MACHINE LEARNING

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EVALUATION

(1) EXAM (40%)
(2) HOMEWORK (30%)
(3) QUESTION AT EVERY LECTURE (30%)

2020/2021

16/09

INTRODUCTION

MACHINE LEARNING

- way of extract rules from data

DATA → [ML] → RULES/PROGRAM

ISSUES

- To large set of rules (REM Pidgeon)
- To small set of rules (REM Rat)

& TOO MUCH reduces learning flexibility

INDUCTIVE PRIOR

- A PRIORI KNOWLEDGE to prevent useless conclusion

INDUCTIVE INFERENCE

- Ability to proceed from general examples to a broadere generalization

KINDS OF LEARNING

SUPERVISED / UNSUPERVISED

- In supervised learning output label are given; Unsupervised instead finds unexpected correlation. (SPAM MAIL vs ANOMALITY DETECTION)

REINFORCHENT

- Labels are given but after a while

COOPERATIVE / INDIFFERENT / ADVERSARIAL TEACHER

- Change the way the data is given from a teacher

BATCH / ONLINE

- Data can be given all together or PASSO PASSO

! ⇒

- we'll se SUPERVISE STATISTICAL BATCH LEARNING WITH PASSIVE TEACHER

STATISTIC & ML

1. Statistic wants to check an HYPOTHESIS (■ smoke affects heart?)
   ML wants to find the HYPOTHESIS (■ what affects heart?)
2. Statistic starts from the Prob. Distr. of data.
   ↳ ASYNTOTICS: if you look at enough data you find a GAUSSIAN
   ML doesn't know the distribution
   ↳ FINITE SAMPLES: you get the distribution you have

STATISTICAL LEARNING FRAMEWORK

- X input space
- Y output space

} (vector of features)

- S dataset of size m (# SAMPLES)

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f and D are unknown to the learner

- f correct labeling function ; D distribution over X
- $A : S \subset (X \times Y)^m \longrightarrow \{\text{functions } X \to Y\}$
  $\Rightarrow A(S) = h$

h is the output of an ML Algo. (is a PREDICTION RULE)

- $h : X \to Y$ called HYPOTESIS or PREDICTOR or CLASSIFIER it can be seen as $A(S)$, so the output of the ML ALGO. A given the DS S

INDICATOR FUNCTION: 0 if prediction is ok it is 0 else is 1

EXPECTED LOSS

- $L_D(h) = \mathbb{E}_D\left(\mathbb{1}_{h(x) \neq y}\right) \in [0,1]$

EMPIRICAL LOSS

- $L_S(h) = \dfrac{\sum_{i=1}^{m} \mathbb{1}_{h(x_i) \neq y_i}}{m}$

$\Rightarrow$ for large number theory, EMPIRICAL LOSS will converge to EXPECTED LOSS.

---

note

- we can write $y = f(x)$ so we can rewrite $L_{D,f}(h) = \mathbb{E}_D(\mathbb{1}_{h(x) \neq f(x)})$; note also that our hope is to find h such that $L_{D,f}(h) < \varepsilon$ with Prob $> 1 - \delta$ because in the best case we can have $L_{D,f}(h) = 0$ (that is obv. impossible)

APPROXIMATE WITH HIGH PROBABILITY

ERP - EMPIRICAL RISK MINIMIZATION

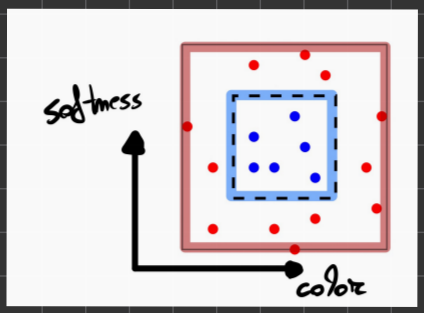
Def

- ERP is the process of finding a PREDICTOR h which minimizes the EMPIRICAL Loss $L_S(h)$.

OVERFITTING

- When a predictor is excellent on the training set but very poor on the distribution

ERM is bad?



