Graph Theory and Optimization
Computational Complexity (in brief)

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Outline

1. Time-complexity Hierarchy
2. 3-SAT
3. Hamiltonian path/cycle
4. Vertex-disjoint paths
5. Proper Coloring
6. Vertex-Cover
7. Approximation algorithms
8. Other NP-hard problems

Time-Complexity very brief introduction

Decision Problem

Input: Instance I
Problem: does I satisfiy Property $\mathscr{P}$?
Output: $solution \in \{Yes, No\}$

ex: Is graph G connected? Does G admit a s,d-flow of value $\geq k$?...

How to evaluate if:

1. a Problem $\mathscr{P}$ is “difficult" or “easy"?
2. an algorithm for solving $\mathscr{P}$ is efficient or not?

(classical) Time-complexity of an algorithm $\mathscr{A}$ for solving $\mathscr{P}$

Number of elementary operations of $\mathscr{A}$ as a function of the size n of the instance *running time in the worst case*

Definitions of elementary operations and size depend on:
Context, Data Structure, Units of measure...

Time-Complexity very brief introduction

Assume that Algorithm $\mathscr{A}$ has time-complexity $f(N)$:

in the worst case, $\mathscr{A}$ executes $f(N)$ operations on an instance of size N

For which size of instances is your problem feasible?

Assume 10^{10} operations (e.g., addition of 2 integers on 64 bits) per second

(it is more than current desktops)

Complexity $f(N)$	maximum size N
$\Theta(N)$	10^{11}
$\Theta(N \log N)$	10^{10}
$\Theta(N^2)$	$4 \cdot 10^5$
$\Theta(N^3)$	4600
$\Theta(N^4)$	560
$\Theta(2^N)$	36
$\Theta(N!)$	13

Table: Approximation of maximum size to obtain an answer in 10 seconds

Problems solvable by an algorithm with polynomial running time are “easy"

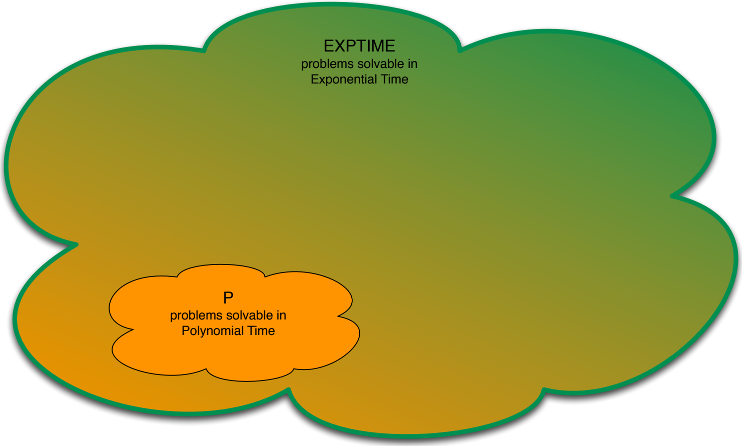
Some polynomial problems

Some examples you may know:

- Sorting n integers: $\Theta(n \log n)$ *heap sort, merge sort*
- Multiplying two $n \times n$ matrices: $O(n^{2.3728639})$ *[Le Gall, 2014]*
- Decide if m-edge graph is connected: $O(m)$ *BFS*
- Compute a shortest path in n-node m-edge graph: $O(m + n \log n)$ *[Dijkstra]*
- Compute min. spanning tree in m-edge graph: $O(m \log m)$ *[Kruskal...]*
- max. flow and min. cut in n-node m-edge graph and max capacity c_{max}: $O(m \cdot n \cdot c_{max})$ *[Ford-Fulkerson]*
- Maximum matching in n-node graph: $O(n^4)$ *[Edmonds 1965]*
  $O(mn^2)$ *[Micali, Vazirani, 1980]*

All these problems can be solved in time **polynomial** in the size of the input
$\Rightarrow$ "generally", they are said "easy"

Complexity Hierarchy (very informal and partial description)



EXPTIME: set of the (decision) problems solvable in exponential-time
P: set of the (decision) problems solvable in polynomial-time $P \subset EXPTIME$

