Algorithms for Concurrent Distributed Systems:

The Mutual Exclusion problem

Concurrent Distributed Systems

Changes to the model from the MPS:

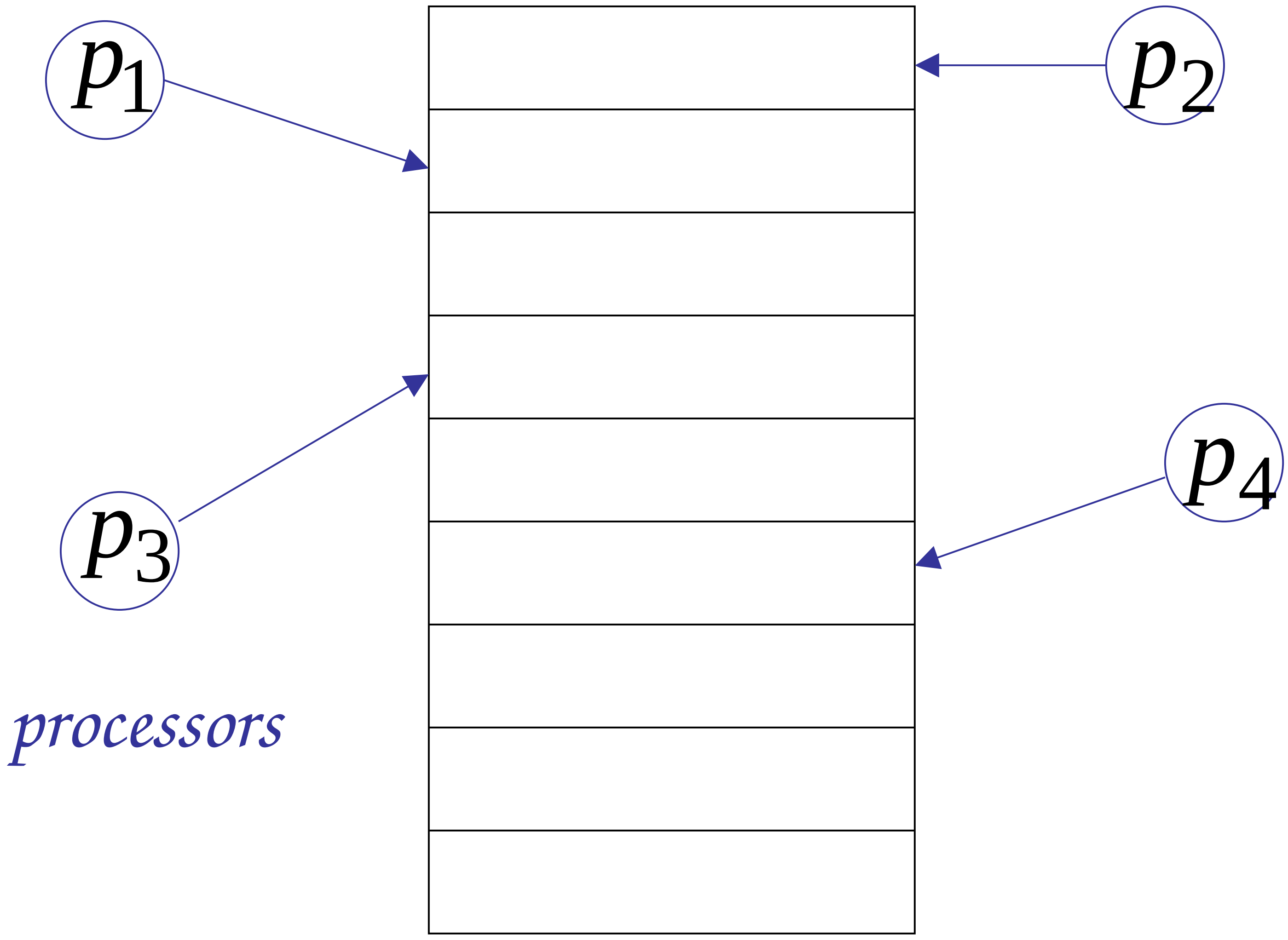
- MPS basically models a distributed system in which processors needs to coordinate to perform a widesystem goal (e.g., electing their leader)
- Now, we turn our attention to concurrent systems, where the processors run in parallel but without necessarily cooperating (for instance, they might just be a set of laptops in a LAN)

Shared Memory System (SMS)

Changes to the model from the MPS:

- No cooperation ⇒ no **communication** channels between processors and no inbuf and outbuf state components
- Processors **notificate** their status via a set of shared variables, instead of passing messages ⇒ no any communication graph!
- Each shared variable has a **type**, defining a set of operations that can be performed **atomically** (i.e., **instantaneously, without interferences)**

Shared Memory



Types of Shared Variables

1. Read/Write
2. Read-Modify-Write
3. Test & Set
4. Compare-and-swap

   .
   .
   .

We will focus on the **Read/Write** type (the simplest one to be realized)

Read/Write Variables

Read(v)

return(v);

Write(v,a)

$v := a$;

In one atomic step a processor can:

