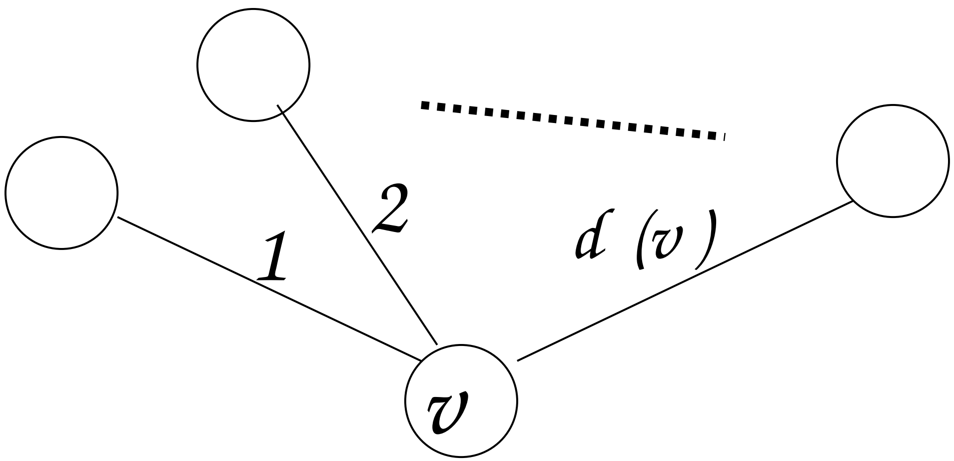
Luby's MIS Distributed Algorithm

Runs in time $O(\log d \cdot \log n)$

with high probability

This algorithm is asymptotically better than the previous one.

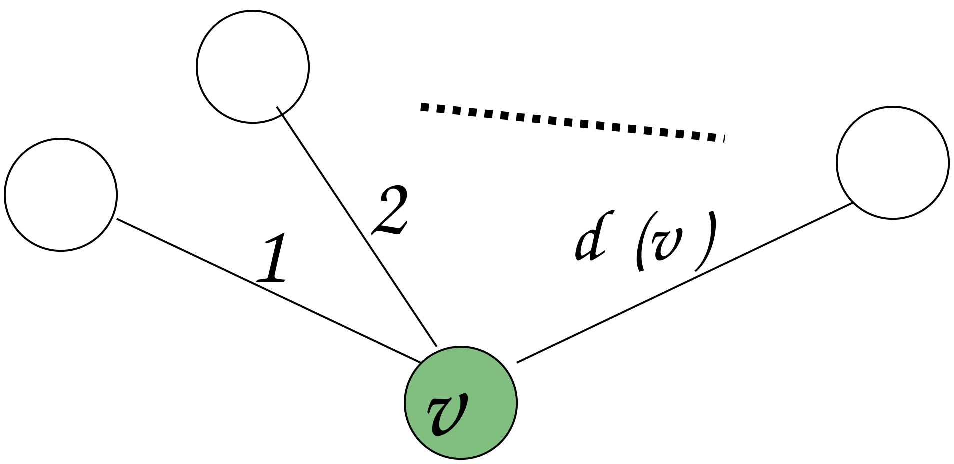
Let $d(v)$ be the degree of node v



At each phase k:

