Graph Theory and Optimization
Parameterized Algorithms

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What is it about?

- Goal:
  - Find “efficient” exact algorithms for difficult problems (NP-hard).
  - For some (NP-hard) problems, the difficulty is not due to the size of the input, but to...
    the structure of the input, the size of the solution...
- Introduction to Parameterized Algorithms through Vertex Cover

A very nice book:

M. Cygan, F.V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, S. Saurabh:

Parameterized Algorithms. Springer 2015, ISBN 978-3-319-21274-6, pp. 3-555

Outline

Reminder on Minimum Vertex Cover

Let $G = (V, E)$ be a graph

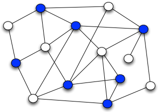
Vertex Cover: set $K \subseteq V$ such that $\forall e \in E$, $e \cap K \neq \emptyset$

set of vertices that "touch" every edge

Finding a Vertex Cover of minimum size is "difficult"

Compute a Min. Vertex Cover is NP-complete

[Garey,Johnson 1979]



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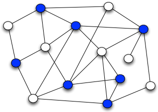

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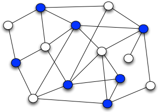

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Naive Exact Algo. for Min. Vertex Cover

input: graph $G = (V, E)$

For $k = 1$ to $|V| - 1$ **do**

  For every set $S \subseteq V$ of size k **do**

    If S is a vertex cover of G

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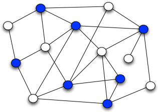

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Time-complexity: $O(2^{|V|}|E|)$

$2^{|V|}$: number of subsets of vertices

$O(|E|)$: time to check if vertex cover

$\Rightarrow$ Exponential in the size of the graph.

Toward “polynomial" algorithms

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Return “No vertex cover of size $\leq k$"

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Remark: the algorithm is still exponential (in the size k of the solution)

Outline

Vertex Cover of size $\leq k$? Two simple Lemmas

$G = (V, E)$ be a graph

$vc(G) =$ min. size of a vertex cover in G

Lemma 1: $vc(G) \leq k \Rightarrow |E| \leq k(|V| - 1)$

small vertex cover $\Rightarrow$ “few" edges

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Lemma 2: Let $\{x,y\} \in E$. $vc(G)=\min\{vc(G\setminus x), vc(G\setminus y)\}+1$

“for any edge xy, any minimum vertex cover contains at least one of x or y..."

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proof:

- Let $S \subseteq V$ be any vertex cover of $G \setminus x$. Then $S \cup \{x\}$ is a vertex cover of G. Hence $vc(G) \leq vc(G \setminus x) + 1$ (symmetrically for $G \setminus y$)
- Let $S \subseteq V$ be any vertex cover of G. At least one of x or y is in S.
  If $x \in S$ then $S \setminus x$ vertex cover of $G \setminus x$. Hence $vc(G \setminus x) \leq vc(G) - 1$.
  Otherwise, if $y \in S$, then $S \setminus y$ vertex cover of $G \setminus y$ and $vc(G \setminus y) \leq vc(G) - 1$.

Vertex Cover of size $\leq k$?   First FPT algorithm

Lemma 2 proves the correctness of the following algorithm

Rec: Branch & Bound Algorithm for computing Minimum size Vertex Cover

input: graph $G = (V, E)$

If $|E| = 0$, *Return* 0.   **Else if** $|E| = 1$, *Return* 1

Else Let $\{x, y\} \in E$, let $A = \mathbf{Rec}(G \setminus x)$, $B = \mathbf{Rec}(G \setminus y)$, *Return* $\min\{A, B\} + 1$

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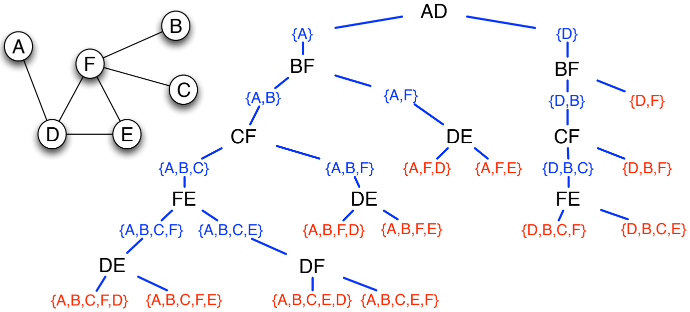

Binary tree of depth $O(|V|)$. Complexity: $O(2^{|V|}|E|)$.

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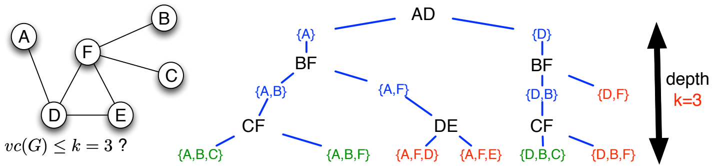
Alg2: Branch & Bound Algorithm for deciding if $vc(G) \leq k$

input: graph $G = (V, E)$, integer $\ell \leq k$.

