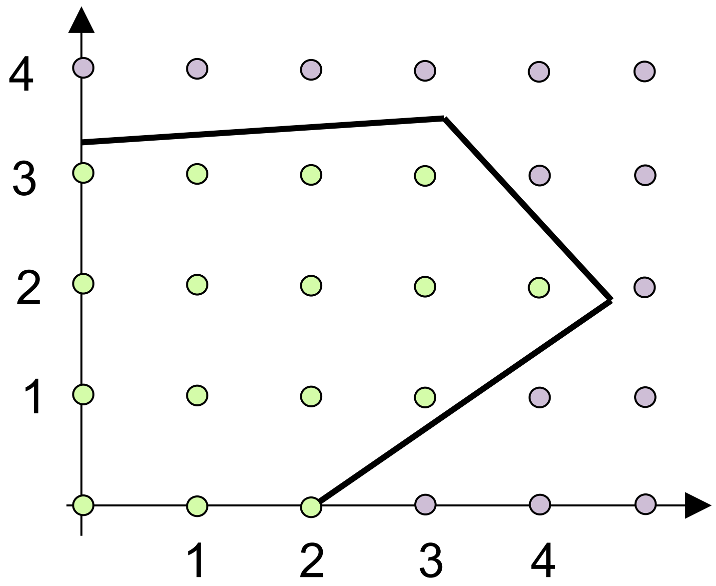
Network Optimization

Part II Branch-and-bound algorithm

Wolsey, Integer Programming Chapter 7

Integer Program: feasible region



$$z^* := \max c'x$$
$$x \in P$$
$$x \in \mathbb{Z}^n$$

$$P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$$

P bounded and not empty

$S = P \cap \mathbb{Z}^n$ set of feasible points

x^* **optimal solution** of value z^*

Linear Relaxation

$$
\begin{aligned}
z^{\mathrm{LP}} &:= \max c' x \\
Ax &\le b \\
x &\ge 0
\end{aligned}
$$

If the optimal solution x^{LP} of the linear relaxation is **integral**, then x^{LP} is also **optimal** for the IP

$$
\begin{cases}
z^{\mathrm{LP}} = c' x^{\mathrm{LP}} \ge z^* \ \text{(Relaxation)} \\
x^{\mathrm{LP}} \in S, \ \text{i.e. } c' x^{\mathrm{LP}} \le z^* \ \text{(Feasibility)}
\end{cases}
\qquad \Rightarrow c' x^{\mathrm{LP}} = z^*
$$

Example: x^{LP} is **integer**

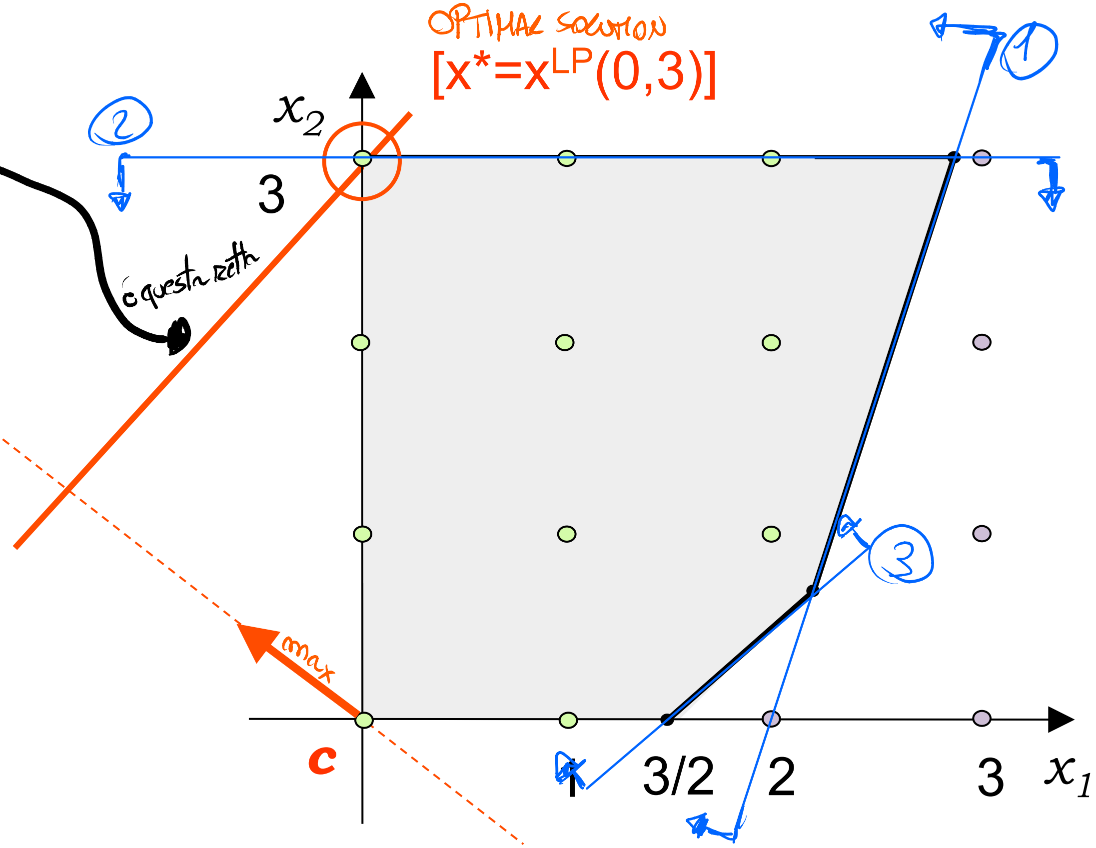
$$\boxed{\max -x_1 + x_2}$$

1 $\quad 7x_1 - 2x_2 \leq 14$

2 $\quad x_2 \leq 3$

3 $\quad 2x_1 - 2x_2 \leq 3$

$\quad x_1, x_2 \geq 0$



⇒ se trovi la soluzione ottima per il problema rilassato, trovi la stessa dell'IP: x*

