# Università Bocconi

# Microeconomics

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Lecture T3

# Chapter 4: Utility

## Road map

- Utility function; definition
- (Cardinal) and ordinal utility
- Relationship between indifference curves and utility functions
- Marginal utility: definition
- **Relationship between MRS and marginal utilities**
- Types of utility functions
    - Cobb Douglas (well behaved preferences, satisfy all 5 axioms)
    - Quasi-linear
    - Perfect substitutes
    - Perfect complements
    - Neutral goods

# Preferences: a reminder

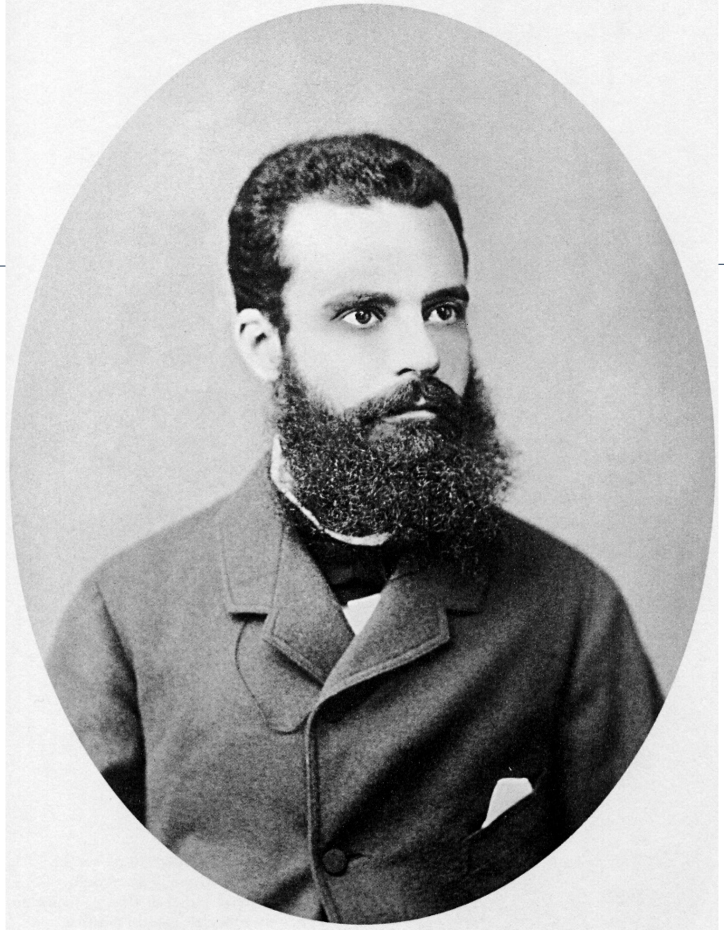
Consider two bundles, *a* and *b* each containing some units of commodities $x_1$ and $x_2$. Then we define the preference relations as follows:

| | |
|---|---|
| $a \succ b$ | *a* is preferred strictly to *b* |
| $a \sim b$ | *a* and *b* are equally preferred |
| $a \succsim b$ | *a* is preferred at least as much as *b* |

Describing preferences with preference relations, indifference curves and indifference maps is intellectually fascinating. Also, the MRS delivers a key concept.

***Problem****:* there is an infinite number of bundles on the real plane. How can we compare each of them and choose «the best alternative» among the ones available? Also, it would take forever to compute the MRS for each indifference curve at each bundle. **Solution???**

# Preferences



approach due to
**Vilfredo Pareto**, 1848-1923

Two (identical) ways of representing individual preferences

indifference curves (*I*)

utility functions (*U*)

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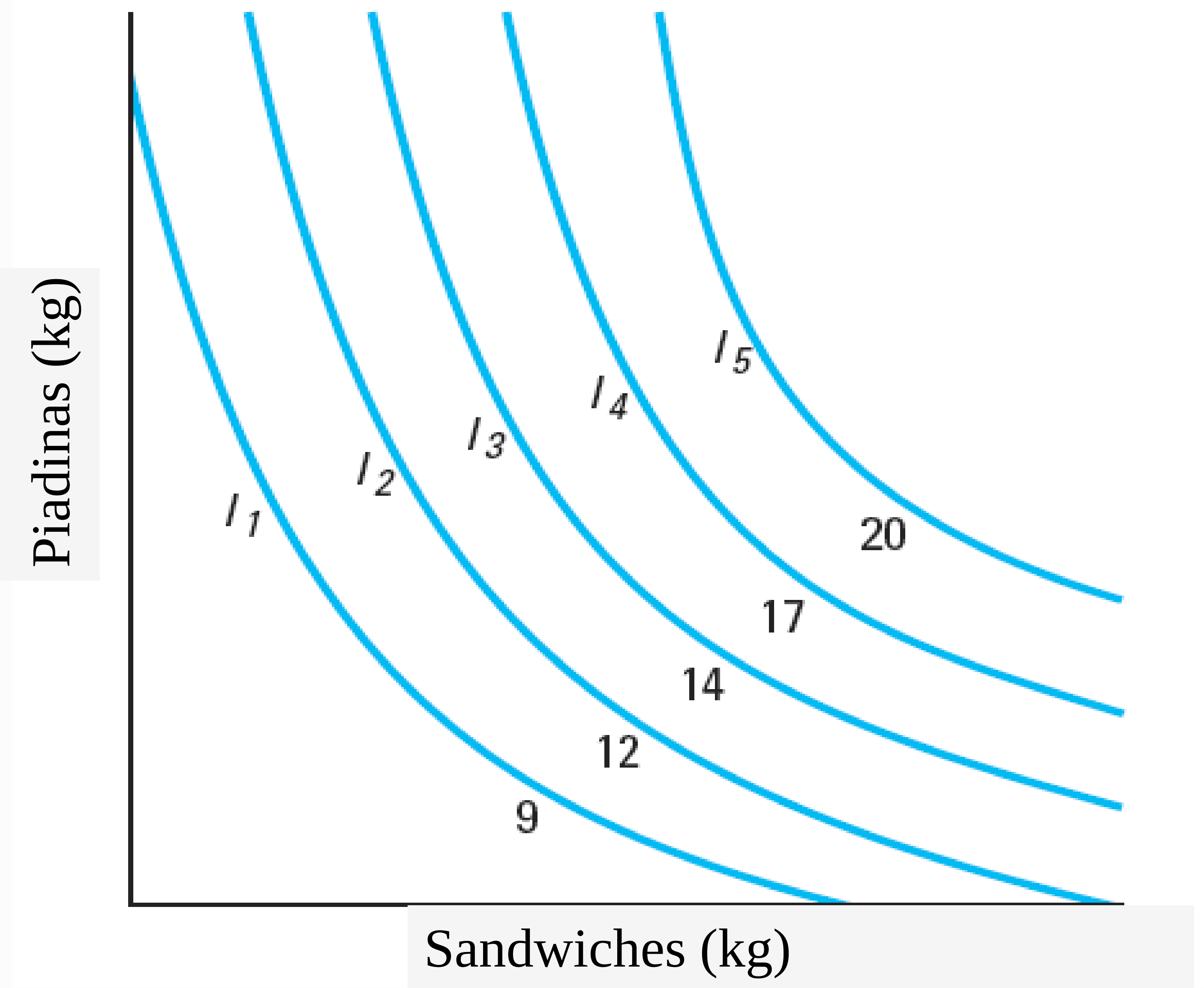
Great and rigorous treatment of this topic in your Math book,
Sections 6.2, 6.8 e 22.1.4

# From indifference curves to utility functions



# Utility function, I

- *Utility* refers to a set of numerical values that reflect the relative rankings of various bundles of goods.
- The *utility function* is the relationship between utility measures and every possible bundle of goods.
- Given a specific utility function, you can graph a specific indifference curve and determine exactly how much utility is gained from specific consumption choices.
- Example: $x_1$ = sandwiches and $x_2$ = piadinas

$$U(x_1, x_2) = \sqrt{x_1 x_2}$$

Bundle *A* contains 16 sandwiches and 9 piadinas: $U(A) = 12$

Bundle *B* contains 13 sandwiches and 13 piadinas: $U(B) = 13$

Thus, $B \succ A$

# Utility function, II

A preference relation that is complete, reflexive, and transitive, can be represented by a utility function.

A utility function $u(x)$ represents a preference relation if and only if:

$$a \succsim b \quad \Leftrightarrow \quad u(a) \geq u(b)$$

$$a \succ b \quad \Leftrightarrow \quad u(a) > u(b)$$

$$a \sim b \quad \Leftrightarrow \quad u(a) = u(b)$$

# Food for thought

## Ordinal utility and cardinal utility

- Information on individuals' preferences can be ordinal or cardinal.

- Ordinal information only tells us if one alternative is better or worse than another.

- Cardinal information tells us how much better or worse one alternative is with respect to another.

- We assume that utility functions summarize ordinal information: the scale we choose for the utility function is arbitrary (and it does not change the underlying preferences).

# Utility function, III

One more time: following Pareto's approach, utility is an ordinal (i.e. ordering) concept.

E.g., if $u(a) = 6$ and $u(b) = 2$ then bundle $a$ is strictly preferred to bundle $b$. But $a$ is **not** preferred three times as much as is $b$!

# Utility functions & indifference curves, I

- Consider the bundles (4, 1), (2, 3), and (2, 2).
- Suppose $(2, 3) \succ (4, 1) \sim (2, 2)$.
- Assign to these bundles any numbers that preserve the preference ordering; e.g.,

$$u(2, 3) = 6 > u(4, 1) = u(2, 2) = 4.$$

- Call these numbers utility levels.

