

Network Optimization

Part I Integer programming Modeling techniques
Wolsey, Integer Programming, Chapter 1



Mixed Integer Linear Program

Data

A : m by n rational matrix

G : m by n rational matrix

c : n -dimensional rational vector

h : p -dimensional rational vector

b : m -dimensional rational vector

Variables

x : n -dimensional vector of variables y : p -dimensional vector of integer variables

$$\max c'x + h'y$$

$$Ax + Gy \leq b$$

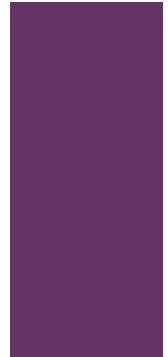
Linear constraints

$$x \geq 0$$

$$y \geq 0, \text{ integer}$$



Integer Linear Program



Data

A : m by n rational matrix

c : n -dimensional rational vector

b : m -dimensional rational vector

Variables

x : n -dimensional vector of integer variables

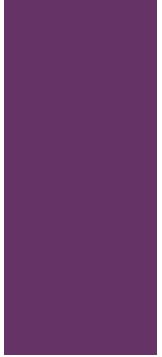
$$\max c'x$$

$$Ax \leq b$$

$$x \geq 0, \text{ integer}$$



Integer Linear Program



Solution

An assignment of values to variables

Feasible Solution

An assignment of values to variables such that all the constraints are satisfied

Objective function value

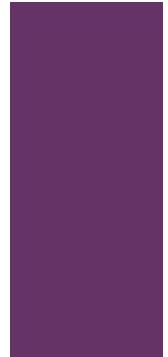
Value of a solution obtained by evaluating the objective function at a given point

Optimal solution

(max) is one whose objective function value is greater than or equal to that of all other feasible solutions.



Binary Integer Program



Data

A : m by n rational matrix

c : n -dimensional rational vector

b : m -dimensional rational vector

Variables

x : n -dimensional vector of $\{0,1\}$ variables

$$\max c'x$$

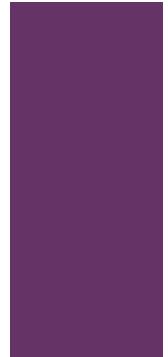
$$Ax \leq b$$

$$x \in \{0, 1\}^n$$



Combinatorial Optimization Problem

↳ RISOLVIBILI ENUMERANDO TUTTE LE SOLUZIONI



Given

A finite set $N = \{1, \dots, n\}$

A vector c of weights $\{c_1, c_2, \dots, c_n\}$ for each $j \in N$

A collection \mathcal{F} of feasible subsets of N → Family of feasible solutions

Find

A minimum (maximum) weight feasible subset

$$\min_{S \subseteq N} \left\{ \sum_{j \in S} c_j : S \in \mathcal{F} \right\} \quad (\max_{S \subseteq N} \left\{ \sum_{j \in S} c_j : S \in \mathcal{F} \right\})$$

S can be represented by an incidence vector $x = (x_1, x_2, \dots, x_n)$

with
$$x_j = \begin{cases} 1 & \text{if } j \in S, \\ 0 & \text{otherwise} \end{cases}$$



An example: the Knapsack Problem

MODELING KNAPSACK AS A BIP

↳ what are steps required to formulate combinatorial opt prob?

Given

A set of items $N = \{1, \dots, n\}$

Each item has associated a weight a_j and a profit c_j

A capacity b

Find

A subset of items such that the sum of weights does not exceed b and the sum of profits is maximized

① Choose your variables

Variables definition

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected,} \\ 0 & \text{otherwise} \end{cases}$$

+

An example: the Knapsack Problem

② find constraints

Constraints definition

$$\sum_{j=1}^n a_j x_j \leq b$$

$$x \in \{0, 1\}^n$$

③ define objective

Objective function

$$\max \sum_{j=1}^n c_j x_j$$



An example: the Knapsack Problem

$$\max 10x_1 + 14x_2 + 12x_3 + 8x_4$$

s.t.

