ISNS

2º PARTE

UNRELATED MACHINE SUM

IN: m UNR $h = 1, \dots, m$; n JOB $J = 1, \dots, n$; $P_{ih} > 0$

OUT: $S = \{S_1, \dots, S_m\}$

GOAL: $\min \left(\sum_{h=1}^{m} MC_h(S) \right)$

ALGO → MSUM
$O(nm)$ $i \to h$ CON MIN P_{ih}
ret S

TH 1: MSUM TROVA OPT AL $\tilde{\Pi}$

Proof:
$\sum_{h=1}^{m} MC_h(S) = \sum_{h=1}^{m} \sum_{i \in S_h} P_{ih}$ $\forall i$

SUPPONIAMO $S' \neq S$ T/C $\exists\, i \to M_{h'}$ T/C $P_{ih'} > P_{ih}$

SE NELLA FORMULA SCAMBIAMO I DUE ABBASSIAMO LA MSUM

$\Rightarrow S$ OPT $\Leftrightarrow$ $J_i \to M_h$ CON MIN P_{ih} $\forall i$

$\Rightarrow$ MSUM TROVA OPT $\square$

UNRELATED JOB SUM $(m = 1)$

IN: M; n JOB $J = 1, \dots, n$; $P_i > 0$

OUT: $S = \{S_1\}$

GOAL: $\min \left(\sum_{i=1}^{n} JC_i(S) \right)$

ALGO → 1-JSUM
$O(n \log n)$ ORDINA JOB CON P_i CRESCENTE, ASSEGNA
ret S

TH 2: 1-JSUM TROVA OPT AL $\tilde{\Pi}$ CON $m = 1$

Proof:
$P_{(j)}(S)$ // P DEL JOB ALLA J-ESIMA POS IN S

$\sum_{i=1}^{n} JC(S) = n P_{(1)}(S) + (n-1) P_{(2)}(S) + \dots + P_{(n)}(S)$

JSUM MINIMA $\Leftrightarrow$ JOB IN ORDINE CRESCENTE

SUPPONI $S' \neq S$ T/C $\exists\, i < J$ IN S T/C $P_{(i)} > P_{(J)}$

$\Rightarrow$ SCAMBIANDOLI JSUM DIMINUISCE

$\Rightarrow$ 1-SUM RITORNA OPT $\square$

UNRELATED JOB SUM $(m=1, \text{WEIGHTS})$

IN M

m JOB $S = 1, \dots, m$

$P_i > 0$

$w_i > 0$

OUT $S = \{s_i\}$ **GOAL** $\min\left(\sum_{i=1}^{m} w_i \cdot JC_i(S)\right)$

ALGO → WEIGHTED 1-JSUM

$O(m \log m)$ ORDINA I JOB IN ORDINE DI w_i/p_i DECRESCENTE, ASSEGNA ret S

TH 3 WEIGHTED 1-JSUM TROVA OPT AL $\sim$

PROOF SUPPONIAMO $S' \neq S$

* IN S' CI SONO ALMENO i_1, i_2 ADIACENTI T/C $\frac{w_{i_1}}{P_{i_1}} < \frac{w_{i_2}}{P_{i_2}}$

CREA S'' SCAMBIANDOLI

SIA t IL TEMPO IN CUI i_1 INIZIA IL PROCESSAMENTO

↳ IN S' LA JSUM DI i_1, i_2:

$$(t + P_{i_1})w_{i_1} + (t + P_{i_1} + P_{i_2})w_{i_2}$$

↳ IN S'' INVECE

$$(t + P_{i_2})w_{i_2} + (t + P_{i_2} + P_{i_1})w_{i_1}$$

$\Rightarrow S' - S'' = P_{i_1} w_{i_2} - P_{i_2} w_{i_1}$

$S' > S''$?

