Graph Theory and Optimization
Introduction on Graphs

Nicolas Nisse

Université Côte d'Azur, Inria, CNRS, I3S, France

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Outline

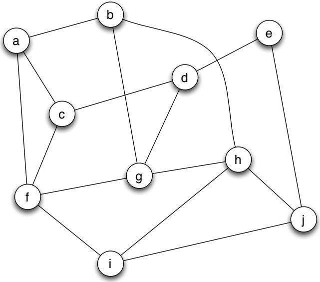
Graph: terminology and notations (Vertex/Edge)

A graph $G = (V, E)$

Vertices: $V = V(G)$ is a finite set *circles*

Edges: $E = V(E) \subseteq \{\{u, v\} \mid u, v \in V\}$ is a binary relation on V

lines between two circles



Example: $G = (V, E)$ with $V = \{a, b, c, d, e, f, g, h, i, j\}$ and
$E = \{\{a,b\},\{a,c\},\{a,f\},\{b,g\},\{b,h\},\{c,f\},\{c,d\},\{d,g\},\{d,e\},\{e,j\},\{f,g\},\{f,i\},\{g,h\},\{h,i\},\{h,j\},\{i,j\}\}$.

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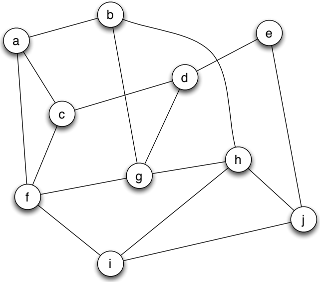

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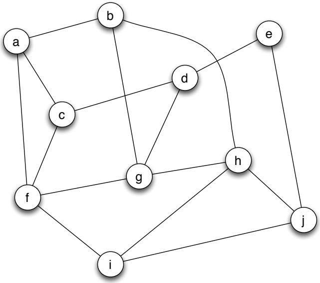
$E = \{\{a, b\}, \{a, c\}, \{a, f\}, \{b, g\}, \{b, h\}, \{c, f\}, \{c, d\}, \{d, g\}, \{d, e\}, \{e, j\}, \{f, g\}, \{f, i\}, \{g, h\}, \{h, i\}, \{h, j\}, \{i, j\}\}$.

Exercise: What is the maximum number of edges of a graph with n vertices?

Outline

Graph: terminology and notations (Neighbor/Degree)

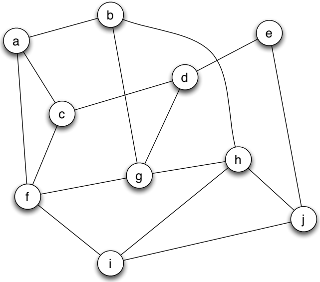


- two vertices $x \in V$ and $y \in V$ are adjacent or neighbors if $\{x, y\} \in E$

  i.e. there is an edge $\{x, y\}$
- $N(x)$: set of neighbors of $x \in V$ **ex:** $N(g) = \{b, d, f, h\} \subseteq V$
- degree of $x \in V$: number of neighbors of x i.e., $deg(x) = |N(x)|$

Exercise: Prove that, for any graph $G = (V, E)$, $\sum_{x \in V} deg(x) = 2|E|$

Graph: terminology and notations (Neighbor/Degree)



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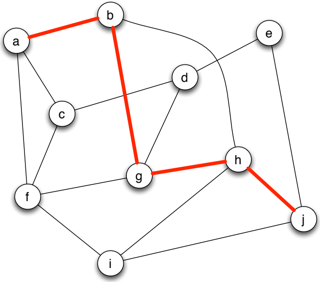
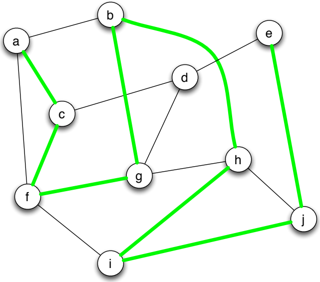
Graph: terminology and notations (Path/Cycle)



- Path: sequence $(v_1, \cdots, v_\ell)$ of distinct vertices such that consecutive vertices are adjacent, i.e., $\{v_i, v_{i+1}\} \in E$ for any $1 \leq i < \ell$
  ex: $P_1 = (a, b, g, h, i)$

Graph: terminology and notations (Path/Cycle)



