MACHINE LEARNING

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EVALUATION

(1) EXAM (40%)
(2) HOMEWORK (30%)
(3) QUESTION AT EVERY LECTURE (30%)

2020/2021

16/09

INTRODUCTION

MACHINE LEARNING

- way of extract rules from data

DATA → [ML] → RULES/PROGRAM

ISSUES

- To large set of rules (REM Pidgeon)
- To small set of rules (REM Rat)

& TOO MUCH reduces learning flexibility

INDUCTIVE PRIOR

- A PRIORI KNOWLEDGE to prevent useless conclusion

INDUCTIVE INFERENCE

- Ability to proceed from general examples to a broadere generalization

KINDS OF LEARNING

SUPERVISED / UNSUPERVISED

- In supervised learning output label are given; Unsupervised instead finds unexpected correlation. (SPAM MAIL vs ANOMALITY DETECTION)

REINFORCHENT

- Labels are given but after a while

COOPERATIVE / INDIFFERENT / ADVERSARIAL TEACHER

- Change the way the data is given from a teacher

BATCH / ONLINE

- Data can be given all together or PASSO PASSO

! ⇒

- we'll se SUPERVISE STATISTICAL BATCH LEARNING WITH PASSIVE TEACHER

STATISTIC & ML

1. Statistic wants to check an HYPOTHESIS (■ smoke affects heart?)
   ML wants to find the HYPOTHESIS (■ what affects heart?)
2. Statistic starts from the Prob. Distr. of data.
   ↳ ASYNTOTICS: if you look at enough data you find a GAUSSIAN
   ML doesn't know the distribution
   ↳ FINITE SAMPLES: you get the distribution you have

STATISTICAL LEARNING FRAMEWORK

- X input space
- Y output space

} (vector of features)

- S dataset of size m (# SAMPLES)

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f and D are unknown to the learner

- f correct labeling function ; D distribution over X
- $A : S \subset (X \times Y)^m \longrightarrow \{\text{functions } X \to Y\}$
  $\Rightarrow A(S) = h$

h is the output of an ML Algo. (is a PREDICTION RULE)

- $h : X \to Y$ called HYPOTESIS or PREDICTOR or CLASSIFIER it can be seen as $A(S)$, so the output of the ML ALGO. A given the DS S

INDICATOR FUNCTION: 0 if prediction is ok it is 0 else is 1

EXPECTED LOSS

- $L_D(h) = \mathbb{E}_D\left(\mathbb{1}_{h(x) \neq y}\right) \in [0,1]$

EMPIRICAL LOSS

- $L_S(h) = \dfrac{\sum_{i=1}^{m} \mathbb{1}_{h(x_i) \neq y_i}}{m}$

$\Rightarrow$ for large number theory, EMPIRICAL LOSS will converge to EXPECTED LOSS.

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note

- we can write $y = f(x)$ so we can rewrite $L_{D,f}(h) = \mathbb{E}_D(\mathbb{1}_{h(x) \neq f(x)})$; note also that our hope is to find h such that $L_{D,f}(h) < \varepsilon$ with Prob $> 1 - \delta$ because in the best case we can have $L_{D,f}(h) = 0$ (that is obv. impossible)

APPROXIMATE WITH HIGH PROBABILITY

ERM - EMPIRICAL RISK MINIMIZATION

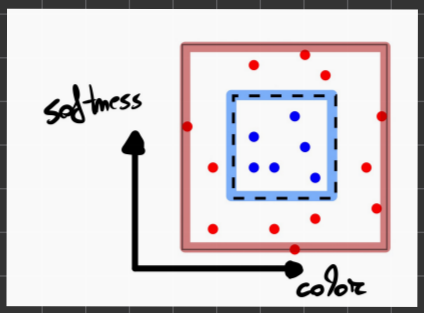
Def

- ERM is the process of finding a PREDICTOR h which minimizes the EMPIRICAL LOSS $L_S(h)$.

OVERFITTING

- When a predictor is excellent on the training set but very poor on the distribution

ERM is bad?



- Uniform D
- AREA( ) = 1
- AREA( ) = 2
- $f(x)$ { 1 if in , 0

Overfitting example

- let $S =$ and define:

$$h_S(x) = \begin{cases} y, & \forall x \in S \\ 0 \end{cases}$$

it means that $h_S(x)$ is perfect for the training set, so $L_S(h_S) = 0$; but, having $\mathbb{P} = 1/2$ of taking an element $x \notin S \Rightarrow L_D(h_S) = 1/2$, so it's bad for the distribution D

- MEMORIZATION ALGO.

other OVERFITTING EXAMPLE

- Suppose: $h_M(x) = \begin{cases} 1 \text{ if } x = x_i \in S \\ \text{TOSS A COIN} \end{cases}$
- the $\mathbb{P}$ of taking an already known SAMPLE from all over the distribution D is ϕ, so you always toss a coin.

$\Rightarrow$ it perfectly explain the dataset, but is shit in general

ERM WITH INDUCTIVE BIAS

⇒ ERM is not bad but learns in a too large set of HYPOTESIS, so it depends from the set of HYPOTESIS you apply it.

Def
- The IDEA is to reduce the SEARCH SPACE of ERP;
- we define, before looking at the data, a set of predictor H called HYPOTESIS CLASS, and we use ERP rule to find $h_{ERM_H} = ERM_H(S) \in \underset{h \in H}{\operatorname{argmin}} L_S(h)$.

INDUCTIVE BIAS
- mean biasing a LEARNER to a finet set of predictor.

?
HOW TO REDUCE THE SEARCH SPACE H?
- UPPERBOUNDING IT'S SIZE

REALIZABILITY ASSUMPTION
- There exist alway $h^* \in H \mid L_{D,f}(h^*) = 0$, it means $h^* = f$

⇒ it implies that, if the set H is finite, then for every S taken from D $L_S(h^*) = 0$, so ERM_H will alway find it.

ISSUE
- We are interested in $L_{D,f}(h_{ERM_H})$, so to be well done S should be enough representative of the underlying dist. D

↓

i.i.d. ASSUMPTION
- All sample of S are INDIPENDENTLY and IDENTICALLY DISTRIBUTED according to D

↳ it implies that the bigger is S the more it is representative of D and f

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NOTE

- $L_{D,f}(h_S)$ is a random variable because the choice of S is random, so can always happen that S would be not representative of D

CONFIDENCE PARAMETER

- δ is the PROB. of getting NON-REPR. S
- $(1-\delta)$

ACCURACY PARAMETER

- ε is needed when to address the QUALITY of a Prediction

⇒ if $L_{D,f}(h_S) > \varepsilon$ FAIL; else if $L_{D,f} \leq \varepsilon$ we got an APPROX. CORRECT PREDICTOR h_S

? IDEA

Can we upp. bound the prob. of sampling S from D in a way that S want be MISLEADING?

We can find the right size $m = |S|$ such that ERM_H won't choose a $h_{ERM_H} = h_S$

Some Defs

- BAD HYPOTHESIS SET → $H_B := \{h \in H \mid L_{D,f}(h) > \varepsilon\}$
- POSSIBLY MISLEADING S →

$$S_p := \{S \mid \exists \underbrace{h_S \in H_B, L_S(h_S) = 0}_{\text{BAD HYPOT. THAT LOOK GOOD ON } S}\}$$

- MISLEADING S →

$$S_M \in \{S \mid \underbrace{L_{D,f}(h_S) > \varepsilon}_{\text{IT HAPPENS ONLY IF } h_S \in H_B}\} = \{S \mid h_S \in H_B\} \subseteq S_p$$

- we can rewrite $S_P = \bigcup_{h_S \in H_B} \{S \mid L_S(h_S) = 0\}$

Prob. of picking S_M

- we now calculate $\mathbb{P}_{D^m}(S_M)$

$$\mathbb{P}_{D^m}(S_M) \leq \mathbb{P}_{D^m}(S_P) = \mathbb{P}_{D^m}\left(\bigcup_{h_S \in H_B} \{S \mid L_S(h_S) = 0\}\right) \leq$$

$$\leq \sum_{h_S \in H_B} \mathbb{P}_{D^m}(\{S \mid L_S(h_S) = 0\}) \dots$$

- Notice that $L_S(h_S) = 0$ if $\forall (x_i, y_i) \in S \; h_S(x_i) = y_i \Rightarrow$

$$\Rightarrow \mathbb{P}(h_S(x_i) = y_i) = 1 - L_{D,f}(h_S) \leq 1 - \varepsilon \Rightarrow$$

($L_{D,f}(h_S) > \varepsilon$ (searching S_M!))

$$\Rightarrow \mathbb{P}(h_S(x_i) = y_i \; \forall i = 1, \dots, m) = \mathbb{P}_{D^m}(\{S \mid L_S(h_S) = 0\}) \leq (1-\varepsilon)^m$$

$$\dots \sum_{h_S \in H_B} \mathbb{P}_{D^m}(\{S \mid L_S(h_S) = 0\}) \leq \sum_{h_S \in H_B} (1-\varepsilon)^m =$$

$$= |H_B|(1-\varepsilon)^m \leq |H| e^{-\varepsilon m}$$

(obv $H_B \subseteq H$)

$\Rightarrow$ - in conclusion $\boxed{\mathbb{P}_{D^m}(S_M) \leq |H_B| e^{-\varepsilon m}}$

BOUND

- we want $\mathbb{P}_{D^m}(S_M) \leq \delta$ so we should choose m carefully:

$$|H_B| e^{-\varepsilon m} \leq \delta \Rightarrow \boxed{m \geq \frac{\log(|H|/\delta)}{\varepsilon}}$$

Conclusion

COROLLARY 2.3 *Let $\mathcal{H}$ be a finite hypothesis class. Let $\delta \in (0,1)$ and $\epsilon > 0$ and let m be an integer that satisfies*

$$m \geq \frac{\log(|\mathcal{H}|/\delta)}{\epsilon}.$$

Then, for any labeling function, f, and for any distribution, $\mathcal{D}$, for which the realizability assumption holds (that is, for some $h \in \mathcal{H}$, $L_{(\mathcal{D},f)}(h) = 0$), with probability of at least $1 - \delta$ over the choice of an i.i.d. sample S of size m, we have that for every ERM hypothesis, h_S, it holds that

$$L_{(\mathcal{D},f)}(h_S) \leq \epsilon.$$

The preceeding corollary tells us that for a sufficiently large m, the $\text{ERM}_{\mathcal{H}}$ rule over a finite hypothesis class will be *probably* (with confidence $1-\delta$) *approximately* (up to an error of ϵ) correct.

