UBINET, Master 2 IFI Algorithms for telecommunications

Final Exam, November 2018

3 hours

No documents are allowed. No computers, cellphones.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **The two parts are independent.** *The points are indicated so you may adapt your effort.*

1 Parameterized Complexity

All notations throughout this part are in red the first time they are introduced.

1.1 Introduction

Question 1 *Give the definition of a vertex cover of a graph $G = (V, E)$.*

In the following, for any graph G, let $vc(G)$ denote the minimum size of a vertex cover in G.

Question 2 (Naive algorithm.) *Propose a naive algorithm that takes a graph G as input and computes $vc(G)$. What is its time-complexity in function of the size n of V?*

The goal of this problem is the design of a fixed parameter tractable (FPT) algorithm for the following decision problem. Let $k \in \mathbb{N}$ be a fixed integer.

Input: a n-node graph G
output: answer Yes if $vc(G) \leq k$ and No otherwise.

Question 3 (FPT.) *Give the definition of a FPT problem.*

Question 4 (Case $k \in \{0, 1\}$.) *Propose an algorithm that takes a connected graph G as input and decides if $vc(G) \leq 1$. What is its time-complexity?*

Question 5 (Integer Linear Programme (ILP).) *Write an ILP that, given a graph $G = (V, E)$ computes $vc(G)$. For every vertex $v \in V$, let $x_v \in \{0, 1\}$ denote the variable corresponding to v. Give the meaning of the variables and of the constraints.*

Question 6 (LP relaxation.)

- *Explain what is the fractional relaxation of the ILP of the previous question.*
- *What is the goal of relaxing an ILP?*

Let $(\mu_v)_{v \in V}$ be an optimal solution of the fractional relaxation of the LP. That is, for every vertex v, let assign the value $\mu_v \geq 0$ to the variable x_v, and such that this assignment satisfies all constraints.

Let $V_0 = \{v \in V \mid \mu_v < 1/2\}$, $V_1 = \{v \in V \mid \mu_v > 1/2\}$ and $V_{1/2} = \{v \in V \mid \mu_v = 1/2\}$.

1.2 Toward a FPT algorithm

1.2.1 Case $V_1 = \emptyset$

Let us assume that G has no isolated vertices[1].

Question 7 *Prove that, if $V_1 = \emptyset$, then $V_{1/2} = V$.*

Question 8 *Prove that, if $V_{1/2} = V$ and $vc(G) \leq k$, then $|V| \leq 2k$.*

[1] An isolated vertex is a vertex adjacent to no edge

1.2.2 Case $V_1 \neq \emptyset$

This subsection aims at proving that there exists an optimal (integral) vertex cover $X \subseteq V$ with $V_1 \subseteq X \subseteq V_1 \cup V_{1/2}$. Let $S^* \subseteq V$ be an optimal (**integral**) vertex cover. Let $X = (S^* \setminus V_0) \cup V_1$.

Question 9 *Prove that X is a vertex cover of G.*

Question 10 *Prove that $|X| = |S^*| - |S^* \cap V_0| + |V_1 \setminus S^*|$. Deduce that $|S^* \cap V_0| \leq |V_1 \setminus S^*|$.*

Let $\epsilon = \min_{v \in V_0 \cup V_1} |\mu_v - 1/2|$. Note that $\epsilon > 0$ is well defined since $V_0 \cup V_1 \neq \emptyset$. For every v, let us define λ_v as follows:

- $\lambda_v = \mu_v - \epsilon$ if $v \in V_1 \setminus S^*$;
- $\lambda_v = \mu_v + \epsilon$ if $v \in V_0 \cap S^*$, and
- $\lambda_v = \mu_v$ otherwise.

Question 11 *Prove that $(\lambda_v)_{v \in V}$ is a solution of the fractional relaxation of the LP.*

Question 12 *What is the value of the solution $(\lambda_v)_{v \in V}$ in function of $|V_0 \cap S^*|, |V_1 \setminus S^*|, \epsilon$ and the value of the solution $(\mu_v)_{v \in V}$? Deduce that $|S^* \cap V_0| = |V_1 \setminus S^*|$*

Question 13 *Deduce that X is an optimal vertex cover of G.*

Question 14 *Conclude that $vc(G) \leq k$ if and only if $vc(G \setminus V_1) \leq k - |V_1|$*

1.3 A FPT algorithm for vertex cover

Let $\mathcal{B}$ denote the algorithm of question 2.

Algorithm 1 A FPT algorithm $\mathcal{A}$ for deciding if a graph has a VERTEX COVER of size $\leq k$

Require: A graph $G = (V, E)$ and an integer k

If $|E| = 0$ **then**
    Return Yes
Else
    Remove isolated vertices
    Let $(x_v = \mu_v)_{v \in V}$ be an optimal fractional solution obtained by LP
    If $\sum_{v \in V} \mu_v > k$
        Return No
    Else
        let $V_1 = \{v \in V \mid \mu_v > 1/2\}$.
        If $V_1 \neq \emptyset$ **then**
            Return $\mathcal{A}(G \setminus V_1, k - |V_1|)$.
        Else
            If $|V| > 2k$ **then**
                Return No.
            Else
                If $\mathcal{B}(G) \leq k$ **then**
                    Return Yes.
                Else
                    Return No.