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq 5$$

$$x \in \{0, 1\}^4$$

	a	c
1	2	10
2	3	14
3	4	12
4	1	8

Capacity: $b = 5$

$$\mathcal{F} = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$$

The family \mathcal{F} can be represented by a characteristic vector x



An example: the Knapsack Problem



Constraints

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq 5$$

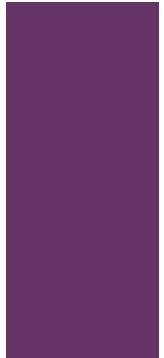
$$x \in \{0, 1\}^4$$

represent the set of feasible solutions

$$x \in \mathcal{F} \rightarrow \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$



Set Covering, Set Packing, Set Partitioning



Given

Two finite sets $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$

A collection of n subsets of M : $\{M_1, M_2, \dots, M_n\}$

A weight c_j for each M_j

A subset $F \subset N$ is a **cover** of M if

$$\bigcup_{j \in F} M_j = M$$

A subset $F \subset N$ is a **packing** of M if

$$M_h \cap M_k = \emptyset \quad \forall h, k, h \neq k$$

A subset $F \subset N$ is a **partition** of M if it is both a cover and a packing of M

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Set Covering, Set Packing, Set Partitioning

$$M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

$$M_1 = \{1, 2, 3, 4\}$$

$$M_2 = \{1, 2, 5, 6, 9, 10\}$$

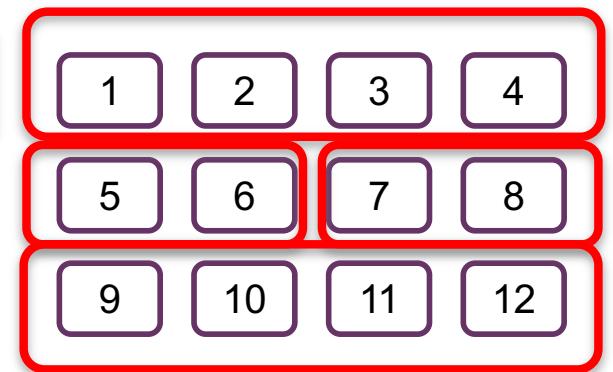
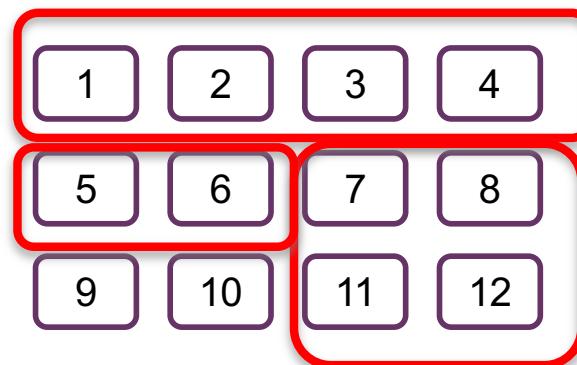
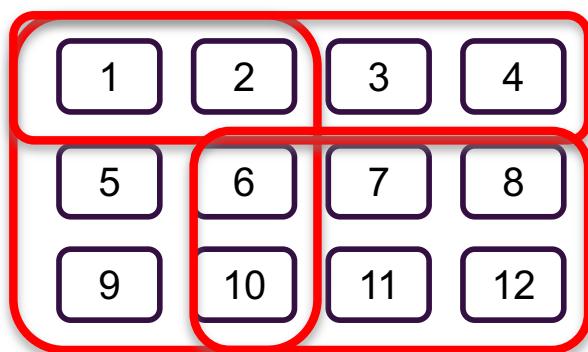
$$M_3 = \{7, 8, 11, 12\}$$

$$M_4 = \{5, 6\}$$

$$M_5 = \{6, 7, 8, 10, 11, 12\}$$

$$M_6 = \{9, 10, 11, 12\}$$

$$M_7 = \{7, 8\}$$



M_1, M_2, M_5 is a **cover**

M_1, M_3, M_4 is a **packing**

M_1, M_4, M_6, M_7 is a **partition**



Set Covering, Set Packing, Set Partitioning

Define

A $m \times n$ incidence matrix A of the family $\{M_j | j \in N\}$

$$a_{ij} = \begin{cases} 1 & \text{if } i \in M_j \\ 0 & \text{otherwise} \end{cases} //$$

A decision variable $x_j, j=1, \dots, n$

$$x_j = \begin{cases} 1 & \text{if } j \in F \\ 0 & \text{otherwise} \end{cases}$$

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Set Covering, Set Packing, Set Partitioning

$$\begin{aligned} & \min c'x \\ \text{s.t. } & Ax \geq 1 \\ & x \in \{0, 1\}^n \end{aligned}$$

Set covering

$$\begin{aligned} & \max c'x \\ \text{s.t. } & Ax \leq 1 \\ & x \in \{0, 1\}^n \end{aligned}$$

Set packing

$$\begin{aligned} & \min(\text{ or } \max) c'x \\ \text{s.t. } & Ax = 1 \\ & x \in \{0, 1\}^n \end{aligned}$$

