Graph Theory and Optimization
Introduction on Duality in LP

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Thank you to F. Giroire for his slides

Motivation

- Finding bounds on the optimal solution. Provides a measure of the "goodness" of a solution.
- Provide certificate of optimality.
- Economic interpretation of the dual problem.

Outline

Duality Theorem: introduction

$$\begin{array}{lrcrcrcrcl}
\text{Maximize} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{Subject to :} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
& & & & & & & x_1, x_2, x_3, x_4 & \geq & 0.
\end{array}$$

Lower bound: a feasible solution, e.g. $(0, 0, 1, 0) \Rightarrow z^* \geq 5$.

What if we want an upper bound?

Duality Theorem: introduction

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\end{array}$$

Second Inequation $\times 5/3$:

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}.$$

Note that (all variables are positive),

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4$$

Hence, a first bound:

$$z^* \leq \frac{275}{3}.$$

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& & & & & & & x_1, x_2, x_3, x_4 & \geq & 0.
\end{array}
$$

Similarly, $2^d + 3^d$ constraints:

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$$

Hence, a second bound:

$$z^* \leq 58.$$

$\rightarrow$ need for a systematic strategy.

Duality Theorem: introduction

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\text{Subject to :} & & & & & & & & & \\
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& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \qquad \boxed{\times y_3} \\
& & & & & & & x_1, x_2, x_3, x_4 & \geq & 0.
\end{array}
$$

Build linear combinations of the constraints. Summing:

$$
(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 \\
+(3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3.
$$

We want left part upper bound of z. We need coefficient of $x_j \geq$ coefficient in z:

$$
\begin{array}{rcrcrcr}
y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
-y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
-y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3.
\end{array}
$$

Duality Theorem: introduction

Build linear combinations of the constraints. Summing:

$$(y_1+5y_2-y_3)x_1+(-y_1+y_2+2y_3)x_2+(-y_1+3y_2+3y_3)x_3$$
$$+(3y_1+8y_2-5y_3)x_4 \leq y_1+55y_2+3y_3.$$

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If the $y_i \geq 0$ and satisfy theses inequations, then

$$4x_1+x_2+5x_3+3x_4 \leq y_1+55y_2+3y_3.$$

In particular,

$$z^* \leq y_1+55y_2+3y_3.$$

Duality Theorem: introduction

Objective: smallest possible upper bound. Hence, we solve the following PL:

$$
\begin{array}{lrcrcrcr}
\text{Minimize} & y_1 & + & 55y_2 & + & 3y_3 & & \\
\text{Subject to:} & & & & & & & \\
 & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
 & & & & & y_1, y_2, y_3 & \geq & 0.
\end{array}
$$

It is the dual problem of the problem.

Outline

The Dual Problem

Primal problem:

$$
\begin{array}{rlcll}
\text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\
\text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \cdots, m) \\
 & x_j & \geq & 0 & (j = 1, 2, \cdots, n)
\end{array}
$$

Its dual problem is defined by the LP problem:

$$
\begin{array}{rlcll}
\text{Minimize} & \sum_{i=1}^{m} b_i y_i & & & \\
\text{Subject to:} & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & (j = 1, 2, \cdots, n) \\
 & y_i & \geq & 0 & (i = 1, 2, \cdots, m)
\end{array}
$$

Weak Duality Theorem

Theorem — **Weak Duality**

If $(x_1, x_2, ..., x_n)$ is feasible for the primal and $(y_1, y_2, ... y_m)$ is feasible for the dual, then

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

Proof:

$$\begin{aligned} \sum_j c_j x_j &\leq \sum_j (\sum_i y_i a_{ij}) x_j & \text{dual definition: } \sum_i y_i a_{ij} \geq c_j \\ &= \sum_i (\sum_j a_{ij} x_j) y_i & \\ &\leq \sum_i b_i y_i & \text{primal definition: } \sum_i x_i a_{ij} \leq b_j \end{aligned}$$

Corollary:

The optimal value of the dual is an upper bound for the optimal value of the primal.

$$\max_{(x_1,\cdots,x_n)\ feasible} \sum_j c_j x_j \leq \min_{(y_1,\cdots,y_m)\ feasible} \sum_i b_i y_i.$$

Weak Duality Theorem

Theorem — **Weak Duality**

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Proof:

$$\begin{aligned} \sum_j c_j x_j &\leq \sum_j (\sum_i y_i a_{ij}) x_j && \text{dual definition: } \sum_i y_i a_{ij} \geq c_j \\ &= \sum_i (\sum_j a_{ij} x_j) y_i && \\ &\leq \sum_i b_i y_i && \text{primal definition: } \sum_i x_i a_{ij} \leq b_j \end{aligned}$$