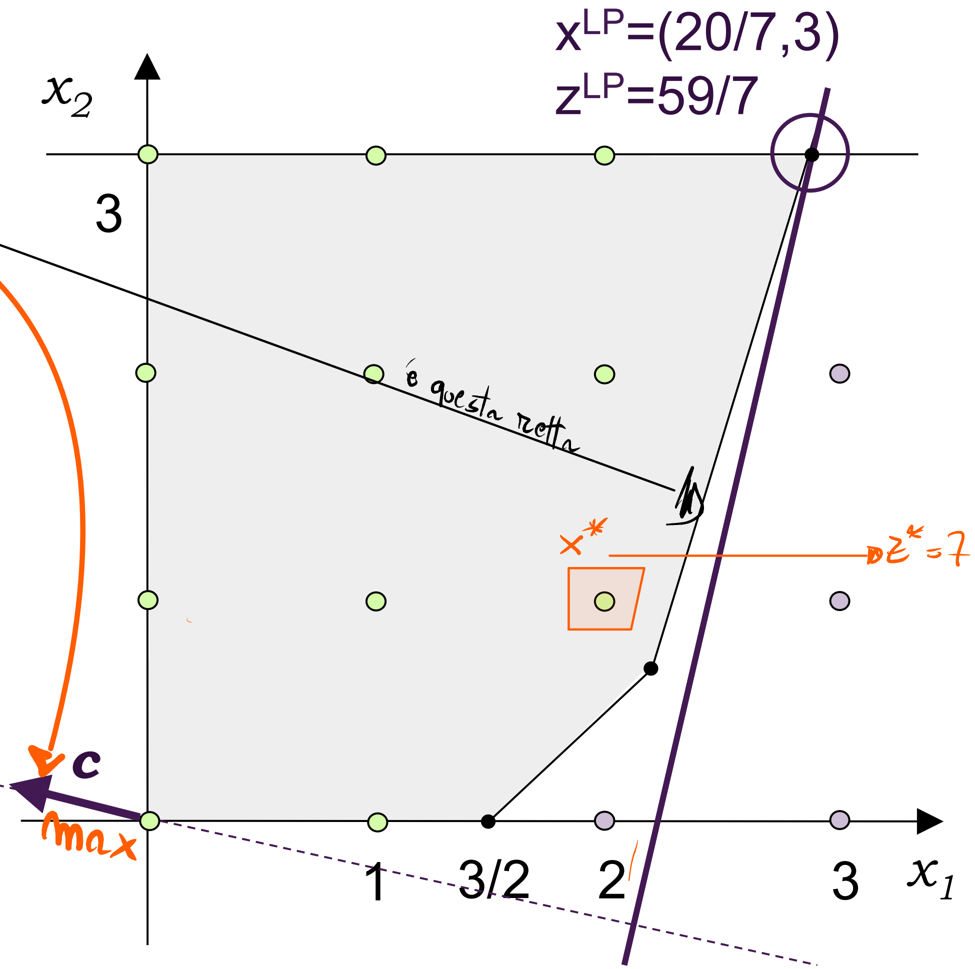
Example: x^{LP} is **fractional**

Questa cambia

$$\boxed{\max 4x_1 - x_2}$$

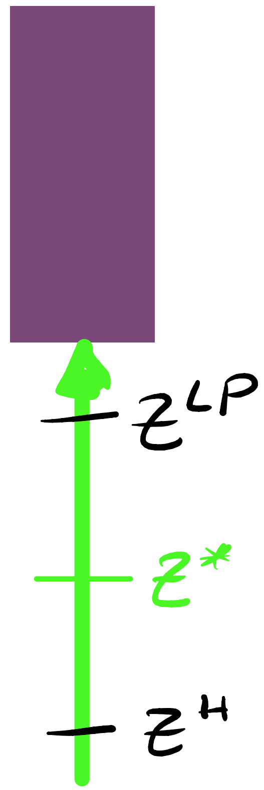
STESSI DI PRIMA + STESSO POLITOPE

$$\begin{cases} 7x_1 - 2x_2 \leq 14 \\ x_2 \leq 3 \\ 2x_1 - 2x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{cases}$$



In questo caso la soluzione rilassata x^{LP} non coincide con la soluzione intera del IP x^*

x^{LP} is **fractional**



$S \subset P$ implies that:

$$z^{\mathrm{LP}} = \max_{x \in P} c'x \geq \max_{x \in S} c'x = z^*$$

z^{LP} is an **upper bound** on z^*

Note
The value z^{H} of **any integer feasible solution** x^{H} is a **lower bound** on z^*

→ Ogni qualsiasi soluzione intera è ovv. un LOW. BOUND per z^*

x^{LP} is **fractional** (minimization case)

$S \subset P$ implies that:

$$z^{\mathrm{LP}} = \min_{x \in P} c'x \leq \min_{x \in S} c'x = z^*$$

z^{LP} is a **lower bound** on z^*

Note
The value z^{H} of **any integer feasible solution** x^{H} is an **upper bound** on z^*

+ x^{LP} is fractional: **branching**

↳ operazione che restringe i bounds di z^*

Given the solution x^{LP}, **select** a **fractional component** x_h and **partition** the original problem into two supbroblems:

$$
\begin{aligned}
z_1^* &:= \max c'x \\
Ax &\leq b \\
x &\geq 0 \\
x_h &\leq \lfloor x_h^{\mathrm{LP}} \rfloor \\
x &\in \mathbb{Z}^n
\end{aligned}
$$

$$
\begin{aligned}
z_2^* &:= \max c'x \\
Ax &\leq b \\
x &\geq 0 \\
x_h &\geq \lceil x_h^{\mathrm{LP}} \rceil \\
x &\in \mathbb{Z}^n
\end{aligned}
$$

x^{LP} is fractional: **branching**

The branching defines two subproblems and two relaxations:

$$P_1 = \{x \in P : x_h \leq \lfloor x_h^{LP} \rfloor\}$$
$$S_1 = P_1 \cap \mathbb{Z}_+^n$$