- Uniform D
- AREA( ) = 1
- AREA( ) = 2
- $f(x)$ { 1 if in , 0

Overfitting example

- let $S =$ and define:

$$h_S(x) = \begin{cases} y, & \forall x \in S \\ 0 \end{cases}$$

it means that $h_S(x)$ is perfect for the training set, so $L_S(h_S) = 0$; but, having $\mathbb{P} = 1/2$ of taking an element $x \notin S \Rightarrow L_D(h_S) = 1/2$, so it's bad for the distribution D

- MEMORIZATION ALGO.

other OVERFITTING EXAMPLE

- Suppose: $h_M(x) = \begin{cases} 1 \text{ if } x = x_i \in S \\ \text{TOSS A COIN} \end{cases}$
- the $\mathbb{P}$ of taking an already known SAMPLE from all over the distribution D is ϕ, so you always toss a coin.

$\Rightarrow$ IT perfectly explain the dataset, but is shit in general

ERM WITH INDUCTIVE BIAS

⇒ ERM is not bad but learns in a too large set of HYPOTESIS, so it depends from the set of HYPOTESIS you apply it.

Def
- The IDEA is to reduce the SEARCH SPACE of ERP;
- we define, before looking at the data, a set of predictor H called HYPOTESIS CLASS, and we use ERP rule to find $h_{ERM_H} = ERM_H(S) \in \underset{h \in H}{\operatorname{argmin}} L_S(h)$.

INDUCTIVE BIAS
- mean biasing a LEARNER to a finet set of predictor.

?
HOW TO REDUCE THE SEARCH SPACE H?
- UPPERBOUNDING IT'S SIZE

REALIZABILITY ASSUMPTION
- There exist alway $h^* \in H \mid L_{D,f}(h^*) = 0$, it means $h^* = f$

⇒ it implies that, if the set H is finite, then for every S taken from D $L_S(h^*) = 0$, so ERM_H will alway find it.

ISSUE
- We are interested in $L_{D,f}(h_{ERM_H})$, so to be well done S should be enough representative of the underlying dist. D

↓

i.i.d. ASSUMPTION
- All sample of S are INDIPENDENTLY and IDENTICALLY DISTRIBUTED according to D

↳ it implies that the bigger is S the more it is representative of D and f

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NOTE

- $L_{D,f}(h_S)$ is a random variable because the choice of S is random, so can always happen that S would be not representative of D

CONFIDENCE PARAMETER

- δ is the PROB. of getting NON-REPR. S
- $(1-\delta)$

ACCURACY PARAMETER

- ε is needed when to address the QUALITY of a Prediction

⇒ if $L_{D,f}(h_S) > \varepsilon$ FAIL; else if $L_{D,f} \leq \varepsilon$ we got an APPROX. CORRECT PREDICTOR h_S

? IDEA

Can we upp. bound the prob. of sampling S from D in a way that S want be MISLEADING?

We can find the right size $m = |S|$ such that ERM_H won't choose a $h_{ERM_H} = h_S$

Some Defs

- BAD HYPOTHESIS SET → $H_B := \{h \in H \mid L_{D,f}(h) > \varepsilon\}$
- POSSIBLY MISLEADING S →

$$S_p := \{S \mid \exists \underbrace{h_S \in H_B, L_S(h_S) = 0}_{\text{BAD HYPOT. THAT LOOK GOOD ON } S}\}$$

- MISLEADING S →

$$S_M \in \{S \mid \underbrace{L_{D,f}(h_S) > \varepsilon}_{\text{IT HAPPENS ONLY IF } h_S \in H_B}\} = \{S \mid h_S \in H_B\} \subseteq S_p$$

- we can rewrite $S_P = \bigcup_{h_S \in H_B} \{S \mid L_S(h_S) = 0\}$

Prob. of picking S_M

- we now calculate $\mathbb{P}_{D^m}(S_M)$

$$\mathbb{P}_{D^m}(S_M) \leq \mathbb{P}_{D^m}(S_P) = \mathbb{P}_{D^m}\left(\bigcup_{h_S \in H_B} \{S \mid L_S(h_S) = 0\}\right) \leq$$

$$\leq \sum_{h_S \in H_B} \mathbb{P}_{D^m}(\{S \mid L_S(h_S) = 0\}) \dots$$

- Notice that $L_S(h_S) = 0$ if $\forall (x_i, y_i) \in S \; h_S(x_i) = y_i \Rightarrow$

$$\Rightarrow \mathbb{P}(h_S(x_i) = y_i) = 1 - L_{D,f}(h_S) \leq 1 - \varepsilon \Rightarrow$$

($L_{D,f}(h_S) > \varepsilon$ (searching S_M!))