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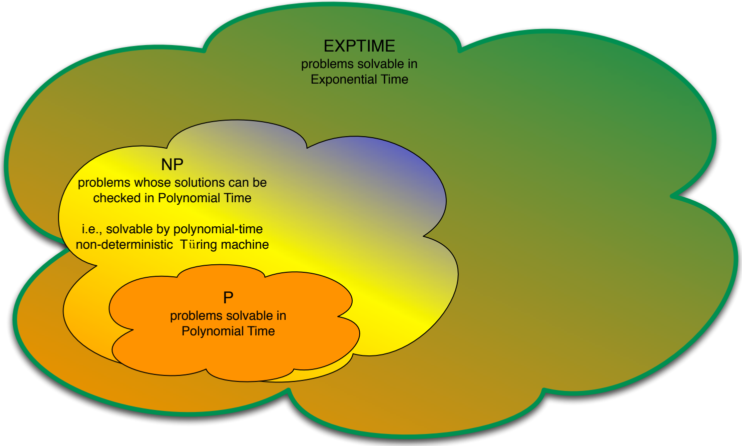

Non-deterministic Polynomial (NP): problems that can be solved in polynomial-time by a Non-deterministic Türing machine $P \subseteq NP$

Equivalently, a solution can be checked in polynomial-time

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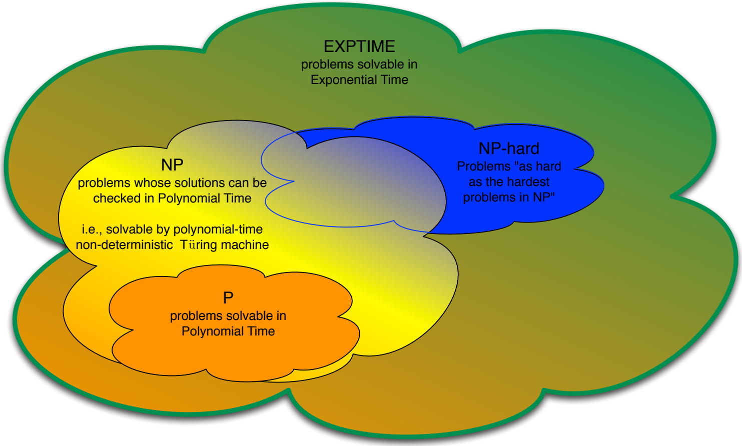

NP-hard: problems that are as “difficult" as the “hardest" problems in *NP*

Solving one of them in polynomial-time would prove that $P = NP$

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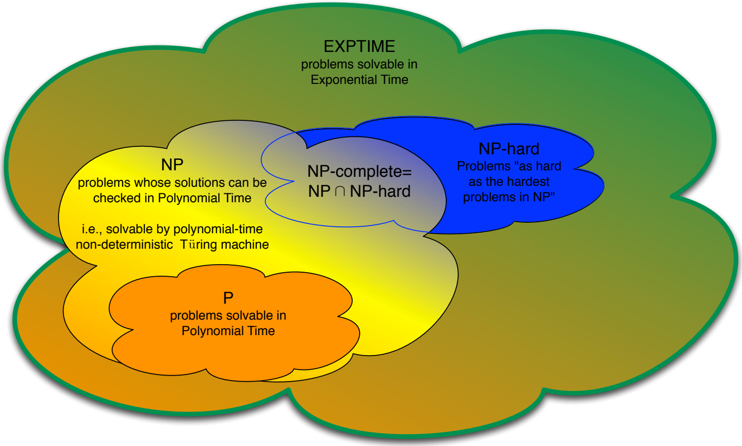

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NP-complete $=$ *NP*-hard $\cap$*NP*