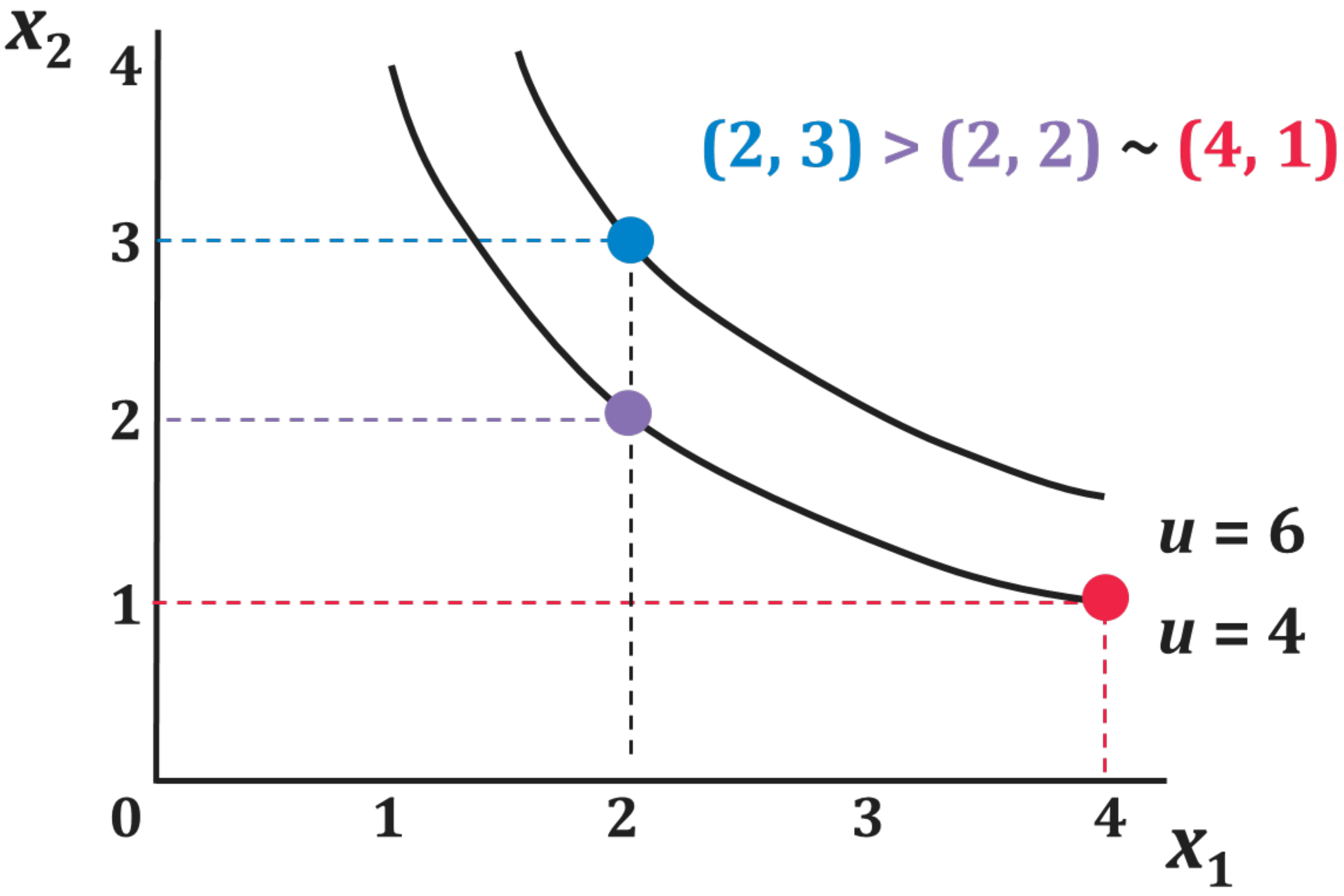
# Utility functions & indifference curves, II

- An indifference curve contains equally preferred bundles.
- Equal preference $\Rightarrow$ same utility level.
- Therefore, all bundles on an indifference curve have the same utility level.

# Utility functions & indifference curves, III

- So bundles (4, 1) and (2, 2) are on the indifference curve with utility level $u = 4$.
- But bundle (2, 3) is on the indifference curve with utility level $u = 6$.
- In an indifference curve diagram, this preference information looks as follows:

# Utility functions & indifference curves, IV

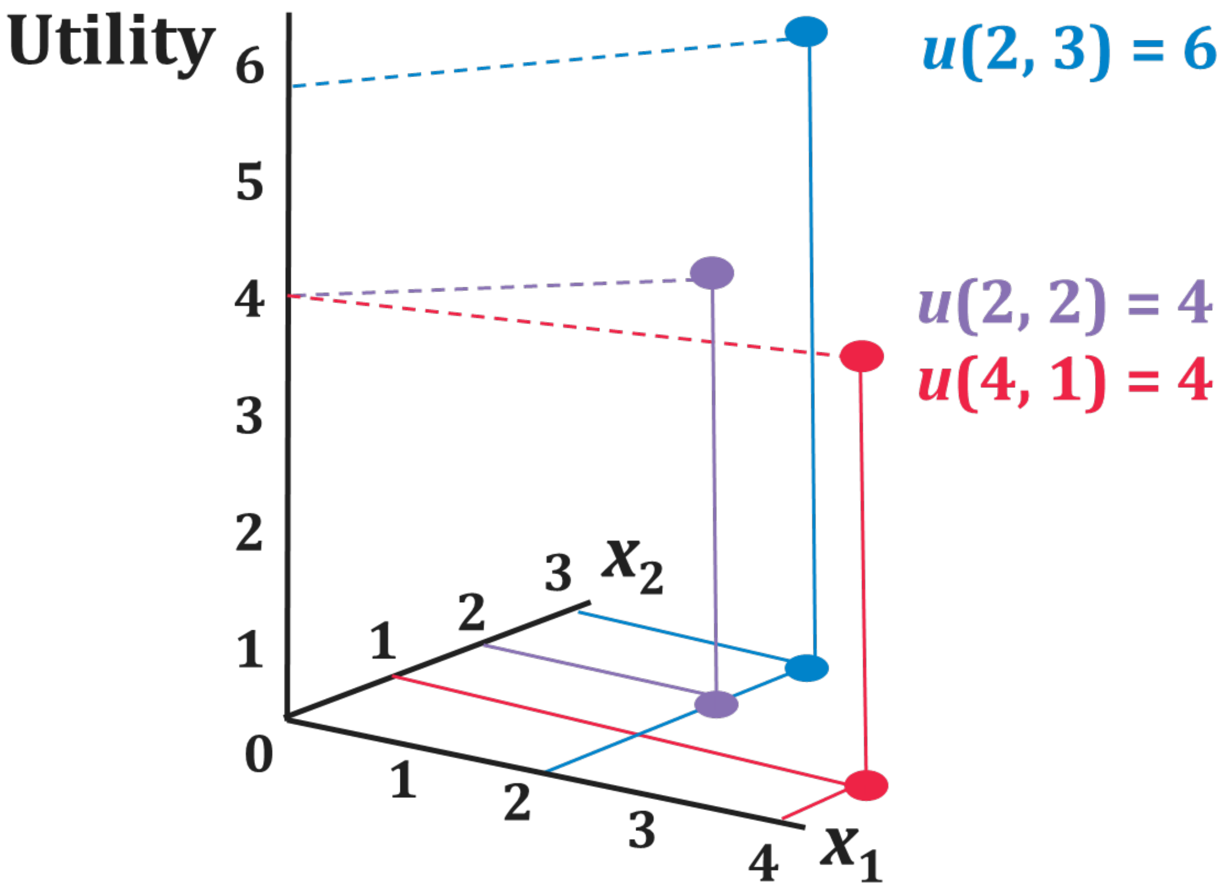


# Utility functions & indifference curves, V

Another way to visualize this same information is to plot the utility level on a vertical axis.

# Utility functions & indifference curves, VI

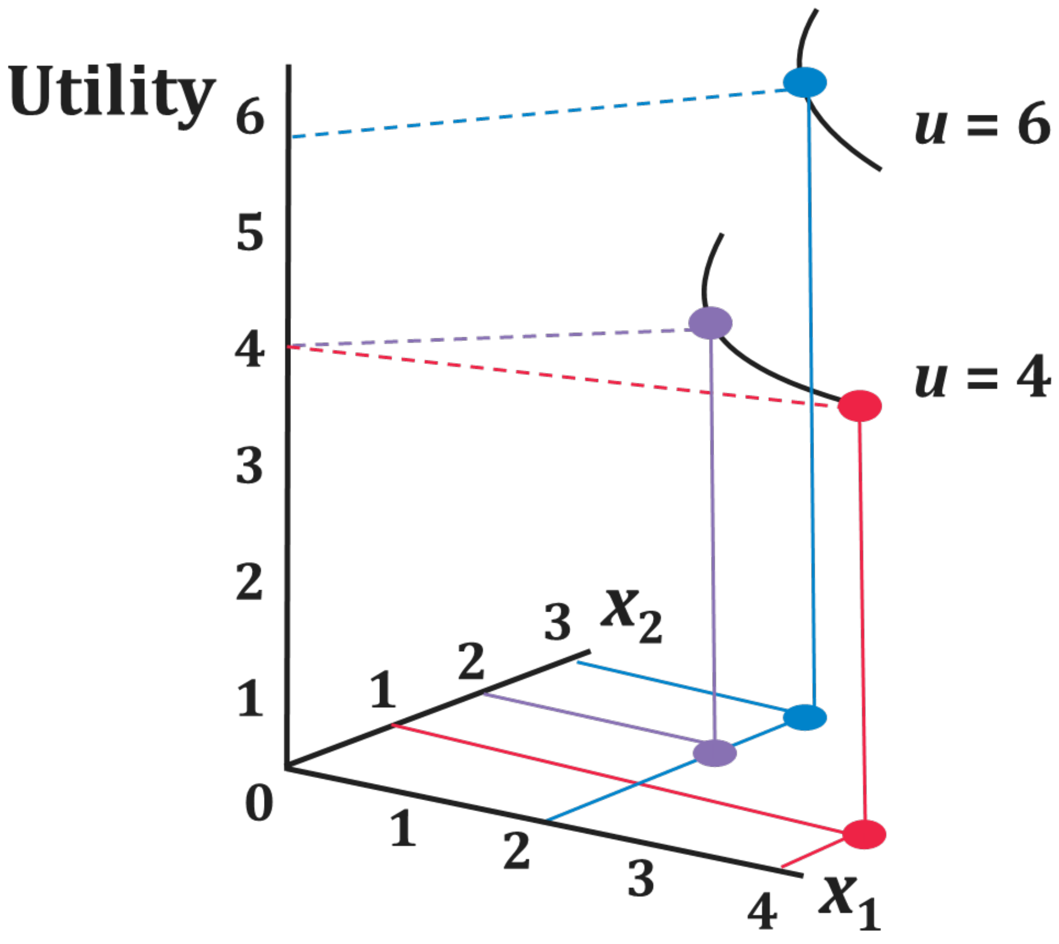
3D plot of consumption & utility levels for 3 bundles



# Utility functions & indifference curves, VII

This 3D visualization of preferences can be made more informative by adding into it the two indifference curves.

