

# Graph Theory and Optimization

## Introduction on Duality in LP

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Thank you to F. Giroire for his slides

# Motivation

- Finding **bounds on the optimal solution**. Provides a measure of the "goodness" of a solution.
- Provide **certificate of optimality**.
- Economic interpretation of the dual problem.

# Outline

- 1 Introduction to duality: find bounds
- 2 Building the dual programme
- 3 Duality
- 4 Certificate of Optimality
- 5 Economical Interpretation

## Duality Theorem: introduction

$$\begin{array}{llllllll}
 \text{Maximize} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 \\
 \text{Subject to :} & & & & & & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 \leq 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 \leq 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 \leq 3 \\
 & & & & & x_1, x_2, x_3, x_4 & \geq & 0.
 \end{array}$$

**Lower bound:** a feasible solution, e.g.  $(0, 0, 1, 0) \Rightarrow z^* \geq 5$ .

What if we want an **upper bound**?

## Duality Theorem: introduction

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 \end{array}$$

Second Inequation  $\times 5/3$ :

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}.$$

Note that (all variables are positive),

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4$$

Hence, a first bound:

$$z^* \leq \frac{275}{3}.$$

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 \end{array}$$

Similarly,  $2^d + 3^d$  constraints:

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$$

Hence, **a second bound:**

$$z^* \leq 58.$$

→ need for a **systematic strategy**.



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 \end{array}$$

 $\times y_1$  $\times y_2$  $\times y_3$ 

Build **linear combinations of the constraints**. Summing:

$$\begin{aligned}
 & (y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 \\
 & + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3.
 \end{aligned}$$

We want left part upper bound of  $z$ . We need coefficient of  $x_j \geq$  coefficient in  $z$ :

$$\begin{array}{rclcl}
 y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3.
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## Duality Theorem: introduction

Build **linear combinations of the constraints**. Summing:

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 \\ + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3.$$

We want left part upper bound of  $z$ . We need coefficient of  $x_j \geq$  coefficient in  $z$ :

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If the  $y_i \geq 0$  and satisfy theses inequations, then

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3.$$

In particular,

$$z^* \leq y_1 + 55y_2 + 3y_3.$$

## Duality Theorem: introduction

**Objective:** smallest possible upper bound. Hence, we solve the following PL:

$$\begin{array}{llllll}
 \text{Minimize} & y_1 & + & 55y_2 & + & 3y_3 \\
 \text{Subject to:} & & & & & \\
 & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
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It is the **dual problem** of the problem.

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# The Dual Problem

Primal problem:

$$\begin{array}{ll}\text{Maximize} & \sum_{j=1}^n c_j x_j \\ \text{Subject to:} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \dots, m) \\ & x_j \geq 0 \quad (j = 1, 2, \dots, n)\end{array}$$

Its **dual problem** is defined by the LP problem:

$$\begin{array}{ll}\text{Minimize} & \sum_{i=1}^m b_i y_i \\ \text{Subject to:} & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad (j = 1, 2, \dots, n) \\ & y_i \geq 0 \quad (i = 1, 2, \dots, m)\end{array}$$

# Weak Duality Theorem

## Theorem

## Weak Duality

If  $(x_1, x_2, \dots, x_n)$  is feasible for the primal and  $(y_1, y_2, \dots, y_m)$  is feasible for the dual, then

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

Proof:

$$\begin{aligned} \sum_j c_j x_j &\leq \sum_j (\sum_i y_i a_{ij}) x_j && \text{dual definition: } \sum_i y_i a_{ij} \geq c_j \\ &= \sum_i (\sum_j a_{ij} x_j) y_i \\ &\leq \sum_i b_i y_i && \text{primal definition: } \sum_j x_j a_{ij} \leq b_i \end{aligned}$$

Corollary:

The optimal value of the dual is an upper bound for the optimal value of the primal.

$$\max_{(x_1, \dots, x_n) \text{ feasible}} \sum_j c_j x_j \leq \min_{(y_1, \dots, y_m) \text{ feasible}} \sum_i b_i y_i.$$



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## Corollary:

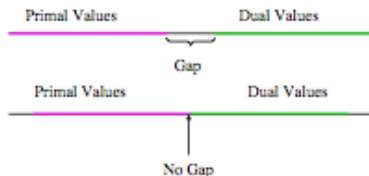
The optimal value of the dual is an upper bound for the optimal value of the primal.

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## Gap or No Gap?