Corollary:

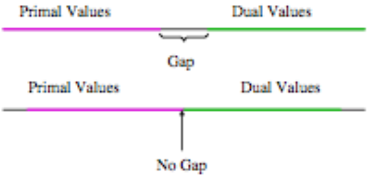
The optimal value of the dual is an upper bound for the optimal value of the primal.

$$\max_{(x_1,\cdots,x_n)\ feasible} \sum_j c_j x_j \leq \min_{(y_1,\cdots,y_m)\ feasible} \sum_i b_i y_i.$$

Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?



Strong Duality Theorem

Theorem — **Strong duality**

If the primal problem has an optimal solution,

$$x^* = (x_1^*, ..., X_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, ..., y_n^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

Relationship between the Primal and Dual Problems

Lemma: The dual of the dual is always the primal problem.

Corollary: + (Strong Duality Theorem) $\Rightarrow$ Primal has an optimal solution iff dual has an optimal solution.
Weak duality: Primal unbounded $\Rightarrow$ dual unfeasible.

			Dual	
		Optimal	Unfeasible	Unbounded
Primal	Optimal	X		
	Unfeasible		X	X
	Unbounded		X	

Relationship between the Primal and Dual Problems

Lemma: The dual of the dual is always the primal problem.

Corollary: + (Strong Duality Theorem) $\Rightarrow$ Primal has an optimal solution iff dual has an optimal solution.
Weak duality: Primal unbounded $\Rightarrow$ dual unfeasible.

		Dual Optimal	Dual Unfeasible	Dual Unbounded
Primal	Optimal	X		
	Unfeasible		X	X
	Unbounded		X	

Application of Duality to Maximum flow

$D = (V, A)$ be a graph with capacity $c : A \to \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a maximum flow from s to t.

$$
\begin{array}{llcrr}
\text{Maximize} & \displaystyle\sum_{(s,u)\in A} f(su) & & & \\
\text{Subject to:} & f(a) & \leq & c(a) & \text{for all } a \in A \\
& \displaystyle\sum_{(v,u)\in A} f(vu) & = & \displaystyle\sum_{(u,v)\in A} f(uv) & \text{for all } v \in V \setminus \{s,t\} \\
& f(a) & \geq & 0 & \text{for all } a \in A
\end{array}
$$

Exercise: Write the dual program

Application of Duality to Maximum flow

Variable y_a per edge constraint; Variable z_v per vertex-constraint

$$R = \sum_{a \in A} f(a) y_a + \sum_{v \in V \setminus \{s,t\}} \Big(\sum_{(v,u) \in A} f(vu) - \sum_{(u,v) \in A} f(uv) \Big) z_v \leq \sum_{a \in A} c(a) y_a$$

Application of Duality to Maximum flow

Variable y_a per edge constraint; Variable z_v per vertex-constraint

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that can be rewritten:

$$R = f(st) y_{st} + \sum_{(s,v) \in A, v \neq t} f(sv)(y_{sv} + z_v) + \sum_{(v,t) \in A, v \neq s} f(vt)(y_{vt} - z_v) +$$

$$\sum_{(u,v) \in A, u \neq s, v \neq t} f(uv)(y_{uv} + z_v - z_u) \leq \sum_{a \in A} c(a) y_a$$

Application of Duality to Maximum flow

Variable y_a per edge constraint; Variable z_v per vertex-constraint

$$R = \sum_{a \in A} f(a) y_a + \sum_{v \in V \setminus \{s,t\}} \Big(\sum_{(v,u) \in A} f(vu) - \sum_{(u,v) \in A} f(uv) \Big) z_v \leq \sum_{a \in A} c(a) y_a$$

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$$\sum_{(u,v) \in A, u \neq s, v \neq t} f(uv)(y_{uv} + z_v - z_u) \leq \sum_{a \in A} c(a) y_a$$

So, to have $\sum_{(s,u) \in A} f(su) \leq R \leq \sum_{a \in A} c(a) y_a$:

$$\begin{array}{lcll} y_a & \geq & 1 & \text{if } a = (s,t) \\ y_a + z_v & \geq & 1 & \text{if } a = (s,v), v \neq t \\ y_a + z_v & \geq & 0 & \text{if } a = (v,t), v \neq s \\ y_a + z_v - z_u & \geq & 0 & \text{if } a = (u,v), u \neq s, v \neq t \end{array}$$