Each node $v \in G_k$ elects itself

with probability $p(v) = \dfrac{1}{2d(v)}$

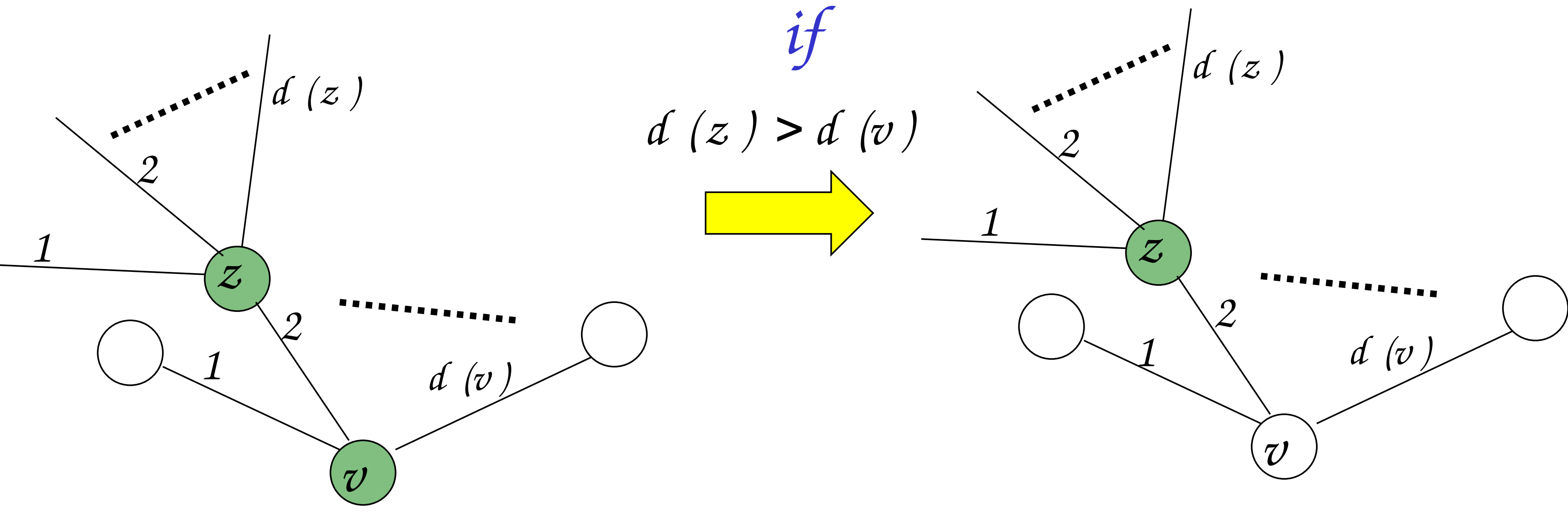


degree of v in G_k

Elected nodes are candidates for the independent set I_k

If two neighbors are elected simultaneously, then the higher degree node wins

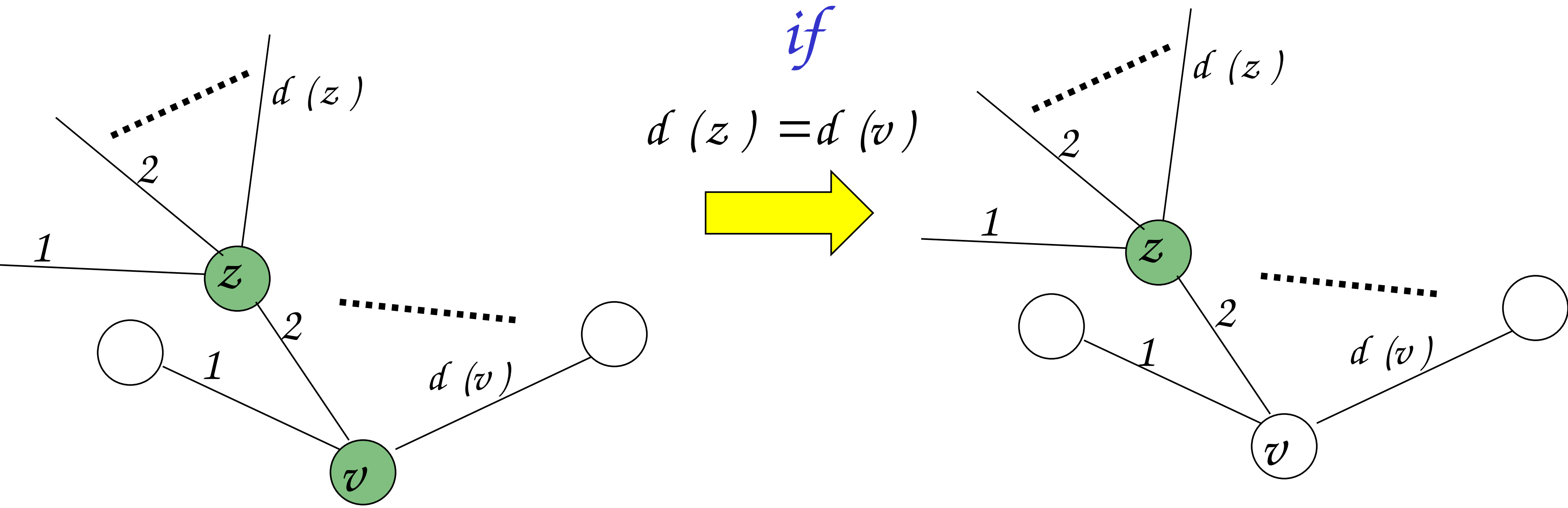
Example:



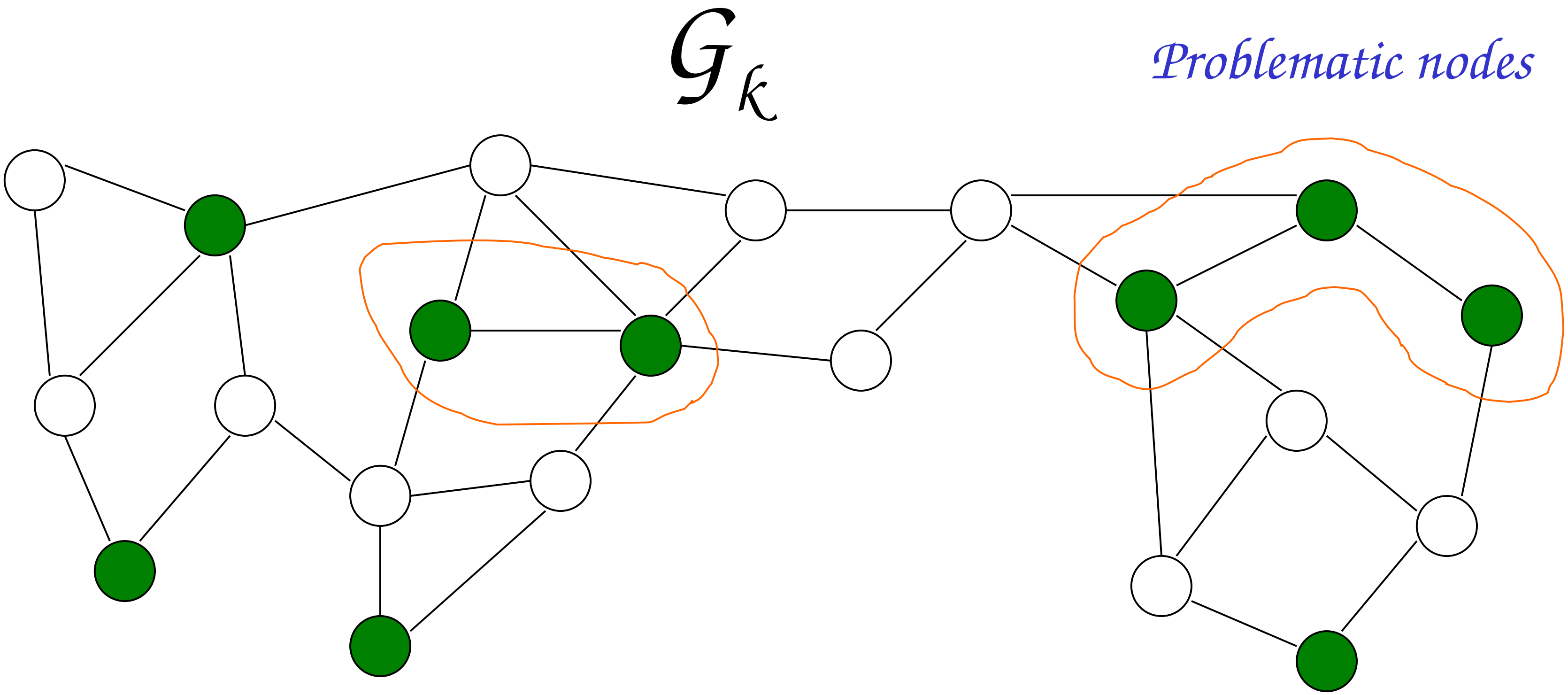
If both have the same degree,

ties are broken arbitrarily

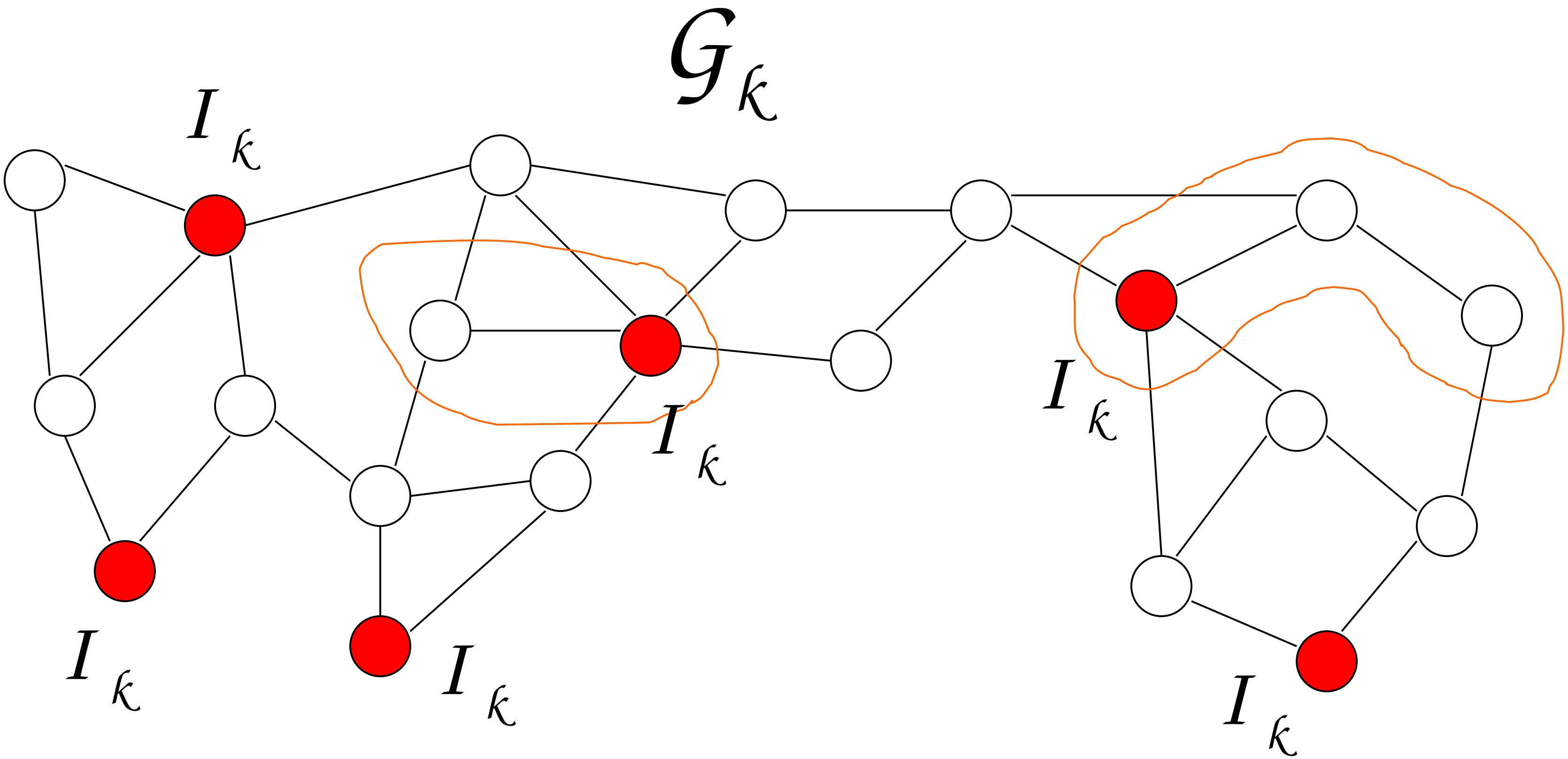
Example:



Using previous rules, problematic nodes are removed



The remaining elected nodes form independent set I_k

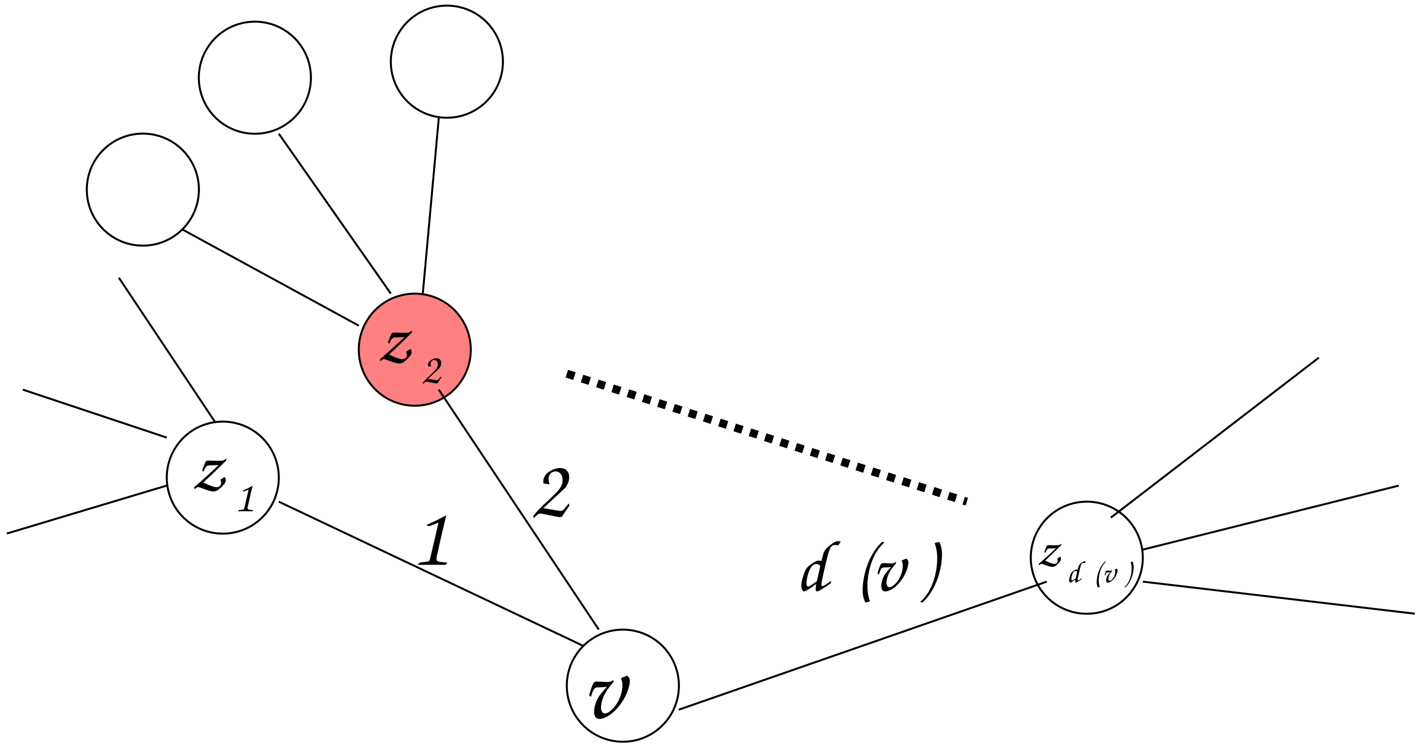


Analysis

Consider phase k

A good event for node v

$\mathcal{H}_v$: at least one neighbor enters I_k



If H_v is true, then $v \in \mathcal{N}(I_k)$ and v will disappear at the end of current phase