# Utility functions & indifference curves, VIII

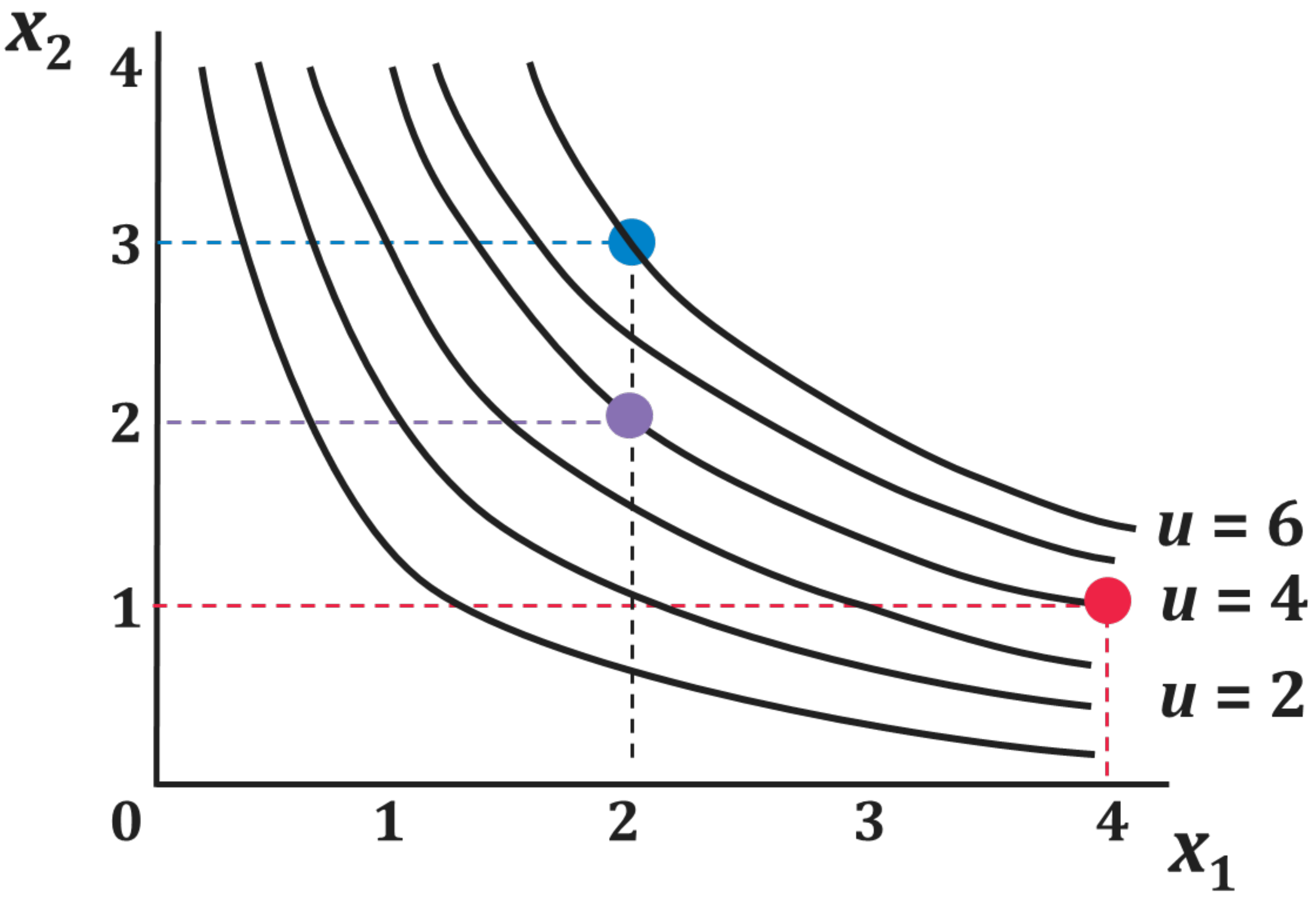



Higher indifference curves contain more preferred bundles.

# Utility functions & indifference curves, IX

Comparing more bundles will create a larger collection of all indifference curves and a better description of the consumer’s preferences.

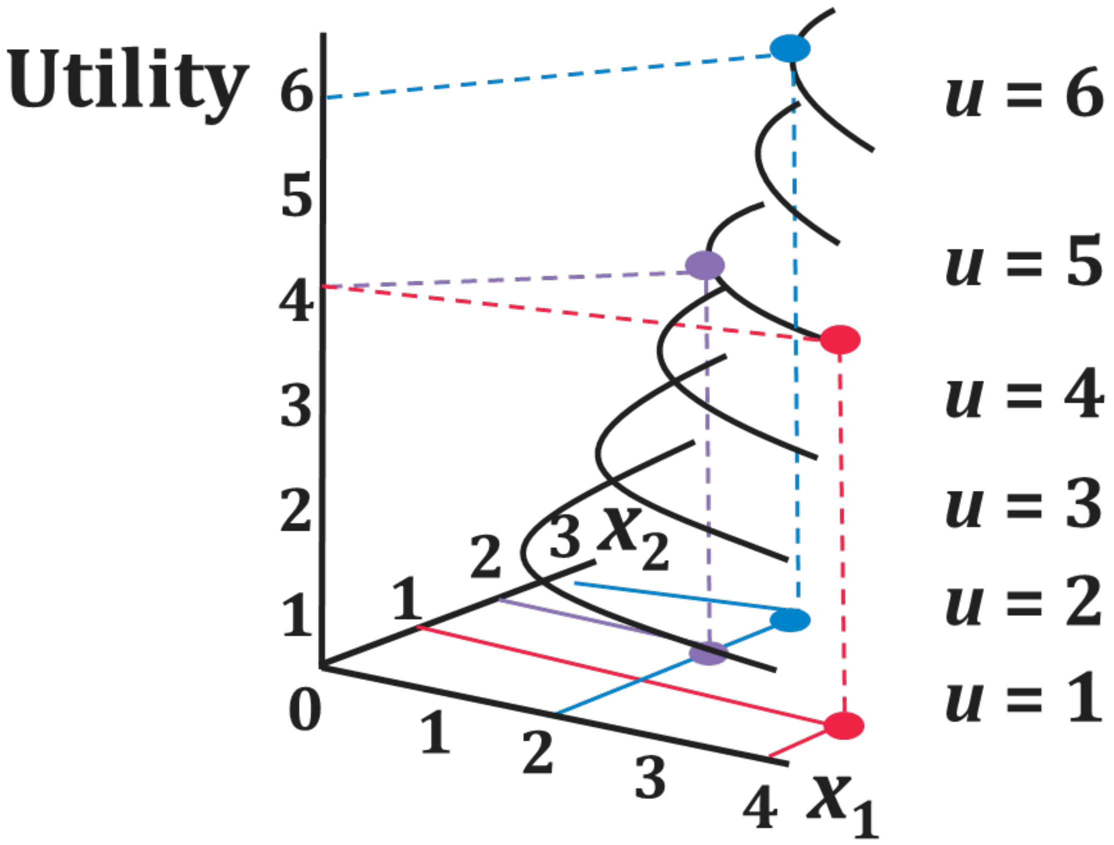
# Utility functions & indifference curves, X



# Utility functions & indifference curves, XI

As before, this can be visualized in 3D by plotting each indifference curve at the height of its utility index.