$\Rightarrow P_{i_1} w_{i_2} - P_{i_2} w_{i_1} > 0 \Rightarrow \frac{P_{i_1} w_{i_2}}{P_{i_2}} > w_{i_1}$

$\Rightarrow \frac{w_{i_2}}{P_{i_2}} > \frac{w_{i_1}}{P_{i_1}}$ *

LA NOSTRA ASSUNZIONE È LA PROVA CHE $\bar{S}$ OPT È $\perp$

$\square$

IDENTICAL JOB SUM

IN: m IDEN. $i = 1, \dots, m$; n JOB $j = 1, \dots, n$; $p_j > 0$

OUT: $S = \{S_1, \dots, S_m\}$

GOAL: $\min \left(\sum_{j=1}^{n} C_j(S) \right)$

ALGO -> JSUM

$O(n \log n)$

ORDINA JOB IN ORDINE DI p_j CRESCENTE, RINOMINA

FOR $j = 1 \dots n$

$j \to h = j \bmod m$

ret S

$O(n) + O(n \log n)$

ASSEGNA SORT

IDENTICAL MACHINE MAKESPAN

IN: m IDENT $h = 1, \dots, m$; n JOB $j = 1, \dots, n$; $p_j > 0$

OUT: $S = \{S_1, \dots, S_m\}$

GOAL: $\min \left(\max_{h=1 \dots m} MC_h(S) \right)$

ALGO -> MAKESPAN (APPROX)

$O(n \log n)$

ORDINA JOB IN ORDINE DI p_j DECRESCENTE, RINOMINA

FOR $j = 1 \dots n$

$j \to h$ t.c. h HA min $T_h(j-1)$

ret S

$O(n) + O(n \log n)$

ASSEGNA SORT

ONLINE IDENTICAL MACHINE MAKESPAN

IN m IDEN. $i=1,\dots,m$ OUT $S=\{S_1,\dots,S_m\}$ GOAL $\min\left(\max_{h=1\dots m} MC_h(S)\right)$
? JOB

ALGO -> LIST (APPROX)

$J=0$
QUANDO ARRIVA JOB:
  $J=J+1$
  ASSEGNA AD h CON min $T_h(J-1)$
ret S

THM LIST È $2-1/m$ APPROX PER IL τ

PROOF. SIA g PROC. TIME DI p_g ULTIMO JOB IN S

SUPPONIAMO t TEMPO IN CUI p_g INIZIA A ESSERE PROCESSATO

SAPENDO

$$\alpha)\ C_{max}(OPT) \geq \frac{\sum_{j=1}^{m} p_j}{m} = \frac{mt + p_g}{m} = \frac{\cancel{m}t}{\cancel{m}} + \frac{p_g}{m} = t + p_g/m$$

E CHE

$$\beta)\ C_{max}(OPT) \geq p_g$$

DATO CHE

$$\gamma)\ C_{max}(SOL) = t + p_g$$

$$\Rightarrow \underset{\substack{\uparrow\\ x = ax + (1-a)x}}{t + p_g} = \underbrace{t + \frac{1}{m}p_g}_{\substack{\uparrow\,\alpha\\ \leq C_{max}}} + \left(1-\frac{1}{m}\right)\underbrace{p_g}_{\substack{\uparrow\,\beta\\ \leq C_{max}}} \leq C_{max}(OPT) + \left(1-\frac{1}{m}\right)C_{max}(OPT) =$$

$$= C_{max}(OPT)\left(2-\frac{1}{m}\right) \Rightarrow$$

$$\Rightarrow C_{max}(SOL) \leq \left(2-\frac{1}{m}\right) C_{max}(OPT) \quad \square$$

K-ENVY-FREE IDENTICAL MACHINE MAKESPAN

IN: m IDENT $h = 1, \dots, m$
n JOB $S = 1, \dots, n$
$P_i > 0$
$K \geq 1$

OUT: $S = \{S_1, \dots, S_m\}$ t/c $MC_{i_1}(S) \leq K \cdot MC_{i_2}(S) \ \forall i_1, i_2 \in h$

GOAL: $\min \left(\max_{h=1 \dots m} MC_h(S) \right)$

TH: K-ENVY-FREENESS PRICE PER IDENTICAL MACHINES È ALMENO $\min\{n, m\} - \varepsilon$ $\forall$ SMALL $\varepsilon > 0$

