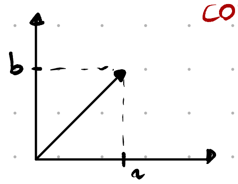
LINEAR ALGEBRA

REFERENCES

- NIELSEN & CHUANG
- QUANTUM COMPUTATION & INFORMATION
- QUANTUM MECHANICS: THE THEORETICAL QUANTUM

COMPLEX NUMBER



$z = a + jb \mid a, b \in \mathbb{R}$

$j^2 = -1$

$z^* = a - jb$

VECTOR SPACE OVER A FIELD ($\mathbb{C}$)

$v_1, v_2 \in V$

$v_1 + v_2 \in V$

$z \cdot v \in V \mid z \in \mathbb{C}, v \in V$

INNER PRODUCT

$(v, u) \in \mathbb{C}$

$(v, u + w) = (v, u) + (v, w)$

$(v, zu) = z(v, u)$

$(v, u) = \underbrace{(u, v)^*}_{\text{CONJUGATE}}$

SCALAR PRODUCT

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

$(x, y) = x \cdot y = x^T y = \sum_{i=1}^{m} x_i y_i$

$\|v\| = \sqrt{(v, v)}$

$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}$

VECTOR SPACE
+
INNER PRODUCT
=
HILBERT SPACE

$f: V \to V$ LINEAR FUNCTION

$f(z_1 v_1 + z_2 v_2) = z_1 f(v_1) + z_2 f(v_2)$

f CAN BE REPRESENTED AS A MATRIX

- BASIS $= \{u_1, \dots, u_m\} \Rightarrow \exists\, z_1, \dots, z_m \in \mathbb{C} \mid v = z_1 u_1 + \dots + z_m u_m \quad \forall v \in V$
- so it follow:

$$f(v_i) = z_1 f(u_1) + \dots + z_m f(u_m) = \begin{bmatrix} f(u_1) & f(u_2) & \dots & f(u_m) \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$$

EIGENVALUES AND EIGENVECTORS

- λ IS AN EIGENVALUE for A w.r.t. the EIGENVECTOR $v \iff Av = \lambda v$
- EIGENVALUES are the solutions of $\det(A - \lambda I) = 0 \Rightarrow$ it has m solutions
- Some solutions can coincide, we say: "λ has ALGEBRAIC MULTIPLICITY #occ"
- HOW MANY INDIPENDENT EIGENVECTOR CAN I FIND FOR λ? GEOMETRIC MULTIPLICITY

  ↳ if $\lambda_1 \mid A v_{1,1} = \underbrace{\lambda_1}_{\text{EIGENVALUE}} \overbrace{v_{1,1}}^{\text{EIGENVECTOR}}$ then every $\alpha \lambda_1$ is valid, but dependent from λ_i
- Let's say that $\forall \lambda_i$ ALG. MULT. = GEOM. MUL.

$$m = 5$$

$$\begin{array}{c|cc|l} \lambda_1 & v_{11}, & v_{12} & \Rightarrow \text{MULT.} = 2 \\ \lambda_2 & v_{21} & & \Rightarrow \text{MULT.} = 1 \\ \lambda_3 & v_{31} & v_{32} & \Rightarrow \text{MULT.} = 2 \end{array}$$

ARE BASIS FOR THE SPACE

- $f \to A$ can be rewritten as function of BASIS

$$A' = \begin{bmatrix} f(v_1) & f(v_2) & \dots \end{bmatrix} = \begin{bmatrix} v_{11}, v_{12}, v_{21}, v_{31}, v_{32} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \emptyset \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ \emptyset & & & & \lambda_3 \end{bmatrix}$$

