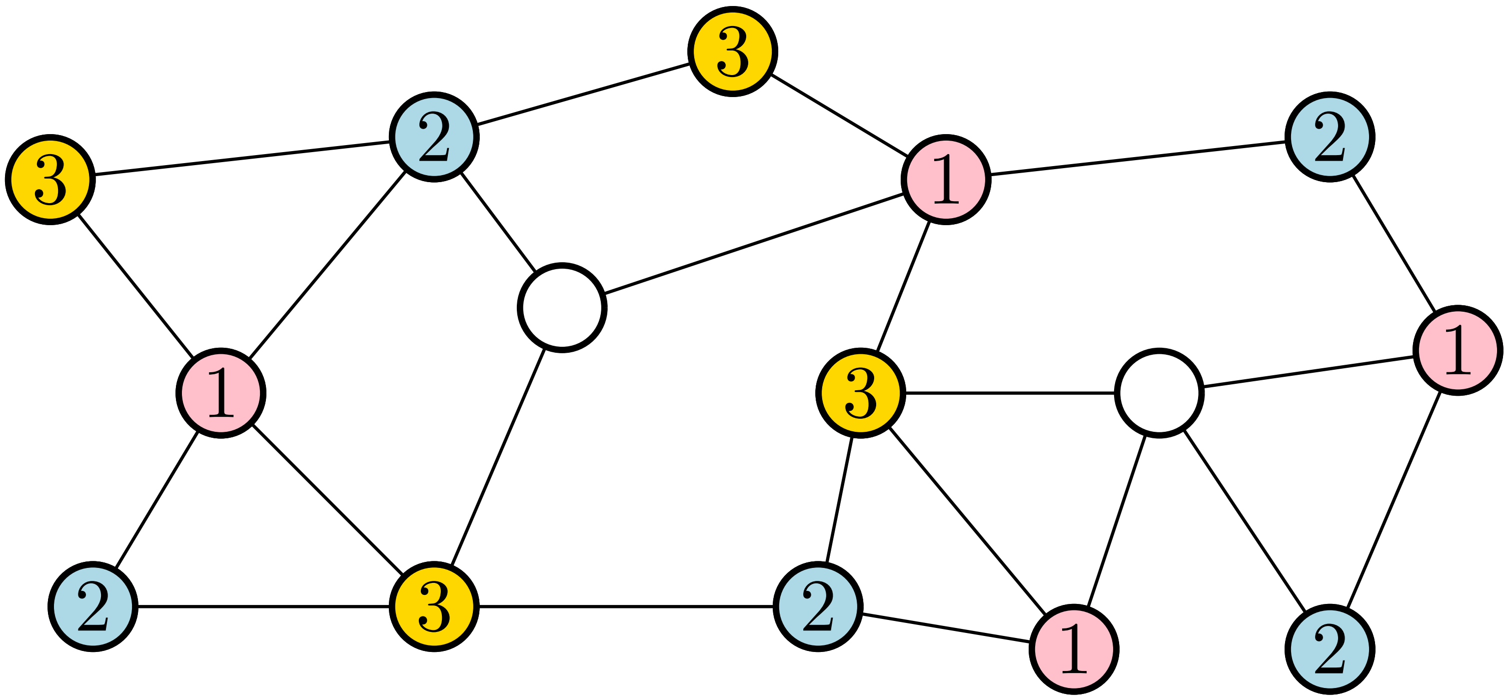
A sample execution

Iteration 2: Find a MIS $\mathcal{I}$ of the uncolored nodes and assign color 2 to the nodes in $\mathcal{I}$



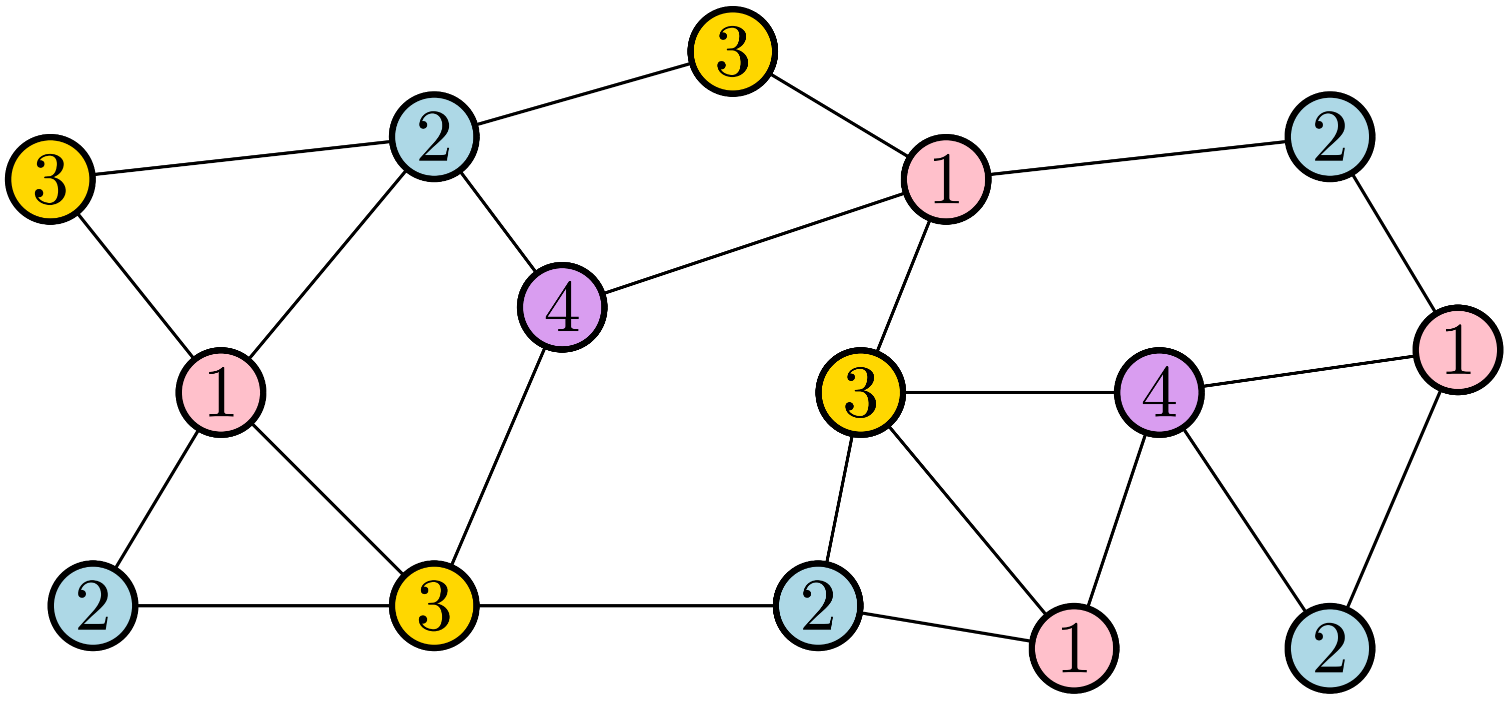
A sample execution

Iteration 3: Find a MIS $\mathcal{I}$ of the uncolored nodes and assign color 3 to the nodes in $\mathcal{I}$



A sample execution

Iteration 4: Find a MIS $\mathcal{I}$ of the uncolored nodes and assign color 4 to the nodes in $\mathcal{I}$



Analysis

Lemma: The algorithm terminates in at most $\Delta + 1$ iterations (i.e., it uses at most $\Delta + 1$ colors).

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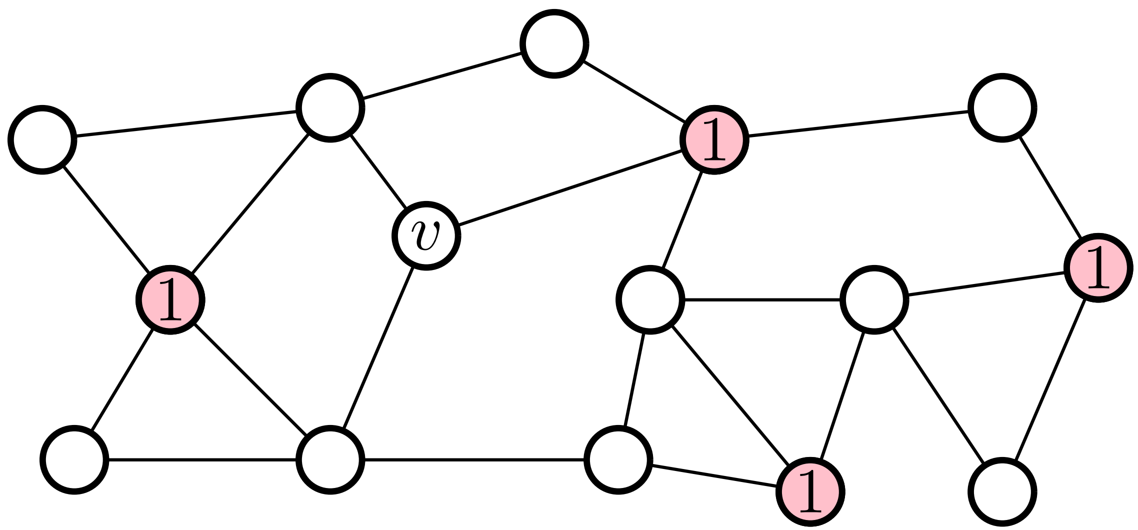
Proof:

- At the end of each iteration, each uncolored node v is adjacent to at least one node in the MIS $\mathcal{I}$.



Analysis

Lemma: The algorithm terminates in at most $\Delta + 1$ iterations (i.e., it uses at most $\Delta + 1$ colors).

