

Graph Theory and Optimization

Introduction on Graphs

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Outline

- 1 Vertex/Edge
- 2 Neighbor/Degree
- 3 Path/Cycle
- 4 Trees
- 5 SubGraph

Graph: terminology and notations (Vertex/Edge)

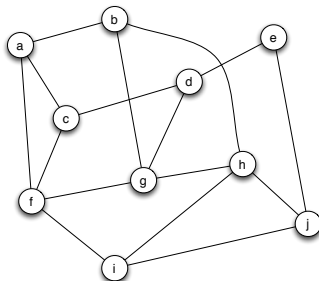
A graph $G = (V, E)$

Vertices: $V = V(G)$ is a finite set

circles

Edges: $E = V(E) \subseteq \{\{u, v\} \mid u, v \in V\}$ is a binary relation on V

lines between two circles



Example: $G = (V, E)$ with $V = \{a, b, c, d, e, f, g, h, i, j\}$ and

$E = \{\{a, b\}, \{a, c\}, \{a, f\}, \{b, g\}, \{b, h\}, \{c, f\}, \{c, d\}, \{d, g\}, \{d, e\}, \{e, j\}, \{f, g\}, \{f, i\}, \{g, h\}, \{h, i\}, \{h, j\}, \{i, j\}\}.$

Graph: terminology and notations (Vertex/Edge)

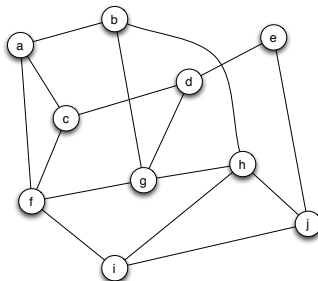
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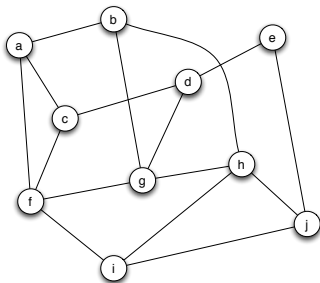
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Exercise: What is the maximum number of edges of a graph with n vertices?

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Graph: terminology and notations (Neighbor/Degree)

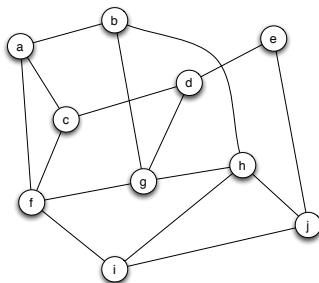


- two vertices $x \in V$ and $y \in V$ are **adjacent** or **neighbors** if $\{x, y\} \in E$
i.e. there is an edge $\{x, y\}$
- $N(x)$: set of neighbors of $x \in V$ **ex:** $N(g) = \{b, d, f, h\} \subseteq V$
- **degree** of $x \in V$: number of neighbors of x i.e., $\text{deg}(x) = |N(x)|$

Exercise: Prove that, for any graph $G = (V, E)$,

$$\sum_{x \in V} \text{deg}(x) = 2|E|$$

Graph: terminology and notations (Neighbor/Degree)



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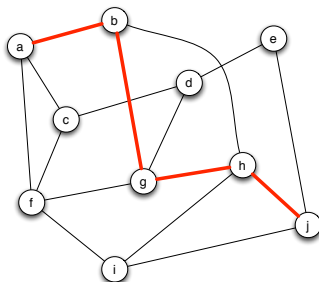
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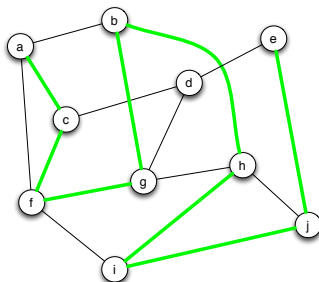
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Graph: terminology and notations (Path/Cycle)



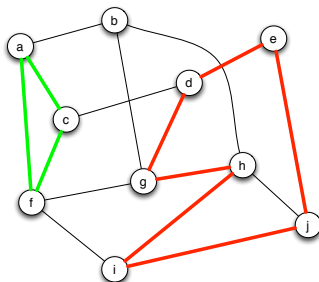
- **Path:** sequence (v_1, \dots, v_ℓ) of distinct vertices such that consecutive vertices are adjacent, i.e., $\{v_i, v_{i+1}\} \in E$ for any $1 \leq i < \ell$
ex: $P_1 = (a, b, g, h, i)$