$$\Rightarrow \mathbb{P}(h_S(x_i) = y_i \; \forall i = 1, \dots, m) = \mathbb{P}_{D^m}(\{S \mid L_S(h_S) = 0\}) \leq (1-\varepsilon)^m$$

$$\dots \sum_{h_S \in H_B} \mathbb{P}_{D^m}(\{S \mid L_S(h_S) = 0\}) \leq \sum_{h_S \in H_B} (1-\varepsilon)^m =$$

$$= |H_B|(1-\varepsilon)^m \leq |H| e^{-\varepsilon m}$$

(obv $H_B \subseteq H$)

$\Rightarrow$ - in conclusion $\boxed{\mathbb{P}_{D^m}(S_M) \leq |H_B| e^{-\varepsilon m}}$

BOUND

- we want $\mathbb{P}_{D^m}(S_M) \leq \delta$ so we should choose m carefully:

$$|H_B| e^{-\varepsilon m} \leq \delta \Rightarrow \boxed{m \geq \frac{\log(|H|/\delta)}{\varepsilon}}$$

Conclusion

COROLLARY 2.3 *Let $\mathcal{H}$ be a finite hypothesis class. Let $\delta \in (0,1)$ and $\epsilon > 0$ and let m be an integer that satisfies*

$$m \geq \frac{\log(|\mathcal{H}|/\delta)}{\epsilon}.$$

Then, for any labeling function, f, and for any distribution, $\mathcal{D}$, for which the realizability assumption holds (that is, for some $h \in \mathcal{H}$, $L_{(\mathcal{D},f)}(h) = 0$), with probability of at least $1 - \delta$ over the choice of an i.i.d. sample S of size m, we have that for every ERM hypothesis, h_S, it holds that

$$L_{(\mathcal{D},f)}(h_S) \leq \epsilon.$$

The preceeding corollary tells us that for a sufficiently large m, the $\text{ERM}_{\mathcal{H}}$ rule over a finite hypothesis class will be *probably* (with confidence $1-\delta$) *approximately* (up to an error of ϵ) correct.

21/09/21

PAC - LEARNABILITY

PAC LEARNABILITY Def 1

. (Given REALIZ. ASS and 0-1 loss funct $\mathbb{1}_{h \neq y}$) H is PAC-LEARNABLE if:

- $\exists A : (X \times Y)^m \to H$
- $\exists m_H : (0,1)^2 \to \mathbb{N}$ // SAMPLE COMPLEXITY how many samples are required to guarantee an approx. correct solution?

They are such that:

$\forall D$ over X, $\forall f$, $\forall \delta \in (0,1)$, $\forall \varepsilon \in (0,1)$ if we get S of m i.i.D. samples according to D from X such that $|S| = m \geq m_H(\varepsilon, \delta)$, then:

$$L_{D,f}(A(S)) < \varepsilon \quad \text{w.p. } (1-\delta)$$

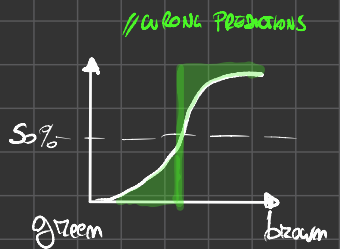
weak points of Def 1

1. REALIZABILITY ASSUMPTION → $\exists f$, $f \in H$
2. 0-1 loss (only binary classification)

=>

- $f(x)$ is wasted and we'll use $P(y|x)$
- We need a THRESHOLD for $P(y|x)$, and put it to 50% is the choiche that reduces $L_{D,P}(h)$

$$h(x) = \begin{cases} 1 & , P(1,x) = 0,5 \\ 0 \end{cases}$$



MARGINAL DIST

- $P(x)$

CONDITIONAL LABELLING PROB

- $P(y|x) = \dfrac{P(x,y)}{P(x)}$

LOSS GENERALIZATION

- There exist different kinds of loss other than the (0-1)-loss:

some losses:

- $\ell(h,(x,y)) \in \mathbb{R}$
- $\ell(h,(x,y)) = \mathbb{1}_{h(x)\neq y}$ ((·)² cause we want $\ell \geq 0$)
- $\ell(h,(x,y)) = (h(x)-y)^2$

- we can generalize them by $\Rightarrow L_{\mathcal{D}}(h) = \mathbb{E}[\ell(h,(x,y))]$

