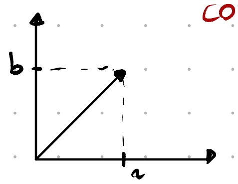
LINEAR ALGEBRA

REFERENCES

- NIELSEN & CHUANG
- QUANTUM COMPUTATION & INFORMATION
- QUANTUM MECHANICS: THE THEORETICAL QUANTUM

COMPLEX NUMBER



$z = a + jb \mid a, b \in \mathbb{R}$

$j^2 = -1$

$z^* = a - jb$

VECTOR SPACE OVER A FIELD ($\mathbb{C}$)

$v_1, v_2 \in V$

$v_1 + v_2 \in V$

$z \cdot v \in V \mid z \in \mathbb{C}, v \in V$

INNER PRODUCT

$(v, u) \in \mathbb{C}$

$(v, u + w) = (v, u) + (v, w)$

$(v, z u) = z(v, u)$

$(v, u) = \underbrace{(u, v)^*}_{\text{CONJUGATE}}$

SCALAR PRODUCT

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

$(x, y) = x \cdot y = x^T y = \sum_{i=1}^{m} x_i y_i$

$\|v\| = \sqrt{(v, v)}$

$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}$

VECTOR SPACE
+
INNER PRODUCT
=
HILBERT SPACE

$f : V \to V$ LINEAR FUNCTION

$f(z_1 v_1 + z_2 v_2) = z_1 f(v_1) + z_2 f(v_2)$

f CAN BE REPRESENTED AS A MATRIX

- BASIS $: \{u_1, \dots, u_m\} \Rightarrow \exists z_1, \dots, z_m \in \mathbb{C} \mid v = z_1 u_1 + \dots + z_m u_m \quad \forall v \in V$
- so it follow:

$$f(v_1) = z_1 f(u_1) + \dots + z_m f(u_m) = \begin{bmatrix} f(u_1) & f(u_2) & \dots & f(u_m) \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$$

EIGEN VALUES AND EIGENVECTORS

- λ is an EIGENVALUE for A w.r.t. the EIGENVECTOR $v \Leftrightarrow Av = \lambda v$
- EIGENVALUES are the solutions of $\det(A - \lambda I) = 0 \Rightarrow$ it has m solutions
- Some solutions can coincide, we say: "λ has ALGEBRAIC MULTIPLICITY #occ"
- HOW MANY INDIPENDENT EIGEN VECTOR CAN I FIND FOR λ? GEOMETRIC MULTIPLICITY

  $\hookrightarrow$ if $\lambda_1 \mid A v_{1,1} = \underset{\text{EIGENVALUE}}{\lambda_1} \overset{\text{EIGENVECTOR}}{v_{1,1}}$ then every $\alpha \lambda_1$ is valid, but dependent from λ_i
- Let's say that $\forall \lambda_i$ ALG.MULT. = GEOM. MUL.

$$\begin{array}{c|c|l} & m=5 & \\ \lambda_1 & v_{11}, \; v_{12} & \Rightarrow \text{MULT} = 2 \\ \lambda_2 & v_{21} & \Rightarrow \text{MULT} = 1 \\ \lambda_3 & v_{31} \;\; v_{32} & \Rightarrow \text{MULT} = 2 \end{array}$$

ARE BASIS FOR THE SPACE

- $f \to A$ can be rewritten as function of BASIS

$$A' = \begin{bmatrix} f(v_1) & f(v_2) & \dots \end{bmatrix} = \begin{bmatrix} v_{11}, v_{12}, v_{21}, v_{31}, v_{32} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \emptyset \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ \emptyset & & & & \lambda_3 \end{bmatrix}$$

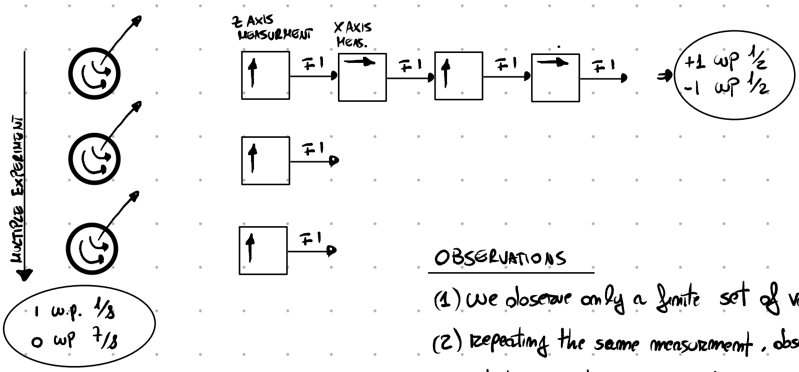
- $A \underbrace{\begin{bmatrix} v_{11}, v_{12}, v_{21}, v_{31}, v_{32} \end{bmatrix}}_{\text{THIS MATRIX IS CALLED } V} = \begin{bmatrix} v_{11}, v_{12}, v_{21}, v_{31}, v_{32} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \emptyset \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ \emptyset & & & & \lambda_3 \end{bmatrix} \Rightarrow A = VDV^{-1}$

- THE CONDITION FOR DIAGONALIZABILITY ARE:
  - if $A \in \mathbb{R}^m \Rightarrow A = A^T$
  - if $A \in \mathbb{C}^m \Rightarrow A = A^T$ AND $A^T = (A^T)^*$ // A is HERMITIAN

- $\forall i$ if ALG.MULT. = GEOM.MULT. You can find a basis made by EIGENVECTORS and you can find even an ORTHONORMAL ONE $\{u_1, u_2, \dots, u_m\} \mid (u_i, u_s) = \begin{cases} 0 & i \neq s \\ 1 & i = s \end{cases}$

INTRODUCTION

SIMPLE EXPERIMENT



OBSERVATIONS

(1) we observe only a finite set of values

(2) repeating the same measurement, observations are consistent

(3) what is maintained doing the same experiment are probabilities to observe a given result

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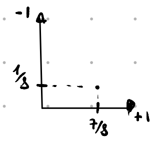
• The description of the state of a particle has to be probabilistic



$$e \rightarrow \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} = \frac{7}{8}|u\rangle + \frac{1}{8}|d\rangle = \frac{7}{8}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{8}\begin{bmatrix}0\\1\end{bmatrix}$$

electron state, up, down

• BRA-KET notation

INNER PRODUCT : $\langle u||v\rangle = \langle u|v\rangle$

$\hookrightarrow \langle u|v_1+v_2\rangle = \langle u|v_1\rangle + \langle u|v_2\rangle$

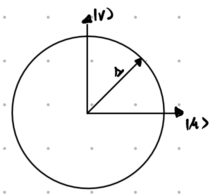
$\hookrightarrow \langle u|v\rangle = \langle v|u\rangle^*$

STATE SPACE REPRESENTATION

• The state of the electron is represented by a vector:

$$|\psi\rangle = \boxed{\alpha_u|u\rangle + \beta_d|d\rangle} \quad | \alpha_u, \beta_d \in \mathbb{C} \quad \Big| \quad \|\alpha_u\|^2 = \text{Prob. to observe } |u\rangle, \quad \|\beta_d\|^2 = \text{Prob. to observe } |d\rangle$$

• Because the sum of probs must be one, and we are in a vector space of complex number we can represent it on:



• we must assume $|u\rangle$ and $|d\rangle$ ORTHOGONAL $\langle d|u\rangle = 0 = \langle u|d\rangle$

EXTRACT PROBABILITY FRO $|\psi\rangle$

• we can so extract the probability of measuring $|u\rangle$

$$\boxed{\begin{aligned} &\langle u|\psi\rangle = \alpha_u \cancel{\langle u|u\rangle} + \beta \cancel{\langle u|d\rangle} = \alpha_u \\ &\langle u|\psi\rangle\langle\psi|u\rangle = \alpha_u^* \alpha_u = \|\alpha_u\|^2 = \text{Prob of measuring } |u\rangle \end{aligned}}$$

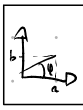
• IS $|\psi\rangle$ ENOUGH TO DESCRIBE THE SYSTEM?

• By observing σ_z we have 2 possibility $|u\rangle$ and $|d\rangle$ // UP/DOWN

• By " σ_x " " $|l\rangle$ and $|r\rangle$ // LEFT/RIGTH

• given $|\psi\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ and $\beta_l|l\rangle + \beta_r|r\rangle$:

CAN WE WRITE $|l\rangle$ AND $|r\rangle$ AS FUNCTION OF u?



