

# Graph Theory and Optimization

## Introduction on Graphs

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# Outline

- 1 Vertex/Edge
- 2 Neighbor/Degree
- 3 Path/Cycle
- 4 Trees
- 5 SubGraph

# Graph: terminology and notations (Vertex/Edge)

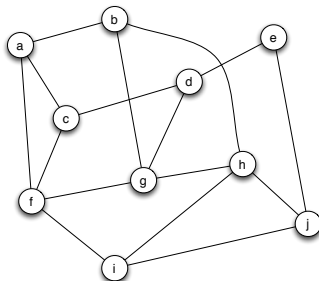
A graph  $G = (V, E)$

**Vertices:**  $V = V(G)$  is a finite set

*circles*

**Edges:**  $E = V(E) \subseteq \{\{u, v\} \mid u, v \in V\}$  is a binary relation on  $V$

*lines between two circles*



**Example:**  $G = (V, E)$  with  $V = \{a, b, c, d, e, f, g, h, i, j\}$  and  
 $E = \{\{a, b\}, \{a, c\}, \{a, f\}, \{b, g\}, \{b, h\}, \{c, f\}, \{c, d\}, \{d, g\}, \{d, e\}, \{e, j\}, \{f, g\}, \{f, i\}, \{g, h\}, \{h, i\}, \{h, j\}, \{i, j\}\}.$

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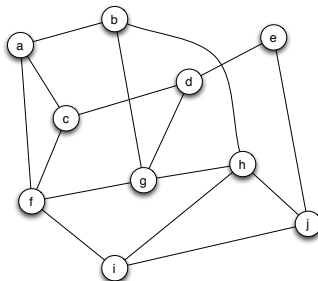
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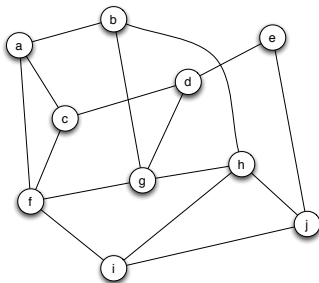
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**Exercise:** What is the maximum number of edges of a graph with  $n$  vertices?

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# Graph: terminology and notations (Neighbor/Degree)

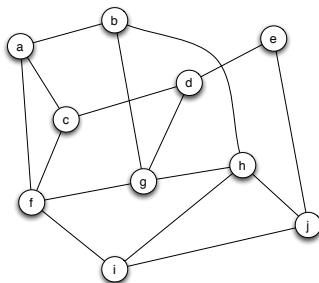


- two vertices  $x \in V$  and  $y \in V$  are **adjacent** or **neighbors** if  $\{x, y\} \in E$   
i.e. there is an edge  $\{x, y\}$
- $N(x)$ : set of neighbors of  $x \in V$  **ex:**  $N(g) = \{b, d, f, h\} \subseteq V$
- **degree** of  $x \in V$ : number of neighbors of  $x$  i.e.,  $\deg(x) = |N(x)|$

**Exercise:** Prove that, for any graph  $G = (V, E)$ ,

$$\sum_{x \in V} \deg(x) = 2|E|$$

# Graph: terminology and notations (Neighbor/Degree)



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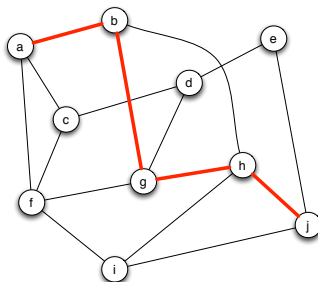
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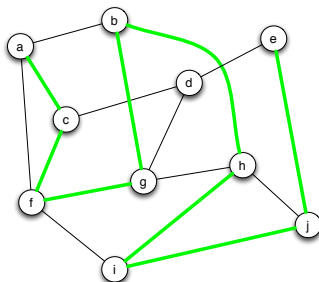


