

# Graph Theory and Optimization

## Examples of (Integer) Linear Programming

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# Outline

- 1 Reminder
- 2 First examples of modelling
- 3 Exercises: understand a LP
- 4 More Examples

## Linear Programme (reminder)

Linear programmes can be written under the **standard form**:

$$\begin{array}{ll}\text{Maximize} & \sum_{j=1}^n c_j x_j \\ \text{Subject to:} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for all } 1 \leq i \leq m \\ & x_j \geq 0 \quad \text{for all } 1 \leq j \leq n.\end{array}$$

- the problem is a **maximization**;
- all constraints are **inequalities** (and not equations);
- all variables  $x_1, \dots, x_n$  are **non-negative**.

Linear Programme (Real variables) can be solved in polynomial-time in the number of variables and constraints (e.g., ellipsoid method)

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## LP Example: Maximum Flow

$D = (V, A)$  be a graph with capacity  $c : A \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

**Problem:** Compute a maximum flow from  $s$  to  $t$ .

**Solution:**  $f : A \rightarrow \mathbb{R}^+$

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**Objective function:** maximize value of the flow

$$\sum_{u \in N^+(s)} f(su)$$

**Constraints:**

- capacity constraints:  $f(a) \leq c(a)$  for each  $a \in A$
- flow conservation:  $\sum_{u \in N^+(v)} f(vu) = \sum_{u \in N^-(v)} f(uv), \forall v \in V \setminus \{s, t\}$

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$$\begin{aligned} \text{Subject to:} \quad f(a) &\leq c(a) && \text{for all } a \in A \\ \sum_{u \in N^+(v)} f(vu) &= \sum_{u \in N^-(v)} f(uv) && \text{for all } v \in V \setminus \{s, t\} \\ f(a) &\geq 0 && \text{for all } a \in A \end{aligned}$$

## Integer Programme Example: Shortest path

$G = (V, E)$  be a graph with length  $\ell : E \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

**Problem:** Compute a shortest path from  $s$  to  $t$ .

**Solution:** A path  $P$  from  $s$  to  $t$   $\Rightarrow$  variables  $x_e$  for each  $e \in E$   
 $x_e = 1$  if  $e \in E(P)$ ,  $x_e = 0$  otherwise.

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$$\begin{array}{ll}
 \text{Minimize} & \sum_{e \in E} \ell(e) x_e \\
 \text{Subject to:} & \sum_{u \in N(s)} x(su) = 1 \\
 & \sum_{u \in N(t)} x(tu) = 1 \\
 & \sum_{u \in N(v)} x(vu) = 2 \quad \text{for all } v \in V \setminus \{s, t\} \\
 & x(e) \in \{0, 1\} \quad \text{for all } e \in E
 \end{array}$$

# Integer Programme Example: Minimum Cut

$G = (V, E)$  be a graph with capacity  $c : E \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

**Problem:** Compute a minimum  $s, t$ -cut

**Solution:** A partition  $(S, T)$  of  $V$  with  $s \in S$  and  $t \in T$

$\Rightarrow$  variables  $x_v$  for each  $v \in V$

$x_v = 1$  if  $v \in S$ ,  $x_v = 0$  otherwise.

$$\text{Minimize} \quad \sum_{\{u,v\} \in E} c(\{u,v\}) |x_u - x_v|$$

Subject to:

$$x_s = 1$$

$$x_t = 0$$

$$x_v \in \{0, 1\} \quad \text{for all } v \in V$$

# Integer Programme Example: Minimum Spanning Tree

$G = (V, E)$  be a graph with weight  $w : E \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

**Problem:** Compute a minimum spanning tree

**Solution:** A spanning tree  $T \Rightarrow$  variables  $x_e$  for each  $e \in E$   
 $x_e = 1$  if  $e \in E(T)$ ,  $x_e = 0$  otherwise.

$$\begin{array}{ll}
 \text{Minimize} & \sum_{e \in E} w(e)x_e \\
 \text{Subject to:} & \sum_{e=\{u,v\} \in E, u \in S, v \notin S} x_e \geq 1 \quad \text{for all } S \subseteq V \\
 & x_e \in \{0, 1\} \quad \text{for all } e \in E
 \end{array}$$

**Remark:** The number of constraints is exponential

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# (Integer) Linear Programme Example: Exercises

$G = (V, E)$  be a graph with weight  $w : E \rightarrow \mathbb{R}^+$ , and  $s, t \in V$ .

What compute the following programmes?

$$\begin{array}{ll}
 \text{Maximize} & \sum_{P \text{ path from } s \text{ to } t} x_P \\
 \text{Subject to:} & \sum_{P, e \in E(P)} x_P \leq w(e) \quad \text{for all } e \in E \\
 & x_P \in \{0, 1\} \quad \text{for all paths } P \text{ from } s \text{ to } t
 \end{array}$$

$$\begin{array}{ll}
 \text{Maximize} & x_t \\
 \text{Subject to:} & x_s = 0 \\
 & x_v \leq x_u + w(\{u, v\}) \quad \text{for all } \{v, u\} \in E \\
 & x_v \geq 0 \quad \text{for all } v \in V
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# Integer Programme Example: Maximum Matching

$G = (V, E)$  be a graph

**Problem:** Compute a maximum matching

**Solution:** a set  $M \subseteq E$  of pairwise disjoint edges

$\Rightarrow$  variables  $x_e$  for each  $e \in E$

$x_e = 1$  if  $e \in M$ ,  $x_e = 0$  otherwise.

$$\begin{array}{ll}
 \text{Maximize} & \sum_{e \in E} x_e \\
 \text{Subject to:} & \sum_{e \in E, v \in e} x_e \leq 1 \quad \text{for all } v \in V \\
 & x_e \in \{0, 1\} \quad \text{for all } e \in E
 \end{array}$$