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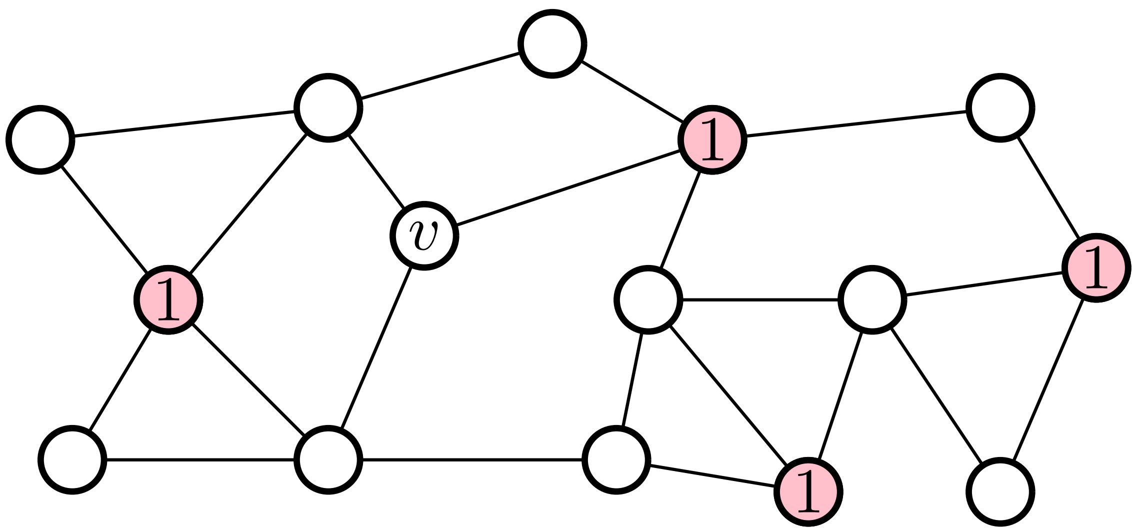

Otherwise $\mathcal{I} \cup \{v\}$ would be an independent set, contradicting the maximality of $\mathcal{I}$.

Analysis

Lemma: The algorithm terminates in at most $\Delta + 1$ iterations (i.e., it uses at most $\Delta + 1$ colors).

Proof:

- At the end of each iteration, each uncolored node v is adjacent to at least one node in the MIS $\mathcal{I}$.
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- After at most Δ iterations the effective degree of all uncolored nodes becomes 0.
- At iteration $\Delta + 1$, each uncolored node enters the MIS $\mathcal{I}$. $\square$

A distributed (randomized) algorithm

Recall: Luby's MIS algorithm runs in $O(\log \Delta \cdot \log n)$ with high probability.

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Using Luby's algorithm to find the required MIS we obtain a $\Delta + 1$ coloring algorithm that runs in time:

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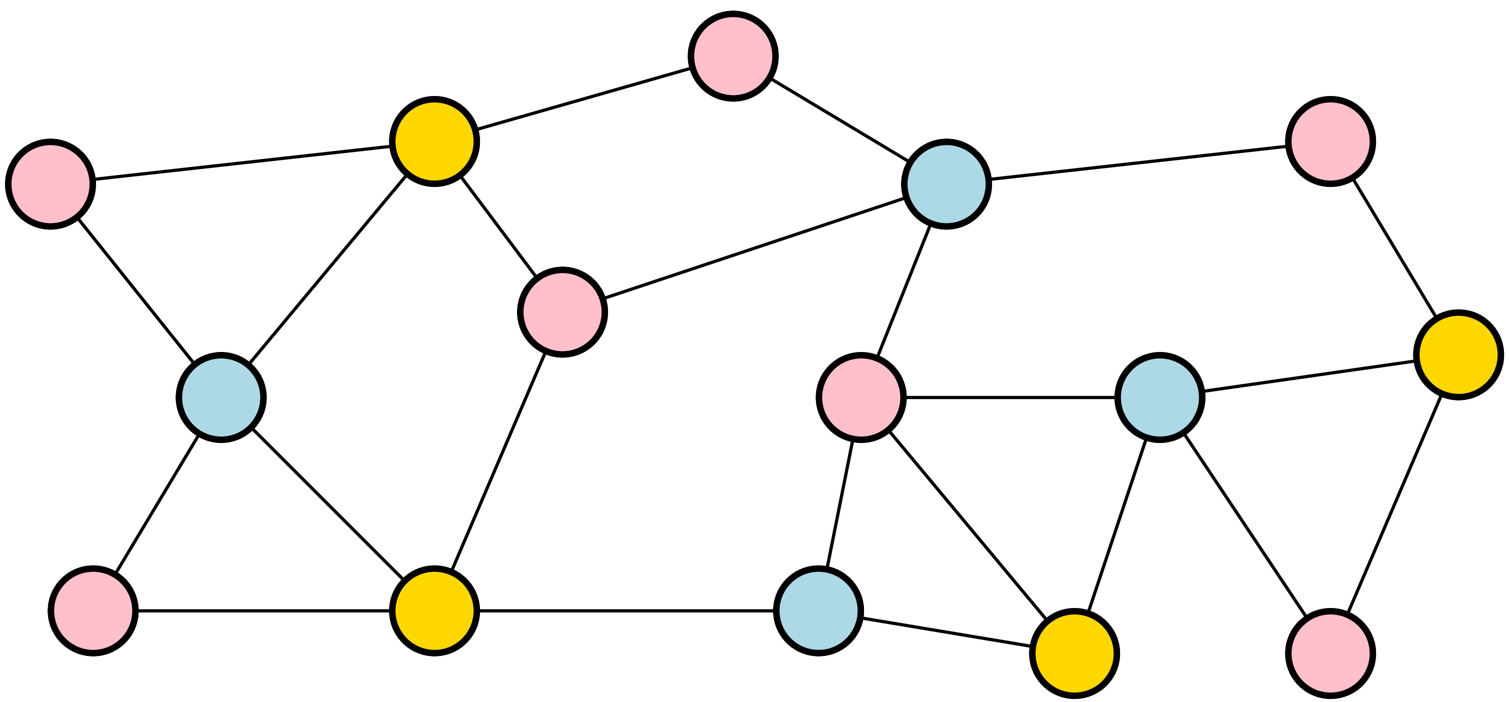
$$O(\underbrace{\Delta}_{\text{At most } \Delta+1 \text{ iterations}} \cdot \underbrace{\log\Delta \cdot \log n}_{\substack{\text{Time to compute a MIS w.h.p.} \\ \text{(using Luby's Algorithm)}}})$$

with high probability.

A faster algorithm (using more colors)

We will design a simple 2Δ-coloring algorithm:

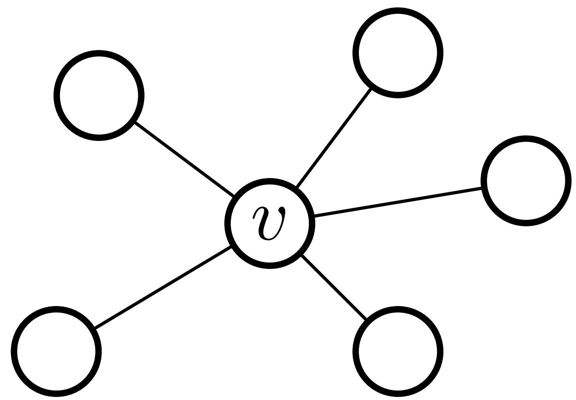
- Distributed
- Randomized
- Running time: $O(\log n)$ with high probability



A distributed 2Δ-coloring algorithm

Each node v mantains a palette of $2\,\delta(v)$ colors:

v's degree in G



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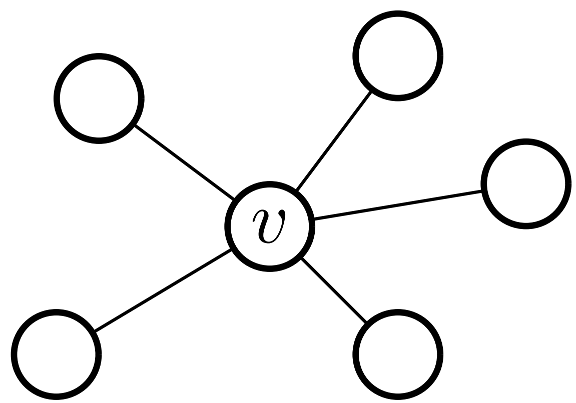
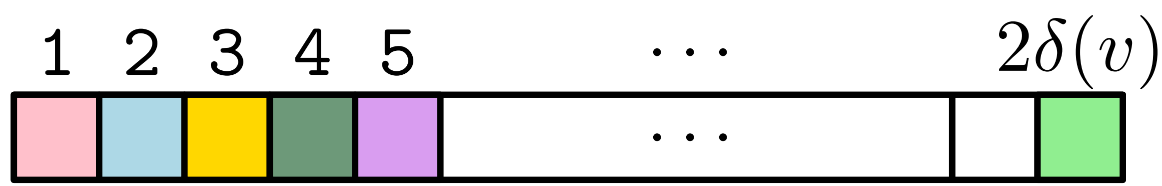


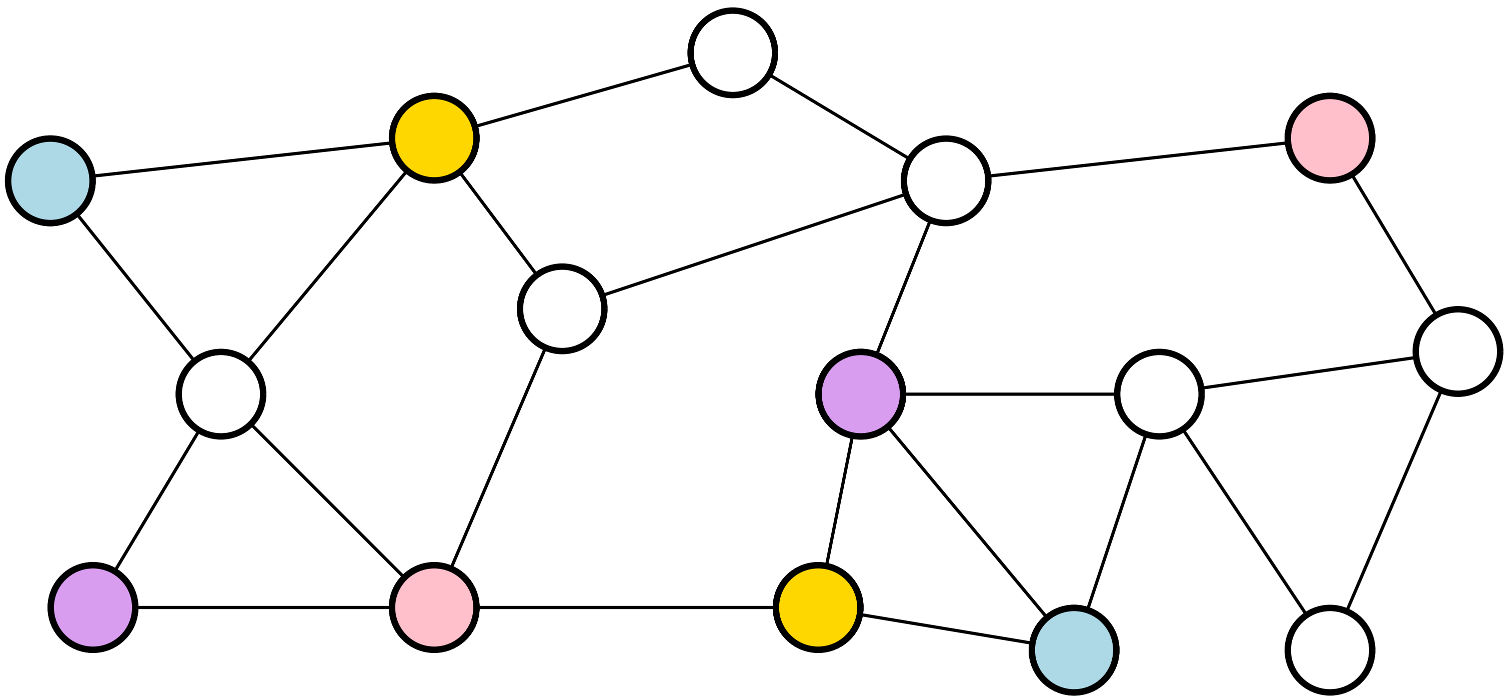
v's palette



Initially all colors are available.