$z = a + ib = \|z\| e^{i\angle z}$

$= e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$|l\rangle = \beta_u|u\rangle + \beta_d|d\rangle \quad // \quad \|\beta_u\|^2 = \|\beta_d\|^2 = 1/2$$

$$|l\rangle = \frac{1}{\sqrt{2}} e^{i\angle\beta_u}|u\rangle + \frac{1}{\sqrt{2}} e^{i\angle\beta_d}|d\rangle = \boxed{\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle}$$

ANGLE in $\mathbb{C}$

$$|r\rangle = \boxed{\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}} e^{i\varphi}|d\rangle}$$

// from $|l\rangle$ we removed PHASE so we must have it on $|r\rangle$

• Obviously $|r\rangle$ and $|l\rangle$ must be ORTHOGONAL

$$\langle r|l\rangle = 0 \Rightarrow \langle \tfrac{1}{\sqrt{2}}u + \tfrac{1}{\sqrt{2}}d \,|\, \tfrac{1}{\sqrt{2}}u + \tfrac{1}{\sqrt{2}}e^{i\varphi}d\rangle = 0$$

$$\langle u+d \,|\, u+e^{i\varphi}d\rangle = 0$$

$$\langle u | u + e^{j\varphi} d \rangle + \langle d | u + e^{j\varphi} | d \rangle = 0$$

$$\langle u | u \rangle + e^{j\varphi} \langle u | d \rangle + \langle d | u \rangle + e^{j\varphi} \langle d | d \rangle = 0$$

$$1 + 0 + 0 + e^{j\varphi} = 0$$

$$1 + e^{j\pi} = 0$$

• In conclusion we have

$$\boxed{\begin{aligned} |g\rangle &= \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}|d\rangle \\ |r\rangle &= \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}} e^{j\pi} |d\rangle = \\ &= \tfrac{1}{\sqrt{2}}|u\rangle - \tfrac{1}{\sqrt{2}}|d\rangle \end{aligned}}$$

• We can do the same for σ_y

$$|i\rangle = \gamma_u |u\rangle + \gamma_d |d\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}} e^{j \lt \gamma_d} |o\rangle$$

$$|o\rangle = \delta_u |u\rangle + \delta_d |d\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}} e^{j \lt \delta_d} |o\rangle$$

$\Rightarrow \langle i | o \rangle = 0$ // ORTHOGONAL

$$\langle u + e^{j\gamma_d} d \mid u + e^{j\delta_d} d \rangle = 0$$

$$1 + e^{j\delta_d} e^{-j\gamma_d} = 1 + e^{j(\delta_d - \gamma_d)} = 0$$

$$\Rightarrow \delta_d = \gamma_d + \pi$$

• TAKING OUT A VARIABLE FROM INNER PRODUCT

$$\langle u | \alpha v \rangle = \alpha \langle u | v \rangle$$

$$\langle \alpha u | v \rangle = (\langle v | \alpha u \rangle)^* = (\alpha \langle v | u \rangle)^* = \alpha^* \langle v | u \rangle^* = \alpha^* \langle u | v \rangle$$

EXPRESS $|i\rangle$ IN TERMS OF $|g\rangle / |r\rangle$

• So we have

$$|i\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}} e^{j\gamma_d} |d\rangle$$

$$\overset{|r\rangle}{|g\rangle} \; \tfrac{1}{\sqrt{2}}|u\rangle \pm \tfrac{1}{\sqrt{2}} |d\rangle$$

$\Rightarrow$ we know that $P(\text{to observ } |\overset{r}{g}\rangle) = \langle i | \overset{r}{g} \rangle \langle \overset{r}{g} | i \rangle = 1/2$

$$\Rightarrow \langle i | \overset{r}{g} \rangle = \tfrac{1}{2} \cdot 1 \pm \tfrac{1}{2} e^{-j\gamma_d} \cdot 1$$

$$\langle \overset{r}{g} | i \rangle = \tfrac{1}{2} \pm \tfrac{1}{2} e^{j\gamma_d}$$

$$\Rightarrow \cancel{\tfrac{1}{2}} (1 \pm e^{-j\gamma}) \tfrac{1}{2} (1 \pm e^{j\gamma_d}) = \cancel{\tfrac{1}{2}}$$

$$(1 + 1 \pm 2\cos \lt \gamma_d) \tfrac{1}{2} = 1$$

$\Rightarrow$ we know $\cos \lt \gamma_d = 0 \;\wedge\; \sin \lt \gamma_d = 1 \Rightarrow \gamma_d = \mp \frac{\pi}{2}$

$$\boxed{\begin{aligned} \gamma_d &= \tfrac{\pi}{2} \\ \delta_d &= -\tfrac{\pi}{2} \end{aligned}}$$

• In conclusion:

$$\boxed{\begin{aligned} &\bullet\ |i\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}\, j\, |d\rangle \\ &\bullet\ |o\rangle = \tfrac{1}{\sqrt{2}}|u\rangle - \tfrac{1}{\sqrt{2}}\, j\, |d\rangle \end{aligned}}$$

• OUR ELECTRON STATE is totally characterized by $\alpha_u|u\rangle + \alpha_d|d\rangle$

$\boxed{\text{QUBIT } |\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$

OBSERVABLE AND MEASURMENT

QUANTUM MECHANICS PRINCIPLE

• OBSERVABLE: something measurable

• After a measurement the system is no more observable

HAVE INTEGER EIGENVALUES

① All observables can be expressed by HERMITIAN (LINEAR) OPERATORS $L \in \mathbb{C}^{m\times m} \,|\, L = L^T$

② The measurement results are the EIGENVALUES → $|0\rangle, |1\rangle$

*③ The Probability to measure λ_i when in state $|\psi\rangle$ is $\sum_{v \in V(\lambda_i)} \langle\psi|v\rangle\langle v|\psi\rangle$

TAKING ORTHONORMAL BASIS MADE BY EIGENVECTORS

$V = \bigcup_{\lambda_i} V(\lambda_i)$

UNION

So Prob to observe λ_i in state $|\psi\rangle$ is

$\sum_{v \in V(\lambda_i)} \langle\psi|v\rangle\langle v|\psi\rangle$

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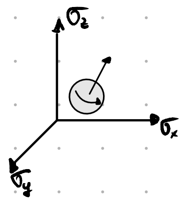
DERIVATION OF PAULI'S GATES

REPRESENTING HERMITIAN OF σ_z



$L_{\sigma_z} = [2 \times 2]$, $\lambda(L_{\sigma_z}) = \{+1, -1\}$, BASIS $= \{|u\rangle, |v\rangle\}$

• we need 2 EIGENVECTORS $|V_1\rangle, |V_2\rangle$ (one for each $\lambda(L_{\sigma_z})$)

① $L_{\sigma_z}|V_1\rangle = +|V_1\rangle \Rightarrow |V_1\rangle = |u\rangle$

② $L_{\sigma_z}|V_{-1}\rangle = -1|V_{-1}\rangle \Rightarrow |V_{-1}\rangle = |d\rangle$

IS $|u\rangle$ AN EIGENVECTOR OF +1?

Proof

• Any vector can be represented by a linear combination of $|V_1\rangle$ and $|V_{-1}\rangle$

• even $|u\rangle$ can be:

$$|u\rangle = \alpha_1|V_1\rangle + \alpha_{-1}|V_{-1}\rangle$$

• so $L_{\sigma_z}|u\rangle = \alpha_1 L_{\sigma_z}|V_1\rangle + \alpha_{-1} L_{\sigma_z}|V_{-1}\rangle = \alpha_1|V_1\rangle - \alpha_{-1}|V_{-1}\rangle$

• we know that in state $|u\rangle$ we can observe only +1 so $\alpha_{-1} = 0$

• it implies that $|u\rangle$ is an EIGENVECTOR relative to the EIGENVALUE +1 ☐

• By the proof $|u\rangle$ and $|d\rangle$ are our basis because $\underbrace{L_{\sigma_z}}_{\text{EIGENVECT RELATIVE TO +1}}|u\rangle = |u\rangle$ $\begin{bmatrix}1\\0\end{bmatrix}$ AND $\underbrace{L_{\sigma_z}}_{\text{EIGENVECTOR RELATIVE TO -1}}|d\rangle = -|d\rangle$ $\begin{bmatrix}0\\1\end{bmatrix}$

• So we can fill L_{σ_z}:

$$L_{\sigma_z} = \begin{bmatrix} (L_{\sigma_z})_{11} & (L_{\sigma_z})_{12} \\ (L_{\sigma_z})_{21} & (L_{\sigma_z})_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

// REPRESENTATION OF σ_z w.r.t. $|u\rangle, |d\rangle$

Representing horizontal for σ_x (w.r.t $|u\rangle, |d\rangle$)

- $|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$

$|r\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

- $|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$

why $\begin{bmatrix}1\\1\end{bmatrix}$?

