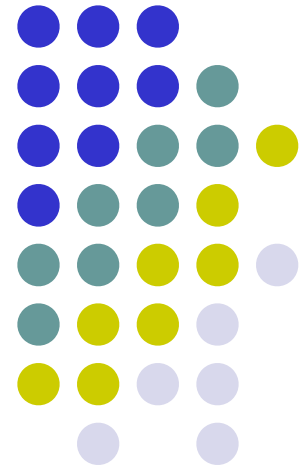
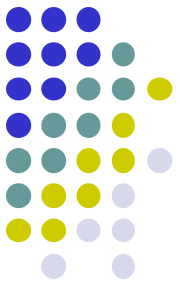


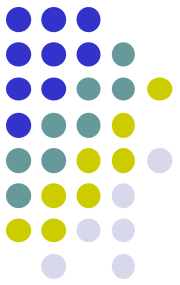
Web Algorithms

Eng. Fabio Persia, PhD



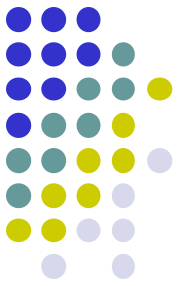


Algorithmic techniques: linear programming



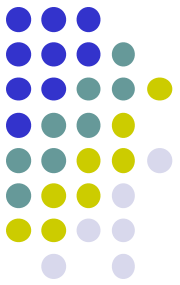
Characteristics

- The problem is formulated as an integer linear program (ILP)
- Integer linear program = linear program + integrality constraints
- There exists a polynomial time algorithm (ellipsoid algorithm) for solving linear problem, but
- Solving an integer linear program is an *NP-HARD* problem
- So what???
- The formulation as ILP makes it possible to use powerful general methods that, according to the properties of the ILP, are able to yield good approximation algorithms:
 - Rounding
 - Primal-dual



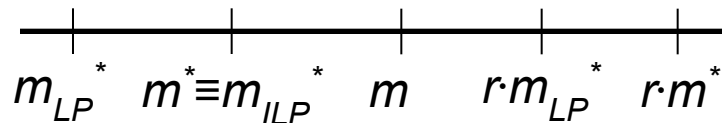
Linear programming: rounding

Characteristics

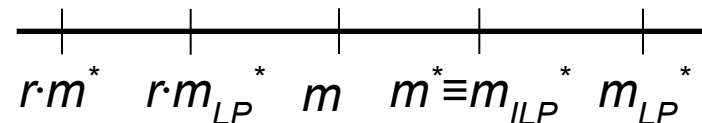


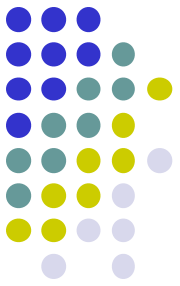
- The problem is formulated as an integer linear program (ILP)
- The linear relaxation (LP) obtained from ILP relaxing integrality constraints, that is substituting them with suitable (on integral) linear constraints
- The obtained solution (optimal for LP) is rounded to a close feasible integral solution for ILP
- The measure m of the obtained solution is then compared with the one of the optimal solution for LP, that is m_{LP}^* , that is a lower (MIN) or upper (MAX) bound for m^*

Min problems:



Max problems

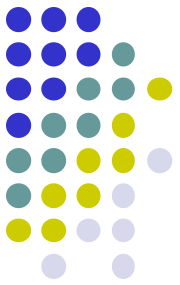




Min Weighted Vertex Cover

- **INPUT:** graph $G=(V,E)$, an integer cost c_j associated to every $v_j \in V$
- **SOLUTION:** $U \subseteq V$ such that $v_j \in U$ or $v_k \in U \forall \{v_j, v_k\} \in E$
- **MEASURE:** Total cost of U , that is

$$\sum_{v_j \in U} c_j$$

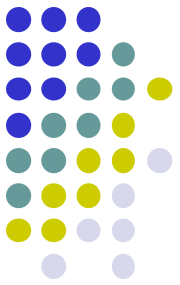


ILP:

$$\min \sum_{j=1}^n c_j \cdot x_j \quad \equiv \text{objective function}$$

$$x_j + x_k \geq 1 \quad \forall \{v_j, v_k\} \in E \quad \equiv \text{constraints}$$

$$x_j \in \{0, 1\} \quad \forall v_j \in V \quad \forall j, 1 \leq j \leq n \quad \equiv \text{integral constraints}$$