A distributed 2Δ-coloring algorithm

- The algorithm works in *phases*
- In each generic phase there are two kinds of nodes: colored and uncolored.



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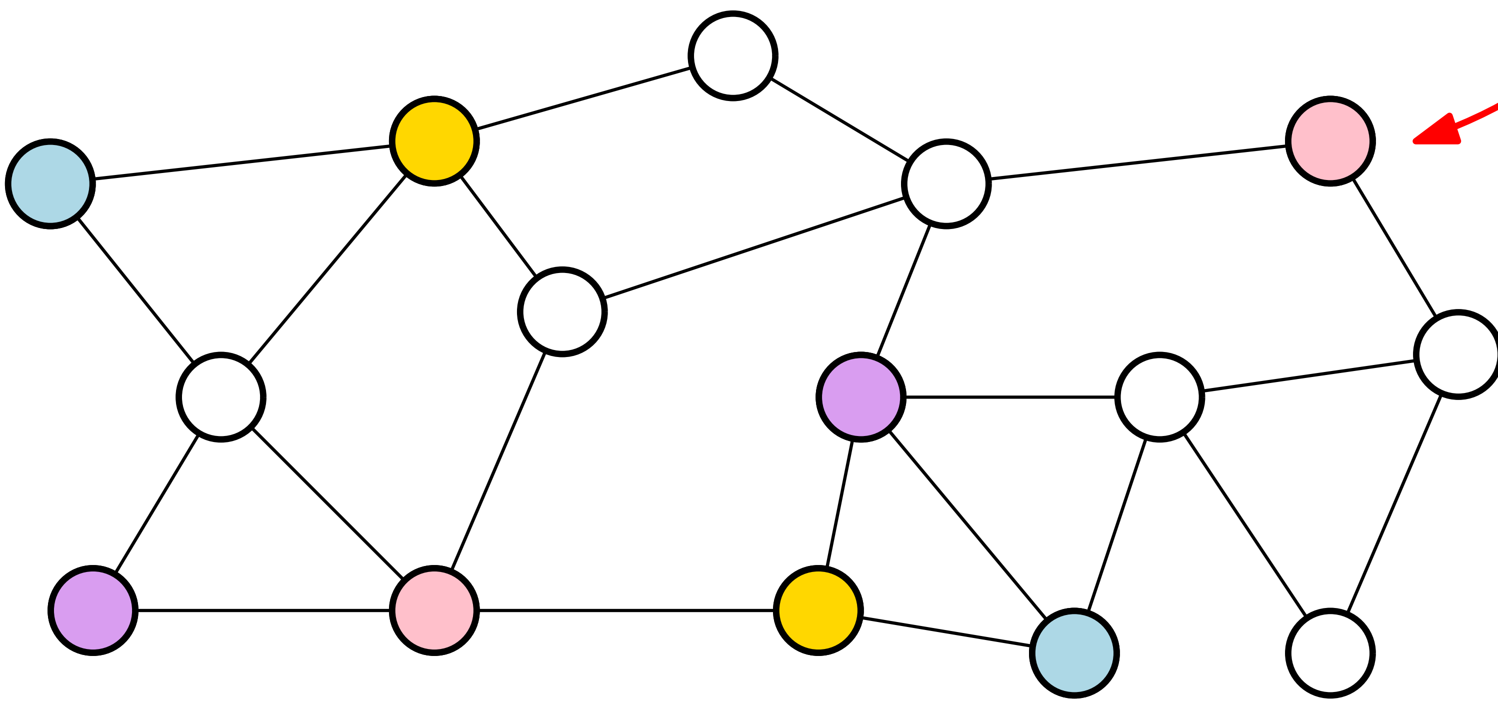

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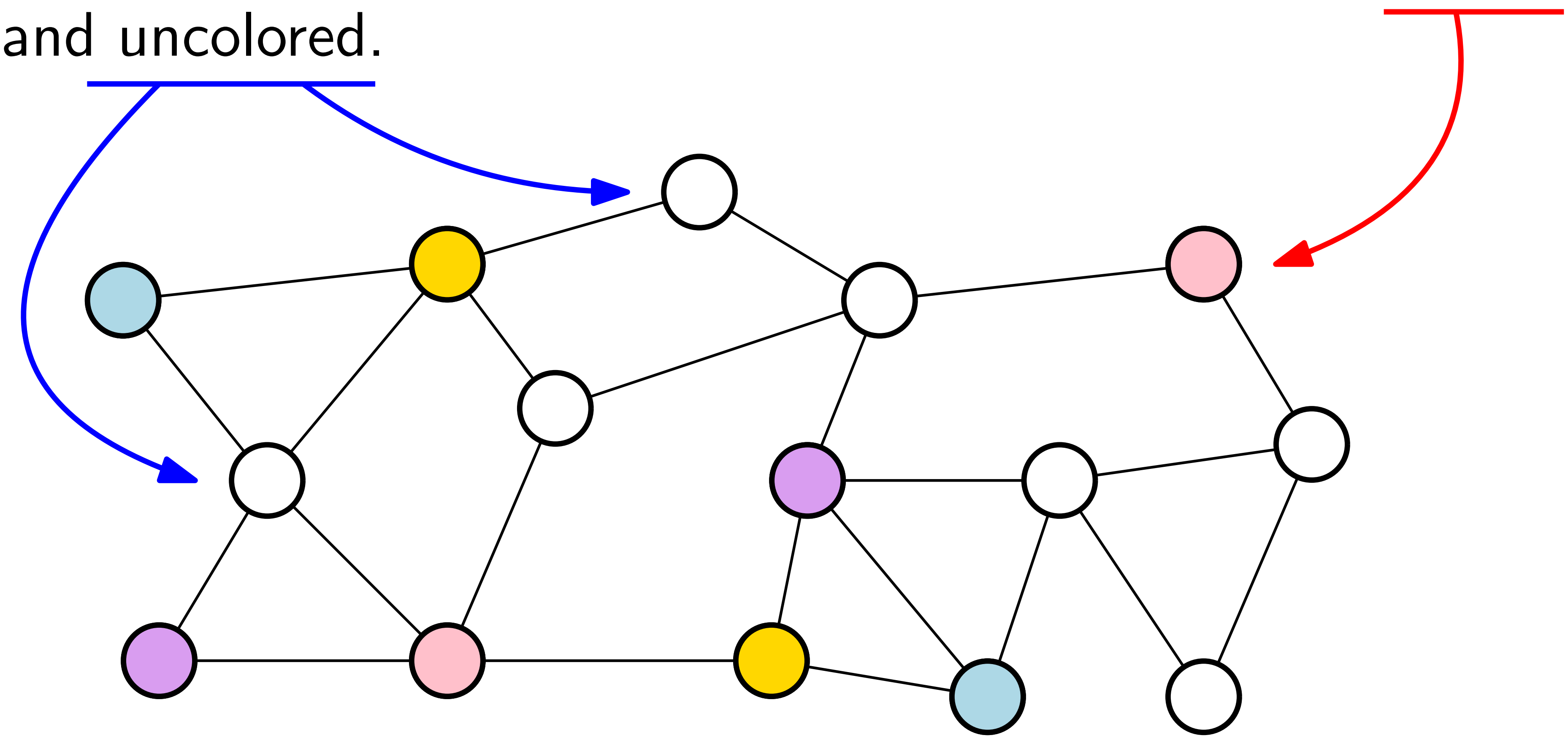

