STUDY OF A DEGREE DISTRIBUTION AND A VERTEX TRAJECTORY IN THE CHUNG-LU MODEL WITH A GENERALIZED ATTACHMENT FUNCTION

SHEEP AND FRIENDS

Abstract.

1. Chung-Lu model with a generalized attachment function

- $G_1 = (\{v_0, v_1\}, \{\{v_0, v_1\}\})$ - initial graph, two vertices joined by an edge
- $G_t = (V_t, E_t)$ - graph at time $t \geq 1$
- $\deg_t(v)$ - degree of vertex v in G_t
- Process (generalized Chung-Lu). At time $t + 1 \geq 2$:
  - with p: add a new vertex v and an edge between v and some vertex from G_t,
  - with $1 - p$: add a new edge between vertices from G_t.

  Endpoints of edges are chosen according two an attachment function $f(x)$, i.e., the probability for $v \in G_t$ to be chosen as a single endpoint of an edge equals
  $$\mathbb{P}[v \text{ is chosen}] = \frac{f(\deg_t(v))}{\sum_{w \in V_t} f(\deg_t(w))}.$$
  When we add a new edge between vertices of G_t it may happen that the same vertex is chosen twice (then we create a loop). Both choices are independent and made according to the same degree distribution, i.e., we update the degrees of vertices only after choosing both edge endpoints.
- Graph G_t consists of
  - t edges ($|E_t| = t$), thus $\sum_{v \in V_t} \deg_t(v) = 2t$,
  - on average pt vertices ($V_t \sim B(t,p)$, thus $|V_t| \sim pt$).

2. Linear attachment function

Let $f(x) = ax + b$, $b > -a$. Note that then
$$\mathbb{P}[v \text{ is chosen}] = \frac{a\deg_t(v) + b}{\sum_{w \in V_{t-1}} (a\deg_t(w) + b)} = \frac{\deg_t(v) + b/a}{\sum_{w \in V_{t-1}} (\deg_t(w) + b/a)}$$
which equals $\mathbb{P}[v \text{ is chosen}]$ for the attachment function $f(x) = x + b/a$. Therefore, w.l.o.g., we are going to consider
$$f(x) = x + c, \qquad c > -1.$$

Date: November 4, 2021.

Key words and phrases. Chung-Lu model, complex network, preferential attachment, attachment function.

This research was partially supported by SNIF .

We have

$$\mathbb{P}[v \text{ is chosen}] = \frac{\deg_t(v)+c}{\sum_{w\in V_t}(\deg_t(w)+c)} \sim \frac{\deg_t(v)+c}{2t+cpt} = \frac{\deg_t(v)+c}{(2+cp)t}.$$

2.1. **Degree distribution.** Let $m_{k,t}$ denote the number of vertices of degree k at time t. Let also

$$\tilde{M}_k = \lim_{t\to\infty} \frac{\mathbb{E}[m_{k,t}]}{t}.$$

Recall that $|V_t| \sim pt$. Thus let

$$M_k = \tilde{M}_k/p$$

and we interpret M_k as the fraction of vertices of degree k. Since $G_1 = (\{v_0,v_1\},\{\{v_0,v_1\}\})$ we have $m_{1,1}=2$. Now, let $\mathcal{F}_t$ be the σ-algebra associated with the probability space at time t. For $t\geq 2$ we get

$$\mathbb{E}[m_{k,t}|\mathcal{F}_{t-1}] = \begin{cases} m_{k,t-1}\left(1-\frac{p(k+c)}{(2+cp)t}-\frac{2(1-p)(k+c)}{(2+cp)t}\right) & \\ \qquad +m_{k-1,t-1}\left(\frac{p(k-1+c)}{(2+cp)t}+\frac{2(1-p)(k-1+c)}{(2+cp)t}\right), & k>1 \\ m_{1,t}=p+m_{1,t-1}\left(1-\frac{p(1+c)}{(2+cp)t}-\frac{2(1-p)(1+c)}{(2+cp)t}\right) & \end{cases}$$

$$= \begin{cases} m_{k,t-1}\left(1-\frac{(2-p)(k+c)}{(2+cp)t}\right)+m_{k-1,t-1}\left(\frac{(2-p)(k-1+c)}{(2+cp)t}\right), & k>1 \\ m_{1,t}=p+m_{1,t-1}\left(1-\frac{(2-p)(1+c)}{(2+cp)t}\right) & \end{cases}$$

which, taking expectation on both sides and by

$$\beta = \frac{2+cp}{2-p},$$

gives

$$\begin{cases} \tilde{M}_1 = \frac{\beta p}{\beta+1+c}, \\ \tilde{M}_k = \tilde{M}_{k-1}\frac{k-1+c}{k+c+\beta}. \end{cases}$$