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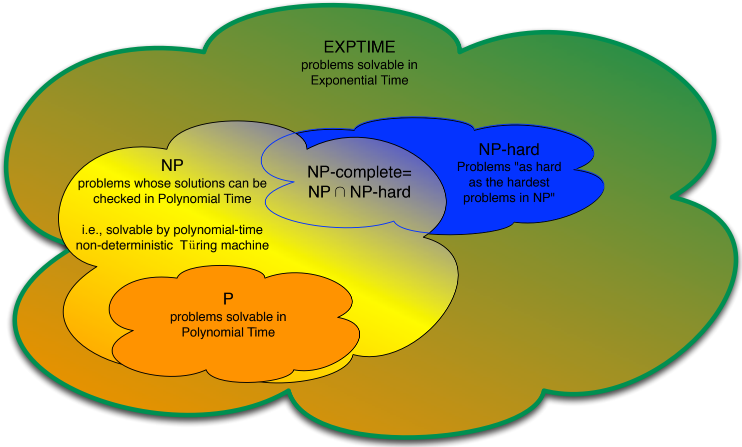

NP-hard problems: <u>we do not know</u> if they can be solved in polynomial-time

Roughly, best existing (known) algorithms (generally) enumerate all possible solutions and take a best one

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Famous NP-complete problems 3-SAT

SAT = Satisfiability

3-SAT is *NP*-complete Cook-Levin Theorem (1971)

3-SAT: Given 3-CNF formula $\Phi(v_1, \cdots, v_n)$ on n Boolean variables
$\exists$? a Boolean assignment $a : \{v_1, \cdots, v_n\} \rightarrow \{0,1\}$ that satisfies Φ?

3-*CNF*= Conjunctive Normal Form, i.e., conjunction of clauses, where a clause is a disjunction of 3 literals.

Ex: $\Phi(a,b,c,d,e) = (a \vee \bar{b} \vee c) \wedge (\bar{e} \vee b \vee \bar{c}) \wedge (\bar{a} \vee \bar{c} \vee e)$
here, the assignment $(a,b,c,d,e) = (1,1,0,0,1)$ satisfies Φ.

example of algorithm for 3-SAT

Try all the 2^n possible assignments

Remarks: 3-SAT is the first problem to be proved NP-hard.
It is often used to prove that other problems are NP-hard.

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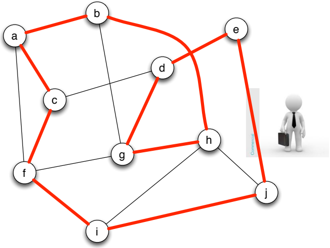
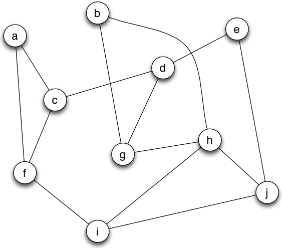
Hamiltonian Path: a spanning path P in a graph G
Hamiltonian Cycle: a spanning cycle C in G

Hamiltonian Path/Cycle is *NP*-complete [Garey, Johnson]

Hamiltonian path/cycle: Given a graph $G = (V,E)$ with n vertices,
$\exists$? an Hamiltonian path/cycle in G?





Application: Travelling Salesman Problem (TSP): want to visit all cities, minimizing the distance he has to cross

Exercice: show that the right graph has no Hamiltonian cycle.

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Longest path/cycle

Exercice: Let $G=(V,E)$ be a graph and $k\in\mathbb{N}$
Prove that deciding if G has a path/cycle of length $\geq k$ is NP-complete

Famous NP-c problems Hamiltonian Path/Cycle

Hamiltonian Cycle: cycle that passes through each vertex (exactly once)

Remark: Problems that “look similar" may be “very different"

Tour≈ “cycle" where vertices may be repeated, but not edges.
Eulerian Tour: Tour that passes through each edge (exactly once)

Euler (1736)

Exercice: Prove that deciding if G admits an Eulerian tour is in P

hint: prove that G admits an Eulerian tour $\Leftrightarrow$ *each vertex has even degree*

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Famous NP-c problems Disjoint paths (multi-flow)

Exercice: Let $G=(V,E)$ be a graph, $S,D \subseteq V$, $k \in \mathbb{N}$
Deciding if it exists k vertex-disjoint paths from S to D is in P

hint: use flow algorithm

Disjoint paths is *NP*-complete [Garey, Johnson]

disjoint paths: Given $G=(V,E)$, $\{s_1,\cdots,s_k\} \subseteq V$ and $\{d_1,\cdots,d_k\} \subseteq V$
$\exists?$ $(P_1,\cdots,P_k)$ pairwise vertex-disjoint paths s.t. P_i path from s_i to d_i