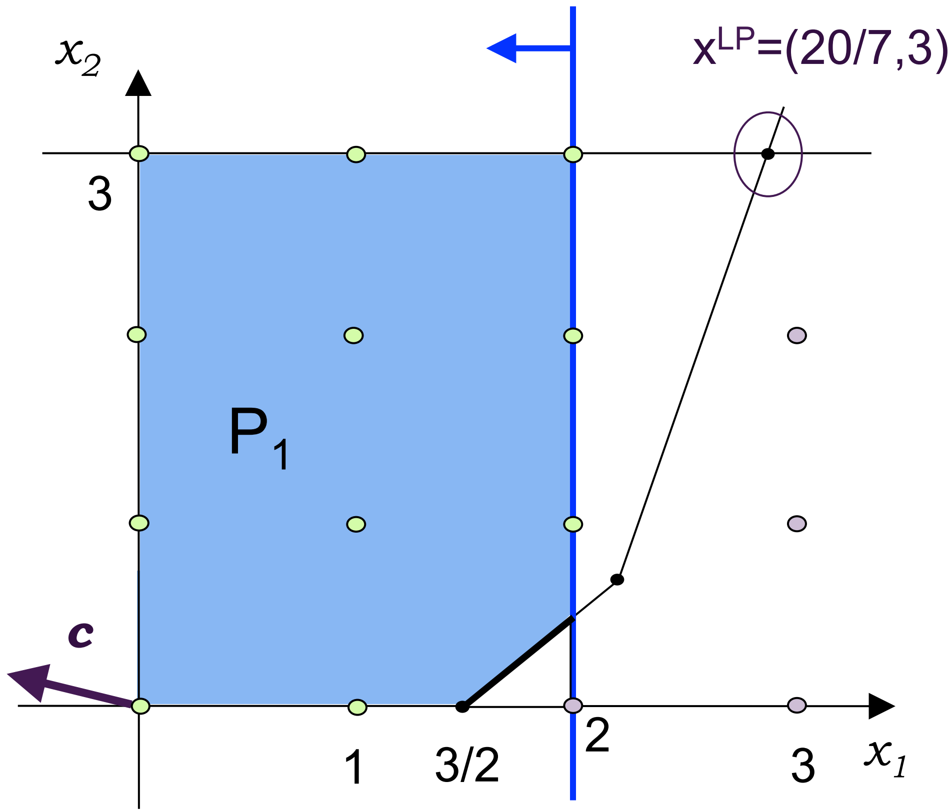
$$P_2 = \{x \in P : x_h \geq \lceil x_h^{LP} \rceil\}$$
$$S_2 = P_2 \cap \mathbb{Z}_+^n$$

Properties

$$x^{\mathrm{LP}} \notin P_1 \text{ and } x^{\mathrm{LP}} \notin P_2$$
$$S_1 \cup S_2 = S, S_1 \cap S_2 = \emptyset$$
$$z^* = \max\{z_1^*, z_2^*\}$$

Example: **subproblem** S_1

$$\max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \leq 2$$
$$x_1, x_2 \geq 0$$



