MALTA 2021–2022: Assignment

Provide your solution before the start of next lecture on October 12th. If you are unable to attend next lecture, then send your solution by email to giovanni.neglia@inria.fr.

Motivate your answers.

Ex. 1 — Show that a finite class has finite VC-dimension.

Ex. 2 — Prove that in the definition of agnostic PAC learnability, when the loss is the 0-1 loss function and the learning algorithm is required to output a hypothesis in H ($A(S) \in H$), it is possible to replace the probabilistic condition

$$\forall \epsilon, \delta \in (0,1),\ \exists\, m_H(\epsilon,\delta) \in \mathbb{N},\ \text{such that } \forall m \geq m_H(\epsilon,\delta),$$
$$L_D(A(S)) - \min_{h \in H} L_D(h) \leq \epsilon,\ \text{with prob. larger than } 1-\delta.$$

with the following condition on the expected value

$$\lim_{m\to\infty} \mathop{\mathbb{E}}_{S \in D^m} \left[L_D(A(S)) - \min_{h \in H} L_D(h) \right] = 0$$

Hints: We have already mapped during the lecture a probabilistic condition to a condition on expected values. The Markov inequality may help you.

Ex. 3 — If you show that H cannot shatter any set of size n, do you need to check if it can shatter a set of size $n' > n$? Why?

Ex. 4 — Consider the class H_k of binary functions over $\mathbb{R}$ which assume value 1 exactly on k points, i.e.,

$$H_k = \Big\{h : \mathbb{R} \to \{0,1\},\ \text{such that } \exists k \text{ distinct values } x_1, x_2, \ldots, x_k \in \mathbb{R},$$
$$\text{such that } h(x) = 1 \text{ if } x \in \{x_1, x_2, \ldots, x_k\} \text{ and } h(x) = 0 \text{ otherwise}\Big\}.$$

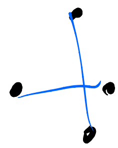
What is the VC-dimension of H?

Ex. 5 — Consider the class of squares in the plane

$$H = \{h_{a,b,r} : \mathbb{R}^2 \to \{0,1\},\ \text{for some } a, b \in \mathbb{R},\ \text{and } r \in \mathbb{R}^+\},$$

where

$$h_{a,b,r}(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 \in [a, a+r] \text{ and } x_2 \in [b, b+r], \\ 0, & \text{otherwise.} \end{cases}$$



1. What is the VC-dimension of H?
   Hint: The following geometric result may help: given any set A of 4 points in the plane, it is always possible to split A in two disjoint sets B and C ($B \cup C = A$, $B \cap C = \emptyset$) such that $\mathcal{CH}(B) \cap \mathcal{CH}(C) \neq \emptyset$, where $\mathcal{CH}(S)$ denotes the convex hull of the set S.
2. What if r is fixed to 1?

Ex. 6 — Consider a binary classification problem over $\mathbb{R}$ and the 0-1 loss function. Let $\mathcal{P}$ denote the class of all polynomials over $\mathbb{R}$. We consider the following hypothesis class:

$$H = \{h(x) = g(p(x)),\ \text{for some } p \in \mathcal{P}\},$$

where $g : \mathbb{R} \to \{0,1\}$ and $g(x) = 1$ if and only if $x > 0$.
Is the class H PAC learnable?

(EX.1) H s.t. $|H|$ IS FINITE $\Rightarrow$ H IS PAC LEARNABLE $\Rightarrow$ VCdim(H < + ∞)

(EX.2) Give, from APAC definition:

$$L_D(A(S)) - \min_{h \in H} L_D(h) \leq \varepsilon \text{ w.p.} \geq 1-\delta$$

for the ε-REPRESENTATIVENESS property:

$$|L_S(h) - L_D(h)| \leq \varepsilon \quad , \forall h \in H$$

we can so fix h^* s.t. we have $\min_{h \in H} L_D(h)$

we can write it as:

$$\left| \frac{1}{m} \sum_{i=1}^{m} \ell(h^*, (x,y)) - \mathbb{E}_{(x,y)\sim D}[\ell(h^*, (x,y))] \right| \leq \varepsilon$$

because we have:

- an aleatory variable $\ell(h^*, (x,y))$
- their expected value $L_D(h^*) = \mathbb{E}_{(x,y)\sim D}[\ell(h^*, (x,y))]$
- the estimation of the medium value $L_S(h) = \frac{1}{m} \sum_{i=1}^{m} \ell(h^*, (x,y))$, $\forall m$
- and it's variance from the expectation ε

from the "WEAK LAW OF GREAT NUMBER" for m that goes to ∞

$$\lim_{m \to \infty} \mathbb{P}\left(\left| \frac{1}{m} \sum_{i=1}^{m} \ell(h^*, (x,y)) - \mathbb{E}_{(x,y)\sim D}[\ell(h^*, (x,y))] \right| \leq \varepsilon \right) = 1 \Rightarrow$$

$$\Rightarrow \lim_{m \to \infty} \mathbb{P}\left(|L_S(h^*) - L_D(h)| \leq \varepsilon \right) = 1$$

- So if with m that goes to ∞ S is always ε-Representative, we can expect that the difference beetween the emp

$$\lim_{m \to \infty} \mathbb{E}_{S \sim D^m}\left[L_D(A(S)) - \min_{h \in H} L_D(h) \right] = 0$$

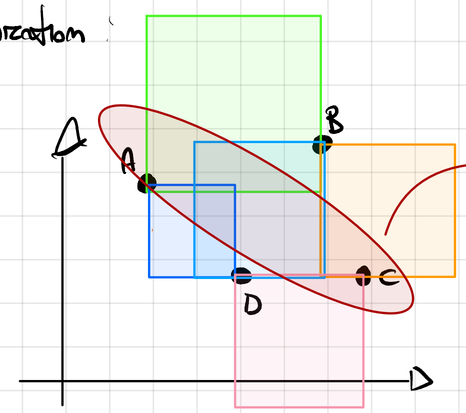
(EX.3) Given $C \subset X$ of size m, if H doesn't shatter C, H cannot shatter C' such that $C \subset C'$ of size $m' > m$ because in order to shatter a set, H must shatter all subset of it.