$$\Rightarrow \begin{cases} L_{\sigma_x}|r\rangle = +|r\rangle \\ L_{\sigma_x}|l\rangle = -|l\rangle \end{cases} \Rightarrow \begin{cases} \frac{1}{\sqrt{2}} L_{\sigma_x}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\ \frac{1}{\sqrt{2}} L_{\sigma_x}\begin{bmatrix}1\\-1\end{bmatrix} = -\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \end{cases} \Rightarrow \begin{cases} (L_{\sigma_x})_{11} + (L_{\sigma_x})_{12} = 1 \\ (L_{\sigma_x})_{21} + (L_{\sigma_x})_{22} = 1 \\ (L_{\sigma_x})_{11} - (L_{\sigma_x})_{12} = -1 \\ (L_{\sigma_x})_{21} - (L_{\sigma_x})_{22} = +1 \end{cases} \Rightarrow L_{\sigma_x} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

L_{σ_y} w.r.t $|u\rangle, |d\rangle$

$$L_{\sigma_x} = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

MULTI QUBITS SYSTEM

- Given 2 independent electrons $|\varphi_1\rangle = \overset{|0\rangle}{\alpha_1}|u_1\rangle + \overset{|1\rangle}{\beta_1}|d_1\rangle$, $|\varphi_2\rangle = \overset{|0\rangle}{\alpha_2}|u_2\rangle + \overset{|1\rangle}{\beta_2}|d_2\rangle$
- We can describe them jointly (a BASIC STATE $|u_1 u_2\rangle, |u_1 d_2\rangle, |d_1 u_2\rangle, |d_1 d_2\rangle$)
  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

Product State $|\varphi_1\rangle \otimes |\varphi_2\rangle$

$$|\varphi_1\rangle \otimes |\varphi_2\rangle = |\varphi_1 \varphi_2\rangle = \alpha_1\alpha_2 \; |u_1 u_2\rangle + \alpha_1 \beta_2 \; |u_1 d_2\rangle + \beta_1 \alpha_2 |d_1 u_2\rangle + \beta_1\beta_2 |d_1 d_2\rangle$$

- So the Probability of measuring:

$$P(|u_1 u_2\rangle) = |\alpha_1 \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2$$

$\Rightarrow$ The joint state of INDEPENDENT QUBIT is the PRODUCT STATE (we can just multiply AMPLITUDES)

$$|\varphi_1\rangle \otimes |\varphi_2\rangle = \sum_{y,x \in \{0,1\}} \alpha_{1x}\alpha_{2y}|xy\rangle$$

- What if 2 electron are entangled?

$$|\varphi_1 \varphi_2\rangle = \alpha_{uu}|u_1 u_2\rangle + \alpha_{ud}|u_1 d_2\rangle + \alpha_{du}|d_1 u_2\rangle + \alpha_{dd}|d_1 d_2\rangle$$

- We require that $|\alpha_{uu}|^2 + |\alpha_{ud}|^2 + |\alpha_{du}|^2 + |\alpha_{dd}|^2 = 1$

- Suppose to observe φ_2 in $|\varphi_1 \varphi_2\rangle$, and get $|u_2\rangle$ we can then describe it by u_2's coefficient
  we must normalize because in the origin the probability 1 was given by summing all α, now we must scale $|\alpha_{uu}|^2$ and $|\alpha_{du}|^2$ to have 1 as sum.

Cause $|\alpha_{uu}|^2 + |\alpha_{ud}|^2 + |\alpha_{du}|^2 + |\alpha_{dd}|^2 = 1$

$\cancel{|\varphi_1\rangle = \alpha_{uu}|u_1\rangle + \alpha_{du}|d_1\rangle}$ $\Rightarrow$ we have $|\alpha_{uu}|^2 + |\alpha_{du}|^2 < 1$ so:

$$\boxed{|\varphi_1\rangle = \frac{\alpha_{uu}}{\sqrt{|\alpha_{uu}|^2 + |\alpha_{du}|^2}}|u_1\rangle + \frac{\alpha_{du}}{\sqrt{|\alpha_{uu}|^2 + |\alpha_{du}|^2}}|d_1\rangle}$$

- 2 QUBITS are MAXIMALLY ENTANGLED if $|\varphi_1 \varphi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, so they collapse to the same state.

• WE HAVE SEEN that given $|\varphi_1\rangle = \alpha_1|\mu_1\rangle + \beta_1|d_1\rangle$, $|\varphi_2\rangle = \alpha_2|\mu_2\rangle + \beta_2|\mu_2\rangle$

PRODUCT STATE

$$|\varphi_1\rangle \otimes |\varphi_2\rangle = |\varphi_1\varphi_2\rangle = \alpha_1\alpha_2\,|\mu_1\mu_2\rangle + \alpha_1\beta_2\,|\mu_1 d_2\rangle + \beta_1\alpha_2|d_1\mu_2\rangle + \beta_1\beta_2|d_1 d_2\rangle$$

• if we measure $|\varphi_2\rangle$ to describe $|\varphi_1\rangle$ we should apply it's STARTING FORMULA
• proof it by applying the entangled formula:

$$|\varphi_1\rangle = \frac{\alpha_1\cancel{\alpha_2}}{\cancel{\sqrt{|\alpha_1\alpha_2|^2+|\beta_1\alpha_2|^2}}_{=\alpha_2}}|\mu_1\rangle + \frac{\beta_1\cancel{\alpha_2}}{\cancel{\sqrt{|\alpha_1\alpha_2|^2+|\beta_1\alpha_2|^2}}_{=\alpha_2}}|d_1\rangle = \alpha_1|\mu_1\rangle + \beta_1|d_1\rangle$$

$\boxtimes$ $|\varphi_1\varphi_2\rangle = \frac{1}{\sqrt{2}}\left(|\mu_1\mu_2\rangle + |d_1 d_2\rangle\right)$

• measure $|\varphi_2\rangle \to |\mu_2\rangle$ then $|\varphi_1\rangle = \frac{\cancel{1/\sqrt{2}}}{\cancel{\sqrt{(1/\sqrt{2})^2}}}|\mu_1\rangle = |\mu_1\rangle$ $\boxtimes$

BELL STATE (EPR)

• MAXIMIZED ENTANGLED STATES

$00 : \|\psi\rangle \to \frac{\|00\rangle + \|11\rangle}{\sqrt{2}}$	$10 : \|\psi\rangle \to \frac{\|10\rangle + \|01\rangle}{\sqrt{2}}$
$01 : \|\psi\rangle \to \frac{\|00\rangle - \|11\rangle}{\sqrt{2}}$	$11 : \|\psi\rangle \to \frac{\|01\rangle - \|10\rangle}{\sqrt{2}}.$

QUANTUM GATES

• A gate transforms QBIT

$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\qquad$ $|\varphi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$

$$|\varphi\rangle \to \boxed{G} \to |\varphi'\rangle$$

$|\varphi'\rangle = G|\varphi\rangle$

• G must preserve the NORM.

$$\||\varphi\rangle\|^2 = (|\varphi\rangle)^\dagger|\varphi\rangle = \langle\varphi|\varphi\rangle = 1$$

$$\|G|\varphi\rangle\|^2 = (G|\varphi\rangle)^\dagger G|\varphi\rangle = (\langle\varphi|G^\dagger)G|\varphi\rangle =$$
$$= \langle\varphi|G^\dagger G|\varphi\rangle = 1$$

• To have $\langle\varphi|G^\dagger G|\varphi\rangle \to$ we must delete $G^\dagger G$ so G must be **UNITARY** $\left(G^\dagger = G^{-1}\right)$

SINGLE QUBIT GATES

- **QUANTUM WIRE** $|\psi\rangle$ ——— $|\psi\rangle$
- **PAULI'S GATE**

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}$$

X → NOT-GATE

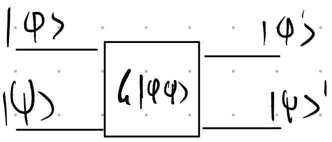
- **HADAMARD GATE**

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Usefull because from a unique state to a superposition
- $H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$
- $H|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

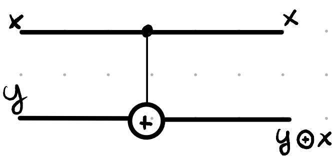
- $H|0\rangle = |+\rangle$
- $H|1\rangle = |-\rangle$

MULTI QBIT GATES



$$G(|\psi\varphi\rangle) = G(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle) =$$
$$= \alpha_{00}\, G(|00\rangle) + \alpha_{01}\, G(|01\rangle) + \alpha_{10}\, G(|10\rangle) + \alpha_{11}\, G(|11\rangle)$$

CNOT

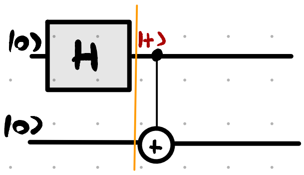


- if x is $|1\rangle$ then the not is done

- Combined with HADAMARD can generate BELL PAIRS

$|0\rangle \otimes |+\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =$

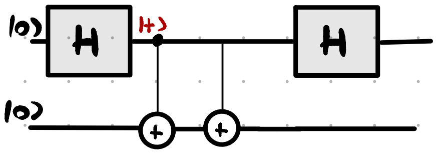
$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ // COMBINED STATE AFTER H