Example: **subproblem** S_2

$$\max 4x_1 - x_2$$

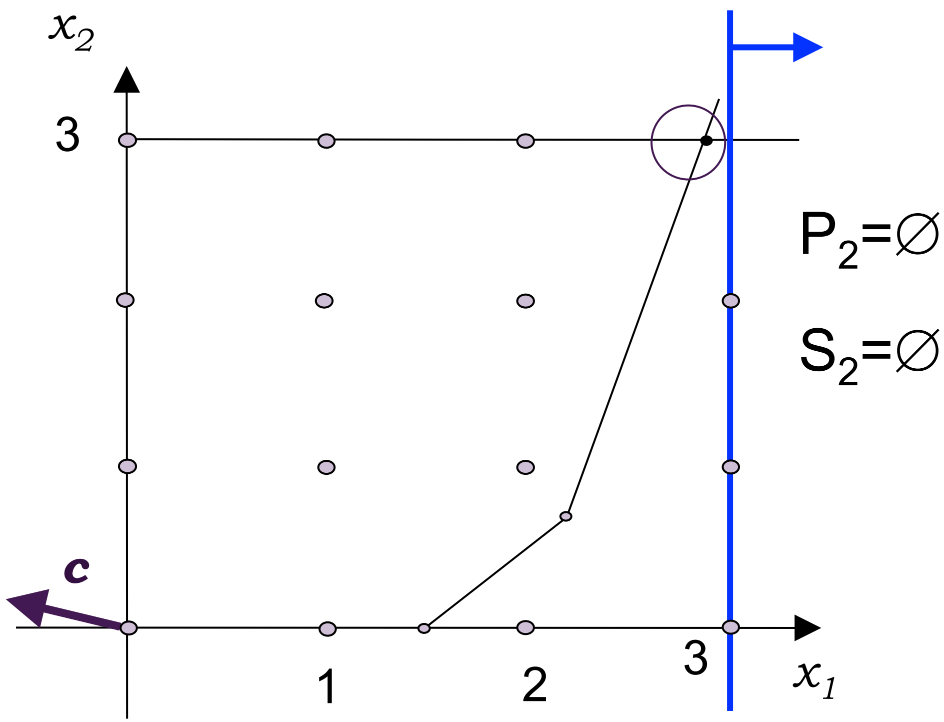
$$7x_1 - 2x_2 \leq 14$$

$$x_2 \leq 3$$

$$2x_1 - 2x_2 \leq 3$$

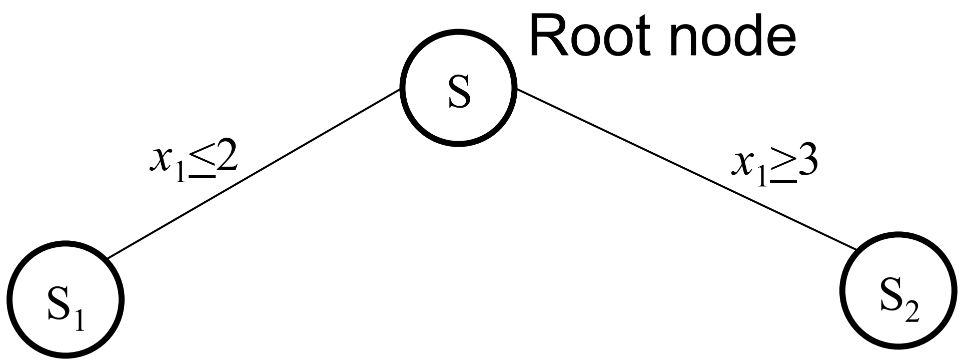
$$x_1 \geq 3$$

$$x_1, x_2 \geq 0$$



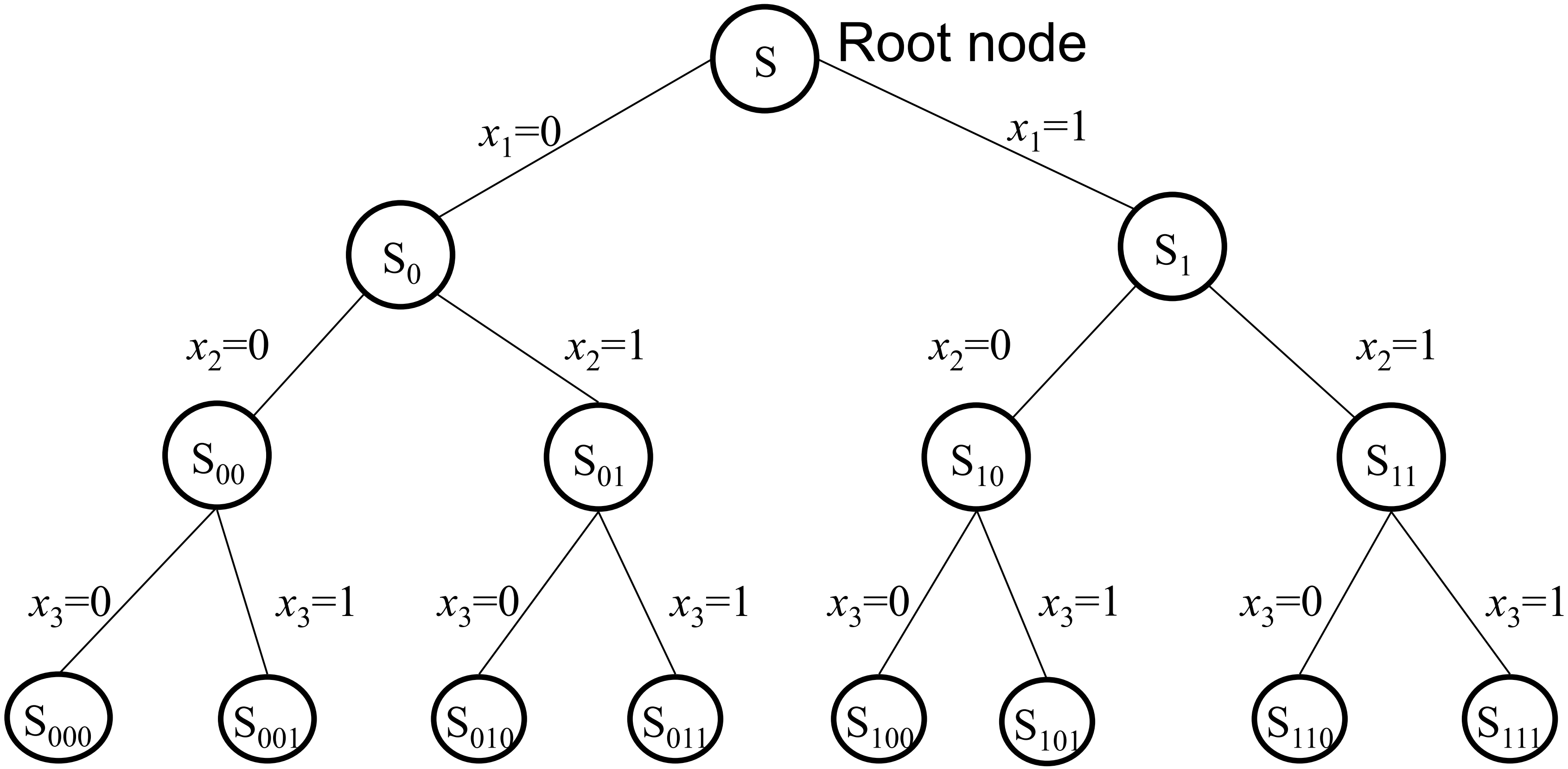
Enumeration tree

Subproblems can be represented by an **enumeration tree**



Enumeration tree

In the **binary case** $x \in \{0,1\}^n$ by applying recursively the partion scheme one can obtain 2^n leaves



+ Combining the scheme with the linear relaxation: **implicit enumeration**

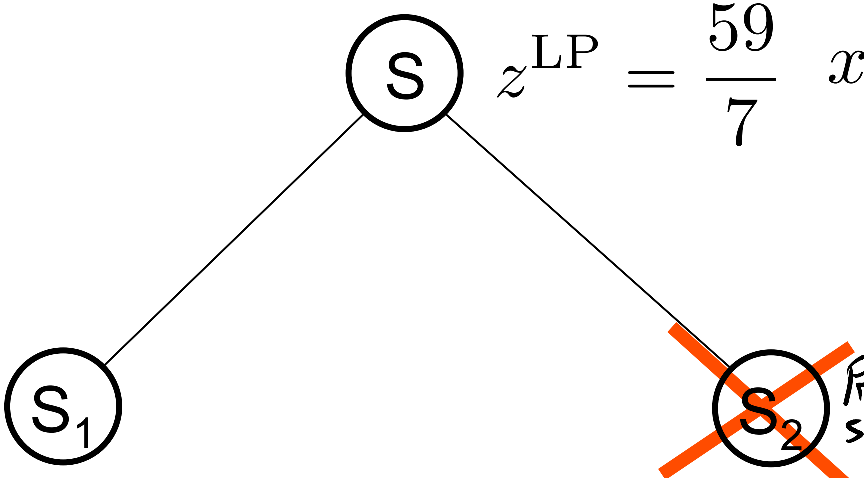
Consider subproblem S_2:

z^H è la miglior sol. intera attualmente conosciuta

$$\boxed{z^{\mathrm{H}} = -\infty}$$

$$z^{\mathrm{LP}} = \frac{59}{7} \qquad x^{\mathrm{LP}} = \left(\frac{20}{7}, 3\right)$$

