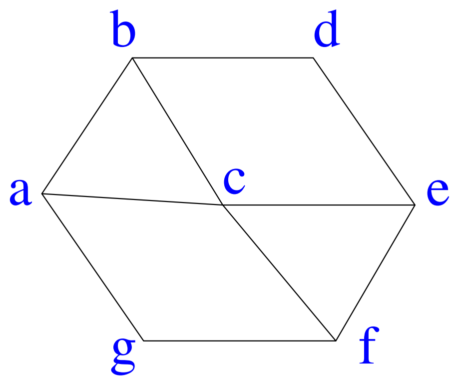
Paths and Walks

$W = (v_1, v_2, \ldots, v_k)$ is a walk in G if $(v_i, v_{i+1}) \in E$ for $1 \leq i < k$.

A path is a walk in which the vertices are distinct.

W_1 is a path, but W_2, W_3 are not.



W_1 = a,b,c,e,d

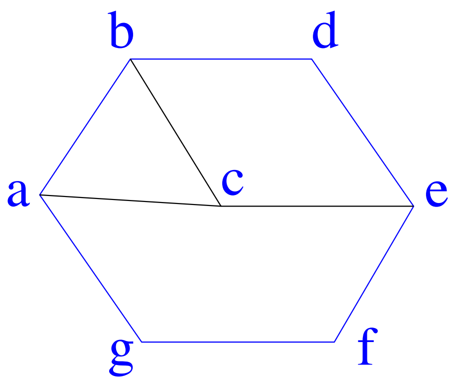
W_2 =a,b,a,c,e

W_3 =g,f,c,e,f

A walk is *closed* if $v_1 = v_k$. A *cycle* is a closed walk in which the vertices are distinct except for v_1, v_k.

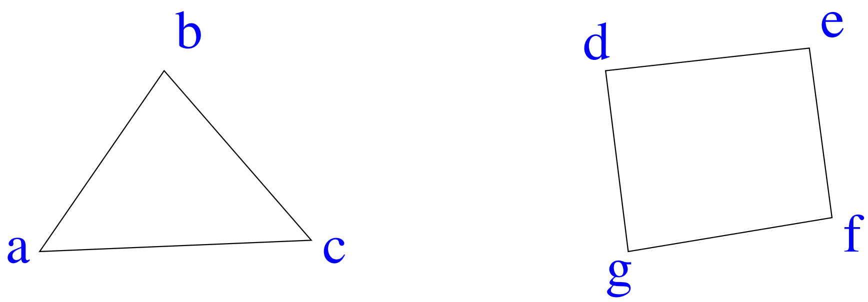
b, c, e, d, b is a cycle.

b, c, a, b, d, e, c, b is not a cycle.



Connected components

We define a relation $\sim$ on V.
$a \sim b$ iff there is a walk from a to b.



$a \sim b$ but $a \not\sim d$.

Claim: $\sim$ is an equivalence relation.

reflexivity $v \sim v$ as v is a (trivial) walk from v to v.

Symmetry $u \sim v$ implies $v \sim u$.
$(u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v implies $(u_k, u_{k-1}, \ldots, u_1)$ is a walk from v to u.

Transitivity $u \sim v$ and $v \sim w$ implies $u \sim w$.

$W_1 = (u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v and $W_2 = (v_1 = v, v_2, v_3, \ldots, v_\ell = w)$ is a walk from v to w imples that $(W_1, W_2) = (u_1, u_2 \ldots, u_k, v_2, v_3, \ldots, v_\ell)$ is a walk from u to w.

The equivalence classes of $\sim$ are called *connected components*.

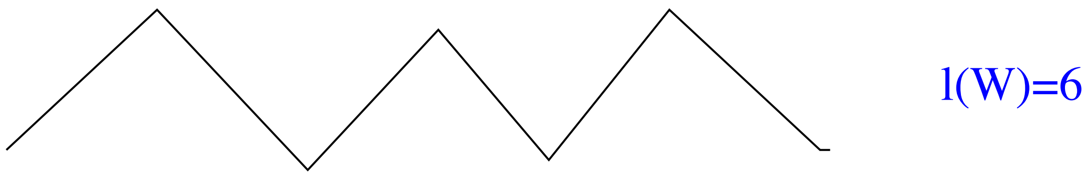
In general $V = C_1 \cup V_2 \cup \cdots \cup C_r$ where $C_1, C_2, \ldots, C_r$ are the connected comonents.

We let $\omega(G) (= r)$ be the number of components of G.

G is *connected* iff $\omega(G) = 1$ i.e. there is a walk between every pair of vertices.

Thus $C_1, C_2, \ldots, C_r$ induce connected subgraphs $G[C_1], \ldots, G[C_r]$ of G

For a walk W we let $\ell(W)$ = no. of edges in W.



Lemma 1 *Suppose W is a walk from vertex a to vertex b and that W minimises ℓ over all walks from a to b. Then W is a path.*

Proof Suppose $W = (a = a_0, a_1, \ldots, a_k = b)$ and $a_i = a_j$ where $0 \leq i < j \leq k$. Then $W' = (a_0, a_1, \ldots, a_i, a_{j+1}, \ldots, a_k)$ is also a walk from a to b and $\ell(W') = \ell(W) - (j - i) < \ell(W)$ – contradiction. $\square$

Corollary 1 *If $a \sim b$ then there is a path from a to b.*

So G is connected $\leftrightarrow \forall a, b \in V$ there is a path from a to b.