Proof

ISTANZA $n = m$, $P_1 = 1 - \varepsilon$, $P_2 = \dots = P_n = 1$

PER $K = 1 \Rightarrow MC_{i_1} \leq MC_{i_2} \ \forall i_1, i_2 \in h$

S CHE ASSEGNA AD ALMENO DUE MACCHINE

M_i È LA MACCHINA CON PIÙ JOB $(|S_i| \geq |S_J| \ \forall S_J \in S)$

↳ CASO 1 $JOB_1 \to M_J \neq M_i$: M_i INVIDIA M_J

↳ CASO 2 $JOB_1 \to M_i$

SIA M_Z LA SECONDA CON PIÙ JOB

M_i INVIDIA M_Z SE $|S_i| > |S_Z|$

M_Z INVIDIA M_i SE $|S_i| = |S_Z|$

1-ENVY-OPT È LA SOLUZIONE IN CUI TUTTI I JOB SONO ASSEGNATI A UNA MACCHINA h t/c

$$MC_h = m - \varepsilon = n - \varepsilon = \min\{n, m\} - \varepsilon$$

$$\Rightarrow \frac{C_{max}(1\text{-ENVY-OPT})}{C_{max}(OPT)} = \min\{n, m\} - \varepsilon$$

$\square$

TH 8 K-ENVY-FREENES PRICE PER IDENTICAL MACHINES É AL PIÚ $\min\{n,m\}, k\geq 1$

Proof

↳ CASO $\min\{n,m\}=n$

OGNI S CHE ASSEGNA TUTTI I JOB A UNA MACCHINA h É K-ENVY-FREE

$C_{max}(S) = n \cdot \max_{i=1..n}\{P_i\}$ // UPP BOUND A C_{max}(K-ENVY-OPT)

$C_{max}(OPT) = \max_{i=1..n}\{P_i\}$ // SENZA K-ENVY-FREENES

$$\Rightarrow \frac{C_{max}(\text{K-ENVY-OPT})}{C_{max}(OPT)} \leq \frac{C_{max}(S)}{C_{max}(OPT)} = \frac{n \cdot \cancel{\max_{i=1..n}\{P_i\}}}{\cancel{\max_{i=1..n}\{P_i\}}} = n$$

↳ CASO $\min\{n,m\}=m$

OGNI S CHE ...

$C_{max}(S) = \sum_{i=1}^{n} P_i$ // UP BOUND ...

$C_{max}(OPT) = \sum_{i=1}^{n} P_i / m$ // SENZA ...

$$\Rightarrow \frac{C_{max}(\text{K-ENVY-OPT})}{C_{max}(OPT)} \leq \frac{C_{max}(S)}{C_{max}(OPT)} = \frac{\cancel{\sum_{i=1}^{n} P_i}}{\cancel{\sum_{i=1}^{n} P_i} / m} = m$$

$\square$

TH9 K-ENVY-FREENESS PRICE PER IDENTICAL MACHINES È AL PIÙ $1+1/k$; $k \geq 2$

Proof ALGO (OPT → k-ENVY-OPT CON $C_{max}(k\text{-ENVY-OPT}) \leq 1+1/k\, C_{max}(OPT)$)

(1) RISCRIVI MC_j DIVIDENDO $p_i / C_{max}(OPT)$

(2) WHILE $\exists j, j'$ t.c. $MC_j(OPT) + MC_{j'}(OPT) \leq 1$

$OPT_j = OPT_j \cup OPT_{j'}$

$OPT_{j'} = \{\emptyset\}$

(3) SIA m' #MACHINES CON ALMENO UN JOB IN OPT: RINOMINA LE MACCHINE IN ORDINE DECRESCENTE DI MC

(4) CREA NUOVO S

IF $MC_{m'}(OPT) < 1/k$

$S_j = OPT_j \quad \forall j \in \{1, \dots, m'-2\}$

$S_{m'-1} = OPT_{m'-1} \cup OPT_{m'}$

$S_{m'} = \{\emptyset\}$

ELSE

$S_j = OPT_j \quad \forall j \in \{1, \dots, m\}$

ANALISI (4) DI S

SE $MC_{m'}(OPT) \geq 1/k \Rightarrow S$ k-ENVY-FREE POICHÉ

$MC_j(OPT) \leq 1,\ j = m'$ E $1 \leq k \cdot MC_{m'}(OPT)$

ALTRIMENTI $MC_{m'}(OPT) > 1$ LA MACCHINA $m'-1$ DIVENTA QUELLA CON MASSIMO MC, SE NON INVIDIA LEI NESSUNO LO FA

$$\Rightarrow MC_{m'-1}(S) > 1 = MC_{m'-1}(OPT) + MC_{m'}(OPT) \leq 2\, MC_{m'-1}(OPT) \leq$$

$$\leq k\, MC_j(S) \quad \forall j \in \{1, \dots, m'-1\};\ k \geq 2 \Rightarrow S \text{ k-ENVY-FREE}$$