21/09/21

PAC - LEARNABILITY

PAC LEARNABILITY Def 1

. (Given REALIZ. ASS and 0-1 loss funct $\mathbb{1}_{h \neq y}$) H is PAC-LEARNABLE if:

- $\exists A : (X \times Y)^m \to H$
- $\exists m_H : (0,1)^2 \to \mathbb{N}$ // SAMPLE COMPLEXITY how many samples are required to guarantee an approx. correct solution?

They are such that:

$\forall D$ over X, $\forall f$, $\forall \delta \in (0,1)$, $\forall \varepsilon \in (0,1)$ if we get S of m i.i.D. samples according to D from X such that $|S| = m \geq m_H(\varepsilon, \delta)$, then:

$$L_{D,f}(A(S)) < \varepsilon \quad \text{w.p. } (1-\delta)$$

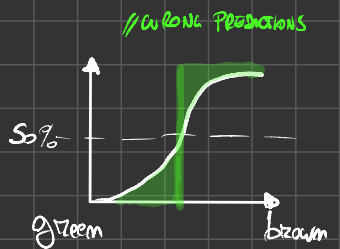
weak points of Def 1

1. REALIZABILITY ASSUMPTION → $\exists f$, $f \in H$
2. 0-1 loss (only binary classification)

⇒

- $f(x)$ is wasted and we'll use $P(y|x)$
- We need a THRESHOLD for $P(y|x)$, and put it to 50% is the choiche that reduces $L_{D,P}(h)$

$$h(x) = \begin{cases} 1 & , P(1,x) = 0,5 \\ 0 \end{cases}$$



MARGINAL DIST

- $P(x)$

CONDITIONAL LABELLING PROB

- $P(y|x) = \dfrac{P(x,y)}{P(x)}$

LOSS GENERALIZATION

- There exist different kinds of loss other than the (0-1)-loss:

some losses:

- $\ell(h,(x,y)) \in \mathbb{R}$
- $\ell(h,(x,y)) = \mathbb{1}_{h(x)\neq y}$ ((·)² cause we want $\ell \geq 0$)
- $\ell(h,(x,y)) = (h(x)-y)^2$

- we can generalize them by $\Rightarrow L_{\mathcal{D}}(h) = \mathbb{E}[\ell(h,(x,y))]$

REAL RISK EVALUATION

- Until now, with realizability ass., we were sure that the best possible $L_{\mathcal{D}}(h)$ was 0 (because $\exists h^* \in H \mid h^* = f$); relaxing this assumption we need the to compare our loss to be the nearest possible to the BEST ONE we can achieve in H.

$$L_{\mathcal{D}}(h) < \min_{h' \in H} L_{\mathcal{D}}(h') + \varepsilon$$

AGNOSTIC PAC LEARNABILITY

Def2

- ~~(Given realiz. ass on 0-1 loss funct $\mathbb{1}_{h\neq y}$)~~ H is AGNOSTIC PAC-LEARNABLE, if:
  - $\exists A : (X \times Y)^m \to H$
  - $\exists m_H : (0,1)^2 \to \mathbb{N}$

w.r.t. the loss ℓ

They are such that:

$\forall \mathcal{D}$ over $(X \times Y)$, ~~$\forall f$~~, $\forall \delta \in (0,1)$, $\forall \varepsilon \in (0,1)$ if we get S of m i.i.d. sampled according to $\mathcal{D}$ from $(X \times Y)$ such that $|S| = m \geq m_H(\varepsilon, \delta)$, ~~then~~:

~~$L_{\mathcal{D},f}(A(S)) < \varepsilon$ w.p. $(1-\delta)$~~

$$L_{\mathcal{D}}(A(S)) \leq \min_{h} L_{\mathcal{D}}(h) + \varepsilon \quad \text{w.p.} \geq 1-\delta$$

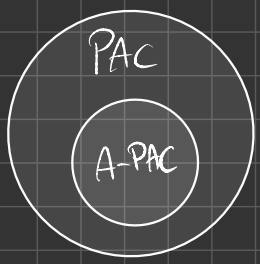
BEST POSSIBLE ERROR OVER H. w/ REALIZABILITY WAS 0 BECAUSE OF $f \in H$

NB

- What is true?

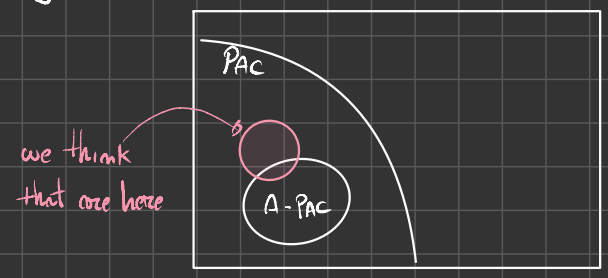
① H is A-PAC $\Rightarrow$ H is PAC

② H is PAC $\Rightarrow$ H is A-PAC



$\Rightarrow$ ①: if H satisfy PAC LEARN. constr, it's the obvious that satisfy also AGNOSTIC PAC LEARN. constr.

Where are finite class of HIPOTHESIS? we know are PAC, but no who they are



ε-REPRESENTATIVE

ε-REPRESENTATIVE DATASET

- S is ε-REPRESENTATIVE if $|L_D(h) - L_S(h)| < \varepsilon \quad \forall h \in H$
- if you have S ε-REPR, then ERM_H finds a good predictor
  - let $h_S = ERM_H(S)$
  - let S be $\varepsilon/2$-REPRESENTATIVE
  - we know by def that

$$L_D(h_S) \leq L_S(h_S) + \varepsilon/2$$
$$\leq L_S(h) + \varepsilon/2 \quad \forall h$$

↕ BY DEF

$$\leq L_D(h) + \varepsilon/2 + \varepsilon/2 \quad \forall h$$
$$\leq \min_{h \in H} L_D(h) + \varepsilon$$

↑ BY A-PAC LEARNABILITY

UNIFORM CONVERGENCE PROPERTY (UC)

- H has UC prop. w.r.t. the loss ℓ $\exists m_H^{UC}: (0,1)^2 \to \mathbb{N}$ $\forall D$, $\forall \varepsilon, \delta \in (0,1)$, if you draw a dataset S i.i.d from D with $|S| = m \geq m_H^{UC}(\varepsilon, \delta)$ then S is ε-REPR. w.p. $\geq 1-\delta$

NOTE

- If H has UC prop. then S is ε-repr. with high prob., so $L_D(h_S) \leq \min_{h \in H} L_D(h) + \varepsilon$, so H is AGNOSTIC PAC LEARNABLE, and $ERM_H(S)$ an AGNOSTIC PAC LEARNER for H

TH

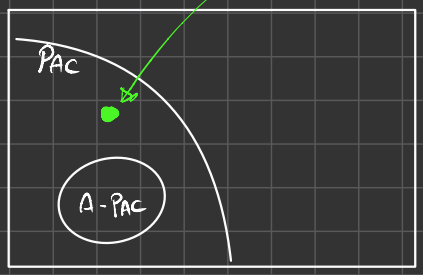
- If H is finite $\Rightarrow$ H has UC prop

Proof

Postponed...

TH

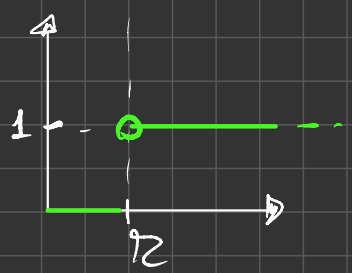
- There is at least an INFINITE PAC CLASS H



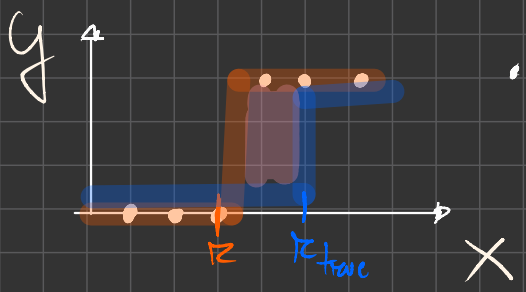
Proof

- Let H be a threshold function

$H = \{ h_\pi, \pi \in [0,1], h_\pi : [0,1] \to \{0,1\}, h_\pi(x) = \begin{cases} 1 & \forall x > \pi \\ 0 & \forall x \leq \pi \end{cases} \}$



- Given the dataset

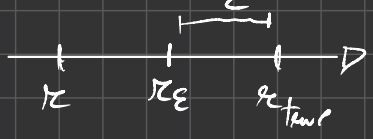


- ERM_H gives me h_z where $z = \max\{x_i \mid y_i = 0\}$

- suppose the $\mathcal{D}$ every time we observe a point in the PINK area we have a loss of 1

$$L_D(h_z) = \mathbb{P}(x \in [z, z_{true}])$$

- now we look a point in $[z, z_{true}]$ such that the error on it is ε



- z_ε: $\mathbb{P}(x \in [z, z_{true}]) = \varepsilon$

- if $z < z_\varepsilon \Rightarrow L_D(h_z) > \varepsilon$ (1)
- if $z \geq z_\varepsilon \Rightarrow L_D(h_z) < \varepsilon$ (2)

$\mathbb{P}(z < z_\varepsilon) =$

$\Rightarrow \mathbb{P}(x \notin [z, z_{true}]) = 1 - \varepsilon$

$= \mathbb{P}(x_i \notin [z, z_{true}] \ \forall i = 1, \dots, m) = \prod_{i=1}^{m} \mathbb{P}(x \notin [z, z_{true}]) =$

