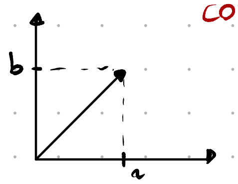
LINEAR ALGEBRA

REFERENCES

- NIELSEN & CHUANG
- QUANTUM COMPUTATION & INFORMATION
- QUANTUM MECHANICS: THE THEORETICAL QUANTUM



COMPLEX NUMBER

$z = a + jb \mid a, b \in \mathbb{R}$

$j^2 = -1$

$z^* = a - jb$

VECTOR SPACE OVER A FIELD ($\mathbb{C}$)

$v_1, v_2 \in V$

$v_1 + v_2 \in V$

$z \cdot v \in V \mid z \in \mathbb{C}, v \in V$

INNER PRODUCT

$(v, u) \in \mathbb{C}$

$(v, u + w) = (v, u) + (v, w)$

$(v, zu) = z(v, u)$

$(v, u) = \underbrace{(u, v)^*}_{\text{CONJUGATE}}$

SCALAR PRODUCT

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

$(x, y) = x \cdot y = x^T y = \sum_{i=1}^{m} x_i y_i$

$\|v\| = \sqrt{(v, v)}$

$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}$

VECTOR SPACE
+
INNER PRODUCT
=
HILBERT SPACE

$f: V \to V$ LINEAR FUNCTION

$f(z_1 v_1 + z_2 v_2) = z_1 f(v_1) + z_2 f(v_2)$

f CAN BE REPRESENTED AS A MATRIX

- BASIS $: \{u_1, \ldots, u_m\} \Rightarrow \exists z_1, \ldots, z_m \in \mathbb{C} \mid v = z_1 u_1 + \ldots + z_m u_m \quad \forall v \in V$
- so it follow:

$$f(v_1) = z_1 f(u_1) + \ldots + z_m f(u_m) = \begin{bmatrix} f(u_1) & f(u_2) & \ldots & f(u_m) \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$$

EIGEN VALUES AND EIGENVECTORS

- λ is an EIGENVALUE for A w.r.t. the EIGENVECTOR $v \iff Av = \lambda v$
- EIGENVALUES are the solutions of $\det(A - \lambda I) = 0 \Rightarrow$ it has m solutions
- Some solutions can coincide, we say: "λ has ALGEBRAIC MULTIPLICITY #occ"
- HOW MANY INDIPENDENT EIGEN VECTOR CAN I FIND FOR λ? GEOMETRIC MULTIPLICITY

  ↳ if $\lambda_1 \mid A v_{1,1} = \underset{\text{EIGENVALUE}}{\lambda_1} \overset{\text{EIGENVECTOR}}{v_{1,1}}$ then every $\alpha\lambda_1$ is valid, but dependent from λ_i
- Let's say that $\forall \lambda_i$ ALG.MULT. = GEOM. MUL.

	$m=5$	
λ_1	v_{11}, v_{12}	⇒ MULT = 2
λ_2	v_{21}	⇒ MULT = 1
λ_3	v_{31} v_{32}	⇒ MULT = 2

ARE BASIS FOR THE SPACE

- $f \to A$ can be rewritten as function of BASIS

$$A' = [f(v_1)\, f(v_2) \ldots] = [v_{11}, v_{12}, v_{21}, v_{31}, v_{32}] \begin{bmatrix} \lambda_1 & & & & \emptyset \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ \emptyset & & & & \lambda_3 \end{bmatrix}$$

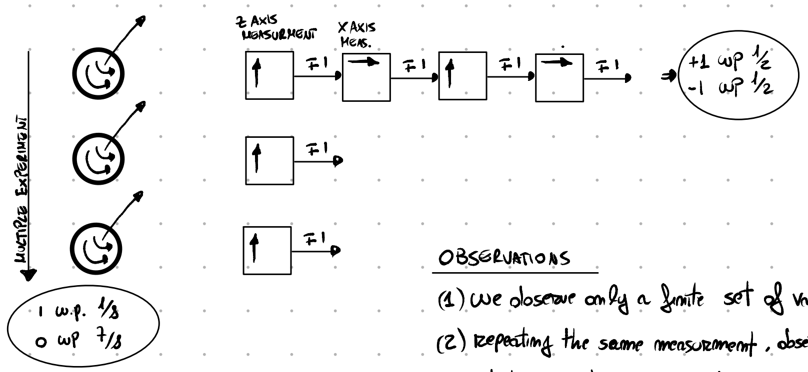
- $$A \underbrace{[v_{11}, v_{12}, v_{21}, v_{31}, v_{32}]}_{\text{THIS MATRIX IS CALLED } V} = [v_{11}, v_{12}, v_{21}, v_{31}, v_{32}] \begin{bmatrix} \lambda_1 & & & & \emptyset \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ \emptyset & & & & \lambda_3 \end{bmatrix} \Rightarrow A = VDV^{-1}$$