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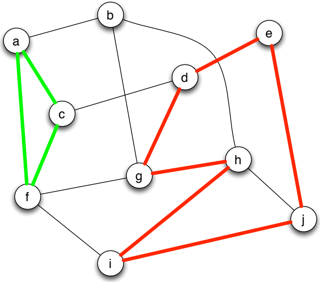
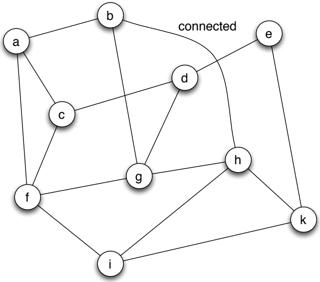
  ex: $P_1 = (a, b, g, h, i)$, $P_2 = (a, c, f, g, b, h, i, j, e)$

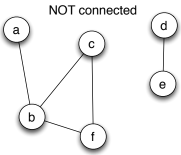
Graph: terminology and notations (Path/Cycle)



- Path: sequence $(v_1, \cdots, v_\ell)$ of distinct vertices such that consecutive vertices are adjacent, i.e., $\{v_i, v_{i+1}\} \in E$ for any $1 \leq i < \ell$
- Cycle: path $(v_1, \cdots, v_\ell)$ such that $\ell \geq 3$ and $\{v_1, v_\ell\} \in E$
  ex: $C_1 = (d, e, j, i, h, g)$, $C_2 = (a, c, f)$

Graph: terminology and notations (Path/Cycle)





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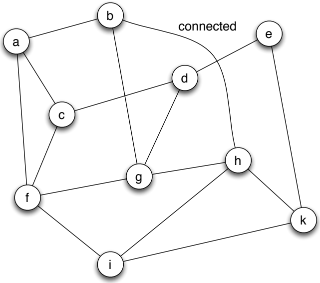
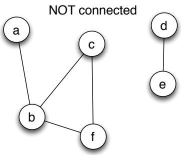
- $G = (V, E)$ is connected if, for every two vertices $x \in V$ and $y \in V$, there exists a path from x to y.

Graph: terminology and notations (Path/Cycle)



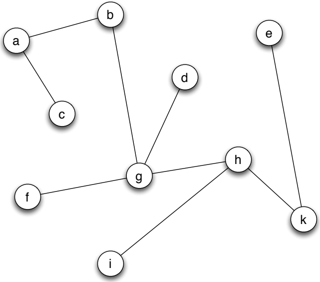


- **Path**: sequence $(v_1, \cdots, v_\ell)$ of distinct vertices such that consecutive vertices are adjacent, i.e., $\{v_i, v_{i+1}\} \in E$ for any $1 \leq i < \ell$
- **Cycle**: path $(v_1, \cdots, v_\ell)$ such that $\ell \geq 3$ and $\{v_1, v_\ell\} \in E$
- $G = (V, E)$ is **connected** if, for every two vertices $x \in V$ and $y \in V$, there exists a path from x to y.

Exercise: Prove that if $|E| < |V| - 1$ then $G = (V, E)$ is NOT connected

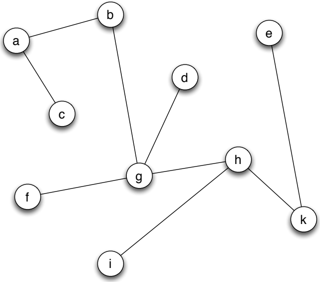
Outline

Graph: terminology and notations (Tree)



- Tree: connected graph without cycles
- Leaf: vertex of degree 1 in a tree

Graph: terminology and notations (Tree)



- Tree: connected graph without cycles
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Trees are important because:

"simple" structure + "minimum" structure ensuring connectivity

Theorem:

$T = (V, E)$ is a tree $\Leftrightarrow$ T connected and $|V| = |E| + 1$

Graph: terminology and notations (Tree)

Theorem: $T = (V, E)$ is a tree $\Leftrightarrow$ T connected and $|V| = |E| + 1$

$\Leftarrow$ By contradiction:

- if T not a tree, then $\exists$ a cycle $(v_1, \cdots, v_\ell)$
- Let T' be obtained from T by removing edge $\{v_1, v_\ell\}$
- T' is connected *"technical" part, to be proved*
- $|E(T')| = |E| - 1 = |V| - 2 = |V(T')| - 2$
- so $|E'| < |V'| - 1$ and T' is not connected by previous Exercise

A contradiction

Graph: terminology and notations (Tree)

Theorem: $T = (V, E)$ is a tree $\Leftrightarrow$ T connected and $|V| = |E| + 1$

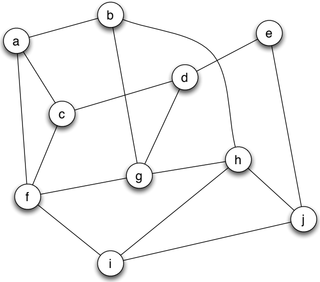
$\Rightarrow$ Induction on $|V|$ OK if $|V| = 1$

