Graph Theory and Optimization
Integer Linear Programming

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Outline

Linear Programme (reminder)

Linear programmes can be written under the standard form:

$$\begin{array}{llcll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for all } 1 \leq i \leq m \\ & x_j & \geq & 0 & \text{for all } 1 \leq j \leq n. \end{array}$$

- the problem is a maximization;
- all constraints are inequalities (and not equations);
- all variables $x_1, \cdots, x_n$ are non-negative.

Linear Programme (Real variables) can be solved in polynomial-time in the number of variables and constraints (e.g., ellipsoid method)

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Integer Linear Programme

Integer Linear programmes:

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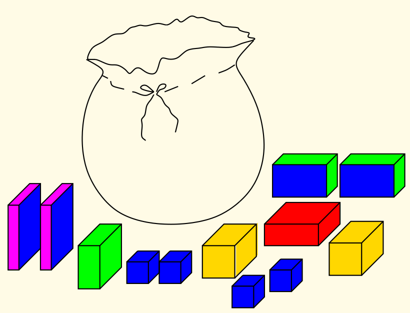
- the problem is a maximization;
- all constraints are inequalities (and not equations);
- all variables $x_1, \cdots, x_n$ are Integers.

Integer Linear Programme is NP-complete in general!

Outline

Knapsack Problem (Weakly NP-hard)



- Data:
  - a knapsack with maximum weight 15 Kg
  - 12 objects with
    - a weight w_i
    - a value v_i
- Objective: which objects should be chosen to maximize the value carried while not exceeding 15 Kg?

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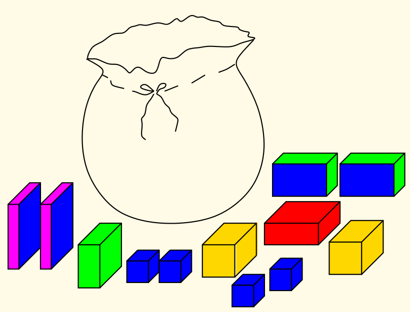

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    - a weight w_i
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- Objective: which objects should be chosen to maximize the value carried while not exceeding 15 Kg?

$$\max \sum_{1 \leq i \leq 12} v_i x_i$$

subject to

$$\sum_{1 \leq i \leq 12} w_i x_i \leq 15$$

$$x_i \in \{0,1\}$$

Minimum Vertex Cover (NP-hard)

Let $G = (V, E)$ be a graph

Vertex Cover: set $K \subseteq V$ such that $\forall e \in E$, $e \cap K \neq \emptyset$

set of vertices that "touch" every edge

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Solution: $K \subseteq V$ $\Rightarrow$ variables x_v, for each $v \in V$
$x_v = 1$ if $v \in K$, $x_v = 0$ otherwise.

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Vertex Coloring (NP-hard)

Let $G=(V,E)$ be a graph

k-Proper coloring: $c : V \rightarrow \{1, \cdots, k\}$ s.t. $c(u) \neq c(v)$ for all $\{u,v\} \in E$.
color the vertices s ($\leq k$ colors) s.t. adjacent vertices receive $\neq$ colors

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Solution: $c: V \rightarrow \{1,\cdots,n\}$ $\Rightarrow$ variables y_j, is color $j \in \{1,\cdots,n\}$ used?
variable c_v^j for color j and vertex v: $c_v^j = 1$ if v colored j, $c_v^j = 0$ otherwise

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Objective function: minimize $\#$ of used colors minimize $\sum_{1 \leq j \leq n} y_j$

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Objective function: minimize $\#$ of used colors minimize $\sum_{1 \leq j \leq n} y_j$

Constraints: each vertex v has 1 color $\sum_{1 \leq j \leq n} c_v^j = 1$

ends of each edge $\{u,v\} \in E$ have $\neq$ colors: $c_v^j + c_u^j \leq 1$ for all $j \in \{1,\cdots,n\}$
color j used if ≥ 1 vertex colored with j $c_v^j \leq y_j$ for all $v \in V$

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Let $G=(V,E)$ be a graph

k-Proper coloring: $c : V \to \{1,\cdots,k\}$ s.t. $c(u) \neq c(v)$ for all $\{u,v\} \in E$.
color the vertices s ($\leq$ k colors) s.t. adjacent vertices receive $\neq$ colors

