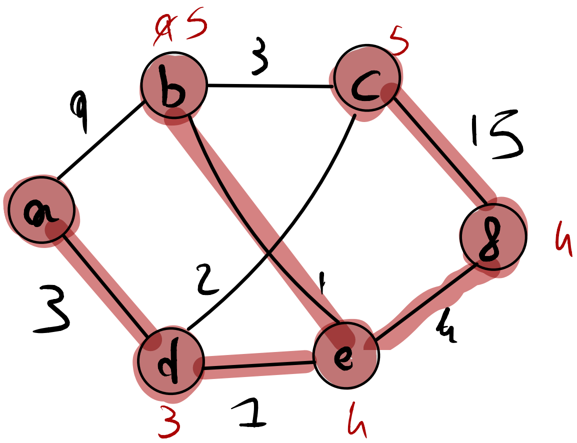
MIDTERM

Exercise 4 (Dijkstra. 4 points, 15 minutes) Consider the graph H depicted in Figure 1.

- Give the definition of a shortest-path tree rooted in a.
- Apply the Dijkstra algorithm on H to compute a shortest-path tree rooted in a and the distance between any vertex and the vertex a.

  You must explain the execution of the algorithm (you may write the table as seen during the lecture). In particular, indicate the order in which vertices are considered during the execution of the algorithm.
- Give the obtained shortest-path tree rooted in a.

- A shortest-path Tree T rooted in a is a Tree s.t. the path in T from a to a generic node i is the shortest $\forall i \in V$



DIJKSTRA

INPUT: $G=(V,E)$, $r \in V$

OUTPUT: $d(v)=d(r,v)\ \forall v \in V$

INIT: $d(r)=0$, $d(v)=\infty\ \forall v \in V \setminus \{r\}$,

S set of seen node, $T=(V_T=\{a\}, E_T=\{\emptyset\})$

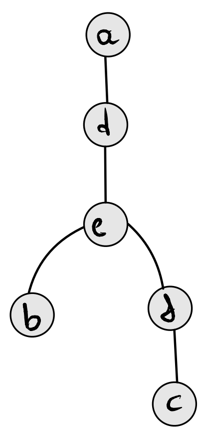
$Parent(v)=\emptyset\ \forall v \in V$, $w_{ij}\ \forall (i,j) \in E$

ALGO: WHILE $(|S| \neq |V|)$

- $v = v \in V \mid d(v) = \min_{u \in V \setminus S} d(u)$
- $V_T = V_T \cup \{v\}$
- $E_T = E_T \cup (v, Parent(v))$
- $S = S \cup \{v\}$
- FOR $(u \in N(v) \setminus S)$
  - if $(d(v,u) < d(u))$
    - $d(u) = d(v) + w_{uv}$
    - $Parent(u) = v$

CHOSEN NODE \ NODES	a	b	c	d	e	f
INIT	0	∞	∞	∞	∞	∞
a	0	9	∞	3	∞	∞
d	0	9	5	3	4	∞
e	0	5	5	3	4	4
f	0	5	5	3	4	4
b	0	5	5	3	4	4
c	0	5	5	3	4	4

SHORTEST PATH TREE



Exercise 5 (Flow. 5 points, 20 minutes) Consider the elementary network flow N depicted in Figure 2 (left) and the initial flow f from s to t in Figure 2 (right).