REAL RISK EVALUATION

- Until now, with realizability ass., we were sure that the best possible $L_{\mathcal{D}}(h)$ was 0 (because $\exists h^* \in H \mid h^* = f$); relaxing this assumption we need the to compare our loss to be the nearest possible to the BEST ONE we can achieve in H.

$$L_{\mathcal{D}}(h) < \min_{h' \in H} L_{\mathcal{D}}(h') + \varepsilon$$

AGNOSTIC PAC LEARNABILITY

Def2

- ~~(Given realiz. ass on 0-1 loss funct $\mathbb{1}_{h\neq y}$)~~ H is AGNOSTIC PAC-LEARNABLE, if:
  - $\exists A : (X \times Y)^m \to H$
  - $\exists m_H : (0,1)^2 \to \mathbb{N}$

w.r.t. the loss ℓ

They are such that:

$\forall \mathcal{D}$ over $(X \times Y)$, ~~$\forall f$~~, $\forall \delta \in (0,1)$, $\forall \varepsilon \in (0,1)$ if we get S of m i.i.d. sampled according to $\mathcal{D}$ from $(X \times Y)$ such that $|S| = m \geq m_H(\varepsilon, \delta)$, ~~then~~:

~~$L_{\mathcal{D},f}(A(S)) < \varepsilon$ w.p. $(1-\delta)$~~

$$L_{\mathcal{D}}(A(S)) \leq \min_{h} L_{\mathcal{D}}(h) + \varepsilon \quad \text{w.p.} \geq 1-\delta$$

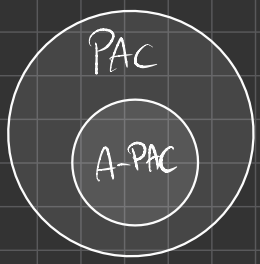
BEST POSSIBLE ERROR OVER H. w/ REALIZABILITY WAS 0 BECAUSE OF $f \in H$

NB

- What is true?

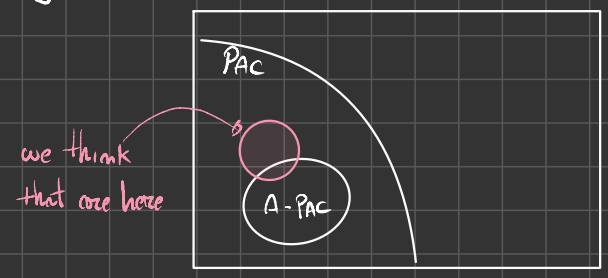
① H is A-PAC $\Rightarrow$ H is PAC

② H is PAC $\Rightarrow$ H is A-PAC



$\Rightarrow$ ①: if H satisfy PAC LEARN. constr, it's the obvious that satisfy also AGNOSTIC PAC LEARN. constr.

Where are finite class of HIPOTHESIS? we know are PAC, but no who they are



ε-REPRESENTATIVE

ε-REPRESENTATIVE DATASET

- S is ε-REPRESENTATIVE if $|L_D(h) - L_S(h)| < \varepsilon \quad \forall h \in H$
- if you have S ε-REPR, then ERM_H finds a good predictor
  - let $h_S = ERM_H(S)$
  - let S be $\varepsilon/2$-REPRESENTATIVE
  - we know by def that

$$L_D(h_S) \leq L_S(h_S) + \varepsilon/2$$
$$\leq L_S(h) + \varepsilon/2 \quad \forall h$$

↕ BY DEF

$$\leq L_D(h) + \varepsilon/2 + \varepsilon/2 \quad \forall h$$
$$\leq \min_{h \in H} L_D(h) + \varepsilon$$

↑ BY A-PAC LEARNABILITY

UNIFORM CONVERGENCE PROPERTY (UC)

- H has UC prop. w.r.t. the loss ℓ $\exists m_H^{UC}: (0,1)^2 \to \mathbb{N}$ $\forall D$, $\forall \varepsilon, \delta \in (0,1)$, if you draw a dataset S i.i.d from D with $|S| = m \geq m_H^{UC}(\varepsilon, \delta)$ then S is ε-REPR. w.p. $\geq 1-\delta$

NOTE

- If H has UC prop. then S is ε-repr. with high prob., so $L_D(h_S) \leq \min_{h \in H} L_D(h) + \varepsilon$, so H is AGNOSTIC PAC LEARNABLE, and $ERM_H(S)$ an AGNOSTIC PAC LEARNER for H

TH

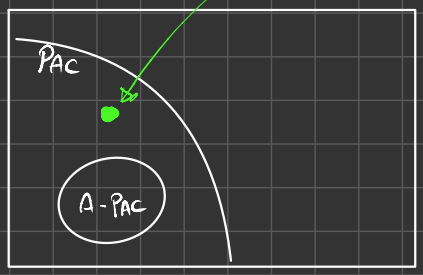
- If H is finite $\Rightarrow$ H has UC prop

Proof

Postponed...