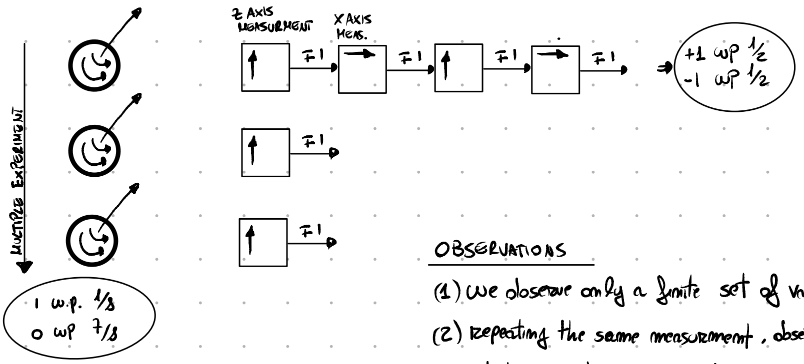
- $$A \underbrace{\begin{bmatrix} v_{11}, v_{12}, v_{21}, v_{31}, v_{32} \end{bmatrix}}_{\text{THIS MATRIX IS CALLED } V} = \begin{bmatrix} v_{11}, v_{12}, v_{21}, v_{31}, v_{32} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \emptyset \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ \emptyset & & & & \lambda_3 \end{bmatrix} \Rightarrow A = VDV^{-1}$$

- THE CONDITION FOR DIAGONALIZABILITY ARE:
  - if $A \in \mathbb{R}^m \Rightarrow A = A^T$
  - if $A \in \mathbb{C}^m \Rightarrow A = A^T$ AND $A^T = (A^T)^*$ // A is HERMITIAN

- $\forall i$ if ALG. MULT. = GEOM. MULT. you can find a basis made by EIGENVECTORS and you can find even an ORTHONORMAL ONE $\{u_1, u_2, \dots, u_m\} \mid (u_i, u_s) = \begin{cases} 0 & i \neq s \\ 1 & i = s \end{cases}$

INTRODUCTION

SIMPLE EXPERIMENT



OBSERVATIONS

(1) we observe only a finite set of values

(2) repeating the same measurement, observations are consistent

(3) what is maintained doing the same experiment are probabilities to observe a given result

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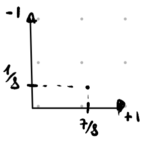
- The description of the state of a particle has to be probabilistic



$$e \rightarrow \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} = \tfrac{7}{8}\underset{\text{up}}{|u\rangle} + \tfrac{1}{8}\underset{\text{down}}{|d\rangle} = \tfrac{7}{8}\begin{bmatrix}1\\0\end{bmatrix} + \tfrac{1}{8}\begin{bmatrix}0\\1\end{bmatrix}$$

electron state

- BRA-KET notation

  INNER PRODUCT : $\langle u||v\rangle = \langle u|v\rangle$

  ↳ $\langle u|v_1+v_2\rangle = \langle u|v_1\rangle + \langle u|v_2\rangle$

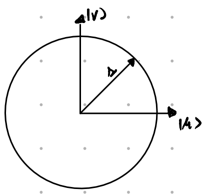
  ↳ $\langle u|v\rangle = \langle v|u\rangle^*$

STATE SPACE REPRESENTATION

- The state of the electron is represented by a vector:

$$|\varphi\rangle = \boxed{\alpha_u|u\rangle + \beta_d|d\rangle} \quad | \alpha_u, \beta_d \in \mathbb{C} \qquad \Big| \begin{array}{l} \|\alpha_u\|^2 = \text{Prob. to observe } |u\rangle, \\ \|\beta_d\|^2 = \text{Prob. to observe } |d\rangle \end{array}$$

- Because the sum of probs must be one, and we are in a vector space of complex number we can represent it on:



- we must assume $|u\rangle$ and $|d\rangle$ ORTHOGONAL $\langle d|u\rangle = 0 = \langle u|d\rangle$

EXTRACT PROBABILITY FRO $|\varphi\rangle$

- we can so extract the probability of measuring $|u\rangle$

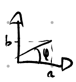
$$\boxed{\begin{array}{l} \langle u|\varphi\rangle = \alpha_u\cancel{\langle u|u\rangle} + \beta\cancel{\langle u|d\rangle} = \alpha_u \\ \langle u|\varphi\rangle\langle\varphi|u\rangle = \alpha_u^*\alpha_u = \|\alpha_u\|^2 = \text{Prob of measuring } |u\rangle \end{array}}$$

- IS $|\varphi\rangle$ ENOUGH TO DESCRIBE THE SYSTEM?

