

Graph Theory and Optimization

Flow: Ford-Fulkerson Algorithm

Max Flow- Min Cut duality

Nicolas Nisse

Université Côte d'Azur, Inria, CNRS, I3S, France

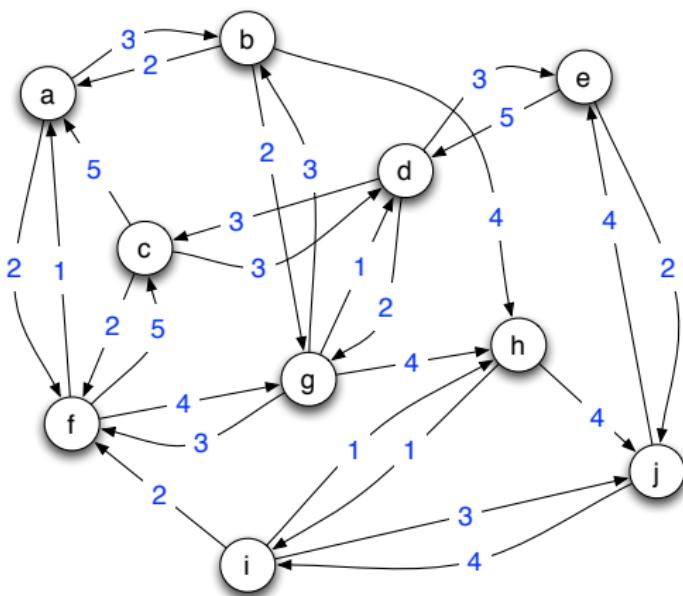
October 2018

Outline

- 1 Transportation Problem
- 2 Elementary Flow Network
- 3 Upper bound on Flow: Cut
- 4 Ford-Fulkerson Algorithm
- 5 Min Cut=Max flow
- 6 Application to Connectivity: Menger Theorem
- 7 Application to Matching

Transportation problem: Modeling

Directed weighted graph: $D = (V, A)$, A : set of arcs, $(x, y) \in A$ ordered pair
arrows

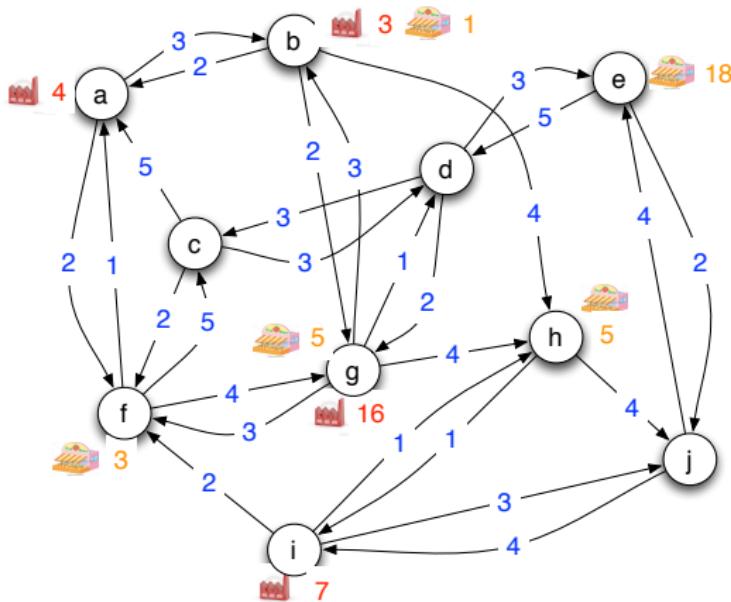


Transportation problem: Modeling

Directed weighted graph: $D = (V, A)$

On vertices: production: $p_{max} : V \rightarrow \mathbb{R}^+$; consumption: $cons_{max} : V \rightarrow \mathbb{R}^+$

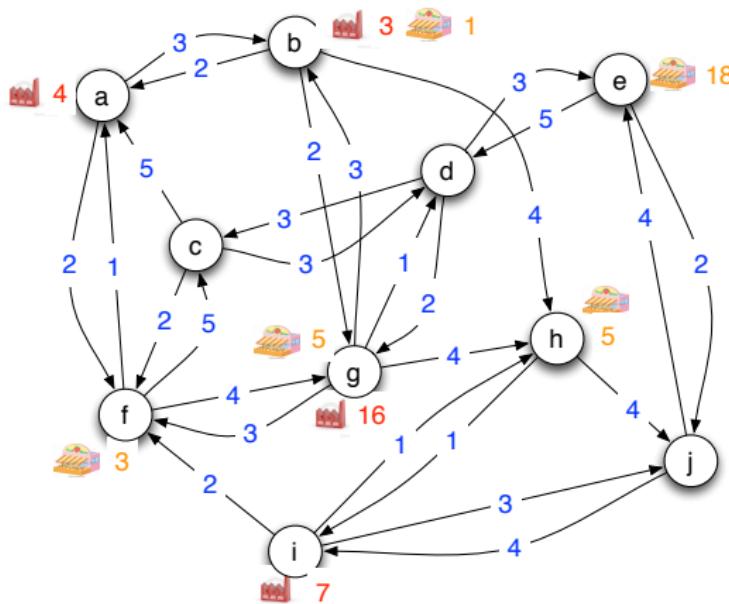
On arcs: capacity: $c : A \rightarrow \mathbb{R}^+$



Transportation problem: Modeling

What is the main amount of goods that can be exchanged?

Actual production: $p : V \rightarrow \mathbb{R}^+$ and actual consumption: $cons : V \rightarrow \mathbb{R}^+$
flow: $f : A \rightarrow \mathbb{R}^+$: satisfies capacity and flow conservation

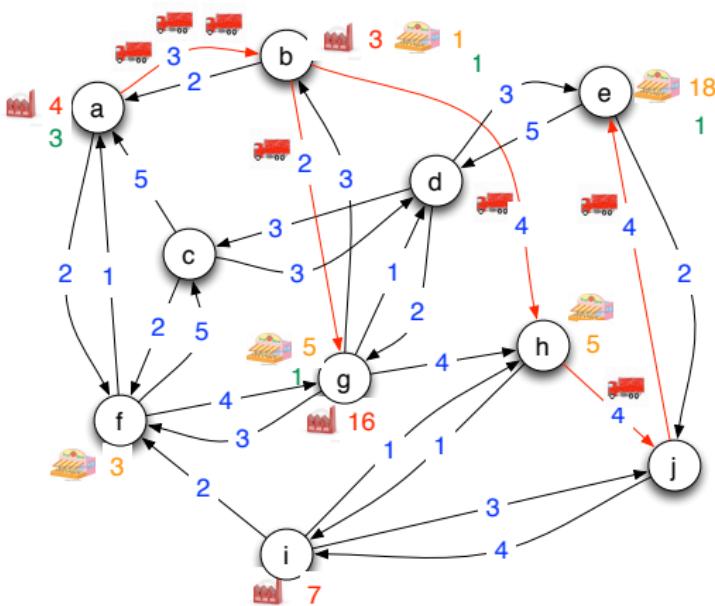


Transportation problem: Modeling

flow: $f : A \rightarrow \mathbb{R}^+$: **feasibility:** $\forall v \in V, p(v) \leq p_{max}(v), cons(v) \leq cons_{max}(v)$

capacity constraint: $\forall a \in A, f(a) \leq c(a)$.

flow conservation: $\forall v \in V, p(v) + \sum_{w \in N^-(v)} f(wv) = c(v) + \sum_{w \in N^+(v)} f(vw)$

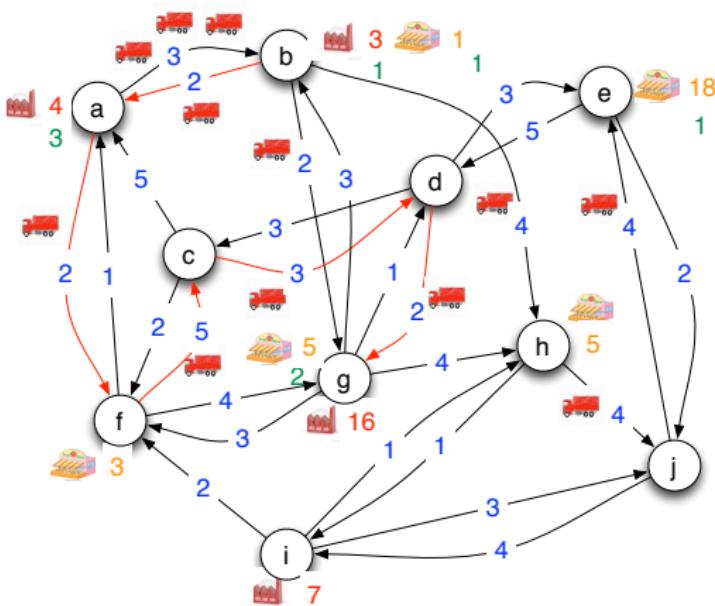


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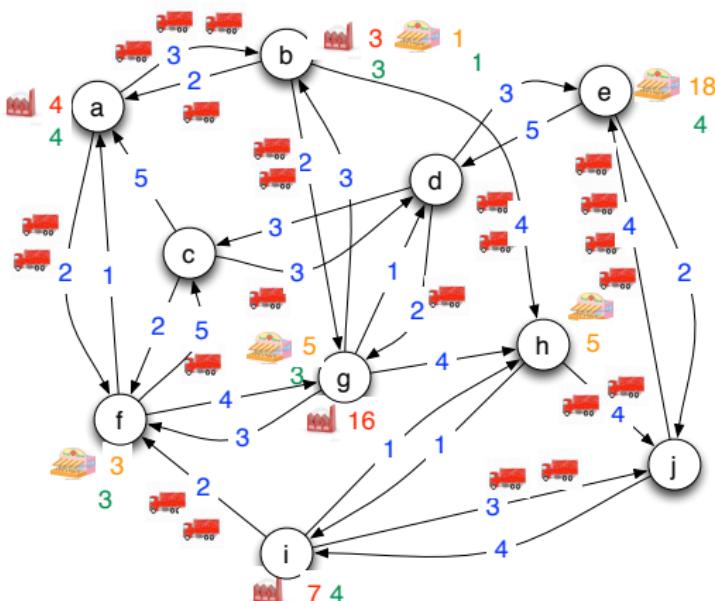


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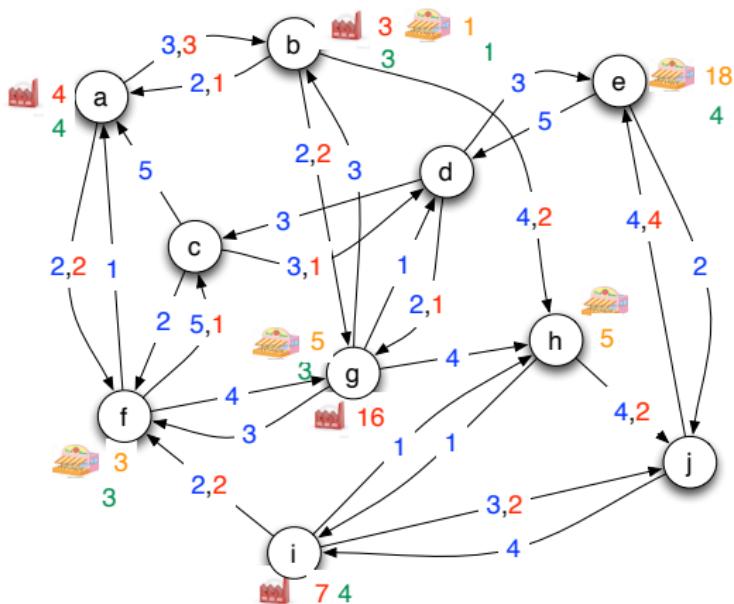


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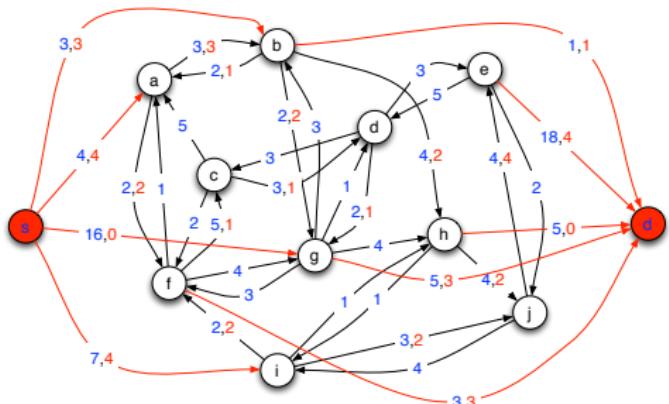
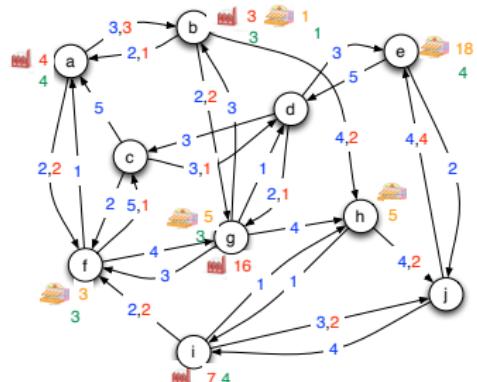
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Transportation problem: Modeling

Simplification: one single source and one single destination



One source s : $\forall v \in V$ add one arc (s, v) with capacity $p_{max}(v)$

One destination d : $\forall v \in V$ add one arc (v, d) with capacity $cons_{max}(v)$