# Graph: terminology and notations (Path/Cycle)



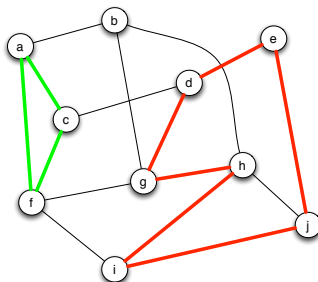
- **Path:** sequence  $(v_1, \dots, v_\ell)$  of distinct vertices such that consecutive vertices are adjacent, i.e.,  $\{v_i, v_{i+1}\} \in E$  for any  $1 \leq i < \ell$   
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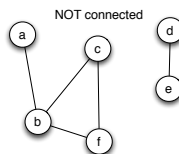
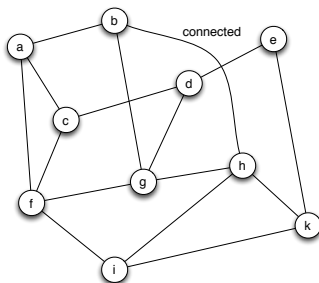
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**ex:**  $P_1 = (a, b, g, h, i)$ ,  $P_2 = (a, c, f, g, b, h, i, j, e)$

# Graph: terminology and notations (Path/Cycle)



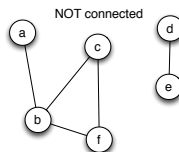
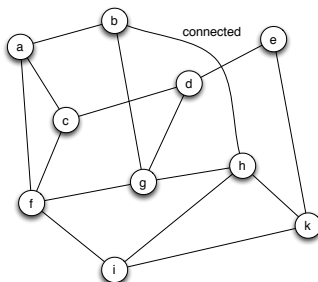
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- **Cycle:** path  $(v_1, \dots, v_\ell)$  such that  $\ell \geq 3$  and  $\{v_1, v_\ell\} \in E$   
**ex:**  $C_1 = (d, e, j, i, h, g)$ ,  $C_2 = (a, c, f)$

# Graph: terminology and notations (Path/Cycle)



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- $G = (V, E)$  is **connected** if, for every two vertices  $x \in V$  and  $y \in V$ , there exists a path from  $x$  to  $y$ .

# Graph: terminology and notations (Path/Cycle)



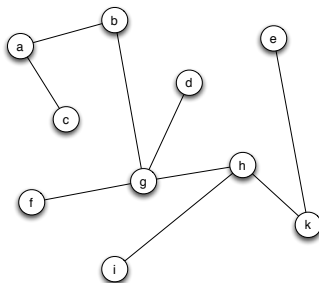
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**Exercise:** Prove that if  $|E| < |V| - 1$  then  $G = (V, E)$  is NOT connected

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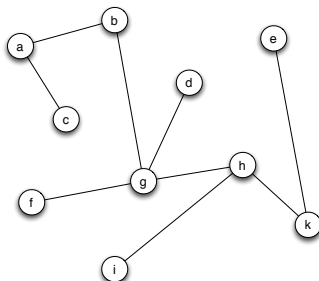
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Trees are important because:

“simple” structure + “minimum” structure ensuring connectivity

**Theorem:**

$T = (V, E)$  is a tree  $\Leftrightarrow T$  connected and  $|V| = |E| + 1$



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**Theorem:**  $T = (V, E)$  is a tree  $\Leftrightarrow T$  connected and  $|V| = |E| + 1$

$\Leftarrow$  By contradiction:

- if  $T$  not a tree, then  $\exists$  a cycle  $(v_1, \dots, v_\ell)$
- Let  $T'$  be obtained from  $T$  by removing edge  $\{v_1, v_\ell\}$
- $T'$  is connected *“technical” part, to be proved*
- $|E(T')| = |E| - 1 = |V| - 2 = |V(T')| - 2$
- so  $|E'| < |V'| - 1$  and  $T'$  is not connected by previous Exercise