TH

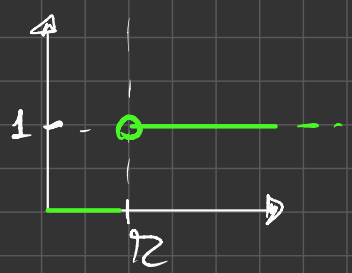
- There is at least an INFINITE PAC CLASS H



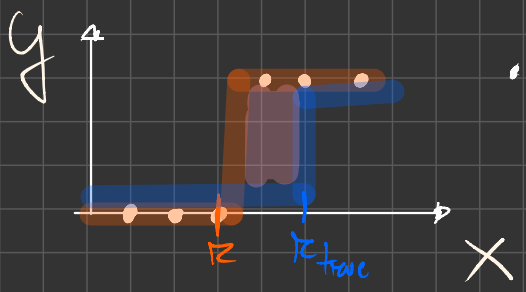
Proof

- Let H be a threshold function

$H = \{ h_\pi, \pi \in [0,1], h_\pi : [0,1] \to \{0,1\}, h_\pi(x) = \begin{cases} 1 & \forall x > \pi \\ 0 & \forall x \leq \pi \end{cases} \}$



- Given the dataset

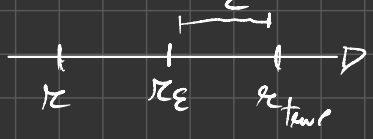


- ERM_H gives me h_z where $z = \max\{x_i \mid y_i = 0\}$

- suppose the $\mathcal{D}$ every time we observe a point in the PINK area we have a loss of 1

$$L_D(h_z) = \mathbb{P}(x \in [z, z_{true}])$$

- now we look a point in $[z, z_{true}]$ such that the error on it is ε



- z_ε: $\mathbb{P}(x \in [z, z_{true}]) = \varepsilon$

- if $z < z_\varepsilon \Rightarrow L_D(h_z) > \varepsilon$ (1)
- if $z \geq z_\varepsilon \Rightarrow L_D(h_z) < \varepsilon$ (2)

$\mathbb{P}(z < z_\varepsilon) =$

$\Rightarrow \mathbb{P}(x \notin [z, z_{true}]) = 1 - \varepsilon$

$= \mathbb{P}(x_i \notin [z, z_{true}] \ \forall i = 1, \dots, m) = \prod_{i=1}^{m} \mathbb{P}(x \notin [z, z_{true}]) =$

$= \prod_{i=1}^{m} (1 - \varepsilon) = (1-\varepsilon)^m \Rightarrow$ we want $, t \leq \delta$

$\Rightarrow$ if we take $m \geq \frac{\log(\delta)}{\log(1-\varepsilon)}$ then $L_D(ERM_S) \leq \varepsilon$

w.p $\geq 1 - \delta$ so H is PAC

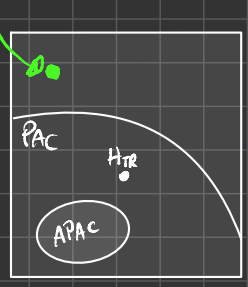
∎

28/04

TH · There is a class $H \notin PAC$

Proof · $H_{\{0,1\}^X} = \{$All binary partition over $X\}$ // Think of it as ALL BINARY NUMBER

· There is not a universal learner (FREE LUNCH TH) ■



TH [NO FREE LUNCH] · $\exists D$ over X and a labeling function f & $\forall A$ learning algorithm, those are s.t. by picking S of $m < |X|/2$ i.i.d. according to D you have

$$L_{D,f}(A(S)) \geq 1/8 \quad \text{w.p.} \geq 1/7 \leftarrow \text{NON NEGLIGIBLE}$$

A

COROLLARY [NO FREE LUNCH] · If $|X| = +\infty$, $H_{\{0,1\}^X}$ is not PAC learnable w.r.t. 0-1 loss

PAC NEGATION · H is not pac if $\forall A$, $\forall m_H : (0,1)^2 \to \mathbb{N}$, $\exists D$ over X and $\exists f \in H$, $\exists \varepsilon_0, \delta_0 \in (0,1)$: if $\exists m \geq m_H(\varepsilon_0, \delta_0)$ $|S| = m$ then EVERY A learn with an error of:

$$L_{D,f}(A(S)) \geq \varepsilon_0 \text{ w.p } \geq 1 - \delta_0$$

→ FROM NO FREE LUNCH TH we can enforce the PAC definition.