Flows are "equivalent" in both networks

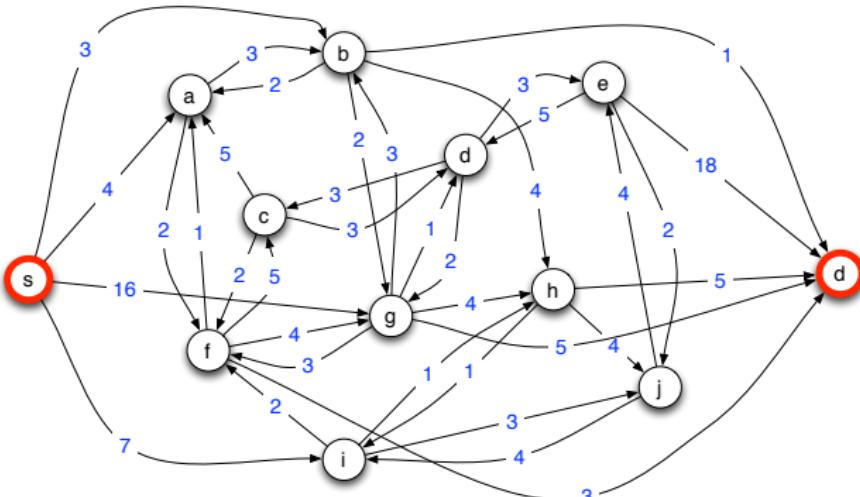
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Elementary Flow Network

Directed graph: $D = (V, A)$, **source** $s \in V$, **destination** $d \in V$, **capacity** $c : A \rightarrow \mathbb{R}^+$.

$$N^-(s) = \emptyset \text{ and } N^+(d) = \emptyset$$



flow $f : A \rightarrow \mathbb{R}^+$ such that : **capacity constraint:** $\forall a \in A, f(a) \leq c(a)$

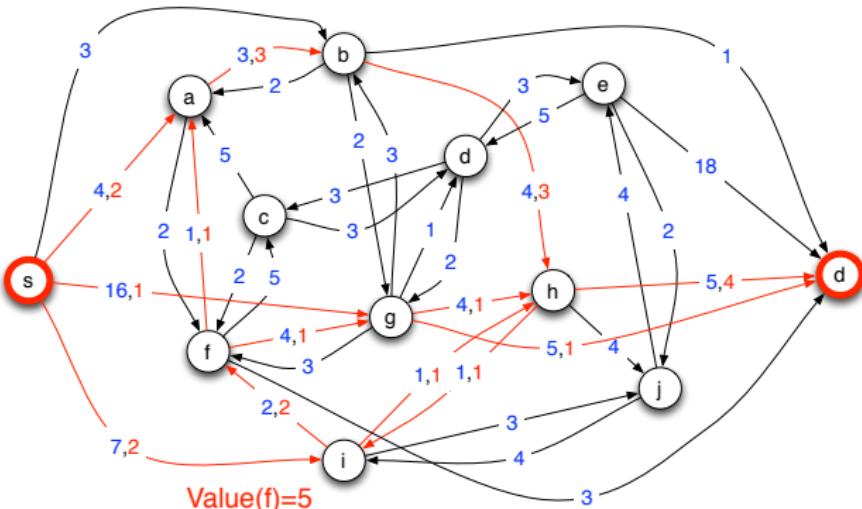
conservation constraint: $\forall v \in V \setminus \{s, d\}, \sum_{w \in N^-(v)} f(wv) = \sum_{w \in N^+(v)} f(vw)$

value of flow: $v(f) = \sum_{w \in N^+(s)} f(sw).$

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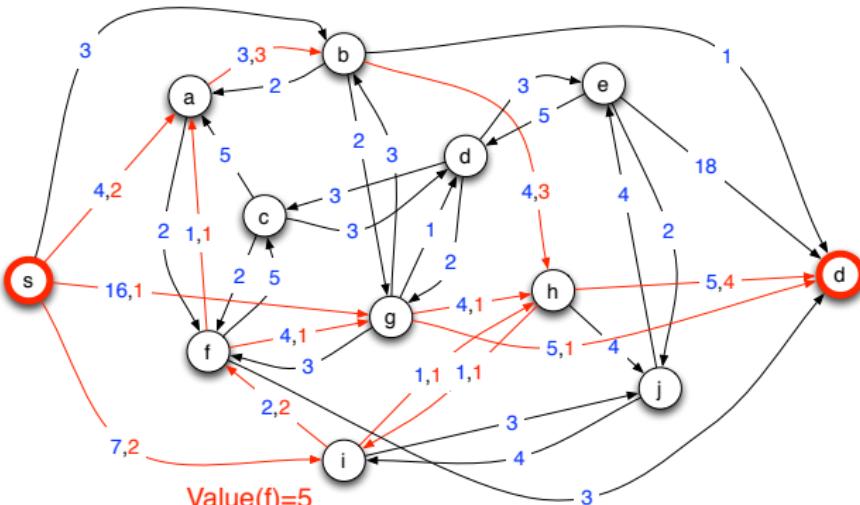
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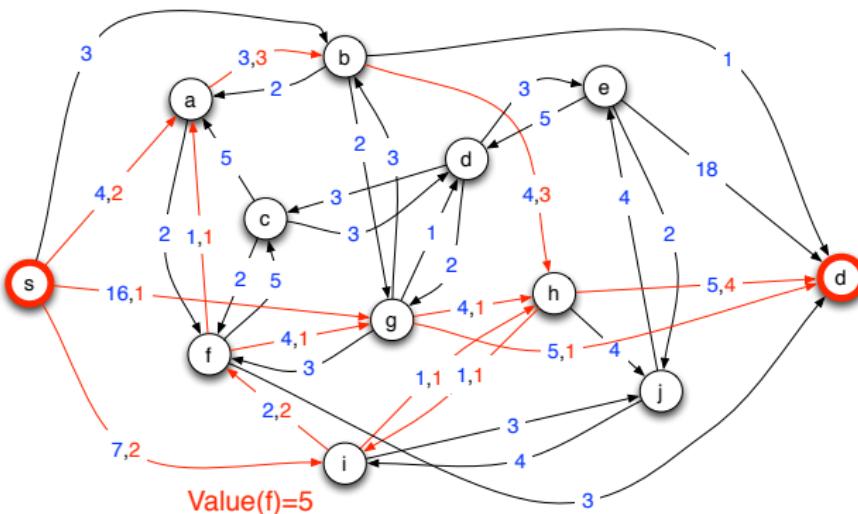
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Exercise: $v(f) = \sum_{w \in N^-(d)} f(wd)$

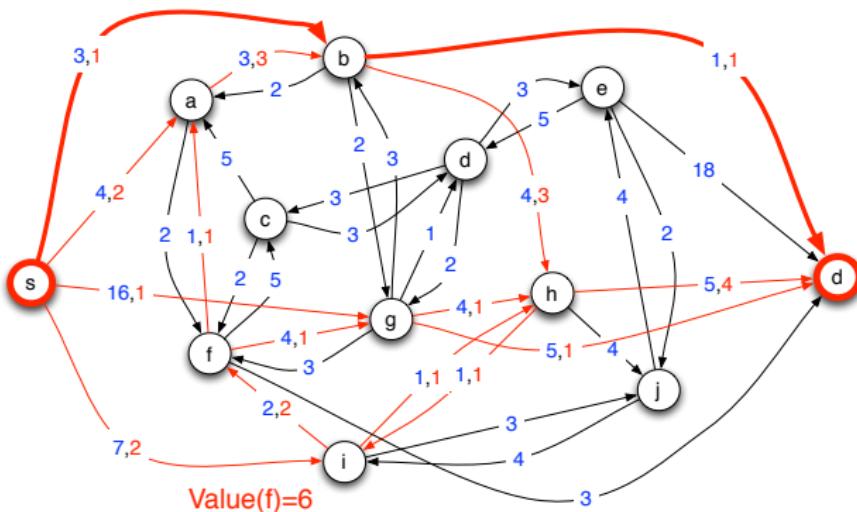
Elementary Flow Network: Max Flow

How to compute a flow with maximum value?



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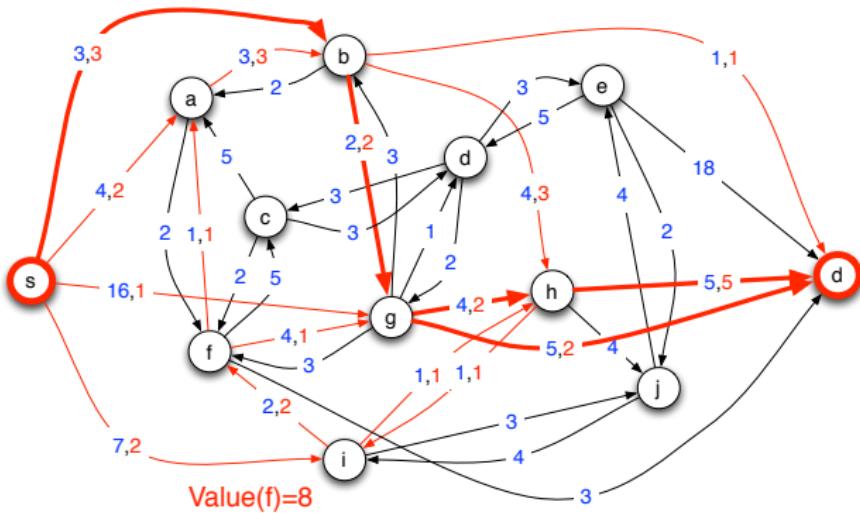
How to compute a flow with maximum value?



Possible to “push” flow along available path

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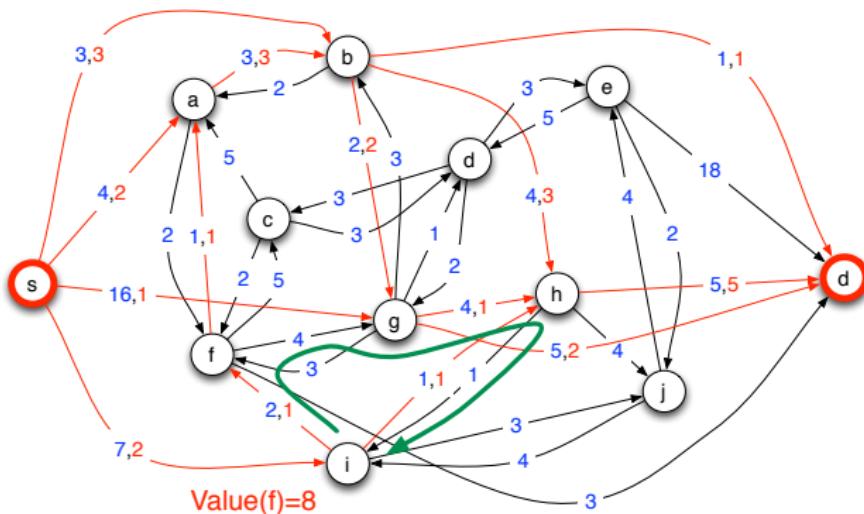
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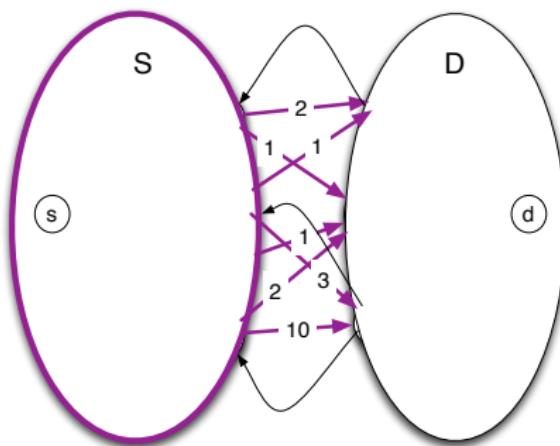
Possible to “push” flow along available path
 May be useful to “remove useless flow”

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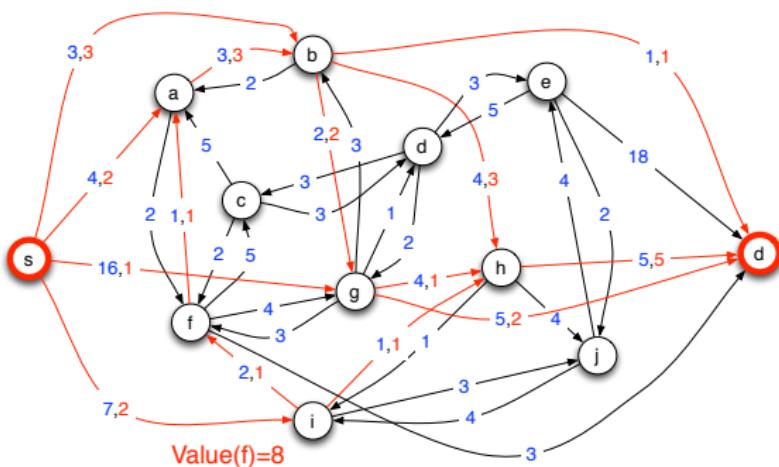
Max Flow, upper bound: Min cut

a cut gives an upper bound on the value of your flow



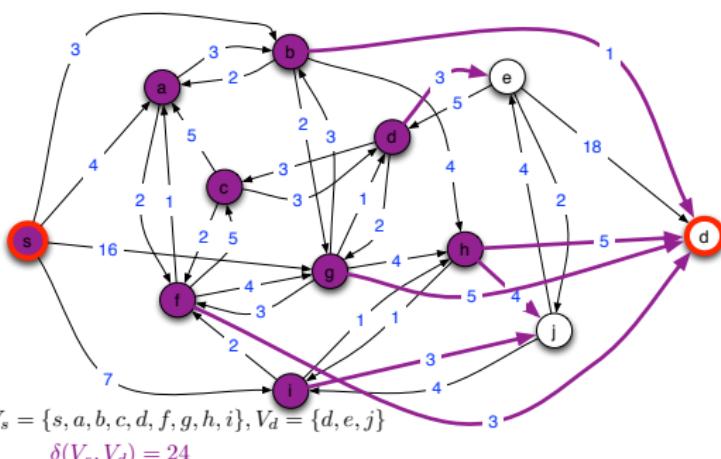
Max Flow, upper bound: Min cut

Is this flow maximum?: a cut gives an upper bound on the value of your flow



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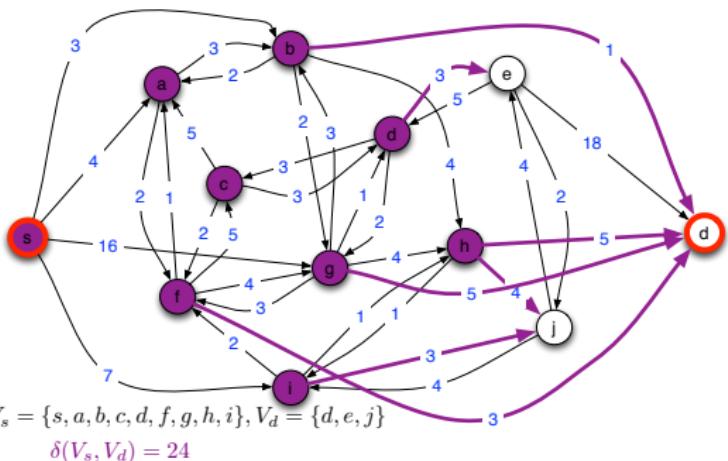


s, d -Cut: partition (V_s, V_d) of V with $s \in V_s$ and $d \in V_d$.