Note that $\beta>1$ (since $c>-1$). Finally we get that the degree distribution follows a power-law with a scaling exponent $\beta+1>2$:

$$M_k = \frac{\beta\cdot\Gamma(1+\beta+c)}{\Gamma(1+c)}\frac{\Gamma(k+c)}{\Gamma(k+c+1+\beta)} \sim \frac{\beta\cdot\Gamma(1+\beta+c)}{\Gamma(1+c)}k^{-(1+\beta)}.$$

2.2. **Vertex trajectory.** Let v be a vertex chosen uniformly at random from V_t and let t_v be a random variable denoting its arrival time. Note that t_v Let $g_v(t) = \mathbb{E}[\deg_t(v)]$ (Expectation counted with respect to which probability space?) Assume that $g_v(t)$ is continuous and consider its one-step change from t to $t+1$:

$$\begin{aligned} g_v'(t) &= p\cdot 1\cdot\frac{g_v(t)+c}{(2+cp)t} + (1-p)\cdot 2\cdot\frac{g_v(t)+c}{(2+cp)t} \\ &= \frac{2-p}{2+cp}\cdot\frac{g_v(t)+c}{t}. \end{aligned}$$

The solution to this differential equation is

$$g_v(t) = const\cdot t^{(2-p)/(2+cp)} - c,$$

which by $\beta = \frac{2+cp}{2-p}$ reads as

$$g_v(t) = const\cdot t^{1/\beta} - c,$$

We get *const* from the initial condition $g_v(t_v) = 1$ which finally gives

$$g_v(t) = (1 + c)(t/t_v)^{1/\beta} - c.$$

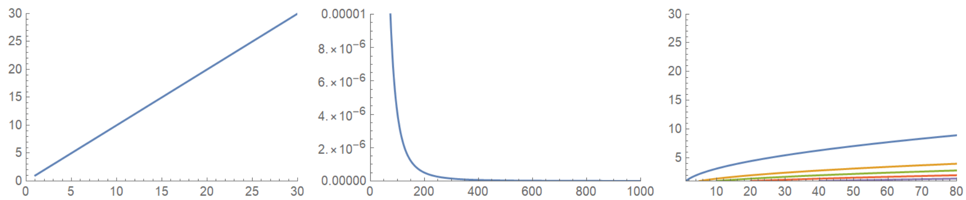

FIGURE 1. Attachment function, degree distribution and vertex trajectories for $p = 1$, $c = 0$, $t_v = 1, 5, 10, 20, 40$.

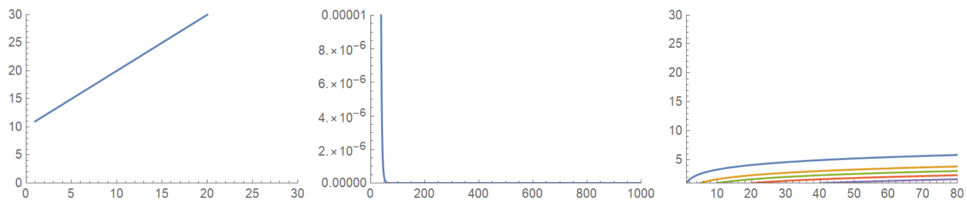

FIGURE 2. Attachment function, degree distribution and vertex trajectories for $p = 1$, $c = 10$, $t_v = 1, 5, 10, 20, 40$.

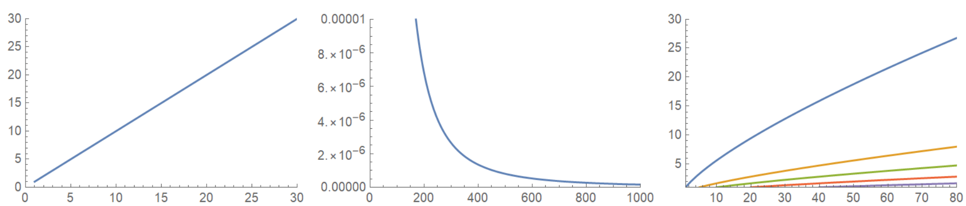

FIGURE 3. Attachment function, degree distribution and vertex trajectories for $p = 0.5$, $c = 0$, $t_v = 1, 5, 10, 20, 40$.