Question 15 *Explain Algorithm $\mathcal{A}$ (Algorithm 1) and prove its correctness.*

Question 16 *Give the time-complexity of algorithm $\mathcal{A}$ in function of the number n of vertices of G and of k (Assume that the time complexity of executing the LP is $O(n^c)$ for some constant $c \in \mathbb{N}$).*

2 Modeling using linear programs

2.1 Traveling Salesman Problem (4 points)

Given a graph G with n vertices whose edges are weighted by a function $w : E(G) \to \mathbb{R}$, a solution to the Traveling Salesman Problem (or TSP) is a cycle of length n going through all the vertices (exactly once) of the graph G whose weight (the sum of the weight of its edges) is minimum.

Below is a formulation of the problem as an integer program.

1) Explain what the variables x_{ij} represent.
2) In two lines, explain what Constraints (1) and (2) are. In one or two lines, explain why they are not sufficient to model the TSP.
3) We now consider the variables $u_i, i = 2, ..., n$ introduced in Constraint (3). Show that $u_i - u_j \leq n - 1, \forall i, j = 2, ..., n$.
4) Show that the variables $u_i = 2, ..., n$ define an order on the vertices of the graph (except for one vertex). Conclude that it is a correct formulation of the TSP.

$$
\begin{array}{rlll}
\min & \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}x_{ij} & & \\
\text{subject to} & & & \\
& \sum_{i=1}^{n} x_{ij} = 1 & (\forall j \in V(G)) & (1) \\
& \sum_{j=1}^{n} x_{ij} = 1 & (\forall i \in V(G)) & (2) \\
& u_i - u_j + nx_{ij} \leq n-1 & (\forall i, j = 2, ...n \text{ and } i \neq j) & (3) \\
& x_{ij} \text{ binary} & (\forall i, j = 1, ...n \text{ and } i \neq j) & \\
& u_i \text{ integer} & (\forall i \in V(G)) &
\end{array}
$$

2.2 Stations and customers. (8 points)

A provider (or operator) wants to connect n customers to the internet. For this purpose, the operator uses an individual wired connection from each customer c to one of the m *stations* (routers, dslams or similar gateways).

Stations and customers are located in the plane (with no obstacle). The n customers are immobile, that is their locations are fixed and known. The *wiring cost* to connect a customer c to station s is $0,1$ euro per meter. For instance, establishing a connection between a customer and a station that are 1 kilometer apart costs 100 euros.

Last, stations do have a bounded capacity, that is, a station can serve at most L customers. The *operator problem* is to choose the connections (such that all customers are connected to a station) in order to minimize the global cost, i.e., the sum of the cost of all n links.

1. **Fixed locations.**

   In this part, we assume that the locations of the stations are also fixed and known. In particular, the distance (in meters) between any customer c and any station s is fixed and denoted by $d(c, s)$.

   - **Question 1.** What is the minimum cost solution when $L \geq n$?

   Now and for all the remaining part of the Problem, we assume that L is a fixed integer, $L > 0$.

   - **Question 2.** Give a necessary and sufficient condition (in function of n, m and L) for the problem to be feasible.

   Let $G = (V, A)$ be a directed graph with a capacity function $c : A \to \mathbb{R}^+$ and a weight function $w : A \to \mathbb{R}^+$. The *weighted flow problem* consists in finding a flow $f : A \to \mathbb{R}^+$ satisfying all usual constraints (capacity, flow conservation) and minimizing $\sum_{a \in A} f(a)w(a)$.

   Let $H = (S \cup C, A)$ be the bipartite graph with $S = \{s_1, \cdots, s_m\}$, $C = \{c_1, \cdots, c_n\}$ and A is the set of all arcs from s_i to c_j, $i \leq m$, $j \leq n$.

- **Question 3.** Add a vertex-source and a vertex-target, and define capacity and weight functions in H, in order to model the operator problem in terms of a weighted flow problem.

- **Question 4.** Give a linear programme to solve the obtained weighted flow problem.

Let $G = (V, E)$ be a graph with a weight function $w : V \to \mathbb{R}^+$ on the vertices, and a weight function $p : E \to \mathbb{R}^+$ on the edges. A *w-matching* is a set F of edges such that, for any vertex $v \in V$, v is incident to at most $w(v)$ edges in F. The weight of a w-matching F is $\sum_{e \in F} p(e)$.

- **Question 5.** Model the operator problem as a weighted w-matching problem. Give a linear programme to solve it.

2. **Fixed locations with distance restriction.**

   Due to signal quality issues one cannot actually connect a point to a station when it is too far way. So we add the following constraint : a station can only be connected to node at distance less than 1 kilometer. We say that a customer *sees* a station if it is at distance less than 1 kilometer from this station.

   - **Question 6.** Adapt the flow model to include this max distance constraint.
   - **Question 7.** Prove that there is no solution to the problem if and only if there is a set of k customers seeing all together less than k/L stations.

3. **Stations are not fixed anymore and no distance restriction.**

   In this part, we come back to the model without any distance restriction, that is a customer can be connected to any station independently of the distance. However, the stations are not fixed anymore. To simplify the problem, we assume that the customers are placed on some vertices of a $k \times k$-grid. To place a station at some vertex of the grid costs 1000 Euros. A station can still serve at most L customers.

   The new problem of the operator is to decide the number of stations, where to place them and how to connect them to the customers. The total cost for the operator is now the wiring cost plus the station cost.

   - **Question 8.** Model the new operator problem as a weighted flow problem. You may use a bipartite graph with one part with k^2 vertices and the other part with n vertices.
   - **Question 9.** Give a linear programme to solve this problem.
   - **Question 10.** Explain why the obtained linear programme is more difficult to solve than the programme in Question 4.