$= \prod_{i=1}^{m} (1 - \varepsilon) = (1-\varepsilon)^m \Rightarrow$ we want $, t \leq \delta$

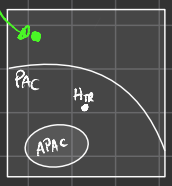
$\Rightarrow$ if we take $m \geq \frac{\log(\delta)}{\log(1-\varepsilon)}$ then $L_D(ERM_S) \leq \varepsilon$

w.p $\geq 1 - \delta$ so H is PAC

∎

28/09

- We will show that There is a class $H_{\{0,1\}^X} = \{$All binary partition over $X\} \mid H_{\{0,1\}^X} \notin PAC$



TH [NO FREE LUNCH]

A

- $\exists D$ over X and a labeling function f, $\forall A$ learning algorithm for binary classification, those are s.t. by picking S of $m < |X|/2$ i.i.d. according to D you have

$$L_{D,f}(A(S)) \geq 1/8 \quad \text{w.p.} \geq 1/7 \leftarrow \text{NON NEGLIGIBLE}$$

PAC NEGATION

- H is not pac if $\forall A$, $\forall m_H : (0,1)^2 \to \mathbb{N}$, $\exists D$ over X and $\exists f \in H$, $\exists \varepsilon_0, \delta_0 \in (0,1)$: if $\exists m \geq m_H(\varepsilon_0, \delta_0) \mid |S| = m$ then EVERY A learn with an error of:

$$L_{D,f}(A(S)) \geq \varepsilon_0 \text{ w.p } \geq 1-\delta_0$$

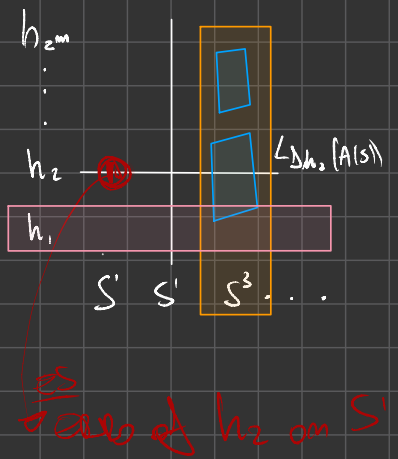
→ FROM NO FREE LUNCH TH we can enforce the ¬PAC definition

- $\exists f$ instead of $\exists f \in H$
- $\varepsilon_0 = 1/3$ and $\delta_0 = 1/7$
- the requirement $\exists m \geq m_H(\varepsilon_0, \delta_0)$ that became $\forall m$

free lunch proof

B

- Pick $D \mid D(x_i) = 1/2^m \ \forall i$
- Let's proof that $\exists f \mid \mathbb{E}_{S \sim D^m}[L_{D,f}(A(S))] \geq 1/4$
- Build a table of $|H|$ rows with every possible S on column



// pick 4 SAMPLES AT A TIME

// the average on each ☐ has an AVG ERROR of 1/4

// the set of a column ☐ has ... 1/4

// the AVG BS too has ERR 1/4

// AVG of elements on RAW ☐ 1/4

⇒ ∃ RAW with AVG ≥ 1/4, H's the RAW of f

↳ Proof if the whole set has AVG ≥ 1/4 there is at least one with AVG ≥ 1/4

- Now show that $A \Rightarrow B$ by showing $\neg A \Rightarrow \neg B$

(A) $L_{D,f}(A(S)) \geq 1/8$ w.p. $\geq 1/7$

(B) $\exists f \mid \mathbb{E}_{S \sim D^m}[L_{D,f}(A(S))] \geq 1/4$

How can $L_{D,f}(A(S)) = 1$? happens w.p. $< 1/7$ so

$$\mathbb{E}_{S \sim D^m}[L_{D,f}(A(S))] < 1 \cdot 1/7 + 1/8 \cdot 6/7 = 1/4$$

COROLLARY [NO FREE LUNCH]

- If $|X| = +\infty$, $H_{\{0,1\}^X}$ is not PAC learnable w.r.t. 0-1 loss

Proof

- Assume H is PAC and choose $\varepsilon < 1/8$ and $\delta < 1/7$
- By PAC def: $L_D(A(S)) \leq \varepsilon$ w.p. $> 1 - \delta$
- By NO-FREE LUNCH: $\exists D \mid L_D(A(S)) > 1/8 > \varepsilon$ w.p. $> 1/7 > \delta$ ⊥

VC DIMENSION

- Finitness of H is a SUFFICIENT but not NECESSARY condition for it's LEARNABILITY

why VC-DIMENSION?

- $PAC \equiv A\text{-}PAC \equiv UC \equiv \{$ everything we can learn via ERM$\} \equiv$
$\equiv VC\text{-}dim(H) < +\infty$

SHATTERING

- Given X, H over X and $A \subset X$ subset of X
- H shatters A if $\forall g: A \to \{0,1\}$ $\exists h \in H \mid h(x) = g(x)$ $\forall x \in A$

shattering example

$H = \{h_1, h_2, h_3, h_4\}$

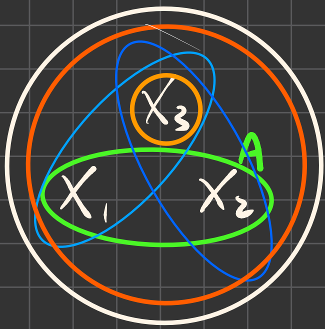
- $A = \{x_1\} \Rightarrow H$ shatter x_1
- $A = \{x_3\} \Rightarrow H$ DOESN'T (you can't get $x_3 = 1$ with $h \in H$)
- $A = \{x_1, x_2\} \Rightarrow H$ shatter A (we have 00, 01, 10, 11 in h)

why shattering

- given a subset of Point, we can always define H in a way that shatter them, so H can be seen as a SUBSET of X:

$\Rightarrow H$ shatters A if $\forall g$ subset of A $\exists h \in H \mid h \cap g = 0$

$H = \{h_1, h_2, h_3, h_4\}$



- To get $\{\emptyset\}$ we get X_3 set cause $\{x_3\} \cap A = \{0\}$
- To get x_1 we get ◯
- To get x_2 we get ◯

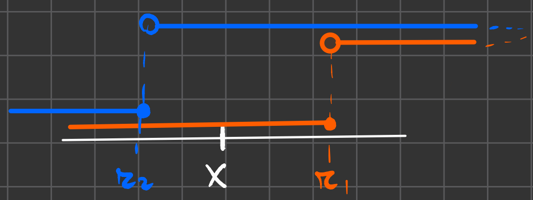
□

VC-dim

- $VC\text{-}dim(H) = \max\{|A| \mid H \text{ shatters } A\}$

Find VC-dim of $H_{THRESH} = \left\{ h_r \mid h_r(x) \begin{cases} 1 & \text{if } x > r \\ 0 \end{cases} \right\}$

//w/ bomst



//w/ set

- we can pick him by taking the set $X \setminus \{x\}$ cause $A \cap X \setminus \{x\} = \{\emptyset\}$

- Now we know $VC\text{-}dim(H) \geq 1$
- we cant do the same for 2 points because there is no way of picking only one of them

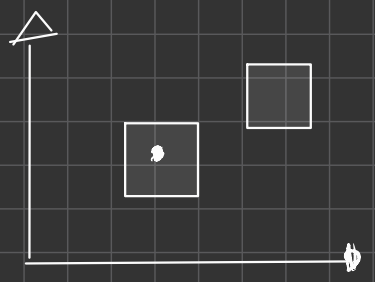
$\Rightarrow VC\text{-}dim(H) = 1$

---

find VC-dim of $H_{rect} = \{h_{a,b,c,d}$ s.t.

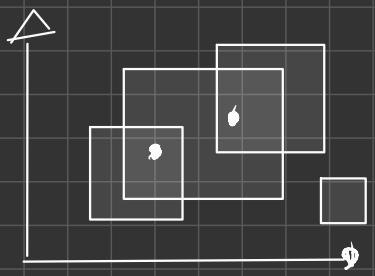
$h_{a,b,c,d} \begin{cases} 1 & \text{if } 0 \le a \le b \text{ and } 0 \le c \le b \\ 0 & \end{cases} \}$

1 point

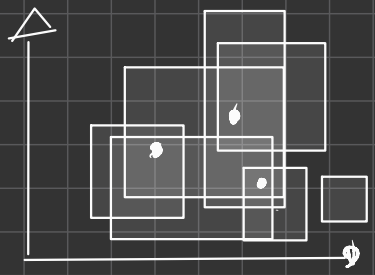


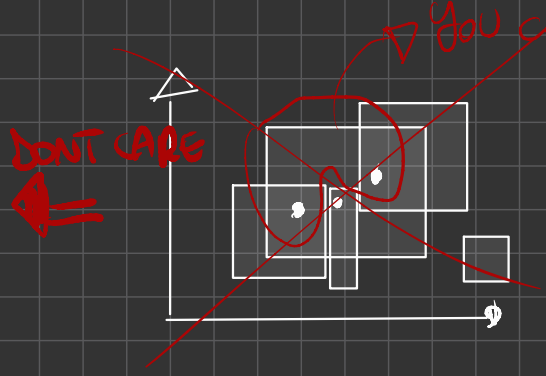
$\Rightarrow$ VC-dim(H_{rect}) ≥ 1

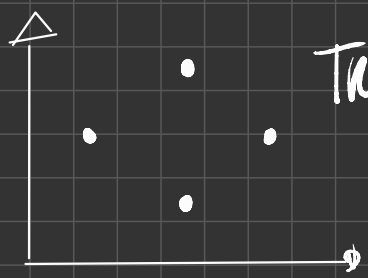
2 point



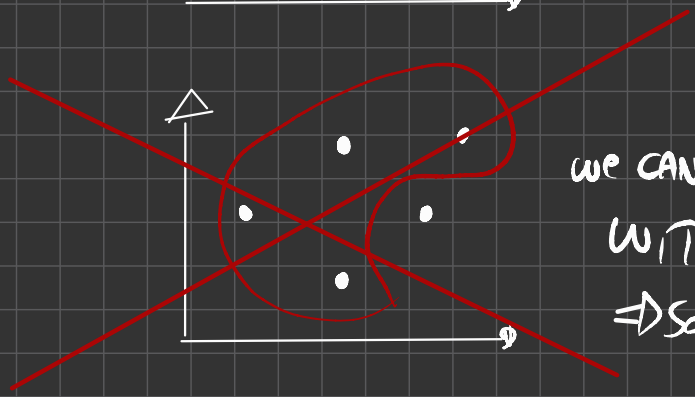
3 point







TRUST, WE CAN!