# Utility functions & indifference curves, XII

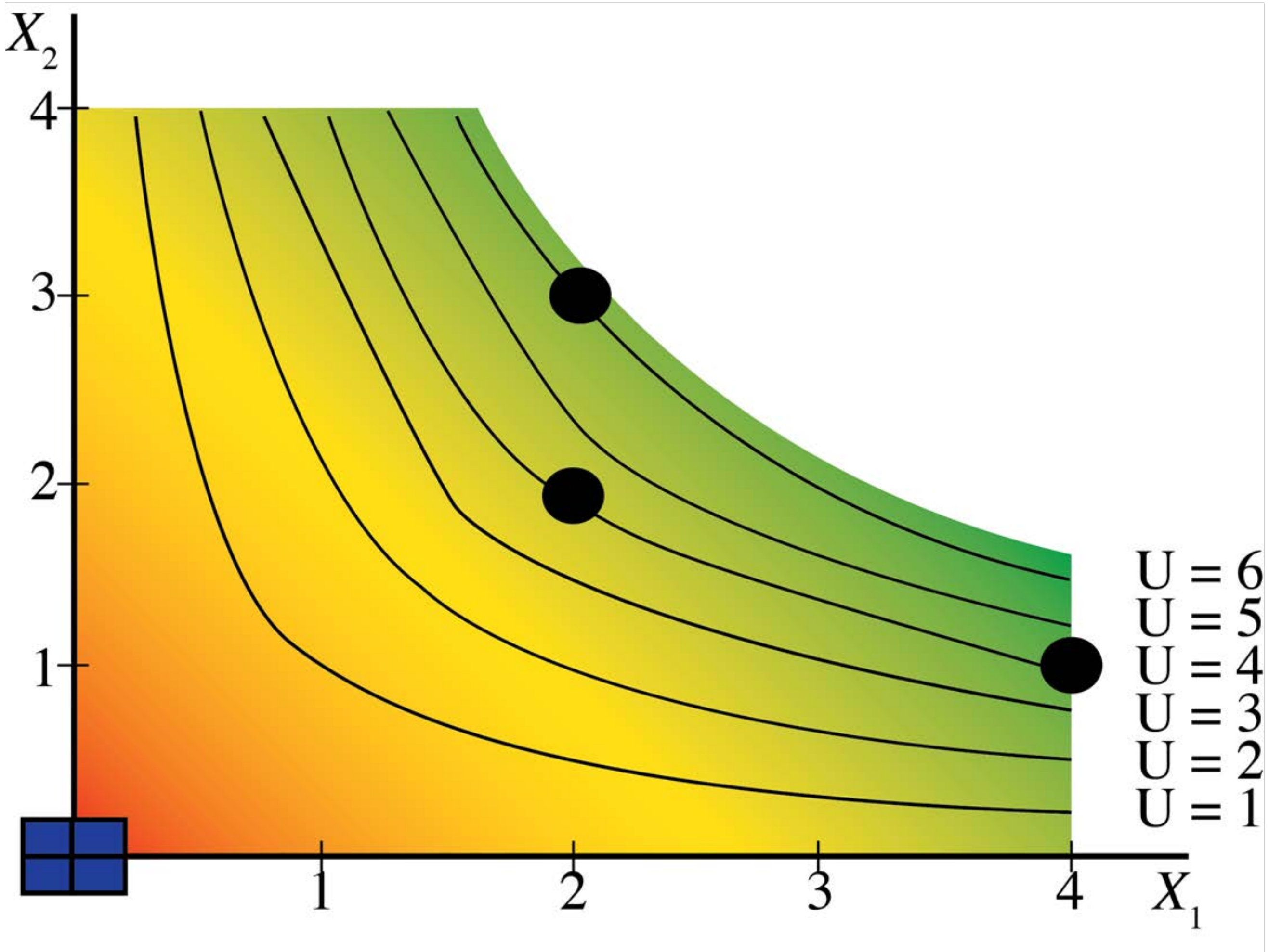


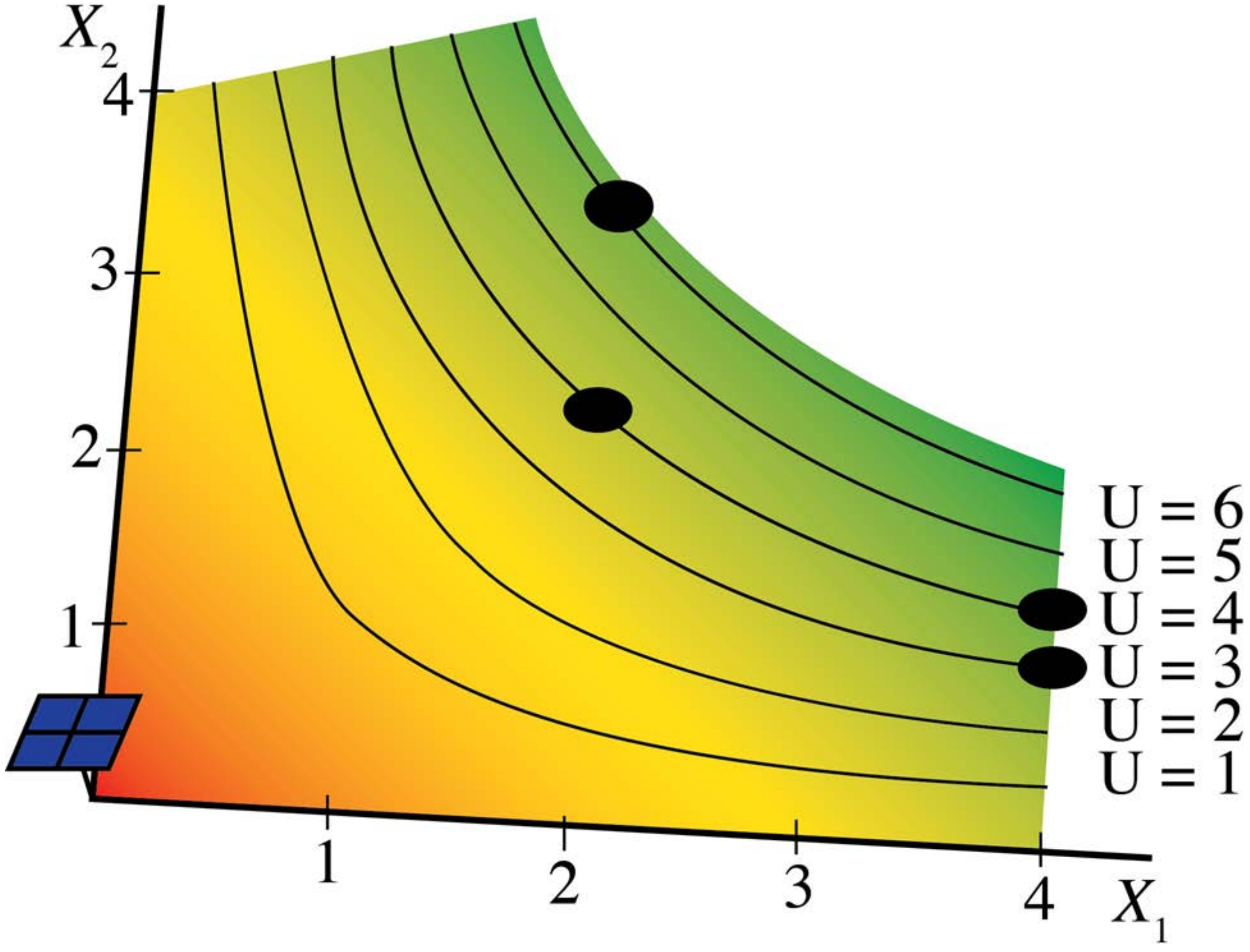
# Utility functions & indifference curves, XIII

- Comparing all possible consumption bundles gives the complete collection of the consumer's indifference curves, each with its assigned utility level.
- This complete collection of indifference curves completely represents the consumer's preferences.

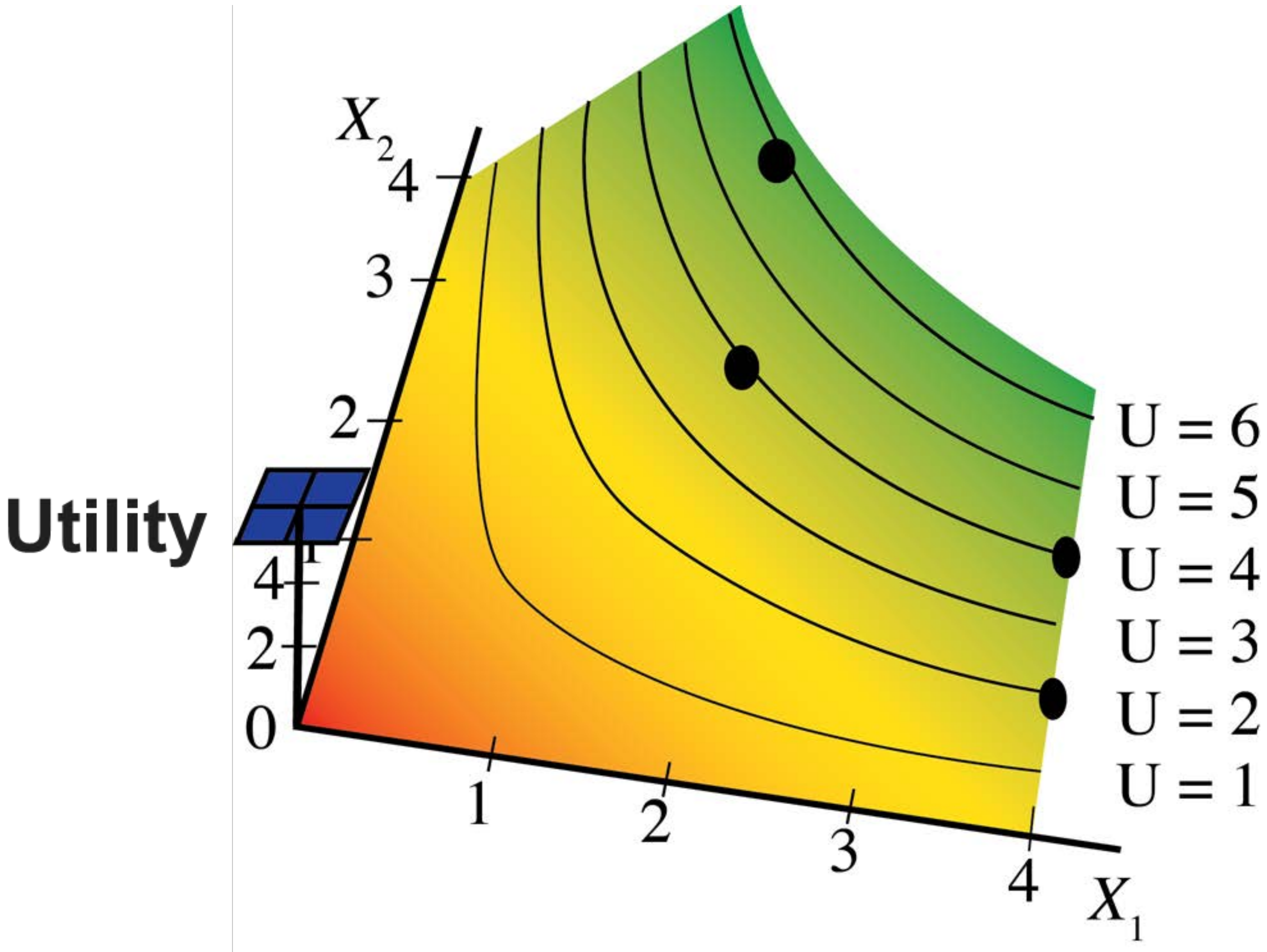
# Utility functions & indifference curves, XIV

The following eight diagrams illustrate the connection between the two-dimensional look at utility functions and indifference curves to the three-dimensional look.