Application of Duality to Maximum flow

The dual of the previous formulation of Max-Flow

$$
\begin{array}{lrcll}
\text{Minimize} & \displaystyle\sum_{a \in A} c(a) y_a & & & \\
\text{Subject to:} & y_a & \geq & 1 & \textit{if } a = (s,t) \\
 & y_a + z_v & \geq & 1 & \textit{if } a = (s,v), v \neq t \\
 & y_a + z_v & \geq & 0 & \textit{if } a = (v,t), v \neq s \\
 & y_a + z_v - z_u & \geq & 0 & \textit{if } a = (u,v), u \neq s, v \neq t \\
 & y_a & \geq & 0 & \text{for all } a \in A \\
 & z_v & \geq & 0 & \text{for all } v \in V
\end{array}
$$

Exercise: Prove it is a LP for the Min-Cut Problem
Deduce the MaxFlow-MinCut Theorem

Exercises

$G = (V,E)$ be a graph with weight $w : E \to \mathbb{R}^+$, and $s,t \in V$.

What compute the following programmes? Give their dual Programme

$$\begin{array}{lrcll}
\text{Maximize} & \displaystyle\sum_{P \text{ path from s to t}} x_P & & & \\
\text{Subject to:} & \displaystyle\sum_{P, e \in E(P)} x_P & \leq & w(e) & \text{for all } e \in E \\
& x_P & \geq & 0 & \text{for all paths } P \text{ from s to t}
\end{array}$$

$$\begin{array}{lrcll}
\text{Maximize} & x_t & & & \\
\text{Subject to:} & x_s & = & 0 & \\
& x_v & \leq & x_u + w(\{u,v\}) & \text{for all } \{v,u\} \in E \\
& x_v & \geq & 0 & \text{for all } v \in V
\end{array}$$

Outline

Complementary Slackness

Theorem **Complementary Slackness**

Let $x_1^*, \dots x_n^*$ be a feasible solution of the primal and $y_1^*, \dots y_n^*$ be a feasible solution of the dual. Then,

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{or} \quad x_j^* = 0 \quad \text{or both}(j = 1, 2, \dots n)$$

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \quad \text{or} \quad y_i^* = 0 \quad \text{or both}(i = 1, 2, \dots m)$$

are necessary and sufficient conditions to have the optimality of x^* and y^*.

Complementary Slackness - Proof

x^* feasible $\Rightarrow b_i - \sum_j a_{ij}x_j \geq 0$.
y^* dual feasible, hence non negative.

Thus

$$(b_i - \sum_j a_{ij}x_j)y_i \geq 0.$$

Similarly,
y^* dual feasible $\Rightarrow \sum_i a_{ij}y_i - c_j \geq 0$.
x^* feasible, hence non negative.

$$(\sum_i a_{ij}y_i - c_j)x_j \geq 0.$$

Complementary Slackness - Proof

$$(b_i - \sum_j a_{ij}x_j)y_i \geq 0 \qquad \text{and} \qquad (\sum_i a_{ij}y_i - c_j)x_j \geq 0$$

By summing, we get:

$$\sum_i (b_i - \sum_j a_{ij}x_j)y_i \geq 0 \qquad \text{and} \qquad \sum_j (\sum_i a_{ij}y_i - c_j)x_j \geq 0$$

Summing + strong duality theorem:

$$\sum_i b_i y_i - \sum_{i,j} a_{ij}x_j y_i + \sum_{j,i} a_{ij}y_i x_j - \sum_j c_j x_j = \sum_i b_i y_i - \sum c_j x_j = 0.$$

Implies: inequalities must be equalities:

$$\forall i, (b_i - \sum_j a_{ij}x_j)y_i = 0 \qquad \text{and} \qquad \forall j (\sum_i a_{ij}y_i - c_j)x_j = 0.$$

$XY = 0$ if $X = 0$ or $Y = 0$. Done.

Complementary Slackness - Proof

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Summing + strong duality theorem:

$$\sum_i b_i y_i - \sum_{i,j} a_{ij} x_j y_i + \sum_{j,i} a_{ij} y_i x_j - \sum_j c_j x_j = \sum_i b_i y_i - \sum c_j x_j = 0.$$

Implies: inequalities must be equalities:

$$\forall i, (b_i - \sum_j a_{ij} x_j) y_i = 0 \qquad \text{and} \qquad \forall j (\sum_i a_{ij} y_i - c_j) x_j = 0.$$

$XY = 0$ if $X = 0$ or $Y = 0$. Done.