Capacity of a s, d -cut: $\delta(V_s, V_d) = \sum_{u \in V_s, v \in V_d} c(uv)$.

Max Flow, upper bound: Min cut

Is this flow maximum?: a cut gives an upper bound on the value of your flow



s, d -Cut: partition (V_s, V_d) of V with $s \in V_s$ and $d \in V_d$.

Capacity of a s, d -cut: $\delta(V_s, V_d) = \sum_{u \in V_s, v \in V_d} c(uv)$.

Theorem: for any network flow $D = (V, A)$, $s, d \in V$ and $c : A \rightarrow \mathbb{R}^+$

For any flow $f : A \rightarrow \mathbb{R}^+$ and s, d -cut (V_s, V_d) , $v(f) \leq \delta(V_s, V_d)$

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Theorem: for any network flow $D = (V, A)$, $s, d \in V$ and $c : A \rightarrow \mathbb{R}^+$

For any flow $f : A \rightarrow \mathbb{R}^+$ and s, d -cut (V_s, V_d) , $v(f) \leq \delta(V_s, V_d)$

Proof:

conservation constraint: $\forall v \in V \setminus \{s, d\}$, $\sum_{w \in N^-(v)} f(wv) = \sum_{w \in N^+(v)} f(vw)$.

sum over all vertices in $V_s \setminus \{s\}$:

$$0 = \sum_{v \in V_s \setminus \{s\}} \left(\sum_{w \in N^-(v)} f(wv) - \sum_{w \in N^+(v)} f(vw) \right) = \sum_{w \in N^+(s)} f(sw) + \sum_{v \in V_s \setminus \{s\}; w \in V_d} f(wv) - \sum_{v \in V_s; w \in V_d} f(vw)$$

So

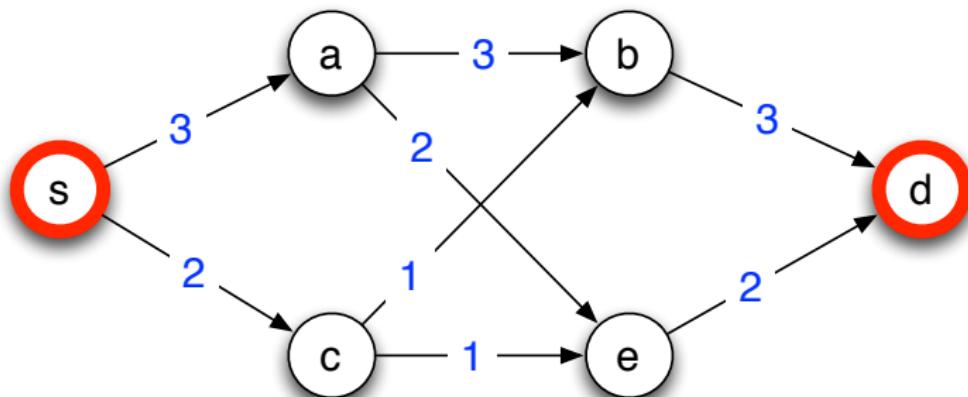
$$v(f) = \sum_{v \in V_s; w \in V_d} f(vw) - \sum_{v \in V_s \setminus \{s\}; w \in V_d} f(wv) \leq \sum_{v \in V_s; w \in V_d} f(vw) \leq \sum_{v \in V_s; w \in V_d} c(vw) = \delta(V_s, V_d)$$

Corollary: Max flow \leq Min Cut

Outline

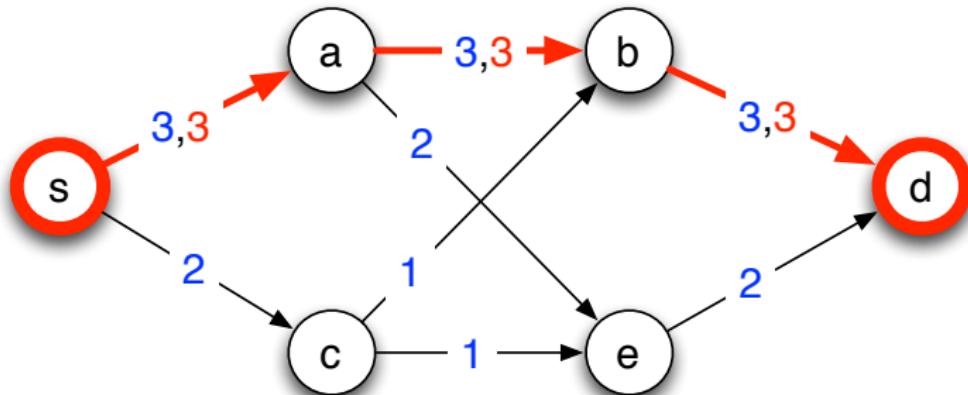
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Algorithm for Max Flow: Intuition



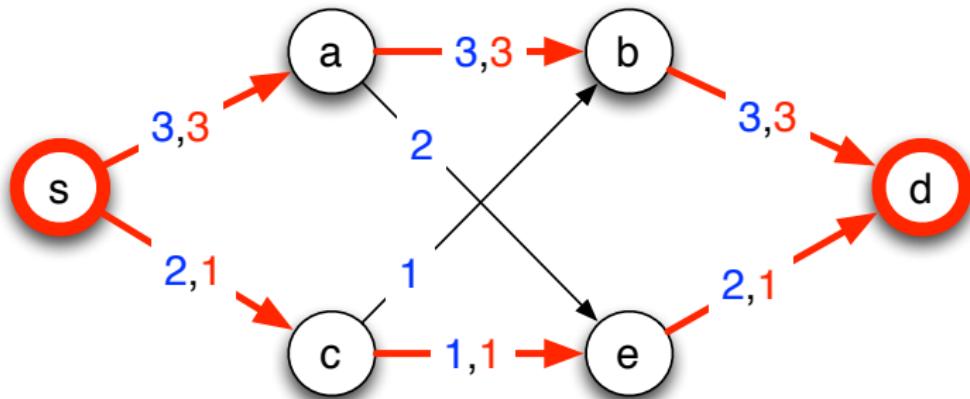
Let us compute a max flow from s to d .

Algorithm for Max Flow: Intuition



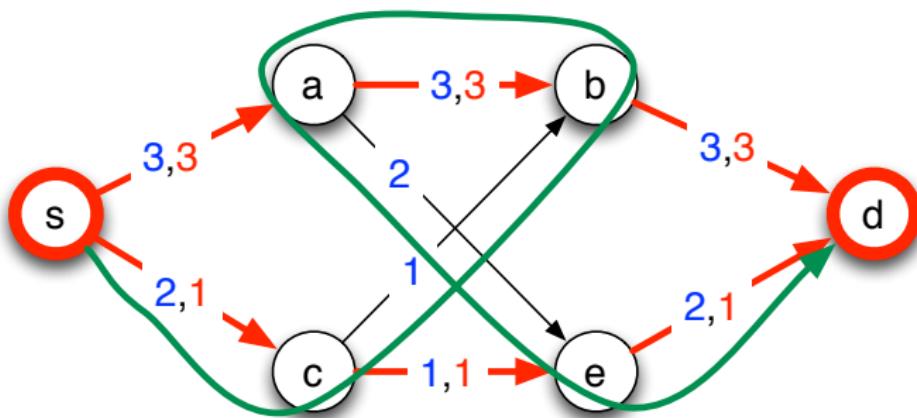
We can “push” 3 units of flow along an available path (s, a, b, d) . $v(f) = 3$.
The only remaining available path is (s, c, e, d)

Algorithm for Max Flow: Intuition



We can “push” 1 units of flow along (s, c, e, d) . $v(f) = 4$.
 No path from s to d remains available, but...

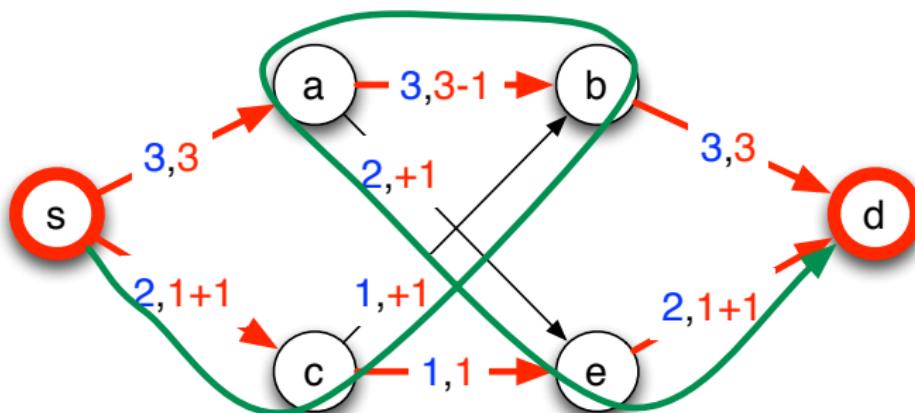
Algorithm for Max Flow: Intuition



We want to "push" some flow along the "path" (s, c, b, a, e, d)

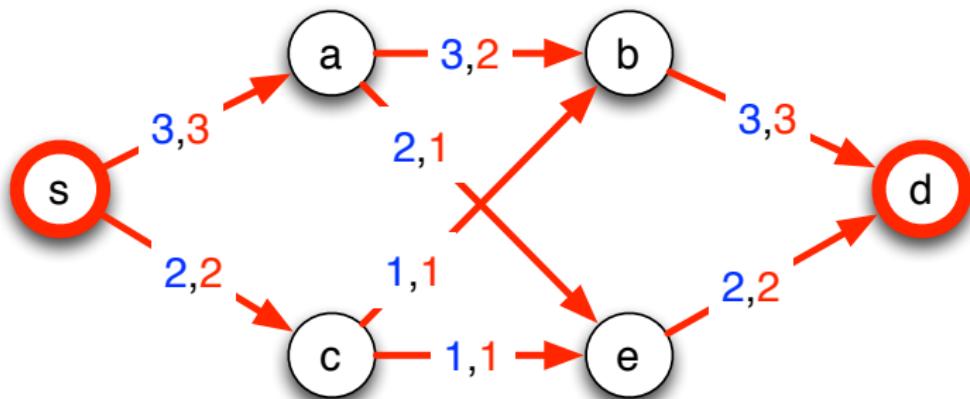
It is NOT a directed path (because $(b, a) \notin A$)

Algorithm for Max Flow: Intuition



Somehow, we "reverse" some flow along the arc (a, b)

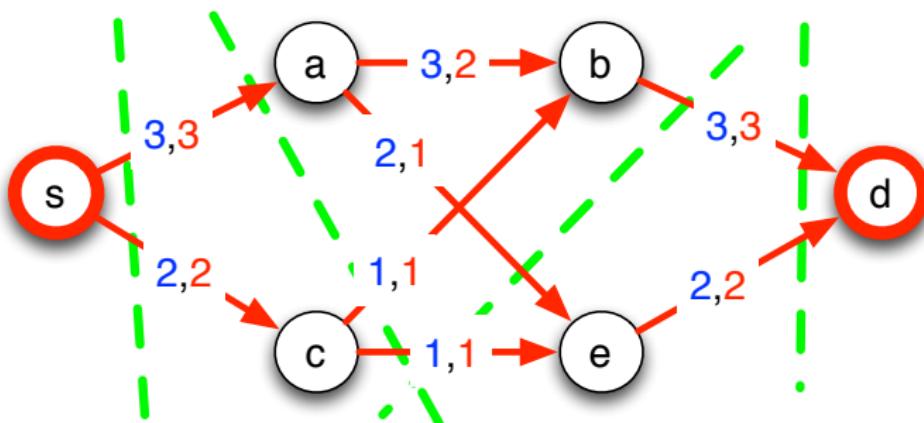
Algorithm for Max Flow: Intuition



So we got a flow with value $v(f) = 5$.