- By observing σ_z we have 2 possibility $|u\rangle$ and $|d\rangle$ // UP/DOWN
- By " σ_x " " $|l\rangle$ and $|r\rangle$ // LEFT/RIGTH

- given $|\varphi\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ and $\beta_l|l\rangle + \beta_r|r\rangle$:

  CAN WE WRITE $|l\rangle$ AND $|r\rangle$ AS FUNCTION OF u?



$z = a + ib = \|z\| e^{i\angle z}$

$= e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$|l\rangle = \beta_u|u\rangle + \beta_d|d\rangle \quad // \quad \|\beta_u\|^2 = \|\beta_d\|^2 = 1/2$$

$$|l\rangle = \tfrac{1}{\sqrt{2}} e^{i\angle\beta_u}|u\rangle + \tfrac{1}{\sqrt{2}} e^{i\angle\beta_d}|d\rangle = \boxed{\tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}|d\rangle}$$

ANGLE in $\mathbb{C}$

$$|r\rangle = \boxed{\tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}} e^{i\varphi}|d\rangle}$$

// from $|l\rangle$ we removed PHASE so we must have it on $|r\rangle$

- Obviusly $|r\rangle$ and $|l\rangle$ must be ORTHOGONAL

$$\langle r|l\rangle = 0 \Rightarrow \langle \tfrac{1}{\sqrt{2}}u + \tfrac{1}{\sqrt{2}}d \,|\, \tfrac{1}{\sqrt{2}}u + \tfrac{1}{\sqrt{2}}e^{i\varphi}d\rangle = 0$$

$$\langle u + d \,|\, u + e^{i\varphi}d\rangle = 0$$

$$\langle u | u + e^{j\varphi} d \rangle + \langle d | u + e^{j\varphi} | d \rangle = 0$$

$$\langle u|u\rangle + e^{j\varphi}\langle u|d\rangle + \langle d|u\rangle + e^{j\varphi}\langle d|d\rangle = 0$$

$$1 + 0 + 0 + e^{j\varphi} = 0$$

$$1 + e^{j\pi} = 0$$

• In conclusion we have

$$\boxed{\begin{aligned} |\rho\rangle &= 1/\sqrt{2}\,|u\rangle + 1/\sqrt{2}\,|d\rangle \\ |\pi\rangle &= 1/\sqrt{2}\,|u\rangle + 1/\sqrt{2}\,e^{j\pi}\,|d\rangle = \\ &= 1/\sqrt{2}\,|u\rangle - 1/\sqrt{2}\,|d\rangle \end{aligned}}$$

• We can do the same for σ_y

$$|i\rangle = \gamma_u\,|u\rangle + \gamma_d\,|d\rangle = 1/\sqrt{2}\,|u\rangle + 1/\sqrt{2}\,e^{j\,\angle\gamma_d}\,|o\rangle$$

$$|o\rangle = \delta_u|u\rangle + \delta_d\,|d\rangle = 1/\sqrt{2}\,|u\rangle + 1/\sqrt{2}\,e^{j\,\angle\delta_d}\,|o\rangle$$

$\Rightarrow \langle i | o \rangle = 0$ // ORTHOGONAL

$$\langle u + e^{j\gamma_d} d \mid u + e^{j\delta_d} d \rangle = 0$$

$$1 + e^{j\delta_d} e^{-j\gamma_d} = 1 + e^{j(\delta_d - \gamma_d)} = 0$$

$$\Rightarrow \delta_d = \gamma_d + \pi$$

• TAKING OUT A VARIABLE FROM INNER PRODUCT

$$\langle u | \alpha v \rangle = \alpha \langle u | v \rangle$$

$$\langle \alpha u | v \rangle = (\langle v | \alpha u \rangle)^* = (\alpha \langle v | u \rangle)^* = \alpha^* \langle v | u \rangle^* = \alpha^* \langle u | v \rangle$$