Set partitioning



Modeling logical conditions by binary variables



Let x, y, z be binary variables

Negation

$$z = \neg x \rightarrow z = 1 - x$$

AND

x	y	$z = x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

$$\rightarrow \begin{cases} z - x \leq 0 \\ z - y \leq 0 \\ x + y - z \leq 1 \end{cases}$$



Modeling logical conditions by binary variables

OR

x	y	$z = x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

$$\rightarrow \begin{cases} z - x \geq 0 \\ z - y \geq 0 \\ x + y - z \geq 0 \end{cases}$$

Exclusive OR

x	y	$z = x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

$$\rightarrow \begin{cases} x - y + z \geq 0 \\ -x + y + z \geq 0 \\ x + y - z \geq 0 \\ z + x + y \leq 2 \end{cases}$$



Modeling dependent decisions



Binary variables can be used to model dependency between two choices.

Suppose x and y are binary variables such that:

$$x = \begin{cases} 1 & \text{if project } x \text{ is selected,} \\ 0 & \text{otherwise} \end{cases} \quad y = \begin{cases} 1 & \text{if project } y \text{ is selected,} \\ 0 & \text{otherwise} \end{cases}$$

Suppose that project x can be selected **only if** project y has already been selected.

This can be expressed by the (linear) constraint

$$x - y \leq 0$$

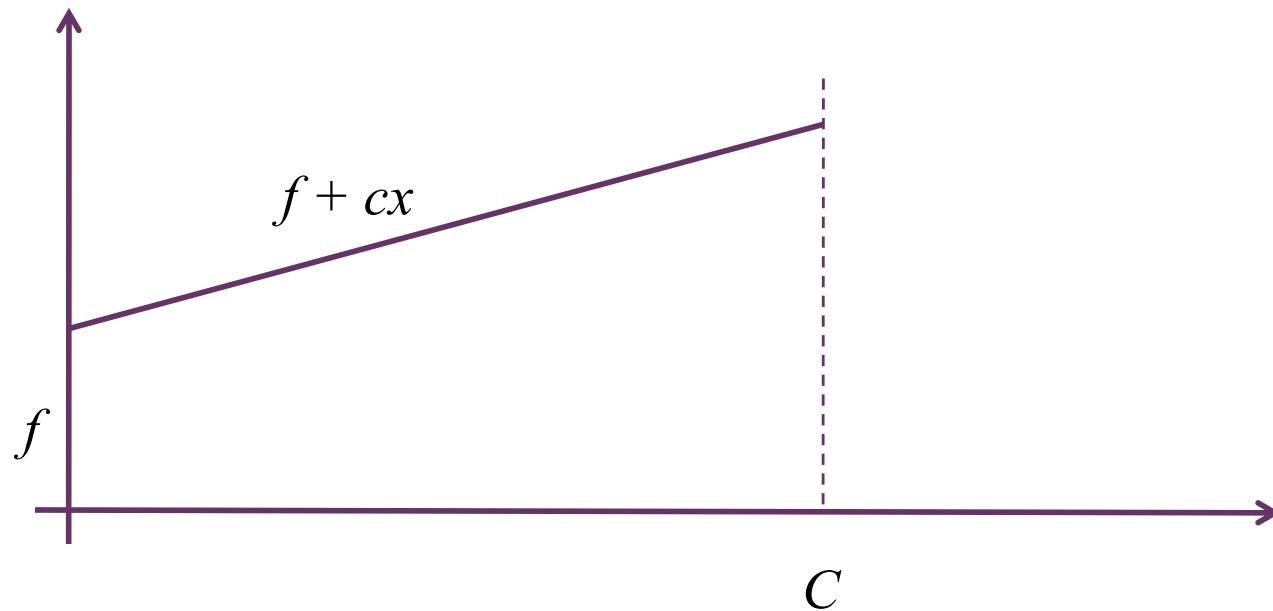


Modeling fixed cost



Consider the following nonlinear objective function ($f, c > 0$)

$$\min h(x) = \begin{cases} f + cx & \text{if } 0 < x \leq C \\ 0 & \text{if } x = 0 \end{cases}$$





Modeling fixed cost



Define an additional binary variable

$$y = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (*)$$

Replace

$$h(x) \rightarrow fy + cx$$

Add the constraints

$$x - Cy \leq 0$$

$$y \in \{0, 1\}$$

Observation: $x = 0$ and $y = 1$ is a feasible solution in contrast to (*), but an optimal solution will always have $y=0$ if $x=0$ (minimization problem)