Exercise: Why is it optimal ?

Algorithm for Max Flow: Intuition



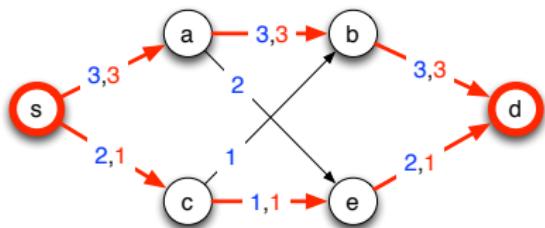
Recall that **Max flow \leq Min Cut**

If there is a flow f and a cut (V_s, V_d) with $v(f) = \delta(V_s, V_d)$, then
 f is maximum and (V_s, V_d) is a minimum cut.

Ford-Fulkerson Algorithm

1st example

Problem here: there is no path where to push flow

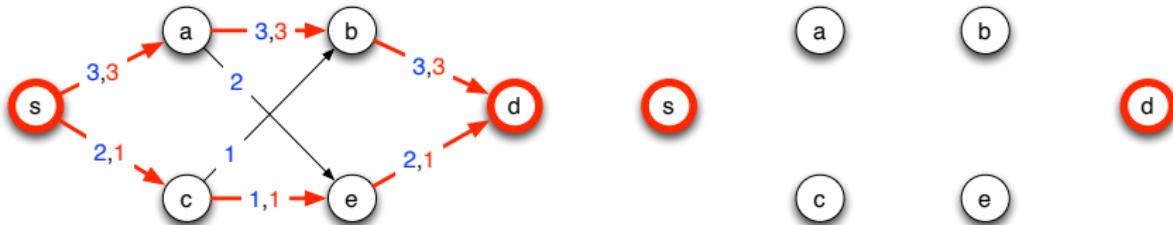


1st Phase of FF-algorithm: Compute an auxiliary graph where to find a path

Ford-Fulkerson Algorithm

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1st Phase of FF-algorithm: Compute an auxiliary graph where to find a path

For all $u, v \in V(G)$, create an arc with capacity $c_{aux}(uv) = c(uv) - f(uv) + f(vu)$

$f(uv)$ current flow from u to v , $f(vu)$ current flow from v to u
 $c(uv) - f(uv)$ is the residual capacity

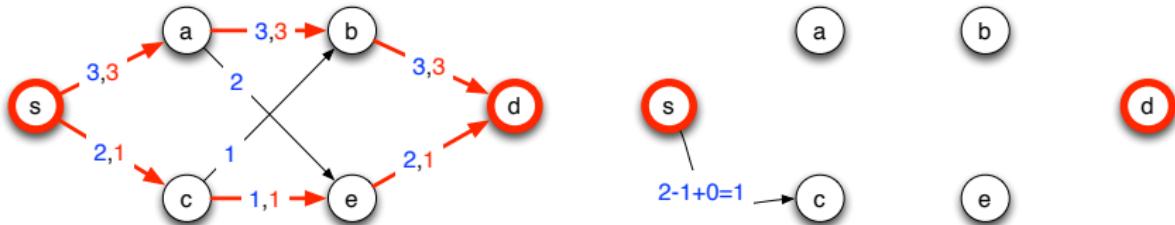
Remarks:

- $c_{aux}(uv)$ may be positive even if $(u, v) \notin A(G)$
- If $(u, v) \notin A(G)$ and $(v, u) \notin A(G)$, then $c_{aux}(uv) = c_{aux}(vu) = 0$

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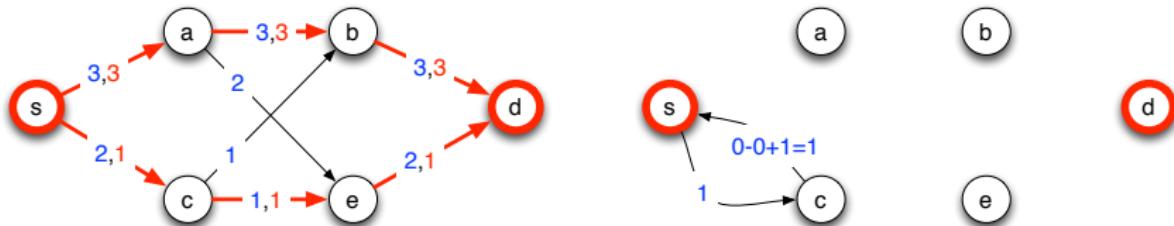
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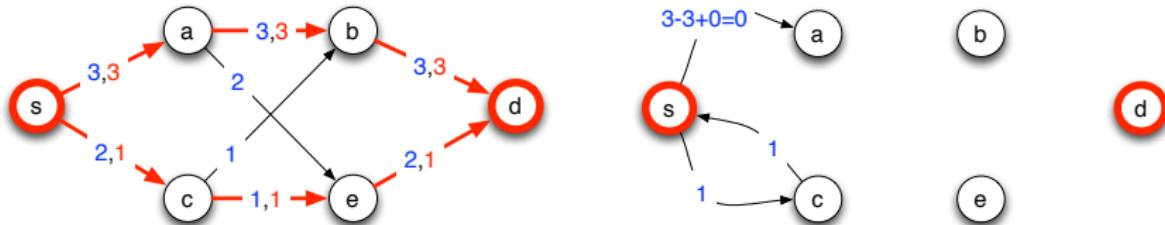
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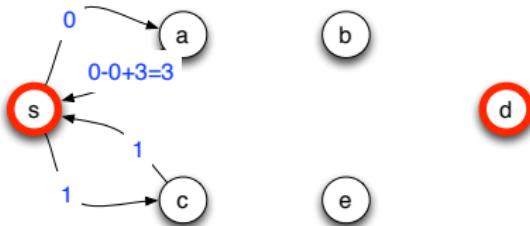
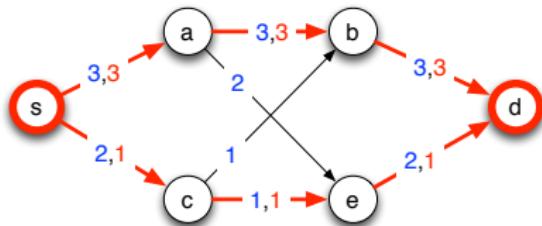
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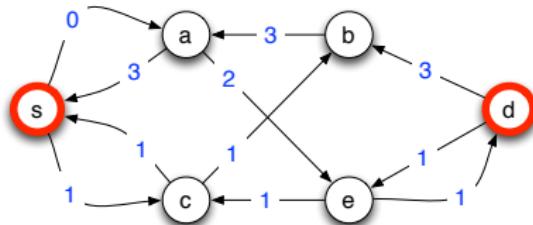
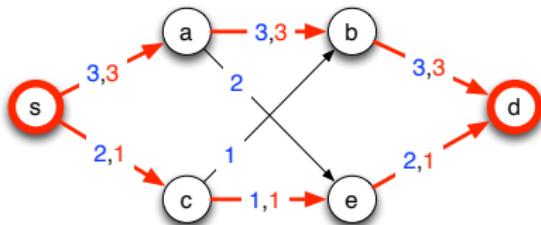
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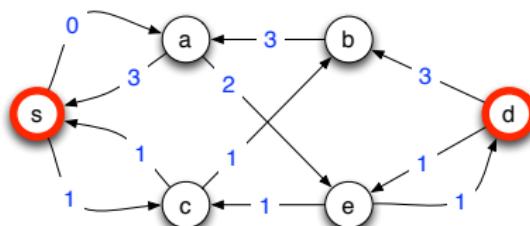
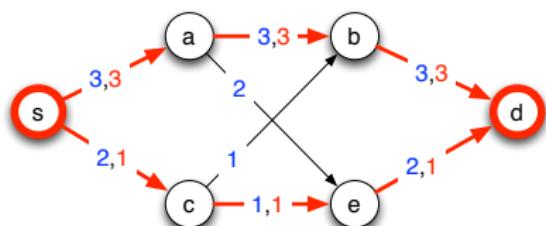
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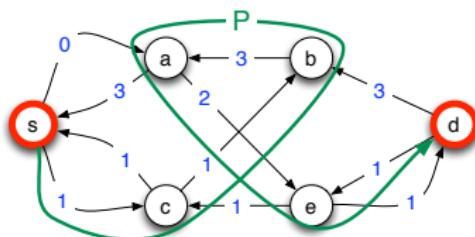
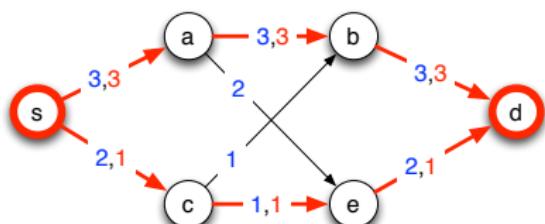


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Ford-Fulkerson Algorithm

1st example

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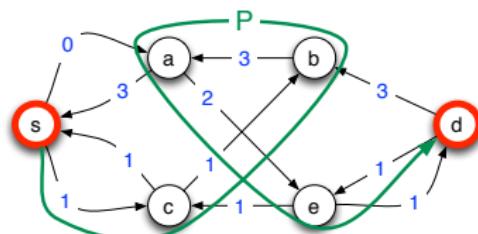
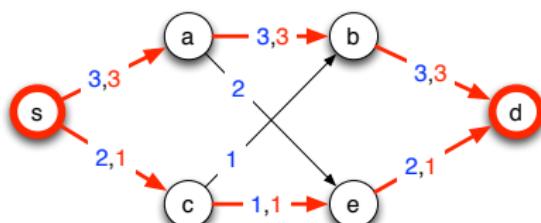
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Here $P = (s, c, b, a, e, d)$ and its minimum capacity is $\varepsilon = 1 > 0$

Ford-Fulkerson Algorithm

1st example

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2nd Phase of FF-algorithm: Look for a path P from s to d in auxiliary graph

Here $P = (s, c, b, a, e, d)$ and its minimum capacity is $\varepsilon = 1 > 0$

We will "push" ε units of flow along P in G

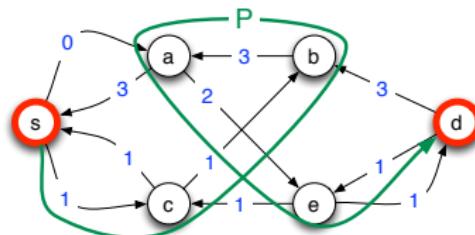
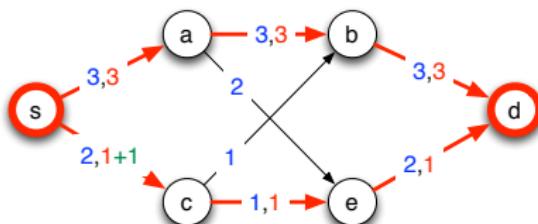
For all arcs (u, v) of P

- Add ε to the current flow of (u, v) if $f(uv) + \varepsilon \leq c(uv)$
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note that $\varepsilon - (c(uv) - f(uv))$ are "lacking to be pushed"
- In the latter case, remove $\varepsilon - (c(uv) - f(uv))$ from the current flow from v to u .