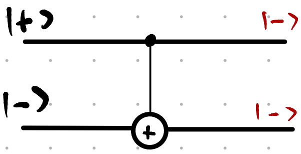
$\text{CNOT}(|0\rangle \otimes |+\rangle) = \text{CNOT}\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right) = \frac{1}{\sqrt{2}}\left(\underbrace{\text{CNOT}(|00\rangle)}_{|00\rangle} + \underbrace{\text{CNOT}(|10\rangle)}_{|11\rangle}\right) =$

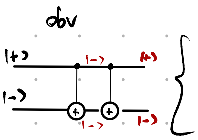
$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

- To invert the combination wit H:



- Anothe example:

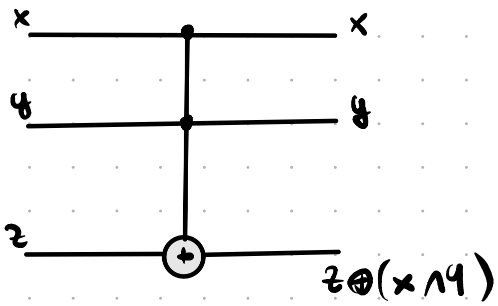




- $|+\rangle \otimes |-\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$
- $G(|+\rangle \otimes |-\rangle) = \frac{1}{2}(|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle)$

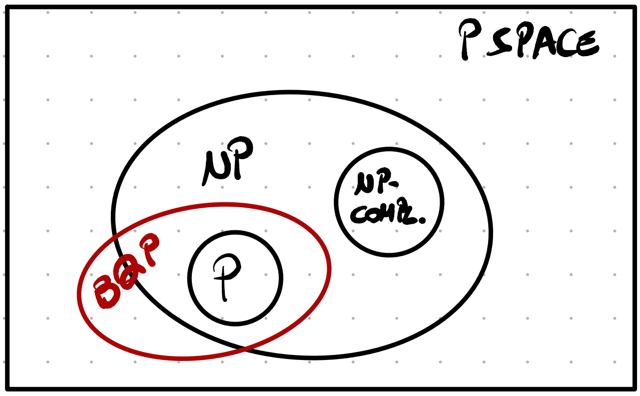
→ So if entangled the second QUBIT can modify the other one

TOFFOLI GATE



- With this gate we can generate every classical binary gate
  - ↳ NAND | $z = 1 \Rightarrow \neg(x \wedge y)$
  - ↳ FANOUT (out x) | $z = 0,\ y = 1$

BQP (BOUNDED ERROR QUANTUM POLYNOMIAL)



QUANTUM WALKS ON GRAPHS

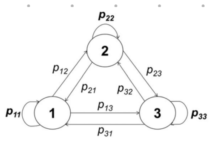
DISCRETE-TIME MARKOV PROCESS (MARKOV CHAIN)

- Sequence of random variables X_n s.t. the value of X_{n+1} depends only from the one of X_n

$$P(X_{i+1}=V_{i+1} \mid X_i=V_i, \_, X_0=V_0) = P(X_{i+1}=V_{i+1} \mid X_i=V_i)$$

- If the RHS not depends from i $\Rightarrow$ HOMOGENEOUS MARKOV CHAIN

S_1 = HAPPY
S_2 = SLEEPY
S_3 = ANGRY



$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$\Rightarrow P(X_{n+1}=S_3) = P(X_{n+1}=S_3 \mid X_n=S_1)P(X_n=S_1) + P(X_{n+1}=S_3 \mid X_n=S_2)P(X_n=S_2) + P(X_{n+1}=S_3 \mid X_n=S_3)P(X_n=S_3)$$

- IRREDUCIBLE MC: If from any state S_i exist a path to any S_j
- APERIODIC MC: If for any S_i, S_j $P(X_n=S_j \mid X_0=S_i)>0$ for all sufficently large n
- ERGODIC: If is BOTH

STATIONARY DISTRIBUTION

- π is a probability distribution that remains unchanged in M.C. as time progresses.
- Is a vector whose entries are probs summing up to 1.
- Given the matrix P:

$$\pi = \pi P \quad // \pi \text{ invariant by } P$$

Given P find π

The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{pmatrix}.$$

The transpose of this matrix has eigenvalues satisfying the equation

$$\det\begin{pmatrix} 0.5-\lambda & 0.25 & 0 \\ 0.5 & 0.5-\lambda & 0.5 \\ 0 & 0.25 & 0.5-\lambda \end{pmatrix} = 0.$$

It follows that $(0.5-\lambda)^3 - 0.125(0.5-\lambda) - 0.125(0.5-\lambda) = (0.5-\lambda)(\lambda^2-\lambda) = 0$. So the eigenvalues are $\lambda=0$, $\lambda=0.5$, and $\lambda=1$. The eigenvalue $\lambda=0$ gives rise to the eigenvector $(1,-2,1)$, the eigenvalue $\lambda=0.5$ gives rise to the eigenvector $(-1,0,1)$, and the eigenvalue $\lambda=1$ gives rise to the eigenvector $(1,2,1)$. The only possible candidate for a stationary distribution is the final eigenvector, as all others include negative values.

Then, the stationary distribution must be $\frac{1}{1+2+1}\cdot(1,2,1) = (\frac{1}{4},\frac{1}{2},\frac{1}{4})$. □

- REVERSIBLE: $\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i,j \in S$

RANDOM WALKS ON GRAPHS

- Let G be UNDIRECTED and WEIGHTED
- $\{X_n\}_{n\geq 0}$ is a RANDOM WALK moving from $i \in V$ to it's neighbours according to the distribution

$$P(X_{n+1}=j \mid X_n=i) = \begin{cases} w_{ij} / \underbrace{\boxed{\sum_{k \sim i} w_{ij}}}_{d_i} & , \text{if } j \in N(i) \\ 0 \end{cases}$$

- Random walks on G are <u>reversible</u>

$$\boxed{\pi_i = d_i / 2m}$$

FIRST RET TIME

- $T_i = \min\{n>0 \mid X_n=i, X_0=i\} \Rightarrow E[T_i] = \frac{1}{\pi_i} = \frac{2m}{d_i}$

DISCRETE-TIME COINED QUANTUM RANDOM WALK

- STATE $|c\rangle|k\rangle$ that EVOLVES as follow
  $\underbrace{|c\rangle}_{\text{COIN}}$

  (1) Apply H to coin state
  (2) Apply shift operator
  $\Rightarrow \begin{cases} S\,|0\rangle|k\rangle = |0\rangle|k-1\rangle \\ S\,|1\rangle|k\rangle = |1\rangle|k+1\rangle \end{cases}$
  (3) Repeat or measure

- EVOLUTION can be described by $U = S(H \otimes I)$:

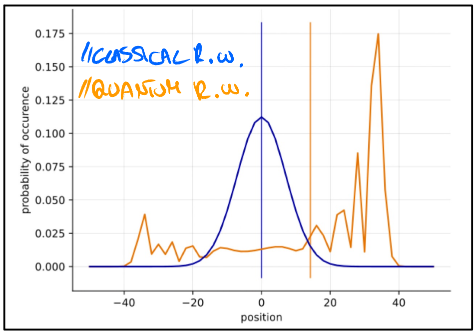
$$\boxed{|\psi(t+1)\rangle = U\,|\psi(t)\rangle = U^{t+1}\,|\psi(0)\rangle}$$

- Starting from $|\psi(0)\rangle = |0\rangle \otimes |0\rangle$:

$$\boxed{\begin{aligned} &|0\rangle \otimes |0\rangle \xrightarrow{H\otimes I} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \\ &\xrightarrow{S} \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |-1\rangle). \end{aligned}}$$

- Measuring after one step we get $k=1$ or $k=-1$ with $1/2$ prob.
- Delaying the measure brings us to :

$P[X_t = k]$



Some deficiencies of coined quantum walk:

- The coined quantum walk requires auxiliary degrees of freedom;
- The coined quantum walk can only be generalized to regular graphs;
- No natural definition of hitting times.