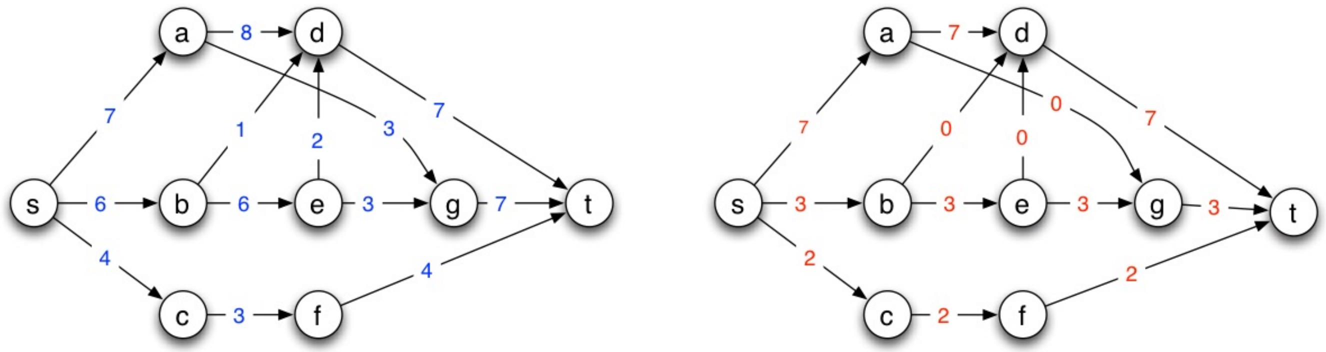


Figure 2: (left) Elementary network flow with arcs' capacity. (right) An initial flow f from s to t: a number close to an arc indicates the amount of flow along it.

(1) • What must be checked to show that f is a flow? What is the value of the initial flow f?

(2) • Apply the Ford-Fulkerson Algorithm to N starting from the flow f. All steps of the execution of the algorithm must be detailed.

For each iteration, draw the auxiliary graph, give the chosen path and the amount of flow that you will push, and draw the network with the new flow.

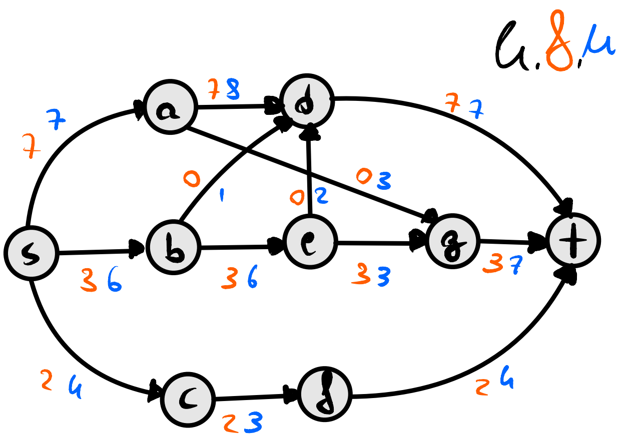
(3) • What is the final value of the flow?

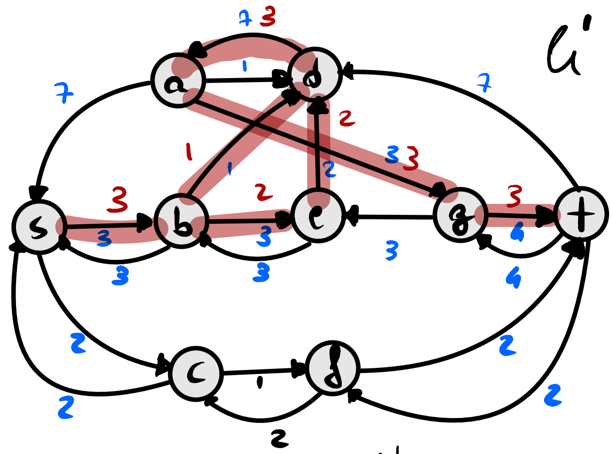
(1) To check if f is a flow we must check if it respect

- CAPACITY constraint : $f_{ij} \le u_{ij} \quad \forall (i,j) \in E$
- BALANCE constraint : $\sum_{\forall i \in N^-(v)} f_{iv} = \sum_{\forall j \in N^+(v)} f_{vj} \quad \forall v \in V$

The initial flow f has value $7+3+2 = 12$

(2)