Ford-Fulkerson Algorithm

1st example

Problem here: there is no path where to push flow



2nd Phase of FF-algorithm: Look for a path P from s to d in auxiliary graph

Here $P = (s, c, b, a, e, d)$ and its minimum capacity is $\varepsilon = 1 > 0$

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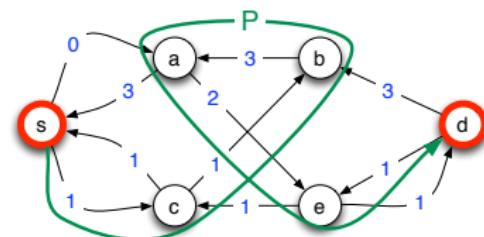
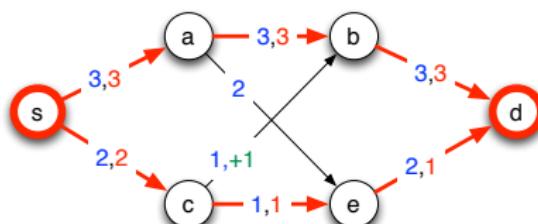
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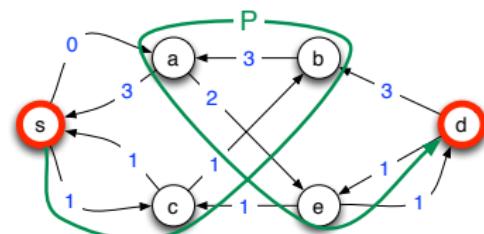
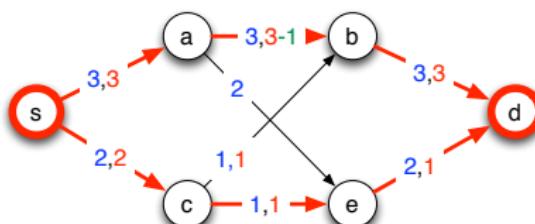
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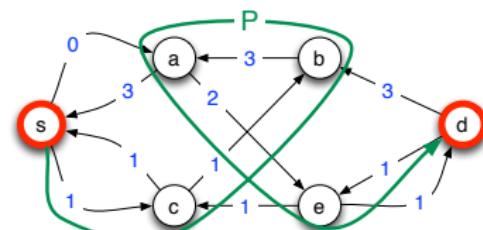
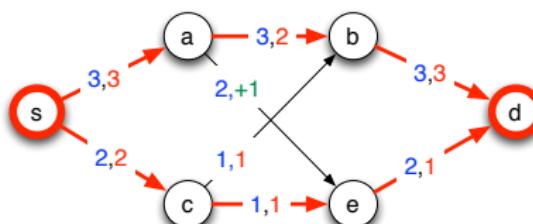
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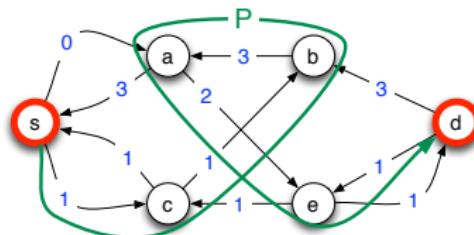
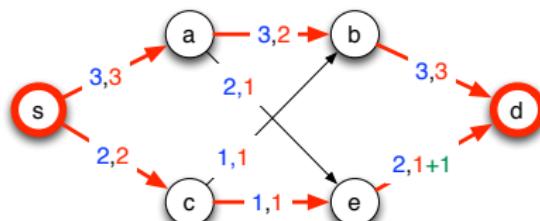
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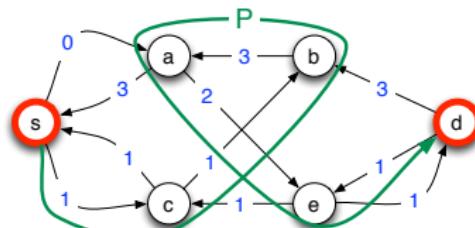
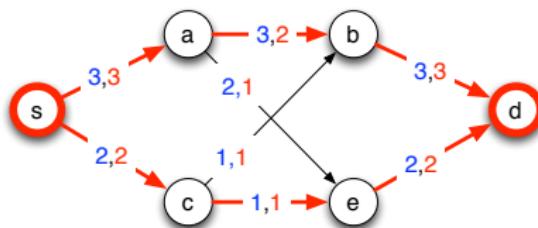
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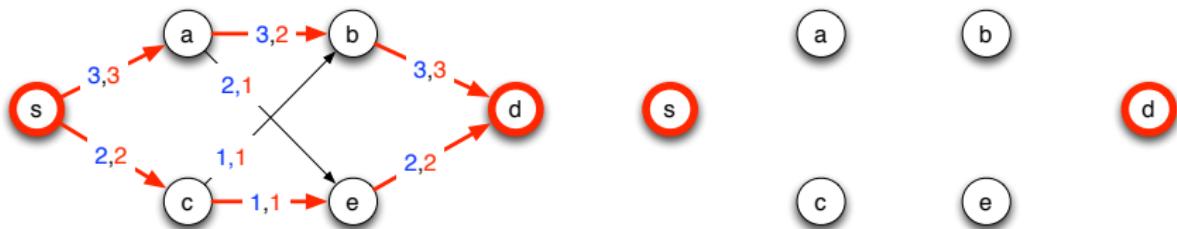
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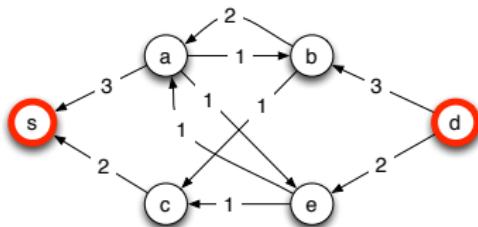
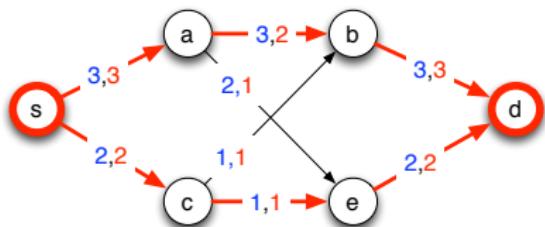


back to the 1st Phase of FF-algorithm: Compute in auxiliary graph

Ford-Fulkerson Algorithm

1st example

Problem here: there is no path where to push flow

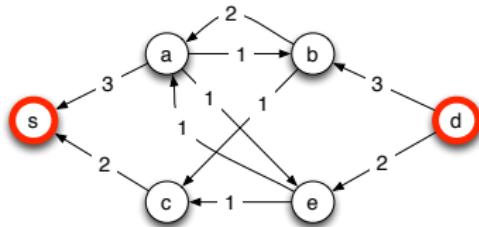
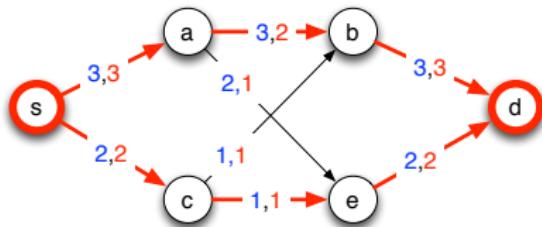


back to the 1st Phase of FF-algorithm: Compute in auxiliary graph

Ford-Fulkerson Algorithm

1st example

Problem here: there is no path where to push flow



back to the 1st Phase of FF-algorithm: Compute in auxiliary graph

Here no path with > 0 capacity from s to d .

The set of nodes reachable from s (here, only s) **defines a cut**

Exercise: What is the capacity of this cut? Why the flow is maximum?

Ford-Fulkerson Algorithm

$D = (V, A)$, $c : A \rightarrow \mathbb{R}^+$ and $s, d \in V$

Let $f : A \rightarrow \mathbb{R}^+$ be a valid flow in D

Initially, f may be null

① Compute an auxiliary graph $D_{aux} = (V, A_{aux})$ and $c_{aux} : A_{aux} \rightarrow \mathbb{R}^+$

② If there is a directed path P from s to d in D_{aux}
 (auxiliary capacity of arcs of P must be > 0)

- Let $\varepsilon > 0$ be the minimum capacity c_{aux} of the arcs of P
- "Push" ε units of flow along P in D

- Go to 1

(Note that the value of the flow has increased)

Else Return f

f is a maximum flow.

Ford-Fulkerson Algorithm

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① Compute an auxiliary graph $D_{aux} = (V, A_{aux})$ and $c_{aux} : A_{aux} \rightarrow \mathbb{R}^+$

$\forall u, v \in V$, add an arc $(u, v) \in A_{aux}$ with capacity

$$c_{aux}(uv) = c(uv) - f(uv) + f(vu).$$

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- Let $\varepsilon > 0$ be the minimum capacity c_{aux} of the arcs of P
- "Push" ε units of flow along P in D

For each arc $(uv) \in A(P)$ of P :

$$f(uv) \leftarrow \min\{c(uv); f(uv) + \varepsilon\}$$

push ε but if it exceeds the capacity of the link

$$f(vu) \leftarrow f(vu) - \max\{0; f(uv) + \varepsilon - c(uv)\}$$

"reverse" some flow

- Go to 1

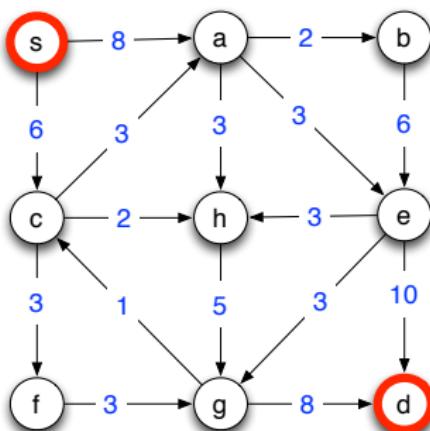
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Ford-Fulkerson Algorithm

Example

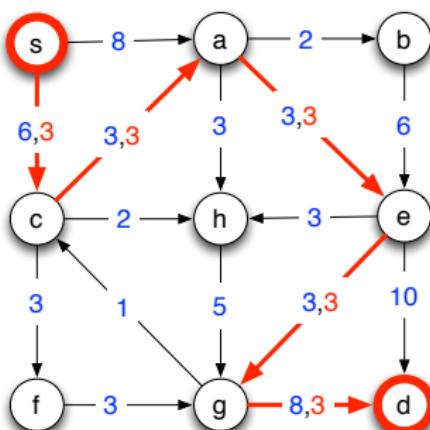


Digraph $D = (V, A)$, capacity $c : A \rightarrow \mathbb{R}^+$

Let us compute a max flow from $s \in V$ to $d \in V$.

Ford-Fulkerson Algorithm

Example

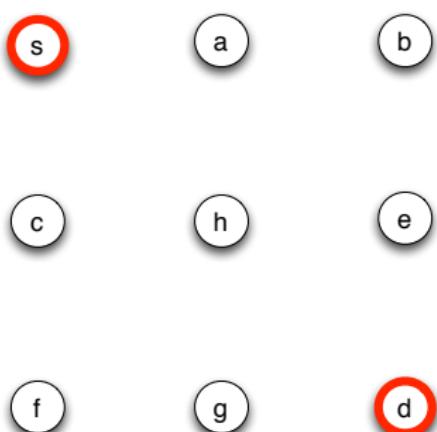
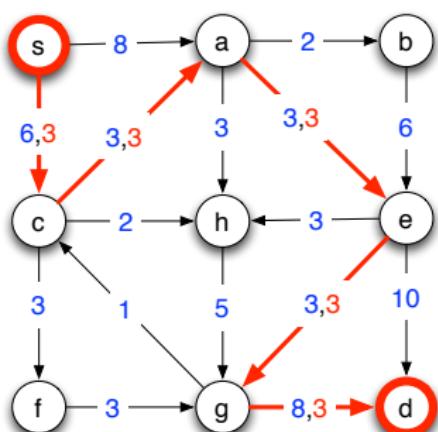


Let us compute a max flow from $s \in V$ to $d \in V$.

start from a given initial flow: in the example $v(f) = 3$.

Ford-Fulkerson Algorithm

Example

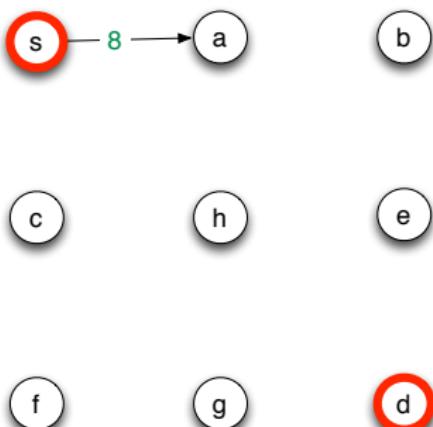
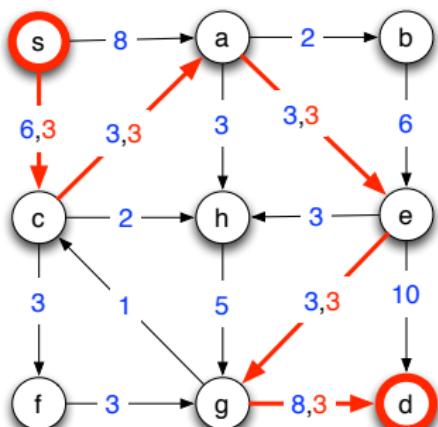


1st step: Compute the first auxiliary digraph D_{aux}

start with same vertices as D

Ford-Fulkerson Algorithm

Example



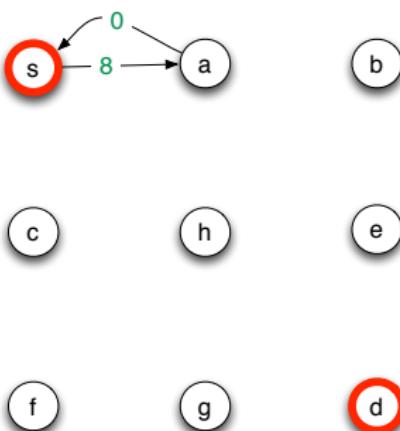
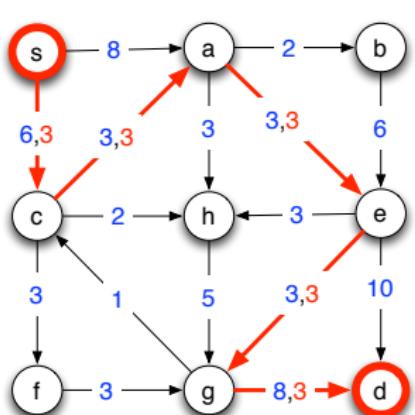
1st step: Compute the first auxiliary digraph D_{aux}

for all $u, v \in V \times V$, $c_{aux}(uv) = c(uv) - f(uv) + f(vu)$:

$$\text{in the example } c_{aux}(sa) = 8 - 0 + 0 = 8$$

Ford-Fulkerson Algorithm

Example

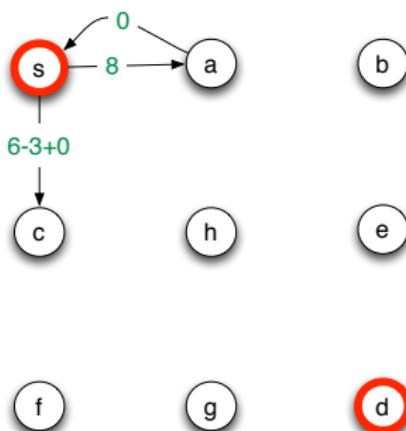
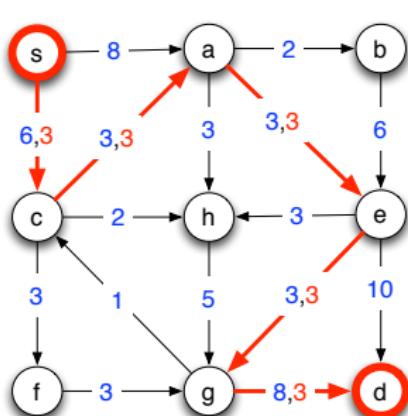