Theorem — **Optimality Certificate**

A feasible solution $x_1^*, \dots x_n^*$ of the primal is optimal iif there exist numbers $y_1^*, \dots y_n^*$ such that

1. they satisfy the complementary slackness condition:

$$\begin{array}{rll} \sum_{i=1}^m a_{ij} y_i^* & = c_j & \text{when } x_j^* > 0 \\ y_j^* & = 0 & \text{when } \sum_{j=1}^n a_{ij} x_j^* < b_i \end{array}$$

2. and $y_1^*, \dots y_n^*$ feasible solution of the dual, that is

$$\begin{array}{rll} \sum_{i=1}^m a_{ij} y_i^* & \geq c_j & \forall j = 1, \dots n \\ y_i^* & \geq 0 & \forall i = 1, \dots, m. \end{array}$$

Example: Verify that $(2,4,0,0,7,0)$ optimal solution of

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & \\
\text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 \\
 & & & & & & & & & & x_1,x_2, & \cdots, x_6 & \geq & 0
\end{array}
$$

First step: Existence of $y_1^*, \dots, y_5^*$, such as

$$
\begin{array}{rll}
\sum_{i=1}^m a_{ij} y_i^* & = c_j & \text{when } x_j^* > 0 \\
y_i^* & = 0 & \text{when } \sum_{j=1}^n a_{ij} x_j^* < b_i
\end{array}
$$

That is

$$
\begin{array}{rcrcrcrcrcr}
2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & = & 18 \\
-6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & = & -7 \\
3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & = & 0 \\
 & & y_2^* & & & & & & & = & 0 \\
 & & & & & & & & y_5^* & = & 0
\end{array}
$$

$(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ is solution.

Example: Verify that $(2,4,0,0,7,0)$ optimal solution of

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & \\
\text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 \\
& -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 \\
& 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 \\
& 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 \\
& 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 \\
& & & & & & & & & & & x_1, x_2, \cdots, x_6 & \geq & 0
\end{array}
$$

Second step: Verify $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ is a solution of the dual.

$$
\begin{array}{rll}
\sum_{i=1}^{m} a_{ij} y_i^* & \geq c_j & \forall j = 1, \dots n \\
y_j^* & \geq 0 & \forall i = 1, \dots, m.
\end{array}
$$

Example: Verify that $(2,4,0,0,7,0)$ optimal solution of

$$
\begin{array}{llllllllllllll}
\text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & \\
\text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 \\
 & & & & & & & & & & & x_1,x_2,\cdots,x_6 & \geq & 0
\end{array}
$$

Second step: Verify $(\frac{1}{3},0,\frac{5}{3},1,0)$ is a solution of the dual.

$$
\begin{array}{rll}
\sum_{i=1}^{m} a_{ij}y_i^* & \geq c_j & \forall j=1,\ldots n \\
y_j^* & \geq 0 & \forall i=1,\ldots,m.
\end{array}
$$

That is, we check

$$
\begin{array}{rrrrrrrrrrrr}
2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & \geq & 18 \\
-6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & \geq & -7 \\
2y_1^* & + & 4y_2^* & + & 5y_3^* & + & 8y^4 & + & 3y_5^* & \geq & 12 \\
7y_1^* & - & 3y_2^* & - & 2y_3^* & + & 7y^4 & + & 6y_5^* & \geq & 5 \\
3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & \geq & 0 \\
8y_1^* & + & 2y_2^* & + & 2y_3^* & + & 3y^4 & 1 & y_5^* & \geq & 8
\end{array}
$$

Example: Verify that $(2,4,0,0,7,0)$ optimal solution of

$$
\begin{array}{lcccccccccccccc}
\text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & \\
\text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 \\
 & & & & & & & & & & & x_1, x_2, \cdots, x_6 & \geq & 0
\end{array}
$$

Second step: Verify $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ is a solution of the dual.

$$
\begin{array}{rll}
\sum_{i=1}^{m} a_{ij} y_i^* & \geq c_j & \forall j = 1, \dots n \\
y_j^* & \geq 0 & \forall i = 1, \dots, m.
\end{array}
$$

That is, we check

$$
\begin{array}{rcrcrcrcrcrcl}
2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & \geq & 18 & & \color{red}{OK} \\
-6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & \geq & -7 & & \color{red}{OK} \\
2y_1^* & + & 4y_2^* & + & 5y_3^* & + & 8y_4^* & + & 3y_5^* & \geq & 12 & & \\
7y_1^* & - & 3y_2^* & - & 2y_3^* & + & 7y_4^* & + & 6y_5^* & \geq & 5 & & \\
3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & \geq & 0 & & \color{red}{OK} \\
8y_1^* & + & 2y_2^* & + & 2y_3^* & + & 3y_4^* & 1 & y_5^* & \geq & 8 & &
\end{array}
$$

Only three equations to check.