Uncapacitated Facility Location



Given

A set $N=\{1, \dots, n\}$ of potential depots (facilities) and a set $M=\{1, \dots, m\}$ of clients

A cost f_j of opening facility j

A cost c_{ij} associated with serving customer i from facility j

Variables definition

$$y_j = \begin{cases} 1 & \text{if facility } j \text{ is open,} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if client } i \text{ is served from facility } j, \\ 0 & \text{otherwise} \end{cases}$$



Uncapacitated Facility Location



Constraints definition

Demand of client i must be satisfied

$$\sum_{j=1}^n x_{ij} = 1 \text{ for } i = 1, \dots, m$$

A client can be served from facility j only if facility j is open

$$x_{ij} \leq y_j \quad \forall i, j$$

Objective function

$$\min \sum_{j=1}^n f_j y_j + \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$



Uncapacitated Facility Location



Formulation

$$\min \sum_{j=1}^n f_j y_j + \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\sum_{j=1}^n x_{ij} = 1 \text{ for } i = 1, \dots, m$$

$$x_{ij} \leq y_j \quad \forall i, j$$

$$x_{ij} \in \{0, 1\}$$

$$y_j \in \{0, 1\}$$



Modeling restricted set of values



x only takes values in a given set $\{a_1, a_2, \dots, a_m\}$.

Model

Introduce m binary variables $y_j, j = 1, \dots, m$ and the following constraints:

$$x = \sum_{j=1}^m a_j y_j,$$

$$\sum_{j=1}^m y_j = 1$$

$$y_j \in \{0, 1\}$$



Piecewise linear cost function



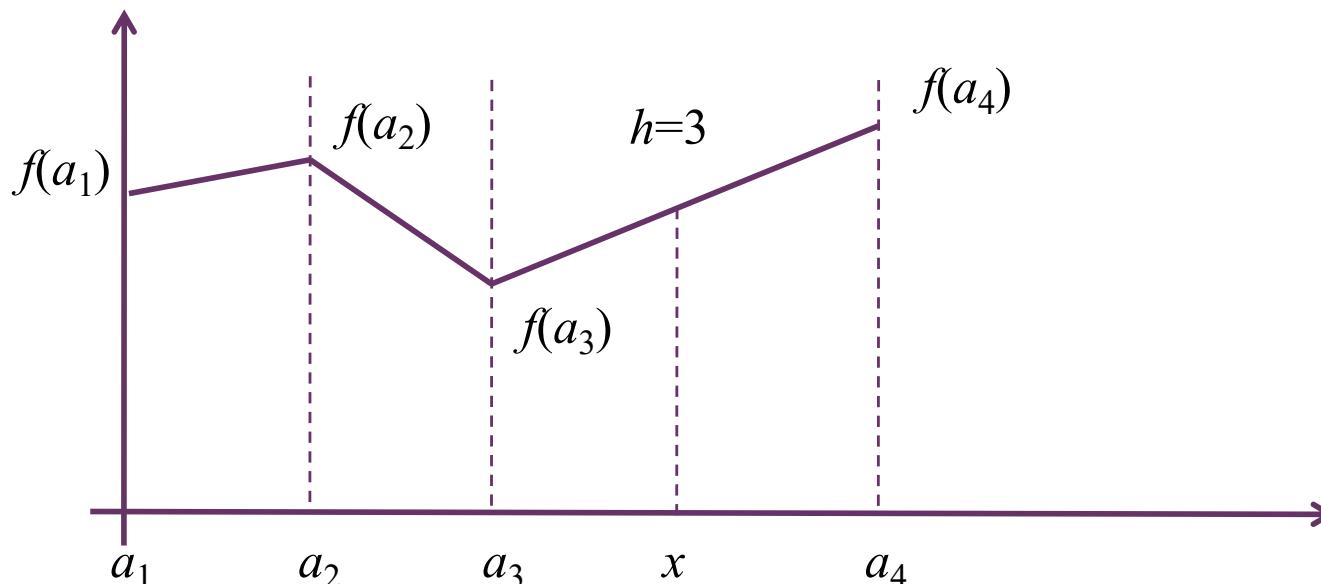
A **piecewise linear** cost function is specified by m ordered pairs $(a_i, f(a_i))$