An important question:

Is there a gap between the **largest primal value** and the **smallest dual value**?



# Strong Duality Theorem

## Theorem

## Strong duality

If the primal problem has an optimal solution,

$$x^* = (x_1^*, \dots, x_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, \dots, y_n^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

# Relationship between the Primal and Dual Problems

**Lemma:** The dual of the dual is always the primal problem.

**Corollary:** + (Strong Duality Theorem)  $\Rightarrow$  Primal has an optimal solution iff dual has an optimal solution.

Weak duality: Primal unbounded  $\Rightarrow$  dual infeasible.

		Dual		
		Optimal	Unfeasible	Unbounded
Primal	Optimal	X		
	Unfeasible		X	X
	Unbounded		X	

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		Optimal	Dual Unfeasible	Unbounded
Primal	Optimal	X		
	Unfeasible		X	
	Unbounded		X	X

## Application of Duality to Maximum flow

$D = (V, A)$  be a graph with capacity  $c : A \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

**Problem:** Compute a maximum flow from  $s$  to  $t$ .

$$\begin{array}{ll}
 \text{Maximize} & \sum_{(s,u) \in A} f(su) \\
 \text{Subject to:} & \sum_{(v,u) \in A} f(vu) \leq \sum_{(u,v) \in A} f(uv) \quad \text{for all } u \in V \setminus \{s, t\} \\
 & f(a) \geq 0 \quad \text{for all } a \in A
 \end{array}$$

**Exercise:** Write the dual program

# Application of Duality to Maximum flow

Variable  $y_a$  per edge constraint; Variable  $z_v$  per vertex-constraint

$$R = \sum_{a \in A} f(a)y_a + \sum_{v \in V \setminus \{s,t\}} \left( \sum_{(v,u) \in A} f(vu) - \sum_{(u,v) \in A} f(uv) \right) z_v \leq \sum_{a \in A} c(a)y_a$$

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that can be rewritten:

$$R = f(st)y_{st} + \sum_{(s,v) \in A, v \neq t} f(sv)(y_{sv} + z_v) + \sum_{(v,t) \in A, v \neq s} f(vt)(y_{vt} - z_v) +$$

$$\sum_{(u,v) \in A, u \neq s, v \neq t} f(uv)(y_{uv} + z_v - z_u) \leq \sum_{a \in A} c(a)y_a$$



# Application of Duality to Maximum flow

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$$\sum_{(u, v) \in A, u \neq s, v \neq t} f(uv)(y_{uv} + z_v - z_u) \leq \sum_{a \in A} c(a)y_a$$

So, to have  $\sum_{(s, u) \in A} f(su) \leq R \leq \sum_{a \in A} c(a)y_a$ :

$$\begin{array}{lll} y_a & \geq & 1 \quad \text{if } a = (s, t) \\ y_a + z_v & \geq & 1 \quad \text{if } a = (s, v), v \neq t \\ y_a + z_v & \geq & 0 \quad \text{if } a = (v, t), v \neq s \\ y_a + z_v - z_u & \geq & 0 \quad \text{if } a = (u, v), u \neq s, v \neq t \end{array}$$

## Application of Duality to Maximum flow

The dual of the previous formulation of Max-Flow

$$\begin{array}{ll}
 \text{Minimize} & \sum_{a \in A} c(a) y_a \\
 \text{Subject to:} & y_a \geq 1 \quad \text{if } a = (s, t) \\
 & y_a + z_v \geq 1 \quad \text{if } a = (s, v), v \neq t \\
 & y_a + z_v \geq 0 \quad \text{if } a = (v, t), v \neq s \\
 & y_a + z_v - z_u \geq 0 \quad \text{if } a = (u, v), u \neq s, v \neq t \\
 & y_a \geq 0 \quad \text{for all } a \in A \\
 & z_v \geq 0 \quad \text{for all } v \in V
 \end{array}$$

**Exercise:** Prove it is a LP for the Min-Cut Problem  
Deduce the MaxFlow-MinCut Theorem

## Exercises

$G = (V, E)$  be a graph with weight  $w : E \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

What compute the following programmes? Give their dual Programme

$$\begin{array}{ll}
 \text{Maximize} & \sum_{P \text{ path from } s \text{ to } t} x_P \\
 \text{Subject to:} & \sum_{P, e \in E(P)} x_P \leq w(e) \quad \text{for all } e \in E \\
 & x_P \geq 0 \quad \text{for all paths } P \text{ from } s \text{ to } t
 \end{array}$$

$$\begin{array}{ll}
 \text{Maximize} & x_t \\
 \text{Subject to:} & x_s = 0 \\
 & x_v \leq x_u + w(\{u, v\}) \quad \text{for all } \{v, u\} \in E \\
 & x_v \geq 0 \quad \text{for all } v \in V
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## Complementary Slackness

