A lower bound to the Price of 1-envy freeness (i.e., k=1)

Theorem 7: The *Price of 1-envy-freeness for identical machines is at least min {n,m}-ε,* for any (small) *ε>0.*

Proof:

- We are going to show an instance where $\frac{C_{\max}(1-Envy-OPT)}{C_{\max}(OPT)} \geq \min\{n,m\}-\varepsilon$

- Consider an instance with *m* machines and *n=m* jobs, such that p_1 =1-ε for some (small) 0<ε<1 and $p_2 = p_3 =....= p_n = 1$

- Notice that since k = 1, in any 1-envy-free scheduling, all the Machine completion time of the non-empty machines (i.e. machines that receive at least one job) must be equal.

…

- INPUT: n jobs, m machines (n=m)

p_i=

J_1	J_2	J_3	…	J_n
1-ε	1	1	1	1

M_1 J_1

M_2 J_2

M_3 J_3

.

.

M_n J_n

0 t

An optimal schedule!

however it is <u>not</u> 1-envy free schedule because machines have different completion time!

In particular all the machines envy M_1

We have to understand what is an optimal 1-envy-free solution!

…

- Let us consider any schedule *S* for our instance that assignes jobs to at least two machines.
- Let M_i be a machine with the maximum number of jobs assigned in *S,* that is $| S_i | \geq | S_j |$ for each $j \neq i$ (notice that there could be more than one machines with the maximum number of jobs assigned).
- *We have two cases:*
  - *Case 1)* J_1 is assigned to some machine $M_j \neq M_i$ in schedule *S* →
  - then M_i envies M_j because the completion time of M_j is strictly smaller than the completion time of M_i. In fact, the number of jobs assigned to M_j is at most the number of jobs assigned to M_i but the completion time of M_j is strictly smaller because J_1 (the only job with processing time 1-ε) is assigned to M_j.

  - Case 2) J_1 is assigned to M_i in schedule *S.* →
  - Then consider the machines with the maximum number of jobs excluding M_i . Let us call such machine M_z
  - It is easy to see that M_i envies M_z or M_z envies M_i .
  - In particular M_i envies M_z if the number of jobs assigned to M_i is strictly greater than the number of jobs assigned to M_z . Otherwise we have that $|S_z|=|S_i|$ and therefore M_z envies M_i because J_1 (the only job with processing time 1-ε) is assigned to M_i.

- Therefore the only feasible 1-envy-free solutions for our instance are the schedulings assigning all the jobs to one machine.
- Such a 1-envy-free scheduling has MAKESPAN equal to $n\text{-}\varepsilon = m\text{-}\varepsilon = \min\{n,m\}\text{-}\varepsilon$

…

- Summarizing, we have an instance where the price of 1-envy-freeness is at least : *min{n,m}-ε*

- *That is,*

$$\frac{C_{\max}(1-Envy-OPT)}{C_{\max}(OPT)}=\frac{\min\{n,m\}-\varepsilon}{1}=\min\{n,m\}-\varepsilon$$

□

An upper bound to the Price of k-envy freeness (k≥1)

Theorem 8: The *Price of k-envy-freeness for identical machines is at most min {n,m} , for any k≥1.*

Proof:

- *Case min{n,m}= n*
  - Any schedule *S* that assignes all the jobs to only one machine is k-envy-free and the MAKESPAN for such solution is at most: $n^* \max_{i=1,\dots,n} \{p_i\}$
  - Clearly the optimal k-envy-free scheduling has MAKESPAN at most such value.
  - *The MAKESPAN* of the optimal solution (non necessarily k-envy-free) is at least: $\max_{i=1,\dots,n} \{p_i\}$

$$\frac{C_{\max}(k-Envy-OPT)}{C_{\max}(OPT)} \leq \frac{C_{\max}(S)}{C_{\max}(OPT)} \leq \frac{n^* \max_{i=1,\dots,n}\{p_i\}}{\max_{i=1,\dots,n}\{p_i\}} = n$$

- *Case min{n,m}= m*
  - Any schedule *S* that assignes all the jobs to only one machine is k-envy-free and the MAKESPAN for such solution is the sum of all the processing times of all the jobs.
  - *The MAKESPAN* of the optimal solution is at least the sum of all the processing times of all the jobs divided by number of machines *m*.

$$\frac{C_{\max}(k-Envy-OPT)}{C_{\max}(OPT)} \leq \frac{C_{\max}(S)}{C_{\max}(OPT)} \leq \frac{\sum_{i=1}^{n} p_i}{\frac{\sum_{i=1}^{n} p_i}{m}} = m$$

□

Value of *k≥2*

- Now we are going to prove a smaller bound to the price of k-envy-freeness for values of *$k \geq 2$.*
- Intuitively, by considering large values of *k*, we are extending the set of scheduling that are k-envy-free.
- Therefore there is hope that we can prove some better (i.e. smaller) bound to the price of k-envy-freeness!

An upper bound to the Price of k-envy freeness ($k \geq 2$)

Theorem 9: The *Price of k-envy-freeness for identical machines is at most $1+1/k$, for any $k \geq 2$.*

Proof:

- We are going to show an algorithm that takes in input an optimal schedule *OPT* for the problem without k-envy-free constraint and transforms *OPT* into a k-envy-free scheduling whose MAKESPAN is at most $1+1/k$ the MAKESPAN of OPT.

- Let us consider the algorithm of the next slide:

…An upper bound to the Price of k-envy freeness (k≥2)

The algorithm:

INPUT: Optimal schedule *OPT* for the problem without k-envy-free constraint.