WE CANT SHATTER 4 POINTS WITHOUT THE 5TH !

$\Rightarrow$ So you proved also for more than 5 points

$\Rightarrow$ VC-dim(H) = 4

FUNDAMENTAL TH OF STATISTICAL LEARNING

TH FTSL

- X, 0-1 LOSS, H

(1) H IS PAC LEARNABLE

(2) ERM_H IS PAC LEARNER

(3) H IS AGNOSTIC PAC LEARNABLE

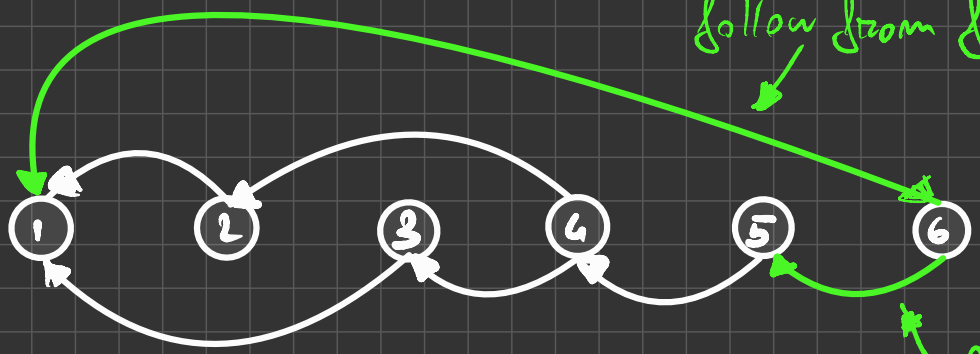
(4) ERM_H IS AGNOSTIC PAC LEARNER

(5) H HAS UC PROPERTY

(6) $VC\text{-}dim(H) < +\infty$

Proof

We don't Know



Codio, non lo vediamo che ce vole troppo

(6-1) Proof

TH: H is PAC $\Rightarrow$ $VC\text{-}dim(H) < +\infty$

Proof.

NEGATED TH:

$VC\text{-}dim(H) > +\infty \Rightarrow H$ is NOT PAC

... vedi Alessio, io non ce la faccio più a sentire Porcodio

5/10/21

TH QUANTITATIVE FTSL

// You can see how VC dim characterise the complexity of different class of learning

. Suppose $d = VCdim(H) < +\infty$, $\exists C_1, C_2 \in \mathbb{R}$ s.t.

(1) H is PAC-LEARNABLE with $m_H(\varepsilon, \delta)$:

$$C_1 \frac{d + \log(1/\delta)}{\varepsilon} \leq m_H(\varepsilon, \delta) \leq C_2 \frac{d \log(1/\varepsilon) + \log(1/\delta)}{\varepsilon}$$

(2) H is APAC-LEARNABLE with $m_H(\varepsilon, \delta)$:

$$C_1 \frac{d + \log(1/\delta)}{\varepsilon^2} \leq m_H(\varepsilon, \delta) \leq C_2 \frac{d + \log(1/\delta)}{\varepsilon^2}$$

(3) H has UC-PROPERTY with $m_H(\varepsilon, \delta)$:

$$C_1 \cdot // \leq m_H^{UC}(\varepsilon, \delta) \leq C_2 \cdot //$$

LINEAR PREDICTORS

- Linear predictors are a FAMILY of HYPOTHESIS CLASS

CLASSES	ALGORITHMS
HALFSPACES	LP & PERCEPTRON
LINEAR REGRESSION PREDICTORS	
LOGISTIC REGRESSION PREDICTORS	LEAST SQUARE

AFFINE FUNCTIONS

- $L_d = \{ h_{w,b} : X \to \langle w, x \rangle + b \mid w \in \mathbb{R}^d \wedge b \in \mathbb{R} \}$
- $h_{w,b}(x) = \langle w, x \rangle + b =$
- we can incorporate the bias b in w

$$w' = (b, w_1, \_, w_d) \in \mathbb{R}^{d+1}, \quad x' = (1, x_1, \_, x_d) \in \mathbb{R}^{d+1}$$

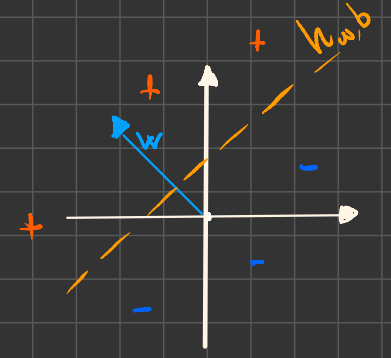
HALFSPACES

- Designed for BINARY CLASSIFICATION problems
- $X = \mathbb{R}^d$, $Y = \{-1, +1\}$

$$HS_d = L_d \circ SIGN = \{ X \to SIGN(h_{w,b}(x)) \mid h_{w,b} \in L_d \}$$

Ex. w/ $d = 2$

EACH HYPOTESIS FORM AN HYPERPLANE PERPENDICULAR TO w



SEPARABLE AND UNSEPARABLE CASES

- Separable mean that we can separate all POSITIVE samples from NEGATIVE with an HYPERPLANE.
- The NON-Separable is NP-HARD

⇒ We have 2 method for IMPLEMENTING ERM FOR HALFSPACES IN SEPARABLE CASE (ie. LP and PERCEPTRON)

LINEAR PROGRAMMING FOR HS

- ERM predictor should have 0 ERROR on the TS, so we looks for a vector w^* s.t.

$$SIGN(\langle w^*, x_i \rangle) = y_i \quad \forall i = 1, \_, m \Rightarrow$$

$$\Rightarrow y_i \langle w^*, x_i \rangle > 0 \quad \forall i = 1, \_, m$$

- Let $\varphi = \min_{i=1,\_,m} \{ y_i \langle w^*, x_i \rangle \}$ and $\bar{w} = \frac{w^*}{\varphi}$ we have

$$y_i \langle \bar{w}, x_i \rangle = \frac{1}{\varphi} y_i \langle w^*, x_i \rangle \geq 1 \quad \forall i = 1, \_, m$$

- So $\exists w \mid y_i \langle w, x_i \rangle \geq 1 \ \forall i = 1 \_ m$
- This is our ERM predictor

- So the ERM LP is:

$$\max_{w \in \mathbb{R}^d} \langle w, u \rangle \quad \text{s.t.} \quad Aw \geq v$$

where

- $A_{m \times d} \mid A_{i,j} = y_i \cdot x_{ij}$
- $v = (1, \_, 1) \in \mathbb{R}^m$
- $u = (0, \_, 0) \in \mathbb{R}^d$ // because every w that satisfy the constraints are equal candidates

---

PERCEPTRON FOR HS

- ERM wants 0 error on the TS so $\text{SIGN}(\langle w^*, x_i \rangle) = y_i \quad \forall i = 1 \_ m$
- At each step t if there is an $x_i \mid \text{SIGN}(\langle w^t, x_i \rangle) \neq y_i$ the algo. update w^t in such a way $w^{t+1} = w^t + y_i x_i$ to accomplish $y_i \langle w^{t+1}, x_i \rangle > 0$

Batch Perceptron

input: A training set $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$
initialize: $\mathbf{w}^{(1)} = (0, \ldots, 0)$
  for $t = 1, 2, \ldots$
    if $(\exists\, i \text{ s.t. } y_i \langle \mathbf{w}^{(t)}, \mathbf{x}_i \rangle \leq 0)$ then
    $\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} + y_i \mathbf{x}_i$
    else
      output $\mathbf{w}^{(t)}$

- At the end of the execution all SAMPLES of TS will be correctly classified

TH

- Given a SEPARABLE TS
- Let $B = \min \{ \|w\| \mid y_i(\langle w, x_i \rangle) \geq 1 \quad \forall i = 1, \_, m \}$
- Let $R = \max_i \{ \|x_i\| \}$

$\Rightarrow (RB)^2$ ITERATIONS

- It stops with $y_i \langle w^t, x_i \rangle > 0 \quad \forall i = 1 \_ m$

VCdim(HS)

- **HOMOGENEOUS HALFSPACE**: is an halfspace that does contain the $\emptyset$-VECTOR

  ⇒ So the induced HYPERPLANE pass through the ORIGIN.