Remark: can be solved in time $f(k)poly(n)$, i.e., in P if k is fixed
[Robertson and Seymour, 1995]

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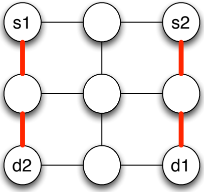
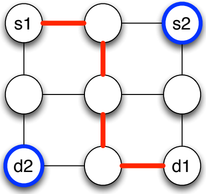


2 disjoint paths from {s1,s2} to {d1,d2}



no 2 disjoint paths from s1 to d1
and from s2 to d2

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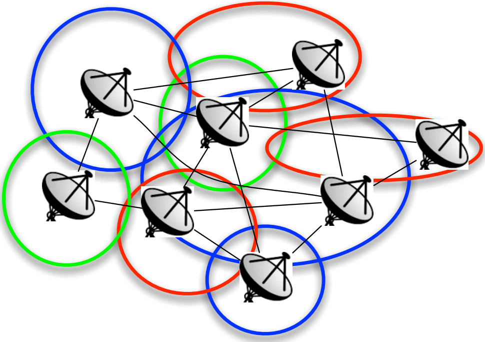
Famous NP-complete problems Coloring

Let $G = (V, E)$ be a graph

k-Proper coloring: $c : V \to \{1, \cdots, k\}$ s.t. $c(u) \neq c(v)$ for all $\{u, v\} \in E$.
color the vertices s ($\leq k$ colors) s.t. adjacent vertices receive $\neq$ colors



Vertices = antennas $\{u, v\} \in E$ iff transmissions of u and v overlap
Color = frequency of transmission

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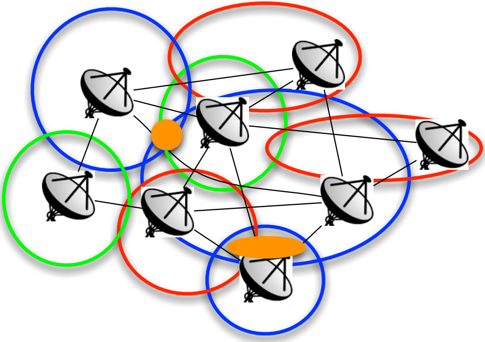

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with the coloring in the example: if you live in orange zone $\Rightarrow$ No WiFi!!

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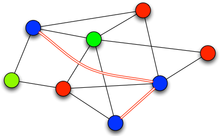
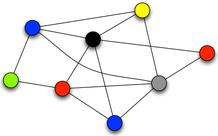
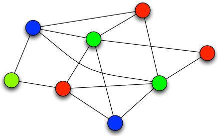


Unproper 3*-coloring (red edges)*



Proper 6*-coloring*



Proper 3*-coloring*

chromatic number $\chi(G)$: min. k such that G admits a k-Proper coloring.

Exercice: Show that $\chi(G) \leq 2$ if and only if G is bipartite.

Coloring is *NP*-complete [Garey, Johnson]

chromatic number: Given $G=(V,E)$ be a graph, $\chi(G) \leq 3$?
even if G is restricted to be a planar graph

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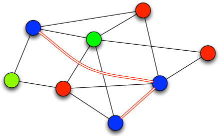

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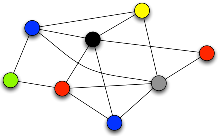

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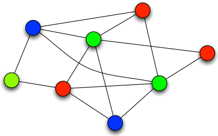

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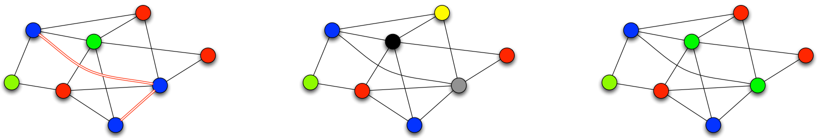

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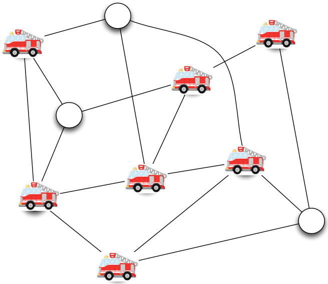
Famous NP-complete problems Vertex Cover