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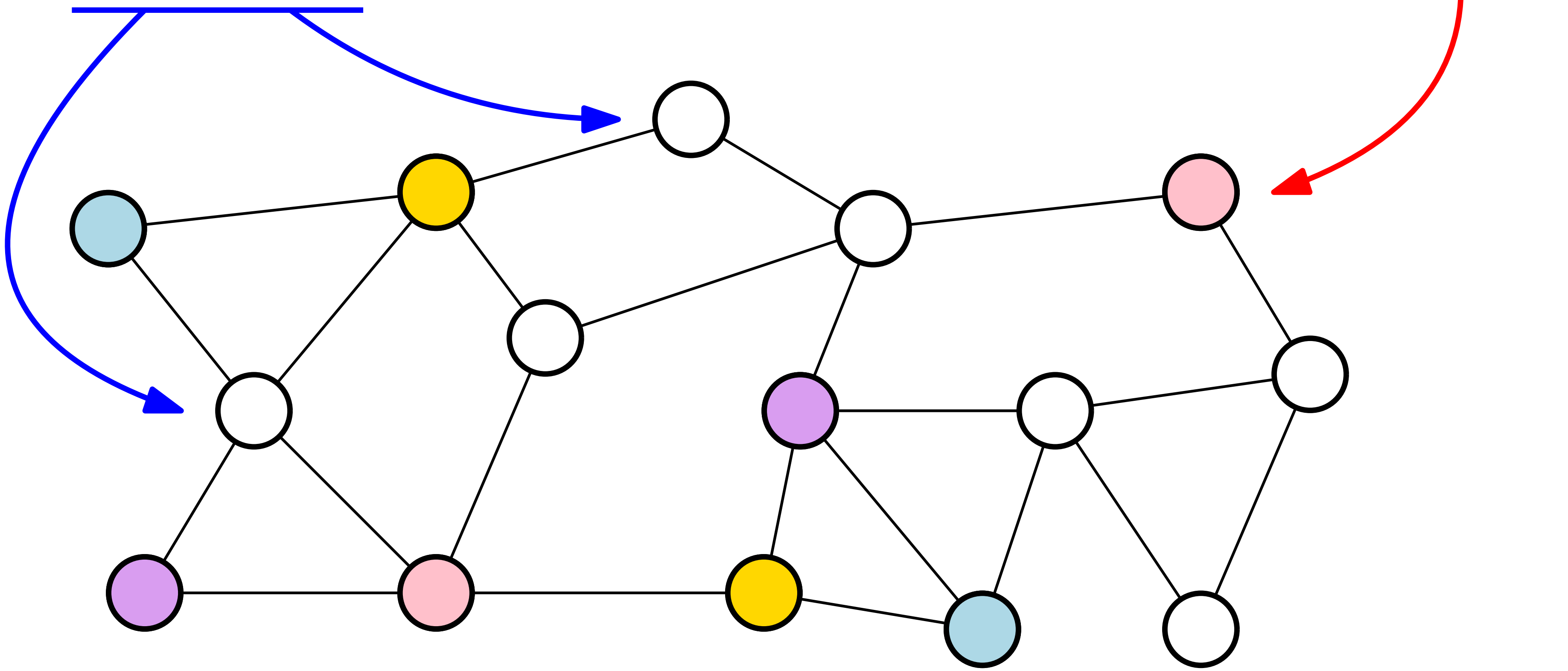
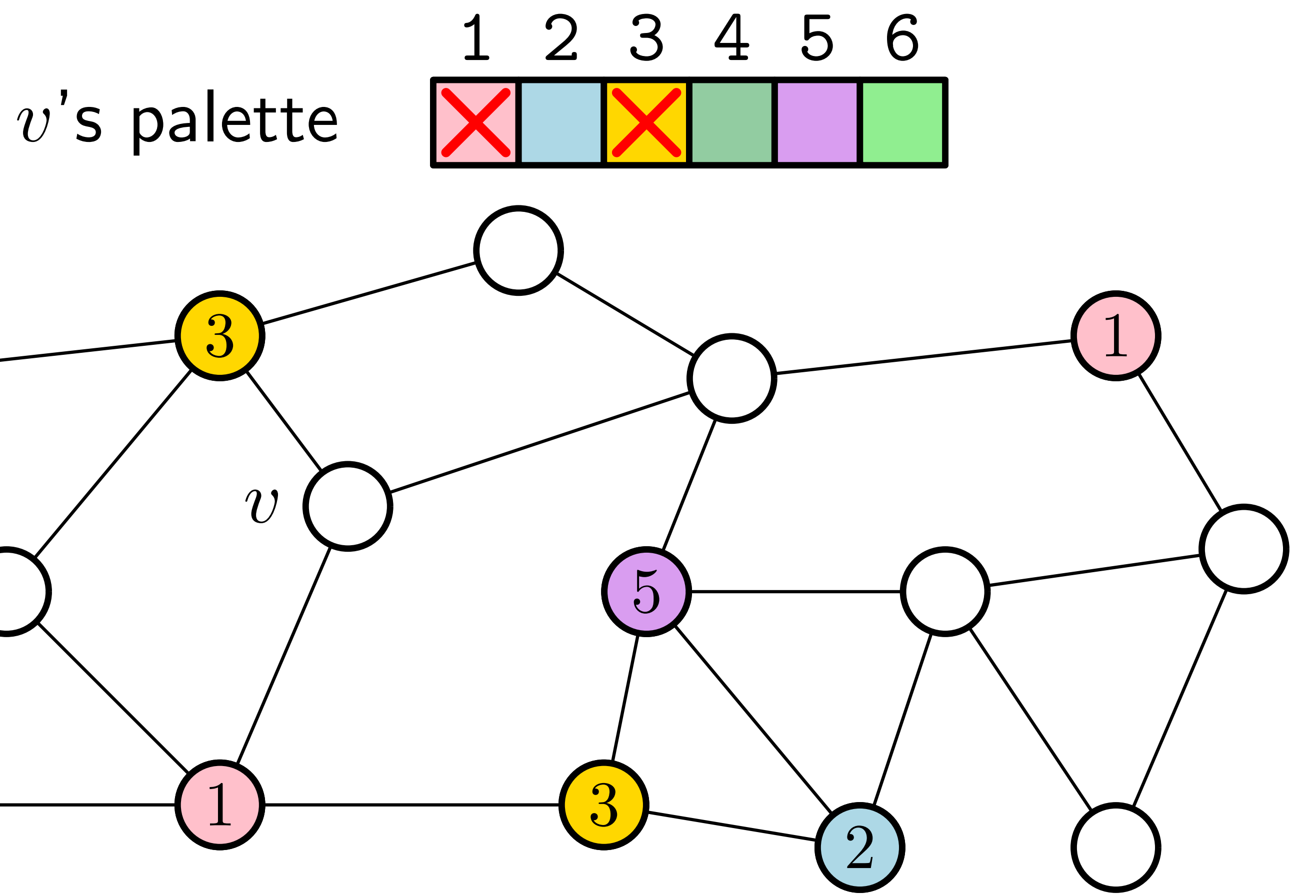


- Colors are **final**.

A distributed 2Δ-coloring algorithm

- Let v be an uncolored node in a generic phase.
- All the colors assigned to v's neighbors are unavailable in v's palette.



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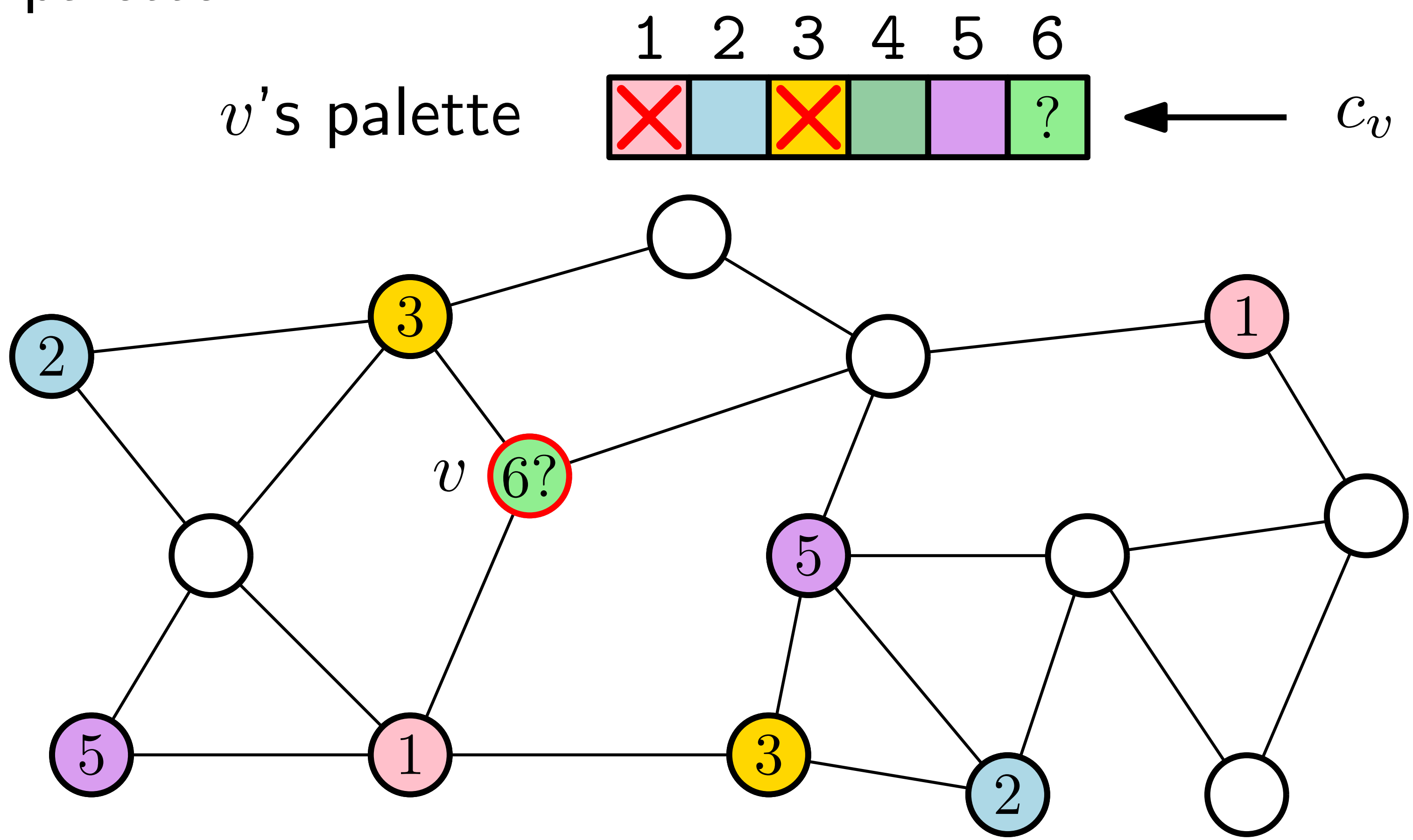