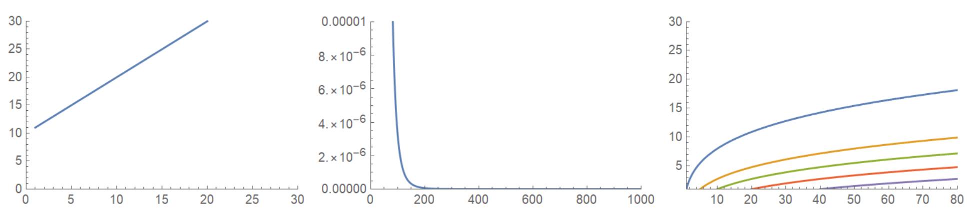

FIGURE 4. Attachment function, degree distribution and vertex trajectories for $p = 0.5$, $c = 10$, $t_v = 1, 5, 10, 20, 40$.

3. Sublinear attachment function

Let

$$f(x) = x^{\gamma} \quad \text{with} \quad 0 \le \gamma < 1.$$

This time we have

$$\mathbb{P}[v \text{ is chosen}] = \frac{(\deg_t(v))^{\gamma}}{\sum_{w \in V_t} (\deg_t(w))^{\gamma}}.$$

Note that

$$pt \sim \sum_{w \in V_t} (\deg_t(w))^0 \le \sum_{w \in V_t} (\deg_t(w))^{\gamma} \le \sum_{w \in V_t} (\deg_t(w))^1 = 2t$$

therefore we assume

$$\sum_{w \in V_t} (\deg_t(w))^{\gamma} \sim \mu t, \quad \text{where} \quad \mu \in [p, 2].$$

Thus we write

$$\mathbb{P}[v \text{ is chosen}] \sim \frac{(\deg_t(v))^{\gamma}}{\mu t}.$$

3.1. **Degree distribution.** Again, $m_{k,t}$ denote the number of vertices of degree k at time t,

$$\tilde{M}_k = \lim_{t \to \infty} \frac{\mathbb{E}[m_{k,t}]}{t},$$

and

$$M_k = \tilde{M}_k / p$$

is interpreted as the fraction of vertices of degree k. Since $G_1 = (\{v_0, v_1\}, \{\{v_0, v_1\}\})$ we have $m_{1,1} = 2$. Recall that $\mathcal{F}_t$ is the σ-algebra associated with the probability space at time t. For $t \ge 2$ we get

$$\begin{aligned}
\mathbb{E}[m_{k,t}|\mathcal{F}_{t-1}] &= \begin{cases} m_{k,t-1}\left(1 - \frac{pk^{\gamma}}{\mu t} - \frac{2(1-p)k^{\gamma}}{\mu t}\right) & \\ \qquad + m_{k-1,t-1}\left(\frac{p(k-1)^{\gamma}}{\mu t} + \frac{2(1-p)(k-1)^{\gamma}}{\mu t}\right), & k > 1 \\ m_{1,t} = p + m_{1,t-1}\left(1 - \frac{p}{\mu t} - \frac{2(1-p)}{\mu t}\right) & \end{cases} \\
&= \begin{cases} m_{k,t-1}\left(1 - \frac{(2-p)k^{\gamma}}{\mu t}\right) + m_{k-1,t-1}\left(\frac{(2-p)(k-1)^{\gamma}}{\mu t}\right), & k > 1 \\ m_{1,t} = p + m_{1,t-1}\left(1 - \frac{(2-p)}{\mu t}\right) & \end{cases}
\end{aligned}$$

which, taking expectation on both sides and by

$$\alpha = \frac{\mu}{2-p},$$

gives

$$\begin{cases} \tilde{M}_1 = \frac{\alpha p}{\alpha + 1}, \\ \tilde{M}_k = \tilde{M}_{k-1} \frac{(k-1)^{\gamma}}{k^{\gamma} + \alpha}. \end{cases}$$