TH

- $VCdim(HS) = d$ for the class of HOMOGENEOUS HALFSPACES in $\mathbb{R}^d$

Proof

- Consider the set $(e_1, \_\_, e_d) \mid e_{is} = 0 \ \forall s \neq i \ \wedge \ e_{ii} = 1$
- This set is shattered by the Hom. HS class, by simply setting w as the labeling $w = (y_1, \_, y_d) \Rightarrow \langle w, e_i \rangle = y_i \ \forall i$
- Let $x_1, \_, x_d, x_{d+1} \in \mathbb{R}^d$ then must exist $a_1, \_, a_{d+1} \in \mathbb{R}$ not all zeros s.t. $\sum_{i=1}^{d+1} a_i \cdot x_i = 0$ (LINEAR DEPENDENCE)
- Let $I := \{i \mid a_i > 0\}$, $J := \{j \mid a_j < 0\}$
- Assume both non empty:

$$\sum_{i \in I} a_i x_i - \sum_{j \in J} |a_j| x_j = 0$$

- Suppose $x_1, \_, x_{d+1}$ are shattered then

$$\exists w \mid \langle w, x_i \rangle > 0 \ \forall i \in I \ \wedge \ \langle w, x_j \rangle < 0 \ \forall j \in J$$

it implies

$$0 < \sum_{i \in I} a_i \langle \mathbf{x}_i, \mathbf{w} \rangle = \left\langle \sum_{i \in I} a_i \mathbf{x}_i, \mathbf{w} \right\rangle = \left\langle \sum_{j \in J} |a_j| \mathbf{x}_j, \mathbf{w} \right\rangle = \sum_{j \in J} |a_j| \langle \mathbf{x}_j, \mathbf{w} \rangle < 0,$$

- if I is empty

$$0 = \sum_{i \in I} a_i \langle \mathbf{x}_i, \mathbf{w} \rangle = \left\langle \sum_{i \in I} a_i \mathbf{x}_i, \mathbf{w} \right\rangle = \left\langle \sum_{j \in J} |a_j| \mathbf{x}_j, \mathbf{w} \right\rangle = \sum_{j \in J} |a_j| \langle \mathbf{x}_j, \mathbf{w} \rangle < 0,$$

- if J is empty

$$0 < \sum_{i \in I} a_i \langle \mathbf{x}_i, \mathbf{w} \rangle = \left\langle \sum_{i \in I} a_i \mathbf{x}_i, \mathbf{w} \right\rangle = \left\langle \sum_{j \in J} |a_j| \mathbf{x}_j, \mathbf{w} \right\rangle = \sum_{j \in J} |a_j| \langle \mathbf{x}_j, \mathbf{w} \rangle = 0,$$

TH • $VCdim(HS) = d+1$ for the class of NON-HOMOGENOUS HALFSPACES in $\mathbb{R}^{d+1}$

Proof • As before consider the set $(0, e_1, \dots, e_d)$ is shattered. Then suppose $x_1, \dots, x_{d+2} \in \mathbb{R}^{d+1}$, we can reach a $\perp$ as before.

ERROR DECOMPOSITION

$$L_D(A(S)) = \underbrace{L_D(A(S)) - \min_{h \in H} L_D(h)}_{\text{ESTIMATION ERROR}} + \underbrace{\min_{h \in H} L_D(h)}_{\text{APPROXIMATION ERROR}}$$

• ESTIMATION ERROR ($\leq \varepsilon$ for PAC LEARNABILITY)

• APPROXIMATION ERROR

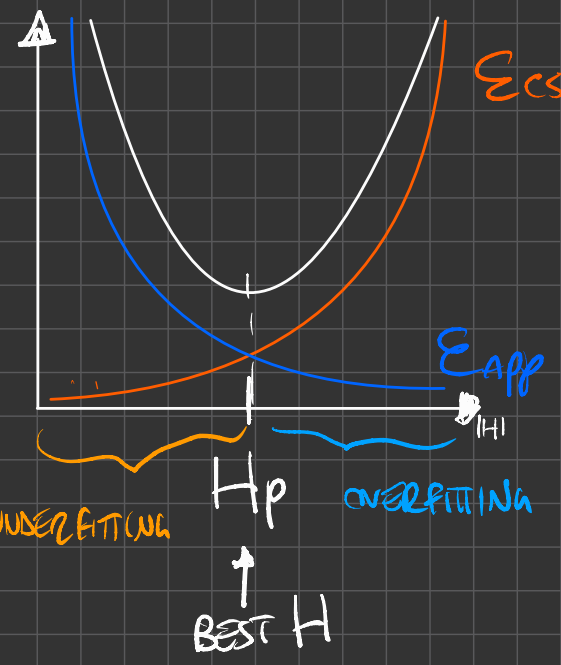
APPROX. ERR • Depends only by $VC(H)$, enlarging $|H|$ decreases ε_{APP} (WITH $\mathcal{S} \in H \Rightarrow \varepsilon_{APP} = 0$)

ESTIMATION ERR • Depends on $VC(H)$ AND $|S| = m$, FROM THE QUANT. FTSL:

$$m_H(\varepsilon, \delta) \approx \frac{d_H + \ln(1/\delta)}{\varepsilon^2} \Rightarrow \varepsilon_{ESTIM} \approx \sqrt{\frac{d_H + \ln(1/\delta)}{m}}$$

NOTE • If $H \subset H'$ we have $\begin{cases} \varepsilon_{APPROX}(H') \leq \varepsilon_{APPROX}(H) \\ \varepsilon_{ESTIMATION}(H) \leq \varepsilon_{ESTIMATION}(H') \end{cases}$

BIAS/COMPLEXITY TRADE-OFF



- What is the best possible error?

$L_D(A(S)) =$

NOTE THAT h IS NOT $\in H$!

$$L_D(A(S)) - \min_{h\in H} L_D(h) + \min_{h\in H} L_D(h) - \min_{h} L_D(h) + \min_{h} L_D(h)$$

ESTIMATION ERROR — APPROXIMATION ERROR — BIAS ERROR (IRREDUCIBLE)

WEAK LEARNABILITY, BOOSTING

- We look for a simpler LEARNABILITY definition leading to more efficient solution.

SHAPIRE HOPE

- Something NP-HARD for PAC LEARN. becomes POLYNOMIAL for WEAK LEARN. (of course they fucked up)

CONCLUSION

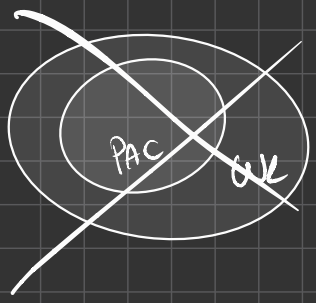
- WL ≡ PAC L.
- SHAPIRE showed that a POLY. ALGO. to WEAKLY LEARN H can be reduced in Polynomial time in a PAC LEARNER for H (che figa)
- The POLY. REDUCTION ALGORITHM is called BOOSTING
- Boosting Allow the learner to have smooth control over BIAS/COMP. TRADEOFF
- Think to a WEAK-LEARNER as an algo. which outputs h_{weak} performing just a bit better than a random guess.
- Boosting is an AGGREGATOR OF h_{weak} hypothesis to approximate predictors for hard to learn classes
- ADABOOST output a an h_{ADA} that is a LINEAR COMB. of h_{weak} hypots

---

WEAK LEARN.

- Assuming $f\in H$ and 0-1 loss, H is γ-WLC if $\exists A: (X\times Y)^m \to H$ and $\exists m_H^{WL}: (0,1)\to\mathbb{N}$ | $\forall D$ over X, $\forall f\in H$, $\forall \delta$ if $S\sim D^m$ iid $|S|\geq m_H^{WL}(\delta)$ then: $L_{D,f}(A(S)) \leq \frac{1}{2}-\gamma$ w.p. $1-\delta$ $\left(\gamma\in(0,\frac{1}{2}\right]$)

- Because WL require a stupid learner it seems $WL \subset PAC\text{-}L$ but $PAC\text{-}L \equiv WL$



TU

- $WL = PAC\text{-}L$

Proof

① $PAC\text{-}L \Rightarrow WL$: $\exists \gamma \in (0, \frac{1}{2}) \mid H$ is γ-WL?

- if $H \in PAC \Rightarrow L_{D,f}(A(S)) < \varepsilon$ w.p. $\geq (1-\delta)$

- suppose $H \in PAC$ is also $H \in WL$:
- we can choose $\gamma = 1/4 \Rightarrow L_{D,f}(A(S)) \leq 1/4 = 1/2 - \gamma$

② $WL \Rightarrow PAC\text{-}L$: Proof $\neg PAC \Rightarrow \neg WL$

- let H NOT PAC $\Rightarrow VCd(H) = +\infty$
- For every $\varepsilon, \delta \Rightarrow m_H(\varepsilon, \delta) \geq \frac{VCd(H) + \log(1/\delta)}{\varepsilon} \leq m_H(\varepsilon, \delta)$
- so $L_D(A(S)) < \varepsilon$ in particular $\varepsilon = 1/2 - \delta$

- In the example we show that computing $ERM_{H'_{DS}}$ (easier than ERM_{H_3}) is a weak learner for H_3 even if $H'_{DS} \subset H_3$.
- Means that every sample labelled from $h \in H'_{DS}$ have an error not fell of at most $1-\gamma$ (obv H'_{DS} must be polynomially bounded by $ERM_{H'_{DS}}$)

$$H_3 = \{ h_{a,b} : \mathbb{R} \to \{-1, +1\} \mid a, b \in \mathbb{R} \cup \{-\infty, +\infty\} \}$$

$$h_{a,b}(x) = \begin{cases} +1, & \text{if } x \in [a,b] \\ -1 \end{cases}$$

$$VCdim(H_3) = 2$$

$$H'_{DS} = \{ h_{a,b} \mid \mathbb{R} \to \{-1,+1\},\ a \in \mathbb{R},\ b \in \{-1,1\} \},$$

$$h_{a,b}(x) = \begin{cases} b & \text{for } x \leq a \\ -b & \text{for } x > a \end{cases}$$

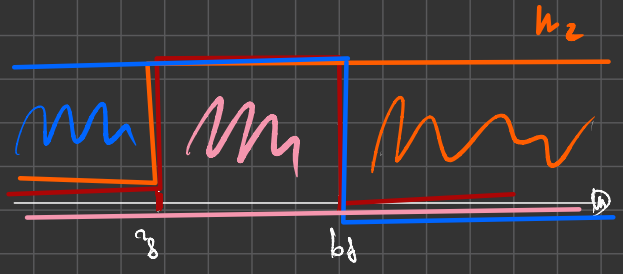
$$VCdim(H'_{DS}) = 2$$

• Notice that $\forall D, f \Rightarrow \exists h \in H'_{DS} \mid L_{D,f}(h) \leq 1/3$

(Also for $f \in H_3$.)