(EX.4) The VC dimension of H_k is k because for a set of $k+1$ elements you can't find a function $h \in H_k$ that set all elements to 1.

(EX.5) The VCdim$(H) = 3$ because with regular squares and 4 points H can't shatter all the 2 points configuration:

$\{A,B\} \to h$
$\{A,D\} \to h$
$\{B,C\} \to h$
$\{B,D\} \to h$
$\{C,D\} \to h$

$\{A,C\} \to \nexists h \in H$



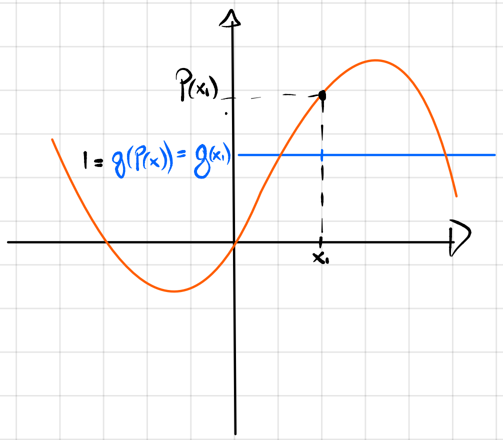
There isn't $h_{red} \in H$ s.t. it cover A and C without covering B and D

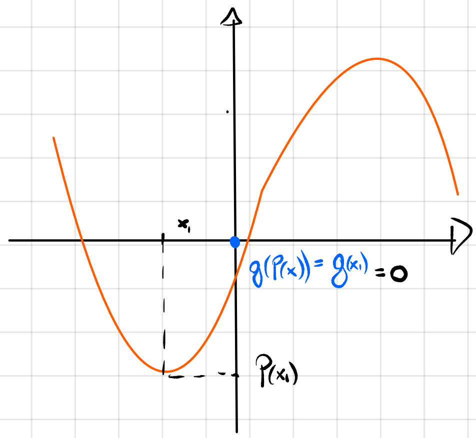
(EX.6) The class H is NOT PAC learnable.

- Let P the class of all polynomials over $\mathbb{R}$
- Let $g: \mathbb{R} \to \{0,1\} \mid g(x) = 1 \iff x > 0$
- Let $H = \{h(x) = g(p(x)),\ p \in P\}$
- For every set of points in $\mathbb{R}$ H can shatter them
- for a single point $x_1 \in \mathbb{R}$ we ha 2 cases:

if $x_1 > 0$ we can choose $P \in P \mid P(x) > 0$ so $h(x) = g(P(x)) = g(x) = 1$

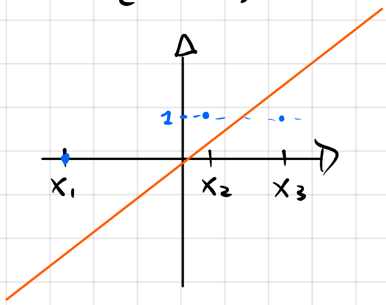
if $x_1 < 0$ we can choose $P \in P \mid P(x) < 0$ so $h(x) = g(P(x)) = g(x) = 1$



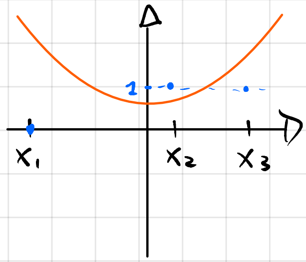


- given any set of point $X = \{x_1 \dots x_m\}$ we can easly find $P \in \mathcal{P}$ such that $\exists h$ matching any given $\{0,1\}^m$ combination associated with X.
- For example with 3 points

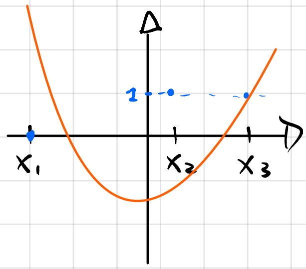
$\{1,1,1\}$



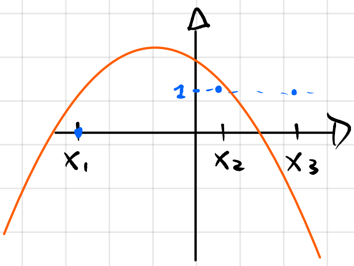
$\{0,1,1\}$



$\{0,0,1\}$

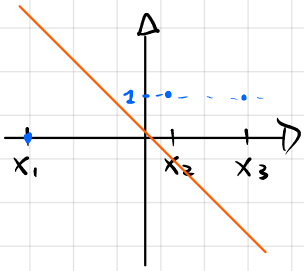


$\{0,1,0\}$



...

$\{0,0,0\}$



- In conclusion $VCdim(\mathcal{H}) = +\infty$