- $\exists f$ instead of $\exists f \in H$
- $\varepsilon_0 = 1/8$ and $\delta_0 = 1/7$
- the requirement $\exists m \geq m_H(\varepsilon_0, \delta_0)$ that became $\forall m$

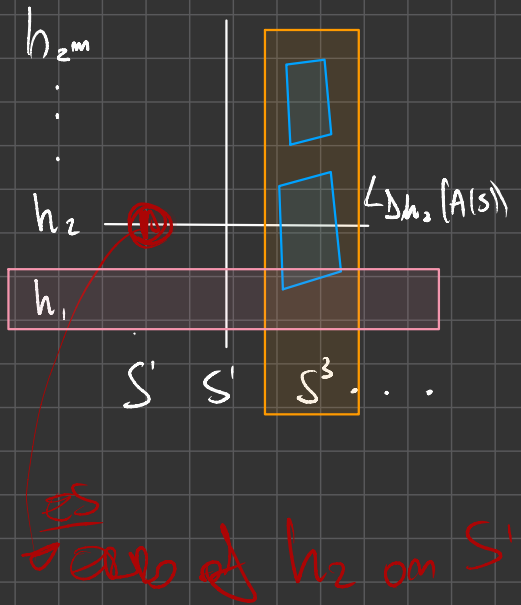
free lunch proof

· Pick D | $D(x_i) = 1/2^m$ $\forall i$

· Let's prove that $\exists f$ | $\mathbb{E}_{S \sim D^m}[L_{D,f}(A(S))] \geq 1/4$

B

· Build a table of $|H|$ rows with every possible S on column



// Pick 4 SAMPLES AT A TIME

// the average on each ■ has an AVG ERROR of 1/4

// the set of a column ■ has ≥ 1/4

// the ch fs too has ERR 1/4

// AVG of elements on row ■ 1/4

$\Rightarrow \exists$ RAW with AVG $\geq 1/4$, H is the RAW of δ

Proof: if the whole set has AVG $\geq 1/4$ there is at least one with AVG $> 1/4$

- Now show that $A \Rightarrow B$ by showing $\neg A \Rightarrow \neg B$

(A) $L_{D,f}(A(S)) \geq 1/8$ w.p. $\geq 1/7$

(B) $\exists \delta \mid \mathbb{E}_{S \sim D^m}[L_{D,f}(A(S))] \geq 1/4$

How COULD $L_{D,\delta}(A(S)) = 1$? happens w.p. $< 1/7$ so

$$\mathbb{E}_{S \sim D^m}[L_{D,\delta}(A(S))] < \overbrace{1 \cdot 1/7 + 1/8 \cdot 6/7}^{} = 1/4$$

VC DIMENSION

why VC-DIMENSION?

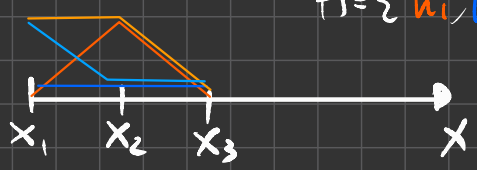
- PAC $\equiv$ A-PAC $\equiv$ UC $\equiv$ {everything we can learn via ERM} $\equiv$ $\equiv$ VC-dim(H) $< +\infty$