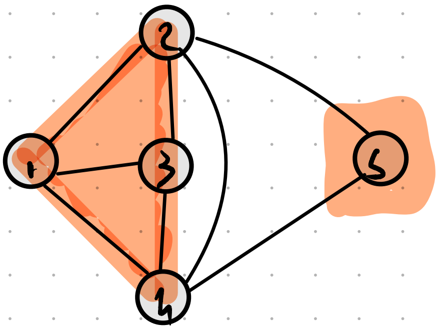
PORTUGAL STAGGERED MODEL

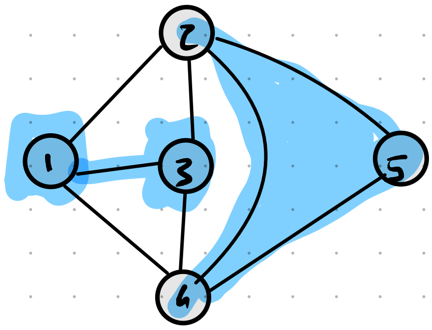
CLIQUE TASSELATION

- Υ is a partition of V into cliques
- $(i,j) \in E(\Upsilon) \Leftrightarrow i,j \in$ same CLIQUES

CLIQUE TASSELATION COVER

- set $\{T_1, \dots, T_k\} \mid E(T_1) \cup \dots \cup E(T_k) = E(G)$

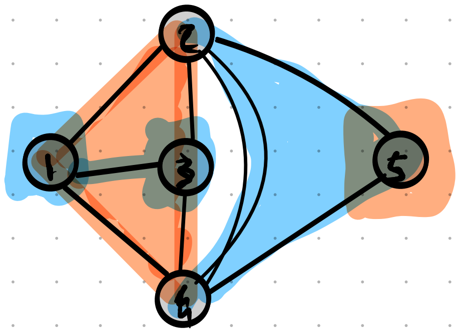




$T_1 = \{\{1,2,3,4\}\ \{5\}\}$

$T_2 = \{\{1,3\}, \{2,4,5\}\}$

$T_1 \cup T_2 = E(G)$

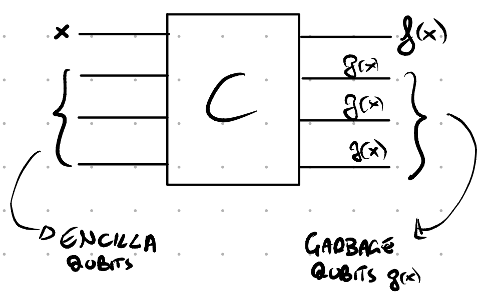


Construction of Portugal's staggered walk model for undirected graphs:

1. Construct a minimal tessellation cover of size k;
2. For each tile α_j construct a unit vector $|\alpha_j\rangle = \frac{1}{\sqrt{|\alpha_j|}} \sum_{v \in \alpha_j} |v\rangle$;
3. For each tessellation $\mathcal{T}_i, i = 1, ..., k$ construct an associate local, Hermitian, unitary operator $H_i = 2 \sum_{j=1}^{p_i} |\alpha_j\rangle \langle\alpha_j| - I$;
4. Take as an evolution operator either $U = H_k \cdots H_1$, or $U = \exp(i\theta_1 H_1) \cdots \exp(i\theta_k H_k)$, with some angles (time-steps) $\theta_i, i = 1, ..., k$.

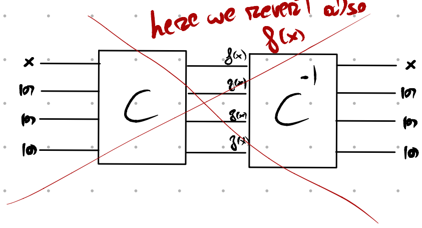
REVERSIBLE GATES/CIRCUIT

- Gates s.t. $GG = I$



- There is a way to revert only garbage qubit?

here we revert also $f(x)$

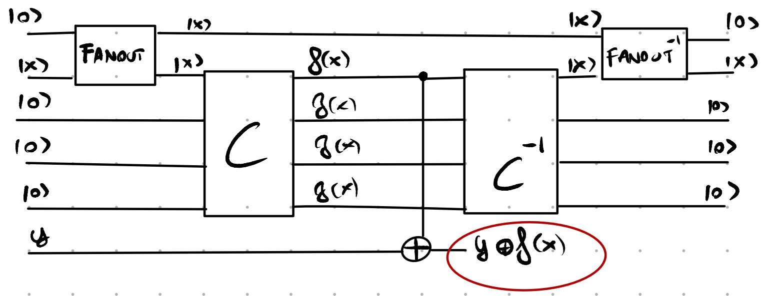


$\Rightarrow$ we'll see

LANDAUER'S PRINCIPLE

- Erasing a 'BIT' consumes at least $K_B T \ln 2$ Joules where $K_B \approx 1.38 \times 10^{-23}\ J/K$

UNCOMPUTATION

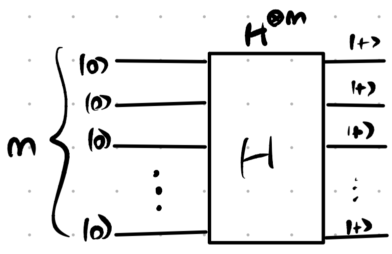


GROVER'S ALGORITHM [96']

- Search in UNSTRUCTURED SPACE of size N
- COMPUTATIONAL COMPLEXITY $O(\sqrt{n})$
- Need an ORACLE that check the correctness of a solution.

$$f(x) = \begin{cases} 1 & \text{, if } x \text{ is the solution}^*: \omega \\ 0 & \text{, else} \end{cases}$$

- How many qubit are needed to store N states? $\Rightarrow \log(N)$
- Encode the search space in n qubits, must be a qubit encoding ω : $|\omega\rangle$
- Apply HADAMARD to have all possible solutions:



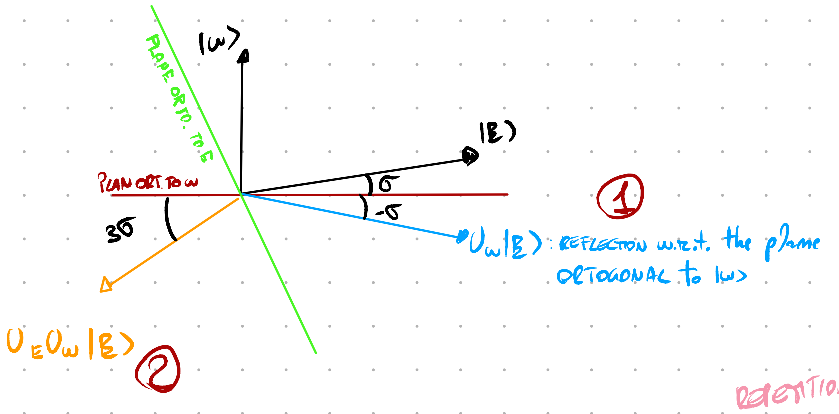
$|\omega\rangle$ is here

$$|+\rangle^{\otimes n} = \frac{1}{(\sqrt{2})^n}\left(|00\ldots0\rangle + |0\ldots1\rangle + \ldots + |11\ldots1\rangle\right)$$

called $|B\rangle$ state ("everywhere")

- The relation between $|E\rangle$ and $|w\rangle$ is that measuring $|E\rangle$ we can get w with prob. $1/(\sqrt{2})^n$
- But thinking about them as vectors, we know that $|E\rangle$ is almost ortogonal to $|w\rangle$

BECAUSE THERE IS A BIT OF $|w\rangle$ IN $|E\rangle$



// Repeating it we increase the probability of measuring $|w\rangle$

- How many repetition of $U_E U_v$? $\theta - r\,2\theta \approx \pi/2 \Rightarrow r = \frac{\theta - \pi/2}{2\theta} = \frac{1}{2} - \frac{\pi}{4\theta}$

REPETITIONS

$$\left(\sin(\theta)^2 = 1/2^n = 1/N \Rightarrow \sin(\theta) = 1/\sqrt{N} \Rightarrow \theta \approx 1/\sqrt{2} \right) \Rightarrow r \approx \frac{\pi}{4}\sqrt{n}$$

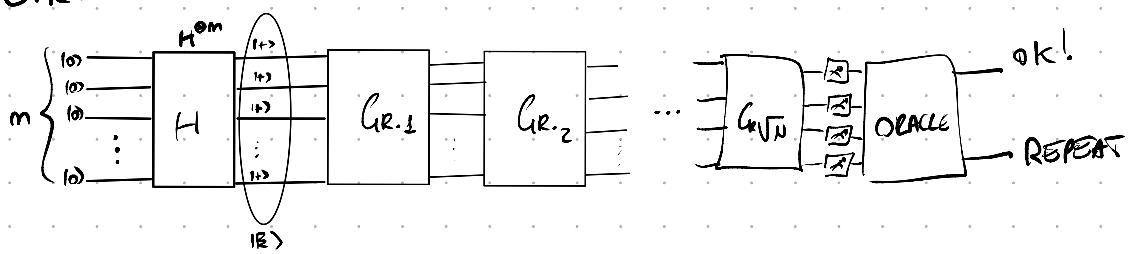
- After $r = O(\sqrt{n})$ operation the vector is almost aligned with $|w\rangle$
- You can stop when the angle between $|w\rangle$ and $|v\rangle$ is at most θ

- Measure $|v\rangle$ what is the prob of finding $|w\rangle$?