ANALISI (3) DI $C_{max}(S)$

SE (3) FALSO $\Rightarrow C_{max}(S) = 1$

ALTRIMENTI $\Rightarrow C_{max}(S) > 1$

$$MC_{m'-1}(S) = \underbrace{MC_{m'-1}(OPT)}_{\leq 1/k} + \underbrace{MC_{m'}(OPT)}_{\leq 1/k} \leq 1 + 1/k$$

$$\Rightarrow \frac{C_{max}(k\text{-ENVY-OPT})}{\cancel{C_{max}(OPT)}} \leq C_{max}(S) = 1 + 1/k$$

□

TH 10 K-ENVY-FREENESS PRICE PER IDENTICAL MACHINES È ALMENO $1+1/k-\varepsilon$ $\forall$ SMALL $\varepsilon>0, k\geq 2$

Proof

MOSTRIAMO UN ISTANZA IN CUI $\dfrac{C_{max}(k\text{-ENVY-OPT})}{C_{max}(OPT)} \geq 1+1/k-\varepsilon$ $\forall_{SMALL}\ \varepsilon>0$

ISTANZA

m MACCHINE

$n=m$ JOBS

$P_1 = 1/k - \varepsilon$

$P_2 = \dots = P_m = 1$

OPT NON NEGATIVO E K-ENVY FREE $\Rightarrow$ (OVVIO) UN JOB A MACCHINA

$C_{max}(OPT) = 1$

IN S K-ENVY-FREE $\Rightarrow P_i$ ASSEGNATO A UN MACCHINA CON GIÀ UN ALTRO JOB

$$C_{max}(S) \geq \underbrace{1}_{P_i} + \underbrace{1/k - \varepsilon}_{P_1}$$

$P_i + P_1$

$$\Rightarrow \frac{C_{max}(k\text{-ENVY-OPT})}{C_{max}(OPT)} \geq \frac{C_{max}(S)}{\cancel{C_{max}(OPT)}} \geq 1+1/k-\varepsilon \qquad \square$$

K-ENVY-FREE UNRELATED MACHINE MAKESPAN

IN m UNR $h=1,\dots,m$
n JOB $j=1,\dots,n$
$p_{ih}>0$
$K \geq 1$

OUT $S=\{S_1,\dots,S_m\}$ T/C $\sum_{i\in S_j} p_{ij} \leq K \sum_{i\in S_{j'}} p_{ij}$

GOAL $\min\left(\max_{h=1\dots m} MC_h(S)\right)$

TH 11 K-ENVY-FREENESS PRICE PER UNRELATED MACHINES È AL PIÙ $(1+1/K)^{\min\{n,m\}-1}$

Proof

ALGO (OPT → K-ENVY-OPT T/C $C_{max}(\text{K-ENVY-OPT}) \leq \left(1+1/K\right)^{\min\{n,m\}-1} C_{max}(OPT)$)

$S = OPT$
WHILE $\exists j, j'$ ch T/C j INVIDIA j'
  $S_j = S_j \cup S_{j'}$
  $S_{j'} = \emptyset$
ret S

(α) #ITERAZIONI $= \underbrace{\min\{n,m\}}_{}-1$

↳ #MACCHINE CON ALMENO UN JOB

(β) SE j INVIDIA j':

$$\sum_{i\in S_{j'}} p_{ij} < \sum_{i\in S_j} p_{ij} / K$$

⇒ A OGNI ITERAZIONE $MC_j(S) = MC_j(S) + \dfrac{MC_j(S)}{K}$ ⇒ CIOÈ CRESCE CON UN FATTORE AL PIÙ $1+1/K$

⇒ DA α, β SEGUE CHE S È AL PIÙ $(1+1/K)^{\min\{n,m\}-1} C_{max}(OPT)$ □

REVENUE MAX ENVY FREE PRICING PROBLEM

IN m ITEMS
m BUYERS
$v_i : \{1 \dots m\} \to \mathbb{R}^+$ VALUTATION
ITEMS — NON NEG R

