PROPERTIES $A \subseteq \mathbb{R}$

- x_0 IS INTERIOR POINT OF A $\Rightarrow$ x_0 IS ACCUMULATION POINT OF A
- x_0 IS A BOUNDARY POINT OF A NOT ISOLATED $\Rightarrow$ x_0 IS ACCUMULATION POINT OF A

NOW THAT WE HAVE PUT A LABEL ON THE POINT, WE CAN DEFINE THE SETS:

def. $A \subseteq \mathbb{R}$ IS SAID OPEN SET WHEN ALL THE POINTS OF A ARE INTERIOR POINTS ($\text{INT} A = A$)

def $A \subseteq \mathbb{R}$ IS SAID CLOSED SET WHEN IT CONTAINS ALL ITS BOUNDARY POINTS ($\partial A \subseteq A$)

REMARK A SET CAN BE NEITHER OPEN OR CLOSED.

ex.

$A = (-2, 5) \cup (8, +\infty)$ IS OPEN

$B = [3, +\infty)$ IS CLOSED $\left(\partial B = \{3\} \subseteq B\right)$

$C = (-\infty, 2) \cup [3, 5]$ IS NOT OPEN AND IS NOT CLOSED

IN FACT 3 IS NOT AN INTERIOR POINT BUT $3 \in C$

2 IS BOUNDARY POINT BUT $2 \notin C$

$D = [-2, 5] \cup \{10\}$ IS CLOSED

REMARK $N_r(x_0)$, $N(+\infty)$, $N(-\infty)$ ARE OPEN SETS.

THEOREM $A \subseteq \mathbb{R}$ IS OPEN $\Longleftrightarrow$ A^c IS CLOSED

THEOREM $A \subseteq \mathbb{R}$ IS CLOSED $\Longleftrightarrow$ A CONTAINS ALL ITS ACCUMULATION POINTS

REMARK IN GENERAL A SET THAT IS OPEN IS NOT CLOSED... BUT THERE ARE TWO EXCEPTIONS:

→ $\mathbb{R}$ AND $\emptyset$ ARE OPEN AND CLOSED AT THE SAME TIME.

CONSIDER $\mathbb{R} = (-\infty, +\infty)$ ALL ITS POINTS ARE INTERIOR POINT, THEN $\mathbb{R}$ IS OPEN.

THEREFORE $\emptyset = \mathbb{R}^c$ IS CLOSED.

THE BOUNDARY OF $\mathbb{R}$ IS $\emptyset$ AND $\emptyset \subseteq \mathbb{R}$ THEN $\mathbb{R}$ CONTAINS ITS BOUNDARY, THEN $\mathbb{R}$ IS CLOSED.

THEREFORE $\emptyset = \mathbb{R}^c$ IS OPEN.

EX. $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

IS MADE BY ISOLATED POINTS

$\partial \mathbb{Z} = \mathbb{Z}$ THEN $\mathbb{Z}$ IS CLOSED IN $\mathbb{R}$

EX.

$$A = \left\{ \frac{1}{n+1}, \ n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$$

FOR SURE A IS NOT OPEN, FOR EXAMPLE THE POINT $\frac{1}{2}$ IS NOT INTERIOR.

IS A CLOSED? THE POINTS OF A ARE ALL ISOLATED, BUT THEY TEND TO GET "VERY CLOSE" TO ZERO.

THE BOUNDARY OF A IS: $\partial A = A \cup \{0\}$

AND $\partial A \nsubseteq A$, THEN A IS NOT CLOSED.

THEOREM

i) THE INTERSECTION OF A FINITE NUMBER OF OPEN SETS IS OPEN

ii) THE UNION OF A FINITE/INFINITE NUMBER OF OPEN SETS IS OPEN

iii) THE INTERSECTION OF A FINITE/INFINITE NUMBER OF CLOSED SETS IS CLOSED

iv) THE UNION OF A FINITE NUMBER OF CLOSED SETS IS CLOSED.

REMARK WHY i) IS NOT VALID FOR AN INFINITE NUMBER OF OPEN SETS?

COUNTEREXAMPLE: $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $\bigcap_n A_n = \{0\}$ CLOSED

WHY iv) IS NOT VALID FOR AN INFINITE NUMBER OF CLOSED SETS?

COUNTEREXAMPLE: $A_n = \left[0, 1 - \frac{1}{n}\right]$, $\bigcup_n A_n = [0, 1)$

IS NOT CLOSED

THERE ARE SETS THAT WE WILL FIND VERY IMPORTANT IN THE FUTURE:

def $A \subseteq \mathbb{R}$ IS SAID $\boxed{\text{COMPACT SET}}$ WHEN A IS CLOSED AND BOUNDED.

ex. $A = [-10, 5]$ IS COMPACT

$B = [-10, 0] \cup [3, 5]$ IS COMPACT

$C = [-20, -10] \cup \{50\}$ IS COMPACT

$D = \{1\} \cup \{3\} \cup \{50\}$ IS COMPACT