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\text{Minimize} & \sum_{1\leq j\leq n} y_j & & & \\
\text{Subject to:} & \sum_{1\leq j\leq n} c_v^j & = & 1 & \text{for all } v \in V \\
 & c_v^j + c_u^j & \leq & 1 & \text{for all } j \in \{1,\cdots,n\}, \{u,v\} \in E \\
 & c_v^j & \leq & y_j & \text{for all } j \in \{1,\cdots,n\}, v \in V \\
 & y_j & \in & \{0,1\} & \text{for all } j \in \{1,\cdots,n\} \\
 & c_v^j & \in & \{0,1\} & \text{for all } j \in \{1,\cdots,n\}, v \in V
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Outline

Integer Programme vs. Linear Programme

Integer Linear programme (*ILP*):

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NP-hard in general

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Fractional Relaxation: Linear Programme

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Polynomial-time solvable

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Polynomial-time solvable

What is the difference between Optimal solutions of *LP* and of *ILP*?

Integer Programme vs. Linear Programme

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Polynomial-time solvable

What is the difference between Optimal solutions of *LP* and of *ILP*?

$$OPT(LP) \geq OPT(ILP) \qquad (\textit{for a maximization problem})$$

$$OPT(LP) \leq OPT(ILP) \qquad (\textit{for a minimization problem})$$

If $OPT(LP)$ is “closed" to $OPT(ILP)$, then solving the Fractional Relaxation (in polynomial-time) gives a good bound for the ILP

Fractional Relaxation of Vertex Coloring

Integer Linear programme (*ILP*):

$$
\begin{array}{lrcl}
\text{Minimize} & \sum_{1\leq j\leq n} y_j & & \\
\text{Subject to:} & \sum_{1\leq j\leq n} c_v^j & = & 1 \\
 & c_v^j + c_u^j & \leq & 1 \\
 & c_v^j & \leq & y_j \\
 & y_j & \in & \{0,1\} \\
 & c_v^j & \in & \{0,1\}
\end{array}
$$

Fractional Relaxation (*LP*):

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$y_{red} = y_{blue} = y_{green} = 1$
$c_a^{red} = c_b^{blue} = c_c^{green} = 1$
$OPT(ILP) = \sum_c y_c = 3$

$y_{red} = y_{blue} = 1/2,\ y_{green} = 0$
$c_a^{red} = c_b^{red} = c_c^{red} = 1/2$
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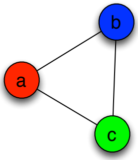

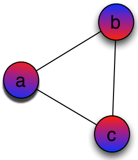

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$OPT(LP) = \sum_c y_c = 1$

Fractional Relaxation of Knapsac

Integer Linear programme (*ILP*):

$$\begin{array}{ll} \textbf{max} & \sum_{1 \le i \le n} v_i x_i \\ \textbf{subject to} & \\ & \sum_{1 \le i \le n} w_i x_i \le W \\ & x_i \in \{0,1\} \end{array}$$

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Example:

- Sac: $W = n$
- Objects:
  - one object (O_1) of weight $n + 0,1$ and value n
  - $n - 1$ objects ($O_2, \cdots, O_n$) of weight 1 and value $1/n$

$x_1 = 0, x_2 = \cdots = x_n = 1$
$OPT(ILP) = \sum_c v_i x_i = (n-1)/n$

$x_1 = \frac{n}{n+0,1}, x_2 = \cdots = x_n = 0$
$OPT(LP) = \sum_c v_i x_i = \frac{n^2}{n+0,1}$

$\Rightarrow$ the ratio between the LP optimal solution and the Integral opt. solution may be arbitrary large

Fractional Relaxation of Knapsac

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Fractional Relaxation (LP):

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Outline

No integrality gap

Integer Linear programme (*ILP*):

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NP-hard in general

Fractional Relaxation: Linear Programme

$$\begin{array}{llll} \text{Max.} & \sum_{j=1}^{n} c_j x_j & & \\ \text{s.t.:} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & \forall 1 \le i \le m \\ & x_j \ge 0 & \forall 1 \le j \le n. \end{array}$$

Polynomial-time solvable

In some cases: $OPT(ILP) = OPT(LP)$.

$\Leftrightarrow$ there exists an integral solution with value $OPT(LP)$.
In such case, Polynomial-time solvable: solve the Fractional Relaxation

Integer Programme Example: Shortest path

$D = (V, A)$ be a digraph with length $\ell : A \rightarrow \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a shortest directed path from s to t.