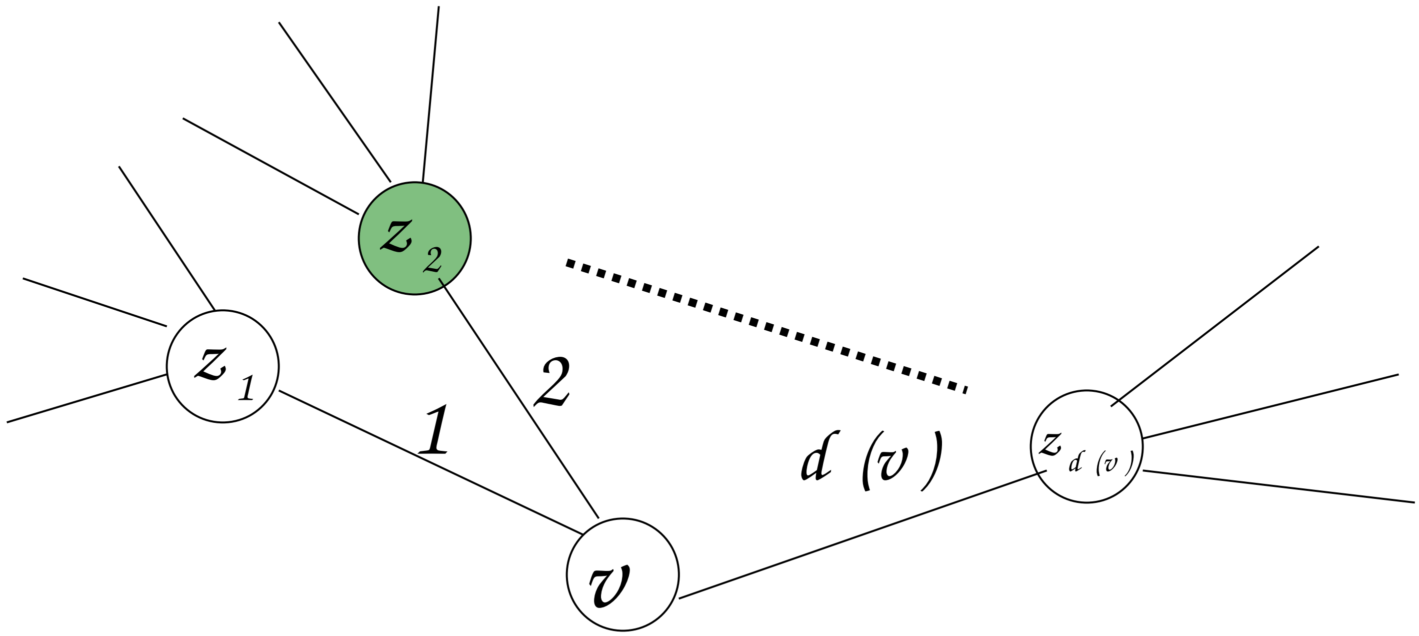
At the end of phase k

LEMMA 1: At least one neighbor of v elects itself with probability at least

$$1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}$$

$$\tilde{d}(v) = \max_{z \in \mathcal{N}(v)} d(z)$$

maximum neighbor degree

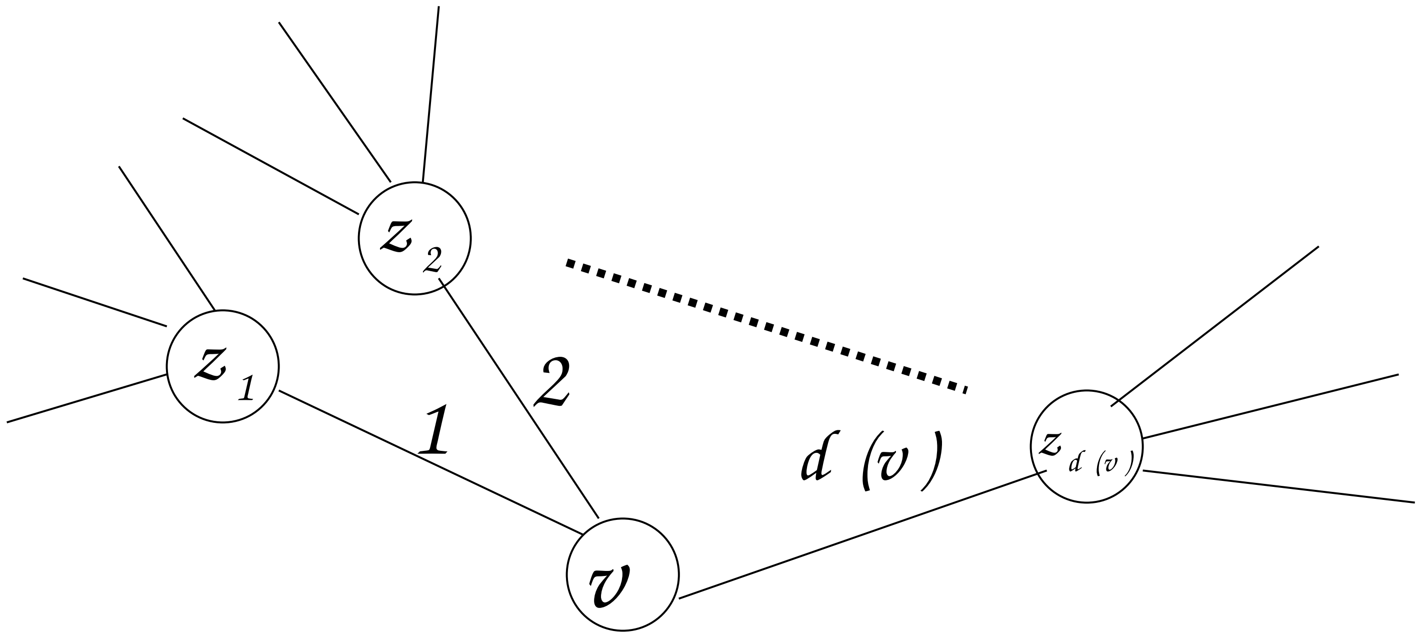


PROOF:

No neighbor of v elects itself with probability

$$\prod_{z \in \mathcal{N}(v)} (1 - p(z))$$

(the elections are independent)



$$\prod_{z \in \mathcal{N}(v)} (1 - p(z)) = \prod_{z \in \mathcal{N}(v)} \left(1 - \frac{1}{2d(z)}\right)$$

$$\leq \left(1 - \frac{1}{2\tilde{d}(v)}\right)^{d(v)} = \left(1 - \frac{1}{2\tilde{d}(v)}\right)^{\frac{2\tilde{d}(v) \cdot d(v)}{2\tilde{d}(v)}} \leq e^{-\frac{d(v)}{2\tilde{d}(v)}}$$

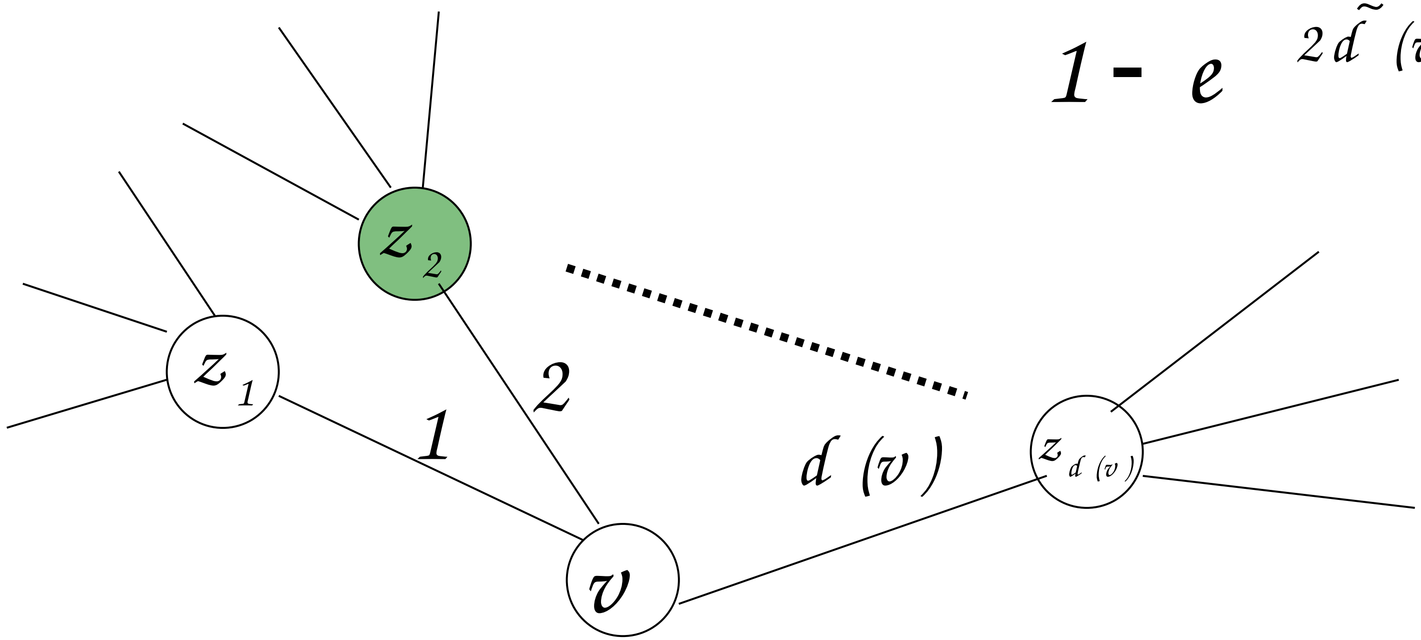
$$\tilde{d}(v) = \max_{z \in \mathcal{N}(v)} d(z)$$