At a given point x in the h -th interval $[a_h, a_{h+1}]$ the function can be evaluated by:

$$f(x) = \lambda_h f(a_h) + \lambda_{h+1} f(a_{h+1})$$

$$\lambda_h + \lambda_{h+1} = 1$$

$$\lambda_h, \lambda_{h+1} \geq 0$$





Piecewise linear cost function



Define **binary variables**

$$y_h = \begin{cases} 1 & \text{if } x \text{ is in the interval } [a_h, a_{h+1}] \\ 0 & \text{otherwise} \end{cases} \quad \text{for } h = 1, \dots, m-1$$

Define **continuous variables**

$$\lambda_h \geq 0 \text{ for } h = 1, \dots, m$$

Objective function

$$\min \sum_{i=1}^m \lambda_i f(a_i)$$



Piecewise linear cost function



Constraints

$$\sum_{i=1}^m \lambda_i = 1,$$

$$\lambda_1 \leq y_1,$$

$$\lambda_i \leq y_{i-1} + y_i, \quad i \in [2, \dots, m-1],$$

$$\lambda_m \leq y_{m-1},$$

$$\sum_{i=1}^{m-1} y_i = 1,$$

This set of constraints implies that there are only two “active” λ -s corresponding to interval in which x lies, i.e.,

$$y_j = 1 \text{ implies } \lambda_i = 0 \quad \forall i \neq j, j+1.$$



Piecewise linear cost function



Formulation

$$\min \sum_{i=1}^m \lambda_i f(a_i)$$

$$\sum_{i=1}^m \lambda_i = 1,$$

$$\lambda_1 \leq y_1,$$

$$\lambda_i \leq y_{i-1} + y_i, \quad i \in [2, \dots, m-1],$$

$$\lambda_m \leq y_{m-1},$$

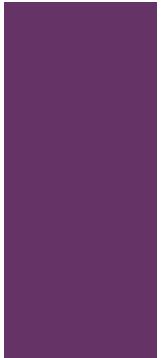
$$\sum_{i=1}^{m-1} y_i = 1,$$

$$\lambda_h \geq 0 \text{ for } h = 1, \dots, m$$

$$y_i = \{0, 1\} \text{ for } i = 1, \dots, m-1$$



Fixed Charge Network Flow Problem



Given

A directed graph $G=(N,A)$ with a source node s

A demand vector

$b \in \mathbb{Z}^{|N|}$ such that: $b_s > 0, b_i \leq 0 \forall i \in N, i \neq \{s\}$ and $\sum_{i \in N} b_i = 0$

A capacity vector $u \in \mathbb{Z}^{|A|}$

A cost vector $c \in \mathbb{Z}^{|A|}$ representing the arcs activation cost

Find

A feasible flow vector (see definition of feasible flow in Module I)

$$0 \leq x \leq u$$

that minimizes the cost of activated arcs



Fixed Charge Network Flow Problem



Variables

Flow variables $0 \leq x_{ij} \leq u_{ij} \quad \forall (i, j) \in A$

Activation variables

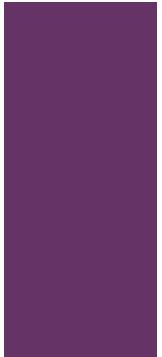
$$y_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is activated,} \\ 0 & \text{otherwise.} \end{cases}$$

Objective function

$$\min \sum_{(i,j) \in A} c_{ij} y_{ij}$$



Fixed Charge Network Flow Problem



Constraints

Flow balance constraints

$$\sum_{(i,j) \in \delta^+(i)} x_{ij} - \sum_{(j,i) \in \delta^-(i)} x_{ji} = b_i \quad \forall i \in N$$

Activation constraints

$$0 \leq x_{ij} \leq u_{ij} y_{ij}$$



Fixed Charge Network Flow Problem



Formulation

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} c_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{(i,j) \in \delta^+(i)} x_{ij} - \sum_{(j,i) \in \delta^-(i)} x_{ji} = b_i \quad \forall i \in N \\ & x_{ij} - u_{ij} y_{ij} \leq 0 \quad \forall (i,j) \in A \\ & x \geq 0 \\ & y \in \{0, 1\}^{|A|} \end{aligned}$$



Disjunctive constraints



Given two constraints

$$a'x \geq b$$

$$e'x \geq f$$

with $a, b, e, f \geq 0$ and $x \geq 0$

Suppose we want to model $a'x \geq b \vee e'x \geq f$
(i.e., at least one of the two constraints is satisfied)