### Theorem

### Complementary Slackness

Let  $x_1^*, \dots, x_n^*$  be a feasible solution of the primal and  $y_1^*, \dots, y_m^*$  be a feasible solution of the dual. Then,

$$\sum_{i=1}^m a_{ij} y_i^* = c_j \quad \text{or} \quad x_j^* = 0 \quad \text{or both} \quad (j = 1, 2, \dots, n)$$

$$\sum_{j=1}^n a_{ij} x_j^* = b_i \quad \text{or} \quad y_i^* = 0 \quad \text{or both} \quad (i = 1, 2, \dots, m)$$

are necessary and sufficient conditions to have the optimality of  $x^*$  and  $y^*$ .

## Complementary Slackness - Proof

$x^*$  feasible  $\Rightarrow b_i - \sum_j a_{ij}x_j \geq 0$ .

$y^*$  dual feasible, hence non negative.

Thus

$$(b_i - \sum_j a_{ij}x_j)y_i \geq 0.$$

Similarly,

$y^*$  dual feasible  $\Rightarrow \sum_i a_{ij}y_i - c_j \geq 0$ .

$x^*$  feasible, hence non negative.

$$(\sum_i a_{ij}y_i - c_j)x_j \geq 0.$$

## Complementary Slackness - Proof

$$(b_i - \sum_j a_{ij}x_j)y_i \geq 0 \quad \text{and} \quad (\sum_i a_{ij}y_i - c_j)x_j \geq 0$$

By summing, we get:

$$\sum_i (b_i - \sum_j a_{ij}x_j)y_i \geq 0 \quad \text{and} \quad \sum_j (\sum_i a_{ij}y_i - c_j)x_j \geq 0$$

Summing + strong duality theorem:

$$\sum_i b_i y_i - \sum_{i,j} a_{ij} x_j y_i + \sum_{j,i} a_{ij} y_i x_j - \sum_j c_j x_j = \sum_i b_i y_i - \sum_j c_j x_j = 0.$$

Implies: inequalities must be equalities:

$$\forall i, (b_i - \sum_j a_{ij}x_j)y_i = 0 \quad \text{and} \quad \forall j, (\sum_i a_{ij}y_i - c_j)x_j = 0.$$

$XY = 0$  if  $X = 0$  or  $Y = 0$ . Done.

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Implies: inequalities must be equalities:

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$XY = 0$  if  $X = 0$  or  $Y = 0$ . Done.

## Theorem

## Optimality Certificate

A feasible solution  $x_1^*, \dots, x_n^*$  of the primal is **optimal** iif there exist numbers  $y_1^*, \dots, y_m^*$  such that

- ① they **satisfy the complementary slackness** condition:

$$\begin{aligned} \sum_{i=1}^m a_{ij} y_i^* &= c_j && \text{when } x_j^* > 0 \\ y_j^* &= 0 && \text{when } \sum_{j=1}^n a_{ij} x_j^* < b_i \end{aligned}$$

- ② and  $y_1^*, \dots, y_m^*$  **feasible solution of the dual**, that is

$$\begin{aligned} \sum_{i=1}^m a_{ij} y_i^* &\geq c_j && \forall j = 1, \dots, n \\ y_i^* &\geq 0 && \forall i = 1, \dots, m. \end{aligned}$$

**Example:** Verify that  $(2, 4, 0, 0, 7, 0)$  optimal solution of

$$\begin{array}{rcllclclclclclclclcl}
 \text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & & & \\
 \text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 & & \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 & & \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 & & \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 & & \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 & & \\
 & & & & & & & & & & & x_1, x_2, \dots, x_6 & \geq & 0 & & 
 \end{array}$$

**First step:** Existence of  $y_1^*, \dots, y_5^*$ , such as

$$\begin{array}{ll}
 \sum_{i=1}^m a_{ij} y_i^* & = c_j \quad \text{when } x_j^* > 0 \\
 y_i^* & = 0 \quad \text{when } \sum_{j=1}^n a_{ij} x_j^* < b_i
 \end{array}$$

That is

$$\begin{array}{rclclclclclclclclclcl}
 2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & = & 18 \\
 -6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & = & -7 \\
 3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & = & 0 \\
 & & y_2^* & & & & & & & = & 0 \\
 & & & & & & & & y_5^* & = & 0
 \end{array}$$

$(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$  is solution.

