Revenue Maximizing Envy-free Pricing Problem

Resources

- As far as the exam is concerned, it is enough studying (and then deeply understanding) the following slides.

- However, if interested, you can find results contained in these slides and much more at this paper:

G. Monaco, P. Sankowski, Q. Zhang: Revenue Maximizing Envy-free Pricing for Homogeneous Resources. *Proceedings of the 24th International Joint Conference on Articial Intelligence (IJCAI), 2015.*

Introduction

- Imagine that we are a company or store in the business of selling products to consumers.
- An important aspect of maximizing the revenue obtained is the pricing of our products: a low price will attract more customers or buyers, while a high price generates more revenue per sold item.
- How, then, should we choose prices optimally?

- This is the pricing problem we are going to study.

Introduction (2)

- We consider the problem of assigning m homogeneous items or resources (that is m identical items) among n buyers where buyers' valuations depend on the number of items they get.
- This multi-unit settings is the simplest case where allocating different number of items to different customers plays a crucial role, i.e., it is arguably the most basic variant of combinatorial auctions.
- Somewhat astonishingly, this very simple setting generates a rich set of problems to study.
- This setting is of practical importance as these items could be homogeneous server grids or supercomputers in computer networks, power supply in manufacture systems, cargo space in transportation industry, hardware resources, SaaS, utility computing, etc.
- Our goal is to design polynomial pricing mechanisms that maximize the seller's revenue and assign items to buyers in a fair manner.

- One of the common notions used to model fair division is *envy-freeness*, where each buyers prefer their allocations to the allocations received by other buyers.

- Envy-freeness dates back to the 20th century Foley (1967); Varian (1974) and is still an intense research topic in mathematics, computer science and economics.

Introduction (3)

- We assume that through market research or interaction with the buyers, the seller knows each buyer's *valuation* for each subset (also called *bundle)* of items, that is the largest amount of money that the customer is willing to pay for that subset.
- If a buyer buys a subset of items, his (or her) *utility* is the difference between the valuation and the purchase price, i.e., the amount of money he "saved" compared to his valuation.

Item pricing problem (INPUT)

- We have m identical and indivisible *items* we want to sell;
- There are *n buyers*. Each buyer has a valuation function $v_i : \{1,2,\ldots,m\} \mapsto \Re^{+}$

that maps the number of items to a non-negative real number.

- For example $v_i(5)$ indicates the valuation that buyer *i* has for receiving 5 items.
- We always suppose that $v_i(0)=0$ for any buyer.

Item pricing problem (OUTPUT)

- Find a feasible and envy-free outcome (X, p) consisting of an allocation vector $X=<x_1,\dots, x_n>$ (where x_i is the number of items we sell to buyers i) and an item price $p\geq 0$ (p is the price for a single item) such that the revenue is maximized.
- The revenue is defined as $p\sum_{i=1}^{n} x_i$

An outcome is feasible if it satisfies the following conditions:

- *supply constraint*: it holds that $\sum_{i=1}^{n} x_i \leq m$
- *individual rationality*: for any $i=1,\dots,n$ it holds that the utility of each buyer is non-negative, i.e., $u_i(x_i, p)\geq 0$ (if the utility is negative then the buyer does not buy any item) where the utility is defined as $u_i(x_i, p)=v_i(x_i) - px_i$

Item pricing problem (OUTPUT) (2)

- We say that buyer *i* envies buyer *i'* if buyer *i* prefers buyer *i'*'s assignment, that is: $u_i(x_i, p) < u_i(x_{i'}, p)$
- An outcome is *envy-free* if no buyer envies other buyers.

Motivations

- Envy-free $\longrightarrow$ buyers have no incentive to change allocations.

- Maximum revenue $\longrightarrow$ seller has no incentive to change prices.

An Example

- m=5, n=4
- The following matrix (that is the INPUT) specifies all the buyers' valuations

|items Buyer|	1	2	3	4	5
Buyer 1	3	4	2	2	7
Buyer 2	2	1	1	4	3
Buyer 3	3	4	3	2	5
Buyer 4	1	2	2	4	3

$v_2(3)$ valuation of buyer 2 for receiving 3 items

An Example (2)

- Let us consider the following OUTPUT:
- $X=<x_1,x_2, x_3,x_4>= <1,1,2,0>; \quad p=2$
- Is this a feasible and envy-free outcome?
  - Supply constraint: $\sum_{i=1}^{n} x_i = 1+1+2+0=4 \leq 5$ ok!
  - Individual rationality:
  - Buyer 1: $u_1(x_1,p)=v_1(x_1) - px_1 \longrightarrow u_1(1,2)=v_1(1) - 2*1= 3-2=1 \geq 0$ *OK!*
  - Buyer 2: $u_2(x_2,p)=v_2(x_2) - px_2 \longrightarrow u_2(1,2)=v_2(1) - 2*1= 2-2=0 \geq 0$ *OK!*
  - Buyer 3: $u_3(x_3,p)=v_3(x_3) - px_3 \longrightarrow u_3(2,2)=v_3(2) - 2*2= 4-4=0 \geq 0$ *OK!*
  - Buyer 4: $u_4(x_4,p)=v_4(x_4) - px_4 \longrightarrow u_4(0,2)=v_4(0) - 2*0= 0-0=0 \geq 0$ *OK!*
  - Envy-freeness: $\xrightarrow{\text{See next slide}}$