A contradiction

# Graph: terminology and notations (Tree)

**Theorem:**  $T = (V, E)$  is a tree  $\Leftrightarrow T$  connected and  $|V| = |E| + 1$

$\Rightarrow$  Induction on  $|V|$

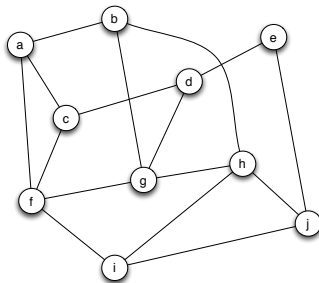
OK if  $|V| = 1$

- Let  $P = (v_1, \dots, v_\ell)$  be a longest path in  $T$  ( $\ell$  max., in particular  $\ell \geq 2$ )
- $v_1$  is a leaf. By contradiction:
  - assume  $\deg(v_1) > 1$ , and  $x \in N(v_1) \setminus \{v_2\}$
  - $x \notin V(P)$  otherwise there is a cycle in  $T$
  - then,  $(x, v_1, \dots, v_\ell)$  path longer than  $P$ , a contradiction
- then  $S = T \setminus \{v_1\}$  is a tree *“technical” part, to be proved*
- $|V(S)| < |V|$  so, by induction  $|V(S)| = |E(S)| + 1$
- $|V| = |V(S)| + 1$  and  $|E| = |E(S)| + 1$ , so  $|V| = |E| - 1$

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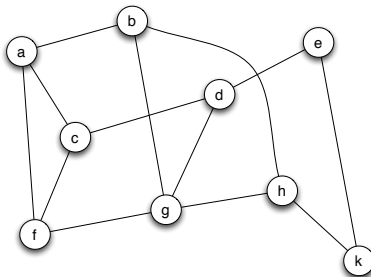
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# Graph: terminology and notations (subgraph)



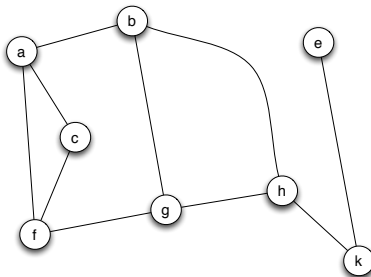
- **Subgraph** of  $G = (V, E)$ : any graph  $H = (V', E')$  with  
 $V' \subseteq V$  and  $E' \subseteq \{\{x, y\} \in E \mid x, y \in V'\}$   
*obtained from  $G$  by removing some vertices and edges*

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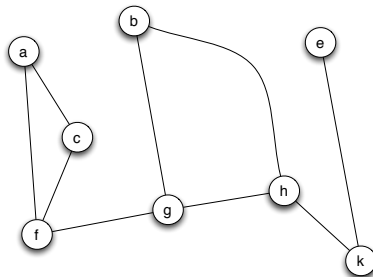
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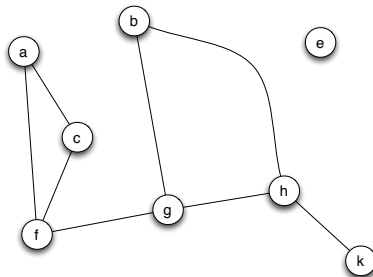
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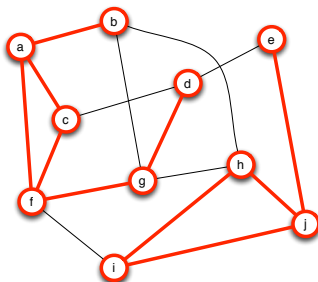
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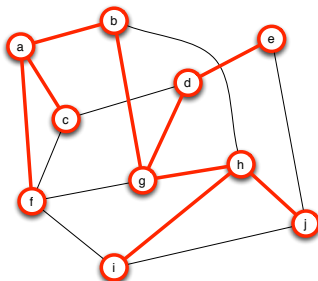


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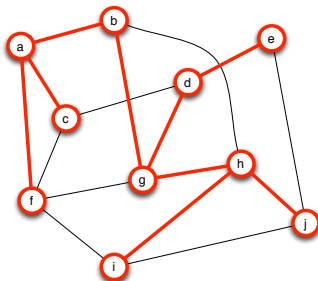
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**Exercise:** A graph  $G$  is connected if and only if  $G$  has a spanning tree

## Summary: To be remembered

All definitions will be important in next lectures

Please remember:

- graph, vertex/vertices, edge
- neighbor, degree
- path, cycle
- connected graph
- tree
- subgraph, spanning subgraph