NUMBER SET $\mathbb{R}$

WE START RECALLING THE SET OF THE REAL NUMBERS ON WHICH WE WILL WORK.

THINKING ABOUT THE NUMBER SETS WE HAVE MET, WE CAN MAKE A LIST:

NATURAL NUMBER SET: $\mathbb{N} = \{0, 1, 2, \dots\}$

INTEGERS: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

RATIONALS: $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$

and we know (it's easy to see) that:

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}}$$

IS $\mathbb{Q}$ LARGE ENOUGH? No!

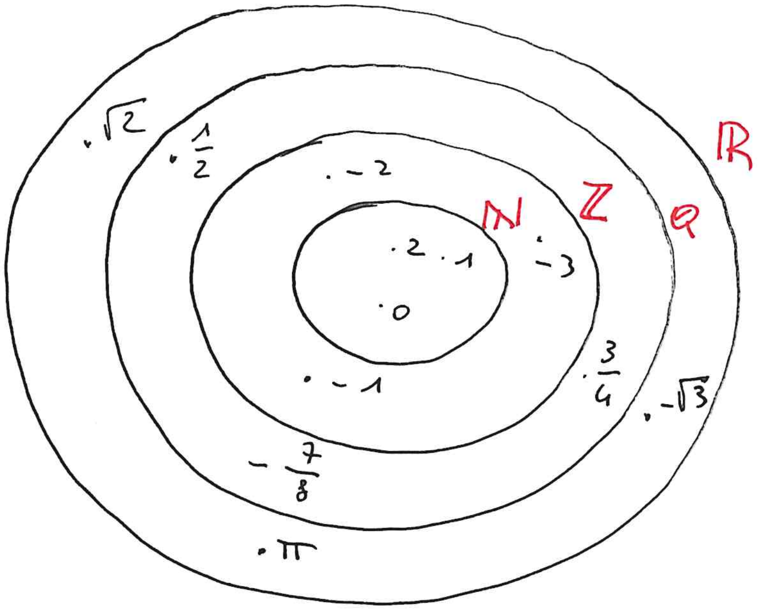
In fact, IF WE WANT TO ANSWER TO THE QUESTION:

"HOW LONG IS THE DIAGONAL OF A SQUARE OF SIDE 1?"

WE HAVE A "STRANGE" NUMBER, THAT IS NOT RATIONAL AND IT IS $\sqrt{2}$ (IT'S IMPOSSIBLE TO WRITE $\sqrt{2}$ LIKE A RATIONAL NUMER $\frac{m}{n}$).

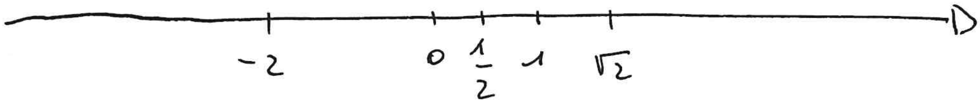
WE HAVE MANY MORE NUMBERS BEYOND THE RATIONALS: $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$, π, ... WE CALL THEM IRRATIONALS.

SO $\mathbb{R}$ IS THE SET THAT CONTAINS ALL THE RATIONAL AND IRRATIONAL NUMBERS.



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$ ... SO $\mathbb{R}$ IS LARGE ENOUGH

GEOMETRICALLY WE CAN USE THE SO CALLED "NUM "REAL LINE" TO REPRESENT THE SET $\mathbb{R}$:



ANY POINT OF THE REAL LINE IS A REAL NUMBER.

STRUCTURES IN $\mathbb{R}$

1) OPERATIONS:

SUM: $x+y \in \mathbb{R} \quad \forall x, y \in \mathbb{R}$ (and $x-y \in \mathbb{R}$)

PRODUCT: $x \cdot y \in \mathbb{R} \quad \forall x, y \in \mathbb{R}$ (and $\frac{x}{y} \in \mathbb{R}$ WHEN $y \neq 0$)

2) ORDER:

TAKEN $x, y \in \mathbb{R}$ WE CAN ALWAYS STATE WHETHER

$x > y$ OR $x < y$ OR $x = y$ ($x \leq y$ OR $x \geq y$)

WE SAY THAT IN $\mathbb{R}$ THERE IS A TOTAL ORDER.

THANKS TO THE NOTION OF ORDER WE CAN DEFINE A SPECIFIC KIND A SUBSET OF $\mathbb{R}$: INTERVALS

def

$A \subseteq \mathbb{R}$ IS CALLED INTERVAL WHEN:

$\forall x, y \in A$ THE SET $\{z \in \mathbb{R} : x \leq z \leq y\} \subseteq A$

ex.

$[a, b] \quad a \leq b$ (THE SQUARE BRACKET MEANS THAT THE ELEMENT IS INCLUDED)

$(a, b) \quad a \leq b$ (THE ROUND BRACKET MEANS THAT THE ELEMENT IS NOT INCLUDED)

$(a, b]$ IS AN INTERVAL

BUT ALSO $[a, +\infty)$

AND $(-\infty, b)$

ex IS $A = (-2, 2) \cup [3, 5)$ AN INTERVAL?

No! BECAUSE IF WE TAKE $\frac{5}{2}$, WE HAVE THAT

$1 \leq \frac{5}{2} \leq 3$ WHERE $1, 3 \in A$ BUT $\frac{5}{2} \notin A$.

ex IS $A = \{5\}$ (ONLY ONE POINT) AN INTERVAL?

YES! $\{5\} = [5, 5]$

ex. IS $A = \emptyset$ (EMPTY SET) AN INTERVAL?