maximum neighbor degree

Therefore, at least one neighbor of v

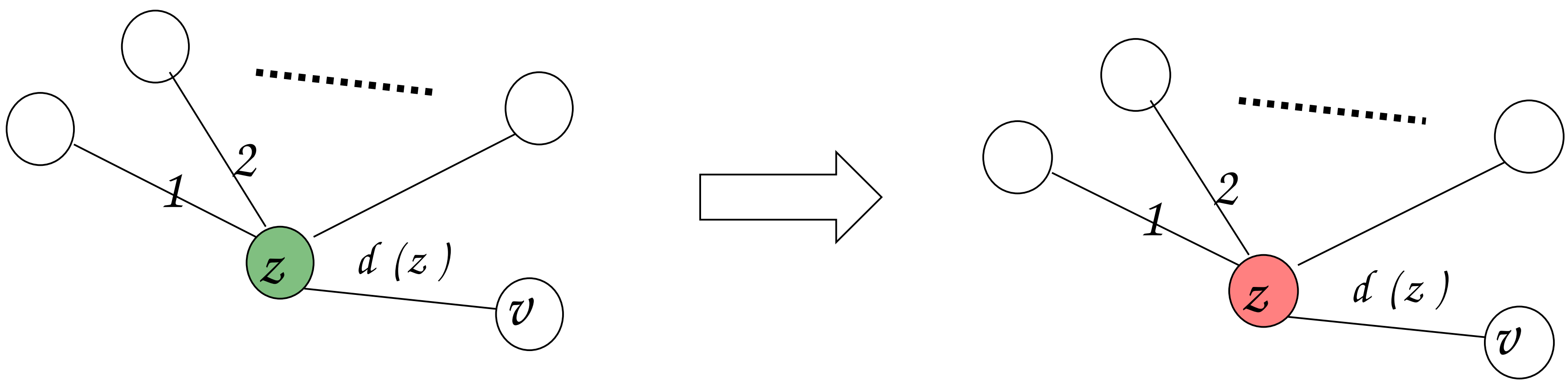
Elects itself with probability at least

$$1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}$$



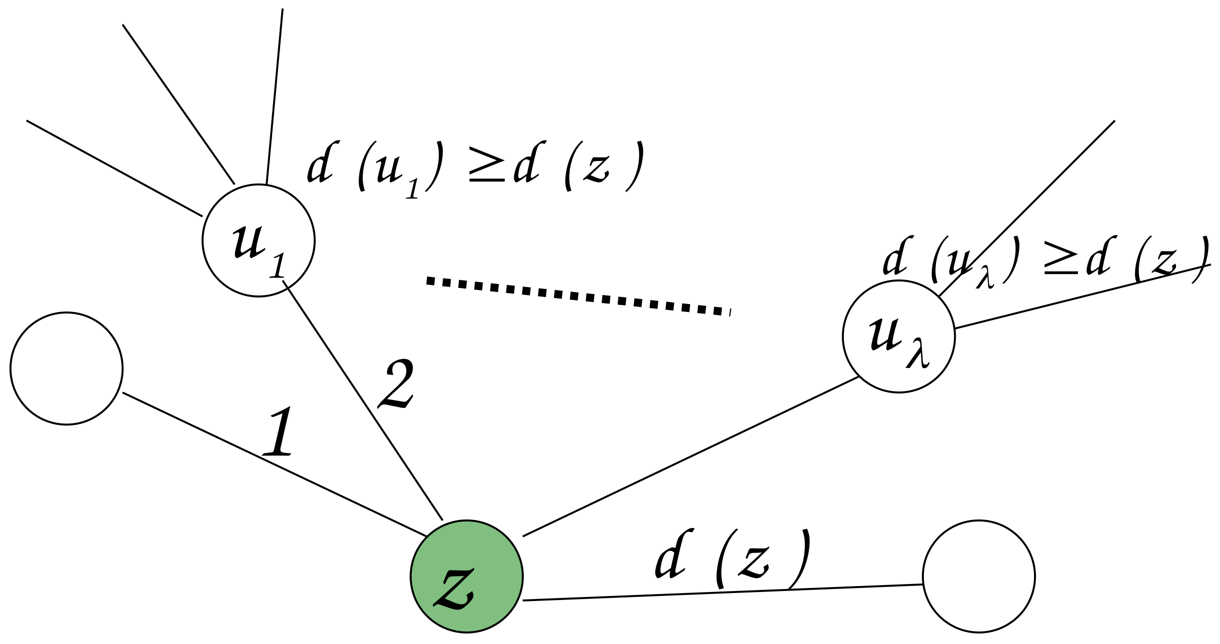
END OF PROOF

LEMMA 2: *If a node* z *elects itself, then it enters* I_k *with probability at least* $\frac{1}{2}$



PROOF:

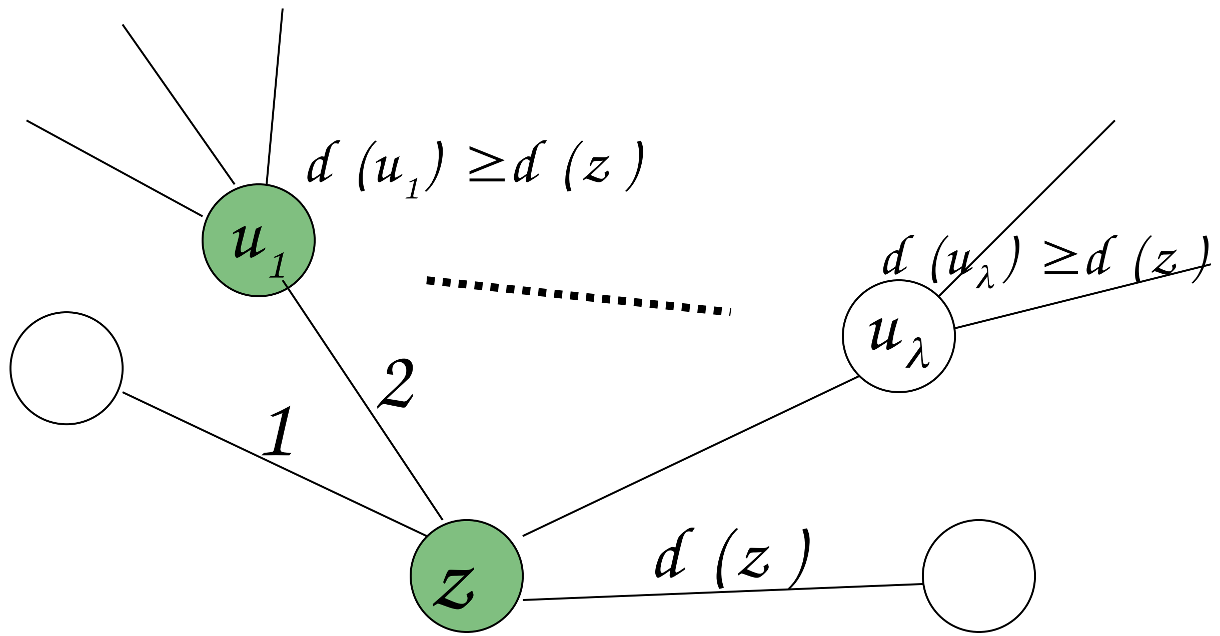
Node z enters I_k if no neighbor of same or higher degree elect itself



Probability that some neighbor of z *with same or higher degree elects itself*

$$P\left[\bigcup_{k}(\text{node } u_k \text{ elects itself})\right]$$

$$\leq \sum_{i=1}^{\lambda} p(u_i) \leq \frac{\lambda}{2d(z)} \leq \frac{d(z)}{2d(z)} = \frac{1}{2}$$

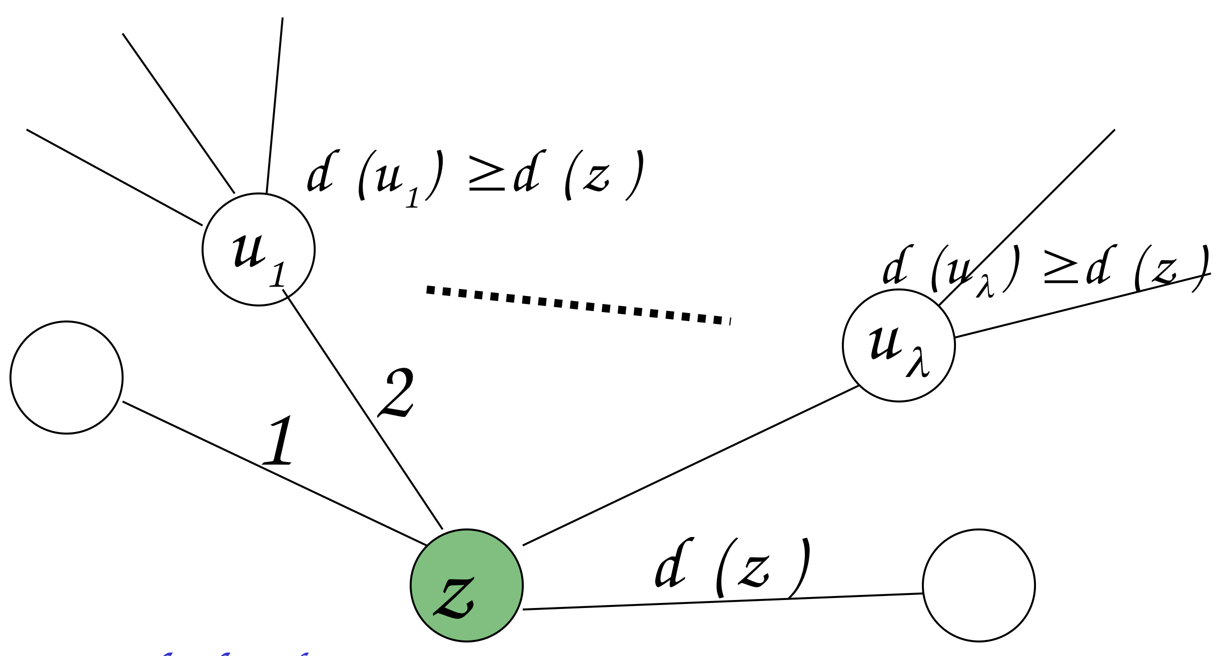


λ *neighbors of same or higher degree*

Probability that that no neighbor of z with same or higher degree elects itself

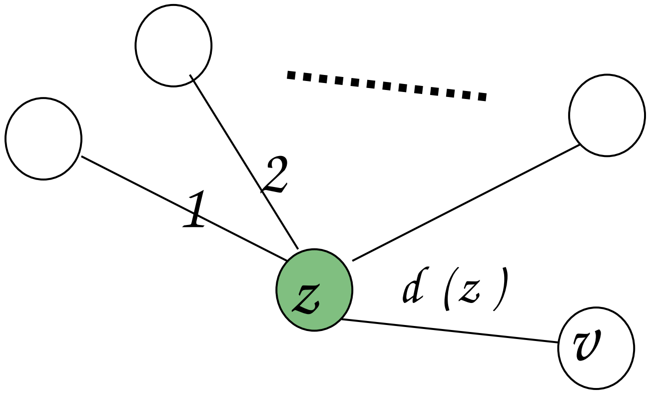
$$P[\bigcap_k (\text{no node } u_k \text{ elects itself})] =$$

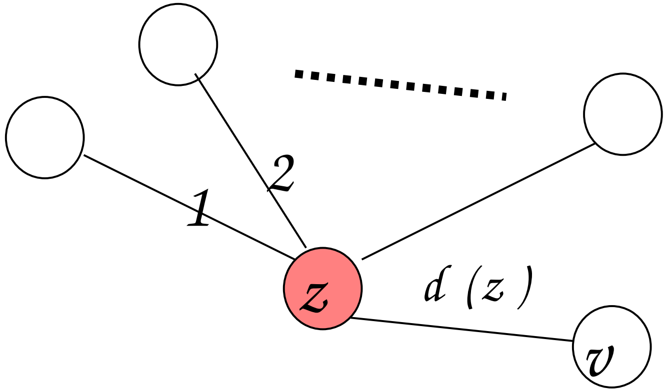
$$1 - P[\bigcup_k (\text{node } u_k \text{ elects itself})] \geq 1 - \frac{1}{2} = \frac{1}{2}$$