$(x_1 \ ,\ x_2)$



Proseguendo da qui non succede nulla di nuovo

$S_1 = S \cap \{x : x_1 \leq 2\}$ $\qquad$ $S_2 = S \cap \{x : x_1 \geq 3\}$

S_2 [P_2] is **empty**: one can avoid to explore the tree from S_2

The node is **pruned by infeasibility**

Combining the scheme with the linear relaxation: **implicit enumeration**

Consider subproblem S_1:

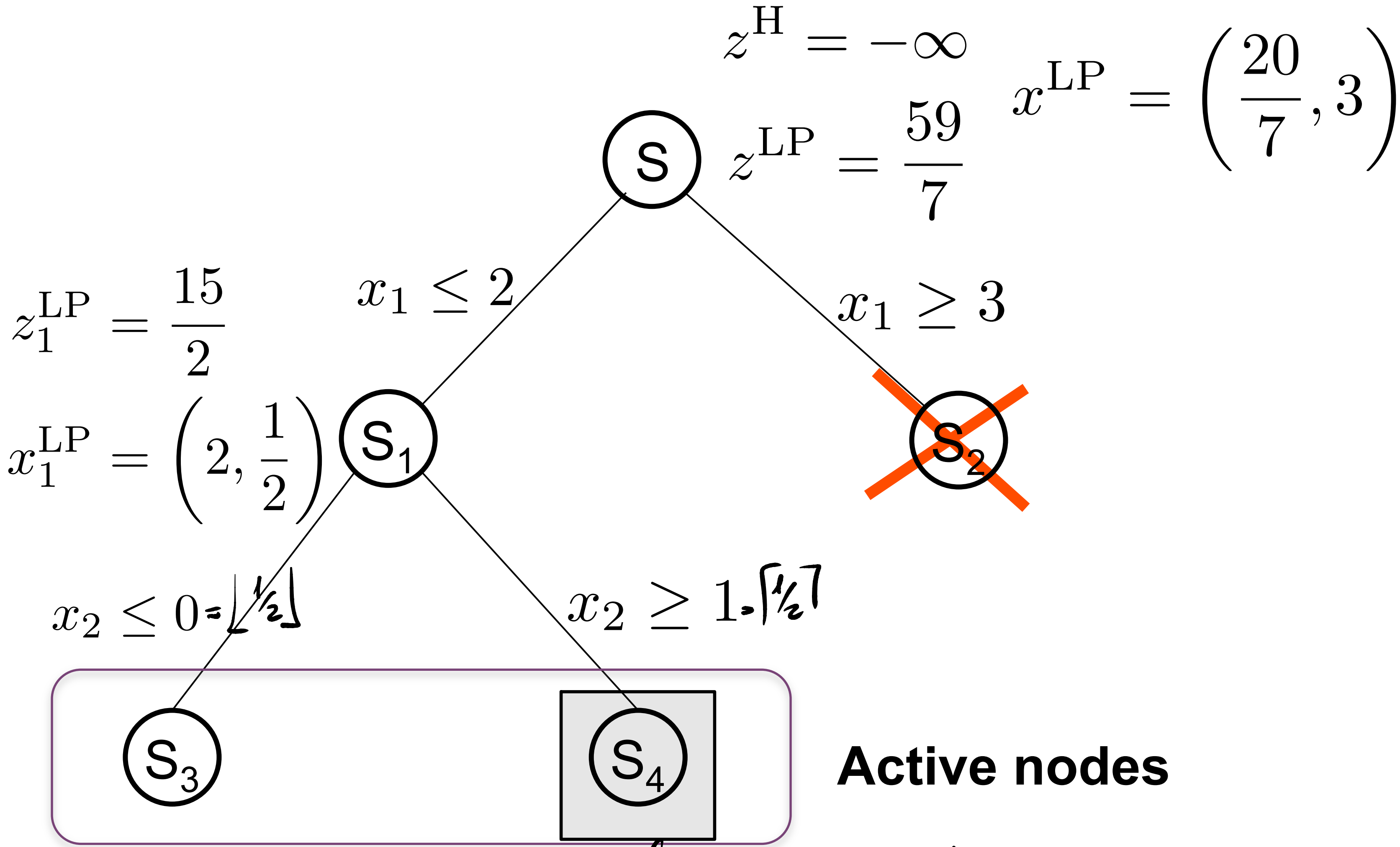
$$
\begin{aligned}
z_1^{\mathrm{LP}} &= \max 4x_1 - x_2 \\
7x_1 - 2x_2 &\le 14 \\
x_2 &\le 3 \\
2x_1 - 2x_2 &\le 3 \\
x_1 &\le 2 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

$$z_1^{\mathrm{LP}} = \frac{15}{2}$$

$$x_1^{\mathrm{LP}} = \left(\underset{x_1}{2}, \underset{x_2}{\frac{1}{2}}\right)$$