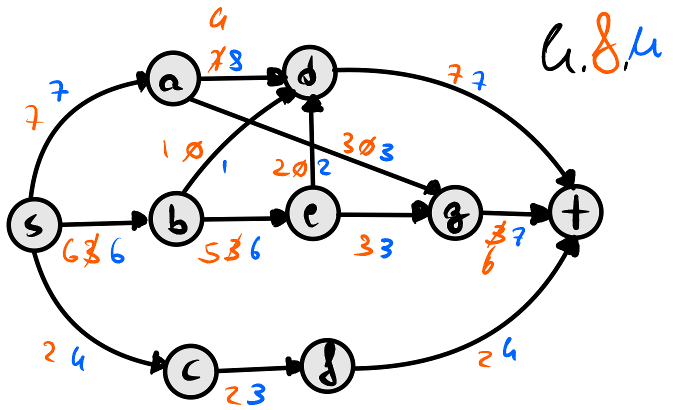


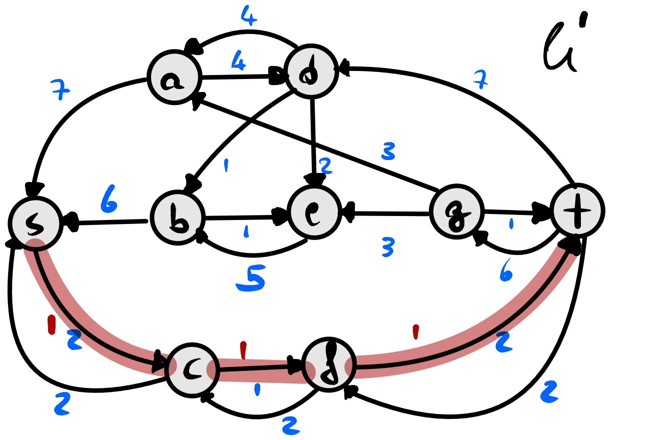


//we compute 2 path in G'.

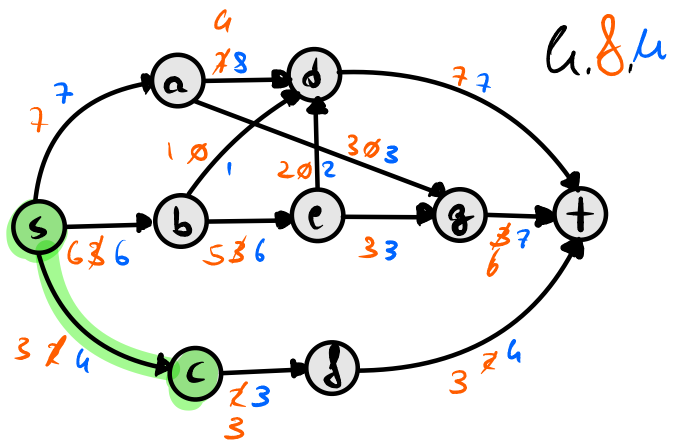
$P_1 = \{s, b, d, a, g, t\}$ where we send 1 flow unit

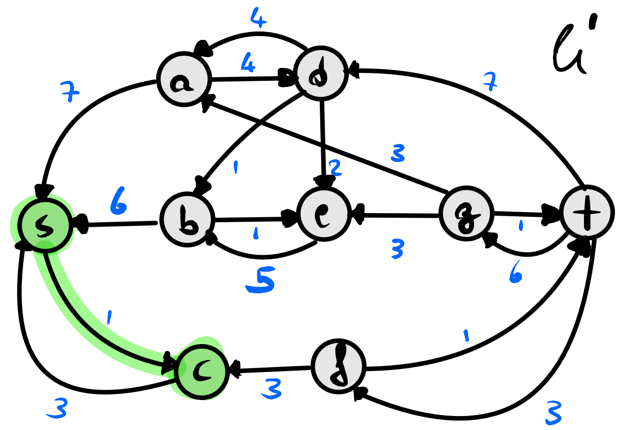
$P_2 = \{s, b, e, a, g, t\}$ " " 2 "





//we compute 1 path
$P_1 = \{s, c, f, t\}$ where we send 1 flow unit





- $\nexists$ an (s,t)-PATH in G' $\Rightarrow$ The flow is maximum and it's value corresponds to the value of the cut, which is minimum:
  - $R = \{v \mid v \text{ reachable from } s \text{ in } G'\} = \{s, c\}$
  - $\bar{R} = V \setminus R$
  - $X = \{(i,j) \in E \mid i \in R \land j \in \bar{R}\} = \{(s,a), (s,b), (c,f)\}$ //ARCS IN MIN-CUT
  - $\delta(X) = \sum_{(i,j) \in X} u_{ij} = 7+6+3 = \max f$

(3) The maximum flow $\max f = \delta(x) = 16$