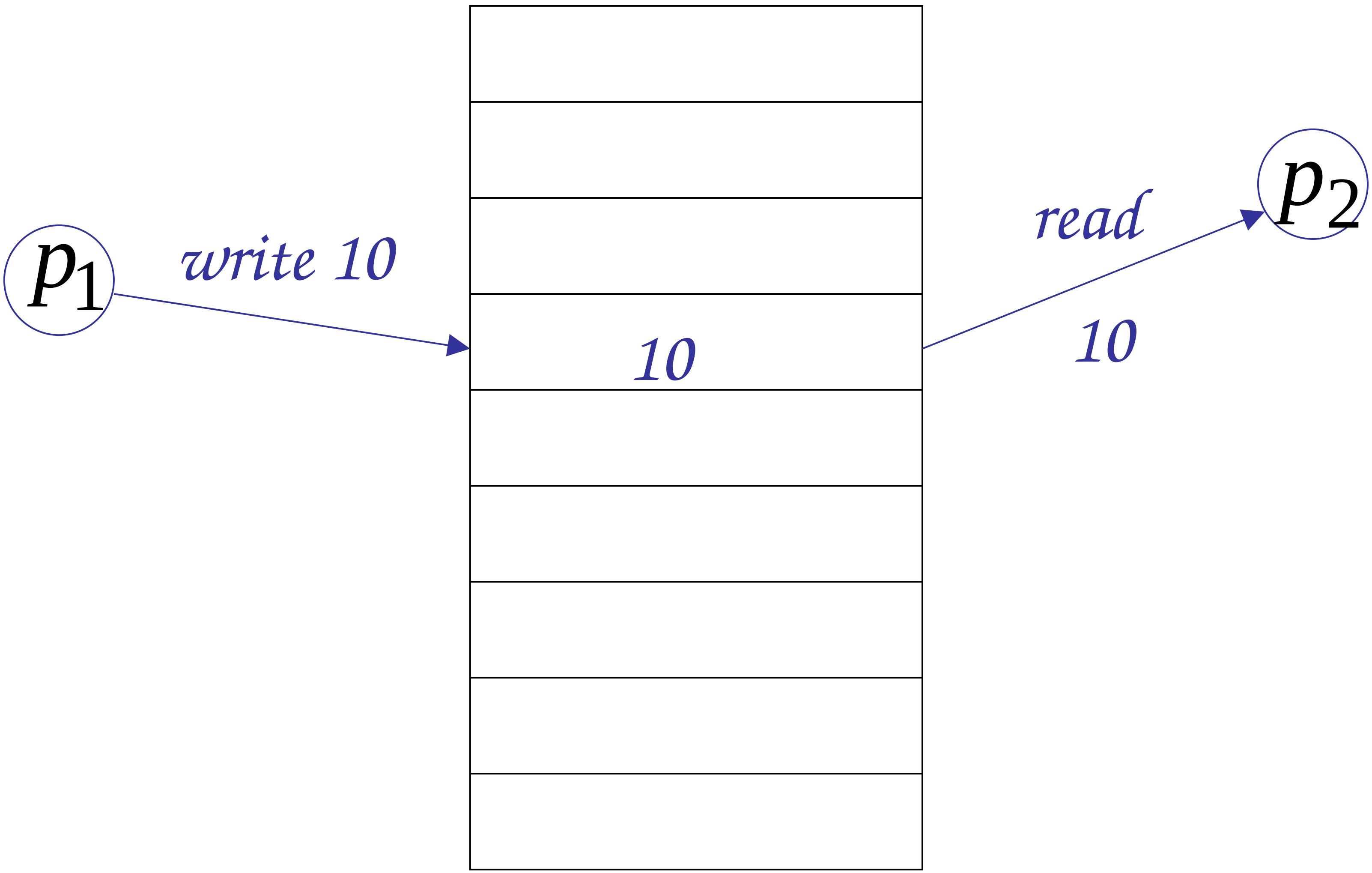
– read the variable, or

– write the variable

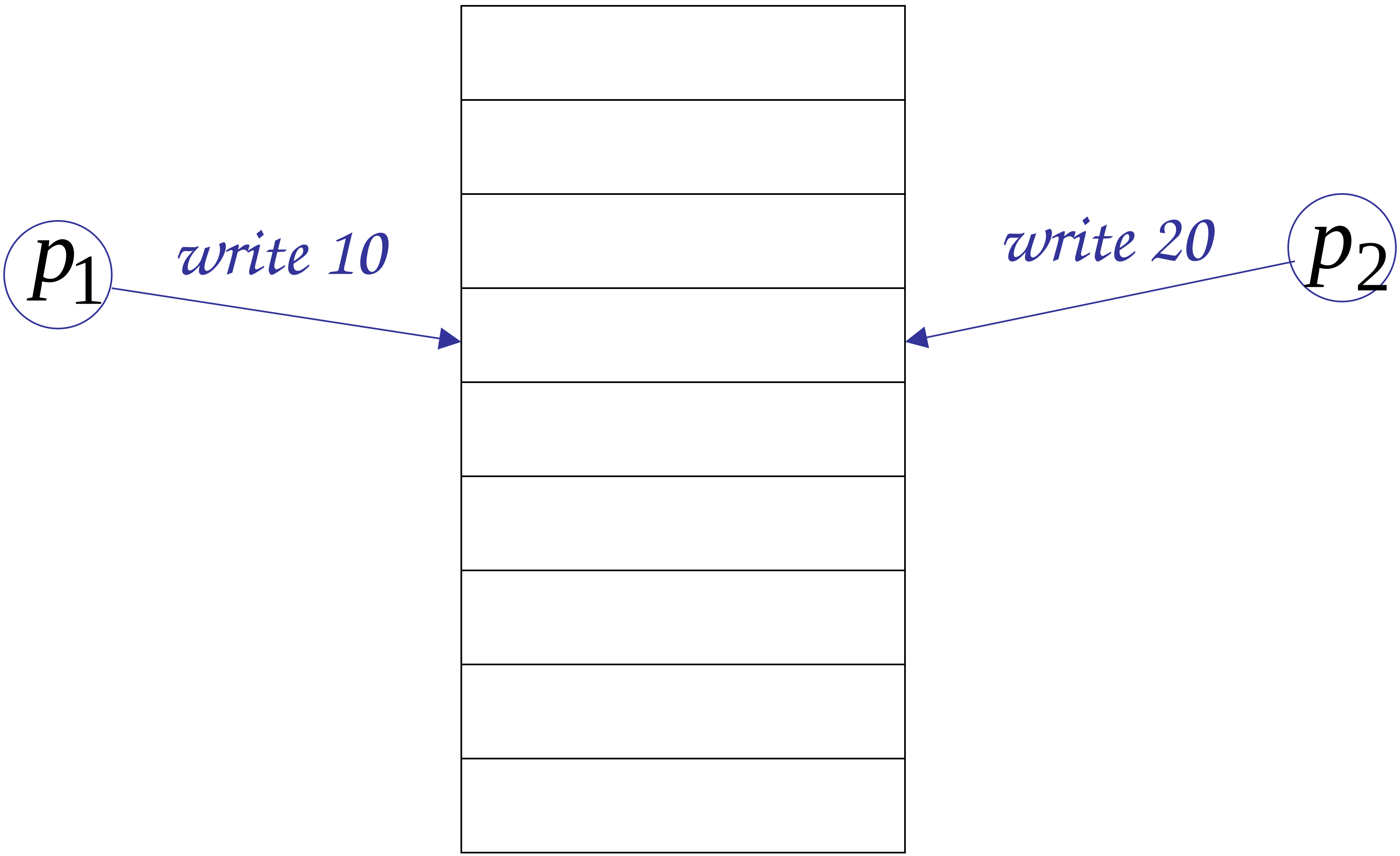
… but not both!

⇒ between a read and a write operation by a given processor (whose written value could be a function of the just read value), some other processor could have changed in the meantime the value of the variable!