Soluzione ritornata da un calcolatore

x_1^{LP} is fractional: **branching**



SIAMO ARRIVATI
POICHE QUI SI TROVA UNA
SOLUZIONE INTERA

Subproblem S_4

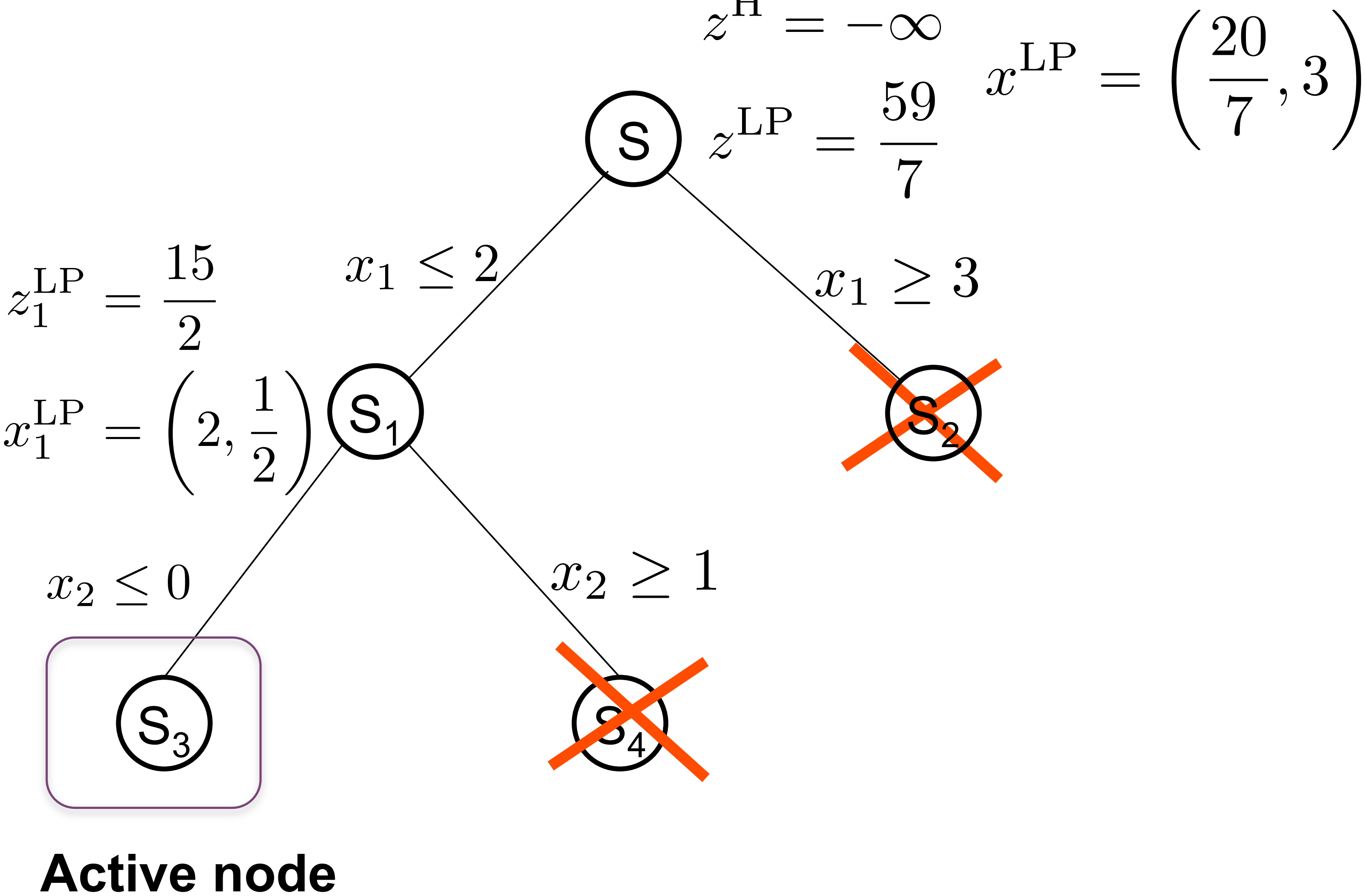
$$z_4^{\mathrm{LP}} = \max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \leq 2$$
$$x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

$$z_4^{\mathrm{LP}} = 7$$
$$x_4^{\mathrm{LP}} = (2, 1)$$

x_4^{LP} is **integer**

1. S_4 is **pruned by optimality**
2. z_4^{LP} is **better than the current incumbent** z^{H}, the incumbent is **updated** to 7

