ex $A = [2,5) \cup (10,30)$

INFIMUM $A = 2$

MINIMUM $A = 2$

SUPREMUM $A = 30$

MAX $A \nexists$

A IS NOT AN INTERVAL

A IS BOUNDED

ex $A = \{ x \in \mathbb{R} : 0 \leq \max(x, -x) \leq 2 \}$

$\left[\max(x,-x) = \begin{cases} x & \text{IF } x \geq -x \\ -x & \text{IF } x < -x \end{cases} \right.$, IT CHOSES THE GREATEST BETWEEN x AND -x]

WE CAN REWRITE $A = [-2, 2]$, THEN:

MIN A = INF $A = -2$

MAX A = SUP $A = 2$

ex $A = \{ x \in \mathbb{R} : \underbrace{\min(x,-x)}_{\text{IT CHOOSES THE SMALLEST BETWEEN } x \text{ AND } -x} \leq -1 \}$

WE HAVE $A = (-\infty, -1] \cup [1, +\infty)$

$\nexists$ SUP A, MAX A, INF A, MIN A

ex.

$$A = \left\{ \frac{1}{m+1} , m \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

$\sup A = \max A = 1$

$\inf A = 0$, $\nexists \min A$ $(0 \notin A)$

ex. $A_m = \left(-\frac{1}{m}, \frac{1}{m}\right)$ COLLECTION OF INTERVALS

$$\left(A_1 = (-1, 1) , A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right), \dots \right)$$

Let $A = \bigcap_m A_m$ THE INTERSECTION OF ALL A_m.

1) DESCRIBE A

2) $\inf A$? $\min A$? $\sup A$? $\max A$?

$$A = (-1, 1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \cap \dots$$

WHERE $A_1 \supset A_2 \supset A_3 \supset \dots$

WE HAVE THAT THE UNIQUE ELEMENT IN COMMON (THE INTERSECTION) IS 0! THEN: $A = \{0\}$

SO $A = [0, 0]$ AND $\inf A = \min A = \sup A = \max A = 0$

TOPOLOGY OF $\mathbb{R}$

WE RECALL THE CONCEPT OF $\boxed{\text{ABSOLUTE VALUE}}$ OF $x \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \quad |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \quad \text{and WE HAVE } |x| \geq 0 .$$

GEOMETRICALLY, $|x|$ IS THE DISTANCE FROM 0 OF $x \in \mathbb{R}$: $d(x, 0) = |x| \quad \forall x \in \mathbb{R}$

NOW WE CAN EXTEND THE CONCEPT OF DISTANCE:

<u>def</u> $\forall x, y \in \mathbb{R}, \quad d(x, y) = |y - x|$

(OR $|x - y|$, IT'S THE SAME)

THANKS TO THE NOTION OF DISTANCE IN $\mathbb{R}$ WE CAN DEFINE THE NEIGHBORHOOD OF THE ELEMENT OF $\mathbb{R}$:

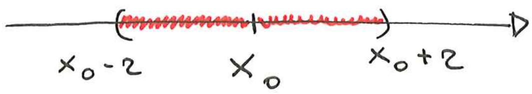
<u>def</u> $x_0 \in \mathbb{R}$, WE CALL NEIGHBORHOOD OF x WITH RADIUS $r > 0$, THE SET:

$$N_r(x_0) = \{ x \in \mathbb{R} : d(x, x_0) < r \}$$

<u>REMARK</u> $d(x, x_0) < r$ MEANS $|x - x_0| < r$,

$-r < x - x_0 < r, \quad x_0 - r < x < x_0 + r$

THEN $N_r(x_0) = (x_0 - r, x_0 + r)$ IS AN INTERVAL WITH EXTREMES EXCLUDED



WE ALSO WILL USE THE NOTATIONS AND THE CONCEPTS OF RIGHT and LEFT NEIGHBORHOOD:

$$N_r^+(x_0) = [x_0, x_0 + r)$$ RIGHT NEIGHBORHOOD

$$N_r^-(x_0) = (x_0 - r, x_0]$$ LEFT NEIGHBORHOOD

NOTICE THAT, DESPITE THE NAMES, $N_r^+(x_0)$ AND $N_r^-(x_0)$ ARE NOT NEIGHBORHOOD (BECAUSE EXTREMES ARE NOT EXCLUDED, IN FACT $x_0 \in N_r^+(x_0), N_r^-(x_0)$)