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$$
\begin{array}{llcll}
\text{Minimize} & \sum_{a \in A} \ell(a) x_a & & & \\
\text{Subject to:} & \sum_{u \in N^+(s)} x(su) & = & 1 & \\
 & \sum_{u \in N^-(t)} x(tu) & = & 1 & \\
 & \sum_{u \in N^+(v)} x(uv) & = & \sum_{u \in N^-(v)} x(vu) & \text{for all } v \in V \setminus \{s, t\} \\
 & x(a) & \in & \{0, 1\} & \text{for all } a \in A
\end{array}
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Integer Programme Example: Shortest path

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& x(a) & \geq & 0 & \text{for all } a\in A
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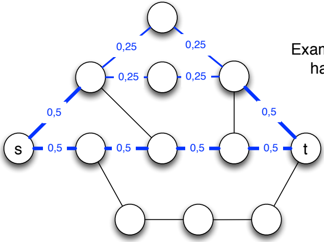
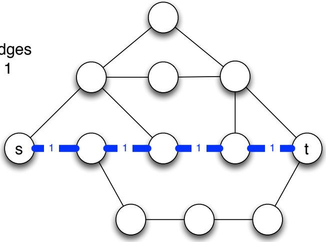
$$



a fractional solution
OPT(LP)=4

Example: all edges have length 1



an integral solution
OPT(ILP)=4

Exercise: Prove that this LP always admits an integral optimal solution

Integer Programme Example: Maximum Matching

$G = (V, E)$ be a graph
Problem: Compute a maximum matching

Solution: a set $M \subseteq E$ of pairwise disjoint edges

$\Rightarrow$ variables x_e for each $e \in E$
$x_e = 1$ if $e \in M$, $x_e = 0$ otherwise.

$$\begin{array}{llcll}
\text{Maximize} & \sum_{e \in E} x_e & & & \\
\text{Subject to:} & \sum_{e \in E, v \in e} x_e & \leq & 1 & \text{for all } v \in V \\
 & x_e & \in & \{0, 1\} & \text{for all } e \in E
\end{array}$$

Exercise: Prove that the fractional relaxation of this ILP always admits an integral optimal solution

Totally unimodular matrices

unimodular matrix: square matrix with determinant $+1$ or -1

totally unimodular matrix: every square non-singular submatrix is unimodular

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NP-hard in general

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Polynomial-time solvable

Theorem [Hoffman,Kruskal, 1956]

If the matrix $A = [a_{ij}]$ is totally unimodular then every *basic* feasible solution (the “corner" of the polytope) is integral

$\Rightarrow$ exist integral optimal solution of the LP

$\Rightarrow$ $OPT(ILP)$ can be computed by solving the Fractional relaxation

Outline

Integer Programme Example: Minimum Spanning Tree

$G = (V, E)$ be a graph with weight $w : E \to \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a minimum spanning tree

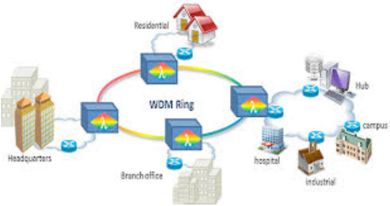
Solution: A spanning tree T $\Rightarrow$ variables x_e for each $e \in E$
$x_E = 1$ if $e \in E(T)$, $x_e = 0$ otherwise.

$$\begin{array}{llccl}
\text{Minimize} & \displaystyle\sum_{e \in E} w(e) x_e & & & \\
\text{Subject to:} & \displaystyle\sum_{e=\{u,v\} \in E, u \in S, v \notin S} x_e & \geq & 1 & \text{for all } S \subseteq V \\
 & x_e & \in & \{0,1\} & \text{for all } e \in E
\end{array}$$

Remark: The number of constraints is exponential

Optical Networks (WDM)

Optical network: optical fiber connecting e.g. routers



Wavelength-division Multiplexing (WDM): technology which multiplexes a number of optical carrier signals onto a single optical fiber by using different wavelengths (i.e., colors) of laser light [Wikipedia]

⇒: different signals on the same link must have different wavelengths (colors)

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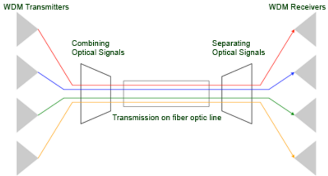