GOAL $\max \left(P \sum_{i=1}^{m} x_i \right)$
// REVENUE

OUT (X, P) T/C $X = \langle X_1 \dots X_m \rangle \in P \geq 0$ — PREZZO
ITEMS VENDUTI AL BUYER i

↳ FEASIBLE OUTCOME SE

(1) SUPPLY CONSTRAINT $\sum_{i=1}^{m} x_i \leq m$

(2) INDIVIDUAL RATIONALITY $u_i(x_i, P) \geq 0$ T/C $u_i = V_i(x_i) - P \cdot x_i$

↳ ENVY FREE OUTCOME

(1) i INVIDIA i' $\iff$ $u_i(x_i, P) < u_i(x_{i'}, P)$

LEMMA A DATO NEARLY-FEASIBLE E ENVY-FREE OUTCOME
$(X_i, P) \Rightarrow \exists (X_i', P')$ FEASIBLE T/C $P_i \geq P$
$\sum_{i=0}^{m} x_i \geq m/2$

Proof

CASO 1: $\exists\ S \in \{1, \dots, m\}$ T/C $m_S \cdot m = m/2$:
COMPUTA $u_i(S, P)$
SIA $m' = \sum_{i=1}^{m} S\ \forall i$ T/C $u_i(S, P) > 0$
SE $m' \leq m$
↳ ASSEGNA IL BUNDLE $x_i = S$ A i T/C $u_i(S, P) > 0$
↳ ASSEGNA I BUNDLE RIMASTI A i ″ $= 0$
$\Rightarrow$ REVENUE $\geq (m/2 \cdot P)$
ALTRIMENTI
↳ INCREMENTA $P \to P'$ IN MODO CHE IL MINIMO $u_i(S, P) > 0$ DIVENTI UGUALE A 0
↳ RIPETI FINCHÉ $m' < m$
↳ GOTO 'SE'

ALGO 1

$\forall J \in \{1, \dots, m\}$ SIA m_J #BUYERS i T/C $j_i = J$

$p = 2$

	1	2	3	4	5	
u_1	1	0	-1	-2	-2	$\to j_1 = 2$
u_2	0	-3	1	-1	-3	$\to J_2 = 3$
u_3	1	0	-1	0	-2	$\to J_3 = 4$
u_4	1	1	-1	-3	-7	$\to J_4 = 2$

J_1, J_4 $\quad$ J_2 $\quad$ J_3

// CERCO IL BUNDLE J_i $\forall$ BUYER i T/C $u_i(x_i, p) \geq 0$ E x_i MASSIMO

$m_1 = 1$ $\quad$ $m_3 = 1$ $\quad$ $m_5 = 0$
$m_2 = 2$ $\quad$ $m_4 = 1$

SIA $z = \underline{\text{ARG}}\text{MAX}_J \left\{ J \cdot \sum_{k=J}^{m} m_k \right\}$

$$\left.\begin{matrix} J_1 = 1 \cdot 4 = 4 & J_3 = 3 \cdot 1 = 3 & J_5 = 0 \\ J_2 = 2 \cdot 4 = 8 & J_4 = 0 & \end{matrix}\right\} \Rightarrow \underline{z = 2}$$

SIA $\overline{m}_J = m_J$ T/C $J \geq z$

$\cancel{m_1}$ $\quad$ $m_3 = 1$ $\quad$ $m_5 = 0$
$m_2 = 2$ $\quad$ $m_4 = 1$

SE $\sum_{j \geq 2}^{m} j \, \overline{m}_j \leq m$ ASSEGNO j ITEMS A CHI LI CHIEDE

$$\sum_{j \geq 2}^{m} j \cdot \overline{m}_j = 2 \cdot 2 + 3 \cdot 1 + 4 \cdot 1 = 11 > m$$