$\Rightarrow (\cos\theta')^2 \geq \cos^2\theta = 1 - \sin^2\theta \approx 1 - 1/N$

- On average we need to run the algo $\boxed{\frac{N}{N-1} \text{ times}}$

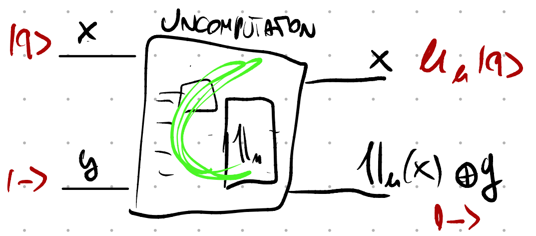
- CIRCUIT



- How we ROTATE?

U_w *

Vogliamo riflettere $|q\rangle$ sul piano ortogonale a $|w\rangle$



$$1\!\!1_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{else} \end{cases}$$

ORACLE

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha'|\mu\rangle + \beta'|\mu_\perp\rangle$$

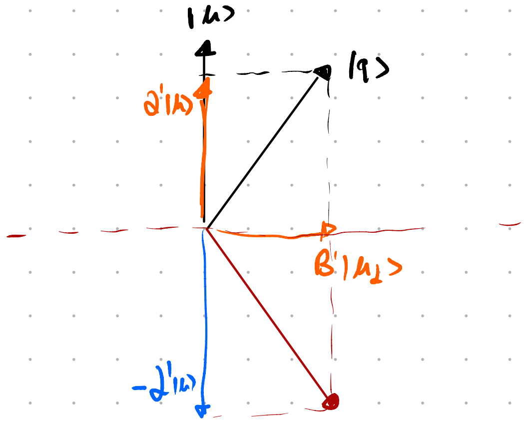
$$|q-\rangle = |q0\rangle - |q1\rangle = \alpha'|\mu 0\rangle + \beta'|\mu_\perp 0\rangle - \alpha'|\mu 1\rangle - \beta'|\mu_\perp 1\rangle$$

$$\xrightarrow{C} \alpha'|\mu 1\rangle + \beta'|\mu_\perp 0\rangle - \alpha'|\mu 0\rangle - \beta'|\mu_\perp 1\rangle =$$

$$= (\beta'|\mu_\perp\rangle - \alpha'|\mu\rangle)|0\rangle \; (\alpha'|\mu\rangle - \beta'|\mu_\perp\rangle)|1\rangle =$$

$$= \beta'|\mu_\perp\rangle - \alpha'|\mu\rangle)(|0\rangle - |1\rangle) =$$

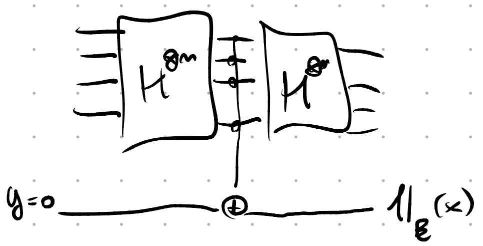
$$= \underbrace{(\beta'|\mu_\perp\rangle - \alpha'|\mu\rangle)|-\rangle}_{R}$$



U_E

⇒ Ruotare su |w⟩ è easy perché abbiamo un oracolo

- How do we rotate on the ortogonal plane of |E⟩?
- We need an oracle for |E⟩

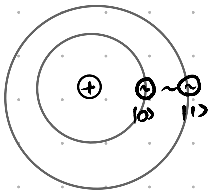


- If we give |E⟩ to this circuit we get |E⟩ in output

* • Put this oracle in the circuit

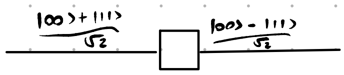
DECOHERENCE

- A qubit, if left alone, can spontaneously change state.



AMPLITUDE DAMPENING ⇒ Change from |0⟩ to |1⟩ or viceversa

PHASE DAMPENING ⇒



- suppose having |q⟩ and after 1h it will be $\begin{cases} |q\rangle \text{ w.p. } 0.9 \\ |q'\rangle \text{ w.p. } 0.1 \end{cases}$

↳ TEMPTATION $\cancel{\sqrt{0.9}\,|q\rangle + \sqrt{0.1}\,|q'\rangle}$ NO because this express my lack of knowledge, the electron knows where it is

- So: $|q\rangle = \alpha|0\rangle + \beta|1\rangle$ w.p. 0.9

$|q'\rangle = \alpha'|0\rangle + \beta'|1\rangle$ w.p. 0.1

- prob. of observing 0? $\Rightarrow$ $\boxed{0.9|\alpha|^2 + 0.1|\alpha'|^2}$
- prob. of observing 1? $\Rightarrow$ $\boxed{0.9|\beta|^2 + 0.1|\beta'|^2}$

↳ TEMPTATION

$$\cancel{\sqrt{0.9}\,|q\rangle + \sqrt{0.1}\,|q'\rangle = (\sqrt{0.9}\,\alpha + \sqrt{0.1}\,\beta)|0\rangle + (\sqrt{0.9}\,\alpha' + \sqrt{0.1}\,\beta')|1\rangle}$$

PROB to observe 0?

$$\cancel{|\sqrt{0.9}\,\alpha + \sqrt{0.1}\,\alpha'|^2}$$

DENSITY MATRIX

- $$\rho = |q\rangle\langle q| = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}[\alpha^* \; \beta^*] = \begin{bmatrix}|\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2\end{bmatrix}$$

- We must mix dens. matr. of $|q\rangle$ and $|q'\rangle$

$$\begin{cases}|q\rangle & \text{w.p. } p \\ |q'\rangle & \text{w.p. } (1-p)\end{cases} \Rightarrow \rho = p|q\rangle\langle q| + (1-p)|q'\rangle\langle q'|$$

$$= \begin{bmatrix}p|\alpha|^2 + (1-p)|\alpha'|^2 & \cdots \\ \cdots & p|\beta|^2 + (1-p)|\beta'|^2\end{bmatrix}$$

↳ MIXED STATE (when I don't know where they are)

↳ All we can measure is on the DIAGONAL.

①

$$\rho = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|$$

②

$$\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle = |+\rangle$$

1 == 2 ? $\Rightarrow$ Apply H

① $\rightarrow$ $\begin{cases}\text{wp } \tfrac{1}{2} & H|0\rangle = |+\rangle \\ \text{wp } \tfrac{1}{2} & H|1\rangle = |-\rangle\end{cases}$ $\Rightarrow$ ① $\neq$ ② !

② $\rightarrow$ wp 1 $\quad H|+\rangle = 0$

FIDELITY

- express the probability that a state still in the original state

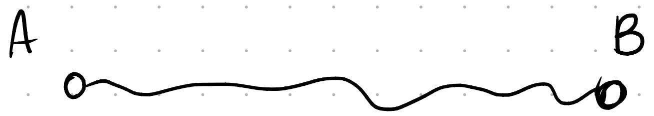
$$|q\rangle \rightarrow \rho = p\,|q\rangle\langle q| + p'\,|q'\rangle\langle q'| + p''\,|q''\rangle\langle q''|$$

- FIDELITY $= p = \boxed{\langle q|\,\rho\,|q\rangle}$

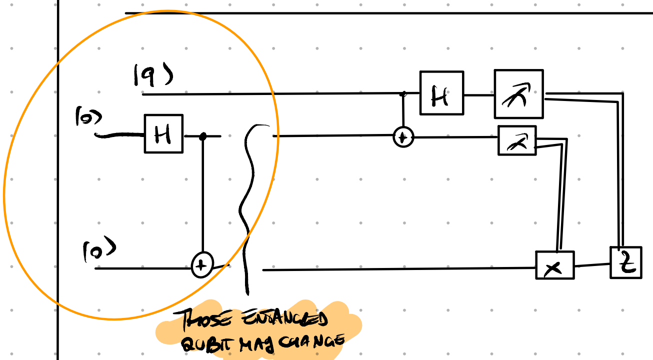
QUBIT TRANSMISSION

- No reliability ⇒ Normally we copy locally the information to do it.
  with qubit the copy would be linked forever to the original.