Graph: terminology and notations (Path/Cycle)



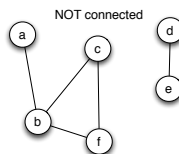
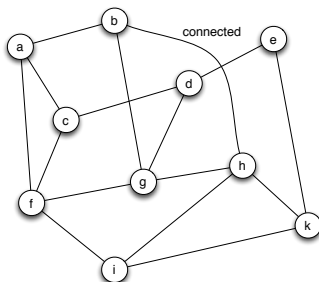
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ex: $P_1 = (a, b, g, h, i)$, $P_2 = (a, c, f, g, b, h, i, j, e)$

Graph: terminology and notations (Path/Cycle)



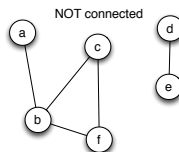
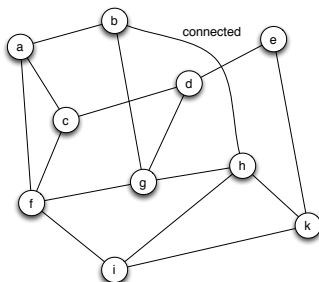
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- **Cycle:** path (v_1, \dots, v_ℓ) such that $\ell \geq 3$ and $\{v_1, v_\ell\} \in E$
ex: $C_1 = (d, e, j, i, h, g)$, $C_2 = (a, c, f)$

Graph: terminology and notations (Path/Cycle)



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- $G = (V, E)$ is **connected** if, for every two vertices $x \in V$ and $y \in V$, there exists a path from x to y .

Graph: terminology and notations (Path/Cycle)



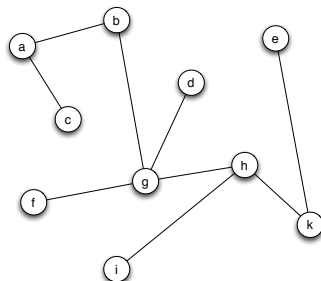
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Exercise: Prove that if $|E| < |V| - 1$ then $G = (V, E)$ is NOT connected

Outline

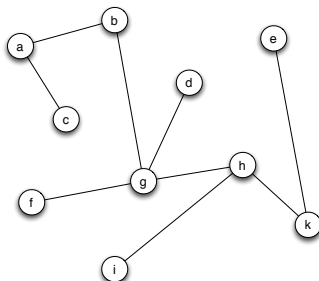
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Graph: terminology and notations (Tree)



- **Tree:** connected graph without cycles
- **Leaf:** vertex of degree 1 in a tree

Graph: terminology and notations (Tree)



- **Tree**: connected graph without cycles
- **Leaf**: vertex of degree 1 in a tree

Trees are important because:

“simple” structure + “minimum” structure ensuring connectivity

Theorem:

$T = (V, E)$ is a tree $\Leftrightarrow T$ connected and $|V| = |E| + 1$

Graph: terminology and notations (Tree)

Theorem: $T = (V, E)$ is a tree $\Leftrightarrow T$ connected and $|V| = |E| + 1$

\Leftarrow By contradiction:

- if T not a tree, then \exists a cycle (v_1, \dots, v_ℓ)
- Let T' be obtained from T by removing edge $\{v_1, v_\ell\}$
- T' is connected *“technical” part, to be proved*
- $|E(T')| = |E| - 1 = |V| - 2 = |V(T')| - 2$
- so $|E'| < |V'| - 1$ and T' is not connected by previous Exercise

A contradiction

Graph: terminology and notations (Tree)

Theorem: $T = (V, E)$ is a tree $\Leftrightarrow T$ connected and $|V| = |E| + 1$