ALTRIMENTI APPLICO LEMMA A A $\langle x_1 = j_1, x_2 = j_2, \dots \rangle$

$\Rightarrow$ LEMMA A $(\langle 2, 3, 4, 2 \rangle)$

1 UNRELATED MACHINE SUM

MSUM $O(n \cdot m)$

$i \to h$ con min P_{ih}

2 UNRELATED JOB SUM $(m=1)$

1-SUM $O(n \log n)$

P_i CRESCENTE

3 UNRELATED JOB SUM $(m=1$, WEIGHTS$)$

WEIGHTED-J-SUM $O(n \log n)$

w_i/p_i DECRESCENTE

4 IDENTICAL JOB SUM

J-SUM $O(n \log n)$

RENAME P_i CRESCENTE, $i \to i \bmod m$

5 IDENTICAL MACHINE MAKESPAN

MAKESPAN (APPROX) $O(n \log n)$

RENAME P_i DECRESCENTE, $i \to h$ CON MIN $T_h(i-1)$

6 IDENTICAL MACHINE MAKESPAN (ONLINE)

LIST (APPROX)

$J=0$, $i \to h$ CON MIN $T_h(J-1)$, $J{+}{+}$

TH 6

LIST È $2-\frac{1}{m}$ APPROX

7 IDENTICAL MACHINE MAKESPAN (K-ENVY-FREE)

S K-ENVY-FREE SSE $MC_i(S) \leq k \cdot MC_J \; \forall i, J \in h$

TH 7 K-ENVY-FREENESS PRICE ALMENO $\min\{n, m\} - \varepsilon \; \forall$ SMALL $\varepsilon > 0$

TH 8 // AL PIÙ $\min\{n, m\} \; \forall k \geq 1$

TH 9 // AL PIÙ $1 + \frac{1}{k} \; \forall k \geq 2$

TH 10 // ALMENO $1 + \frac{1}{k} - \varepsilon \; \forall$ SMALL $\varepsilon > 0, \forall k \geq 2$

8 UNRELATED MACHINE MAKESPAN (K-ENVY-FREE)

S K-ENVY-FREE SSE $\sum_{i \in S_J} P_{iJ} = k \cdot \sum_{i \in S_{J'}} P_{iJ}$

TH 11 K-ENVY-FREENESS PRICE AL PIÙ $\left(1+\frac{1}{k}\right)^{\min\{n,m\}-1}$

9 REVENUE MAX PRICING-PROBLEM (ENVY-FREE)

FEASIBLE OUTCOME

(1) SUPPLY CONSTRAINT $\sum_{i=1}^{m} x_i \leq m$

(2) INDIVIDUAL OUTCOME $h_i(x, p) \geq 0$

ENVY-FREE OUTCOME

LEMMA A DATO (x, p) NEARLY-FEASIBLE E ENVY FREE

DA (x', p) T/C $P' \geq P$ E $\sum_{i=1}^{m} x_i \geq m/2$

2

Exercise 1:

Describe the Scheduling Unrelated Job SUM (minimization) problem without weights with one machine (m=1). Show a polynomial time algorithm that finds a schedule that minimizes the Job SUM and formally prove the performance (i.e., optimality) of such an algorithm.
Return an optimal schedule for the following instance with 7 jobs:

Processing times: $p_1=6$; $p_2=2$; $p_3=4$; $p_4=3$; $p_5=1$; $p_6=1$; $p_7=5$.

3

Exercise 4:

Describe the Scheduling Unrelated Job SUM (minimization) problem with weights and one machine (m=1). Show a polynomial time algorithm that finds a schedule that minimizes the Job (weighted) SUM for the case where there is only one machine, and formally prove the performance (i.e., optimality) of such an algorithm.

Return an optimal schedule for the following instance with 10 jobs:

Processing times: $p_1=2$; $p_2=3$; $p_3=4$; $p_4=6$; $p_5=1$; $p_6=7$; $p_7=1$; $p_8=8$; $p_9=5$; $p_{10}=5$.