λ neighbors of same or higher degree

Thus, if z elects itself, it enters I_k with probability at least $\frac{1}{2}$



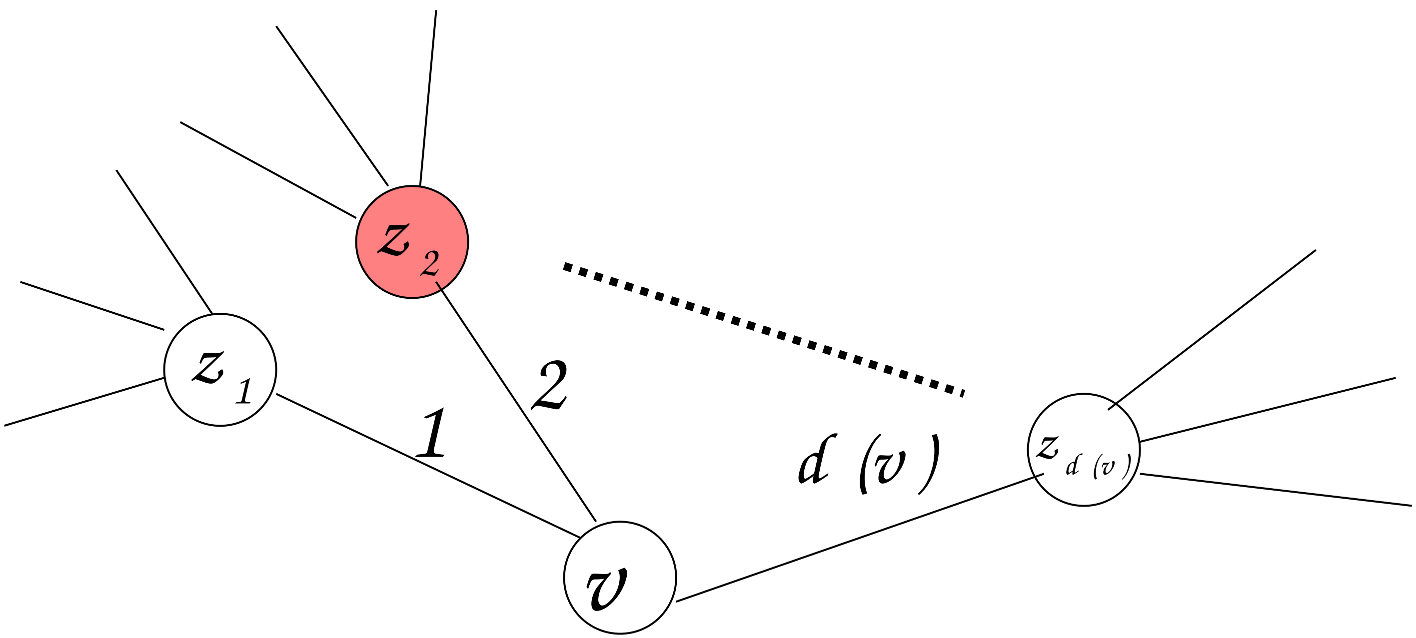


END OF PROOF

LEMMA 3:

$$\mathcal{P}[\mathcal{H}_v] \geq \frac{1}{2}\left(1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}\right)$$

$\mathcal{H}_v$: at least one neighbor of v enters I_k

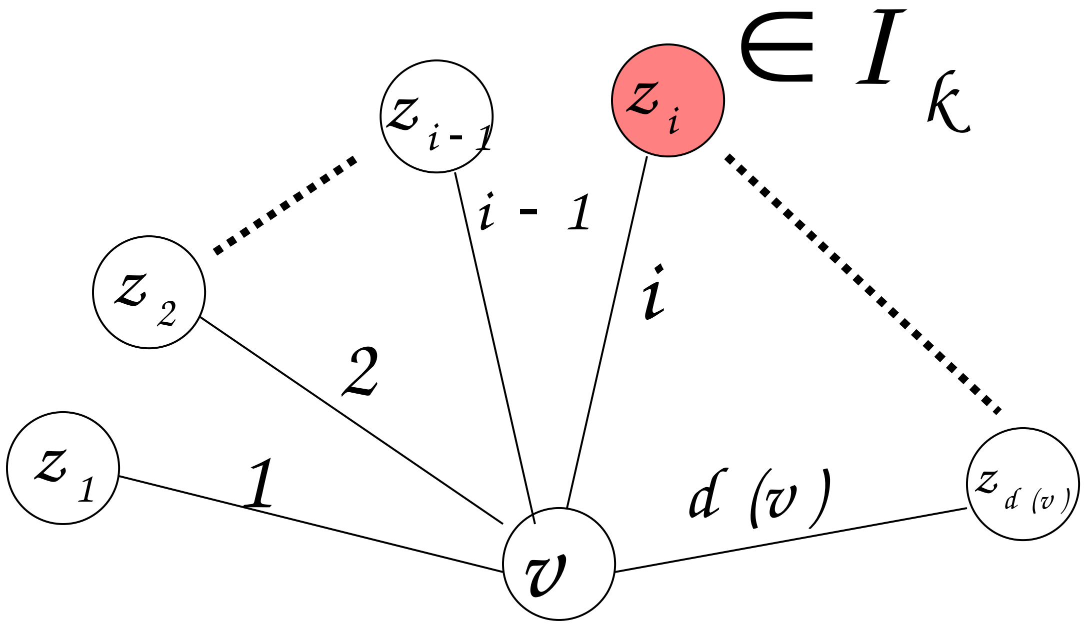


PROOF:

New event γ_i : neighbor z_i is in I_k

and no node $z_1, z_2, \ldots, z_{i-1}$

is elected



The events $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{d(v)}$

are mutually exclusive

$$\mathcal{P}\left[\bigcup_{1 \le i \le d(v)} \mathcal{Y}_i\right] = \sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{Y}_i]$$

It holds:

$$\mathcal{P}[\mathcal{H}_v] \geq \mathcal{P}\left[\bigcup_{1 \leq i \leq d(v)} \mathcal{Y}_i\right]$$

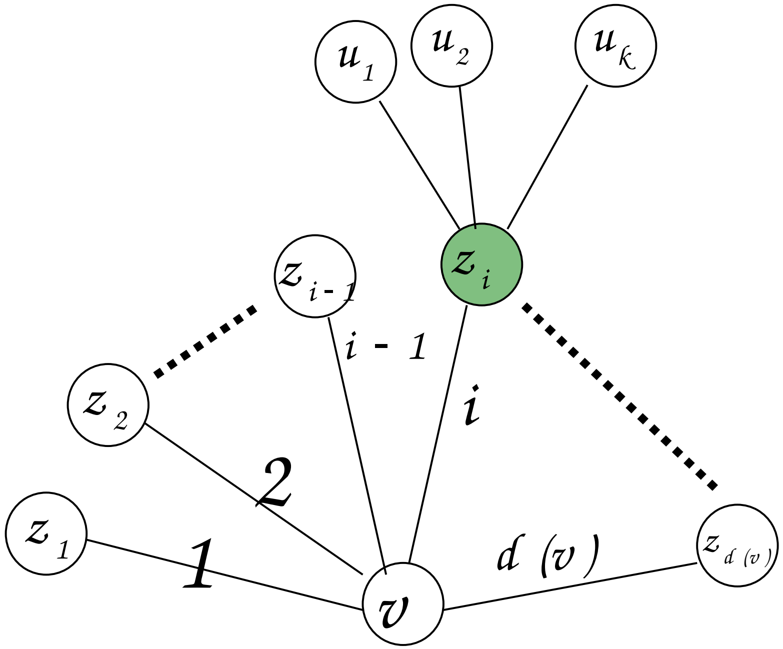
Therefore:

$$\mathcal{P}[\mathcal{H}_v] \geq \mathcal{P}\left[\bigcup_{1 \leq i \leq d(v)} \mathcal{Y}_i\right] = \sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{Y}_i]$$

$$\mathcal{P}[\mathcal{Y}_i] = \mathcal{P}[\mathcal{A}_i] \cdot \mathcal{P}[\mathcal{B}_i]$$