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Ford-Fulkerson Algorithm

Example



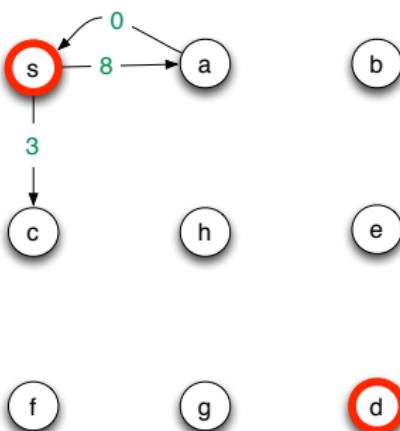
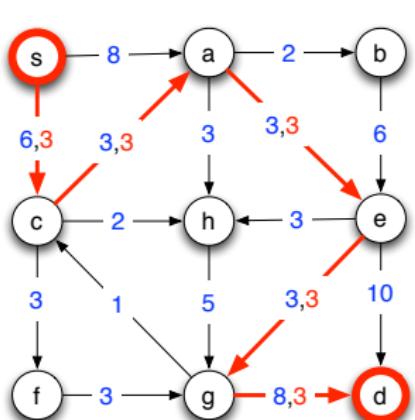
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for all $u, v \in V \times V$, $c_{aux}(uv) = c(uv) - f(uv) + f(vu)$:

$$\text{in the example } c_{aux}(sc) = 6 - 3 + 0 = 3$$

Ford-Fulkerson Algorithm

Example

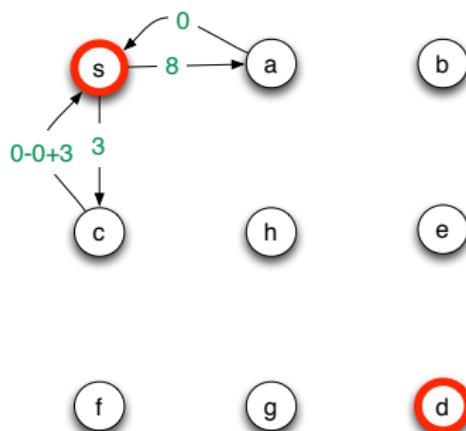
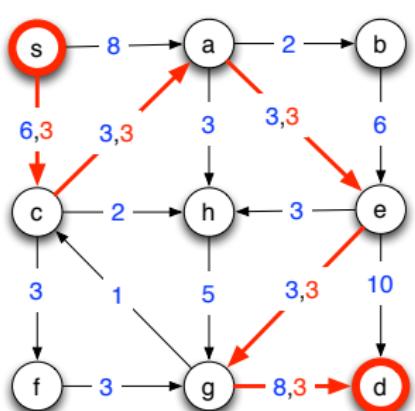


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Ford-Fulkerson Algorithm

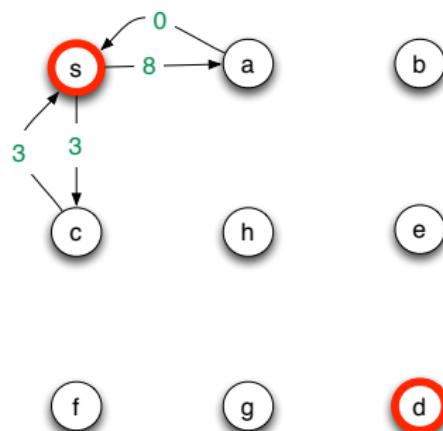
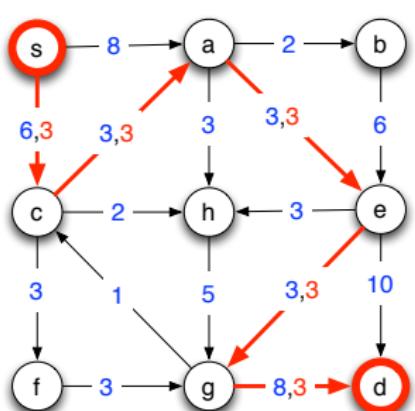
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Ford-Fulkerson Algorithm

Example



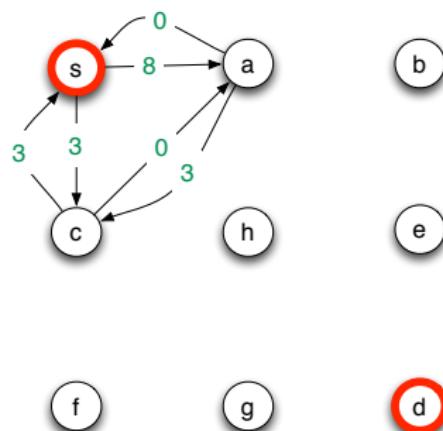
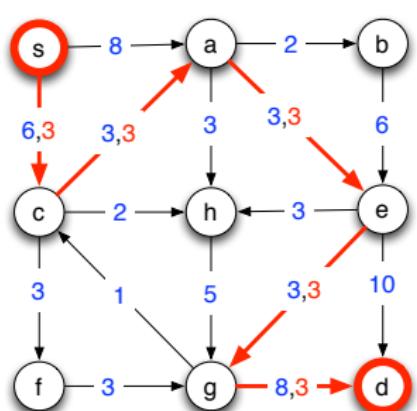
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Ford-Fulkerson Algorithm

Example



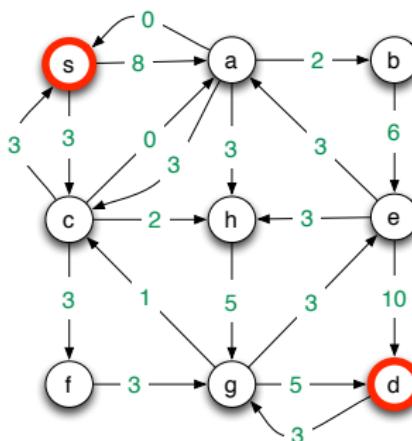
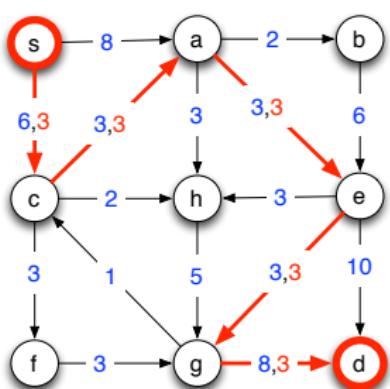
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$$\text{for all } u, v \in V \times V, c_{aux}(uv) = c(uv) - f(uv) + f(vu):$$

$$\text{in the example } c_{aux}(ca) = 3 - 3 + 0 = 0 \text{ and } c_{aux}(ac) = 0 - 0 + 3 = 3$$

Ford-Fulkerson Algorithm

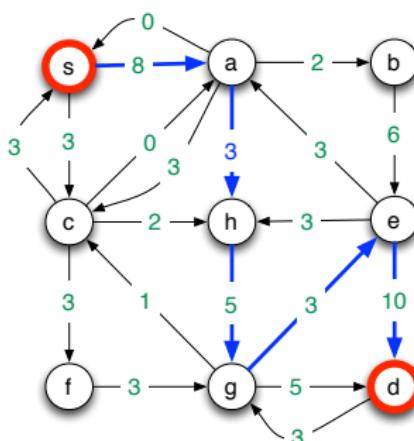
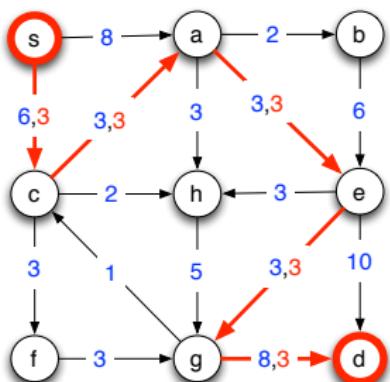
Example



1st step: Compute the first auxiliary digraph D_{aux}
 for all $u, v \in V \times V$, $c_{aux}(uv) = c(uv) - f(uv) + f(vu)$:
 and so on

Ford-Fulkerson Algorithm

Example

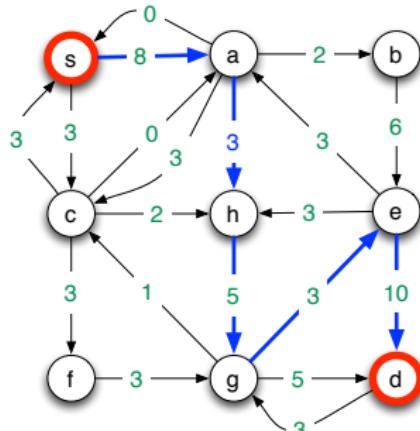
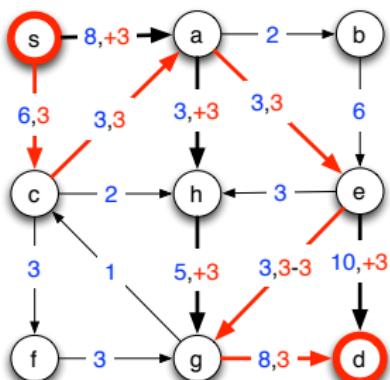


2nd step: Find in D_{aux} a directed s,d -path with positive capacity

in the example $P = (s, a, h, g, e, d)$ and $\varepsilon = \min_{arc \in P} c_{aux}(arc) = 3$

Ford-Fulkerson Algorithm

Example



2nd step: Push ε units of flow along P in D . For each arc (uv) of P :

$$f(uv) \leftarrow \min\{c(uv); f(uv) + \varepsilon\}$$

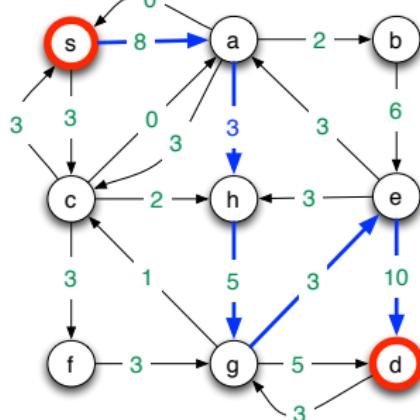
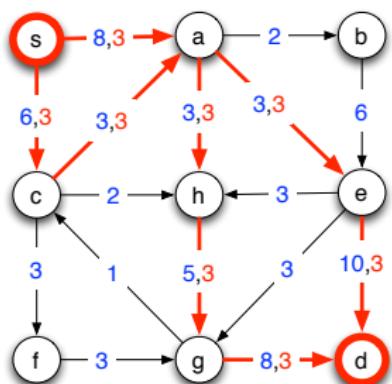
ex: $f(sa) = \min\{8, 0 + 3\} = 3$
 $f(ge) = \min\{0, 0 + 3\} = 0$

$$f(vu) \leftarrow f(vu) - \max\{0; f(uv) + \varepsilon - c(uv)\}$$

ex: $f(eg) = 3 - \max\{0, 3 + 3 - 3\} = 0$

Ford-Fulkerson Algorithm

Example

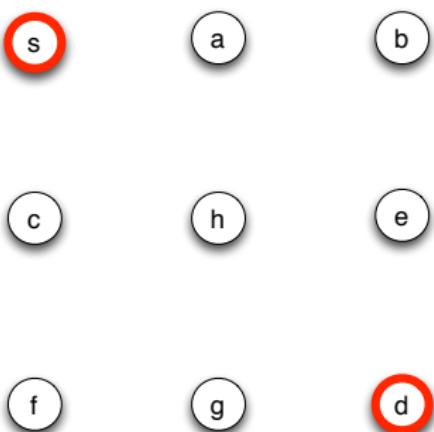
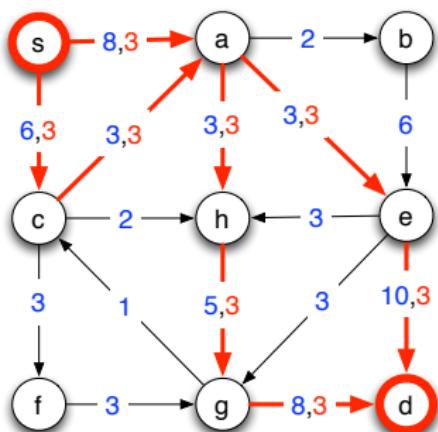


The flow has been increased:

$$v(f) = 6$$

Ford-Fulkerson Algorithm

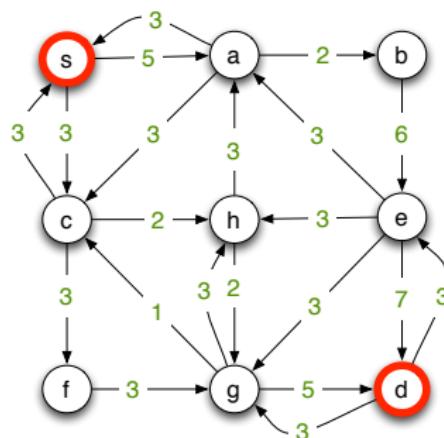
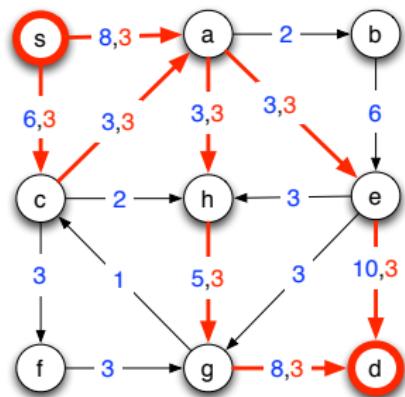
Example



1st step: We have to compute the auxiliary digraph
starting from the new **current flow**