**Example:** Verify that  $(2, 4, 0, 0, 7, 0)$  optimal solution of

$$\begin{array}{rcllclclclclclclclcl}
 \text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & & & \\
 \text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 & & \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 & & \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 & & \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 & & \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 & & \\
 & & & & & & & & & & & x_1, x_2, \dots, x_6 & \geq & 0 & & 
 \end{array}$$

**Second step:** Verify  $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$  is a solution of the dual.

$$\begin{aligned}
 \sum_{i=1}^m a_{ij} y_i^* &\geq c_j & \forall j = 1, \dots, n \\
 y_j^* &\geq 0 & \forall j = 1, \dots, m.
 \end{aligned}$$

**Example:** Verify that  $(2, 4, 0, 0, 7, 0)$  optimal solution of

$$\begin{array}{rcccccccccccccccl}
 \text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & & & & \\
 \text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 \\
 & & & & & & & & & & & x_1, x_2, \dots, x_6 & \geq & 0
 \end{array}$$

**Second step:** Verify  $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$  is a solution of the dual.

$$\begin{array}{ll}
 \sum_{i=1}^m a_{ij}y_i^* & \geq c_j \quad \forall j = 1, \dots, n \\
 y_j^* & \geq 0 \quad \forall j = 1, \dots, m.
 \end{array}$$

That is, we check

$$\begin{array}{rcccccccccccl}
 2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & \geq & 18 \\
 -6y_1^* & - & y_2^* & - & 3y_3^* & + & & + & 2y_5^* & \geq & -7 \\
 2y_1^* & + & 4y_2^* & + & 5y_3^* & + & 8y_4^* & + & 3y_5^* & \geq & 12 \\
 7y_1^* & - & 3y_2^* & - & 2y_3^* & + & 7y_4^* & + & 6y_5^* & \geq & 5 \\
 3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & \geq & 0 \\
 8y_1^* & + & 2y_2^* & + & 2y_3^* & + & 3y_4^* & + & y_5^* & \geq & 8
 \end{array}$$

**Example:** Verify that  $(2, 4, 0, 0, 7, 0)$  optimal solution of

$$\begin{array}{rcccccccccccccccl}
 \text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & & & & \\
 \text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 & & & \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 & & & \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 & & & \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 & & & \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 & & & \\
 & & & & & & & & & & & x_1, x_2, \dots, x_6 & \geq & 0 & & & 
 \end{array}$$

**Second step:** Verify  $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$  is a solution of the dual.

$$\begin{array}{rcl}
 \sum_{i=1}^m a_{ij}y_i^* & \geq & c_j \quad \forall j = 1, \dots, n \\
 y_j^* & \geq & 0 \quad \forall i = 1, \dots, m.
 \end{array}$$

That is, we check

$$\begin{array}{rcccccccccccl}
 2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & \geq & 18 & \text{OK} \\
 -6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & \geq & -7 & \text{OK} \\
 2y_1^* & + & 4y_2^* & + & 5y_3^* & + & 8y_4^* & + & 3y_5^* & \geq & 12 & \\
 7y_1^* & - & 3y_2^* & - & 2y_3^* & + & 7y_4^* & + & 6y_5^* & \geq & 5 & \\
 3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & \geq & 0 & \text{OK} \\
 8y_1^* & + & 2y_2^* & + & 2y_3^* & + & 3y_4^* & + & y_5^* & \geq & 8 & 
 \end{array}$$

Only three equations to check.

**Example:** Verify that  $(2, 4, 0, 0, 7, 0)$  optimal solution of

Max	$18x_1$	$-$	$7x_2$	$+$	$12x_3$	$+$	$5x_4$	$+$	$8x_6$		
st:	$2x_1$	$-$	$6x_2$	$+$	$2x_3$	$+$	$7x_4$	$+$	$3x_5$	$+$	$8x_6 \leq 1$
	$-3x_1$	$-$	$x_2$	$+$	$4x_3$	$-$	$3x_4$	$+$	$x_5$	$+$	$2x_6 \leq -2$
	$8x_1$	$-$	$3x_2$	$+$	$5x_3$	$-$	$2x_4$	$+$		$2x_6 \leq 4$	
	$4x_1$			$+$	$8x_3$	$+$	$7x_4$	$-$	$x_5$	$+$	$3x_6 \leq 1$
	$5x_1$	$+$	$2x_2$	$-$	$3x_3$	$+$	$6x_4$	$-$	$2x_5$	$-$	$x_6 \leq 5$
									$x_1, x_2, \dots, x_6 \geq 0$		$\geq 0$