SHATTERING

- Given X, H over X and $A \subset X$:
- H shatters A if $\forall g : A \to \{0,1\}$ $\exists h \in H \mid h(x) = g(x)$ $\forall x \in A$

shattering example

$H = \{h_1, h_2, h_3, h_4\}$

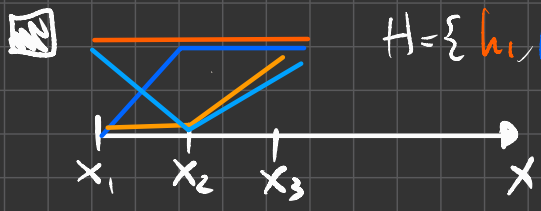


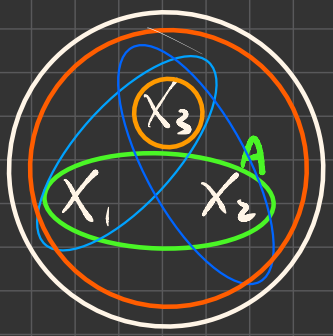
- $A = \{x_1\} \Rightarrow H$ shatter x_1
- $A = \{x_3\} \Rightarrow H$ DOESN'T (you can't get $x_3 = 1$ with $h \in H$)
- $A = \{x_1, x_2\} \Rightarrow H$ shatter A (we have 00, 01, 10, 11 in h)

why shattering

- given a subset of Point, we can always define H in a way that shatter them, so H can be seen as a SUBSET of X:

$\Rightarrow$ H shatters A if $\forall$ subset of A $\exists h \in H \mid h \cap A = \emptyset$





- To get $\{\emptyset\}$ we get x_3 set cause $\{x_3\} \cap A = \{0\}$
- To get x_1 we got ◯
- To get x_2 we get ◯

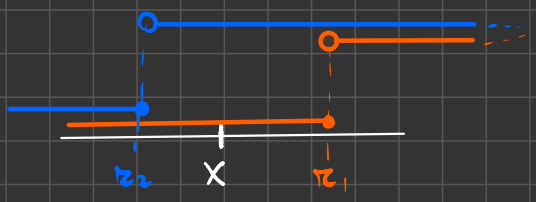
□

VC-dim

- $\text{VC-dim}(H) = \max\{|A| \mid H \text{ shatters } A\}$

Find VC-dim of $H_{\text{THRESH}} = \{h_{x_2} \mid h_{x_2}(x) \begin{cases} 1 & \text{if } x > x_2 \\ 0 & \end{cases}\}$

raw/bunch



real set

- we can pick him by taking the set $X \setminus \{x\}$ cause

$A \cap X \setminus \{x\} = \{\emptyset\}$

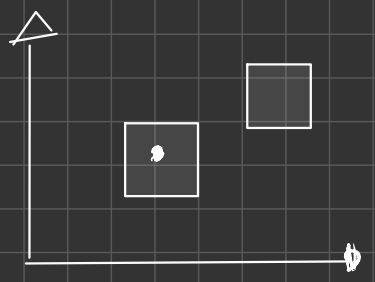
- Now we know $\text{VC-dim}(H) \geq 1$
- we can't do the same for 2 points because there is no way of picking only one of them

$\Rightarrow \text{VC-dim}(H) = 1$

find VC-dim of $H_{rect} = \{h_{a,b,c,d}$ s.t.

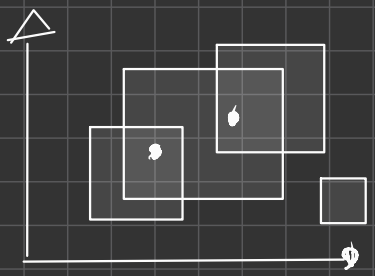
$h_{a,b,c,d} \begin{cases} 1 & \text{if } 0 \le a \le b \text{ and } 0 \le c \le b \\ 0 & \end{cases} \}$