Let $G = (V,E)$ be a graph

Vertex Cover: set $K \subseteq V$ such that $\forall e \in E$, $e \cap K \neq \emptyset$

set of vertices that "touch" every edge



Application: each street must be protected by a fire station

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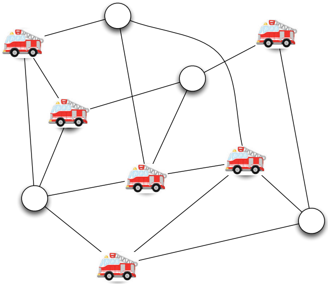

Problem: Min. number of fire stations to protect each street?

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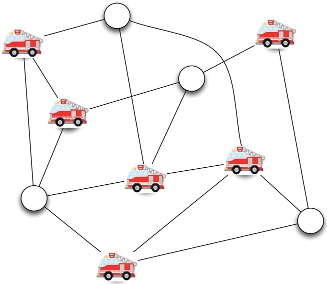

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Min Vertex Cover is *NP*-complete [Garey, Johnson]

Vertex Cover: Given $G=(V,E)$ be a graph, $k \in \mathbb{N}$,

$\exists$? $K \subseteq V$ a vertex cover of G such that $|K| \leq k$?

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How to actually handle NP-hard problems?

NP-hard: No Polynomial-time algorithms known!!!

worst case complexity vs. practice

exponential-time algorithms may be efficient on practical instances

Consider particular instances

Problem $\mathscr{P}$ may be NP-complete in a set $\mathscr{I}$ of instances

but polynomial-time solvable in $\mathscr{I}' \subset \mathscr{I}$

Ex: for any bipartite graph G,

$$\kappa(G) = min.\ vertex\ cover(G) = max.\ matching(G) = \mu(G)$$

c-Approximation algorithms $\mathscr{A}$: polynomial-time algorithm s.t.

for any instance I, $\mathscr{A}$ returns a solution with value

for minimization problem: $OPT(I) \leq value(\mathscr{A}) \leq c \cdot OPT(I)$

for maximization problem: $OPT(I)/c \leq value(\mathscr{A}) \leq OPT(I)$

Exercice: Give a 2-approximation algorithm for Vertex-Cover

hint: show that, for any graph G, $\mu(G) \leq \kappa(G) \leq 2\mu(G)$

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Some other famous NP-complete problems

Max. Independent set is NP-complete

Independent set: Given a graph $G = (V, E)$, $k \in \mathbb{N}$, $\exists$? stable set $S \subseteq V$ of size $\geq k$ in G?

Min. Feedback Vertex Set (FVS) is NP-complete

FVS: Given a digraph $D = (V, A)$, $k \in \mathbb{N}$, $\exists$? $F \subseteq V$ such that $D \setminus F$ is acyclic?

Min. Set Cover is NP-complete

Set Cover: set E, family of subsets $\mathscr{S} = \{\mathscr{S}_1, \cdots, \mathscr{S}_\ell\} \subseteq 2^E$, $k \in \mathbb{N}$
$\exists$? $Y \subseteq \mathscr{S}$, $\bigcup_{S \in Y} S = E$, $|\mathscr{S}| \leq k$?

Min. Hitting Set is NP-complete

Hitting Set: set E, family of subsets $\mathscr{S} = \{\mathscr{S}_1, \cdots, \mathscr{S}_\ell\} \subseteq 2^E$, $k \in \mathbb{N}$
$\exists$? $H \subseteq E$, $H \cap \mathscr{S}_i$ for any $i \leq \ell$, $|H| \leq k$?

Partition is *weakly* NP-complete

Partition: Given set $X = \{x_1, \cdots, x_n\}$ of integers ,
$\exists$? partition (A, B) of X such that $\sum_{x \in A} x = \sum_{x \in B} x$?

Summary: To be remembered

- *P*, *NP*, *NP*-hard, *NP*-complete
- 3-SAT, Hamiltonian path, Coloring, Vertex Cover, ...
- Approximation algorithm