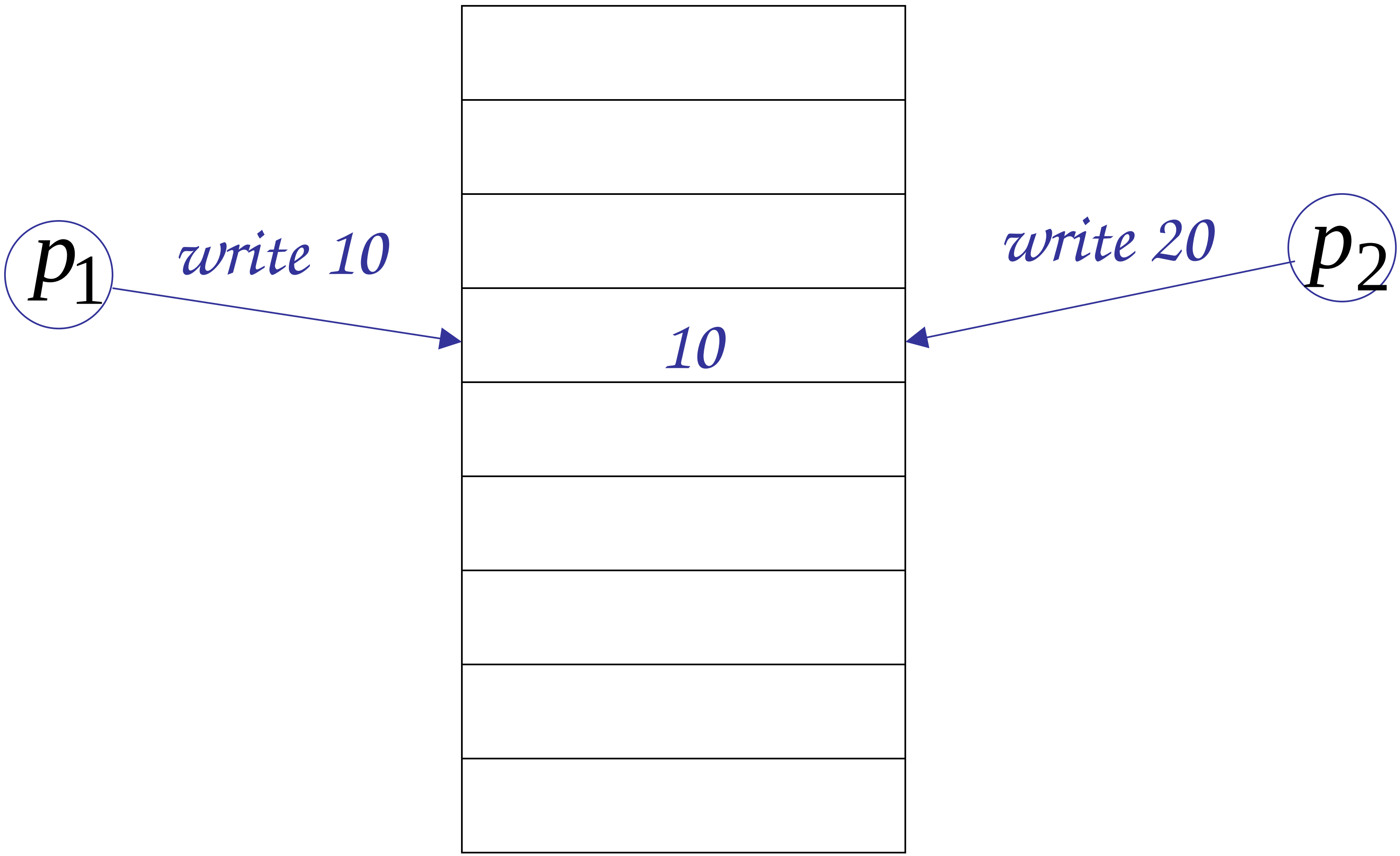
An example



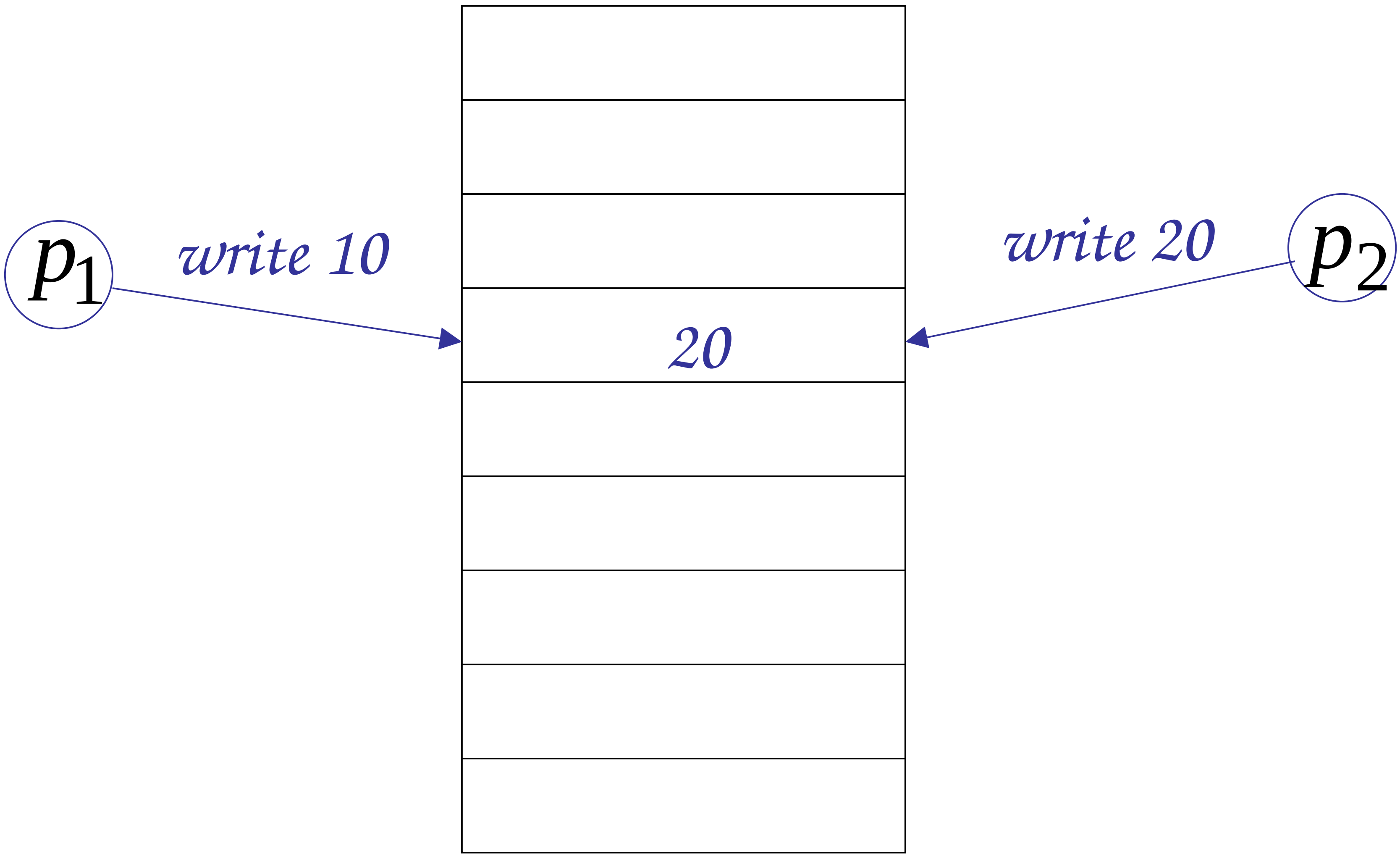
Simultaneous writes



Simultaneous writes are scheduled: Possibility 1



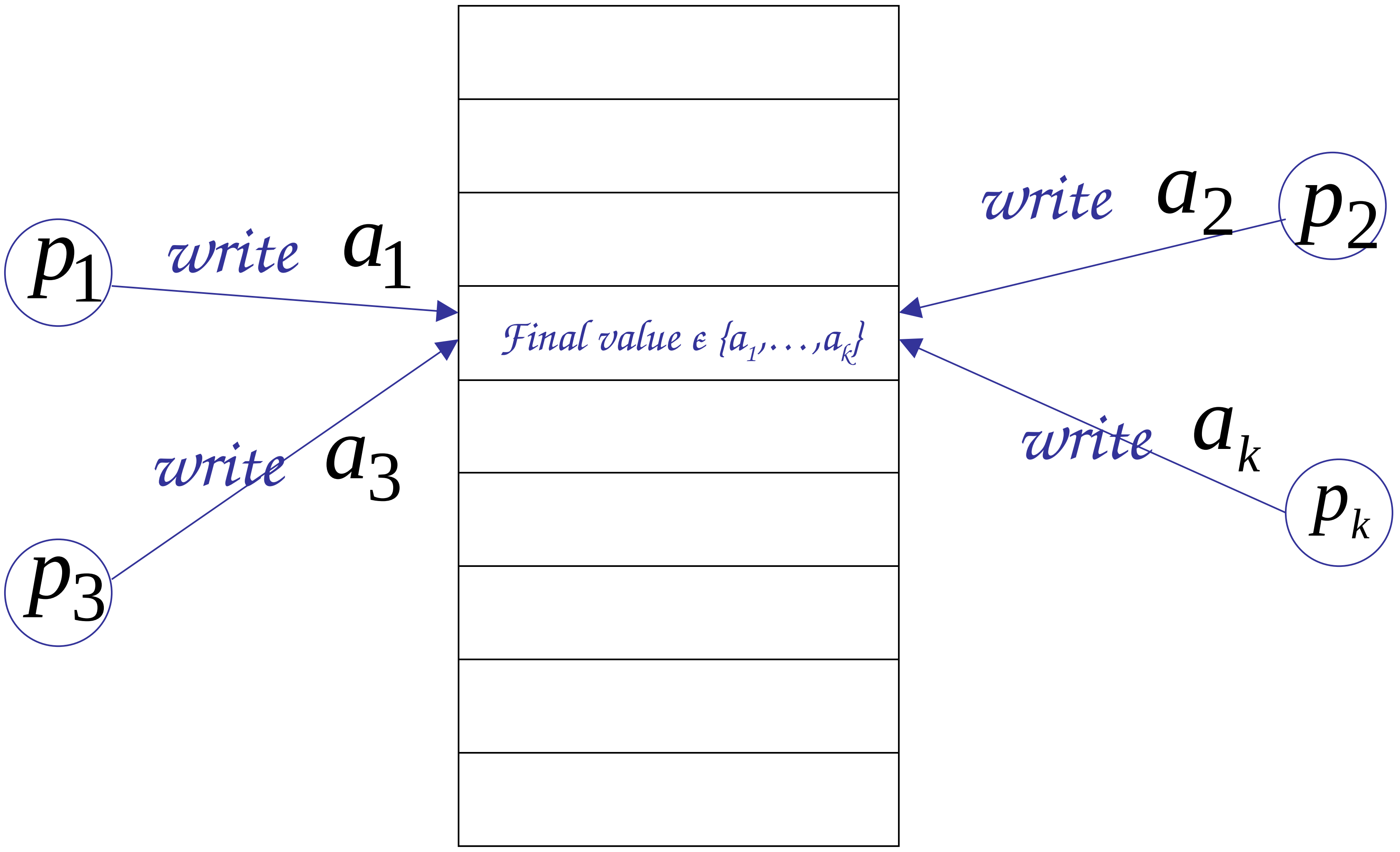
Simultaneous writes are scheduled: Possibility 2



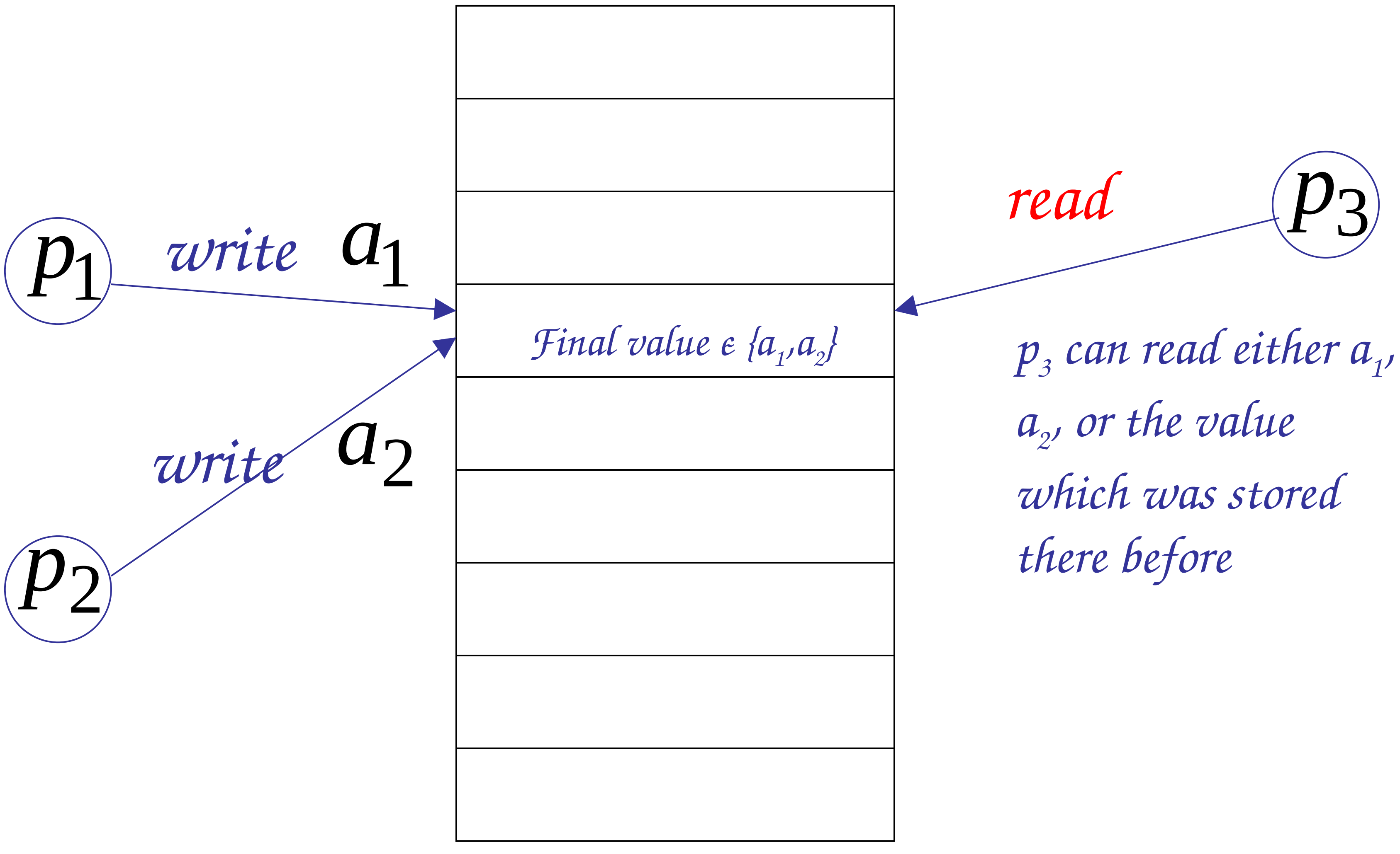
$\Rightarrow$ *the surviving value is arbitrary!*