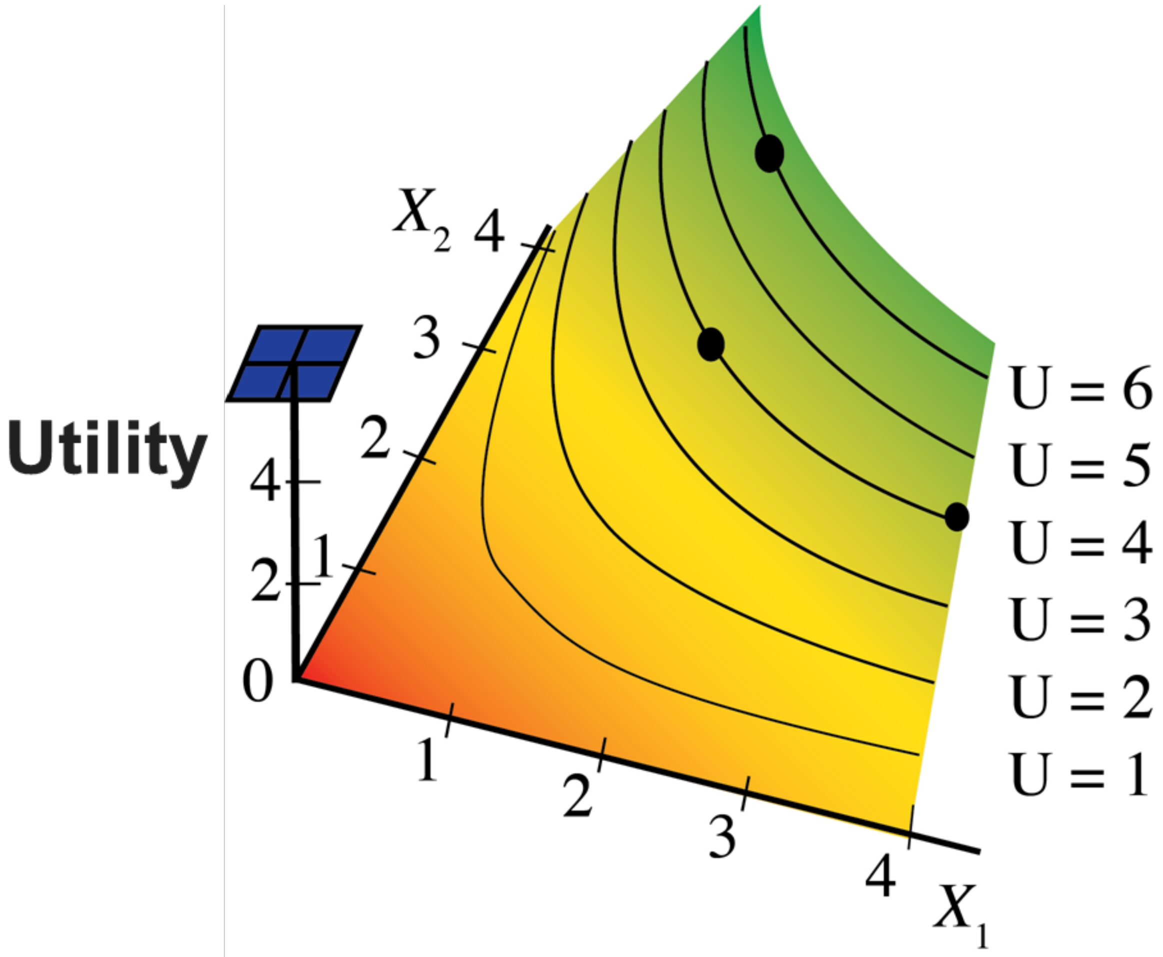

$X_2$
4
3
2
1
U = 6
U = 5
U = 4
U = 3
U = 2
U = 1
1
2
3
4
$X_1$

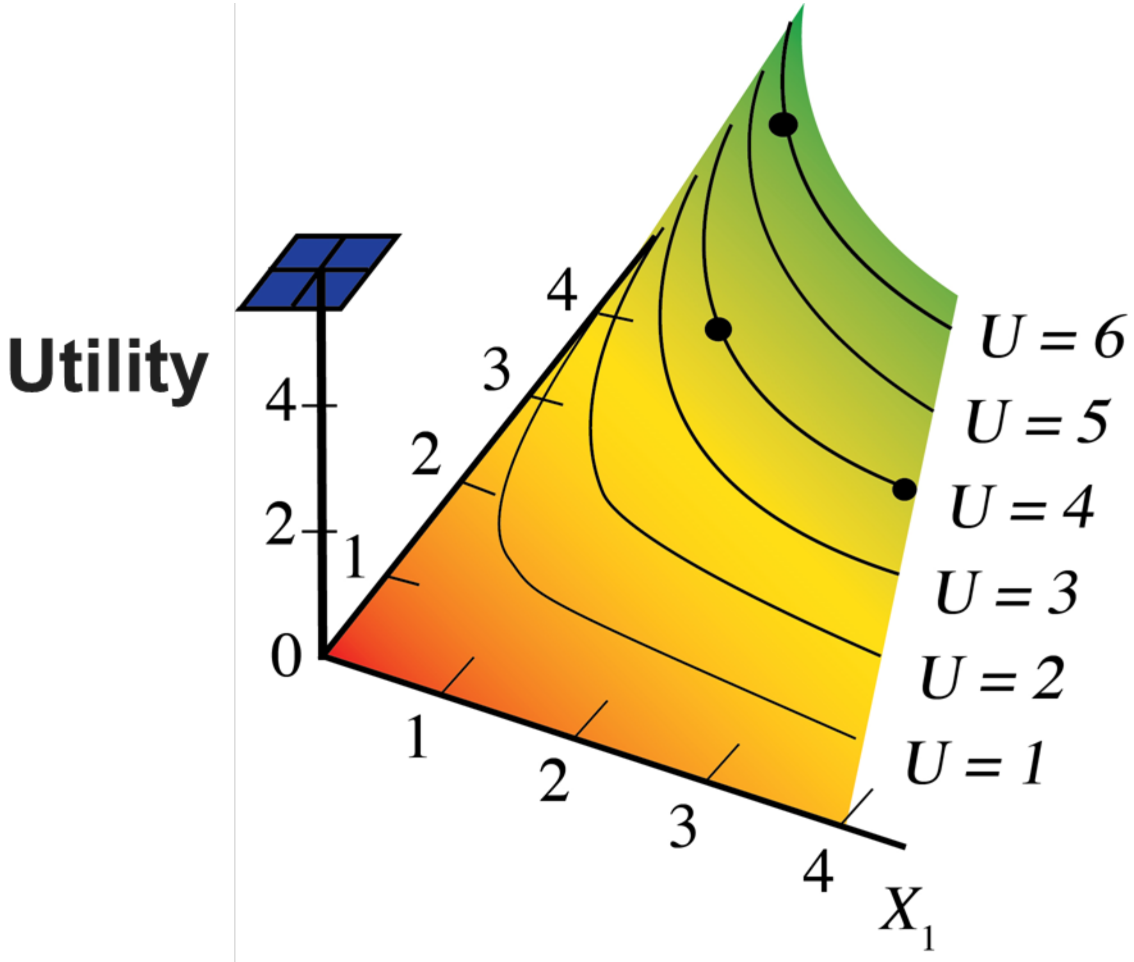


$X_2$
4
3
2
1
1
2
3
4
$X_1$
U = 6
U = 5
U = 4
U = 3
U = 2
U = 1



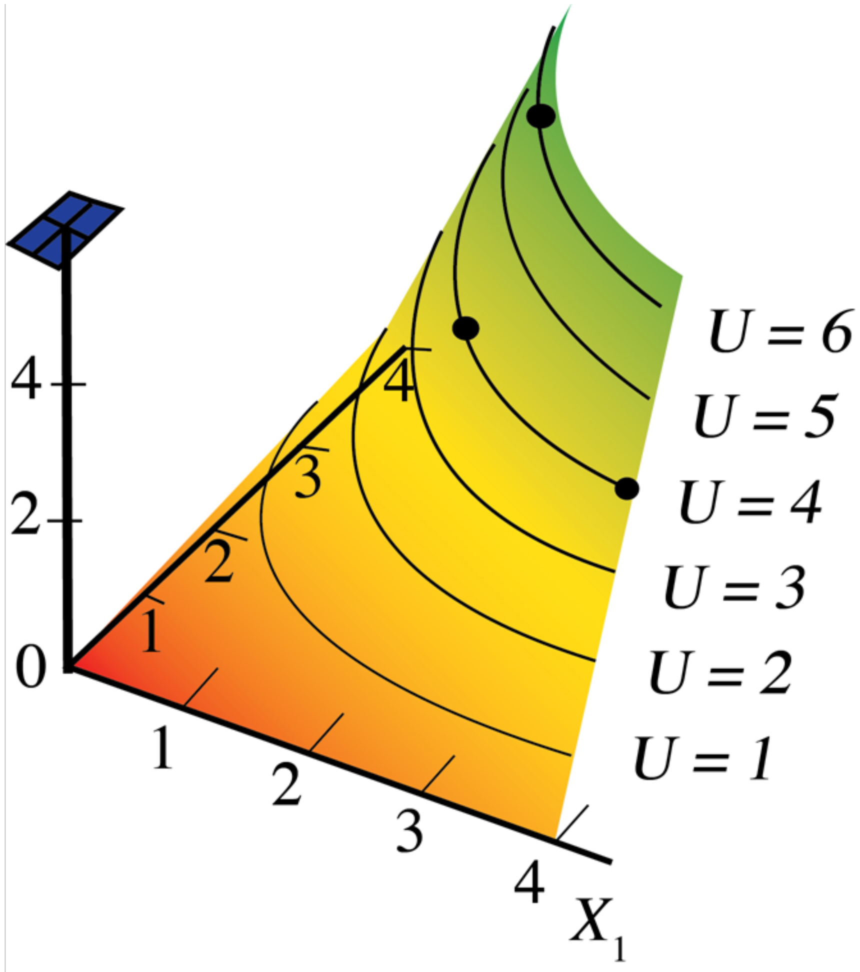
Utility
$X_2$
$X_1$
U = 6
U = 5
U = 4
U = 3
U = 2
U = 1