Weights: $w_1=5$; $w_2=6$; $w_3=3$; $w_4=6$; $w_5=3$; $w_6=8$; $w_7=9$; $w_8=2$; $w_9=2$; $w_{10}=1$.

x3

4

Part II:

Exercise 1:

Return an optimal schedule for the following instance of the Scheduling identical Job SUM (minimization) problem with 4 machines and 10 jobs with the following processing times:

Processing times: $p_1=2$; $p_2=4$; $p_3=6$; $p_4=3$; $p_5=5$; $p_6=2$; $p_7=9$; $p_8=13$; $p_9=15$; $p_{10}=1$.

x4

6

Exercise 2:

Describe the Online Scheduling Identical Machine MAKESPAN (minimization) problem. Show the algorithm LIST that computes an approximated schedule and formally prove the performance of such an algorithm.

x4

7

TH 7

Exercise 2:

Formally prove that the *Price of 1-envy-freeness for identical machines is at least min {n,m}-ε,* for any (small) *ε>0.*

TH 8

Exercise 2:

Formally prove that The *Price of k-envy-freeness* for identical machines is at most *min {n,m}, for any k≥1.*

TH 9

Exercise 5:

Formally prove that the *Price of k-envy-freeness for identical machines is at most 1+1/k, for any k≥2.*

x2

Exercise 3:

Given the following instance of the 3-Envy-free Scheduling Identical Machine MAKESPAN (minimization) problem with 4 machines and 10 jobs (processing times are given below). Consider the following schedule S. Is S a 3-envy-free scheduling? Justify your answer.

Jobs processing times: $p_1=5$; $p_2=5$; $p_3=2$; $p_4=2$; $p_5=1$; $p_6=1$; $p_7=1$; $p_8=2$; $p_9=3$; $p_{10}=3$.

The schedule S is: $S_1=\{j_1; j_2\}$ $S_2=\{j_3; j_4\}$ $S_3=\{j_5; j_6; j_7\}$ $S_4=\{j_8; j_9; j_{10}\}$.

If S is not 3-envy-free then return a 3-envy-free scheduling S' whose MAKESPAN is at most 4/3 times the MAKESPAN of the scheduling S, by using the algorithm of theorem 9 (Recall *Theorem 9*: The *Price of k-envy-freeness for identical machines is at most 1+1/k, for any k≥2*).

x4

TH 10

Exercise 5:

Formally prove that *The Price of k-envy-freeness for identical machines is at least 1 + 1/k - ε, for any (small) ε>0, and for any k≥2.*

x2

8

K-ENVY-FREENES

Exercise 3:

Define the k-Envy-free Scheduling Unrelated Machine MAKESPAN (minimization) problem.

Consider the following schedule *S* for the Scheduling Unrelated Machine setting with 3 machines and 5 jobs with the following processing times:

M\J	J_1	J_2	J_3	J_4	J_5
M_1	3	5	5	3	1
M_2	2	4	8	4	2
M_3	2	4	5	5	6

The schedule is: $S_1=\{j_1; j_2\}$ $S_2=\{j_3\}$ $S_3=\{j_4; j_5\}$

Is *S* a 2-envy free schedule? Justify your answer.

Is *S* a 3-envy free schedule? Justify your answer.

TH 11

Exercise 4:

Formally prove that *The Price of k-envy-freeness for unrelated machines is at most* $(1+1/k)^{\min\{n,m\}-1}$, *for any k≥1.*

x2

Exercise 1:

Consider the following schedule *S* for the Scheduling Unrelated Machine setting with 5 machines and 8 jobs with the following processing times:

M\J	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
M_1	2	1	2	1	2	1	1	2
M_2	2	2	8	1	3	3	1	4
M_3	2	1	2	2	3	1	2	2
M_4	1	2	2	3	4	3	4	3
M_5	2	1	2	1	2	1	4	4

The schedule is: $S_1=\{j_1;j_2\}$ $S_2=\{j_3\}$ $S_3=\{j_4;j_5\}$ $S_4=\{j_6;j_7\}$ $S_5=\{j_8\}$

- Is *S* a 2-envy free schedule? Justify your answer.
- Is *S* a 3-envy free schedule? Justify your answer.
- If S is not 2-envy-free then return a 2-envy-free scheduling S' whose MAKESPAN is at most $(3/2)^4$ times the MAKESPAN of scheduling S, by using Theorem 11.

9

PRICING

Exercise 3:

Consider the following instance of the Item Pricing problem with 4 items and 3 buyers and the following buyers' valuations:

\ Items Buyers\	1	2	3	4
1	2	3	4	5
2	2	2	3	4
3	1	2	3	6

In the following questions if at least one property is not satisfied then just answer "no" by showing which property is not satisfied.

a) Is the following output a feasible and envy-free outcome? $X=\langle x_1,x_2,x_3\rangle=\langle 1,1,1\rangle$; $p=2$.
b) Is the following output a feasible and envy-free outcome? $X=\langle x_1,x_2,x_3\rangle=\langle 2,1,1\rangle$; $p=1$.
c) Is the following output a feasible and envy-free outcome? $X=\langle x_1,x_2,x_3\rangle=\langle 3,0,1\rangle$; $p=1$.