• Take $f \in H_3 \setminus H'_{DS}$ can divide $\mathbb{R}$ in 3 regions; ($\forall f \in H_3 \wedge f \in H'_{DS} \Rightarrow L_{D,f}(h) = 0$)

• Obviously there is $h \in H'_{DS}$ that fail in 1 of the 3 regions:



- $h_1, h_2, h_3 \in H'_{DS}$
- $f \in H_3$

$L_{D,f}(h_1) = 1/3$
$L_{D,f}(h_2) = 1/3$
$L_{D,f}(h_3) = 1/3$

• $\exists h \mid L_{D,f}(h) \leq 1/3$

• It implies H'_{DS} APAC because:

$$L_{D,f}(A(S)) \leq \min_{h \in H'_{DS}} L_{D,f}(h) + \varepsilon \leq 1/3 + \varepsilon$$

• To prove that $ERM_{H'_{DS}}$ is WL for H_3 we must find $\varepsilon, \gamma \mid 1/3 + \varepsilon = 1 - \gamma$ so for $\varepsilon = 1/12 = \gamma$ we have $1/3 + 1/12 = 1/2 - 1/12$

$\Rightarrow H'_{DS}$ is $1/2$-WEAK LEARNER for H_3

Note • Now with BOOSTING ALGO. we can transform it to a PAC LEARNER for H_3.

BOOSTING ALGO

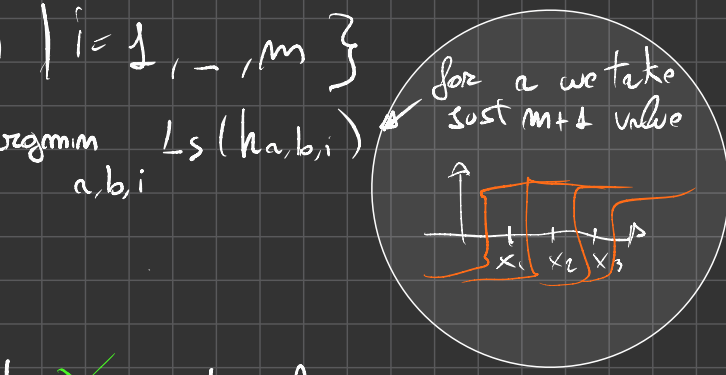
H^d_{DS} means that x_i has only one feature

- $ERM_{H^d_{DS}}$ CAN be implemented in Polynomial time

$$H^d_{DS} = \{ h_{a,b,i} : \mathbb{R}^d \to \{-1,+1\} \mid a \in \mathbb{R}, b \in \{-1,+1\} \}$$

$$h_{a,b,i} = \begin{cases} b & \text{if } x_i < a \\ -b \end{cases}$$

- Given $S = \{(x_i, y_i) \mid i = 1, \dots, m\}$
- $ERM_{H^d_{DS}} = \underset{h \in H^d_{DS}}{\operatorname{argmin}} L_S(h) = \underset{a,b,i}{\operatorname{argmin}} L_S(h_{a,b,i})$



- The complexity is:

COMPLEXITY

$$\underbrace{(m+1)}_{|Q|} \cdot \underbrace{2}_{|b|} \cdot \underbrace{d}_{|i|} \cdot \underbrace{m}_{\text{to compute } L_S(h_{a,b,i})} + \underbrace{d\, m \log m}_{\text{ORDER ALL } d \text{ sets of point}}$$

we can update it instead of recalculate at every step

$$\Rightarrow md + md \log m$$

- QUA BOH, NON SO PERCHÉ MA:

$$ERM_{H^d_{DS}} = \underset{h}{\operatorname{argmin}} \sum_{i=1}^{m} \underbrace{D_i}_{\text{PROBABILITY of guess } y_i} \underbrace{\ell(h_i(x_i, y_i))}_{\text{cs } \mathbb{1}(h(x_i) \neq y_i)}$$

- Perché

$$L_D(h) = \sum_{i=1}^{m} D_i \ell(h_i(x_i, y_i))$$

- so if $D = \{1/m, \dots, 1/m\} \Rightarrow L_D(h) = L_S(h)$

ADA BOOST

- To learn over H_m it uses a γ-WL over H_B WL

$$ERM_{H_B} = \underset{h \in H_B}{\operatorname{argmin}} \sum_{i=1}^{m} D_i \, \ell(h_i(x_i, y_i))$$

↳ $L_D(h) = L_S(h)$

$D_i^{(1)} = 1/m$ for $i = 1 \dots m$

For $(t = 1 \dots T)$

- $h_t = WL(D^{(t)}, S)$
- $\varepsilon_t = \sum_{i=1}^{m} D_i^{(t)} \, \ell(h_i, (x_i, y_i))$
- $w_t = 2 \ln\left(\frac{1}{\varepsilon_t} - 1\right)$
- $D^{t+1} = \dfrac{\exp(-w_t y_i h_t(x_i)) D_i^t}{\sum_{j=1}^{m} \exp(-w_t y_j h_t(x_j)) D_j^t}$

Get i.i.d. and $D^{(t)}$ samples from S

$L_D(h_t) = \varepsilon_t \le \frac{1}{2} - \gamma$

Assigning a weight to h_t inversely proportional to ε_t

Decrease D_i for i that one well guessed and increase D_i for others

→ ADA will focus on problematic samples of S most found

// POSITIVE PAPAYA HAVE POSITIVE NUMERATO, ELSE NEGATIVE (ENFORCE THE WEIGHTS!)

RETURN $h = \text{sign}\left(\sum_{t=1}^{T} w_t h_t\right)$

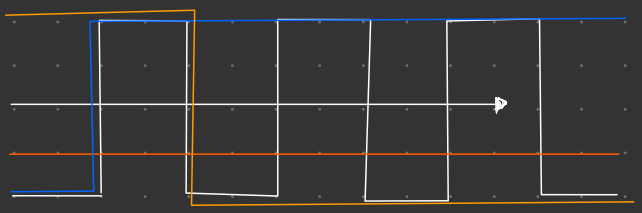
TH · If WL outputs h with $\sum D_i^t \, \ell(h_i(x_i, y_i)) \le \frac{1}{2} - \gamma$ w.p. δ then ADABOOST outputs h_{ADA} with $L_S(h_{ADA}) \le e^{-2\gamma^2 T}$ w.p $(1 - \gamma T)$

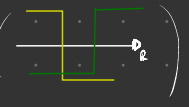
Note for $T \to \infty$ then $L_S(h) = 0$ ↳ overfitting!

19/10

- By applying ADABOOST to $B = DS'$ how much can $h_{ADA}(x) = \text{sign}\left(\sum_{t=1}^{T} w_t h_t(x)\right)$ become complex?
- Define weak hypothesis to sum up:

$h_1(x) = -1$ | $w_1 = \frac{1}{2}$
$h_2(x) = \text{sign}(x - x_1)$ | $w_2 = 1$
$h_3(x) = -\text{sign}(x - x_2)$ | $w_3 = 1$



- Summing those: $h_{ADA} = w_1 h_1 + w_2 h_2 + w_3 h_3$
- Notice that also if B is very poor  we can scale $h \in B$ by weights w and summing them to obtain more complex functions (h_{ADA})
- $VCdim(H_{ADA(B)}) \leq \tilde{O}(T \cdot VCdim(B))$

Given an what is the smallest ε I can get

- $\forall \varepsilon, \delta \quad \forall m \geq m_H^{UC}(\varepsilon, \delta) \Rightarrow |L_S(A(S)) - L_D(A(S))| \leq \varepsilon \quad w.p \geq (1-\delta) \Rightarrow m = m_H^{UC}(\varepsilon, \delta) \propto \frac{\log(1/\delta) + VCdim(H)}{\varepsilon^2} \Rightarrow$
- $\Rightarrow \varepsilon = \sqrt{\frac{VCdim(H) + \log(1/\delta)}{m}} \Rightarrow \boxed{|L_S(A(S)) - L_D(A(S))| \leq \sqrt{\frac{VCdim(H_{ADA}) + \log(1/\delta)}{m}}}$

- LEARNABLE?
  $\exists A \Rightarrow VCdim(H) < +\infty$
- EFFICIENTLY LEARNABLE?
  (1) BOOSTING
  (2) CONVEXITY

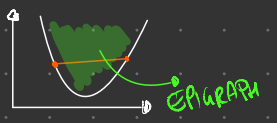
CONVEXITY

CONVEX SET

- $X \subset \mathbb{R}^n$ is convex $\Leftrightarrow \alpha x + (1-\alpha) y \in X \quad \forall x, y \in X \quad \forall \alpha \in [0,1]$

CONVEX f

- $f: X \to \mathbb{R}$ convex $\Leftrightarrow f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \forall x, y \in X, \forall \alpha \in [0,1], \forall X \subset \mathbb{R}^n \mid X$ convex set
- We can also say that f is convex if EPIGRAPH(f) is convex



- The nice property is LOCAL MIN $\equiv$ GLOBAL MIN

How check convexity of f

- $f: \mathbb{R} \to \mathbb{R}$, if $f''(x) \geq 0 \ \forall x \Rightarrow f$ convex
- generalizing $f: \mathbb{R}^n \to \mathbb{R}$, $f' \to \nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$ (DERIVATIVE WRT TO x_i), $H_f = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_2 \partial x_1 & \cdots & \partial^2 f/\partial x_1 \partial x_n \end{bmatrix}$ (SECOND DERIVATIVE WRT x_i)
- $(H_f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$
- ∇f: GRADIENT OF f
- H_f is POSITIVE 'SEMIDEFINITE' $\forall x \quad x^T H_f x \geq 0$ and $\forall y \quad y^T H_f y \geq 0$

Some props.