Thus we get

$$\begin{aligned}
M_k &= \frac{\alpha}{k^{\gamma}} \prod_{j=1}^{k} \left(\frac{j^{\gamma}}{\alpha + j^{\gamma}}\right) = \frac{\alpha}{k^{\gamma}} \prod_{j=1}^{k} \left(1 + \frac{\alpha}{j^{\gamma}}\right)^{-1} \\
&\sim \alpha \cdot k^{-\gamma} \cdot \exp\left\{-\frac{\alpha}{1-\gamma} k^{1-\gamma}\right\}
\end{aligned}$$

which decays faster than any power but not exponentially fast. The degree distribution in this model by $p=1$ was obtained by Krapivski et al. [3] and Krapivski and Redner [2]. (For the detailed derivation of the formula for M_k and its asymptotics check the lemmas in the Appendix.)

3.2. **Vertex trajectory.** Let t_v denote the arrival time of a vertex v. Let $g_v(t) = \mathbb{E}[\deg_t(v)]$ (Expectation counted with respect to which probability space?) Assume that $g_v(t)$ is continuous and consider its one-step change from t to $t+1$:

$$\begin{aligned} g_v'(t) &= p \cdot 1 \cdot \frac{(g_v(t))^\gamma}{\mu t} + (1-p) \cdot 2 \cdot \frac{(g_v(t))^\gamma}{\mu t} \\ &= \frac{2-p}{\mu} \cdot \frac{(g_v(t))^\gamma}{t}. \end{aligned}$$

The solution to this differential equation is

$$g_v(t) = \left((1-\gamma)\frac{2-p}{\mu} \ln t + const \right)^{1/(1-\gamma)}$$

which by $\alpha = \frac{\mu}{2-p}$ reads as

$$g_v(t) = \left(\frac{1-\gamma}{\alpha} \ln t + const \right)^{1/(1-\gamma)}.$$

We get $const$ from the initial condition $g_v(t_v) = 1$ which finally gives

$$g_v(t) = \left(\frac{1-\gamma}{\alpha} \ln(t/t_v) + 1 \right)^{1/(1-\gamma)}.$$

4. Superlinear attachment function

Consider

$$f(x) = x^\gamma \quad \text{with} \quad \gamma > 1.$$

Attachment function	**Degree distribution**	**Vertex trajectory**
$f(x) = x + c$ $c > -1$ $\beta = \frac{2+cp}{2-p}$	$M_k = \frac{\beta \cdot \Gamma(1+\beta+c)}{\Gamma(1+c)} \cdot \frac{\Gamma(k+c)}{\Gamma(k+c+1+\beta)}$ $\sim \frac{\beta \cdot \Gamma(1+\beta+c)}{\Gamma(1+c)} \cdot k^{-(1+\beta)}$	$g_v(t) = (1+c)(t/t_v)^{1/\beta} - c$
$f(x) = x^\gamma$ $0 \le \gamma < 1$ $\alpha = \frac{\mu}{2-p}$ *	$M_k = \frac{\alpha}{k^\gamma} \prod_{j=1}^k \left(\frac{j^\gamma}{\alpha + j^\gamma} \right)$ $\sim \alpha \cdot k^{-\gamma} \cdot \exp\left\{ -\frac{\alpha}{1-\gamma} k^{1-\gamma} \right\}$	$g_v(t) = \left(\frac{1-\gamma}{\alpha} \ln(t/t_v) + 1 \right)^{1/(1-\gamma)}$
$f(x) = x^\gamma$ $\gamma > 1$	one vertex of degree $\Theta(t)$ other vertices of degree $O(1)$	——

* We assume that $\sum_{w \in V_t} (\deg_t(w))^\gamma \sim \mu t$, where $\mu \in [p, 2]$.

Appendix

Lemma 1. $M_k = \frac{\alpha}{k^\gamma} \prod_{j=1}^k \left(1 + \frac{\alpha}{j^\gamma}\right)^{-1}$.