LEMMA A

Exercise 4:

Consider the following instance of the Item Pricing problem with 4 items and 4 buyers and the following buyers' valuations.

\ Items Buyers\	1	2	3	4
1	4	6	6	8
2	2	4	6	6
3	3	5	7	7
4	2	4	6	6

Consider the following outcome (X,p) : $X=\langle x_1,x_2,x_3,x_4\rangle=\langle 1,2,2,0\rangle$; $p=2$.

Is (X,p) a nearly-feasible and envy-free outcome? Motivate your answer.

If (X,p) is indeed a nearly-feasible and envy-free outcome then Apply Lemma A to it and return the corresponding outcome.

CASO Ø

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Exercise 5:

Consider an instance of the Item Pricing problem with 5 items and 5 buyers with the following buyers' valuations.

\ Items Buyers\	1	2	3	4	5
1	2	6	8	10	12
2	4	7	8	10	13
3	3	6	8	11	14
4	4	7	8	11	14
5	5	5	8	10	11

Consider the following outcome (X,p) : $X=\langle x_1,x_2,x_3,x_4,x_5\rangle=\langle 0,1,2,2,1\rangle$; $p=3$.

Is (X,p) a nearly-feasible and envy-free outcome? Motivate your answer.

If (X,p) is indeed a nearly-feasible and envy-free outcome then Apply Lemma A to it and return the corresponding outcome.

CASO 1

$\hookrightarrow m' < m$

*

Exercise 3:

Consider an instance of the Item Pricing problem with 7 items and 6 buyers with the following buyers' valuations.

\Items Buyers\	1	2	3	4	5	6	7
1	4	8	9	10	12	13	14
2	3	8	9	10	12	12	12
3	2	9	12	11	13	15	19
4	5	8	9	11	11	11	11
5	3	6	9	10	10	12	14
6	1	7	10	11	15	16	16

Consider the following outcome (X,p) : $X=\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle=\langle 2,2,3,1,0,2\rangle$; $p=3$.

Is (X,p) a nearly-feasible and envy-free outcome? Motivate your answer.

If (X,p) is indeed a nearly-feasible and envy-free outcome then Apply Lemma A to it and return the corresponding outcome.

CASO 1

$\hookrightarrow m' > m$

*

Exercise 5:

Consider an instance of the Item Pricing problem with 7 items and 5 buyers with the following buyers' valuations.

\ Items Buyers\	1	2	3	4	5	6	7
1	2	4	6	8	9	12	15
2	3	5	7	8	10	11	14
3	3	5	7	10	11	12	13
4	3	7	8	10	12	12	12
5	3	5	7	9	9	10	10

a) Consider the following outcome (X,p) : $X=\langle x_1,x_2,x_3,x_4,x_5\rangle=\langle 0,1,2,3,4\rangle$; $p=2$.
- Is (X,p) a nearly-feasible and envy-free outcome? Motivate your answer.
- If (X,p) is indeed a nearly-feasible and envy-free outcome then Apply Lemma A to it and return the corresponding outcome.

b) Consider the following outcome (X,p) : $X=\langle x_1,x_2,x_3,x_4,x_5\rangle=\langle 1,1,1,2,3\rangle$; $p=2$.
- Is (X,p) a nearly-feasible and envy-free outcome? Motivate your answer.
- If (X,p) is indeed a nearly-feasible and envy-free outcome then Apply Lemma A to it and return the corresponding outcome.

CASO 2

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* x 6 TOTALI

ALGO 1

Exercise 5:

Consider the following instance of the Item Pricing problem with 7 items and 5 buyers

\ Items Buyers\	1	2	3	4	5	6	7
1	3	8	10	11	16	17	20
2	2	8	8	12	14	15	19
3	6	7	8	11	14	15	15
4	4	7	9	10	11	12	13
5	1	6	7	8	9	10	10

and show the execution of ALGORITHM1, just for the price $p=3$.

x3