EXPRESS $|i\rangle$ IN TERMS OF $|\rho\rangle / |\pi\rangle$

• So we have

$$|i\rangle = 1/\sqrt{2}\,|u\rangle + 1/\sqrt{2}\,e^{j\gamma_d}\,|d\rangle$$

$$\overset{|\pi\rangle}{|\rho\rangle}\ 1/\sqrt{2}\,|u\rangle \mp 1/\sqrt{2}\,|d\rangle$$

$\Rightarrow$ we know that $P(\text{to observ } |\overset{\pi}{\rho}\rangle) = \langle i | \overset{\pi}{\rho} \rangle \langle \overset{\pi}{\rho} | i \rangle = 1/2$

$$\Rightarrow \langle i | \overset{\pi}{\rho} \rangle = 1/2 \cdot 1 \mp 1/2\, e^{-j\gamma_d} \cdot 1$$

$$\langle \overset{\pi}{\rho} | i \rangle = 1/2 \mp 1/2\, e^{j\gamma_d}$$

$$\Rightarrow \cancel{1/2}\,(1 \mp e^{-j\gamma})\ 1/2\,(1 \mp e^{j\gamma_d}) = \cancel{1/2}$$

$$(1 + 1 \mp 2\cos\angle\gamma_d)\,1/2 = 1$$

$\Rightarrow$ we know $-\cos\angle\gamma_d = 0 \ \wedge\ \sin\angle\gamma_d = 1 \Rightarrow \angle\gamma_d = \mp\frac{\pi}{2}$

$$\boxed{\begin{aligned} \gamma_d &= \frac{\pi}{2} \\ \delta_d &= -\frac{\pi}{2} \end{aligned}}$$

• In conclusion:

$$\boxed{\begin{aligned} &\bullet\ |i\rangle = 1/\sqrt{2}\,|u\rangle + 1/\sqrt{2}\ j\,|d\rangle \\ &\bullet\ |o\rangle = 1/\sqrt{2}\,|u\rangle - 1/\sqrt{2}\ j\,|d\rangle \end{aligned}}$$

- OUR ELECTRON STATE is totally characterized by $d_u|u\rangle + d_d|d\rangle$

QUBIT $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

OBSERVABLE AND MEASURMENT

QUANTUM MECHANICS PRINCIPLE

- OBSERVABLE: something measurable
- After a measurement the system is no more obsevable

HAVE INTEGER EIGENVALUES

1. All observables can be expressed by HERMITIAN (LINEAR) OPERATORS $L \in \mathbb{C}^{m \times m} \mid L = L^T$
2. The measurement results are the EIGENVALUES $\Rightarrow |0\rangle, |1\rangle$
3. *The probability to measure λ_i when in state $|\psi\rangle$ is $\sum_{v \in V(\lambda_i)} \langle\psi|v_i\rangle\langle v_i|\psi\rangle$

TAKING ORTHONORMAL BASIS MADE BY EIGENVECTORS

$V = \bigcup_{\lambda_i} V(\lambda_i)$

UNION

so Prob to observe λ_i in state $|\psi\rangle$ is

$\sum_{v \in V(\lambda_i)} \langle\psi|v\rangle\langle v|\psi\rangle$

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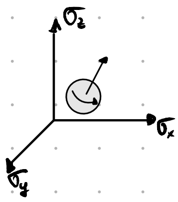
DERIVATION OF PAULI'S GATES

REPRESENTING HERMITIAN OF σ_z



$L_{\sigma_z} = [2^2]$, $\lambda(L_{\sigma_z}) = \{+1, -1\}$, BASIS $= \{|u\rangle, |v\rangle\}$

- we need 2 EIGENVECTORS $|V_1\rangle, |V_2\rangle$ (one for each $\lambda(L_{\sigma_z})$)
  1. $L_{\sigma_z}|V_1\rangle = +|V_1\rangle \Rightarrow |V_1\rangle = |u\rangle$
  2. $L_{\sigma_z}|V_{-1}\rangle = -1|V_{-1}\rangle \Rightarrow |V_{-1}\rangle = |d\rangle$

IS $|u\rangle$ AN EIGENVECTOR OF +1?