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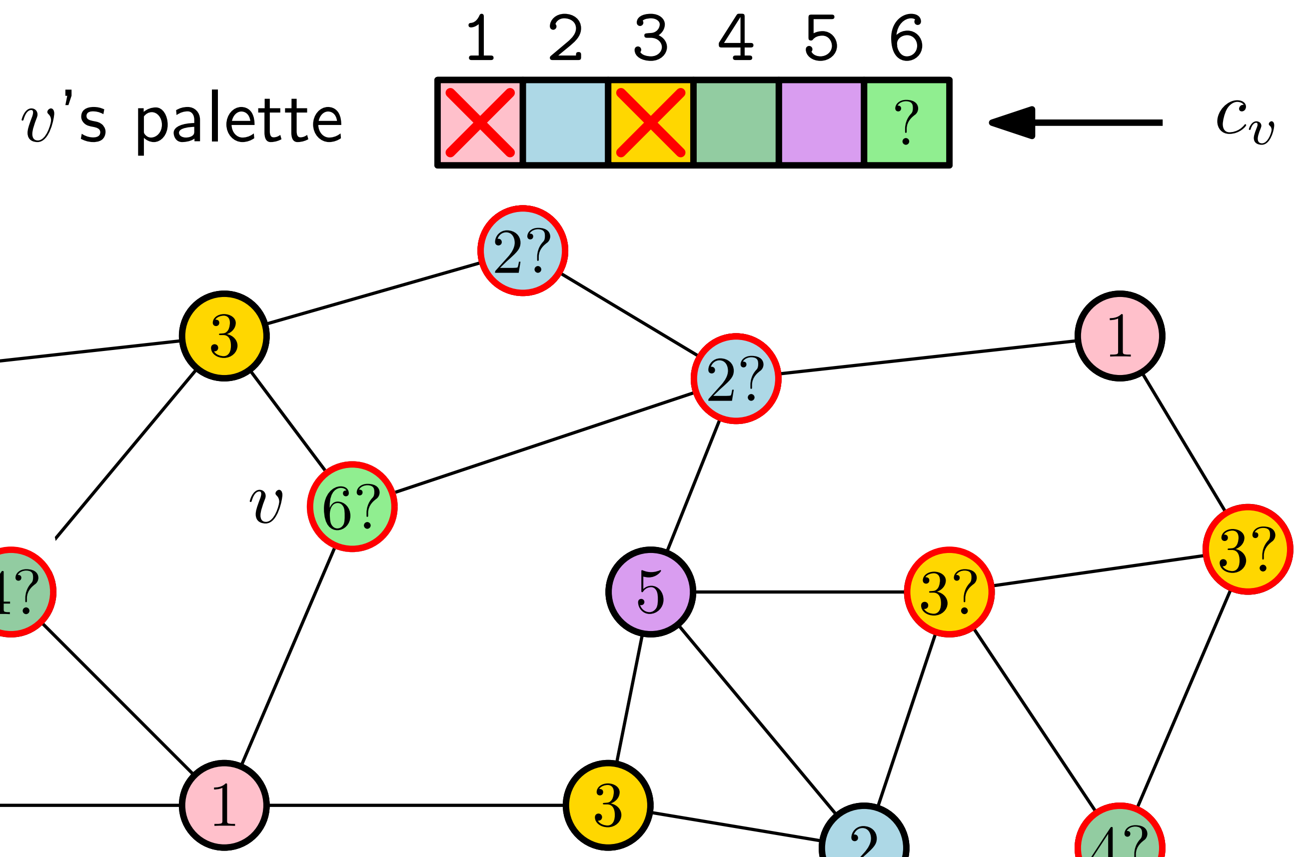

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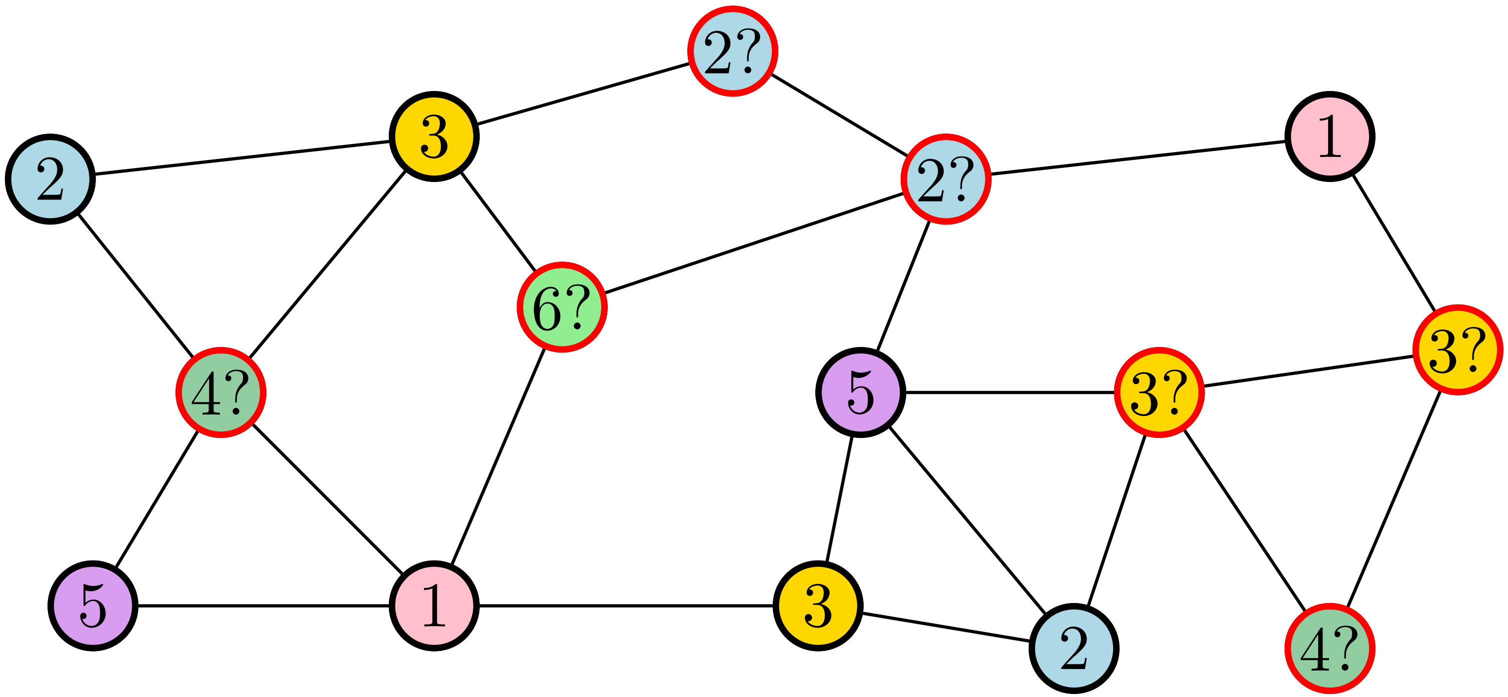

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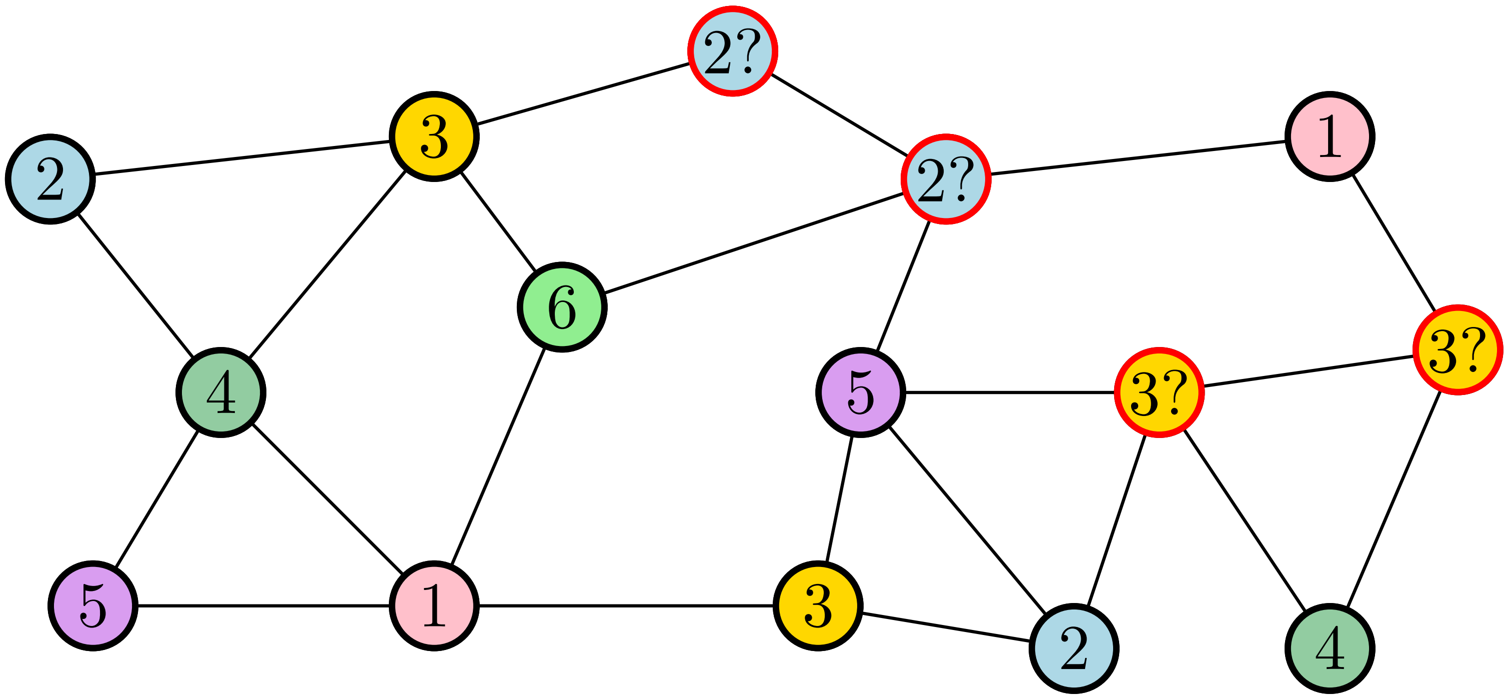