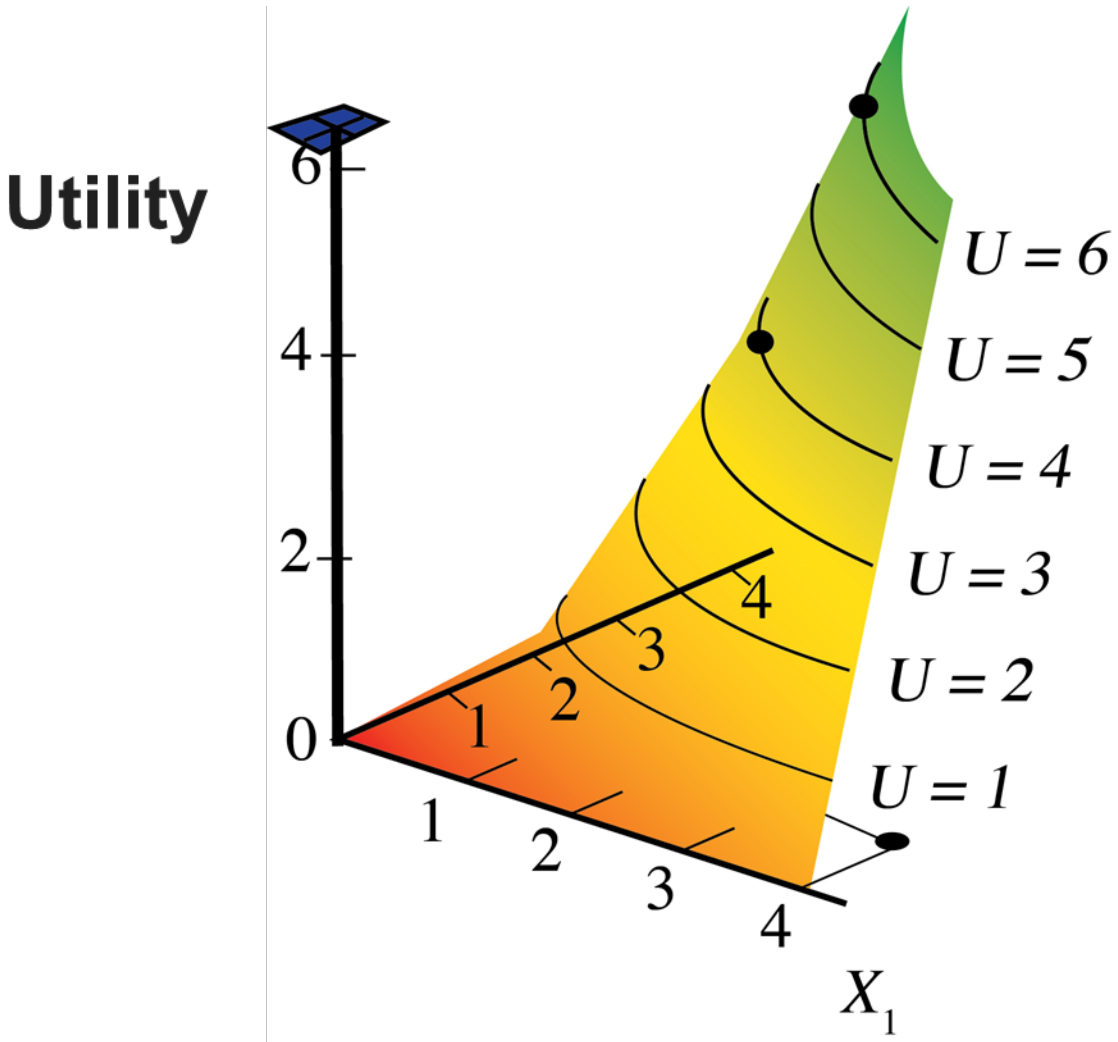

Utility
4
2
0
$X_2$
4
3
2
1
1
2
3
4
$X_1$
U = 6
U = 5
U = 4
U = 3
U = 2
U = 1



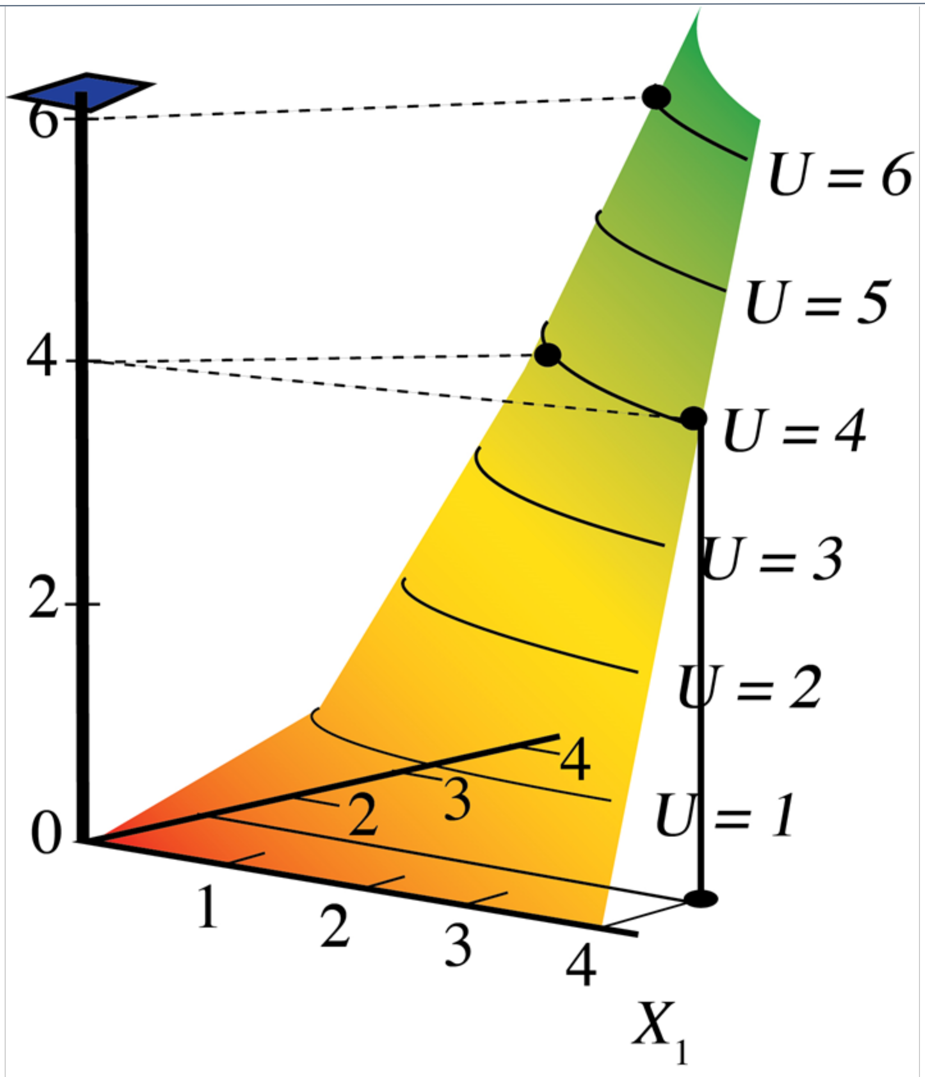
Utility
4
2
0
1
2
3
4
1
2
3
4
$X_1$
$U = 6$
$U = 5$
$U = 4$
$U = 3$
$U = 2$
$U = 1$

**Utility**




Utility
6
4
2
0
1
2
3
4
1
2
3
4
X1
U = 6
U = 5
U = 4
U = 3
U = 2
U = 1

Utility




# Utility functions & indifference curves, XV

The collection of all indifference curves for a given preference relation is an **indifference map.**

An indifference map is equivalent to a **utility function**.

# Utility function, IV

**There is no unique utility function representation of a preference relation**.

Suppose

$$u(x_1, x_2) = x_1 x_2$$

represents a preference relation.

Again consider the bundles (4, 1), (2, 3), and (2, 2).

# Utility functions, V

- $u(x_1, x_2) = x_1 x_2,$
- $u(2, 3) = 6;\ u(4, 1) = 4;\ u(2, 2) = 4.$

➢ Since $u(2, 3) > u(4, 1) = u(2, 2),$

➢ $(2, 3) \succ (4, 1) \sim (2, 2).$

# Utility functions, VI

- $u(x_1, x_2) = x_1 x_2 \to (2, 3) \succ (4, 1) \sim (2, 2).$

- Define $$V = u^2.$$

➢ Then, $V(x_1, x_2) = {x_1}^2 {x_2}^2$ and $V(2, 3) = 36 > V(4, 1) = V(2, 2) = 16.$

➢ So again, $$(2, 3) \succ (4, 1) \sim (2, 2).$$

- $V$ preserves the same order as $u$ and, therefore, represents the same preferences.

# Utility functions, VII

- $u(x_1, x_2) = x_1 x_2 \rightarrow (2, 3) \succ (4, 1) \sim (2, 2).$

- Define $W = 2u + 10.$

➢ Then, $W(x_1, x_2) = 2x_1x_2 + 10$ and $W(2, 3) = 22 > W(4, 1) = W(2, 2) = 18.$

➢ So again, $(2, 3) \succ (4, 1) \sim (2, 2).$

$W$ preserves the same order as $u$ and, therefore, represents the same preferences.

# Utility functions, VIII

If

- $u$ is a utility function that represents a preference relation

  <u>and</u>

- $f$ is a strictly increasing function

then $$V = f(u)$$

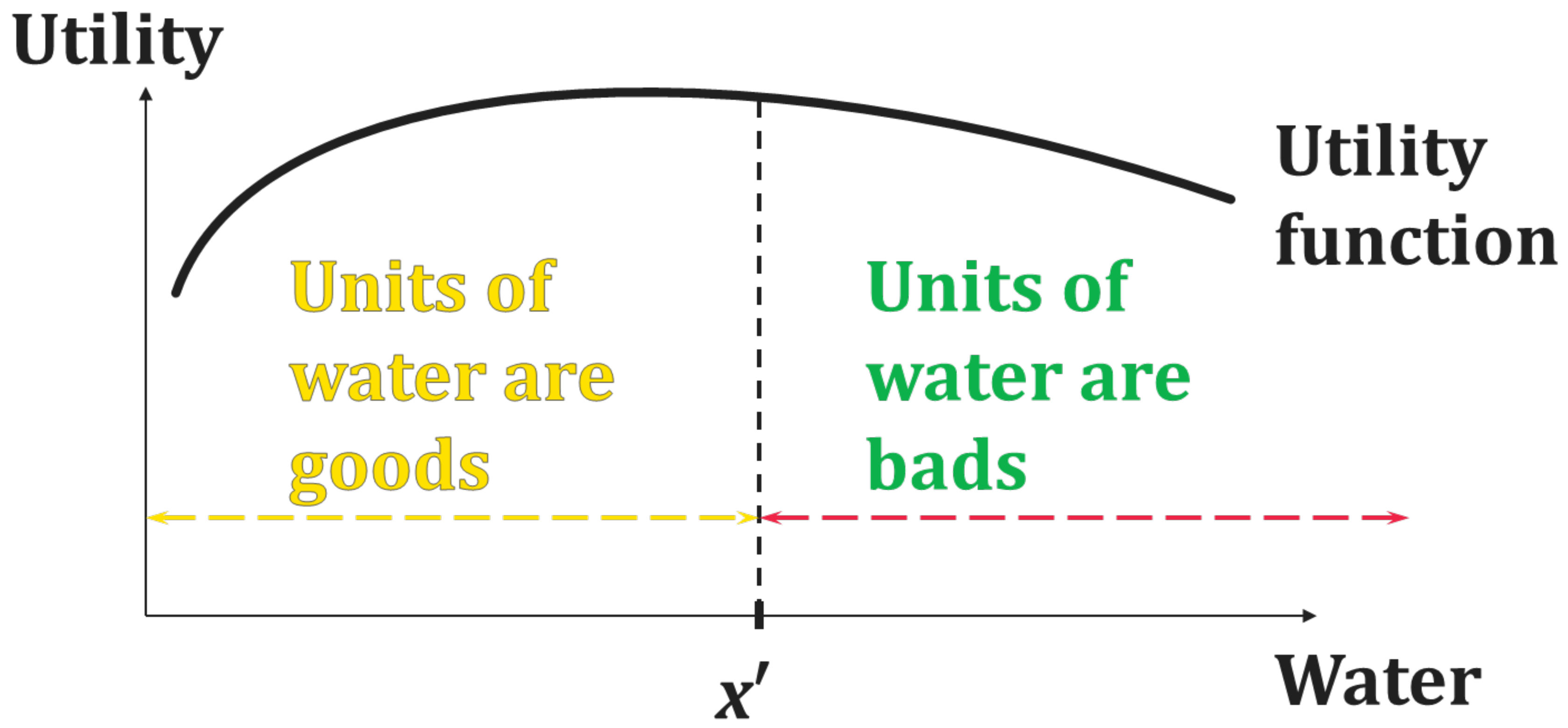
is also a utility function representing the same preference relation.

- This is known as a "monotonic transformation."

# Goods, Bads, and Neutrals, I

- A **good** is a commodity which increases utility with additional consumption.
- A **bad** is a commodity which decreases utility with additional consumption.
- A **neutral** is a commodity which does not change utility with additional consumption.

# Goods, Bads, and Neutrals, II




Around $x'$ units, a little extra water is a neutral.