Ford-Fulkerson Algorithm

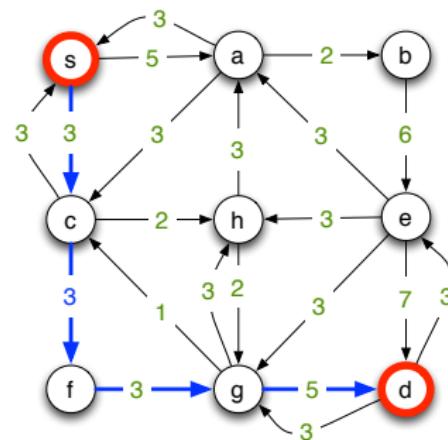
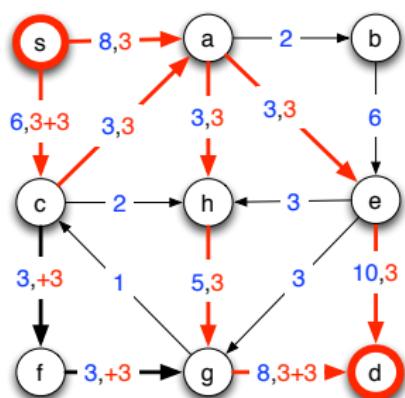
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Ford-Fulkerson Algorithm

Example

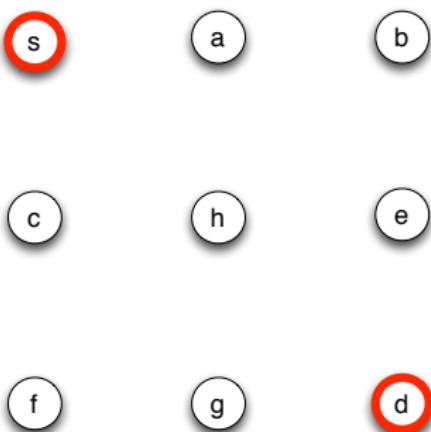
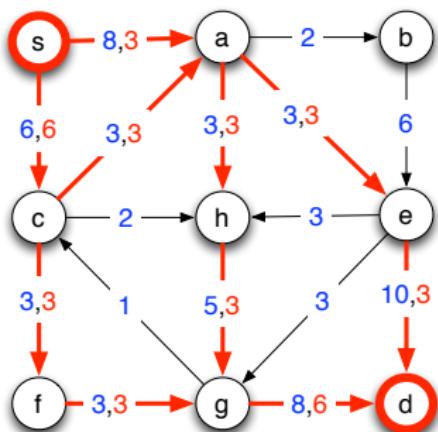


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Ford-Fulkerson Algorithm

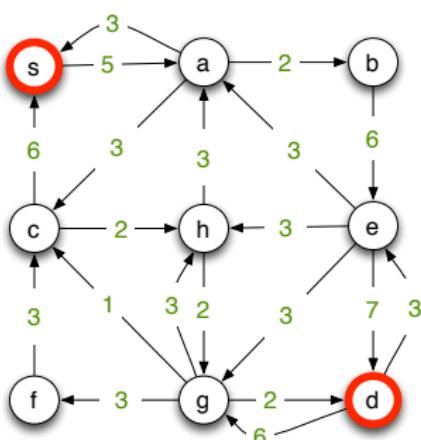
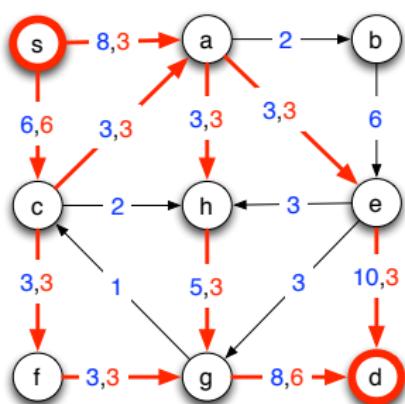
Example



1st step: We have to compute the auxiliary digraph
starting from the new **current flow** $v(f) = 9$

Ford-Fulkerson Algorithm

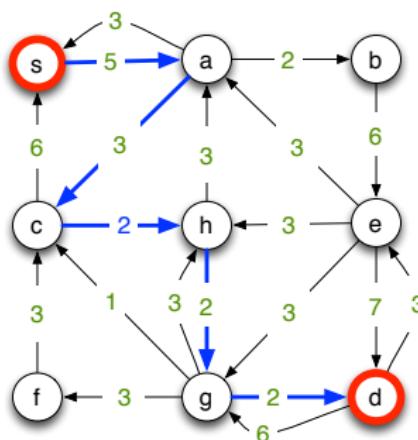
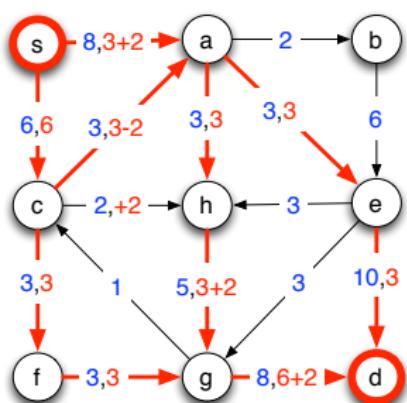
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Ford-Fulkerson Algorithm

Example

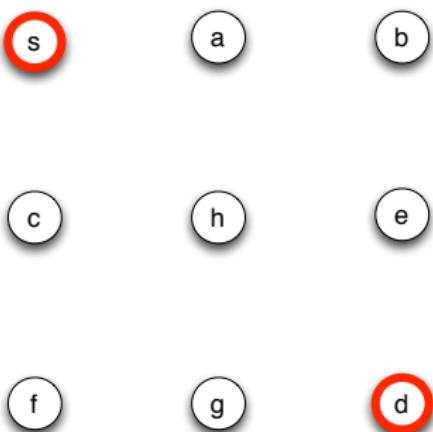
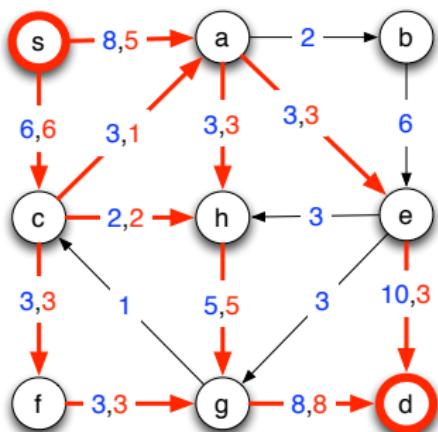


2^{nd} step: Find in D_{aux} a directed s,d -path with positive capacity

in the example $P = (s, a, c, h, g, d)$ and $\varepsilon = \min_{arc \in P} c_{aux}(arc) = 2$
and Push ε units of flow along P in D

Ford-Fulkerson Algorithm

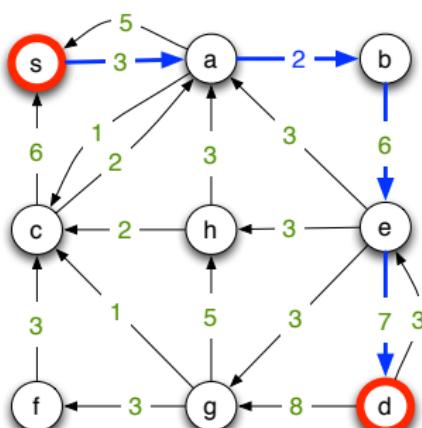
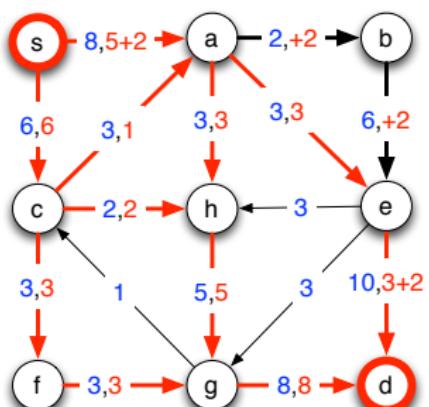
Example



1st step: We have to compute the auxiliary digraph
starting from the new **current flow** $v(f) = 11$

Ford-Fulkerson Algorithm

Example



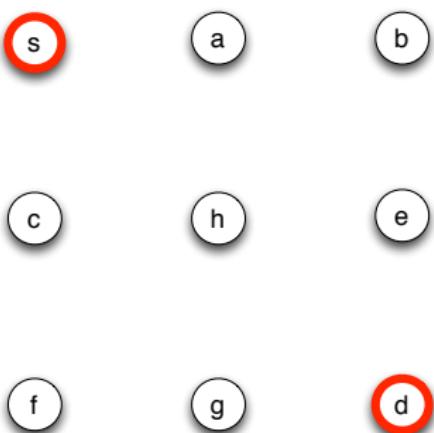
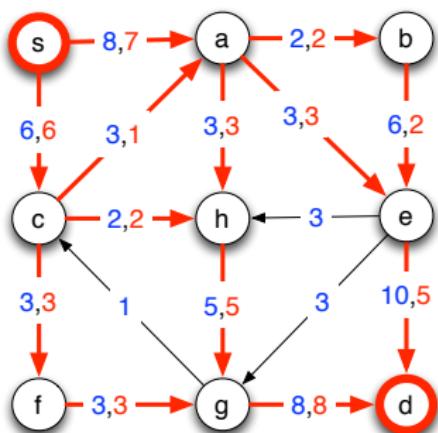
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Ford-Fulkerson Algorithm

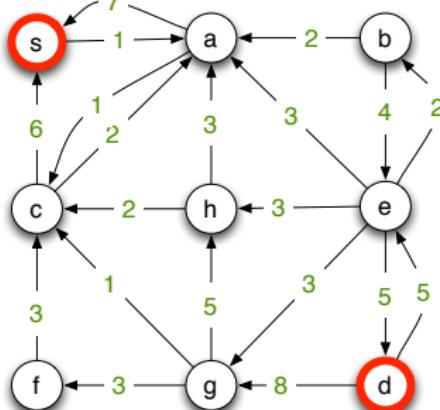
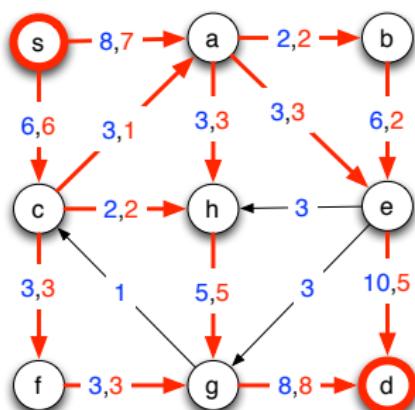
Example



1st step: We have to compute the auxiliary digraph
starting from the new **current flow** $v(f) = 13$

Ford-Fulkerson Algorithm

Example

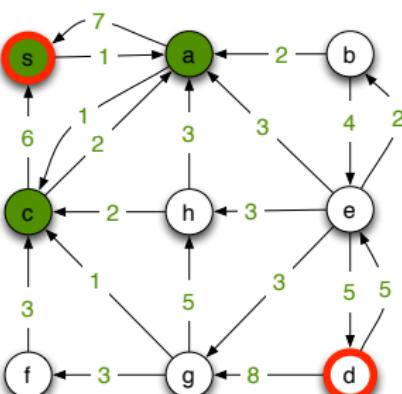
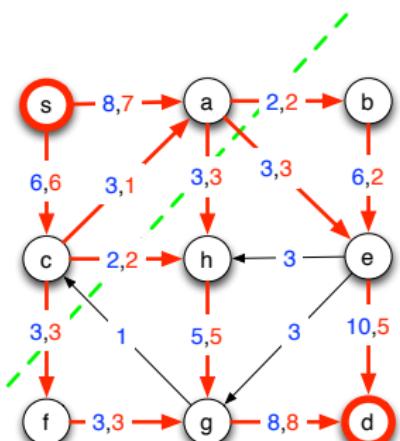


2nd step: Find in D_{aux} a directed s,d -path with positive capacity
in the example, there is no such path !!

Exercise: Prove that the **current flow** is maximum

Ford-Fulkerson Algorithm

Example



Aux. digraph defines a cut: $V_s = \{\text{vertices reachable from } s\}; V_d = V \setminus V_s$.
 $\delta(V_s, V_d) = 13 = v(f) \Rightarrow f \text{ is maximum.}$

Outline

- 1 Transportation Problem
- 2 Elementary Flow Network
- 3 Upper bound on Flow: Cut
- 4 Ford-Fulkerson Algorithm
- 5 Min Cut=Max flow
- 6 Application to Connectivity: Menger Theorem
- 7 Application to Matching

Ford-Fulkerson Algorithm: Min Cut=Max flow

Exercise: Let $f : A \rightarrow \mathbb{R}^+$ be computed by FF-algorithm

- 1 Prove that f is a valid flow
- 2 Prove that f is a maximum flow

idea of proof of 2: FF-algorithm returns a flow $f : A \rightarrow \mathbb{R}^+$
 it also returns a cut with capacity $v(f)$.