Proof

- Any vector can be represented by a linear combination of $|V_1\rangle$ and $|V_{-1}\rangle$
- even $|u\rangle$ can be:

$$|u\rangle = \alpha_1|V_1\rangle + \alpha_{-1}|V_{-1}\rangle$$

- so $L_{\sigma_z}|u\rangle = \alpha_1 L_{\sigma_z}|V_1\rangle + \alpha_{-1}L_{\sigma_z}|V_{-1}\rangle = \alpha_1|V_1\rangle - \alpha_{-1}|V_{-1}\rangle$
- we know that in state $|u\rangle$ we can observe only +1 so $\alpha_{-1} = 0$
- it implies that $|u\rangle$ is an EIGENVECTOR relative to the EIGENVALUE +1

- By the proof $|u\rangle$ and $|d\rangle$ are a basis because $\underbrace{L_{\sigma_z}|u\rangle}_{\text{EIGENVECT RELATIVE TO } +1} = |u\rangle$ $\begin{bmatrix}1\\0\end{bmatrix}$ AND $\underbrace{L_{\sigma_z}|d\rangle}_{\text{EIGENVECTOR RELATIVE TO } -1} = -|d\rangle$ $\begin{bmatrix}0\\1\end{bmatrix}$
- So we can fill L_{σ_z}:

$$L_{\sigma_z} = \begin{bmatrix} (L_{\sigma_z})_{11} & (L_{\sigma_z})_{12} \\ (L_{\sigma_z})_{21} & (L_{\sigma_z})_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

// REPRESENTATION OF σ_z w.r.t. $|u\rangle, |d\rangle$

Representing horizontal for σ_x (w.r.t $|u\rangle, |d\rangle$)

- $|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$
- $|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$

$|r\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

why $\begin{bmatrix}1\\1\end{bmatrix}$?

$$\Rightarrow \begin{array}{l} L_{\sigma_x}|r\rangle = +|r\rangle \\ L_{\sigma_x}|l\rangle = -|l\rangle \end{array} \Rightarrow \begin{array}{l} \frac{1}{\sqrt{2}} L_{\sigma_x}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\ \frac{1}{\sqrt{2}} L_{\sigma_x}\begin{bmatrix}1\\-1\end{bmatrix} = -\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \end{array} \Rightarrow \begin{array}{l} (L_{\sigma_x})_{11} + (L_{\sigma_x})_{12} = 1 \\ (L_{\sigma_x})_{21} + (L_{\sigma_x})_{22} = 1 \\ (L_{\sigma_x})_{11} - (L_{\sigma_x})_{12} = -1 \\ (L_{\sigma_x})_{21} - (L_{\sigma_x})_{22} = +1 \end{array} \Rightarrow L_{\sigma_x} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

L_{σ_y} w.r.t $|u\rangle, |d\rangle$

$L_{\sigma_x} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$

MULTI QUBITS SYSTEM

- Given 2 independent electrons $|\varphi_1\rangle = \overset{|0\rangle}{\alpha_1}|u_1\rangle + \overset{|1\rangle}{\beta_1}|d_1\rangle$, $|\varphi_2\rangle = \overset{|0\rangle}{\alpha_2}|u_2\rangle + \overset{|1\rangle}{\beta_2}|d_2\rangle$
- We can describe them jointly (4 BASIC STATE $|u_1 u_2\rangle, |u_1 d_2\rangle, |d_1 u_2\rangle, |d_1 d_2\rangle$)
  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

Product State $|\varphi_1\rangle\otimes|\varphi_2\rangle$

$$|\varphi_1\rangle\otimes|\varphi_2\rangle = |\varphi_1 \varphi_2\rangle = \alpha_1\alpha_2\ |u_1 u_2\rangle + \alpha_1\beta_2\ |u_1 d_2\rangle + \beta_1\alpha_2|d_1 u_2\rangle + \beta_1\beta_2|d_1 d_2\rangle$$

- So the probability of measuring:

$$P(|u_1 u_2\rangle) = |\alpha_1\alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2$$

$\Rightarrow$ The joint state of INDEPENDENT QUBIT is the PRODUCT STATE (we can just multiply AMPLITUDES)

$$|\varphi_1\rangle\otimes|\varphi_2\rangle = \sum_{y,x\in\{0,1\}} \alpha_{1x}\alpha_{2y}|xy\rangle$$