Introduce a binary variable y and impose

$$a'x \geq by$$

$$e'x \geq f(1 - y)$$

$$y \in \{0, 1\}$$



Disjunctive constraints



In general, given,

$$a'_i x \geq b_i \quad i = \{1, \dots, m\}$$

if you want that at least k out of m constraints are satisfied, you can impose

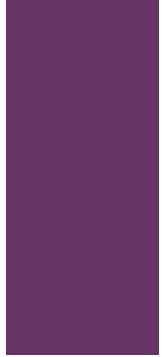
$$a'_i x \geq b_i y_i \quad i = \{1, \dots, m\}$$

$$\sum_{i=1}^m y_i \geq k$$

$$y_i \in \{0, 1\}$$



Disjunctive constraints



Single machine scheduling problem: n jobs, with processing time p_j . Let $t_j > 0$ be the start time of job j .

Consider the pair of jobs h and k . One has:

$$t_k \geq t_h + p_h \text{ if } h \text{ precedes } k$$

$$t_h \geq t_k + p_k \text{ if } k \text{ precedes } h$$

Introduce the binary variable

$$y_{hk} = \begin{cases} 1 & \text{if } h \text{ precedes } k, \\ 0 & \text{otherwise.} \end{cases}$$



$$t_k \geq t_h + p_h - M(1 - y_{hk})$$

$$t_h \geq t_k + p_k - My_{hk}$$

where M is a constant large enough. The reasoning can be easily extended to the whole set of jobs.



Minimum Spanning Tree



Given

A symmetric graph $G=(V,E)$ and a cost $c_e \geq 0$ for each edge in E

Find

A minimum cost spanning tree T

Notation

$E(S)$: set of edges induced by $S \subset V$

Cutset $\delta(S) = \{\{i,j\} \in E | i \in S, j \notin S\}$

Variables

$$x_e = \begin{cases} 1 & \text{if edge } e \text{ is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

Constraints

T is spanning

$$\sum_{e \in E} x_e = |V| - 1$$



Minimum Spanning Tree



Connectivity

Can be imposed by one of the sets of constraints

Subtour inequalities

$$\sum_{e \in E(S)} x_e \leq |S| - 1$$

$$\forall S \subset V, 2 < |S| \leq |V| - 1$$

Cutset inequalities

$$\sum_{e \in \delta(S)} x_e \geq 1$$

$$\forall S \subset V, S \neq \emptyset, V$$

Observation

Both sets contains an exponential number (in $|V|$) of constraints



Minimum Spanning Tree



Formulation 1

$$\begin{aligned} & \min \sum_{e \in E} c_e x_e \\ \text{s.t. } & \sum_{e \in E} x_e = |V| - 1, \\ & \sum_{e \in E(S)} x_e \leq |S| - 1 \quad \forall S \subset V, 2 < |S| \leq |V| - 1, \\ & x \in \{0, 1\}^{|E|} \end{aligned}$$

Formulation 2

$$\begin{aligned} & \min \sum_{e \in E} c_e x_e \\ \text{s.t. } & \sum_{e \in E} x_e = |V| - 1, \\ & \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subset V, S \neq \emptyset, V \\ & x \in \{0, 1\}^{|E|} \end{aligned}$$



Traveling Salesman Problem



Given

A symmetric graph $G=(V,E)$ and a cost $c_e \geq 0$ for each edge in E

Find

A Hamiltonian tour (a cycle that visits all nodes) of minimum cost

Decision variable

$$x_e = \begin{cases} 1 & \text{if edge } e \text{ is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

Property of a Hamiltonian tour

$$\sum_{e \in \delta(\{i\})} x_e = 2$$



Traveling Salesman Problem



Cut property

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall S \subset V, S \neq \emptyset, V$$

Cutset formulation

$$\begin{aligned} & \min \sum_{e \in E} c_e x_e \\ \text{s.t. } & \sum_{e \in \delta(\{i\})} x_e = 2, \\ & \sum_{e \in \delta(S)} x_e \geq 2 \quad \forall S \subset V, S \neq \emptyset, V \\ & x \in \{0, 1\}^{|E|} \end{aligned}$$



Traveling Salesman Problem



Subtour inequalities

$$\sum_{e \in \delta(S)} x_e \leq |S| - 1 \quad \forall S \subset V, S \neq \emptyset, V$$

Subtour formulation

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t. } \sum_{e \in \delta(\{i\})} x_e = 2,$$

$$\sum_{e \in \delta(S)} x_e \leq |S| - 1 \quad \forall S \subset V, S \neq \emptyset, V$$

$$x \in \{0, 1\}^{|E|}$$