⇒ SOL: ENTANGLEMENT



$$|q_A q_B\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



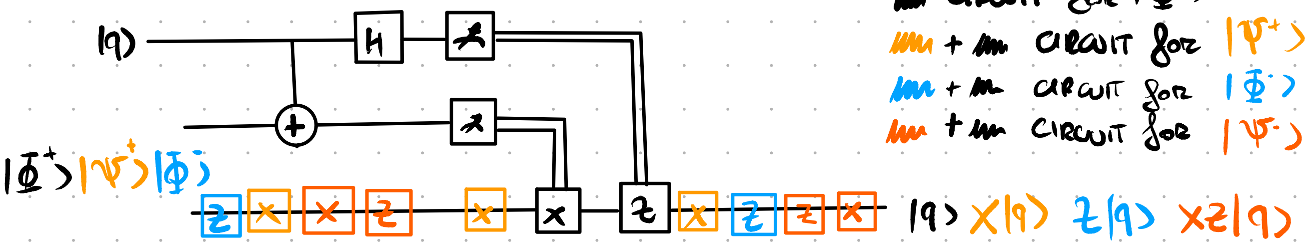
$\Phi^+ = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad \Phi^- = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

$\Psi^+ = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ $\quad \Psi^- = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

So we represent this by the DENSITY MATRIX

$$\rho = A|\Phi^+\rangle\langle\Phi^+| + B|\Psi^+\rangle\langle\Psi^+| + C|\Psi^-\rangle\langle\Psi^-| + D|\Phi^-\rangle\langle\Phi^-|$$

TELEPORTATION



CIRCUIT for $|\Phi^+\rangle$
+ CIRCUIT for $|\Psi^+\rangle$
+ CIRCUIT for $|\Phi^-\rangle$
+ CIRCUIT for $|\Psi^-\rangle$

- $IX|\Psi^+\rangle = |\Phi^+\rangle$
- $IZ|\Phi^-\rangle = |\Phi^+\rangle$
- $I(ZX)|\Psi^-\rangle = |\Phi^+\rangle$

$$\rho = A|\Phi^+\rangle\langle\Phi^+| + B|\Psi^+\rangle\langle\Psi^+| + C|\Psi^-\rangle\langle\Psi^-| + D|\Phi^-\rangle\langle\Phi^-|$$

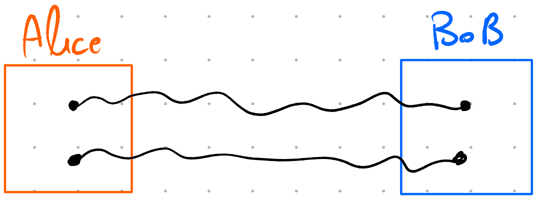
AFTER TELEPORT:

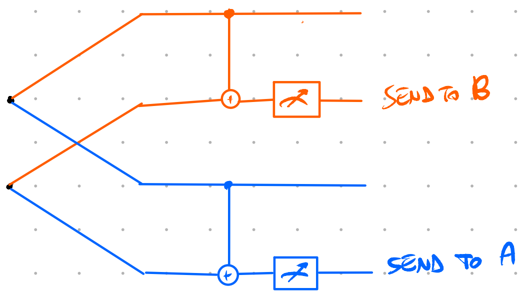
$$\rho = A|q\rangle\langle q| + B|\bar{q}\rangle\langle\bar{q}| + C|\bar{q}^*\rangle\langle\bar{q}^*| + D|q^*\rangle\langle q^*|$$

↳ $A = \langle q|\rho|q\rangle$: FIDELITY

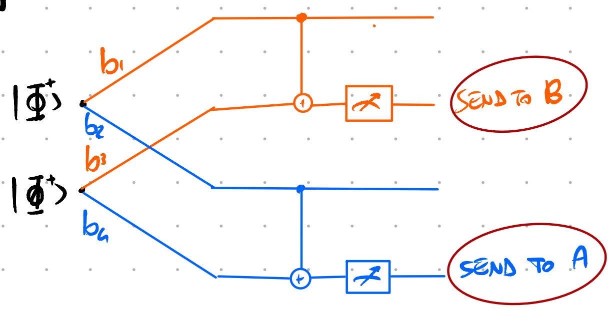
PURIFICATION

- INCREASE FIDELITY IN ENTANGLED QUBIT with 2 qubit





- if they agree on their sended measure they are pretty sure that the qubit 1 still in the rigth state.



$(|00\rangle + |11\rangle)(|00\rangle + |11\rangle) =$

$= |0000\rangle + |1100\rangle + |0011\rangle + |1111\rangle \Rightarrow$

$\xrightarrow{\text{CNOT}_A} (|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle)$ $b_1 b_2 b_3 b_4$

$\xrightarrow{\text{CNOT}_B} (|0000\rangle + |0011\rangle + |1111\rangle + |1100\rangle)$

$\Rightarrow$ on top we still have $|\Phi^+\rangle$

• Imagine $|\Phi^+\rangle$ can only swap to $|\Psi^+\rangle$

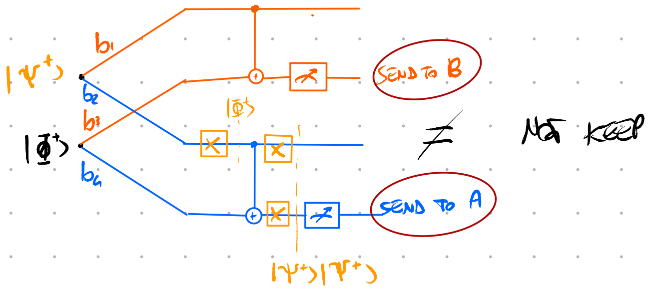
$$P_1 = F|\Phi^+\rangle + (1-F)|\Phi^+\rangle$$
$$P_2 = F|\Phi^+\rangle + (1-F)|\Phi^+\rangle$$

• At the end:

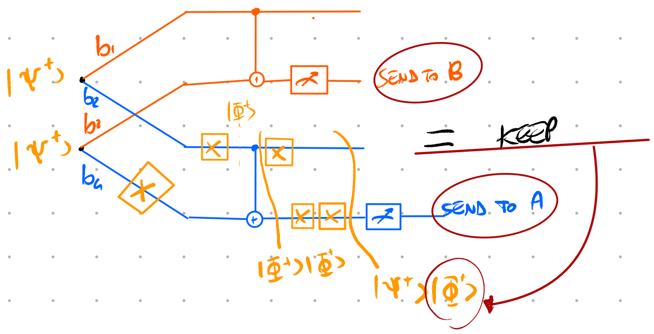
① wp F^2: $|\Phi^+\rangle|\Phi^+\rangle \longrightarrow |\Phi^+\rangle$

② wp $F(1-F)$: $|\Phi^+\rangle|\Psi^+\rangle \longrightarrow$ NOT KEEP

③ wp $(1-F)F$: $|\Psi^+\rangle|\Phi^+\rangle \longrightarrow$ NOT KEEP



④ wp $(1-F)^2$: $|\Psi^+\rangle|\Psi^+\rangle \longrightarrow |\Psi^+\rangle$ keep wrongly!



$\Rightarrow$ If the pair is kept do we have high fidelity?

• Prob (KEEP) $= F^2 + (1-F)^2$

• Prob (KEEP PAIR is $|\Phi^+\rangle$ | I kept the pair) $= \dfrac{P(\text{PAIR is } |\Phi^+\rangle)}{P(\text{kept the pair})} =$

$$= \frac{F^2}{F^2 + (1-F)^2}$$

• Prob (kept PAIR is $|\Psi^2\rangle$ | I kept the pair) $= \dfrac{F^2}{(1-F)^2 + F^2}$

THE FIDELITY IS INCREASED

$$\Rightarrow \rho_{out} = \frac{F^2}{F^2 + (1-F)^2}|\Phi^+\rangle\langle\Phi^+| + \frac{F^2}{(1-F)^2 + F^2}|\Psi^+\rangle\langle\Psi^+|$$

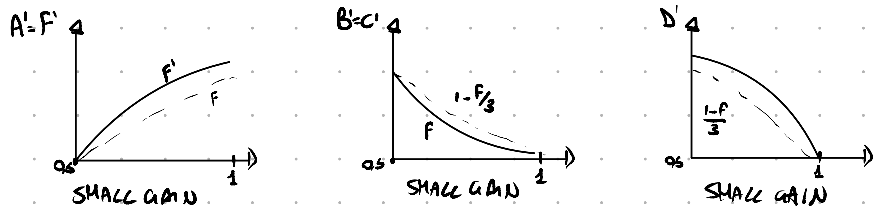
$\hookrightarrow$ New FIDELITY

- We focused only on $|\Psi^+\rangle$ what happen in other cases?
- Given $\rho_i = A|\Phi^+\rangle\langle\Phi^+| + B|\Psi^+\rangle\langle\Psi^+| + C|\Psi^-\rangle\langle\Psi^-| + D|\Phi^-\rangle\langle\Phi^-|$
- we get as out $\rho_i = A'|\Phi^+\rangle\langle\Phi^+| + B'|\Psi^+\rangle\langle\Psi^+| + C'|\Psi^-\rangle\langle\Psi^-| + D'|\Phi^-\rangle\langle\Phi^-|$
- so $A' = \dfrac{A_1 A_2}{Prob(Keep)}$ $B' = \dfrac{B_1 B_2 + C_1 C_2}{Prob(keep)}$ $C' = \dfrac{B_1 C_2 + B_2 C_1}{Prob(keep)}$ $D' = \dfrac{A_1 D_2 + A_2 D_1}{Prob(Keep)}$
- where $Prob(keep) = A_1 A_2 + B_1 B_2 + C_1 C_2 + B_1 C_2 + B_2 C_1 + A_1 D_2 + A_2 D_1$

WERNER STATES

$$\rho = F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}\left(|\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-| + |\Phi^-\rangle\langle\Phi^-|\right)$$

- What A', B' and C' are if the input is the WERNER STATE?