x_1^{LP} is fractional: **branching**



Subproblem S_3

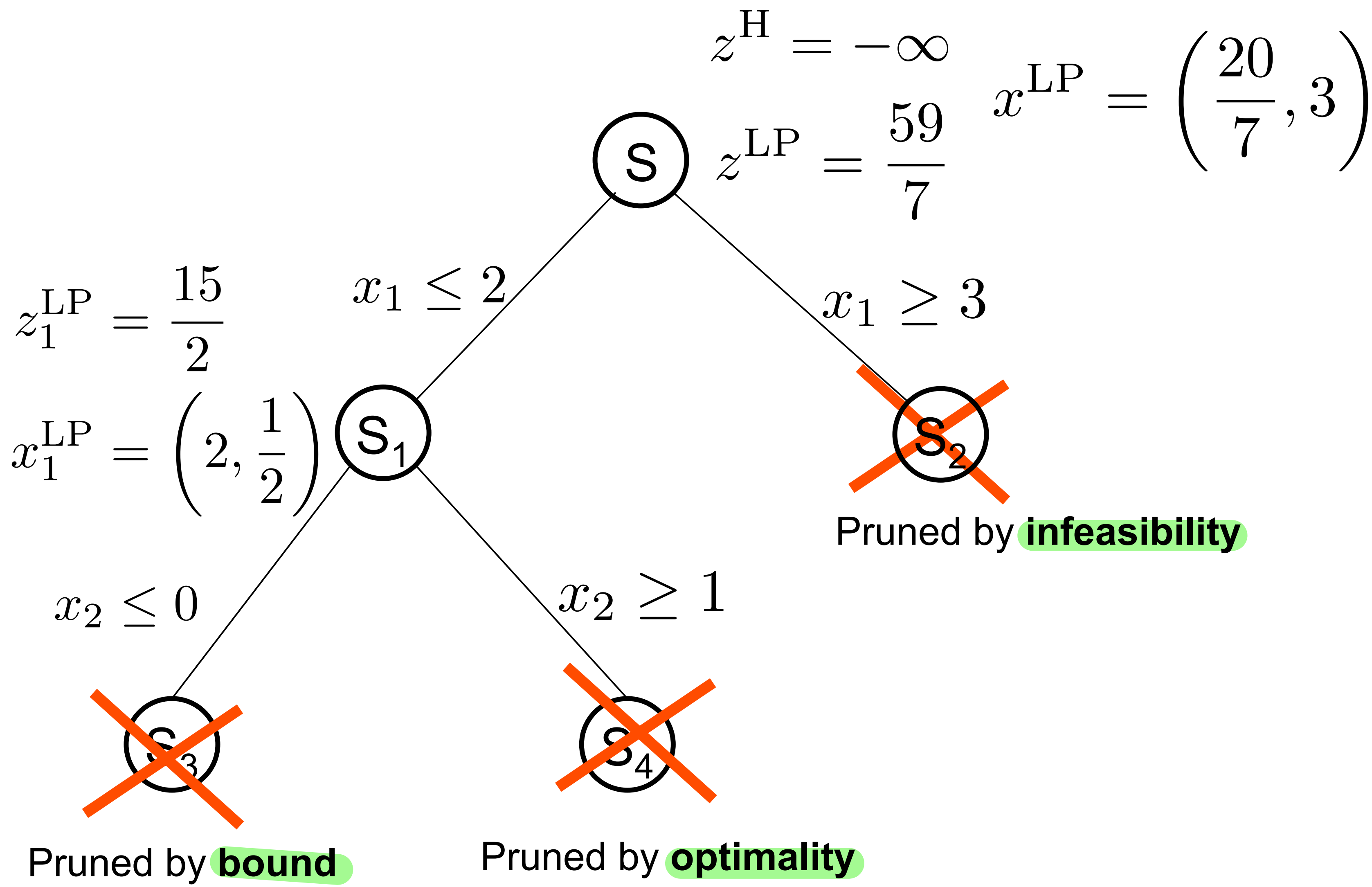
$$z_3^{\mathrm{LP}} = \max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \geq 2$$
$$x_2 \leq 0$$
$$x_1, x_2 \geq 0$$

$$z_3^{\mathrm{LP}} = 6$$
$$x_3^{\mathrm{LP}} = \left(\frac{3}{2}, 0\right)$$

x_3^{LP} is **fractional** but $z_3^{LP} < z^H = 7$

S_3 can be **pruned by bound**

x_1^{LP} is fractional: **branching**



↳ z_4^{LP} è migliore di z_3^{LP}, non ha senso proseguire per questo branche

Pruning by infeasibility

Let S_t be the current subproblem and P_t be its linear relaxation

If P_t is **empty** then S_t is empty

Consequence

The node corresponding to S_t can be **pruned by infeasibility**

Pruning by optimality

If x_t^{LP} is **integral** then it is the **optimal** for the subproblem S_t.

Consequence

The node corresponding to S_t can be **pruned by optimality**.

Note

If x_t^{LP} is integral, then it is **feasible** also for S

If z_t^{LP} improves the current incumbent z^H, then the incumbent is **updated** $z^H = z^{LP}$