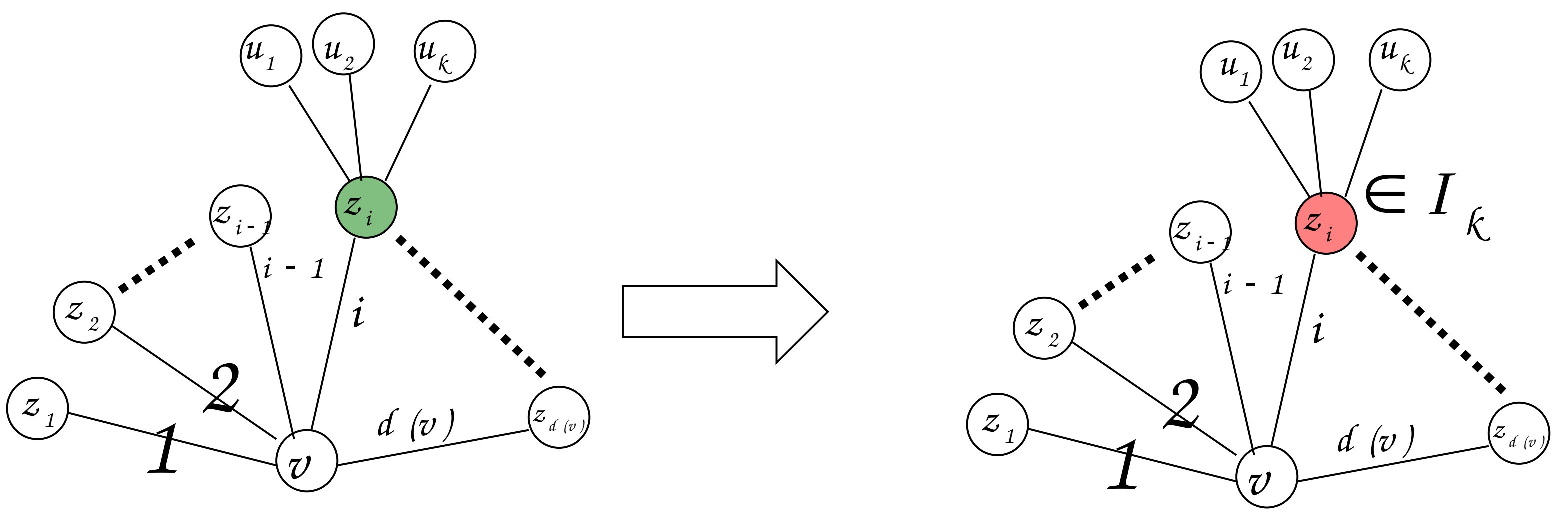
$\mathcal{A}_i$: z_i elects itself and no node $z_1, z_2, \ldots, z_{i-1}$ elects itself

$\mathcal{B}_i$: after z_i elects itself, it enters I_k



$\mathcal{B}_i$: after z_i elects itself, it enters I_k

$$\mathcal{P}[\mathcal{B}_i] \geq \frac{1}{2}$$ (from Lemma 2)



$$\mathcal{P}[\mathcal{H}_v] \geq \sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{Y}_i] = \sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{A}_i]\mathcal{P}[\mathcal{B}_i] \geq \frac{1}{2}\sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{A}_i]$$

$$\left(\mathcal{P}[\mathcal{B}_i] \geq \frac{1}{2} \right)$$

$\mathcal{A}_i$: z_i elects itself

and no node $z_1, z_2, \ldots, z_{i-1}$ elects itself

The events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{d(v)}$ are mutually exclusive

$$\mathcal{P}[\text{at least one neighbor of } v \text{ is elected}]$$

$$= \mathcal{P}[\bigcup_{1 \le i \le d(v)} \mathcal{A}_i] = \sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{A}_i]$$

We showed earlier (Lemma 1) that:

$$P[\text{at least one neighbor of } v \text{ is elected}] \geq 1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}$$

Therefore:

$$\sum_{i=1}^{d(v)} P[A_i] \geq 1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}$$

Therefore node v *disappears in phase* k

with probability at least

$$\mathcal{P}[\mathcal{H}_v] \geq \frac{1}{2}\sum_{i=1}^{d(v)} \mathcal{P}[\mathcal{A}_i] \geq \frac{1}{2}\left(1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}\right)$$

END OF PROOF

Let d_k be the maximum node degree in the graph $\mathcal{G}_k$

Suppose that in $\mathcal{G}_k$ $d(v) \geq \dfrac{d_k}{2}$

Then, $\tilde{d}(v) \leq 2d(v)$

$$\mathcal{P}[\mathcal{H}_v] \geq \frac{1}{2}\left(1 - e^{-\frac{d(v)}{2\tilde{d}(v)}}\right) \geq \frac{1}{2}\left(1 - e^{-\frac{1}{4}}\right) = c$$

constant

Thus, in phase k

a node with degree $d(v) \geq \frac{d_k}{2}$

disappears

with probability at least c

(thus, nodes with high degree

will disappear fast)

Consider a node v which in initial graph G

has degree $d(v) \geq \frac{d}{2}$

Suppose that the degree of v remains

at least $\frac{d}{2}$ for the next ϕ phases

Node v does not disappear

within ϕ phases with probability at most

$$(1-c)^{\phi}$$

Take $\phi = 3 \log_{1-c} \frac{1}{n}$

Node v does not disappear

within ϕ phases with probability at most

$$(1-c)^{\phi} = (1-c)^{3 \log_{1-c} \frac{1}{n}} = \frac{1}{n^3}$$

Thus, within $3 \log_{1-c} \frac{1}{n}$ phases

v either disappears

or its degree gets less than $\frac{d}{2}$

with probability at least $1 - \frac{1}{n^3}$

Therefore,

by the end of $3\log_{1-c}\frac{1}{n}$ *phases*

there is no node of degree higher than $\frac{d}{2}$

with probability at least (ineq. 2)

$$\left(1-\frac{1}{n^3}\right)^n \geq 1-\frac{1}{n^2}$$

In every $3 \log_{1-c} \frac{1}{n}$ phases,

the maximum degree of the graph

reduces by at least half,

with probability at least $1 - \frac{1}{n^2}$

Maximum number of phases until degree drops to 0 (MIS has formed)

$$\log d \cdot 3\log_{1-c}\frac{1}{n} = O(\log d \cdot \log n)$$

with probability at least (ineq. 2)

$$\left(1-\frac{1}{n^2}\right)^{\log d} \geq \left(1-\frac{1}{n^2}\right)^{n} \geq 1-\frac{1}{n}$$

Total number of phases:

$$O(\log d \cdot \log n)$$

with high probability

Time duration of each phase: $O(1)$

Total time: $O(\log d \cdot \log n)$