- After PURIFICATION we want $A' > B' > C' > D'$ so if for ex. $D' > B'$ we can SWAP THEM

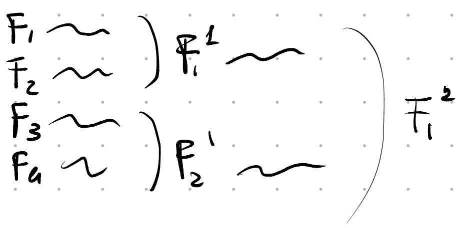
$A' - \Phi^+$ $B' - \Psi^+$

$C' - \Psi^-$ $D' - \Phi^-$

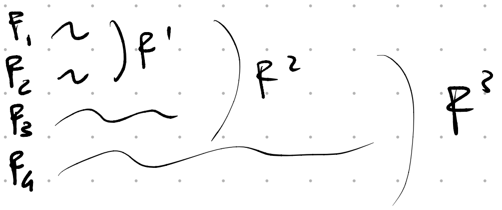
Applying H_1 to first qubit and H_2 to the second swaps from Ψ^+ to Φ^-

PURIFICATION APPROACHES

SYMMETRIC APPROACH

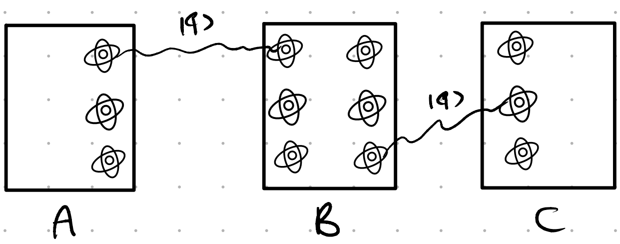


PUMPING



ENTANGLEMENT TELEPORTATION

HOP BY HOP



ENTANGLEMENT SWAPPING

(1) ENTANGLE 2 QUBIT

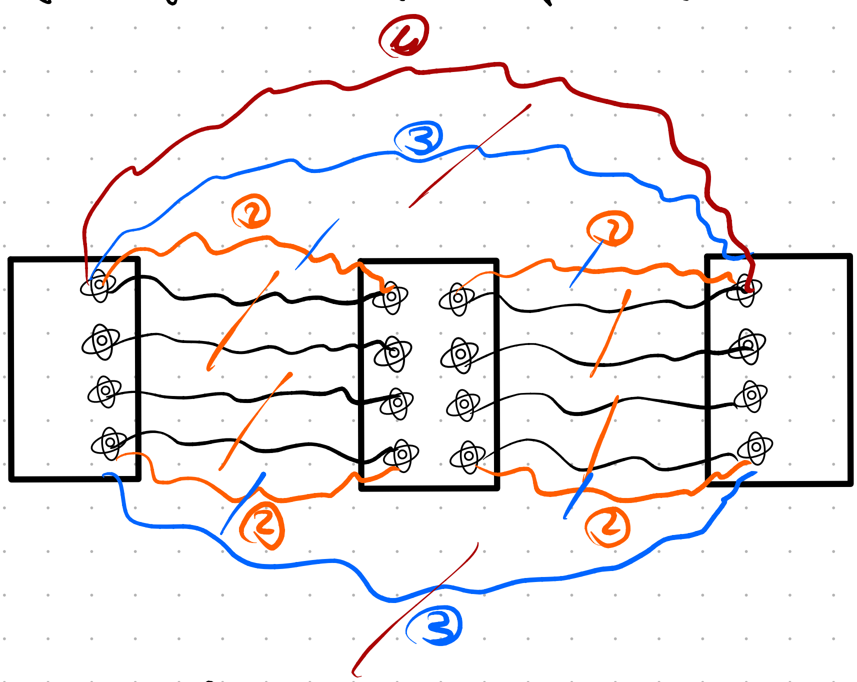
(2) SEND ONE BY HOP BY HOP TO C

(3) NOW THERE IS AN (A-C)-CHANNEL

⇒ The fidelity is F^2 in both cases

NESTED PURIFICATION

- Fidelity F increase by the # hops ⇒ n hops mean F^n
- You can purify one of 2 channels by breaking one of the two

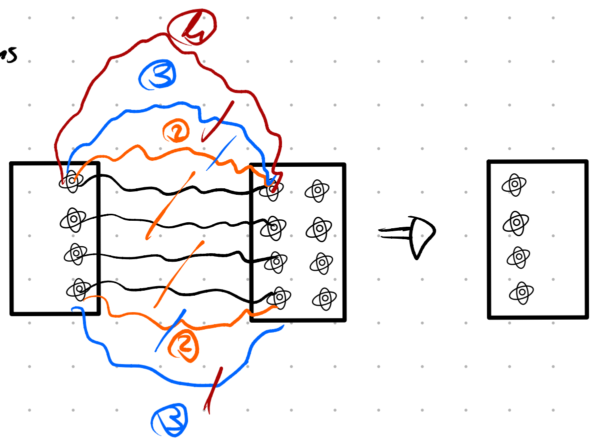


- With N hops we use $\log_2 N$ operation an for each step we need the double of the previous

$$R_i = 2R_{i-1}$$

$$R_T = 2^{\log_2 N} = N \quad // \text{ \#Round poly in = Hops}$$

- With Hop by Hop we need 2^N operations



QUANTUM ERROR CODE

- REPLICATION CODE

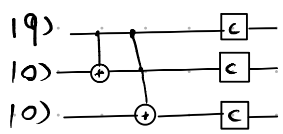
(1) No cloning th.

(2) Recv. can't look at what recvd. ( )

(3) Continuous information $\alpha, \beta \in \mathbb{R}$

- The best we can do is, from $|q\rangle = \alpha|0\rangle + \beta|1\rangle$ build $|q'\rangle = \alpha|000\rangle + \beta|111\rangle$ and send this qubits in parallel?

How to generate them?



$\Rightarrow$ $(1-P)^3(\alpha|000\rangle + \beta|111\rangle) +$
$P^3\,(\alpha|001\rangle + \beta|110\rangle +$
$+\ \alpha|010\rangle + \beta|101\rangle +$
$+\ \alpha|100\rangle + \beta|110\rangle +$ CORRECTABLE CASES
$+ \dots)$ BROKEN CASES

$\Rightarrow$

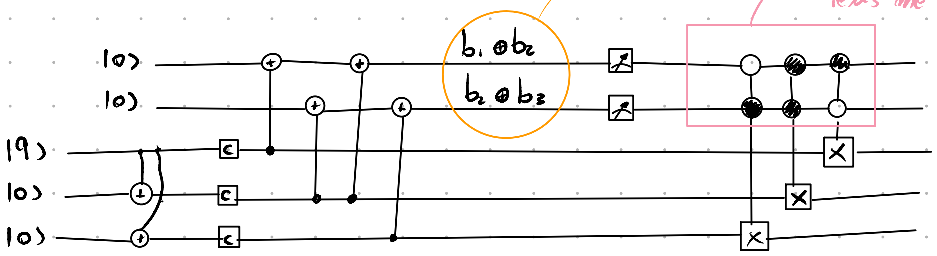
	$Z_1 Z_2$	$Z_2 Z_3$
$a\|000\rangle + b\|111\rangle$	+1	+1
$a\|001\rangle + b\|110\rangle$	+1	-1
$a\|010\rangle + b\|101\rangle$	-1	-1
$a\|100\rangle + b\|011\rangle$	-1	+1

when $Z_i Z_j = \mp 1$?

BIT FLIP ERROR CODE

$a \oplus b = 0$ if $a = b$

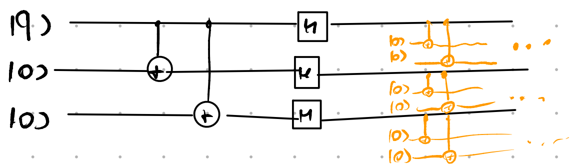
the previous table tells me which bit to flip



PHASE FLIP CODE

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$$

- We can reduce the problem to the Bit flip one $a|{+}{+}{+}\rangle + b|{-}{-}{-}\rangle$



// we can add the previous code there