# Cobb-Douglas utility function

Any utility function of the form

$$u(x_1, x_2) = {x_1}^a \, {x_2}^b \qquad \text{with } a > 0 \text{ and } b > 0$$

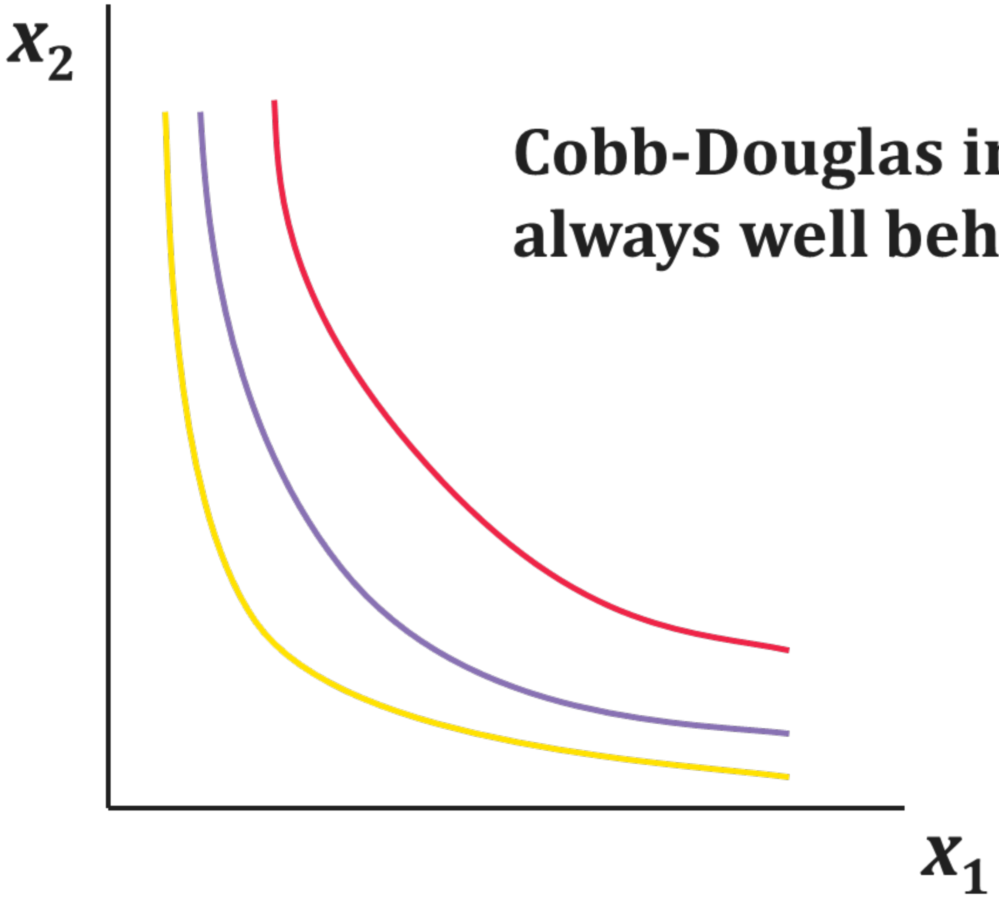
is called a "Cobb-Douglas" utility function.

E.g.,

$$u(x_1, x_2) = {x_1}^{1/2} \, {x_2}^{1/2} \quad (a = b = 1/2)$$

$$V(x_1, x_2) = x_1 \, {x_2}^3 \quad (a = 1, b = 3)$$

# Cobb-Douglas indifference curves




**Cobb-Douglas indifference curves are always well behaved (convex, monotonic).**

# Quasi-linear utility function

A utility function of the form

$$u(x_1, x_2) = f(x_1) + x_2$$

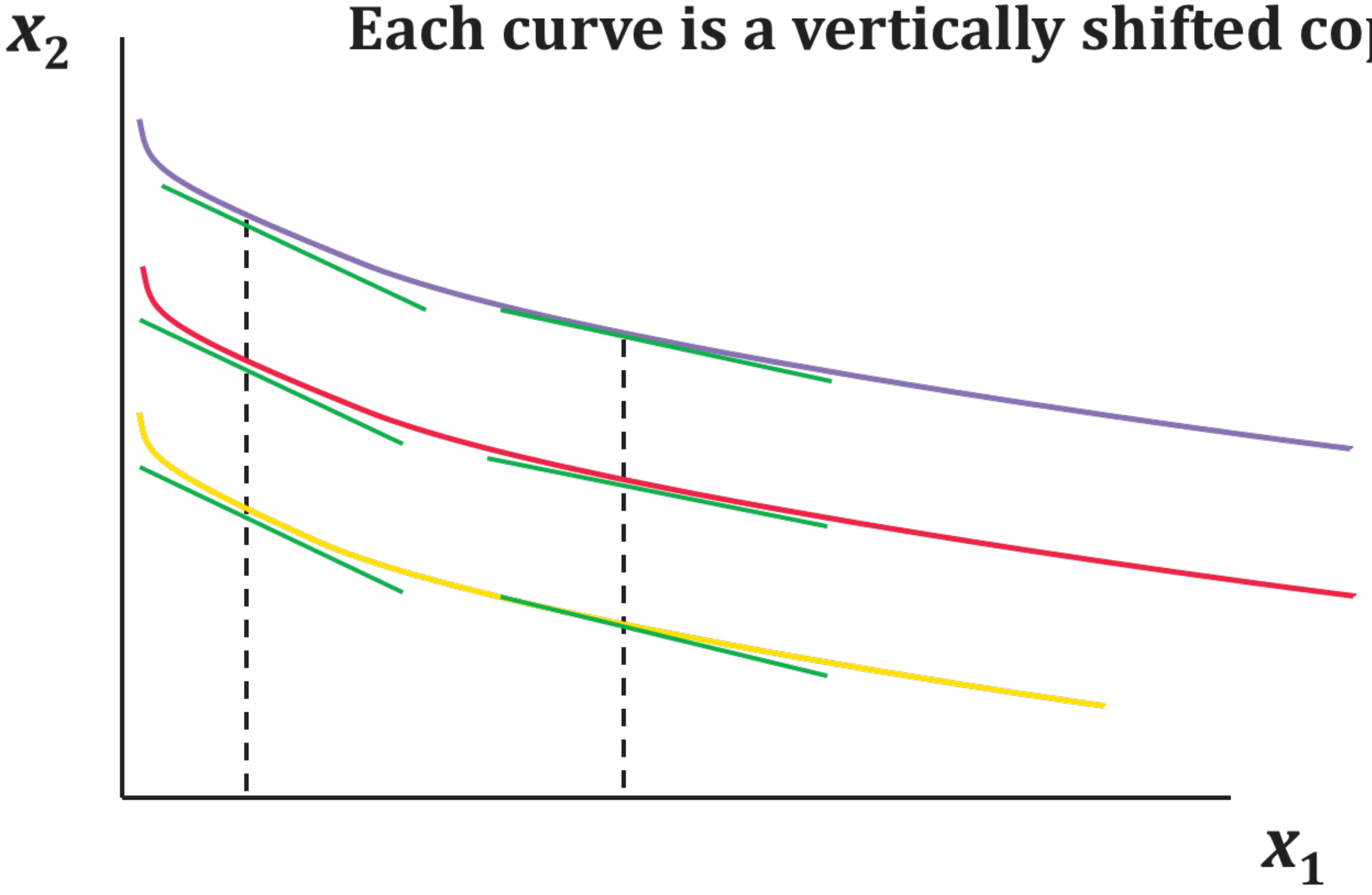
is linear in just $x_2$ and is called quasi-linear.

Example:

$$u(x_1, x_2) = 2x_1^{1/2} + x_2.$$

# Quasi-linear indifference curves

**Each curve is a vertically shifted copy of the others.**



# Utility function for perfect substitutes

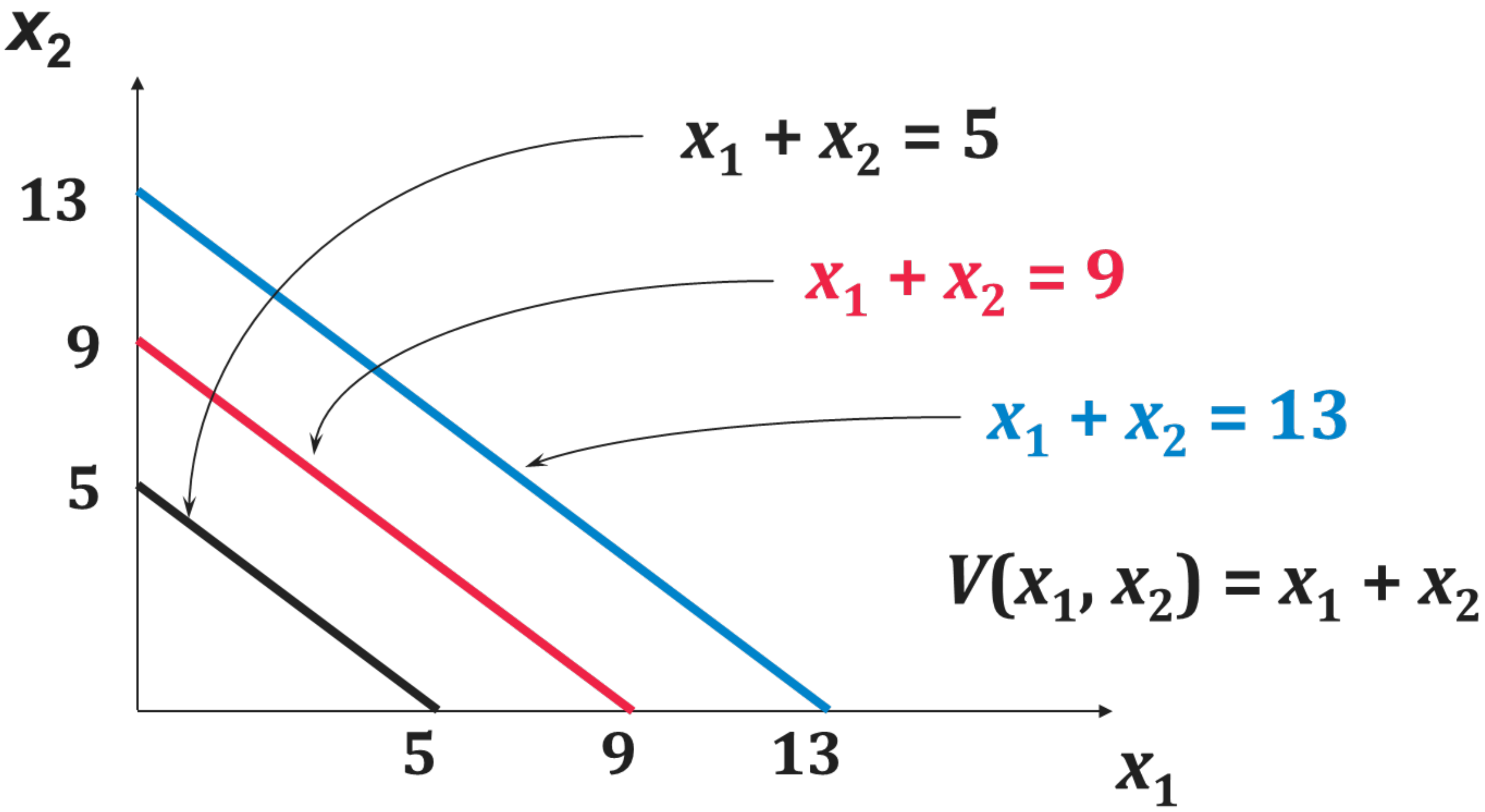
Consider this utility function:

$$V(x_1, x_2) = x_1 + x_2.$$

This utility function is an example of two goods that are perfect substitutes.

What do the indifference curves for this perfect substitutes utility function look like?

# Indifference curves for perfect substitutes




All are linear and parallel.