Optical Networks (WDM) RWA problem

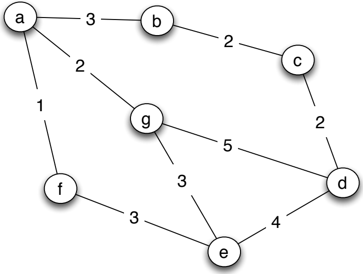
RWA: Routing and Wavelentgh Assignment Problem

Given a graph $G = (V, E)$ with capacity on links, and a traffic-demand matrix T, where $T[u, v]$ is the amount of traffic that must transit from u to v, for any $u, v \in V$.
Find a set of paths and one wavelength assignment for each path such that:

- all demands are routed
- capacity of each link cannot be exceeded
- total number of wavelength is as small as possible

demand matrix:

	a	b	c	d	e	f	g
a	0	0	**1**	**1**	0	**1**	0
b	0	0	0	0	0	**2**	0
c	0	0	0	0	0	0	0
d	0	0	0	0	0	0	**5**
e	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0
g	0	**1**	0	0	0	0	0



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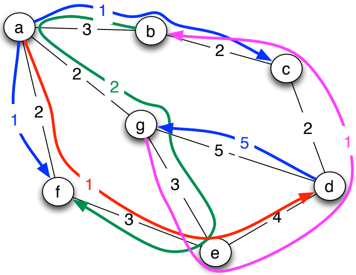
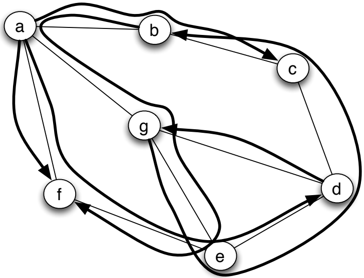


Optical Networks (WDM) RWA problem

Let us simplify the problem ⇒ consider only Wavelength assignment

WA: Wavelentgh Assignment Problem

Given a graph $G = (V, E)$ with capacity on links, and a set of paths
Give One color to each path s.t. no two paths with the same color cross a same link
Minimize the number of colors



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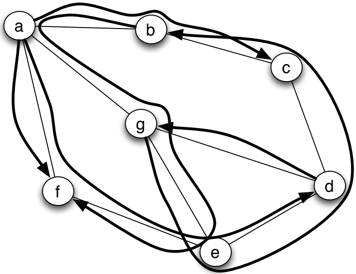

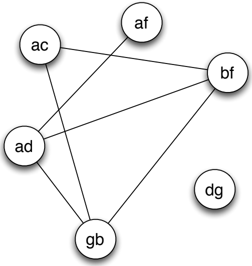

It is the problem of PROPER COLORING in graphs !!

the “simplified" problem is already NP-complete :(

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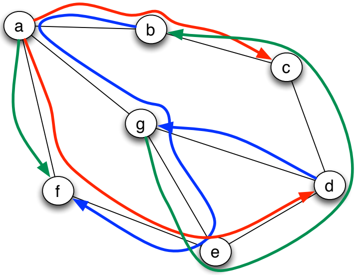

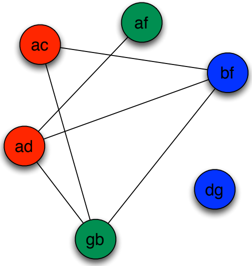

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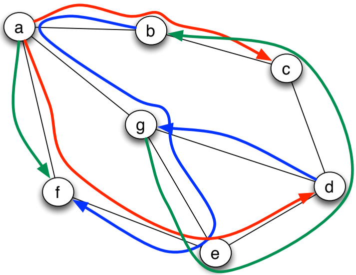

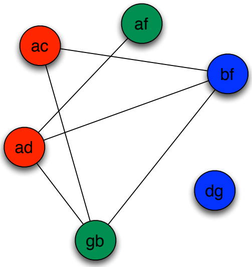

Exercise: Write a (Integer) Linear Programme that solves the RWA problem

Summary: To be remembered

- ILP allow to model many problems
- there may be a huge integrality gap
  (between $OPT(LP)$ and $OPT(ILP)$).
- if no integrality gap (e.g., totally unimodular matrices)
  $\Rightarrow$ Fractional Relaxation gives Optimal Integral Solution