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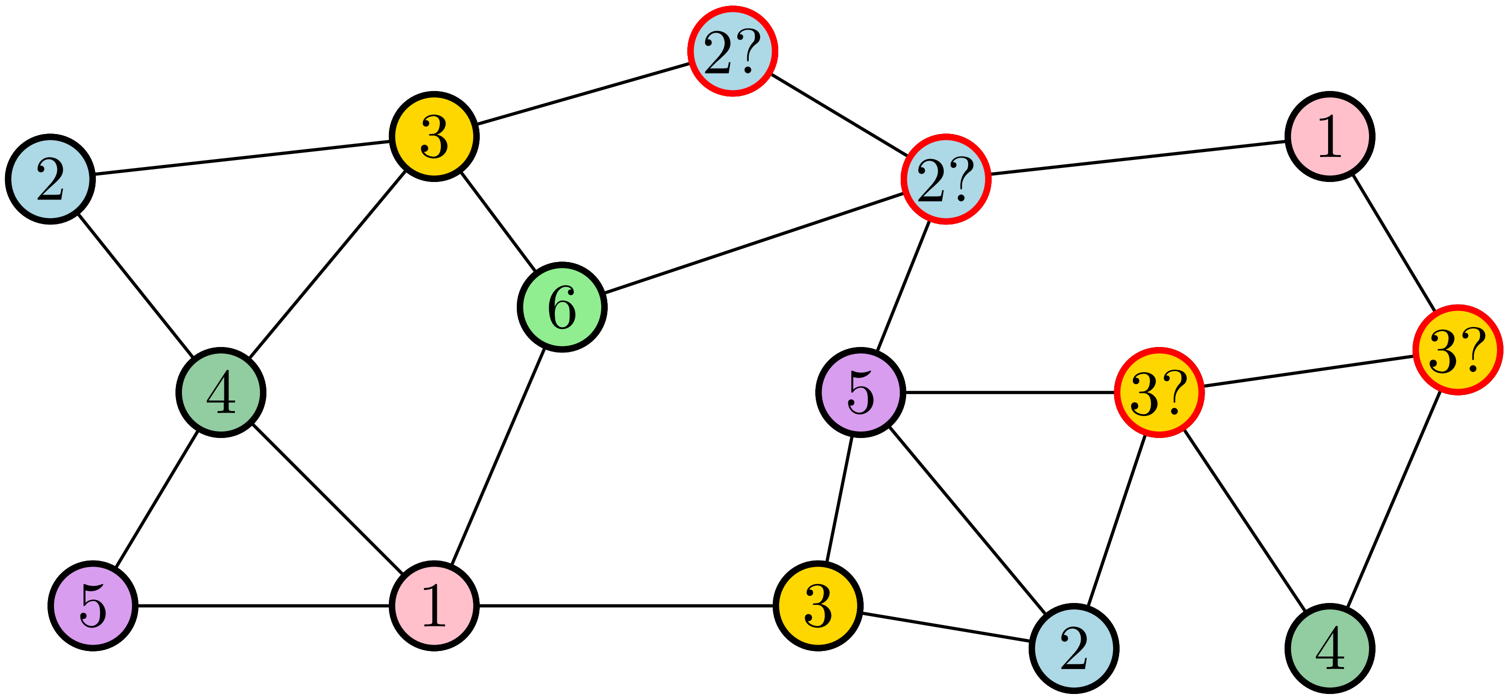

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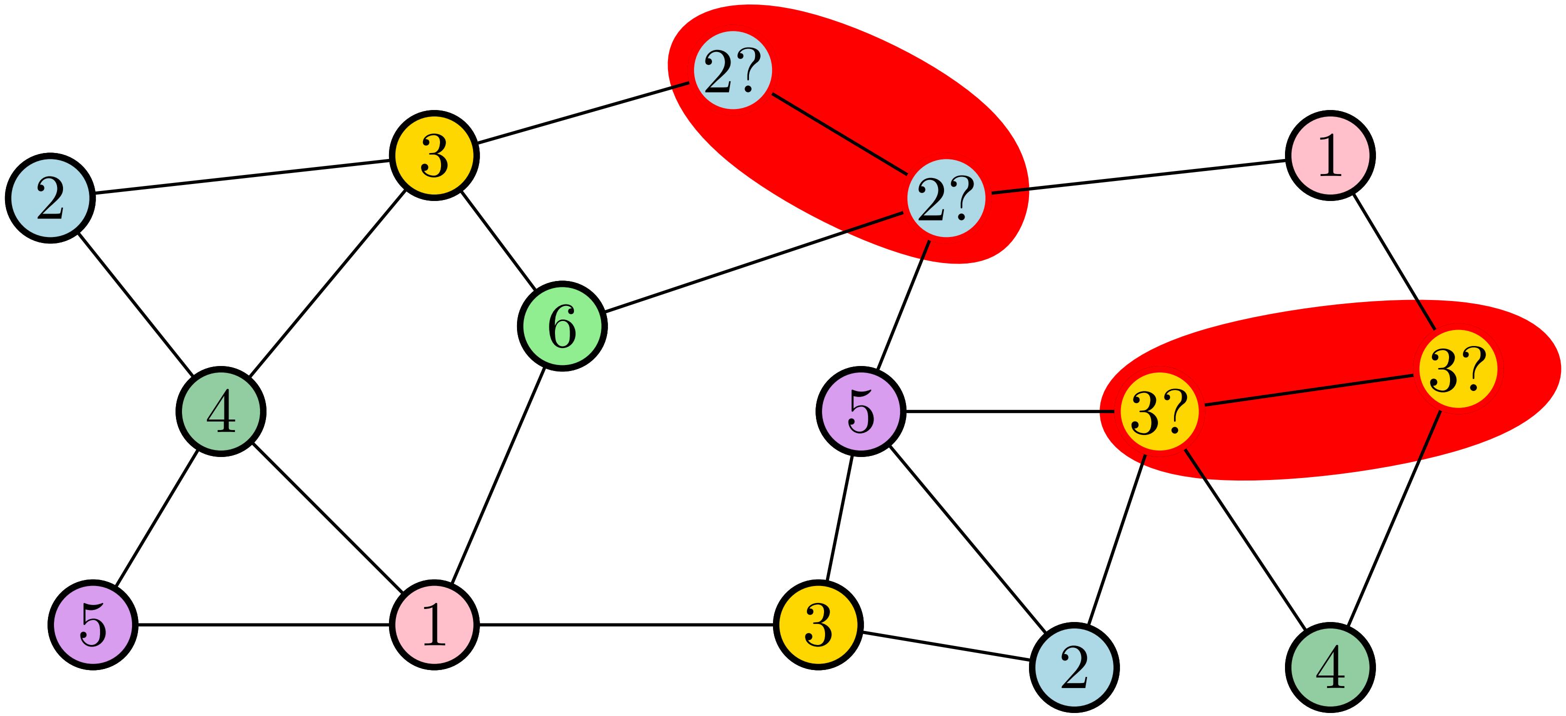

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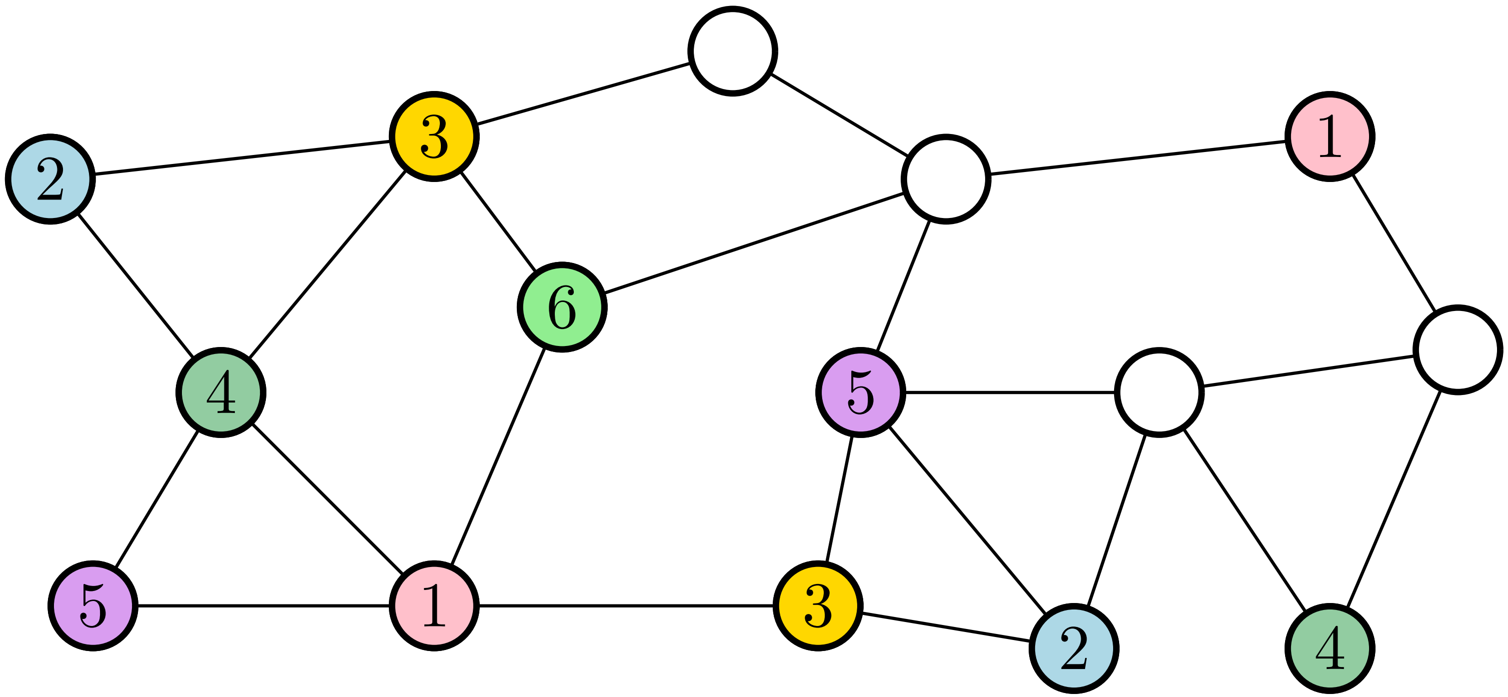