Pruning by bound

x_t^{LP} is **fractional** and $z^{LP}{}_t \leq z^H$

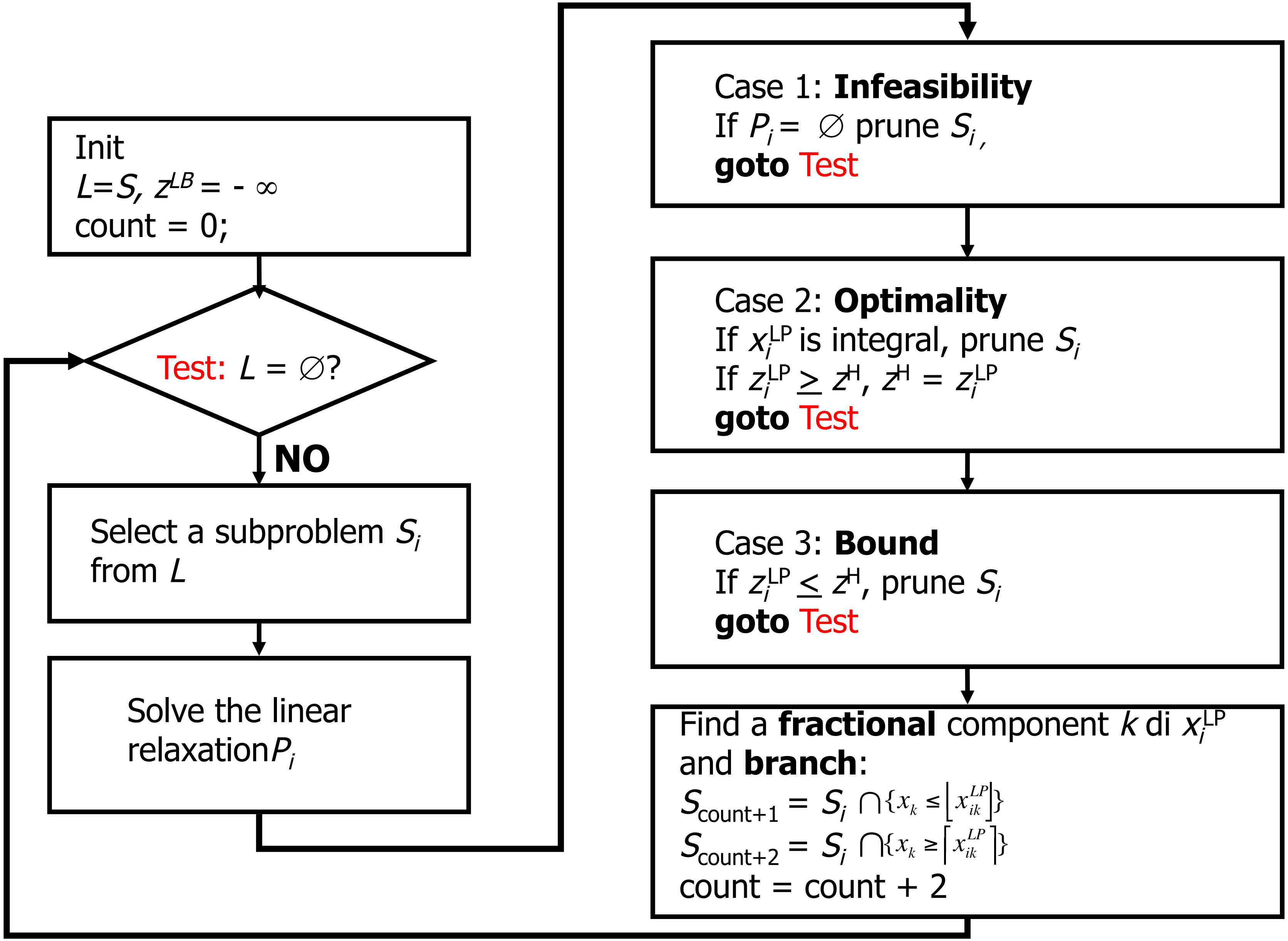
Consequence

S_t can be **pruned by bound**.

Branching

If x_t^{LP} is **fractional** and $z^{LP}{}_t > z^H$ then **branch**

LP-based Branch-and-Bound



{0,1} example

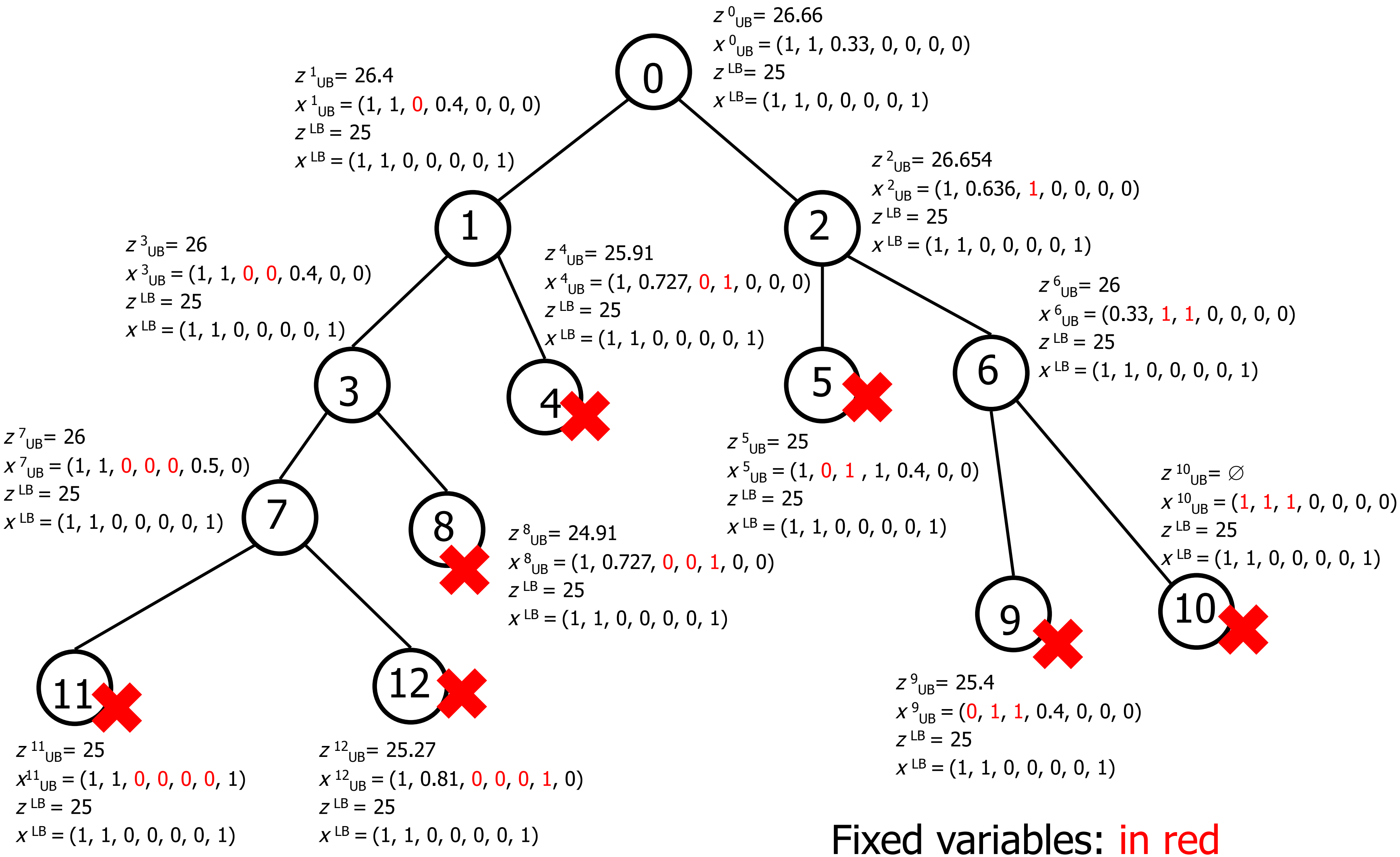
(KNAPSACK WITH 7 VARS)

$$\max \; 9x_1+15x_2+8x_3+6x_4+5x_5+4x_6+x_7$$

s.t.

$$6x_1+11x_2+6x_3+5x_4+5x_5+4x_6+x_7 \le 19$$

$$x \in \{0,1\}^7$$



Fixed variables: in red