- THE CONDITION FOR DIAGONALIZABILITY ARE:
  - if $A \in \mathbb{R}^m \Rightarrow A = A^T$
  - if $A \in \mathbb{C}^m \Rightarrow A = A^T$ AND $A^T = (A^T)^*$ // A is HERMITIAN

- $\forall i$ if ALG.MULT. = GEOM.MULT. You can find a basis made by EIGENVECTORS and you can find even an ORTHONORMAL ONE $\{u_1, u_2, \ldots, u_m\} \mid (u_i, u_s) = \begin{cases} 0 & i \neq s \\ 1 & i = s \end{cases}$

INTRODUCTION

SIMPLE EXPERIMENT



OBSERVATIONS

(1) we observe only a finite set of values

(2) repeating the same measurement, observations are consistent

(3) what is maintained doing the same experiment are probabilities to observe a given result

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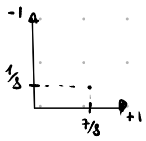
• The description of the state of a particle has to be probabilistic



$$e \rightarrow \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} = \tfrac{7}{8}\,\underset{\text{UP}}{|u\rangle} + \tfrac{1}{8}\,\underset{\text{DOWN}}{|d\rangle} = \tfrac{7}{8}\begin{bmatrix}1\\0\end{bmatrix} + \tfrac{1}{8}\begin{bmatrix}0\\1\end{bmatrix}$$

electron state

• BRA-KET notation

INNER PRODUCT : $\langle u||v\rangle = \langle u|v\rangle$

$\hookrightarrow \langle u|v_1+v_2\rangle = \langle u|v_1\rangle + \langle u|v_2\rangle$

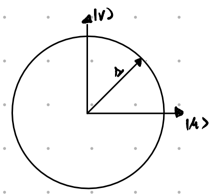
$\hookrightarrow \langle u|v\rangle = \langle v|u\rangle^*$

STATE SPACE REPRESENTATION

• The state of the electron is represented by a vector:

$$|\psi\rangle = \boxed{\alpha_u|u\rangle + \beta_d|d\rangle} \quad | \; \alpha_u, \beta_d \in \mathbb{C} \qquad \begin{array}{l} \|\alpha_u\|^2 = \text{Prob. to observe } |u\rangle, \\ \|\beta_d\|^2 = \text{Prob. to observe } |d\rangle \end{array}$$

• Because the sum of probs must be one, and we are in a vector space of complex number we can represent it on:



• we must assume $|u\rangle$ and $|d\rangle$ ORTHOGONAL $\langle d|u\rangle = 0 = \langle u|d\rangle$

EXTRACT PROBABILITY FRO $|\psi\rangle$

• we can so extract the probability of measuring $|u\rangle$

$$\boxed{\begin{array}{l} \langle u|\psi\rangle = \alpha_u\cancel{\langle u|u\rangle} + \beta\cancel{\langle u|d\rangle} = \alpha_u \\ \langle u|\psi\rangle\langle\psi|u\rangle = \alpha_u^*\,\alpha_u = \|\alpha_u\|^2 = \text{Prob of measuring } |u\rangle \end{array}}$$

• IS $|\psi\rangle$ ENOUGH TO DESCRIBE THE SYSTEM?

• By observing σ_z we have 2 possibility $|u\rangle$ and $|d\rangle$ // UP/DOWN

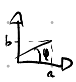
• By " σ_x " " $|l\rangle$ and $|r\rangle$ // LEFT/RIGTH

• given $|\psi\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ and $\beta_l|l\rangle + \beta_r|r\rangle$:

CAN WE WRITE $|l\rangle$ AND $|r\rangle$ AS FUNCTION OF u?

$z = a + jb = \|z\|\, e^{j\angle z}$

$= e^{j\varphi} = \cos\varphi + \sin\varphi$



$$|l\rangle = \beta_u|u\rangle + \beta_d|d\rangle \quad // \quad \|\beta_u\|^2 = \|\beta_d\|^2 = 1/2$$

$$|l\rangle = \tfrac{1}{\sqrt{2}}\, e^{j\angle\beta_u}|u\rangle + \tfrac{1}{\sqrt{2}}\, e^{j\angle\beta_d}|d\rangle = \boxed{\tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}|d\rangle}$$

ANGLE in C

$$|r\rangle = \boxed{\tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}\, e^{j\varphi}|d\rangle}$$