- Let $P = (v_1, \cdots, v_\ell)$ be a longest path in T (ℓ max., in particular $\ell \geq 2$)
- v_1 is a leaf. By contradiction:
  - assume $deg(v_1) > 1$, and $x \in N(v_1) \setminus \{v_2\}$
  - $x \notin V(P)$ otherwise there is a cycle in T
  - then, $(x, v_1, \cdots, v_\ell)$ path longer than P, a contradiction
- then $S = T \setminus \{v_1\}$ is a tree *"technical" part, to be proved*
- $|V(S)| < |V|$ so, by induction $|V(S)| = |E(S)| + 1$
- $|V| = |V(S)| + 1$ and $|E| = |E(S)| + 1$, so $|V| = |E| - 1$

Outline

1. Vertex/Edge
2. Neighbor/Degree
3. Path/Cycle
4. Trees
5. SubGraph

Graph: terminology and notations (subgraph)

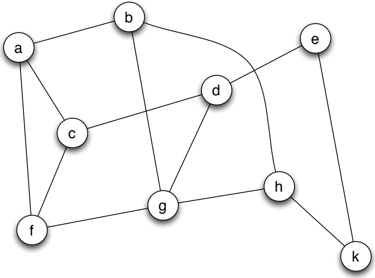


- Subgraph of $G=(V,E)$: any graph $H=(V',E')$ with

$$V' \subseteq V \text{ and } E' \subseteq \{\{x,y\} \in E \mid x,y \in V'\}$$

obtained from G by removing some vertices and edges

Graph: terminology and notations (subgraph)



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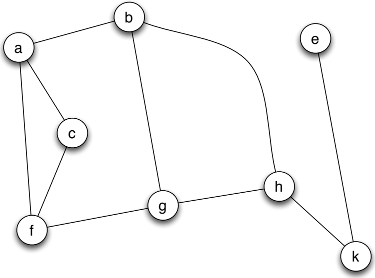

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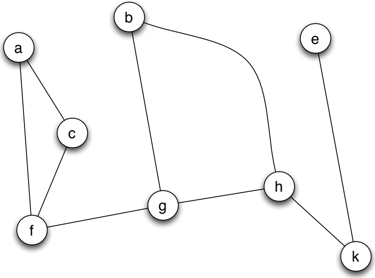

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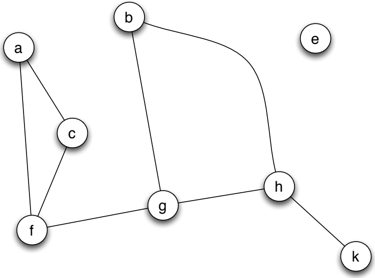

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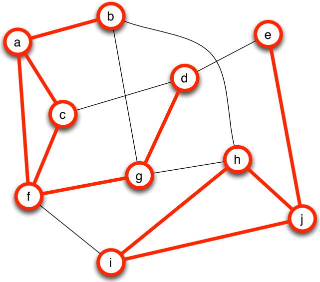
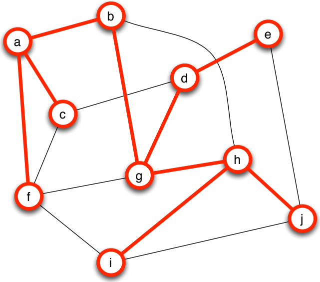


- Subgraph: $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq \{\{x,y\} \in E \mid x,y \in V'\}$
- Spanning subgraph of G: subgraph $H = (V', E')$ where $V' = V$

obtained from G by removing only some edges

Graph: terminology and notations (subgraph)



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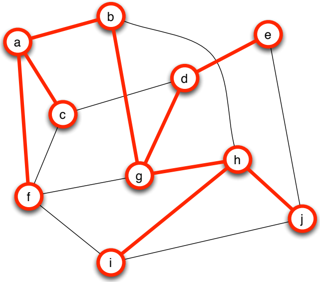
- Spanning subgraph of G: subgraph $H = (V', E')$ where $V' = V$

  obtained from G by removing only some edges
- Spanning tree of G: spanning subgraph $H = (V, E')$ with H a tree

Graph: terminology and notations (subgraph)



- Subgraph: $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq \{\{x,y\} \in E \mid x,y \in V'\}$
- Spanning subgraph of G: subgraph $H = (V', E')$ where $V' = V$

  obtained from G by removing only some edges
- Spanning tree of G: spanning subgraph $H = (V, E')$ with H a tree

Exercise: A graph G is connected if and only if G has a spanning tree

Summary: To be remembered

All definitions will be important in next lectures
Please remember:

- graph, vertex/vertices, edge
- neighbor, degree
- path, cycle
- connected graph
- tree
- subgraph, spanning subgraph