# Utility function for perfect complements

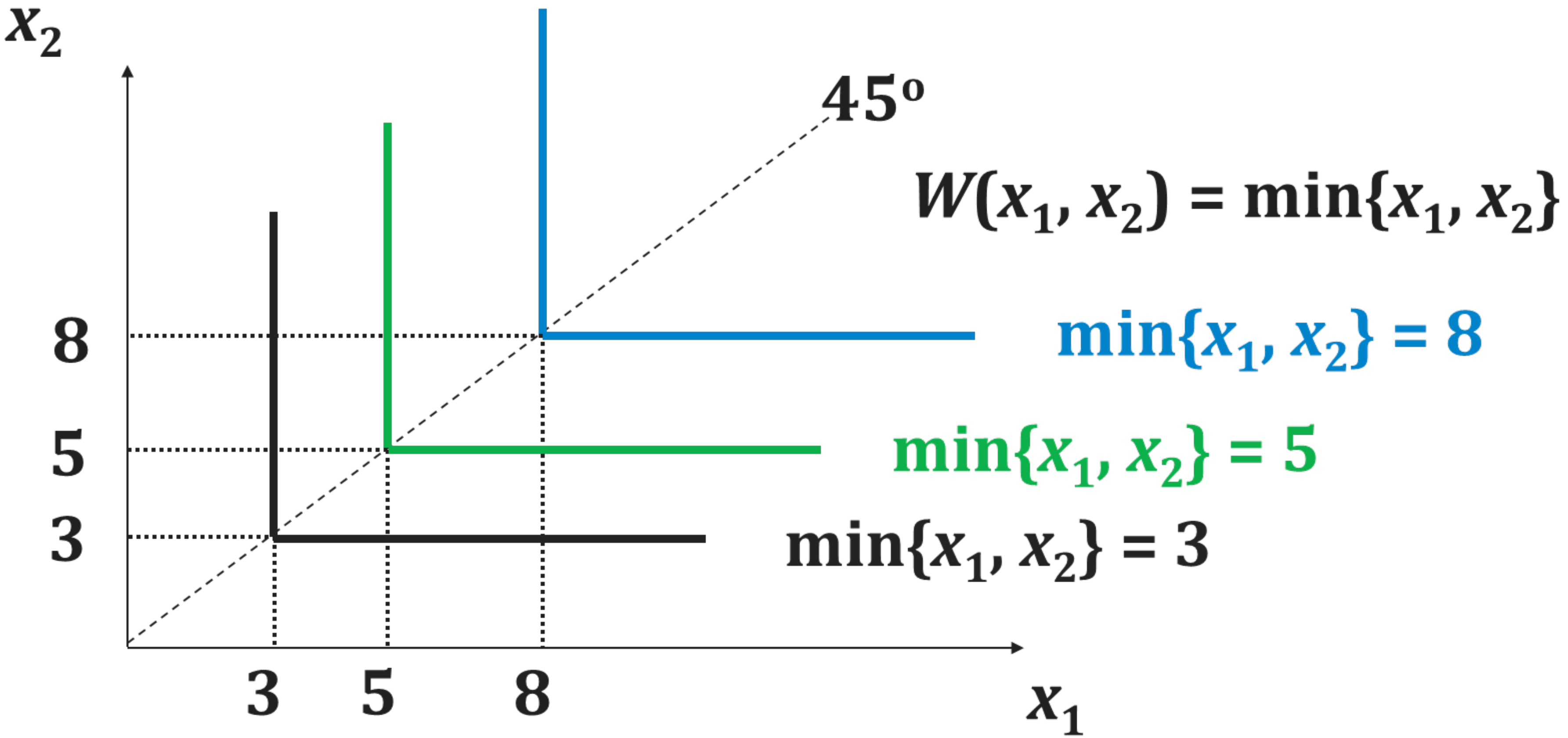
Consider

$$W(x_1, x_2) = \min\{x_1, x_2\}.$$

This utility function is an example of two goods that are perfect complements.

What do the indifference curves for this perfect complements utility function look like?

# Indifference curves for perfect complements




All are right-angled with vertices on a ray from the origin.

# Marginal (incremental) utility

## Definition

Marginal utility of $x_i$ is the change in the utility an individual gets if he/she gets one more or one less unit of good $x_i$

Mathematically:

$$MU_{x_i} = \frac{\Delta U}{\Delta x_i}$$

For «small» changes in $x_i$,
the marginal utility is the slope of the utility function with respect to good $x_i$

$$MU_{x_i} = \frac{\partial U}{\partial x_i}$$

# Marginal Utility and MRS

- Infinitesimal variations of $x_1$ (= $dx_1$) make total utility vary as follows: $MU_{x1}\, dx_1$
- Infinitesimal variations of $x_2$ (= $dx_2$) make total utility vary as follows: $MU_{x2}\, dx_2$
- By definition, on the same indifference curve

$$\mathrm{MU}_{x1} dx_1 = -\mathrm{MU}_{x2} dx_2$$

- Rearranging, we get $\dfrac{dx_2}{dx_1} = -\dfrac{MU_{x1}}{MU_{x2}}$
- Remembering that by definition $\mathrm{MRS} = \dfrac{dx_2}{dx_1}$

we obtain

$$\boxed{\mathrm{MRS} = -\frac{MU_{x1}}{MU_{x2}}}$$

# Marginal Utility and MRS:

## example, Cobb Douglas utility function

Suppose

$$u(x_1, x_2) = x_1 x_2.$$

Then

$$\frac{\partial u}{\partial x_1} = (1)(x_2) = x_2 \quad \text{and}$$

$$\frac{\partial u}{\partial x_2} = (1)(x_1) = x_1.$$

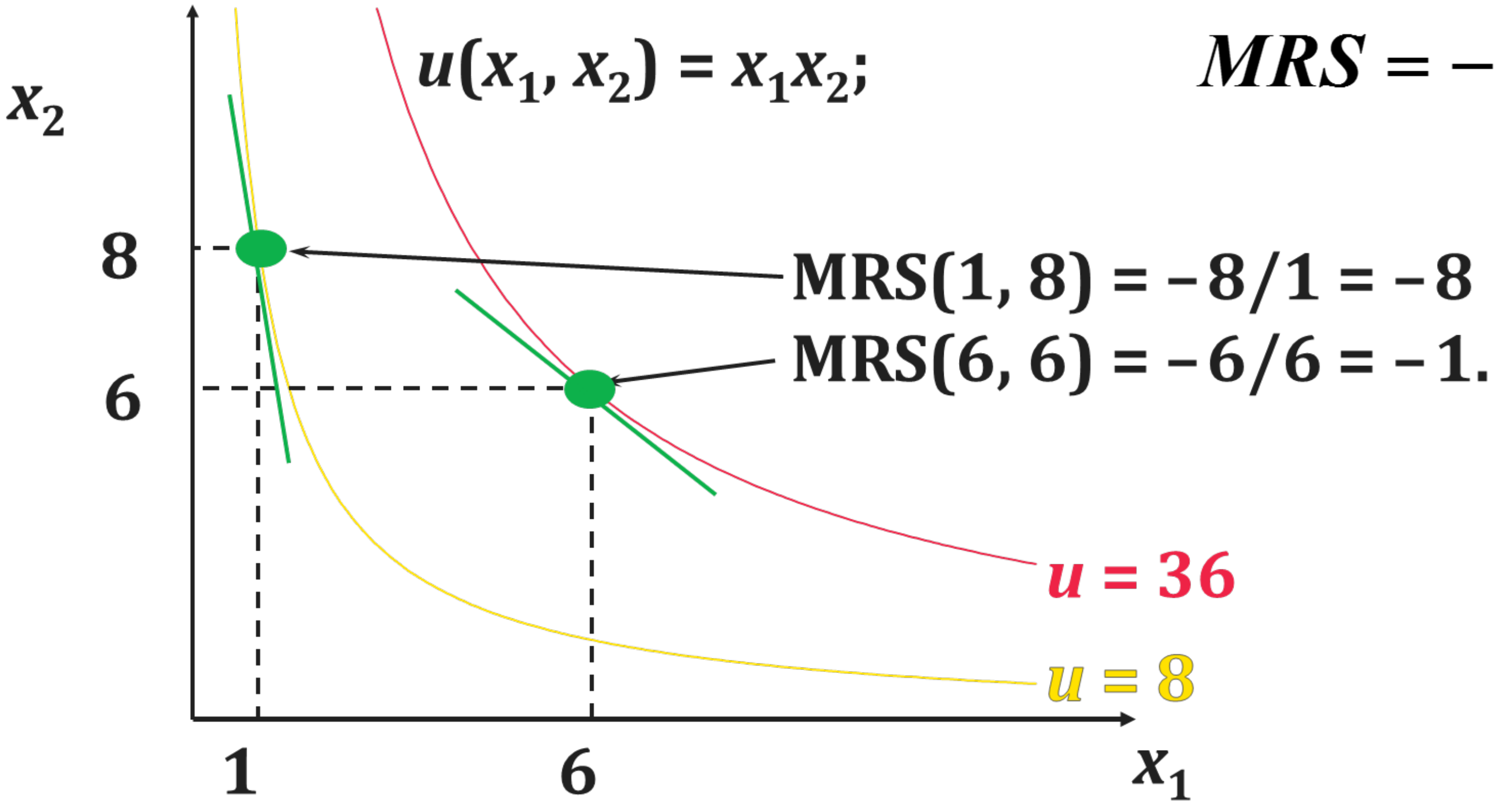
So,

$$MRS = \frac{dx_2}{dx_1} = -\frac{\partial u / \partial x_1}{\partial u / \partial x_2} = -\frac{x_2}{x_1}.$$

# Marginal Utility and MRS:

## example, Cobb Douglas preferences

$$MRS = -\frac{x_2}{x_1}$$



# MRS and monotonic transformations of the utility function, I

- As previously discussed, applying a monotonic transformation to a utility function representing a preference relation simply creates another utility function representing the same preference relation.

- Question: what happens to marginal rates of substitution when a monotonic transformation is applied?

# **MRS and monotonic transformations of the utility function:** example

For $u(x_1, x_2) = x_1 x_2$ the $\text{MRS} = -x_2/x_1$.

Create $V = u^2$; i.e., $V(x_1, x_2) = {x_1}^2 {x_2}^2$.

What is the MRS for *V*?

$$MRS = \frac{dx_2}{dx_1} = -\frac{\partial V / \partial x_1}{\partial V / \partial x_2} = -\frac{2x_1 {x_2}^2}{2{x_1}^2 x_2} = -\frac{x_2}{x_1}$$

Which is **the same** as the MRS for *u* !!!

# MRS and monotonic transformations of the utility function, II

- More generally, if $V = f(u)$

where $f$ is a strictly increasing function

- then the MRS is unchanged by this positive monotonic transformation.

# Chapter 4:
# Take home message

Which is the key concept we learned when studying the utility function?