- What if 2 electron are entangled?

$$|\varphi_1 \varphi_2\rangle = \alpha_{uu}|u_1 u_2\rangle + \alpha_{ud}|u_1 d_2\rangle + \alpha_{du}|d_1 u_2\rangle + \alpha_{dd}|d_1 d_2\rangle$$

- We require that $|\alpha_{uu}|^2 + |\alpha_{ud}|^2 + |\alpha_{du}|^2 + |\alpha_{dd}|^2 = 1$

- Suppose to observe φ_2 in $|\varphi_1\varphi_2\rangle$, and get $|u_2\rangle$ we can then describe it by u_2's coefficient we must normalize because in the origin the probability 1 was given by summing all 4, now we must scale $|\alpha_{uu}|^2$ and $|\alpha_{du}|^2$ to have 1 as sum.

Cause $|\alpha_{uu}|^2 + |\alpha_{ud}|^2 + |\alpha_{du}|^2 + |\alpha_{dd}|^2 = 1$

$\cancel{|\varphi_1\rangle = \alpha_{uu}|u_1\rangle + \alpha_{du}|d_1\rangle}$ $\Rightarrow$ we have $|\alpha_{uu}|^2 + |\alpha_{du}|^2 < 1$ **so:**

$$\boxed{|\varphi_1\rangle = \frac{\alpha_{uu}}{\sqrt{|\alpha_{uu}|^2 + |\alpha_{du}|^2}}|u_1\rangle + \frac{\alpha_{du}}{\sqrt{|\alpha_{uu}|^2 + |\alpha_{du}|^2}}|d_1\rangle}$$

- 2 QUBITS are MAXIMALLY ENTANGLED if $|\varphi_1\varphi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, so they collapse to the same state.

• WE HAVE SEEN that given $|\varphi_1\rangle = \alpha_1|\mu_1\rangle + \beta_1|d_1\rangle$, $|\varphi_2\rangle = \alpha_2|\mu_2\rangle + \beta_2|\mu_2\rangle$

PRODUCT STATE

$$|\varphi_1\rangle \otimes |\varphi_2\rangle = |\varphi_1 \varphi_2\rangle = \alpha_1\alpha_2 |\mu_1\mu_2\rangle + \alpha_1\beta_2 |\mu_1 d_2\rangle + \beta_1\alpha_2 |d_1\mu_2\rangle + \beta_1\beta_2 |d_1 d_2\rangle$$

• if we measure $|\varphi_2\rangle$ to describe $|\varphi_1\rangle$ we should apply it's STARTING FORMULA

• proof it by applying the entangled formula:

$$|\varphi_1\rangle = \frac{\alpha_1\cancel{\alpha_2}}{\underbrace{\cancel{\sqrt{|\alpha_1\alpha_2|^2+|\beta_1\alpha_2|^2}}}_{=\alpha_2}}|\mu_1\rangle + \frac{\beta_1\cancel{\alpha_2}}{\underbrace{\cancel{\sqrt{|\alpha_1\alpha_2|^2+|\beta_1\alpha_2|^2}}}_{=\alpha_2}}|d_1\rangle = \alpha_1|\mu_1\rangle + \beta_1|d_1\rangle$$

$$|\varphi_1 \varphi_2\rangle = \frac{1}{\sqrt{2}}\left(|\mu_1\mu_2\rangle + |d_1 d_2\rangle\right)$$

• measure $|\varphi_2\rangle \to |\mu_2\rangle$ then $|\varphi_1\rangle = \frac{\cancel{1/\sqrt{2}}}{\cancel{\sqrt{(1/\sqrt{2})^2}}}|\mu_1\rangle = |\mu_1\rangle$

BELL STATE (EPR)

• MAXIMIZED ENTANGLED STATES

$$00 : |\psi\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad 10 : |\psi\rangle \to \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$01 : |\psi\rangle \to \frac{|00\rangle - |11\rangle}{\sqrt{2}} \qquad 11 : |\psi\rangle \to \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