If $|E| > 0$ and $\ell = 0$, *Return* ∞. **Else if** $|E| = 0$, *Return* 0.
Else if $|E| = 1$, *Return* 1
Else Let $\{x, y\} \in E$, let $A =$ **Alg2**$(G \setminus x, \ell - 1)$, $B =$ **Alg2**$(G \setminus y, \ell - 1)$, *Return* $\min\{A, B\} + 1$



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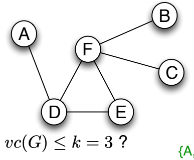

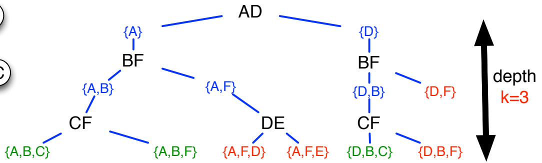

Binary tree of depth $O(k)$: Complexity: $O(2^k|E|)$. By Lem. 1, $|E| = O(k|V|)$,

Alg2 decides if $vc(G) \leq k$ in time $O(2^k \cdot k|V|)$ (linear in $|G|$)

$|V|$ and k are "separated" $\Rightarrow$ Fixed Parameterized Tractable (FPT)

Outline

Parameterized Complexity in brief

Parameterized Problem

A parameterized problem is a language $L \subseteq \Sigma^* \times \Sigma^*$, where Σ is a finite alphabet. The first component corresponds to the input. The second component is called the parameter of the problem.

Class FPT

A parameterized problem is fixed-parameter tractable (FPT) if it can be determined in time $f(k) \cdot |x|^{O(1)}$ whether $(x, k) \in L$, where f is a computable function only depending on k.
The corresponding complexity class is called FPT.

In other words:

Given a (NP-hard) problem with input of size n and a parameter k, a **FPT algorithm** runs in time $f(k) \cdot n^{O(1)}$ for some computable function f.

Examples: k-Vertex Cover, k-Longest Path...

Outline

Vertex Cover of size $\leq k$? Data Reduction

A general GOOD idea: Data reduction

Find simple rules to reduce the size of the input

From input G, compute (in polynomial-time) another instance G' s.t.

$|G'| < |G|$ and a solution for G can be deduced from a solution for G'.

Hence, it is sufficient to solve the problem on the (smaller) instance G'

Back to k-Vertex Cover:

Lemma 3: Let $G = (V, E)$ and $v \in V$ with degree $> k$.
Then v belongs to any vertex cover S of size at most k

Rule: If G has a vertex v of degree $> k$, $vc(G) \leq k \Leftrightarrow vc(G \setminus v) \leq k - 1$.

Lemma 4: $G = (V, E)$. If $vc(G) \leq k$ and no vertex of degree $> k$
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proof: Indeed, if $v \notin S$, all its neighbors must belong to it and $|S| > k$.

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proof: Each of the $\leq k$ vertices of a Vertex Cover covers at most k edges.

Vertex Cover of size $\leq k$? First Kernelization algorithm

Alg3: Kernelization Algorithm for deciding if $vc(G) \leq k$

input: graph $G = (V, E)$, integer $\ell \leq k$.

Remove isolated vertices
If $|E| = 0$, *Return TRUE*. **Else if** $\ell = 0$, *Return FALSE*
Else if no vertex of degree $> \ell$ and $|V| > \ell^2$, *Return FALSE*
Else if no vertex of degree $> \ell$, Apply $Alg2(G, \ell)$
Else Let v be a vertex of degree $> \ell$. Apply $Alg3(G \setminus v, \ell - 1)$.

While there is a “high" degree node, add it to the solution. When there are no such nodes, either it remains too much edges to have a small vertex cover. Otherwise, apply brute force algorithm (e.g., $Alg2$) to the remaining “small" graph

Time-complexity: $O(2^k \cdot k^2 + |V| \cdot k)$ (It is a FPT algorithm!!)

$O(|V| \cdot k)$: find at most k vertices of “high" degree **Reduction Rule**
$O(2^k \cdot k^2)$: application of $Alg2$ to a graph with $O(k^2)$ edges **“Brute Force"**

Kernelization: Apply reduction rule(s) until the instance has constant (only dependent on k) size. Then apply “brute force"

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Vertex Cover Comparison of previous algorithms

Problem: Let $k \in \mathbb{N}$ be a fixed integer. Given $G = (V, E)$, $vc(G) \leq k$?

	time-complexity	numerical example $\lvert V \rvert = 10^4$ and $k = 10$
brute-force for Min. Vertex Cover	$O(\lvert E \rvert \cdot 2^{\lvert V \rvert})$	$>> 10^{3000}$
brute-force, k fixed ($Alg1$)	$O(\lvert E \rvert \lvert V \rvert^k)$	10^{48}
bounded Branch & Bound ($Alg2$)	$O(2^k \cdot k \lvert V \rvert)$	10^8
first kernelization ($Alg3$)	$O(2^k \cdot k^2 + k \lvert V \rvert)$	$2 \cdot 10^5$

Outline

Kernelization

Problem Kernel

Let L be a parameterized problem, that is, L consists of (I, k), where I is the problem instance and k is the parameter.
Reduction to a problem kernel then means to replace instance (I, k) by a "reduced" instance (I', k') (called problem kernel) such that

1. $k' \leq k$, $|I'| \leq g(k)$ for some function g only depending on k,
2. $(I, k) \in L$ if and only if $(I', k') \in L$, and
3. reduction from (I, k) to (I', k') has to be computable in polynomial time.