LP:

(Linear relaxation)

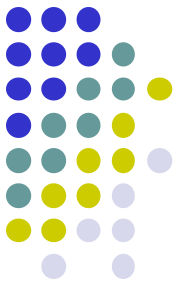
$$\min \sum_{j=1}^n c_j \cdot x_j$$

$$x_j + x_k \geq 1 \quad \forall \{v_j, v_k\} \in E$$

~~$$x_j \leq 1 \quad \forall v_j \in V$$~~

(superfluous, why?)

$$x_j \geq 0 \quad \forall v_j \in V$$



Algorithm Round-Vertex-Cover

Begin

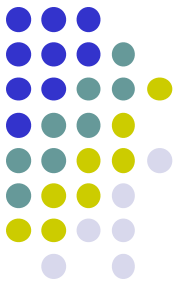
Determine the ILP associated to the input instance.

Solve the linear relaxation LP of the ILP and let $\langle x_1^*, \dots, x_n^* \rangle$ be the resulting optimal solution of LP.

$\forall v_j$ let $x_j = 1$ if $x_j^* \geq \frac{1}{2}$ and $x_j = 0$ if $x_j^* < \frac{1}{2}$.

Return the cover U associated to $\langle x_1, \dots, x_n \rangle$, that is such that
$$U = \{ v_j \in V \mid x_j = 1 \}.$$

End



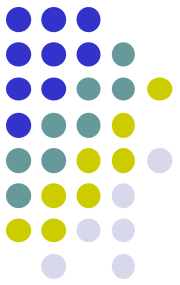
Theorem: Round-Vertex-Cover is a 2-approximation algorithm

Proof. It is sufficient to show that:

1. x_1, \dots, x_n is feasible for the ILP (it satisfies all the constraints), that is that U is a cover

1. $\frac{m}{m_{PL}^*} \leq 2$ and thus also

$$\frac{m}{m^*} \leq \frac{m}{m_{PL}^*} \leq 2$$



- *Let us prove fact 1)*

By the feasibility of $\langle x_1^*, \dots, x_n^* \rangle$ for LP, for every edge $\{v_j, v_k\} \in E$ it is $x_j^* + x_k^* \geq 1$, that is $x_j^* \geq 0.5$ or $x_k^* \geq 0.5$, so that $x_j = 1$ or $x_k = 1$, and thus $x_j + x_k \geq 1$ is satisfied in ILP.

- *Let us prove fact 2)*

$$m = \sum_{j=1}^n c_j \cdot x_j \leq \sum_{j=1}^n c_j \cdot 2 \cdot x_j^* = 2 \cdot \sum_{j=1}^n c_j \cdot x_j^* = 2 \cdot m_{LP}^*$$

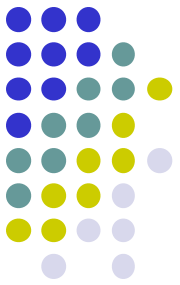
By the rounding:
 $x_j \leq 2 \cdot x_j^*$

m_{LP}^*

that is

$$\frac{m}{m^*} \leq \frac{m}{m_{LP}^*} \leq 2.$$

□



Min Weighted Set Cover

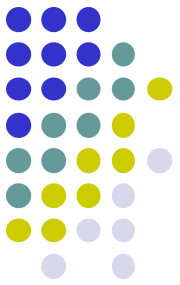
- **INPUT:** Universe $U=\{o_1, \dots, o_n\}$ of n objects, family $\hat{S}=\{S_1, \dots, S_h\}$ of h subsets of U , integer cost c_j associated to every $S_j \in \hat{S}$
- **SOLUTION:** Cover if U , that is subfamily $\hat{C} \subseteq \hat{S}$ such that

$$\bigcup_{S_j \in \hat{C}} S_j = U$$

- **MEASURE:** Total cost of the cover, that is

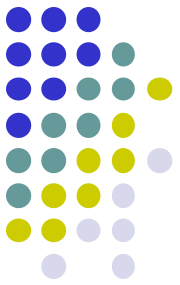
$$\sum_{S_j \in \hat{C}} c_j$$

f = max frequency of an object in the subsets of \hat{S} , that is every object occurs in at most f subsets.