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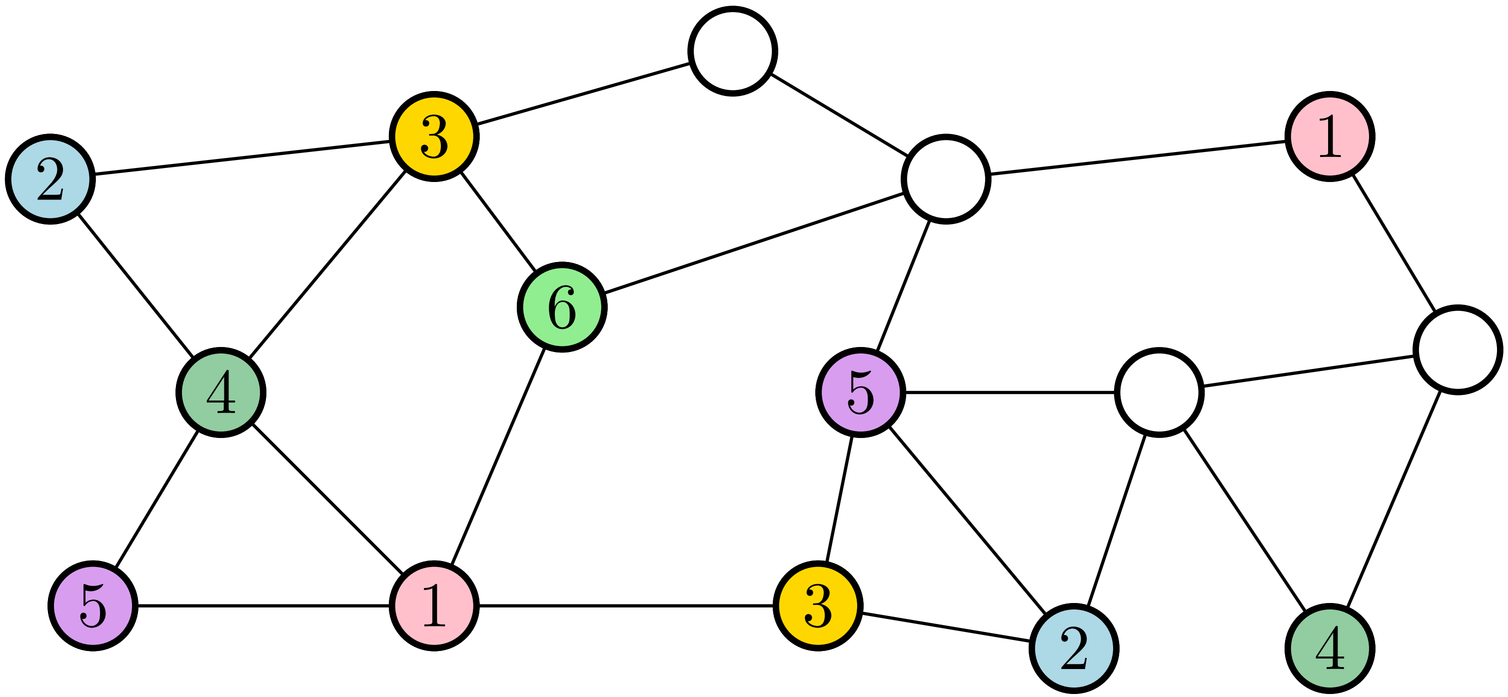
All uncolored nodes advance to the next phase (and try again with another candidate color)



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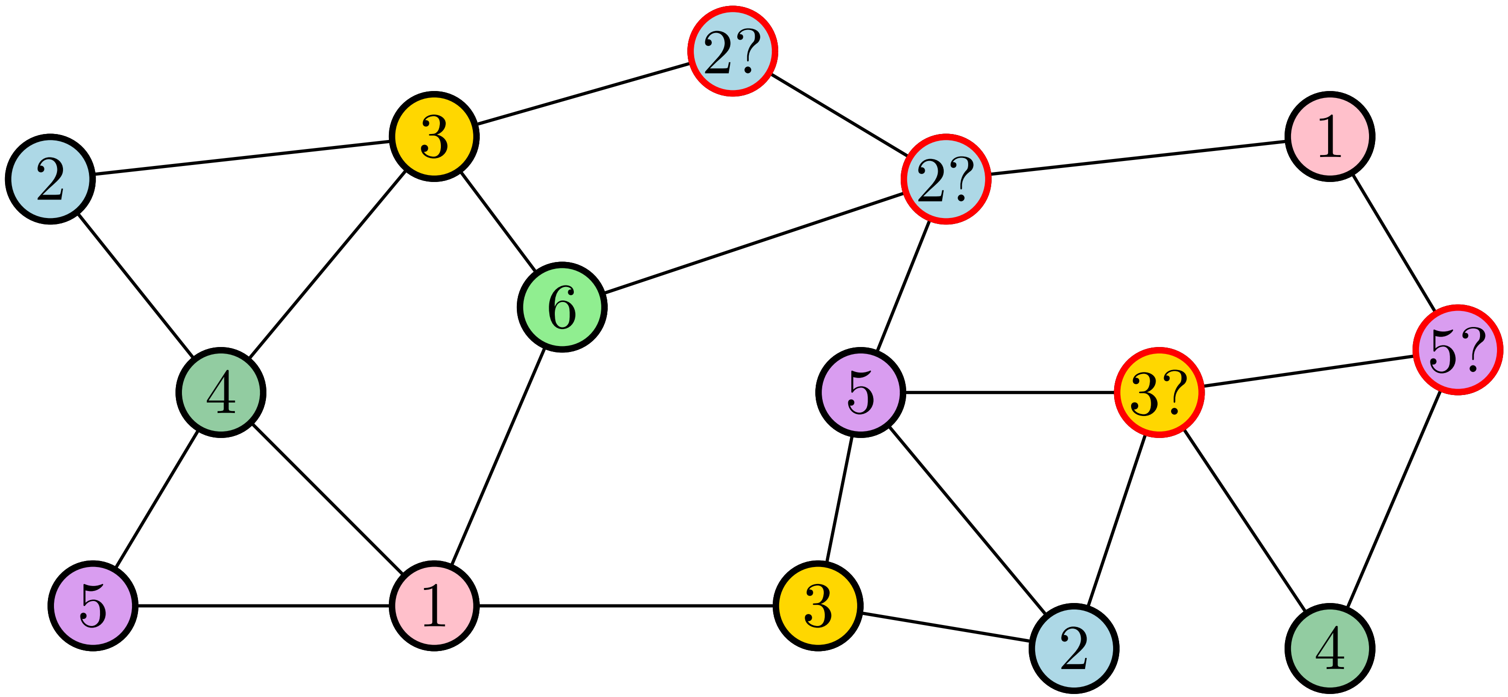

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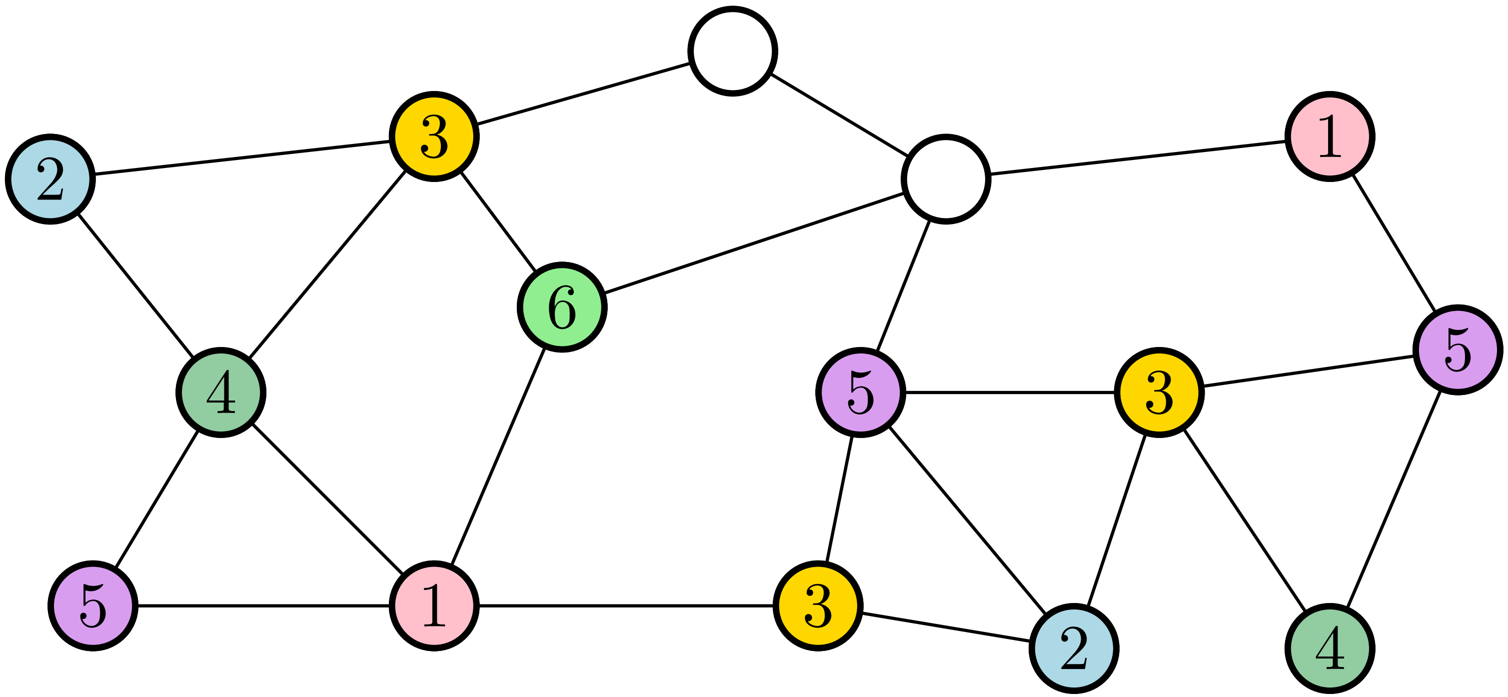