Example: Verify that $(2,4,0,0,7,0)$ optimal solution of

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{Max} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & \\
\text{st:} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \leq & 1 \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \leq & -2 \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \leq & 4 \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \leq & 1 \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \leq & 5 \\
 & & & & & & & & & & x_1,x_2, & \cdots, x_6 & \geq & 0
\end{array}
$$

Second step: Verify $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ is a solution of the dual.

$$
\begin{array}{rll}
\sum_{i=1}^{m} a_{ij} y_i^* & \geq c_j & \forall j = 1, \dots n \\
y_j^* & \geq 0 & \forall i = 1, \dots, m.
\end{array}
$$

That is, we check

$$
\begin{array}{rcrcrcrcrcl}
2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & \geq & 18 & \quad \textit{OK} \\
-6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & \geq & -7 & \quad \textit{OK} \\
2y_1^* & + & 4y_2^* & + & 5y_3^* & + & 8y_4^* & + & 3y_5^* & \geq & 12 & \\
7y_1^* & - & 3y_2^* & - & 2y_3^* & + & 7y_4^* & + & 6y_5^* & \geq & 5 & \\
3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & \geq & 0 & \quad \textit{OK} \\
8y_1^* & + & 2y_2^* & + & 2y_3^* & + & 3y_4^* & 1 & y_5^* & \geq & 8 &
\end{array}
$$

Only three equations to check.

OK. The solution $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ is optimal.

Outline

Signification of Dual Variables

$$\begin{array}{lrcll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \cdots, n) \end{array} \qquad \begin{array}{lrcll} \text{Minimize} & \sum_{i=1}^{m} b_i y_i & & & \\ \text{Subject to:} & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & (j = 1, 2, \cdots, n) \\ & y_i & \geq & 0 & (i = 1, 2, \cdots, m) \end{array}$$

Signification can be given to variables of the dual problem (dimension analysis):

- x_j: production of a product j (chair, ...)
- b_i: available quantity of resource i (wood, metal, ...)
- a_{ij}: unit of resource i per unit of product j
- c_j: net benefit of the production of a unit of product j

Signification of Dual Variables

$$
\begin{array}{llcll}
\text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\
\text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \cdots, m) \\
 & x_j & \geq & 0 & (j = 1, 2, \cdots, n)
\end{array}
\qquad
\begin{array}{llcll}
\text{Minimize} & \sum_{i=1}^{m} b_i y_i & & & \\
\text{Subject to:} & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & (j = 1, 2, \cdots, n) \\
 & y_i & \geq & 0 & (i = 1, 2, \cdots, m)
\end{array}
$$

Signification can be given to variables of the dual problem (dimension analysis):

- x_j: production of a product j (chair, ...)
- b_i: available quantity of resource i (wood, metal, ...)
- a_{ij}: unit of resource i per unit of product j
- c_j: net benefit of the production of a unit of product j

unit of resource i/unit of product j   **euros/unit of product j**   **euros/unit of resource i**

$$a_{1j}y_1 + \dots + a_{nj}y_n > c_j$$

$\rightarrow y_i$ euro by unit of resource i.

Marginal cost of resource i.

Signification of Dual Variables

Theorem: If the LP admits at least one optimal solution, then there exists $\varepsilon > 0$, with the property: If $|t_i| \leq \varepsilon \; \forall i = 1, 2, \cdots, m$, then the LP

$$\begin{array}{rrcll} \text{Max} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i + t_i & (i = 1, 2, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \cdots, n). \end{array}$$

has an optimal solution and the optimal value of the objective is

$$z^* + \sum_{i=1}^{m} y_i^* t_i$$

with z^* the optimal solution of the initial LP and $(y_1^*, y_2^*, \cdots, y_m^*)$ the optimal solution of its dual.

Summary: To be remembered

- How to compute a Dual Programme.
- Weak/Strong duality Theorem.
- Optimality certificate (Complementary Slackness).