NEIGHBORHOOD OF $+\infty$ AND $-\infty$

EVEN IF $+\infty$ and $-\infty$ ARE NOT REAL NUMBERS, WE CAN DEFINE THE NEIGHBORHOODS OF THEM:

$$N(+\infty) = (a, +\infty) \quad \text{WITH } a \in \mathbb{R}$$

(ex. $(100, +\infty)$, $(3, +\infty)$, $(-20, +\infty)$, ...)

$$N(-\infty) = (-\infty, a) \quad \text{WITH } a \in \mathbb{R}$$

(ex. $(-\infty, -50)$, $(-\infty, 0)$, $(-\infty, 8)$, ...)

NOW, THROUGH THE CONCEPT OF NEIGHBORHOOD, WE CAN DEFINE A POINT OF $\mathbb{R}$ WITH RESPECT TO A SET $A \subseteq \mathbb{R}$.

<u>def</u>: Let $x_0 \in \mathbb{R}$ AND $A \subseteq \mathbb{R}$. WE SAY THAT:

i) x_0 IS AN <u>INTERIOR POINT</u> OF A IF THERE EXISTS $N_r(x_0)$ SUCH THAT $N_r(x_0) \subseteq A$.

ii) x_0 IS AN <u>EXTERIOR POINT</u> OF A IF THERE EXISTS $N_r(x_0)$ SUCH THAT $N_r(x_0) \subseteq A^c$.

iii) x_0 IS A <u>BOUNDARY POINT</u> OF A IF $\forall N_r(x_0)$ WE HAVE THAT $N_r(x_0) \cap A \neq \emptyset$ AND $N_r(x_0) \cap A^c \neq \emptyset$

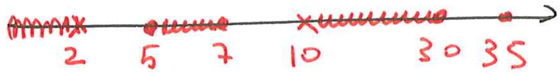
iv) x_0 IS AN <u>ACCUMULATION POINT</u> OF A IF $\forall N_r(x_0)$ WE HAVE THAT $A \cap \underbrace{N_r(x_0) \setminus \{x_0\}}_{\substack{\text{NEIGHBORHOOD} \\ \text{OF } x_0 \text{ WITHOUT } x_0 \\ (x_0 - r,\, x_0) \cup (x_0,\, x_0 + r)}} \neq \emptyset$.

<u>REMARK</u> THERE IS A PARTICULAR CASE OF BOUNDARY POINTS THAT ARE ISOLATED POINTS:

x_0 IS ISOLATED POINT OF A IF $\exists N_r(x_0)$ SUCH THAT $N_r(x_0) \cap A = \{x_0\}$

ex.

$$A = (-\infty, 2) \cup [5, 7] \cup (10, 30] \cup \{35\}$$



0 IS AN INTERIOR POINT OF A

0 IS AN ACCUMULATION POINT OF A

9 IS AN EXTERIOR POINT OF A

2 IS A BOUNDARY POINT OF A

2 IS AN ACCUMULATION POIN OF A

35 IS AN ISOLATED POINT

⋮

REMARK GIVEN THE SET A:

- THE SET OF THE INTERIOR POINTS OF A IS INDICATED WITH INT A (READ AS "INTERIOR OF A")
- THE SET OF THE EXTERIOR POINTS OF A IS INT A^c
- THE SET OF THE BOUNDARY POINTS OF A IS INDICATED WITH ∂A (READ AS "BOUNDARY OF A")
- THE SET OF THE ACCUMULATION POINTS OF A IS INDICATED WITH A' (READ AS "DERIVED SET OF A")

ex $A = (-\infty, 2) \cup [5, 7] \cup (10, 30) \cup \{35\}$

$$\text{INT } A = (-\infty, 2) \cup (5, 7) \cup (10, 30)$$

$$\partial A = \{2, 5, 7, 10, 30, 35\}$$

$$A' = (-\infty, 2] \cup [5, 7] \cup [10, 30]$$