- G CONNECTED if $|E| = |V| - 1$
- $T = (V, E)$ TREE ⟺ CONNECTED AN $|V| = |E| + 1$

- BFS

$O(|E|)$

IN: $G = (V, E)$, $r \in V$

OUT: $d(v) = d(r, v)$ $\forall v \in V$

INIT $d(r) = 0$, $d(v) = \infty$ $\forall v$

$H = \{r\}$, $S = \{\emptyset\}$

$T = \{V_T = \{r\}, E_T = \{\emptyset\}\}$

WHILE $(H \neq \emptyset)$ {
- $v = head(H)$
- FOR $(u \in N(v) \setminus H \cup S)$ {
  - $d(u) = d(v) + 1$
  - $V_T = V_T \cup \{u\}$
  - $E_T = E_T \cup \{u\}$
  - ADD_TAIL$(H, \{u\})$
- $H = H \setminus v$
- $S = S \cup v$

DIJKSTRA

$O(|E| + |V| \log |V|)$

IN: $G = (V, E)$ WEIGHTED, $r \in V$

OUT: $d(v) = d(r, v)$ $\forall v \in V$

INIT $d(r) = 0$ $d(v) = +\infty$ $\forall v \in V \setminus r$

$H = \{\emptyset\}$, $S = \{\emptyset\}$, $T = (V, E_T = \{\emptyset\})$

$Parent(v) = \emptyset$ $\forall v \in V$

WHILE $(S \neq V)$
- $v = v \mid d_v$ minimum $\forall v \in V$
- $E_T = E_T \cup \{Parent(v), v\}$
- $S = S \cup \{v\}$
- FOR $(u \in N(v) \setminus S)$
  - IF $(d(u) > d(v) + w_{vu})$
    - $d(u) = d(v) + w_{vu}$
    - $Parent(u) = v$

KRUSKAL

$O(|E| \log |E|)$

IN: $G = (V, E)$ positive weighted

OUT: MST

$T = (V, E_T = \{\emptyset\})$

let $(e_1, \ldots, e_m)$ be a non decr. order of $e_i \in E$ w.r.t. w_{e_i}

FOR $(i = 1 \ldots m)$
- IF ($E_T \cup e_i$ NOT CREATE CYCLES)
  - $E_T = E_T \cup e_i$

MAX MATCHING

- **MATCHING** $M \subseteq E \mid e \cap f = \emptyset \ \forall e, f \in M$
- LEMMA $|M| \leq \frac{|V|}{2}$
- M-ALTERNATING $P = (e_1 \dots e_m) \mid (e_i \in M \wedge e_{i+1} \notin M) \vee (e_i \notin M \wedge e_{i+1} \in M) \ \forall e_i, e_{i+1}$
- M-AUGMENTING P is M-ALTERNATING $\wedge\ e_1 \notin M \wedge e_2 \notin M$
- TH BERG M MAX $\Leftrightarrow \nexists$ M-AUGMENTING P
- ALGO
  Given M
  WHILE ($\exists P$ M-AUGMENTING)
  AUGMENT (M)

  $O(\overbrace{m^3}^{\text{EDMOND}} + \overbrace{m/2}^{\text{BY LEMMA}}) = O(m^4)$

- BIPARTITE GRAPH $G = (V_1 + V_2, E) \mid V_1, V_2$ STABLES

MIN VERTEX COVER

- VC $Q \subseteq V \mid i \in Q \vee j \in Q \ \forall (i,j) \in E$
- KONING IN BIP. GRAPH $|M^*_{MAX}| = |VC^*_{MIN}|$
- LEMMA $m(G) \leq vc(G) \leq 2m(G) \underbrace{\leq 2vc(G)}_{\text{PLUS}}$

ALGO
  $M = \{\emptyset\}$
  WHILE ($\exists e \in E \mid M \cup e$ is matching)
    ADD e to M
  return $V(M)$ // vertices touched by M

$O(E)$

$V(M) = 2|M| \leq 2VC \Rightarrow 2$ APPROX

APPROXIMATION ALGOS

MIN : SOL $\leq C \cdot OPT$, $C \geq 1$
MAX : SOL $\geq C \cdot OPT$, $C \leq 1$

FLOW

$\max \delta(x)$

$\sum_{u \in N^-(v)} x_{uv} - \sum_{w \in N^+(v)} x_{vw} \quad \forall v \in V$

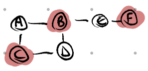
$x_{uv} \leq u_{uv} \quad , \forall (u,v) \in E$

- MAX-FLOW / MIN-CUT
  $\Rightarrow \delta(x^*) = \delta(R^*)$
- FF ALGO $O(\delta(x^*) \cdot |E|)$

SET-COVER

- Given $U = \{e_1, \dots, e_m\}$, $S = \{S_1, \dots, S_k\} \mid S_i \subseteq U$, find $S^* \subseteq S$ that cover all $e_i \in U$
- f is the max #occurrences of e_i
- RELAXATION $U = \{e, f, g\}$, $S_1 = \{e, f\}$, $S_2 = \{f, g\}$, $S_3 = \{g, e\}$
  - LP SOL $= \frac{3}{2}$