(1) f, g convex $\Rightarrow \max(f, g)$ is convex
(2) f, g convex AND g mon decreasing $\Rightarrow g(f(x))$ is convex
(3) f linear, g convex $\Rightarrow g(f(x))$ convex
(4) f, g convex $\Rightarrow \alpha f + \beta g$ is convex $\forall \alpha, \beta \geq 0$

CONVEX LEARNING PROBLEM

$ERM_H = \underset{h \in H}{\text{argmin}} \, L_S(h) = \frac{1}{m} \sum_{i=1}^{m} \ell(h, (x_i, y_i))$

1) $H = \{h_w : w \in \mathbb{R}^n\}$
2) $\ell(h_w(x, y))$ convex $\forall$:

Redefine ERM_H

CONVEX OPT. PROB.

- By (4) we can define $H = \{h_w, w \in \mathbb{R}^d\}$ so all classes with a real parameter (like $H_{\text{threshold}}$ . . .)
- So in $H = \{h_w / w \in \mathbb{R}^d\}$ we can look for a set of parameter $w \in \mathbb{R}^d$ instead of h: // W ARE PARAMS OF LINEAR FUNCTIONS

$$ERM_H = h_S \in \underset{h \in H}{\operatorname{argmin}}\, L_S(h) \equiv w_S \in \underset{w \in \mathbb{R}^d}{\operatorname{argmin}}\, L_S(w)$$

We want $L_S(w)$ to be convex in w

$$L_S(w) = \frac{1}{m} \sum_{i=1}^{m} \ell(w, (x_i, y_i))$$

- So if ℓ is convex in $w \Rightarrow L_S$ is convex in w

Note ℓ is convex in w if given $\ell = w^2 + (x-y)$ you don't care about the other variables

Conclusion

- if ℓ convex in w and $w \in W$ / W convex set then ERM is a CONVEX OPT. PROBLEM

LINEAR REGRESSION (EXAMPLE)

- You can use a LINEAR CLASSIFIER for REGRESSION problem (GUESS A NUMBER)

$$y = \langle w, x \rangle = w \cdot x = w^T x = \sum_{i=1}^{m} w_i x_i$$

- The loss we use is LEAST SQUARE $\quad \ell(h, (x, y)) = (h(x) - y)^2$

$$\ell(h_w, (x, y)) = (\langle w, x \rangle - y)^2 \quad \text{// CONVEX (this is linear in } w\text{)}$$

$$\ell(h_{w,b}, (x, y)) = \underbrace{(\langle w, x \rangle + b - y)^2}_{h_{w,b}} \quad \text{// CONVEX}$$

PROP (3) → 12 is convex

- So given $L_S(h_{w,b}) = \frac{1}{m} \sum_{i=1}^{m} (\langle w, x_i \rangle + b - y_i)^2$
- if the derivative (the gradient ∇ in higher dimensions)

$$\ell'(x) = 0 \rightarrow \nabla \ell = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- then derive w.r.t. every variable

$$\frac{\partial L_S}{\partial b} = \frac{1}{m} \sum_{i=1}^{m} 2(\langle w, x_i \rangle + b - y_i) = 0$$

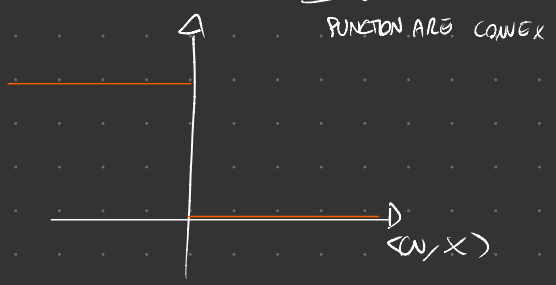
$$\frac{\partial L_S}{\partial w_s} = \frac{1}{m} \sum_{i=1}^{m} 2(\langle w, x_i \rangle + b - y_i) x_{is} = 0 \qquad \forall s = 1, \ldots, d \;\; // \text{Attribute } s \text{ of } x_i$$

BINARY CLASSIFICATION (EXAMPLE)

W/ HALF SPACES

- $\ell_{0,1}(h, (x_i, y_i)) = \mathbb{1}_{h(x) = y_i}$
- $h_w(x) = \operatorname{sgn}(\langle w, x \rangle)$
- $\ell_{0,1}(h_w, (x, y)) = \mathbb{1}_{y \langle w, x \rangle \leq 0}$ // NOT CONVEX

NOTE NO PIECEWISE FUNCTION ARE CONVEX



SURROGATE LOSS FOR 0-1 LOSS

- Lets find a convex function ℓ that works the same as $\ell_{0,1}$

$$\ell(h_{w,}(x,y)) \geq \ell_{0-1}(h_{w,}(a,b))$$

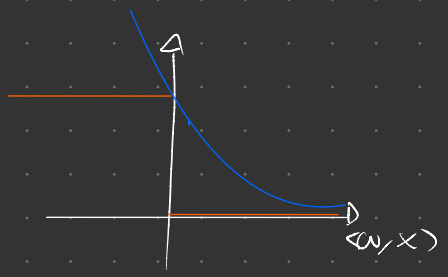
- We can for example take ℓ as:

$$(e^{-x})^2 = e^{-x} \geq 0 \Rightarrow \log_2(1+\exp(-\langle w,x\rangle \cdot y))$$

① LOGISTIC LOSS

$$\Rightarrow \boxed{\ln(1+\exp(-\langle w,x\rangle \cdot y))} \text{ called LOGISTIC LOSS,}$$

that is a SURROGATE FUNCTION for 0-1 loss



- MINIMIZING LOG. LOSS means solving LOG. REGR. PROBLEM

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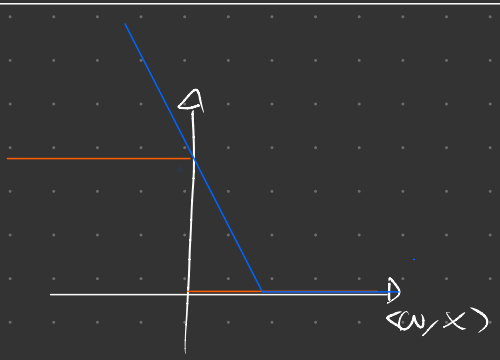
② HINGE LOSS

- Another surrogate loss for 0-1 loss is the blue function.

$$\max(0, -\langle w,x\rangle y + 1) = \text{HINGE LOSS}$$



$$L_D^{0-1}(h_S) =$$

$$\underbrace{L_D^{0-1}(h_S) - L_D^{SURR}(h_S)}_{<0} + \underbrace{L_D^{SURR}(h_S) - \min_{h\in H} L_D^{SURR}(h)}_{\text{ESTIMATION ERROR}} + \underbrace{\min_{h\in H} L_D^{SURR}(h) - \min_{h\in H} L_D^{0-1}(h)}_{\text{OPT ERROR}} + \underbrace{\min_{h\in H} L_D^{0-1}(h)}_{\text{APP ERROR}}$$

- So $L_D^{0-1}(h_S) \leq \varepsilon_{est} + \varepsilon_{opt} + \varepsilon_{app}$ (+ BIAS SE TE VA!)

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MODEL SELECTION

LINEAR CLASSIFIER

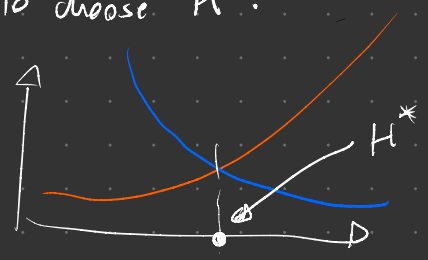
- $h_{w,b} = \langle w,x\rangle + b = \sum_{i=1}^{m} w_i x_i + b$

↳ IS LINEAR IN w AND b (NOT x)

ALSO

$h_{w,b}(x) = w_1 x_1 + w_2 x_2 + w_3 x^2 + b$ IS LINEAR IN w BUT QUADRATIC IN x

- How to choose H?



1) STRUCTURAL RISK MINIMIZATION (SRM)

NON UNIFORM LEARNABILITY

- H is not Uniform Learnable (NUL) if

$\exists A : (X \times Y)^m \to H$

$\exists m_H : (0,1)^2 \times H \to \mathbb{N}$ // NON UNIFORM w.r.t. h → DEPENDS FROM h

Then $\forall D, \forall \varepsilon, \delta \in (0,1)$, $\forall \delta \in (0,1)$, $\forall h \in H$, sampling $|S| = m$ iid sample according to D, if $m \geq m_H(\varepsilon, \delta, h)$ then

$$L_D(A(S)) \leq L_D(h) + \varepsilon \quad \text{w.p. } 1-\delta$$

COR. • H is NUL $\iff H = \bigcup_{n=1}^{\infty} H_n \mid VC(H_n)$ is finite

---

- Given the class of odd polynomials

$$H = \{ \text{SIGN}(P(x)) \mid P(x) \text{ is Polynomials} \}$$

- $VCdim(H) = +\infty$

⇒ H can be rewritten as $H = \bigcup_{n=1}^{\infty} H_n \mid VCd(H_n)$ finite

where H_n:

$$H_n = \{ \text{SIGN}(P(x)) \mid P(x) \text{ Polynomial with degree at most } n \}$$

⇒ H is NUL

CONCLUSION

- If we have a NUL-learner then we INDUCTIVELY solve the model selection problem when $H = \bigcup_n H_n$ ∧ $VCd(H_n) < \infty$ ⇒ We need an ALGO. that learns in a non uniform way (NO ERM!)