1. Rescale all the machine completion times of *OPT* in a way that $C_{max}(OPT)=1$. We can get it by dividing all the processing times p_i by the value of the MAKESPAN of the optimal solution *OPT (this is just to make the proof easier).*
2. While *there exists a pair of machines (j,j') such that* $MC_j(OPT) + MC_{j'}(OPT) \leq 1$ *then:*
   - $OPT_j = OPT_j \cup OPT_{j'}$; (*$OPT_j$ is the set of jobs assigned to machine j in OPT*)
   - $OPT_{j'} = \emptyset$;
3. End while. (Let m' be the number of machines with at least one job assigned);
4. Renumber (rename) the machines in non-increasing order of machine completion time, such that $MC_1(OPT) \geq MC_2(OPT) \geq \ldots \geq MC_{m'}(OPT)$.
5. *Create a new assignment S as follow:*
   - *If* $MC_{m'}(OPT) < 1/k$ *then*
     - ✓ $S_j = OPT_j$ *for each* $j=1,\ldots,m'-2$;
     - ✓ $S_{m'-1} = OPT_{m'-1} \cup OPT_{m'}$;
     - ✓ $S_{m'} = \emptyset$;
   - *else*
     - ✓ $S_j = OPT_j$ *for each* $j=1,\ldots,m'$;
   - *End If.*
6. Return *S.*

… An upper bound to the Price of k-envy freeness (k≥2)

- We first prove that the schedule S returned by the algorithm is k-envy-free.
- At line 4 of the algorithm, $M_{m'}$ is the machine with the smallest machine completion time.
- We have two cases:
  1. - (at line 5 of the algorithm), if $MC_{m'}(OPT) \geq 1/k$ (that is the "else" branch of the If at line 5) then the returned S is k-envy free because all the other machines but $M_{m'}$ have machine completion time at most 1, and clearly $1 \leq k^* MC_{m'}(OPT)$
  2. - (at line 5 of the algorithm), if $MC_{m'}(OPT)<1/k$ then the algorithms moves all the jobs of machine $M_{m'}$ to machine $M_{m'-1}$, obtaining a new schedule that we are going to prove it is k-envy-free.

     - notice that machine $M_{m'-1}$ gets a machine completion time larger than 1 in S and therefore it is the machine with largest completion time in S. It means that if machine $M_{m'-1}$ is not envious, then all the other machines are not envious as well.

     - $MC_{m'-1}(S) = MC_{m'-1}(OPT) + MC_{m'}(OPT) \leq 2{*}MC_{m'-1}(OPT) \leq k{*}MC_{m'-1}(OPT) \leq k{*}\ MC_j(S)$ *for any* $j=1,\dots,m'-1$*, and for any* $k\geq 2$*.*

Thus we conclude that S is k-envy-free.

… An upper bound to the Price of k-envy freeness ($k \geq 2$)

- Now we analyse the MAKESPAN of the returned schedule *S*.
- The MAKESPAN of the optimal solution *OPT* is 1 (recall in the algorithm we rescale all the processing time so that the MAKESPAN of *OPT* is 1).
- We have two cases:
  1. (at line 5 of the algorithm), if $MC_{m'}(OPT) \geq 1/k$ (that is the "else" branch of the If at line 5) then in the returned scheduling *S* all the machines have machine completion time at most 1 and therefore it is optimal.
  2. (at line 5 of the algorithm), if $MC_{m'}(OPT) < 1/k$ then the algorithms returns a solution *S* whose MAKESPAN is given by machine $M_{m'-1}$ that gets a machine completion time larger than 1 in *S.*

… An upper bound to the Price of k-envy freeness (k≥2)

- In such a case we have that:

$$MC_{m'-1}(S) = MC_{m'-1}(OPT) + MC_{m'}(OPT) \leq MC_{m'-1}(OPT) + 1/k \leq 1 + 1/k.$$

- By recalling that the MAKESPAN of the optimal solution *OPT* is 1 (recall in the algorithm we rescale all the processing time so that the MAKESPAN of *OPT* is 1), we get that:

$$\frac{C_{\max}(k-Envy-OPT)}{C_{\max}(OPT)} \leq \frac{C_{\max}(S)}{C_{\max}(OPT)} \leq \frac{1+1/k}{1} = 1+1/k$$

□

A lower bound to the Price of k-envy freeness (k≥2)

Theorem 10: The *Price of k-envy-freeness for identical machines is at least $1 + 1/k - \varepsilon$, for any (small) $\varepsilon>0$, and for any $k\geq 2$.*

Proof:

- We are going to show an instance where $\frac{C_{\max}(k-Envy\_OPT)}{C_{\max}(OPT)} \geq 1+\frac{1}{k}-\varepsilon$ for any small $\varepsilon>0$.

- Consider an instance with *m* machines and *n=m* jobs, such that $p_1 = 1/k - \varepsilon$ for some $\varepsilon>0$ and $p_2 = p_3 = \ldots = p_n = 1$.
- It is easy to see that an optimal solution *OPT* (without the k-envy-free constraint) assigns a single job to each machine and the MAKESPAN is 1.
- Notice that such *OPT* is not k-envy-free, since each machine with completion time 1 envies the machine of completion time $1/k - \varepsilon$ (in fact $1 > k*(1/k - \varepsilon)$).
- Indeed, any k-envy-free scheduling is forced to assign to some machine the job j_1 together with at least another job j_i for a MAKESPAN of at least $1 + 1/k - \varepsilon$.
- *We get*

$$\frac{C_{\max}(k-Envy-OPT)}{C_{\max}(OPT)} \geq \frac{1+\frac{1}{k}-\varepsilon}{1} = 1+\frac{1}{k}-\varepsilon \qquad \square$$