\Rightarrow Induction on $|V|$

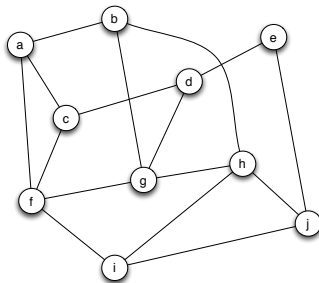
OK if $|V| = 1$

- Let $P = (v_1, \dots, v_\ell)$ be a longest path in T (ℓ max., in particular $\ell \geq 2$)
- v_1 is a leaf. By contradiction:
 - assume $\deg(v_1) > 1$, and $x \in N(v_1) \setminus \{v_2\}$
 - $x \notin V(P)$ otherwise there is a cycle in T
 - then, (x, v_1, \dots, v_ℓ) path longer than P , a contradiction
- then $S = T \setminus \{v_1\}$ is a tree *“technical” part, to be proved*
- $|V(S)| < |V|$ so, by induction $|V(S)| = |E(S)| + 1$
- $|V| = |V(S)| + 1$ and $|E| = |E(S)| + 1$, so $|V| = |E| - 1$

Outline

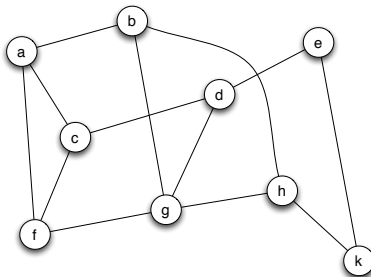
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Graph: terminology and notations (subgraph)



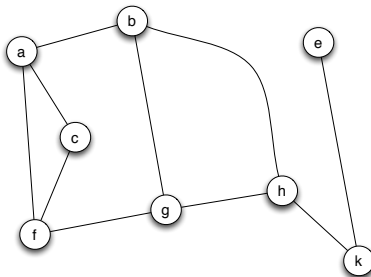
- **Subgraph** of $G = (V, E)$: any graph $H = (V', E')$ with
 $V' \subseteq V$ and $E' \subseteq \{\{x, y\} \in E \mid x, y \in V'\}$
obtained from G by removing some vertices and edges

Graph: terminology and notations (subgraph)



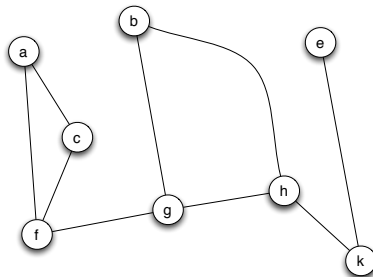
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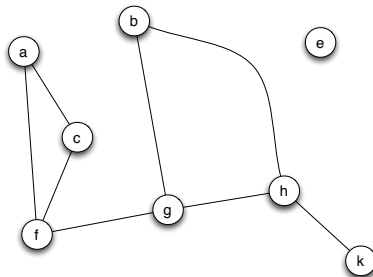
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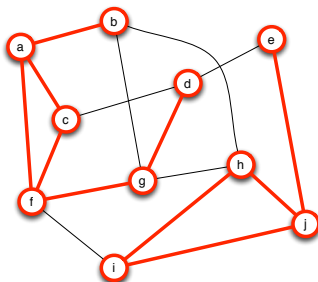
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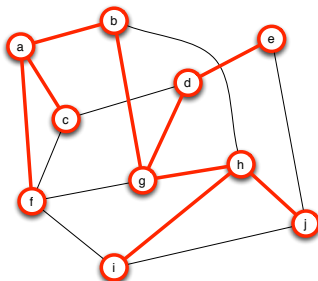
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Graph: terminology and notations (subgraph)



- **Subgraph:** $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq \{\{x, y\} \in E \mid x, y \in V'\}$
- **Spanning** subgraph of G : subgraph $H = (V', E')$ where $V' = V$
obtained from G by removing only some edges

Graph: terminology and notations (subgraph)



- **Subgraph**: $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq \{\{x, y\} \in E \mid x, y \in V'\}$
- **Spanning** subgraph of G : subgraph $H = (V', E')$ where $V' = V$
obtained from G by removing only some edges
- **Spanning tree** of G : spanning subgraph $H = (V, E')$ with H a tree

Summary: To be remembered

All definitions will be important in next lectures

Please remember:

- graph, vertex/vertices, edge
- neighbor, degree
- path, cycle
- connected graph
- tree
- subgraph, spanning subgraph