Simultaneous writes are scheduled:

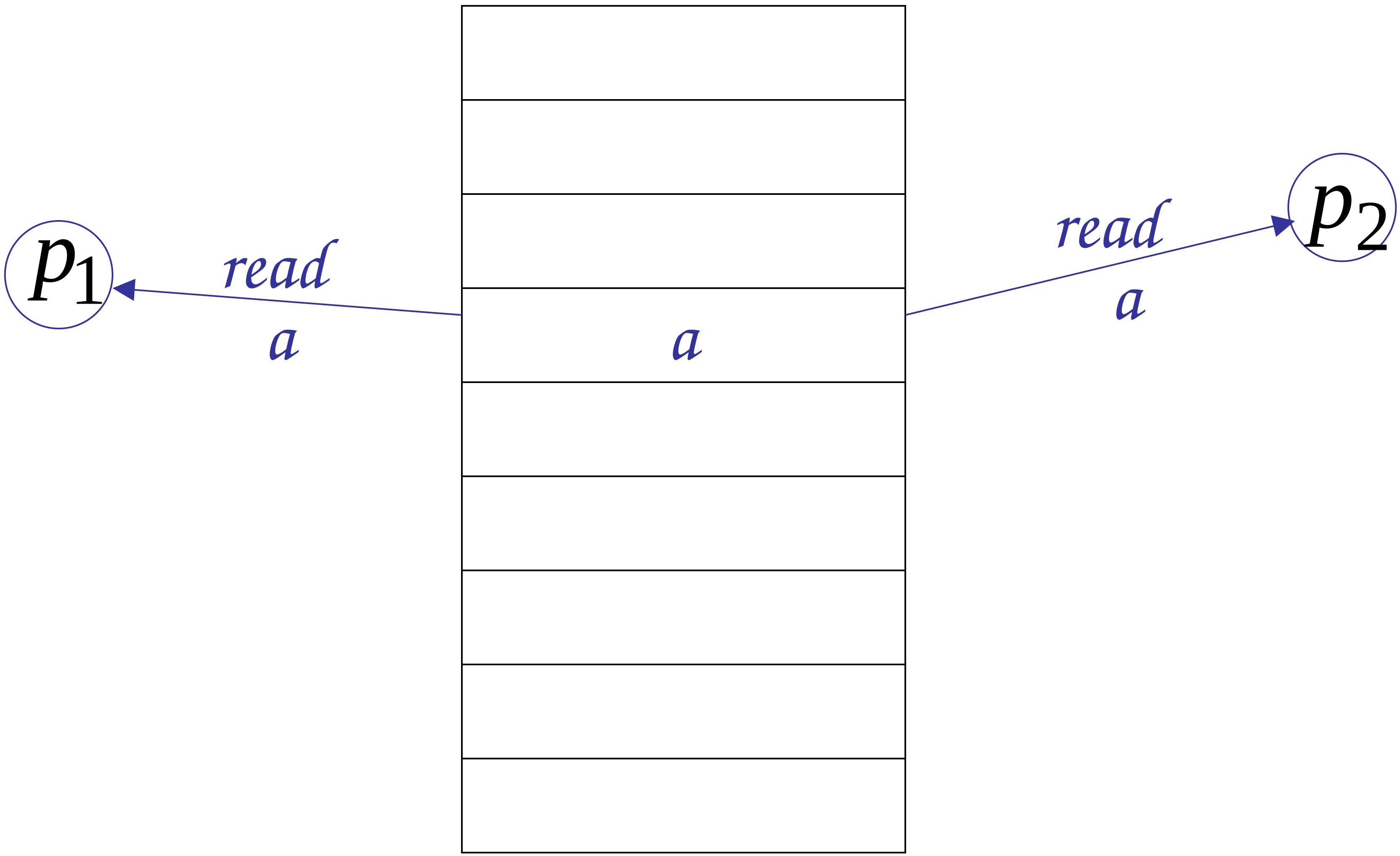
In general:



Simultaneous reads and writes are also scheduled



Simultaneous Reads: no problem!



All read the same value

Computation Step in the Shared Memory System

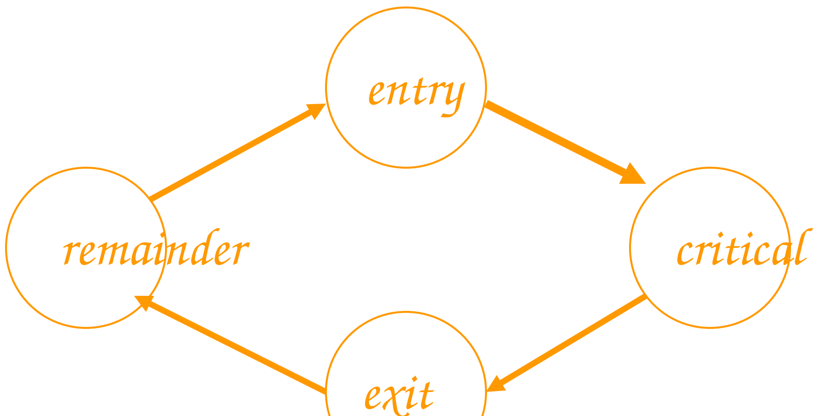
- When processor p_i takes a step:
  - p_i 's state in the old configuration specifies which shared variable is to be accessed and with which operation
  - operation is done: shared variable's value in the new configuration (possibly) changes according to the operation
  - p_i 's state in the new configuration changes according to its old state and the result of the operation

The mutual exclusion (mutex) problem

- The main challenge in managing concurrent systems is coordinating access to resources that are shared among processes
- Assumptions on the SMS (similarly to the MPS):
  - Non-anonymous (ids are in [0..n-1])
  - Non-uniform
  - Asynchronous

Mutex code sections

- Each processor's code is divided into four sections:



  - **entry:** synchronize with others to ensure mutually exclusive access to the …
  - **critical:** use some resource; when done, enter the…
  - **exit:** clean up; when done, enter the…
  - **remainder:** not interested in using the resource

Mutex Algorithms

- A mutual exclusion algorithm specifies code for entry and exit sections to ensure:
  - mutual exclusion: at most one processor is within the critical section at any time, and
  - some kind of liveness condition, i.e., a guarantee on the use of the critical section (under the general assumption that no processor stays in its critical section forever). There are three commonly considered ones:

Mutex Liveness Conditions

- ***no deadlock:*** *if a processor is in its entry section at some point of time, then later some processor (i.e., maybe another one) is in its critical section (notice that a processor can be starved/locked in this situation)*
- ***no lockout:*** *if a processor is in its entry section at some point of time, then later the same processor is in its critical section (but maybe it will be overtaken an unbounded number of times by some other processor)*
- ***bounded waiting:*** *no lockout + while a processor is in its entry section, it can be overtaken in entering into the critical section only a bounded number of times by any other processor*

These conditions are increasingly stronger: bounded waiting ⇒ no lockout ⇒ no deadlock

Complexity Measure for Mutex

- Main complexity measure of interest for shared memory mutex algorithms is amount of shared space needed.
- Space complexity is affected by:
  - how powerful is the type of the shared variables (recall we only focus on Read/Write type)
  - how strong is the liveness condition to be satisfied (no deadlock/no lockout/bounded waiting)

Mutex Results Using R/W

Liveness Condition	upper bound	lower bound
no deadlock		n booleans
no lockout	$3(n-1)$ booleans (for $n=2^k$) (tournament algorithm)	
bounded waiting	n booleans + n unbounded integers (bakery algorithm)	

The Bakery Algorithm (L. Lamport, 1974)

- *Guaranteeing:*
  - *Mutual exclusion*
  - *Bounded waiting*
- *Using 2n shared read/write variables*
  - *booleans* `Choosing[i]` *: initially false, written by p_i and read by others*
  - *(unbounded) integers* `Number[i]` *: initially 0, written by p_i and read by others*