An Example (3)

- <u>Envy-freeness</u>:
  - Buyer 1: $u_1(x_1, p) = u_1(x_2, p)$ *because* $x_1 = x_2$ (buyer 1 does not envy buyer 2)
    $u_1(x_1, p) = 1;\quad u_1(x_3, p) = v_1(2) - 2*2 = 4-4 = 0$ (buyer 1 does not envy buyer 3)
    $u_1(x_1, p) = 1;\quad u_1(x_4, p) = v_1(0) - 2*0 = 0-0 = 0$ (buyer 1 does not envy buyer 4)
  - Buyer 2: $u_2(x_2, p) = u_2(x_1, p)$ *because* $x_1 = x_2$ (buyer 2 does not envy buyer 1)
    $u_2(x_2, p) = 0;\quad u_2(x_3, p) = v_2(2) - 2*2 = 1-4 = -3$ (buyer 2 does not envy buyer 3)
    $u_2(x_2, p) = 0;\quad u_2(x_4, p) = v_2(0) - 2*0 = 0-0 = 0$ (buyer 2 does not envy buyer 4)
  - Buyer 3: $u_3(x_3, p) = 0;\quad u_3(x_1, p) = v_3(1) - 2*1 = 3-2 = 1$ (buyer 3 envies buyer 1)
    $u_3(x_3, p) = 0;\quad u_3(x_2, p) = v_3(1) - 2*1 = 3-2 = 1$ (buyer 3 envies buyer 2)
    $u_3(x_3, p) = 0;\quad u_3(x_4, p) = v_3(0) - 2*0 = 0-0 = 0$ (buyer 3 does not envy buyer 4)
  - Buyer 4: $u_4(x_4, p) = 0;\quad u_4(x_1, p) = v_4(1) - 2*1 = 1-2 = -1$ (buyer 4 does not envy buyer 1)
    $u_4(x_4, p) = 0;\quad u_4(x_2, p) = v_4(1) - 2*1 = 1-2 = -1$ (buyer 4 does not envy buyer 2)
    $u_4(x_4, p) = 0;\quad u_4(x_3, p) = v_4(2) - 2*2 = 2-4 = -2$ (buyer 4 does not envy buyer 3)

Then the outcome is not Envy-free! Since there is at least one buyer (i.e., buyer 3) that is envious.

An Example (4)

- Let us consider the following OUTPUT (still for the instance defined in slide 10):
- $X = \langle x_1, x_2, x_3, x_4 \rangle = \langle 1,0,1,0 \rangle$; $p=3$
- Is this a feasible and envy-free outcome?
  - Supply constraint: $\sum_{i=1}^{n} x_i = 1+0+1+0 = 2 \leq 5$ ok!
  - Individual rationality:
  - Buyer 1: $u_1(x_1, p) = v_1(x_1) - px_1$ $u_1(1,3) = v_1(1) - 3*1 = 3-3 = 0 \geq 0$ *OK!*
  - Buyer 2: $u_2(x_2, p) = v_2(x_2) - px_2$ $u_2(0,3) = v_2(0) - 3*0 = 0-0 = 0 \geq 0$ *OK!*
  - Buyer 3: $u_3(x_3, p) = v_3(x_3) - px_3$ $u_3(1,3) = v_3(1) - 3*1 = 3-3 = 0 \geq 0$ *OK!*
  - Buyer 4: $u_4(x_4, p) = v_4(x_4) - px_4$ $u_4(0,3) = v_4(0) - 3*0 = 0-0 = 0 \geq 0$ *OK!*
  - Envy-freeness: *you can easily check that this is an envy-free solution. In fact buyer 2 and buyer 4 have valuations smaller than 3 for receiving one item.*

What is the revenue? $p\sum_{i=1}^{n} x_i = 3*2 = 6$

Is this an optimal solution? I.e., does there exist a different envy-free outcome that achieves strictly higher revenue?

An Example (5)

- Let us consider the following OUTPUT (still for the instance defined in slide 10):
- $X = \langle x_1, x_2, x_3, x_4 \rangle = \langle 5,0,0,0 \rangle$; $p=7/5$
- Is this a feasible and envy-free outcome?
  - Supply constraint: $\sum_{i=1}^{n} x_i = 5+0+0+0 = 5 \leq 5$ ok!
  - Individual rationality:
  - Buyer 1: $u_1(x_1, p) = v_1(x_1) - px_1$ $u_1(5,7/5) = v_1(5) - 7/5*5 = 7-7=0 \geq 0$ *OK!*
  - Buyer 2: $u_2(x_2, p) = 0$ *OK!*
  - Buyer 3: $u_3(x_3, p) = 0$ *OK!*
  - Buyer 4: $u_4(x_4, p) = 0$ *OK!*
  - Envy-freeness*: you can easily check that this is an envy-free solution. In fact all the buyers but 1 have valuations smaller than 7 for receiving five items.*

What is the revenue? $p\sum_{i=1}^{n} x_i = 7/5*5 = 7$

Is this an optimal solution? I.e., does there exist a different envy-free outcome that achieves strictly higher revenue?

An Example (6)

- There exists at least an envy-free solution that achieves revenue 8 (HOMEWORK).