YES! $\emptyset = (a, a)$

ex. IS $A = \mathbb{R}$ AN INTERVAL?

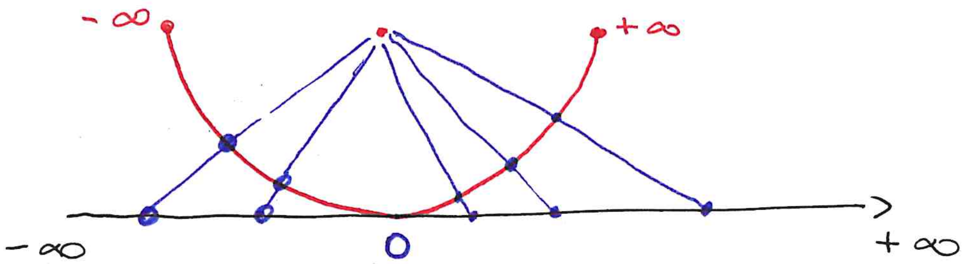
YES! $\mathbb{R} = (-\infty, +\infty)$

WE HAVE USED THE SYMBOLS "$+\infty$" AND "$-\infty$" IN A VERY SIMPLE WAY, LIKE "$+\infty$ MEANS: SO FAR ON THE RIGHT" AND "$-\infty$ MEANS: SO FAR ON THE LEFT".

IN FACT $+\infty$ AND $-\infty$ ARE NOT REAL NUMBERS!

THERE IS A GRAPHICAL WAY "TO TOUCH" $\pm\infty$.

CONSIDER A SEMI-CIRCLES AND A LINE (THE REAL LINE):



WE HAVE A ONE TO ONE RELATION BETWEEN THE POINTS OF THE SEMICIRCLES AND THE LINE, SO THE LEFT AND RIGHT EXTREMES OF THE SEMI-CIRCLE ARE, RESPECTIVELY, $-\infty$ AND $+\infty$.

UPPER AND LOWER BOUNDS

def. CONSIDER $A \subseteq \mathbb{R}$ AND $A \neq \emptyset$.

1) A NUMBER $h \in \mathbb{R}$, IF IT EXISTS, SUCH THAT

$x \leq h \ \forall x \in A$ IS CALLED UPPER BOUND OF A.

2) A NUMBER $k \in \mathbb{R}$, IF IT EXISTS, SUCH THAT

$x \geq k \ \forall x \in A$ IS CALLED LOWER BOUND of A.

ex $A = (-2, 5]$

- WE HAVE THAT 5 IS AN UPPER BOUND, BUT ALSO 7, 10, 25...
  ARE UPPER BOUNDS
- WE HAVE THAT -2 IS A LOWER BOUND of A, BUT ALSO
  $0, -3, -5 \ldots$ ARE LOWER BOUNDS

SO, IF THERE EXISTS AN UPPER / LOWER BOUND THEN WE HAVE INFINITELY MANY OTHERS.

ex $A = (-\infty, 2) \cup (3, 5]$

A HAS AN UPPER BOUND $(5, 6, \ldots)$ BUT HAS NOT A LOWER BOUND ex

$$A = \left\{ \frac{1}{n+1} \quad \forall n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

1 IS AN UPPER BOUND OF A

0 IS A LOWER BOUND OF A

def. $A \subseteq \mathbb{R}$, $A \neq \emptyset$ IS SAID TO BE:

a) BOUNDED FROM ABOVE IF IT HAS AN UPPER BOUND

b) BOUNDED FROM BELOW IF IT HAS A LOWER BOUND

c) BOUNDED IF IT HAS UPPER AND LOWER BOUND.

def. Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$. A NUMBER $x^* \in A$, IF IT EXISTS, IS CALLED:

a) MAXIMUM OF A WHEN $x^* \geq x \quad \forall x \in A$ (max A)

b) MINIMUM OF A WHEN $x^* \leq x \quad \forall x \in A$ (min A)

REMARK IT LOOKS LIKE THE DEFINITION OF UPPER AND LOWER BOUNDS BUT HERE WE HAVE THE REQUEST $x^* \in A$.

ex $(-2, 5]$ 5 IS MAXIMUM OF A

THERE IS NO MINIMUM OF A

ex $[0, 4) \cup (7, 10]$ 10 IS MAXIMUM OF A
0 IS MINIMUM OF A

THEOREM

Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$. A HAS, IF THEY EXIST, AT MOST A MAXIMUM AND MINIMUM.

PROOF. (ONLY FOR MAX, FOR THE MIN IS QUITE THE SAME)

BY CONTRADICTION: LET $x_1, x_2 \in A$, $x_1 \neq x_2$ TWO MAXIMA FOR A.

BY DEFINITION $x_1 \geq x \ \forall x \in A$ AND $x_2 \geq x \ \forall x \in A$

BECAUSE $x_1, x_2 \in A$ THEN $x_1 \geq x_2$ AND $x_2 \geq x_1$.

THEREFORE $x_1 = x_2$.

def. Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$. WE SAY THAT:

1) THE SUPREMUM OF A, IF IT EXISTS, IS THE MINIMUM OF THE UPPER BOUNDS OF A ($\sup A$).

2) THE INFIMUM OF A, IF IT EXISTS, IS THE MAXIMUM OF THE LOWER BOUND OF A ($\inf A$).

ex. $A = (3, +\infty)$ NO MAX, NO SUPREMUM (THERE ARE NO UPPER BOUNDS)

INFIMUM $A = 3$

NO MIN ($3 \notin A$)