Bakery Algorithm

Code for entry section of p_i:

```
Choosing[i] := true
Number[i] := max{Number[0],...,
                         Number[n-1]} + 1
Choosing[i] := false
for j := 0 to n-1 (except i) do
      wait until Choosing[j] = false
      wait until Number[j] = 0 or
       (Number[j],j) > (Number[i],i)
endfor
```

Doorway subsectio *(DS)*

Bakery subsection (BS)

Semaphores

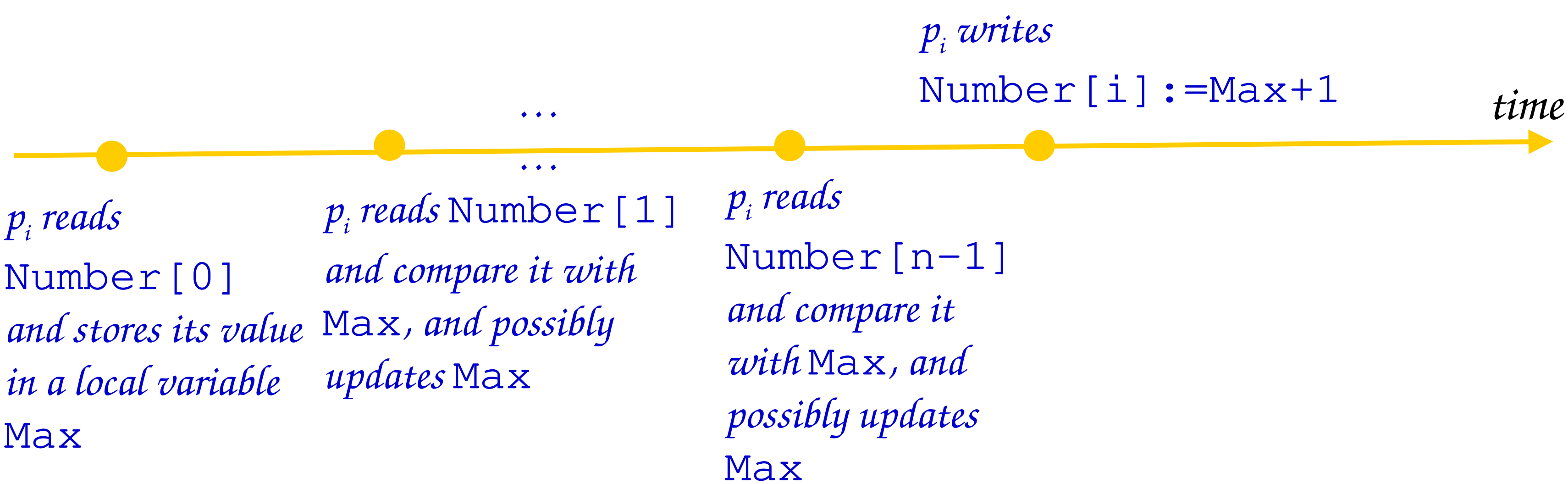
Ticket of p_i

Code for exit section of p_i:

```
Number[i] := 0
```

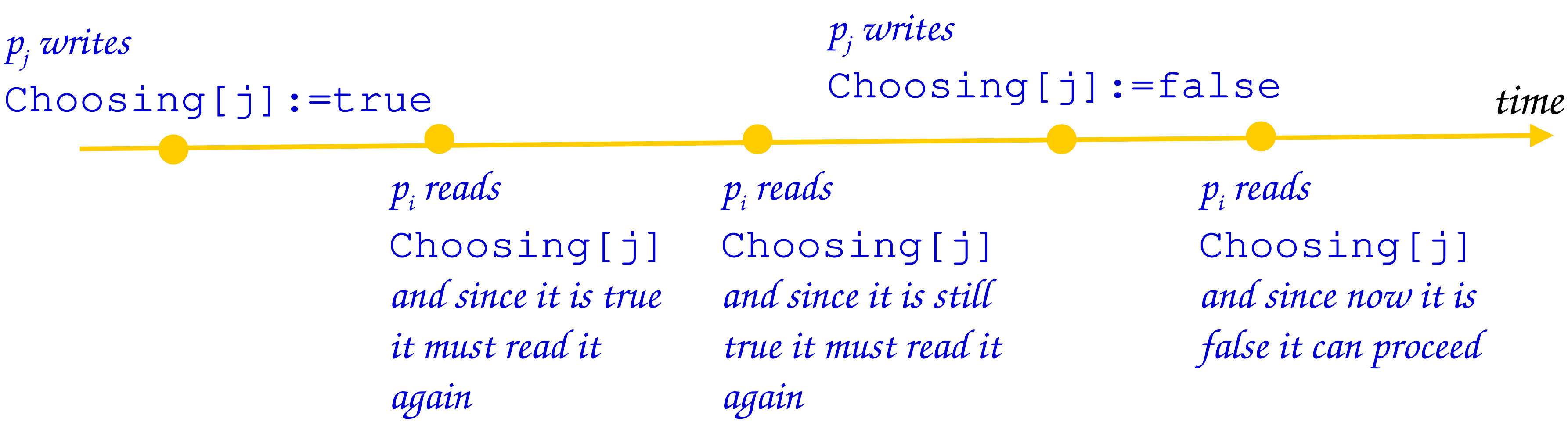
The max operation

The *max* instruction will look like as follows on the timeline of p_i:



The semaphore

The ***wait until*** *instruction will look like as follows on the timeline of* p_i*:*



BA Provides Mutual Exclusion

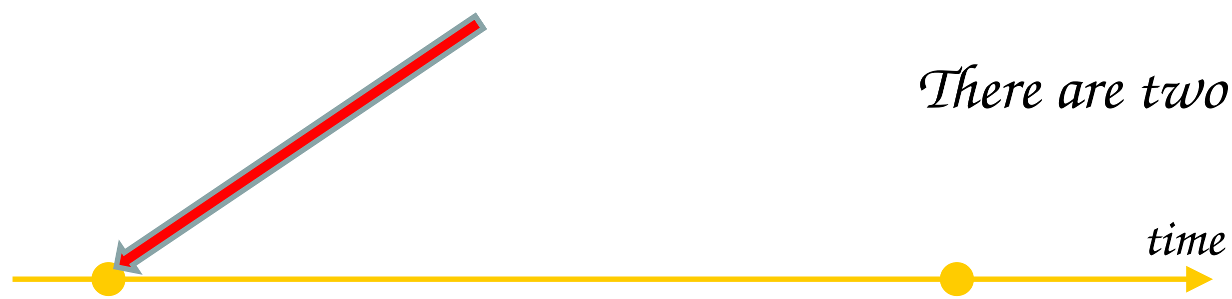
Lemma 1: *If p_i is in the critical section (CS), then* `Number`*[i] > 0.*

Proof: *Trivial: in the doorway section it always takes a number>0.*

Lemma 2: *If p_i is in the CS and* `Number`*[k] ≠ 0 (k ≠ i), then (*`Number`*[k],k) > (*`Number`*[i],i).*

Proof: *Observe that a chosen number changes only after exiting from the CS, and that a number is ≠ 0 iff the corresponding processor is either in the entry (bakery) section or in the CS. Now, since p_i is in the CS, it passed the second wait statement for j=k.*

There are two cases:



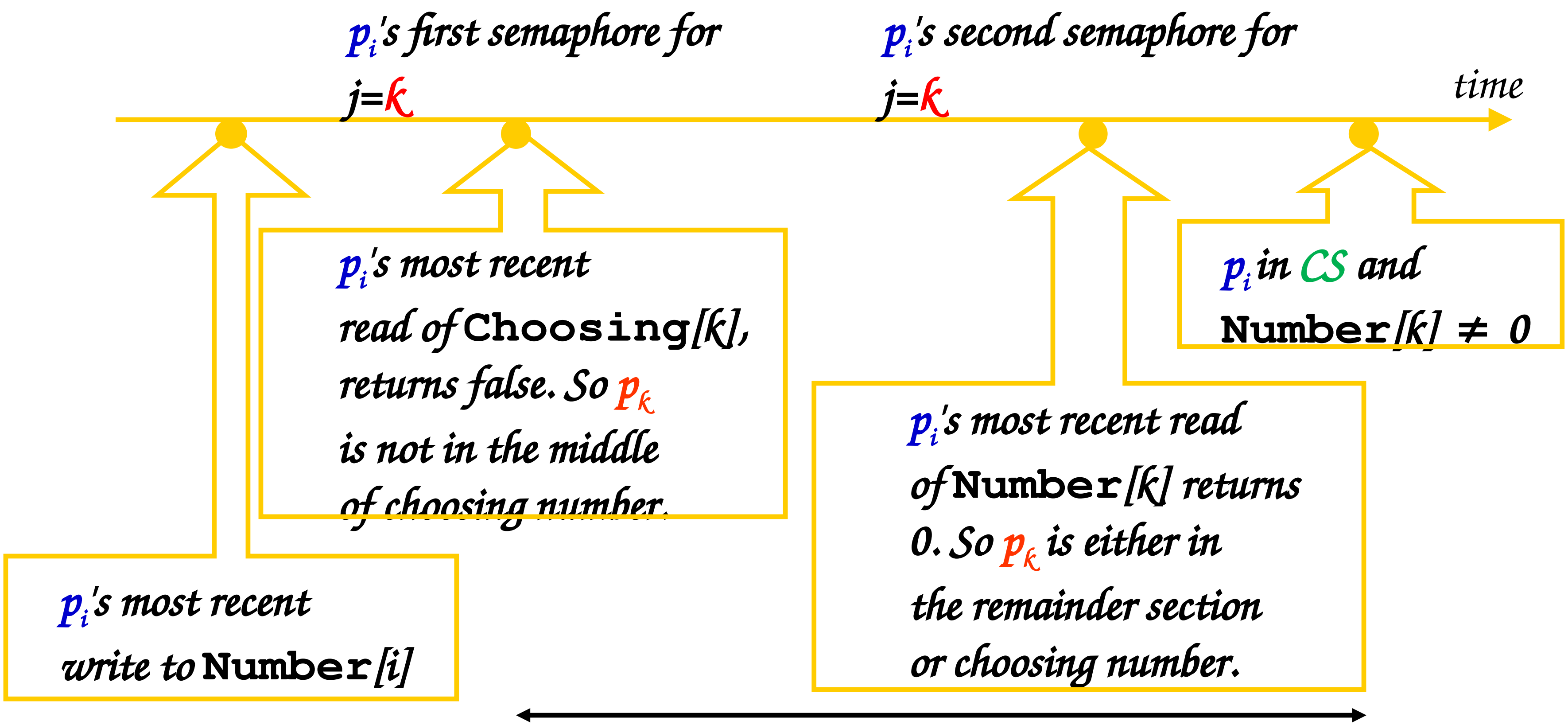
p_i's most recent read of* `Number`*[k] (second semaphore)