// from $|l\rangle$ we removed PHASE so we must have it on $|r\rangle$

• Obviusly $|r\rangle$ and $|l\rangle$ must be ORTHOGONAL

$$\langle r|l\rangle = 0 \Rightarrow \langle \tfrac{1}{\sqrt{2}}u + \tfrac{1}{\sqrt{2}}d \,|\, \tfrac{1}{\sqrt{2}}u + \tfrac{1}{\sqrt{2}}e^{j\varphi}d\rangle = 0$$

$$\langle u+d \,|\, u+e^{j\varphi}d\rangle = 0$$

$$\langle u|u+e^{j\varphi}d\rangle + \langle d|u+e^{j\varphi}|d\rangle = 0$$
$$\langle u|u\rangle + e^{j\varphi}\langle u|d\rangle + \langle d|u\rangle + e^{j\varphi}\langle d|d\rangle = 0$$
$$1 + 0 + 0 + e^{j\varphi} = 0$$
$$1 + e^{j\pi} = 0$$

• In conclusion we have

$$\boxed{\begin{aligned} |l\rangle &= \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}|d\rangle \\ |r\rangle &= \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}e^{j\pi}|d\rangle = \\ &= \tfrac{1}{\sqrt{2}}|u\rangle - \tfrac{1}{\sqrt{2}}|d\rangle \end{aligned}}$$

• We can do the same for σ_y

$$|i\rangle = \gamma_u|u\rangle + \gamma_d|d\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}e^{j\gamma_d}|o\rangle$$
$$|o\rangle = \delta_u|u\rangle + \delta_d|d\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}e^{j\delta_d}|o\rangle$$

$\Rightarrow \langle i|o\rangle = 0$ // ORTHOGONAL

$$\langle u+e^{j\gamma_d}d \,|\, u+e^{j\delta_d}d\rangle = 0$$
$$1 + e^{j\delta_d}e^{-j\gamma_d} = 1 + e^{j(\delta_d-\gamma_d)} = 0$$
$$\Rightarrow \delta_d = \gamma_d + \pi$$

• TAKING OUT A VARIABLE FROM INNER PRODUCT

$$\langle u|\alpha v\rangle = \alpha\langle u|v\rangle$$
$$\langle \alpha u|v\rangle = (\langle v|\alpha u\rangle)^* = (\alpha\langle v|u\rangle)^* = \alpha^*\langle v|u\rangle^* = \alpha^*\langle u|v\rangle$$

EXPRESS $|i\rangle$ IN TERMS OF $|l\rangle / |r\rangle$

• So we have

$$|i\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}e^{j\gamma_d}|d\rangle$$
$$|l\rangle\ (|r\rangle)\quad \tfrac{1}{\sqrt{2}}|u\rangle \pm \tfrac{1}{\sqrt{2}}|d\rangle$$

$\Rightarrow$ we know that $P(\text{to observe } |l\rangle) = \langle i|l\rangle\langle l|i\rangle = 1/2$

$$\Rightarrow \langle i|l\rangle = \tfrac{1}{2}\cdot 1 \pm \tfrac{1}{2}e^{-j\gamma_d}\cdot 1$$
$$\langle l|i\rangle = \tfrac{1}{2} \pm \tfrac{1}{2}e^{j\gamma_d}$$

$$\Rightarrow \cancel{\tfrac{1}{2}}\left(1 \pm e^{-j\gamma}\right)\tfrac{1}{2}\left(1 \pm e^{j\gamma_d}\right) = \cancel{\tfrac{1}{2}}$$

$$(1 + 1 \pm 2\cos\gamma_d)\tfrac{1}{2} = 1$$

$\Rightarrow$ we know $\pm\cos\gamma_d = 0 \wedge \sin\gamma_d = 1 \Rightarrow \gamma_d = \pm\frac{\pi}{2}$

$$\boxed{\begin{aligned} \gamma_d &= \tfrac{\pi}{2} \\ \delta_d &= -\tfrac{\pi}{2} \end{aligned}}$$

• In conclusion:

$$\boxed{\begin{aligned} &\bullet\ |i\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}\, j\,|d\rangle \\ &\bullet\ |o\rangle = \tfrac{1}{\sqrt{2}}|u\rangle - \tfrac{1}{\sqrt{2}}\, j\,|d\rangle \end{aligned}}$$

- OUR ELECTRON STATE is totally characterized by $\alpha_u|u\rangle + \alpha_d|d\rangle$

$\boxed{\text{QUBIT } |\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$

OBSERVABLE AND MEASURMENT

- OBSERVABLE: something measurable
- After a measurement the system is no more observable

QUANTUM MECHANICS PRINCIPLE

1. All observables can be expressed by HERMITIAN (LINEAR) OPERATORS $L \in \mathbb{C}^{n\times n} \mid L = L^T$
2. The measurement results are the EIGENVALUES
3. * The Probability to measure λ_i when in state $|\psi\rangle$ is $\sum_{v \in V(\lambda_i)} \langle\psi|v_i\rangle\langle v_i|\psi\rangle$

*
$L = L^T$
Pick ORTHONORMAL BASIS OF EIGENVECTORS
from $\lambda_i \to V(\lambda_i) = \{v_{i,j}\}$
$V = \bigcup V(\lambda_i)$
BASIS