A *Kernelization* algorithm consists in

1. reduce the size of the instance I in time polynomial in $|I| = n$
2. solve the problem on the reduced instance I' with size $O(g(k))$

Time-complexity: $O(f(g(k)) + n^{O(1)})$
where function f is the time-complexity for solving the problem on I' (e.g., brute force)

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FPT vs. Kernelization

Theorem: [Bodlaender et al. 2009]

A parameterized problem is FPT if and only if it is decidable and has a kernelization algorithm.

proof: $\Leftarrow$ see previous slide ("decidable" implies that function f exists)
$\Rightarrow$ Kernelization: Apply the FPT algorithm. The kernel is the answer $\in \{YES, NO\}$.

It is desirable (if possible) to compute "small" kernel, e.g.,

- linear kernel $g(k) = O(k)$
- quadratic kernel $g(k) = O(k^2)$

  Example: *Alg*3 for Vertex Cover
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In what follows: kernelization algorithm for Vertex Cover with linear kernel

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In what follows: kernelization algorithm for Vertex Cover with linear kernel

FPT vs. Kernelization

Theorem: [Bodlaender et al. 2009]

A parameterized problem is FPT if and only if it is decidable and has a kernelization algorithm.

proof: $\Leftarrow$ see previous slide ("decidable" implies that function f exists)
$\Rightarrow$ Kernelization: Apply the FPT algorithm. The kernel is the answer $\in \{YES, NO\}$.

It is desirable (if possible) to compute "small" kernel, e.g.,

- linear kernel $g(k) = O(k)$
- quadratic kernel $g(k) = O(k^2)$

  Example: $Alg3$ for Vertex Cover
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Outline

Back to Fractional Relaxation for Vertex Cover

Let $G=(V,E)$ be a graph

Integer Linear programme (*ILP*) for Vertex Cover:

$$\begin{array}{llcll} \text{Min.} & \sum_{v\in V} x_v & & & \\ \text{s.t.:} & x_v + x_u & \geq & 1 & \forall \{u,v\}\in E \\ & x_v & \in & \{0,1\} & \forall v \in V \end{array}$$

Fractional relaxation (*LP*) for Vertex Cover:

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Theorem: From Fractional to Integral Solution

Let $(x_v)_{v\in V}$ be a fractional optimal solution.
$V_0=\{v\in V \mid x_v<1/2\}$, $V_1=\{v\in V \mid x_v>1/2\}$ and $V_{1/2}=\{v\in V \mid x_v=1/2\}$
There exists a Minimum (Integral) vertex cover S such that $V_1\subseteq S\subseteq V_1\cup V_{1/2}$

Corollary: reduction Rule using LP for Vertex Cover

Let $(x_v)_{v\in V}$ be a fractional optimal solution.
Then $vc(G)\leq k$ if and only if $vc(G\setminus V_1)\leq k-|V_1|$.

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proof: S^* be an optimal (integral) solution of Vertex Cover. Let $S=(S^*\setminus V_0)\cup V_1$. Clearly S is a vertex cover. By contradiction, if S is not optimal, $|S^*\cap V_0|<|V_1\setminus S^*|$. Let $v\in V_1\cup V_0$ be a vertex with x_v as close as possible from $1/2$ (exists by assumption). Let $\varepsilon=|x_v-1/2|$. Remove ε to x_w for any $w\in V_1\setminus S^*$ and add ε to x_w for any $w\in V_0\cap S^*$. We get a smaller feasible fractional solution, a contradiction.

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Linear Kernel for Vertex Cover

Alg4: Linear Kernel for $vc(G) \leq k$

input: graph $G = (V, E)$, integer $\ell \leq k$.

- **If** $|E| = 0$, *Return TRUE*
- Remove isolated vertices
- Let $(x_v)_{v \in V}$ be an optimal solution obtained by LP
- **If** optimal fractional solution $> \ell$, *Return FALSE*
- **Else** let $V_1 = \{v \in V \mid x_v > 1/2\}$.
  - **If** $V_1 \neq \emptyset$ then *Return* $Alg4(G \setminus V_1, \ell - |V_1|)$.
  - **Else** Apply $Alg2(G, \ell)$

While possible, apply LP and add to the solution the vertices w with $x_w > 1/2$.

When it is not possible anymore, then all vertices v are such that $x_v = 1/2$ (check it).

Hence $|V| \leq 2k$ (Linear kernel).

Then, apply brute force algorithm (e.g., $Alg2$) to the remaining "small" graph

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Outline

Take Aways

- Parameterized Problem: input (size n) $+$ parameter k
- FPT algorithm: in time $f(k)n^{O(1)}$
- Kernelization: Data reduction
- Kernelization $\Leftrightarrow$ FPT
- Linear Kernel for Vertex Cover