p_i in CS and* `Number`*[k] ≠ 0

Case 1: returns 0

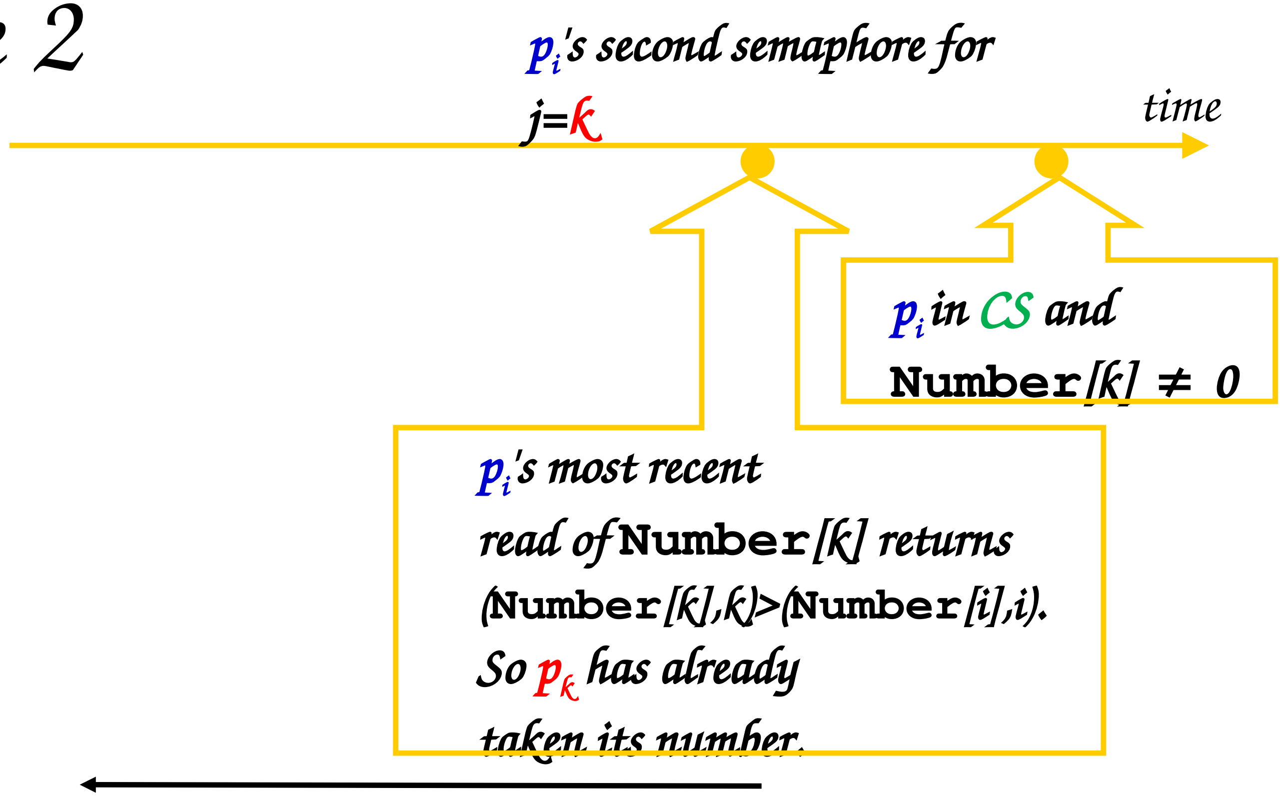
Case 2: returns (*`Number`*[k],k)>(*`Number`*[i],i)

Case 1



So p_k starts choosing its number in this interval, sees p_i's number, and then will choose a larger one (i.e., **Number**[k] > (**Number**[i]); so, it will never enter in CS before than p_i, which means that its number does not change all over the time p_i is in the CS, and so the claim is true

Case 2

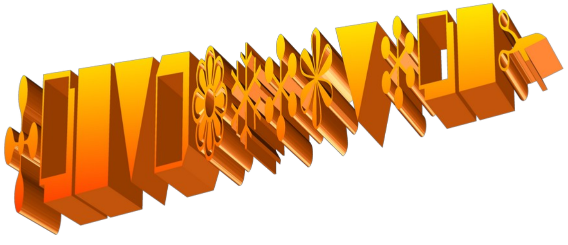


So p_k chooses Number[k] $\geq$ Number[i] in this interval, and does not change it until p_i exits from the CS, since it cannot overtake p_i. Indeed, p_k will be stopped by p_i either in the first wait statement (in case p_k finished its choice before than p_i and p_i is still choosing its number), or in the second one (since (Number[i],i)<(Number[k],k)). Thus, it will remain (Number[i],i)<(Number[k],k) until p_i finishes its CS, and the claim follows.

END of PROOF

Mutual Exclusion for BA

- ***Mutual Exclusion:*** *Suppose* p_i *and* p_k *are simultaneously in CS,* $i \neq k$.
  - *By Lemma 1, both have number > 0.*
  - *Since* `Number`*[k],* `Number`*[i]* $\neq 0$*, by Lemma 2*
    - *(*`Number`*[k],k) > (*`Number`*[i],i) and*
    - *(*`Number`*[i],i) > (*`Number`*[k],k)*



No Lockout for BA

- Assume in contradiction there is a starved processor.
- Starved processors must be stuck at the semaphores, not while choosing a number.
- Starved processors can be stuck only at the second semaphore, since sooner or later the `Choosing` variable of each processor will become false
- Let p_i be a starved processor with smallest (`Number`[i],i).
- Any processor entering entry section after that p_i chose (i.e., wrote) its number, will choose a larger number, and therefore cannot overtake p_i
- Every processor with a smaller ticket eventually enters CS (not starved) and exits, setting its number to 0. So, in the future, its number will be either 0 or larger than `Number`[i]
- Thus p_i cannot be stuck at the second semaphore forever by another processor.

What about bounded waiting?

YES: It is easy to see that any processor in the entry section can be overtaken at most once by any other processor (and so in total it can be overtaken at most $n-1$ times).

Space Complexity of BA

- Number of shared variables is $2n$
- `Choosing` variables are booleans
- `Number` variables are unbounded: as long as the CS is occupied and some processor enters the entry section, the ticket number increases
- Is it possible for an algorithm to use less shared space?