**Second step:** Verify  $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$  is a solution of the dual.

$$\sum_{i=1}^m a_{ij}y_i^* \geq c_j \quad \forall j = 1, \dots, n$$

$$y_j^* \geq 0 \quad \forall i = 1, \dots, m.$$

That is, we check

$2y_1^*$	$-$	$3y_2^*$	$+$	$8y_3^*$	$+$	$4y_4^*$	$+$	$5y_5^*$	$\geq$	18	OK
$-6y_1^*$	$-$	$y_2^*$	$-$	$3y_3^*$	$+$		$+$	$2y_5^*$	$\geq$	-7	OK
$2y_1^*$	$+$	$4y_2^*$	$+$	$5y_3^*$	$+$	$8y_4^*$	$+$	$3y_5^*$	$\geq$	12	
$7y_1^*$	$-$	$3y_2^*$	$-$	$2y_3^*$	$+$	$7y_4^*$	$+$	$6y_5^*$	$\geq$	5	
$3y_1^*$	$+$	$y_2^*$			$-$	$y_4^*$	$-$	$2y_5^*$	$\geq$	0	OK
$8y_1^*$	$+$	$2y_2^*$	$+$	$2y_3^*$	$+$	$3y_4^*$	$1$	$y_5^*$	$\geq$	8	

Only three equations to check.

OK. The solution  $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$  is optimal.



# Outline

- 1 Introduction to duality: find bounds
- 2 Building the dual programme
- 3 Duality
- 4 Certificate of Optimality
- 5 Economical Interpretation

# Signification of Dual Variables

$$\begin{array}{ll} \text{Maximize} & \sum_{j=1}^n c_j x_j \\ \text{Subject to:} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \dots, m) \\ & x_j \geq 0 \quad (j = 1, 2, \dots, n) \end{array}$$

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^m b_i y_i \\ \text{Subject to:} & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad (j = 1, 2, \dots, n) \\ & y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{array}$$

**Signification can be given** to variables of the dual problem (**dimension analysis**):

- $x_j$ : production of a product  $j$  (chair, ...)
- $b_i$ : available quantity of resource  $i$  (wood, metal, ...)
- $a_{ij}$ : unit of resource  $i$  per unit of product  $j$
- $c_j$ : net benefit of the production of a unit of product  $j$

# Signification of Dual Variables

$$\begin{array}{ll} \text{Maximize} & \sum_{j=1}^n c_j x_j \\ \text{Subject to:} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \dots, m) \\ & x_j \geq 0 \quad (j = 1, 2, \dots, n) \end{array}$$

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^m b_i y_i \\ \text{Subject to:} & \sum_{j=1}^n a_{ij} y_i \geq c_j \quad (j = 1, 2, \dots, n) \\ & y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{array}$$

**Signification can be given** to variables of the dual problem (**dimension analysis**):

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unit of resource  $i$ /unit of product  $j$     euros/unit of product  $j$     euros/unit of resource  $i$

$$a_{1j} y_1 + \dots + a_{nj} y_n > c_j$$

→  $y_i$  euro by unit of resource  $i$ .  
Marginal cost of resource  $i$ .

## Signification of Dual Variables

**Theorem:** If the LP admits at least one optimal solution, then there exists  $\varepsilon > 0$ , with the property: If  $|t_i| \leq \varepsilon \forall i = 1, 2, \dots, m$ , then the LP

$$\begin{array}{ll} \text{Max} & \sum_{j=1}^n c_j x_j \\ \text{Subject to:} & \sum_{j=1}^n a_{ij} x_j \leq b_i + t_i \quad (i = 1, 2, \dots, m) \\ & x_j \geq 0 \quad (j = 1, 2, \dots, n). \end{array}$$

has an optimal solution and the **optimal value of the objective is**

$$z^* + \sum_{i=1}^m y_i^* t_i$$

with  $z^*$  the optimal solution of the initial LP and  $(y_1^*, y_2^*, \dots, y_m^*)$  the optimal solution of its dual.

## Summary: To be remembered

- How to compute a **Dual Programme**.
- **Weak/Strong duality Theorem**.
- **Optimality certificate** (Complementary Slackness).