TH if $H = \bigcup_n H_n \mid H_n$ has UC Prop. ⇒ H is NUL

SRM (NOT LEARNER)

- Given $H = \bigcup_n H_n$ with hedt
- $n(h) = \min\{n \mid h \in H_n\}$ // in $H_{\text{polynomial}}$: $h(x) = \text{sign}(x^3 - 2)$ has $n(h) = 3$
- Define $\varepsilon_n : \mathbb{N} \times (0,1) \to (0,1)$

$$\varepsilon_n(m, \delta) = \min\{\varepsilon \mid m^{UC}_{H_n}(\varepsilon, \delta) \leq m\} \Rightarrow$$

$$m^{UC}_{H_n}(\varepsilon, \delta) \approx c\,\frac{VCd(H_n) - \ln(1/\delta)}{\varepsilon^2}$$

$$\Rightarrow \min\left\{\varepsilon \mid \varepsilon \geq \sqrt{c\,\frac{VCd(H_n) - \ln(1/\delta)}{m}}\right\} = \varepsilon_n(m, \delta)$$

- Introduce a weight for each H_n $\begin{cases} \cdot\ H_n \to \omega_n \geq 0 \\ \cdot\ \sum_n \omega_n \leq 1 \end{cases}$

IN PRACTICE IF $VC(H_n) \leq VC(H_{n+1})$ we give weights $\omega_n \geq \omega_{n+1}$

- We know that $|L_D(h) - L_S(h)| \leq \varepsilon_n(m, \delta) \leq \varepsilon_n(m, \omega_n \delta)$
- $L_D(h) \leq L_S(h) + \min_{n \mid h \in H_n} \varepsilon_n(m, \omega_n \delta)$ ⇒ SRM WANT TO MINIMIZE THIS QUANTITATY

$$\Rightarrow \min_{h \in H} L_S(h) + \sqrt{c\,\frac{d_{n(h)} + \ln(1/\delta\,\omega_n)}{m}}$$

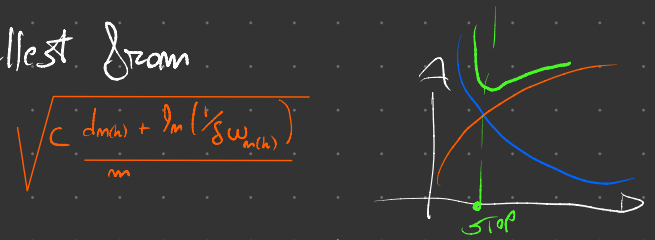
SRM ALGO. MINIMIZZA QUESTA MERDA

QUESTO TERMINE TI IMPEDISCE DI PRENDERE CLASSI TROPPO COMPLESSE

Apply SRM as an algo

- Apply ERM_{H_n} $\forall n$ ⇒ we have reduced SRM to ERM, so if ERM_{H_n} is Polynomial $\forall n$, then SRM is too.
- We take the smallest $\hat{h}_{SRM}$

$$\min_{h \in H} L_S(h) + \sqrt{c\,\frac{d_{n(h)} + \ln(1/\delta\,\omega_{n(h)})}{m}}$$



we know that the quantity decrease until it start to increase. when it does we stop

ASSIGN WEIGHTS

① way

- $h \to |h|$ = # bits to code h
- $H = \bigcup_n \{h_n\}$ $\quad w_n = 1/2^{|h_n|}$

- $\varepsilon_n(m, \delta) = \min\{\varepsilon \mid m^{UC}_{H_n}(\varepsilon, \delta) \le m\}$

$$m^{UC}_{H_n} = \frac{\log|H_n| + \log 1/\delta}{\varepsilon^2} = \frac{0 + \log 1/\delta}{\varepsilon^2} \Rightarrow \varepsilon_n(m,\delta) = \sqrt{\frac{\log(1/\delta)}{m}}$$

(composed by one function $H_n = \{h_n\}$)

② way

$$\Rightarrow \min_h L_S(h) + \sqrt{\frac{\log 1/\delta}{m}} = \min_h L_S(h) + \sqrt{\frac{\log|h| + \log 1/\delta}{m}}$$

$\Rightarrow w_n \propto 1/n^2$ (proportional)

---

2) VALIDATION/TEST SET

- Split S into : TS, testS, ValidationS
  - TS → TO FIND AN HYPOTHESIS
  - testS → TEST HOW GOOD IS YOUR MODEL
  - ValidationS → TO SOLVE MODEL SELECTION PROBLEM

- Given $h = h \in \operatorname{argmin}_{h'} L_{TS}(h')$ we check if

$$L_D(h) \approx \underbrace{L_{test}(h)}_{\text{ERROR ON testS}} = \frac{1}{m_{test}} \sum_{(x,y)\in testS} \ell(h,(x,y))$$

- Then for h we look at

$$|L_{test}(h) - L_D(h)| \le \varepsilon_{test} = \sqrt{\frac{\ln 2|H| + \ln 1/\delta}{2m_{test}}}$$

- If we haven't found already h we are in model selection Problem $H_1, H_2, \dots, H_n$ so we find ERM_{H_i} $\forall i = 1, \dots, n$

- You have to choose from: $\{h_1, h_2 \dots h_m\}$
- we must compare $L_D(h_1), \_, L_D(h_m)$ but not having D we compute L over Validations

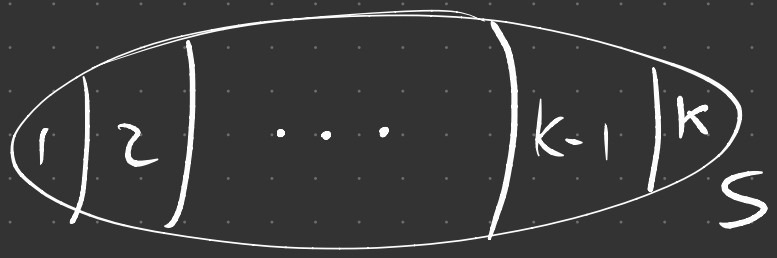
$$L_{VALS}(h_1), \_, \underbrace{L_{VALS}(h_m)}_{\frac{1}{m_{VALS}} \sum_{(x,y) \in VALS} \ell(h_m(x,y))}$$

- So $h \in \underset{i=1\_m}{\operatorname{argmin}} L_{VALS}(h)$

- We lastly check

$$|L_{Vals}(h_i) - L_D(h_i)| \leq \sqrt{\frac{\log_n(2|H|/\delta)}{2m_{Vals}}}$$

K-FOLD VALIDATION

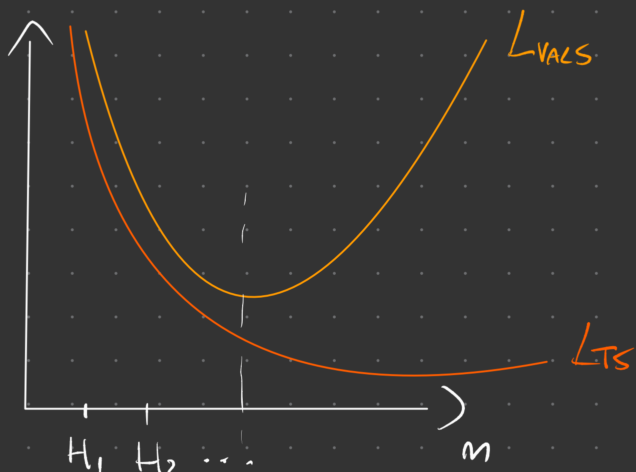


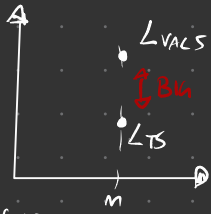
ALGO

For $i = 1, \_, m$ // m classes
  For $j = 1, \_, k$
    $h_{i,j} = ERM_{H_i}(S_j)$
    $\ell_{i,j} = L_{S_j}(h_{i,j})$
  $\ell_i = \frac{1}{k} \sum_{j=1}^{k} \ell_{i,j}$

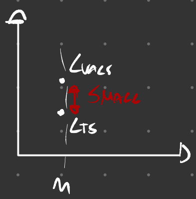
$i^* = \underset{i=1\_m}{\operatorname{argmin}} \ell_i$

$h = ERM_{H_{i^*}}(S)$

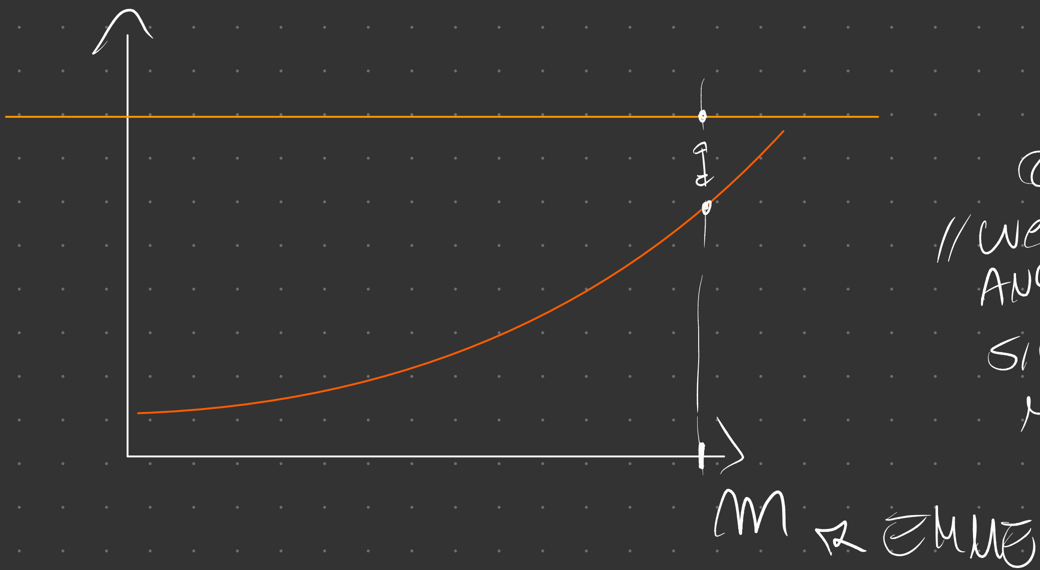




we are in overfitting
1) DECREASE m
2) AUGMENT m



if L_{TRAIN} BIG → UNDERFITTING
if L_{TRAIN} SMALL → OK!



OVERFITTING
// WE ARE NOT LEARNING ANYTHING, WE MUST SIMPLIFY THE MODEL (PICK SIMPLER CLASS)

Maria