QUANTUM GATES

• A gate transforms QBIT

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\qquad$ $|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$

$$|\psi\rangle \to \boxed{G} \to |\psi'\rangle$$

$|\psi'\rangle = G|\psi\rangle$

• G must preserve the NORM.

$$\| |\psi\rangle \|^2 = (|\psi\rangle)^\dagger |\psi\rangle = \langle\psi|\psi\rangle = 1$$

$$\| G|\psi\rangle \|^2 = (G|\psi\rangle)^\dagger G|\psi\rangle = (\langle\psi|G^\dagger) G|\psi\rangle =$$

$$= \langle\psi|G^\dagger G|\psi\rangle = 1$$

• To have $\langle\psi|G^\dagger G|\psi\rangle$ → we must delete $G^\dagger G$ so G must be **UNITARY** $(G^\dagger = G^{-1})$

SINGLE QUBIT GATES

- **QUANTUM WIRE** $|\psi\rangle$ —— $|\psi\rangle$
- **PAULI'S GATE**

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \underbrace{X}_{\text{NOT-GATE}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}$$

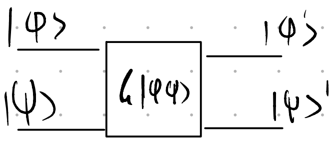
- **HADAMARD GATE**

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Usefull because from a unique state to a superposition
- $H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

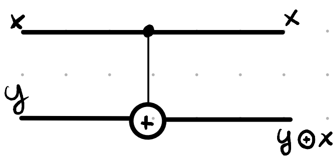
- $H|0\rangle = |+\rangle$
- $H|1\rangle = |-\rangle$

MULTI QBIT GATES



$$G(|\psi,\varphi\rangle) = G(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle) =$$
$$= \alpha_{00}\, G(|00\rangle) + \alpha_{01}\, G(|01\rangle) + \alpha_{10}\, G(|10\rangle) + \alpha_{11}\, G(|11\rangle)$$

CNOT

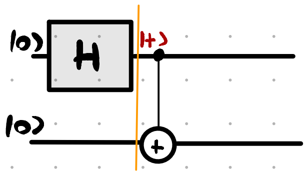


- if x is $|1\rangle$ then the not is done

- Combined with HADAMARD can generate BELL PAIRS

$|0\rangle \otimes |+\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =$

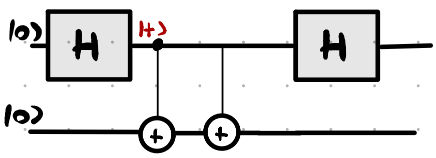
$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ // COMBINED STATE AFTER H



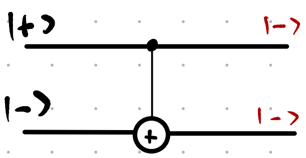
$$CNOT(|0\rangle \otimes |+\rangle) = CNOT\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right) = \frac{1}{\sqrt{2}}\left(\underbrace{CNOT(|00\rangle)}_{|00\rangle} + \underbrace{CNOT(|10\rangle)}_{|11\rangle}\right) =$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- To invert the combination wit H:

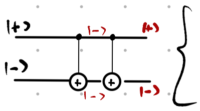


- Anothe example:



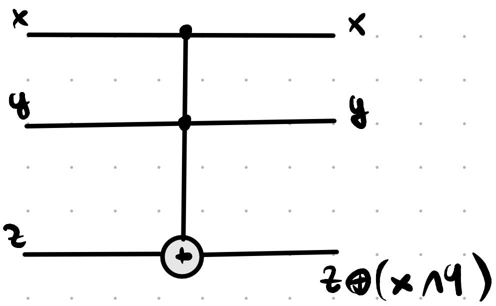
- $|+\rangle \otimes |-\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$
- $G(|+\rangle \otimes |-\rangle) = \frac{1}{2}(|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)$

obv



→ So if entangled the second QUBIT can modify the other one

TOFFOLI GATE



- With this gate we can generate every classical binary gate
  - ↳ NAND | $z = 1 \Rightarrow \neg(x \wedge y)$
  - ↳ FANOUT (out x) | $z = 0,\ y = 1$

BQP (BOUNDED ERROR QUANTUM POLYNOMIAL)