Proof. We have

$$\begin{cases} \tilde{M}_1 = \frac{\alpha p}{\alpha+1}, \\ \tilde{M}_k = \tilde{M}_{k-1} \frac{(k-1)^\gamma}{k^\gamma+\alpha} \end{cases}$$

thus

$$\begin{aligned} \tilde{M}_k &= \tilde{M}_1 \frac{1^\gamma 2^\gamma \dots (k-1)^\gamma}{(2^\gamma+\alpha)(3^\gamma+\alpha)\dots(k^\gamma+\alpha)} \\ &= \frac{\alpha p}{\alpha+1} \cdot \frac{1^\gamma+\alpha}{k^\gamma} \cdot \frac{1^\gamma 2^\gamma \dots k^\gamma}{(1^\gamma+\alpha)(2^\gamma+\alpha)\dots(k^\gamma+\alpha)} \\ &= \frac{\alpha p}{k^\gamma} \prod_{j=1}^k \frac{j^\gamma}{j^\gamma+\alpha}. \end{aligned}$$

Recall that $\tilde{M}_k = M_k/p$. We get

$$M_k = \frac{\alpha}{k^\gamma} \prod_{j=1}^k \left(1 + \frac{\alpha}{j^\gamma}\right)^{-1}.$$

□

Lemma 2. $M_k \sim \alpha \cdot k^{-\gamma} \cdot \exp\left\{-\frac{\alpha}{1-\gamma} k^{1-\gamma}\right\}$

Proof. By Lemma 1 we have

$$M_k = \frac{\alpha}{k^\gamma} \prod_{j=1}^k \left(1 + \frac{\alpha}{j^\gamma}\right)^{-1}.$$

We start with a rough calculations just to get the intuition (one can find them in [1]). Recall that $\gamma < 1$ and note that

$$\log \prod_{j=1}^k \left(1 + \frac{\alpha}{j^\gamma}\right)^{-1} = -\sum_{j=1}^k \log\left(1 + \frac{\alpha}{j^\gamma}\right) \sim -\sum_{j=1}^k \frac{\alpha}{j^\gamma} \sim -\frac{\alpha}{1-\gamma} k^{1-\gamma}$$

which results in $M_k \sim \alpha \cdot k^{-\gamma} \cdot \exp\left\{-\frac{\alpha}{1-\gamma} k^{1-\gamma}\right\}$.

Now, let us treat it in more details. We will show, using Stolz-Cesàro theorem 1, that

$$\lim_{k\to\infty} \frac{\sum_{j=1}^k \log\left(1 + \frac{\alpha}{j^\gamma}\right)}{\frac{\alpha}{1-\gamma} k^{1-\gamma}} = 1.$$

Define $a_n = \sum_{j=1}^n \log\left(1 + \frac{\alpha}{j^\gamma}\right)$ and $b_n = \frac{\alpha}{1-\gamma} n^{1-\gamma}$. The sequence $(b_n)_{n\geq 1}$ is strictly increasing and divergent. By L'Hôpital's rule we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} &= \lim_{n\to\infty} \frac{\log\left(1 + \frac{\alpha}{(n+1)^\gamma}\right)}{\frac{\alpha}{1-\gamma}((n+1)^{1-\gamma} - n^{1-\gamma})} \\
&\overset{0/0}{=} \lim_{n\to\infty} \frac{-\frac{1}{1+\alpha/(n+1)^\gamma} \cdot \alpha\gamma \cdot (n+1)^{-\gamma-1}}{\frac{\alpha}{1-\gamma}(1-\gamma)((n+1)^{-\gamma} - n^{-\gamma})} \\
&= \lim_{n\to\infty} -\gamma \frac{1/(n+1)}{1 - (1 + \frac{1}{n})^\gamma} \\
&\overset{0/0}{=} \lim_{n\to\infty} -\gamma \frac{-1/(n+1)^2}{-\gamma(1+\frac{1}{n})^{\gamma-1}(-1/n^2)} = 1.
\end{aligned}$$

By 1 we conclude that $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{k\to\infty} \frac{\sum_{j=1}^k \log\left(1+\frac{\alpha}{j^\gamma}\right)}{\frac{\alpha}{1-\gamma} k^{1-\gamma}} = 1.$ □

Theorem 1 (Stolz-Cesàro theorem)**.** *Let $(b_n)_{n\geq 1}$ be a strictly monotone and divergent sequence. If the following limit exists and equals g*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = g.$$

References

[1] Rick Durrett. *Random Graph Dynamics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2006.

[2] P. L. Krapivsky and S. Redner. Organization of growing random networks. *Phys. Rev. E*, 63:066123, May 2001.

[3] P. L. Krapivsky, S. Redner, and F. Leyvraz. Connectivity of growing random networks. *Phys. Rev. Lett.*, 85:4629–4632, Nov 2000.