1 point

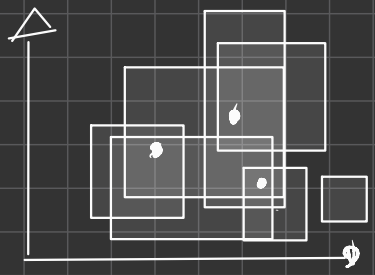


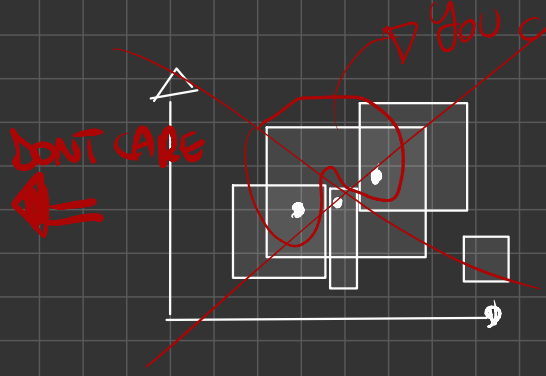
$\Rightarrow$ VC-dim(H_{rect}) ≥ 1

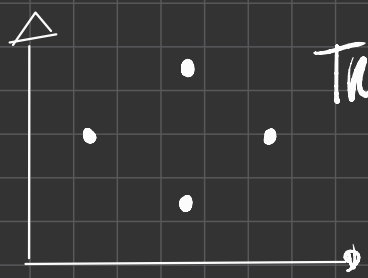
2 point



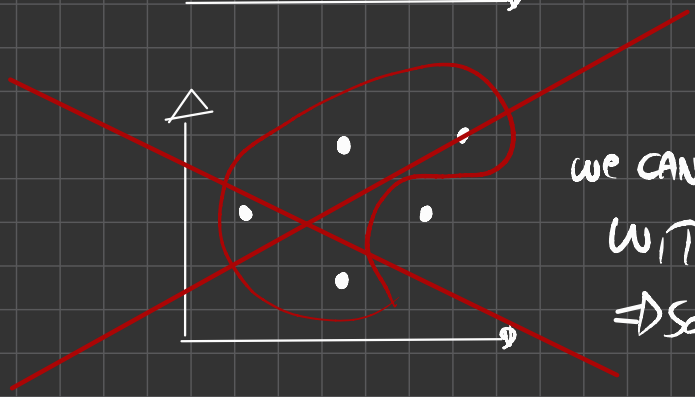
3 point







TRUST, WE CAN!



WE CANT SHATTER 4 POINTS WITHOUT THE 5TH !

$\Rightarrow$ So you proved also for more than 5 points

$\Rightarrow$ VC-dim(H) = 4

FUNDAMENTAL TH OF STATISTICAL LEARNING

TH · X, 0-1 LOSS, H

(1) H IS PAC LEARNABLE

(2) ERM_H IS PAC LEARNER

(3) H IS AGNOSTIC PAC LEARNABLE

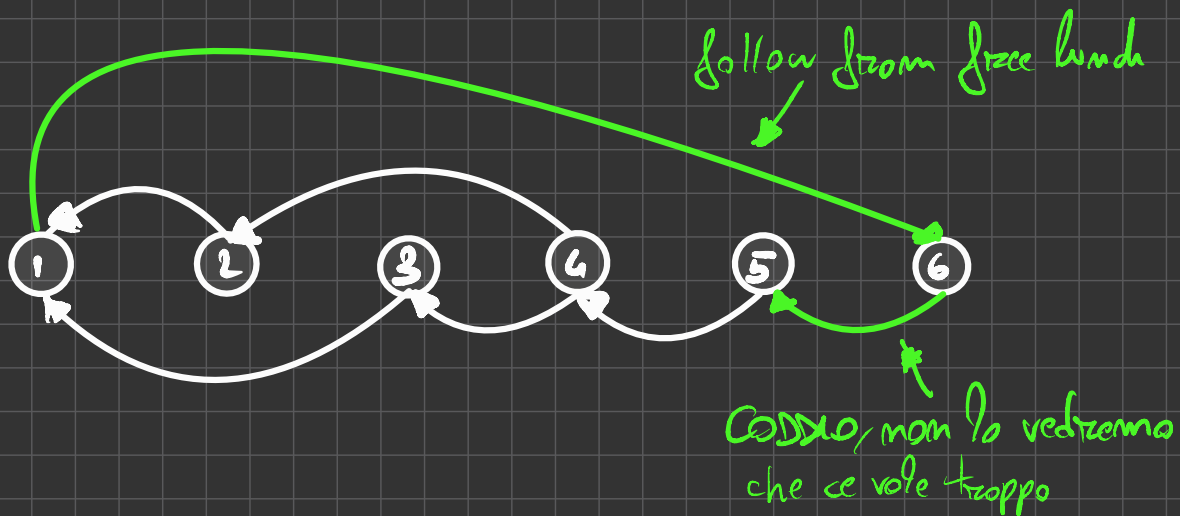
(4) ERM_H IS AGNOSTIC PAC LEARNER

(5) H HAS UC PROPERTY

(6) $VC\text{-}dim(H) < +\infty$

Proof

we don't Know



(6-1) Proof

TH: H is PAC $\Rightarrow$ $VC\text{-}dim(H) < +\infty$

Proof.

NEGATED TH:

$VC\text{-}dim(H) > +\infty \Rightarrow H$ is NOT PAC