Bounded-space 2-Processor Mutex Algorithm with no deadlock (J.L. Peterson, 1981)

- *Start with a bounded-variables algorithm for 2 processors with no deadlock, then extend to no lockout, then extend to n processors.*
- *Use 2 binary shared read/write variables (intuition: if p_i wants to enter into the CS, then it sets* `W[i]` *to 1):*

`W[0]` *: initially 0, written by p_0 and read by p_1*

`W[1]` *: initially 0, written by p_1 and read by p_0*

- *Asymmetric (or non-homogenous) code: p_0 always has priority over p_1*

Bounded-space 2-Processor Mutex Algorithm with no deadlock

Code for p_0's entry section:

```
.
.
W[0] := 1
.
.
wait until W[1] = 0
```

Semaphore

Code for p_0's exit section:

```
.
W[0] := 0
```

Bounded-space 2-Processor Mutex Algorithm with no deadlock

Code for p_1 's entry section:

```
1   W[1] := 0
2   wait until W[0] = 0
3   W[1] := 1
4   .
5      if (W[0] = 1) then goto Line 1
6   .
```

Semaphore

Code for p_1 's exit section:

```
7   .
8   W[1] := 0
```

Analysis

- *Satisfies mutual exclusion: processors use* `W` *variables to make sure of this (a processor enters only when its own* `W` *variable is set to 1 and the other* `W` *variable* ***is seen*** *to be 0; notice that when p_1 is in the CS and p_0 is waiting at Line 5 in the entry, then both* `W[0]` *and* `W[1]` *are equal to 1, while if p_0 is in the CS and p_1 is waiting at Line 2 in the entry, then* `W[0]=1`*, while* `W[1]=0`*)*
- *Satisfies no-deadlock: if p_0 wants to enter, it cannot be locked by p_1 (since p_1 will be forced to set* `W[1]` *:=0)*
- *But unfair w.r.t. p_1 (it can remain locked, if p_0 sets* `W[0]` *to 1 continuously between line 3 and 5 of p_1 execution)*

⇒ *Fix it by having the processors alternate in having the priority*

Bounded-space 2-Processor Mutex Algorithm with no lockout

Uses 3 binary shared read/write variables and is symmetric:

- `W[0]`: *initially 0, written by p_0 and read by p_1*
- `W[1]`: *initially 0, written by p_1 and read by p_0*
- `Priority`: *initially 0, written and read by both*

Bounded-space 2-Processor Mutex Algorithm with no lockout

Code for p_i's entry section:

```
1   W[i] := 0
2   wait until W[1-i] = 0 or Priority = i
3   W[i] := 1
4   if (Priority = 1-i) then
5     if (W[1-i] = 1) then goto Line 1
6   else wait until (W[1-i] = 0)
```

Semaphores

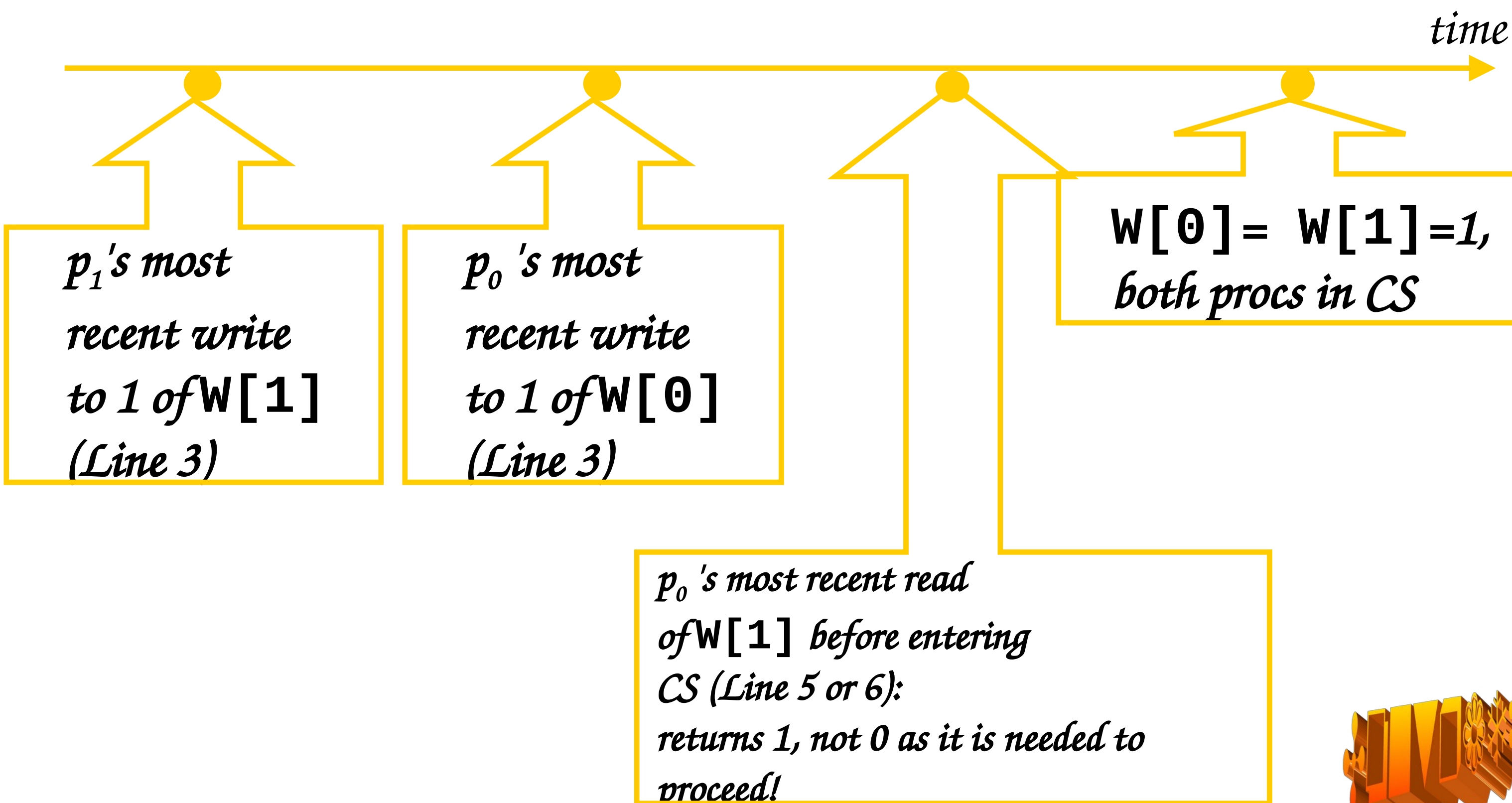
Code for p_i's exit section:

```
7   Priority := 1-i
8   W[i] := 0
```

Analysis: ME

Mutual Exclusion:

- *Suppose in contradiction p_0 and p_1 are simultaneously in CS, and then their* `W[]` *variables are set to 1.*
- *W.l.o.g., assume p_1 last write of* `W[1]` *before entering CS happens* ***not later*** *than p_0 last write of* `W[0]` *before entering CS*

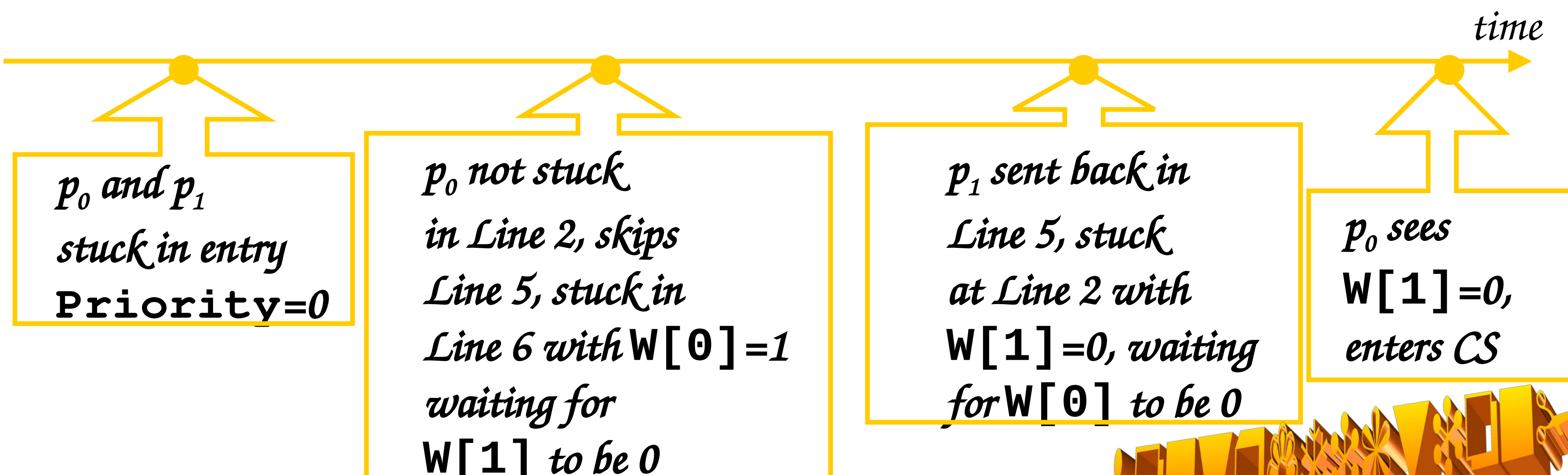


Analysis: No-Deadlock

- *Useful for showing no-lockout.*
- *If one processor ever stays in the remainder section forever, then its* `W[]` *variable will constantly be equal to 0, and so the other processor cannot be starved (it cannot be stuck at Line 5 or 6)*
- *So any deadlock would starve both processors in the entry section*