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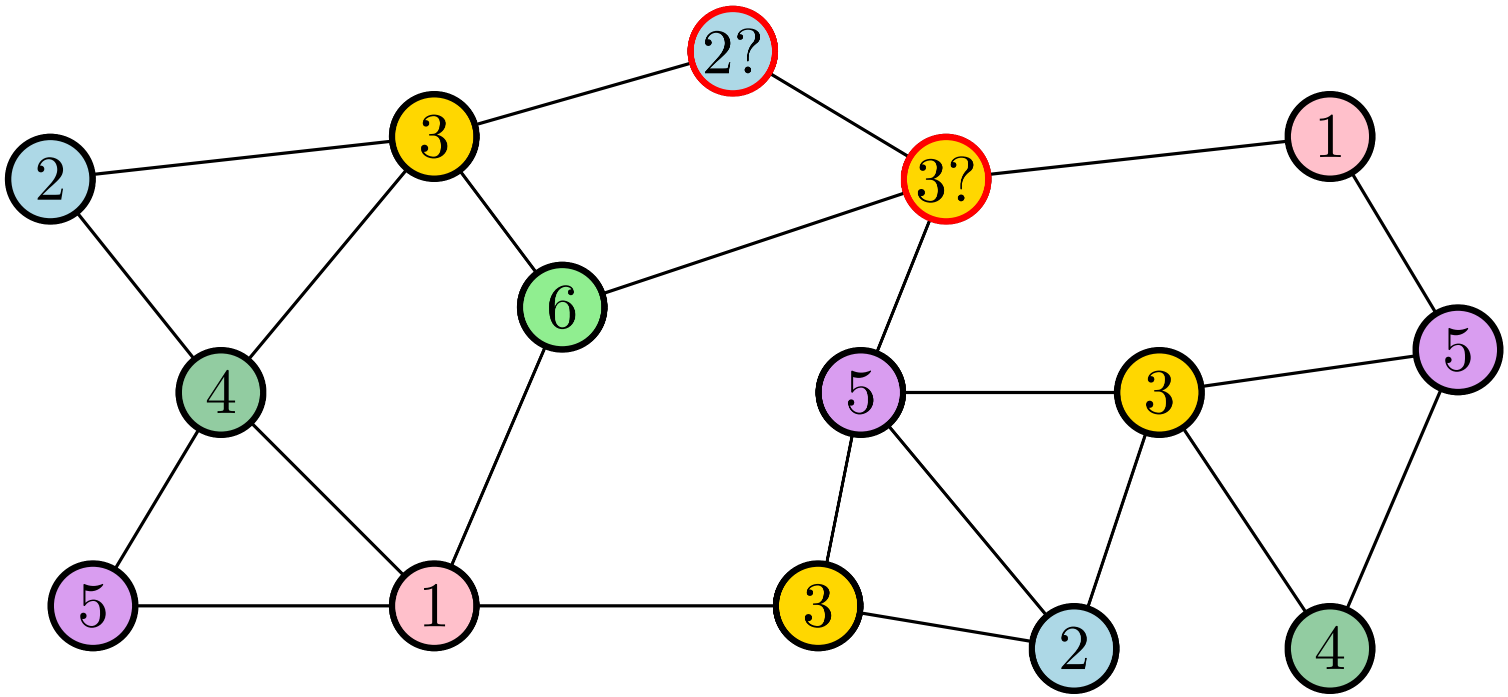

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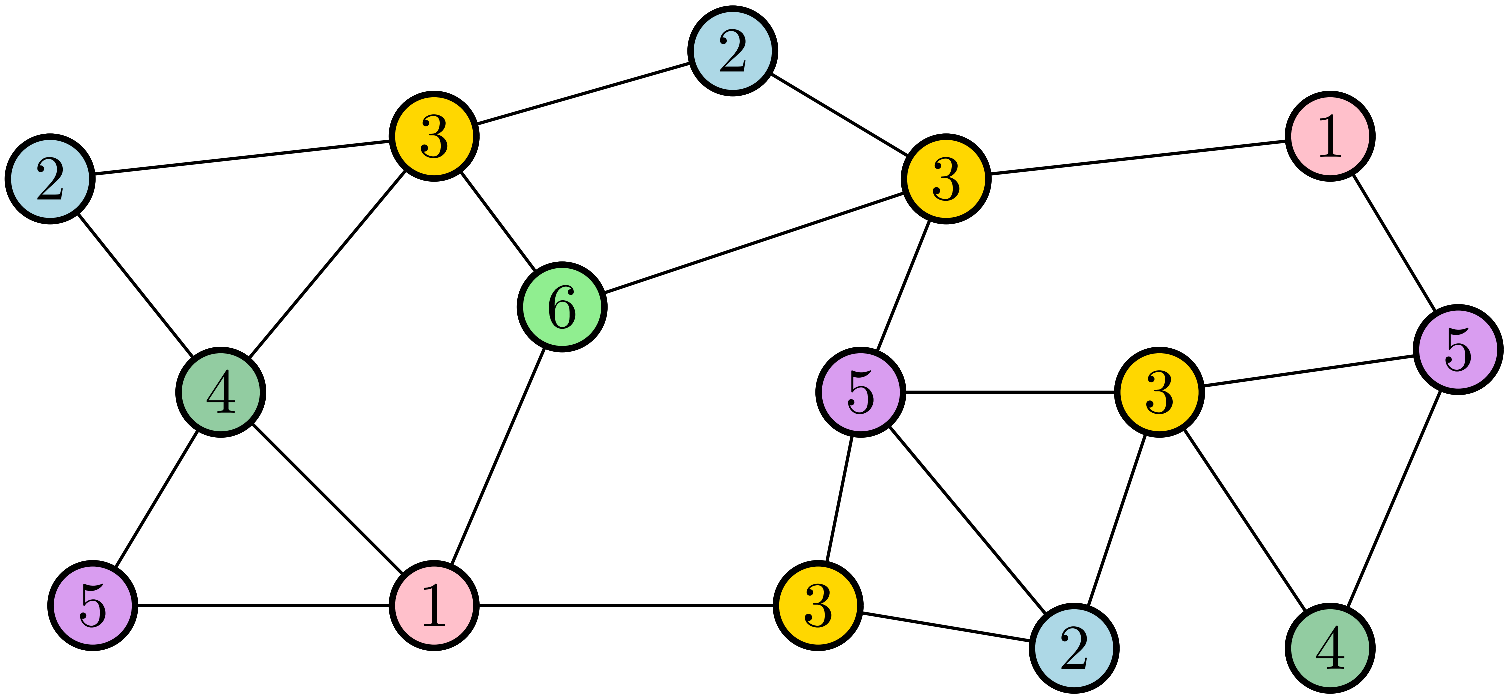


...until all nodes are colored.

A distributed 2Δ-coloring algorithm

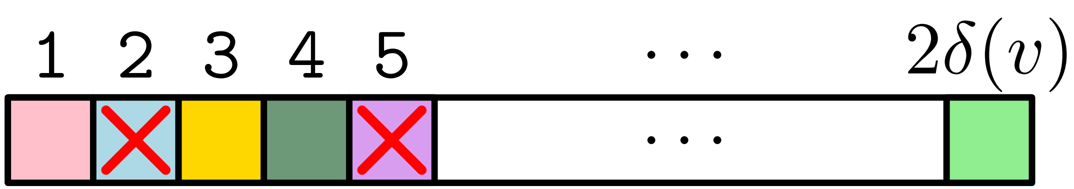
Algorithm for node v:

- **While v is uncolored** (iteration# $=$ phase#):
  - Pick a color c_v uniformly at random from the available palette colors
  - Send c_v to uncolored neighbors (and receive neighbors' colors)
  - **If** some uncolored neighbor u chose $c_u = c_v$:
    - Reject color c_v (do nothing)
  - **Else**
    - Accept color c_v
    - Inform neighbors about color c_v
      (neighbors mark color c_v as unavailable in their palettes)

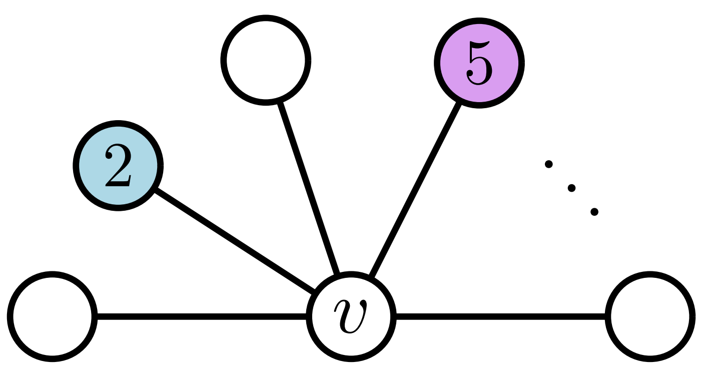
Analysis

Consider a generic uncolored node v in a generic phase k.

Let $A(v)$ be the set of available colors in v's palette at the beginning of phase k.



$$A(v) = \{1, 3, 4, \dots\}$$



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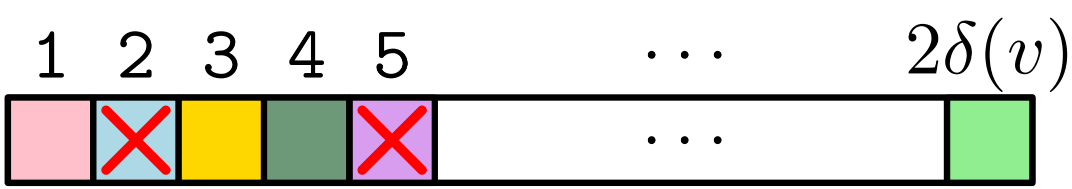
Let $U(v) = \{c_u : (v, u) \in E\}$ be the set of **candidate colors** chosen by the neighbors of v that were uncolored at the beginning of phase k.



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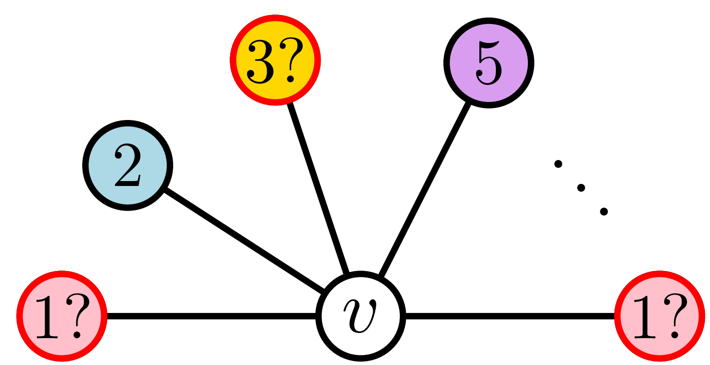

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Let $A'(v)$ be the set of available colors in v's palette that are not chosen as candidates by any uncolored neighbor of v.

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$$|A'(v)| = |A(v) \setminus U(v)| \geq |A(v)| - |U(v)|$$

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$$\begin{aligned} |A'(v)| = |A(v) \setminus U(v)| &\geq |A(v)| - |U(v)| \\ &\geq (\underbrace{2\delta(v)}_{\text{Colors in } v\text{'s palette}} - \underbrace{(\delta(v) - \ell)}_{\text{Colored neighbors of } v}) - \underbrace{\ell}_{|U(v)| \leq \ell} \end{aligned}$$

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The probability that the candidate color c_v chosen by v is accepted is:

Number of successful choices for c_v

$$\frac{|A'(v)|}{|A(v)|}$$

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Number of available choices for c_v

Probability that v **succeeds** during phase k: at least $\frac{1}{2}$.
(becomes colored)

Analysis

Probability that v **succeeds** in a (generic) phase: at least $\frac{1}{2}$.

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With probability at least $1-\frac{1}{n}$ all nodes succeed within $2\log(n)$ phases.

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The algorithm computes a valid 2Δ-coloring in time $O(\log n)$, with high probability.