VERTEX COVER LP



$U = \{AB, AC, \dots, EF\}$

$S_A = \{AB, AC\}$
$S_B = \{AB, BD, BE\}$
$\vdots$
$S_F = \{EF\}$

$$\min \sum_{i=1}^{k} x_i \cdot c_i$$

$$\text{s.t.} \quad \sum_{i \mid e \in S_i} x_i \geq 1 \qquad \forall e \in U$$

$$x_i \geq 0 \qquad \forall i = 1, \dots, k$$

DUALITY

PRIMAL

$$\max \quad 2x_1 + 3x_2 + 2x_3 + 3x_4$$
$$\text{s.t.} \quad 2x_1 + x_2 + 3x_3 + 2x_4 \leq 8$$
$$3x_1 + 2x_2 + 2x_3 + x_4 \leq 7$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$\Rightarrow$

DUAL

$$\min \quad 8y_1 + 7y$$
$$\text{s.t.} \quad 2y_1 + 3y_2 \geq 2$$
$$y_1 + 2y_2 \geq 3$$
$$3y_1 + 2y_2 \geq 2$$
$$2y_1 + y_2 \geq 3$$
$$y_1, y_2 \geq 0$$

OPTIMALITY CHECK

Given x^*:

(1) $\sum_{j=1}^{m} a_{ij} \cdot x_j^* < b$ implies $y_i^* = 0$ $\forall i = 1 \dots m$

(2) solve system of constraints of D associated with $x_j^* > 0 \Rightarrow y^*$

(3) If y^* feasible for $D \Rightarrow x^*$ optimal

COMPLEMENTARY SLACKNESS

SIMPLEX

$$\max \quad 2x_1+3x_2+2x_3+3x_4$$
$$\text{s.t.} \quad 2x_1+x_2+3x_3+2x_4 \le 8$$
$$3x_1+2x_2+2x_3+x_4 \le 7$$
$$x_1,x_2,x_3,x_4 \ge 0$$

① SLACK VARS

$$x_5 = 8-2x_1-x_2-3x_3-2x_4$$
$$x_6 = 7-3x_1-2x_2-2x_3-x_4$$

② BUILD DICTIONARY

$$x_5 = 8-2x_1-x_2-3x_3-2x_4$$
$$x_6 = 7-3x_1-2x_2-2x_3-x_4$$
$$z = \quad 2x_1+\underbrace{3x_2}+2x_3+3x_4$$

HIGHEST COEFFICIENT
$\Rightarrow x_2$ ENTER THE BASIS

③ WHO EXIT?

$x_5 \ge 0 \Rightarrow x_2 \le 8$

$x_6 \ge 0 \Rightarrow x_2 \le 7/2$ → STRONGEST CONSTRAINT

④ SWAP x_6 W/ x_2

$$x_5 = 8-2x_1-\left(\tfrac{7}{2}-\tfrac{3}{2}x_1-x_3-\tfrac{1}{2}x_4-\tfrac{1}{2}x_6\right)-3x_3-2x_4$$
$$x_2 = \tfrac{7}{2}-\tfrac{3}{2}x_1 \quad -\tfrac{1}{2}x_6-x_3-\tfrac{1}{2}x_4$$
$$z = \quad 2x_1+3\left(\tfrac{7}{2}-\tfrac{3}{2}x_1-x_3-\tfrac{1}{2}x_4-\tfrac{1}{2}x_6\right)+2x_3+3x_4$$

$\Rightarrow$

$$x_5 = \tfrac{9}{2}-\tfrac{1}{2}x_1-2x_3-\tfrac{3}{2}x_4+\tfrac{1}{2}x_6$$
$$x_2 = \tfrac{7}{2}-\tfrac{3}{2}x_1-x_3-\tfrac{1}{2}x_4-\tfrac{1}{2}x_6$$
$$z = \tfrac{21}{2}-\tfrac{5}{2}x_1-x_3+\tfrac{3}{2}x_4-\tfrac{3}{2}x_6$$

LP-FORMULATION

CLIQUE

$$\max \quad \sum_{i=1}^{m} x_i$$
$$\text{s.t.} \quad x_i+x_j \le 1 \quad \forall (i,j)\notin E$$
$$x_i \in \{0,1\} \quad \forall i=1,\dots,m$$

VC W/ SET-COVER

$$\min \quad \sum_{i=1}^{k} x_i \cdot c_i$$
$$\text{s.t.} \quad \sum_{i\,|\,e\in S_i} x_i \ge 1 \quad \forall e\in U$$
$$x_i \ge 0 \quad \forall i=1,\dots,k$$

FLOW

$$\max \quad \sum_{(u,v)\in E} f_{uv}$$
$$\sum_{u\in N^-(v)} f_{uv} = \sum_{w\in N^+(v)} f_{vw} \quad \forall v\in V$$
$$f_{uv} < u_{uv} \quad , \forall (u,v)\in E$$

MAX-MATCHING

$$\max \quad \sum_{(i,j)\in E} x_{ij}$$
$$\text{s.t.} \quad \sum_{v\in N(u)} x_{uv} \le 1 \quad \forall v\in V$$
$$x_{ij}\in\{0,1\} \quad \forall (i,j)\in E$$

SHORTEST-PATH

$$\min \quad \sum_{(i,j)\in A} x_{ij}\, c_{ij}$$
$$\text{s.t.} \quad \sum_{w\in N^+(v)} x_{vw} - \sum_{u\in N^-(v)} x_{uv} = \begin{cases} 1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{else} \end{cases} \quad ,\forall v\in V$$
$$\sum_{u\in N^+(v)} x_{vu} \le 1 \quad ,\forall v\in V$$