Min Weighted Set Cover

Given a set of elements $\{1, 2, \dots, n\}$ (called the **universe**) and a collection S of m sets whose **union** equals the universe, the set cover problem is to identify the smallest sub-collection of S whose union equals the universe. For example, consider the universe $U = \{1, 2, 3, 4, 5\}$ and the collection of sets $S = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$. Clearly the union of S is U . However, we can cover all of the elements with the following, smaller number of sets: $\{\{1, 2, 3\}, \{4, 5\}\}$.

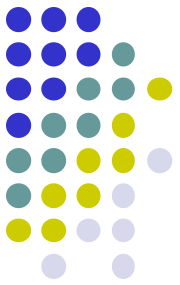


ILP:

$$\min \sum_{j=1}^h c_j \cdot x_j \quad \equiv \text{objective function}$$

$$\sum_{S_j | o_i \in S_j} x_j \geq 1 \quad \forall o_i \in U \quad \equiv \text{constraints}$$

$$x_j \in \{0, 1\} \quad \forall S_j \in \hat{S} \quad \equiv \text{integral constraints}$$



LP:

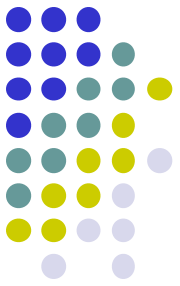
$$\min \sum_{j=1}^h c_j \cdot x_j$$

$$\sum_{S_j | o_i \in S_j} x_j \geq 1 \quad \forall o_i \in U$$

~~$$x_j \leq 1 \quad \forall S_j \in \hat{S}$$~~

(superfluous, why?)

$$x_j \geq 0 \quad \forall S_j \in \hat{S}$$



Algorithm Round-Set-Cover

Begin

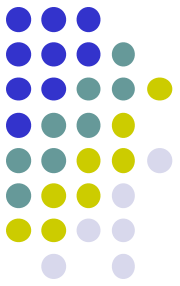
Determine the ILP associated to the input instance.

Solve the linear relaxation LP of the ILP and let $\langle x_1^*, \dots, x_n^* \rangle$ be the resulting optimal solution of LP.

$\forall S_j$ let $x_j=1$ if $x_j^* \geq 1/f$ and $x_j=0$ if $x_j^* < 1/f$.

Return the resulting cover, i.e. $\hat{C} = \{ S_j \in \hat{S} \mid x_j = 1 \}$.

End

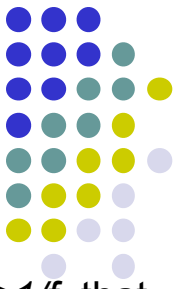


Theorem: Round-Set-Cover is f -approximating ($f \geq 1$)

Proof. It is sufficient to show that:

1. x_1, \dots, x_n is feasible for the ILP

1. $\frac{m}{m_{LP}^*} \leq f$ (and thus also $\frac{m}{m^*} \leq \frac{m}{m_{LP}^*} \leq f$)



- Let us prove claim 1)

By the feasibility of $\langle x_1^*, \dots, x_n^* \rangle$ for LP, $\forall o_i \in U$

$$\sum_{S_j | o_i \in S_j} x_j^* \geq 1$$

and since the summation has at most f terms, there must exist S_j containing o_i s.t. $x_j^* \geq 1/f$, that is such that $x_j = 1$, and thus

- Let us prove claim 2)

$$\sum_{S_j | o_i \in S_j} x_j \geq 1.$$

that is

$$m = \sum_{j=1}^h c_j \cdot x_j \leq \sum_{j=1}^h c_j \cdot f \cdot x_j^* = f \cdot \sum_{j=1}^f c_j \cdot x_j^* = f \cdot m_{LP}^*$$

By the rounding:
 $x_j \leq f x_j^*$

$$\frac{m}{m^*} \leq \frac{m}{m_{LP}^*} \leq f.$$

□