Theorem: $\text{Max Flow} = \text{Min Cut}$ *(duality)*

For any digraph D , $c : A(D) \rightarrow \mathbb{R}^+$ and $s, d \in V(D)$:
 maximum value of a flow from s to d = minimum capacity of a s, d -cut

Ford-Fulkerson Algorithm: Complexity

Remark: the FF-algorithm may not terminate

Exercise: Assume capacities are integral $c : A \rightarrow \mathbb{N}$

- Prove that FF-algorithm always terminates
- Prove that the maximum value of a flow is integral

Complexity with integral capacities

FF-algorithm terminates in time $O(v_{max}|A(D)|)$

Ford-Fulkerson Algorithm: Complexity

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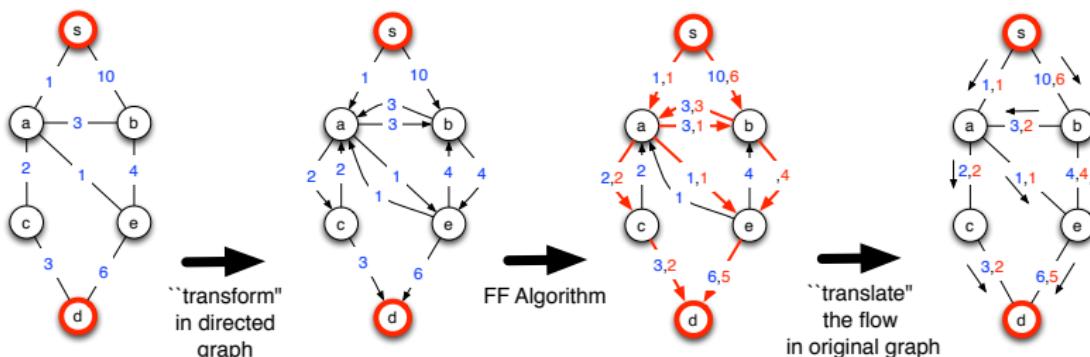
Ford-Fulkerson Algorithm: undirected graphs

In undirected graph $G = (V, E)$

$f : V \times V \rightarrow \mathbb{R}^+$ is defined on ordered pairs of vertices

Flow conservation: for any $v \in V \setminus \{s, d\}$, $\sum_{u \in N(v)} f(uv) = \sum_{u \in N(v)} f(vu)$

Capacity: for any $e = \{u, v\} \in E$, $f(uv) + f(vu) \leq c(\{u, v\})$.



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Ford-Fulkerson Algorithm: Application to Connectivity

Edge-disjoint Paths:

$$G = (V, E), s, d \in V, k \in \mathbb{N}$$

Exercise: $\exists k$ edge-disjoint paths from s to d in G

$\Leftrightarrow \exists$ a s, d -flow with value k in G (with capacity 1 on edges)

Idea: use the fact that there exists an integral maximum flow

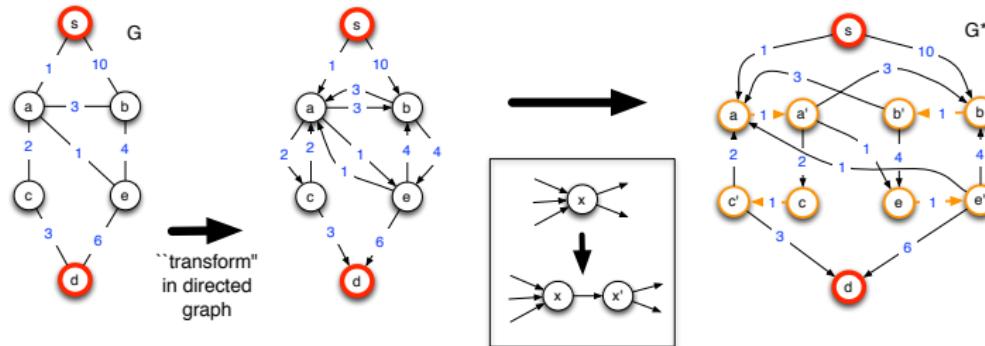
Ford-Fulkerson Algorithm: Application to Connectivity

Vertex-disjoint Paths:

$$G = (V, E), s, d \in V, k \in \mathbb{N}$$

Exercise: $\exists k$ (internally) vertex-disjoint paths from s to d in G

$\Leftrightarrow \exists$ a s, d -flow with value k in G^* (see Picture)



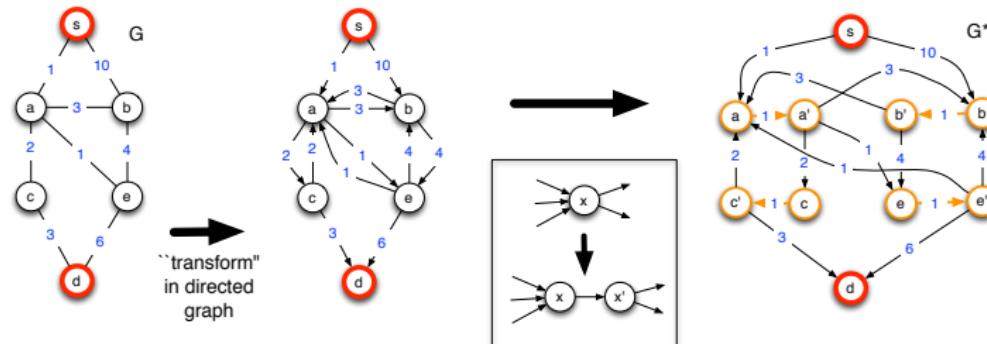
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$\Leftrightarrow \exists$ a s, d -flow with value k in G^* (see Picture)



$S \subseteq V$ is a s, d -separator if s and d in different components of $G \setminus S$

Menger's Theorem (1927):

(duality)

For any graph $G = (V, E)$, $s, d \in V$ non-adjacent, $k \in \mathbb{N}$:

$\exists k$ internally-vertex-disjoint paths from s to $d \Leftrightarrow \nexists s, d$ -separator of size $< k$.

Outline

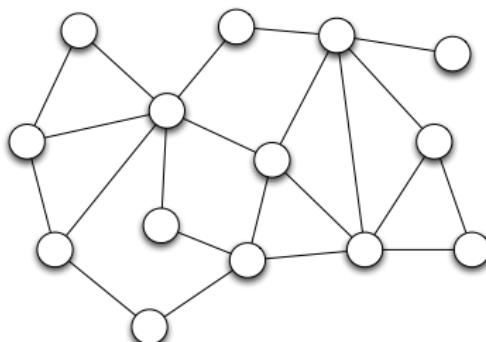
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Ford-Fulkerson Algorithm: Application to Matching

Let $G = (V, E)$ be a graph.

Matching: A set M of pairwise disjoint edges in a graph

$$(M \subseteq E)$$



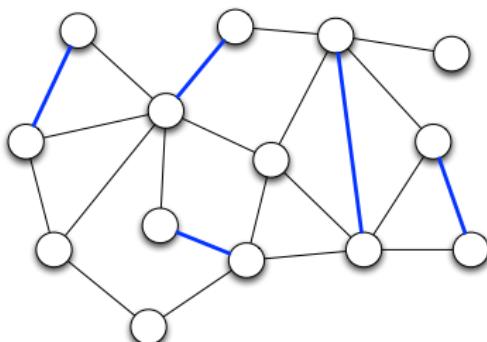
Problem: Compute a matching of maximum cardinality

Ford-Fulkerson Algorithm: Application to Matching

Let $G = (V, E)$ be a graph.

Matching: A set M of pairwise disjoint edges in a graph

$$(M \subseteq E)$$



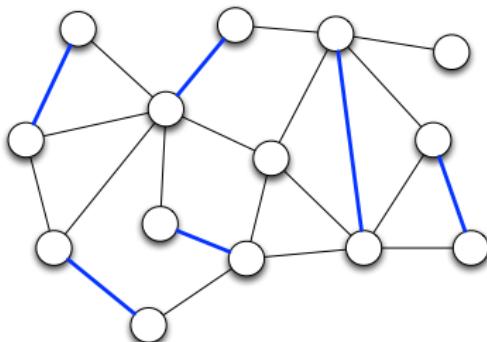
Question: is this matching maximum?

Ford-Fulkerson Algorithm: Application to Matching

Let $G = (V, E)$ be a graph.

Matching: A set M of pairwise disjoint edges in a graph

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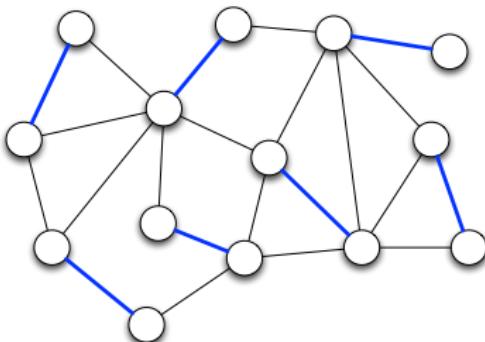
Question: and this one?

Ford-Fulkerson Algorithm: Application to Matching

Let $G = (V, E)$ be a graph.

Matching: A set M of pairwise disjoint edges in a graph

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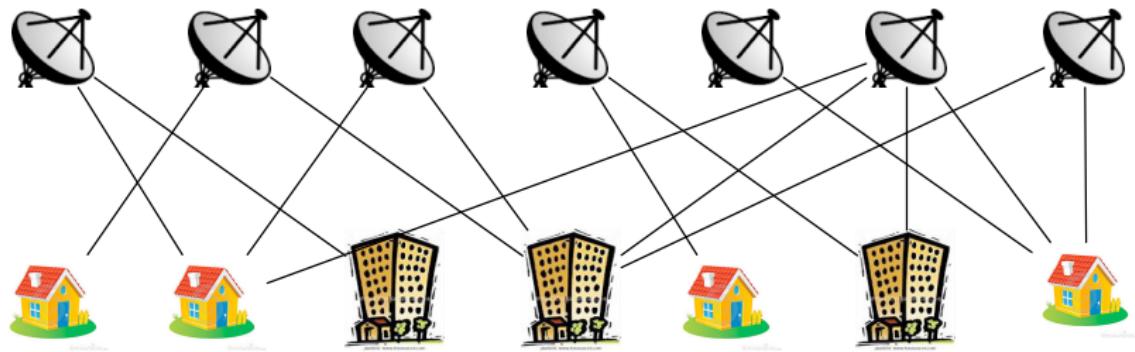


Question: and this one?

Exercise: Prove that any matching M is such that $|M| \leq \lfloor |V|/2 \rfloor$

Ford-Fulkerson Algorithm: Application to Matching

Example of application of matching:

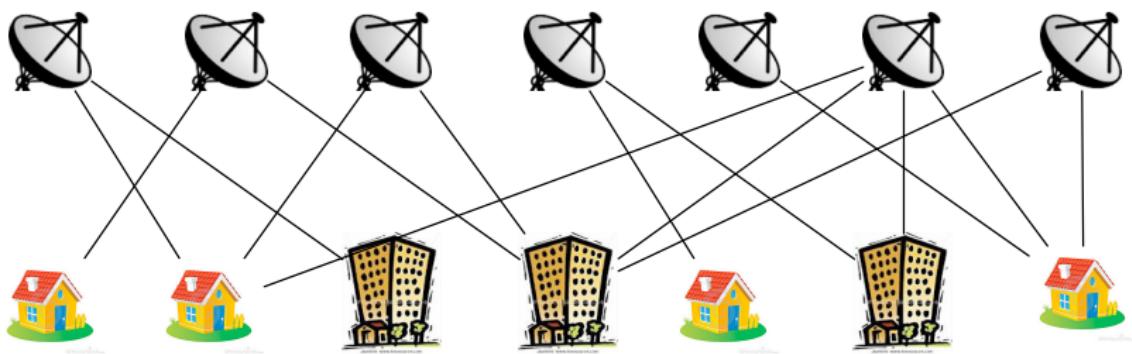


stable set: set of vertices pairwise non-adjacent

Bipartite graph: $G = (V, E)$ and $V = A \cup B$ can be partitioned into 2 stable sets A and B .

Ford-Fulkerson Algorithm: Application to Matching

Example of application of matching:



stable set: set of vertices pairwise non-adjacent

Bipartite graph: $G = (V, E)$ and $V = A \cup B$ can be partitioned into 2 stable sets A and B .

Matching in bipartite graphs

$$G = (A \cup B, E)$$

- **Hall's Theorem** (1935): \exists matching saturating $A \Leftrightarrow \forall S \subseteq A, |N(S)| \geq |S|$.
- **Hungarian Method** [Kuhn, 1955]: compute a maximum matching in time $O(n^3)$

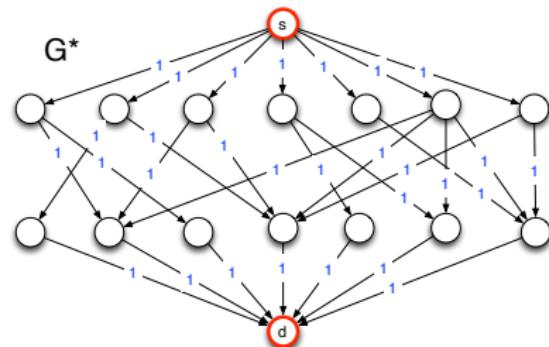
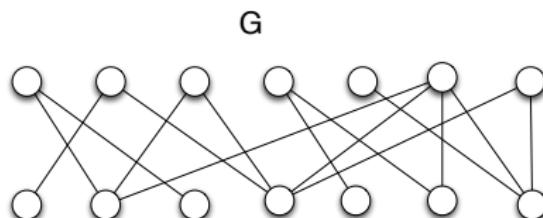
Ford-Fulkerson Algorithm: Application to Matching

$G = (A \cup B, E)$ a bipartite graph, $k \in \mathbb{N}$

Exercise: Prove that there exists a matching of size $\geq k$

\Leftrightarrow exists a s,d -flow of value $\geq k$ in G^* (see Picture)

Idea: use the fact that there exists an integral maximum flow



Summary: To be remembered

- flow, cut
- Ford-Fulkerson algorithm $O(|E| \text{flow}_{\max})$ for rational capacities
- **Min Cut = Max Flow**
- Menger Theorem (max # disjoint paths = min size of separator)
- Matching (Hungarian method, Edmonds algorithm, Hall Theorem)