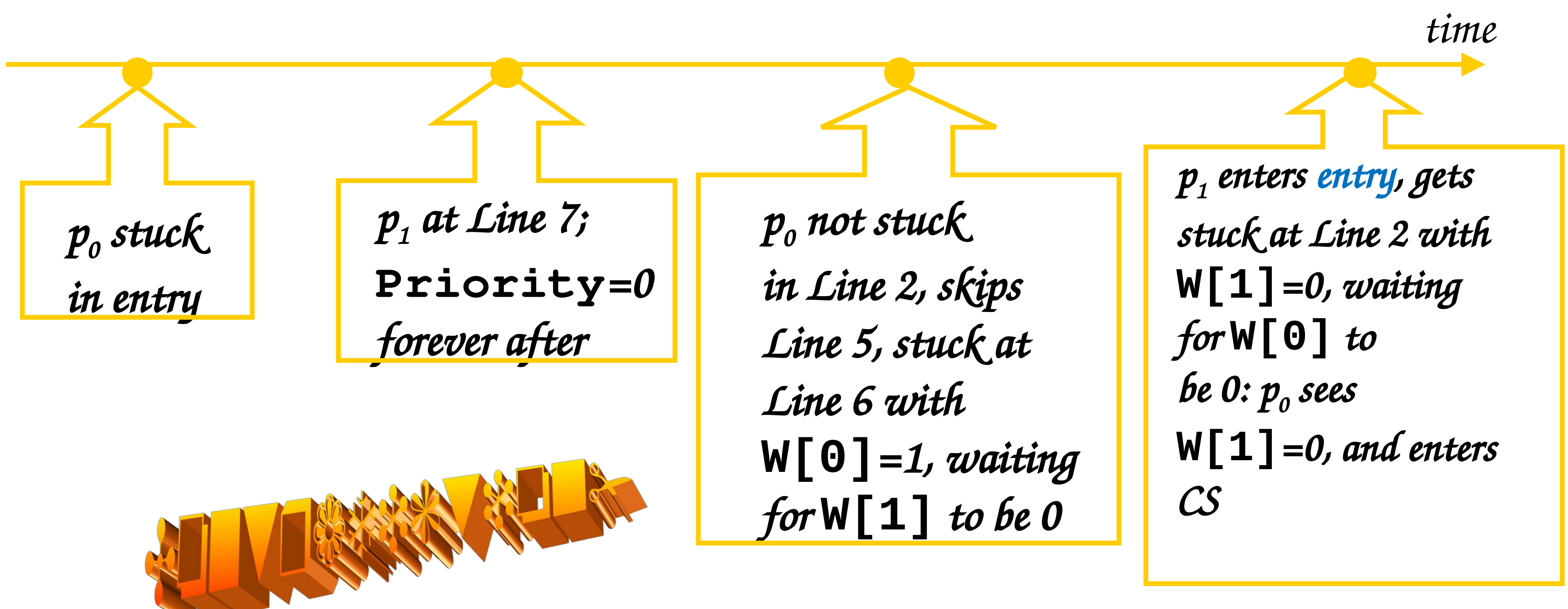
Analysis: No-Deadlock

- *Suppose in contradiction there is deadlock, and w.l.o.g., suppose* `Priority` *gets stuck at 0 after both processors are stuck in their entry sections (indeed* `Priority` *cannot be changed within the entry section):*



Analysis: No-Lockout

- Suppose in contradiction p_0, w.l.o.g., is starved.
- Since there is no deadlock, p_1 enters CS infinitely often.
- The first time p_1 executes Line 7 in exit section after p_0 is stuck in entry, `Priority` gets stuck at 0 (only p_0 can set `Priority` to 1)



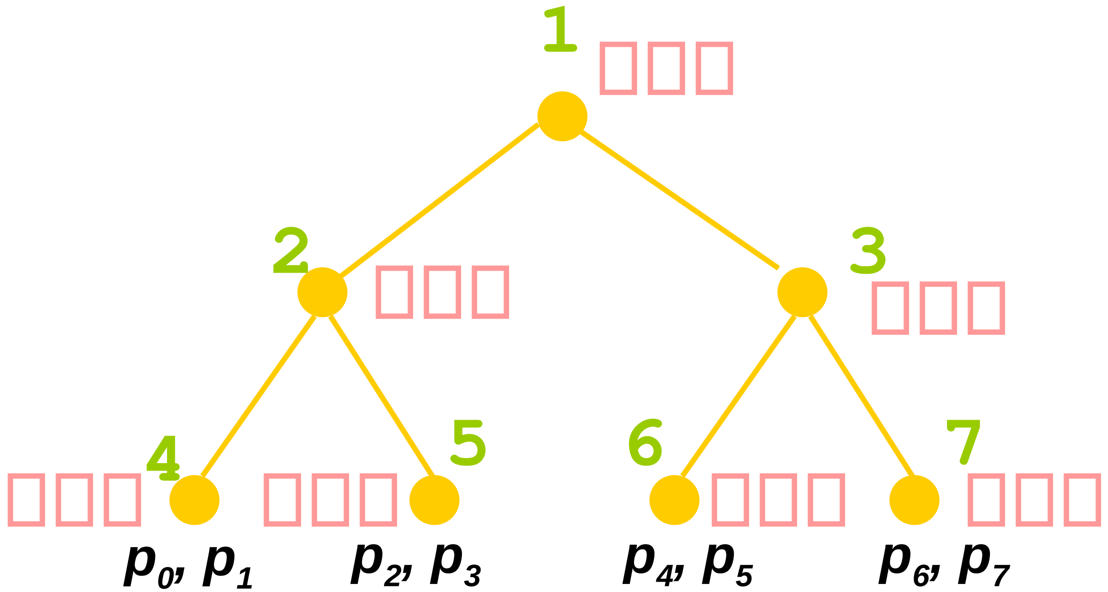
Bounded Waiting?

- *NO: A processor, even if having priority, might be overtaken repeatedly (in principle, an unbounded number of times) when it is in between Line 2 and 3.*

Bounded-space *n*-Processor Mutex Algorithm with no lockout

- Can we get a bounded-space no-lockout mutex algorithm for $n>2$ processors?
- Yes! For the sake of simplicity, assume that $n=2^k$, for some $k>1$.
- Based on the notion of a tournament tree: complete binary tree with $n-1$ nodes
  - tree is conceptual only! does not represent message passing channels
- A copy of the 2-processor algorithm is associated with each tree node
  - includes separate copies of the 3 shared variables

Tournament Tree



We label the tree nodes from top to down and from left to right, from 1 to n-1; then, associate processor p_i, i=0,…, n-1, with the leaf node labelled $2^{k-1} + \lfloor i/2 \rfloor$, where $k = \log n$ (recall that $n=2^k$). Notice that, in general, if $n \neq 2^k$, then we complete the tree by adding "dummy" leaves

Tournament Tree Mutex Algorithm

- *Each processor begins entry section at the associated leaf (2 processors per leaf)*
- *A processor proceeds to next level in the tree by winning the 2-processor competition for current tree node:*
  - *on left side, plays role of p_0*
  - *on right side, plays role of p_1*
- *When a processor wins the 2-processor algorithm associated with the tree root, it enters CS.*

The entry code

procedure Node(v: integer; $side$: 0..1)

L: $want^v_{side} := 0$

wait until ($want^v_{1-side} = 0$ or $priority^v = side$)

$want^v_{side} := 1$

if ($priority^v = 1 - side$) then

  if ($want^v_{1-side} = 1$) then goto L

else wait until ($want^v_{1-side} = 0$)

Entry

if ($v = 1$) then /* at the root */

  ⟨Critical Section⟩

Exit

  else Node($\lfloor v/2 \rfloor$, v mod 2)

$want^v_{side} := 0$

$priority^v := 1 - side$

end procedure

More on TT Algorithm

- Code is recursive
- p_i begins at tree node v labelled $2^{k-1} + \lfloor i/2 \rfloor$, playing role of $p_{i \bmod 2}$, where $k = \log n$.
- After winning at node v, "critical section" for node v is
  - entry code for all nodes on path from $\lfloor v/2 \rfloor$ to root
  - real critical section
- Finally, executes exit code for all nodes on path from root to v (in each of these nodes, gives priority to the other side and sets its want variable to 0)

Analysis

- ***Correctness***: based on correctness of 2-processor algorithm and tournament structure:
  - Mutual exclusion for TT algorithm follows from ME for 2-processors algorithm at tree root.
  - No-lockout for tournament algorithm follows from no-lockout for the 2-processor algorithms at all nodes of tree
- ***Space Complexity:*** 3(n-1) boolean read/write shared variables.
- ***Bounded Waiting?***

No, as for the 2-processor algorithm.

Homework

Consider the mutex problem on a synchronous DS of 8 processors (with ids in 0..7). Show an execution of the tournament tree algorithm by assuming the following:

1. *Initially, all the want and priority variables are equal to 0;*
2. *The system is totally synchronous, i.e., lines of code are executed simultaneously by all processors;*
3. *Throughout the entry section, a processor ends up a round either if it wins the competition (and possibly it enters the CS), or if it executes 7 lines of codes;*
4. *If a node enters the CS at round k, then it exits at round $k+1$;*
5. *Throughout the exit section, a processor ends up a round after having executed the exit code for a node of the tree;*
6. *p_0, p_1, p_3, p_5 and p_6 decide to enter the CS in round 1, while the remaining processors decide to enter the CS in round 2.*

Hints: 16 rounds until the last processor completes the exit section; entering sequence is p_0, p_5, p_3, p_6, p_1, p_4, p_2, p_7