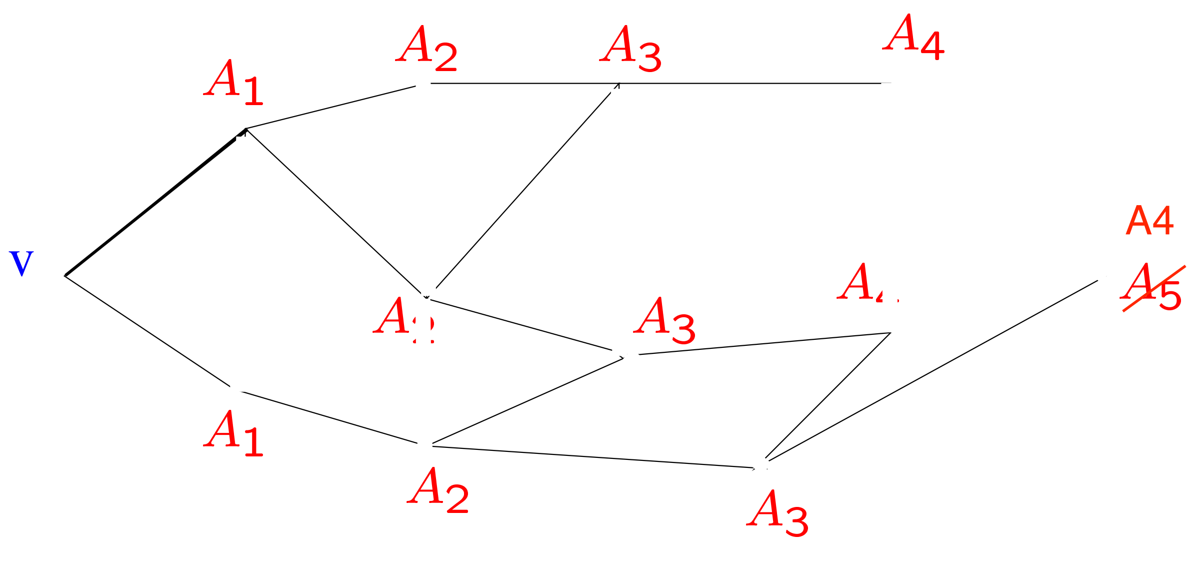
Breadth First Search – BFS

Fix $v \in V$. For $w \in V$ let

$d(v,w) =$ length of shortest path from v to w.

For $t = 0, 1, 2, \ldots$, let

$$A_t = \{w \in V : \; d(v,w) = t\}.$$



$A_0 = \{v\}$ and $v \sim w \leftrightarrow d(v,w) < \infty$.

In BFS we construct $A_0, A_1, A_2, \ldots$, by

$$A_{t+1} = \{w \notin A_0 \cup A_1 \cup \cdots \cup A_t : \exists \text{ an edge } (u,w) \text{ such that } u \in A_t\}.$$

$$\text{Note}: \quad \text{no edges } (a,b) \text{ between } A_k \text{ and } A_\ell \text{ for } \ell - k \geq 2, \text{ else } w \in A_{k+1} \neq A_\ell. \tag{1}$$

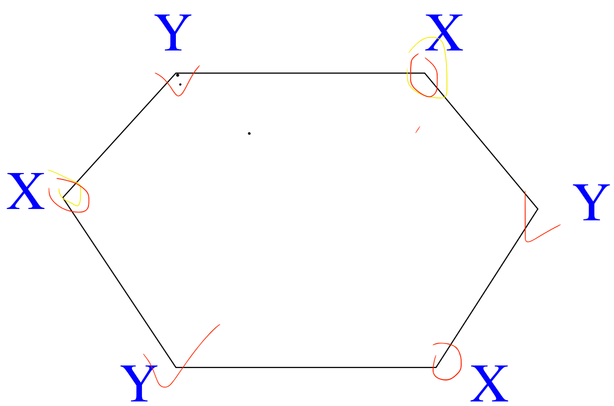
In this way we can find all vertices in the same component C as v.

By repeating for $v' \notin C$ we find another component etc.

Characterisation of bipartite graphs

Theorem 1 *G is bipartite $\leftrightarrow$ G has no cycles of odd length.*

Proof $\rightarrow$: $G = (X \cup Y, E)$.



Typical Cycle

Suppose $C = (u_1, u_2, \ldots, u_k, u_1)$ is a cycle. Suppose $u_1 \in X$. Then $u_2 \in Y, u_3 \in X, \ldots, u_k \in Y$ implies k is even.

if bipartite then k is even, this direction of the proof is trivial here we want to show that if it has no cycles of odd length then it is bipartite (more complicated than the previous proof…)

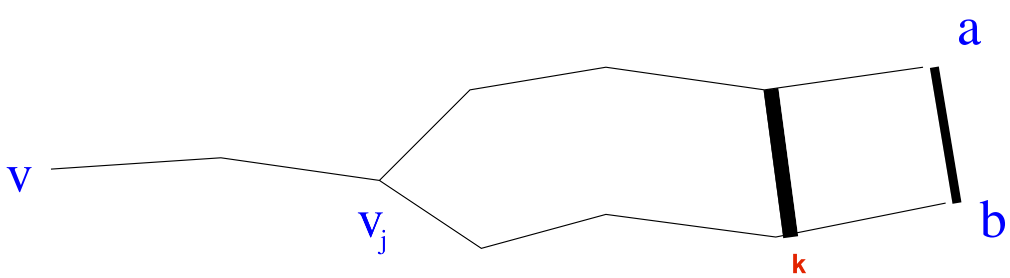
$\leftarrow$ Assume G is connected, else apply following argument to each component.
Choose $v \in V$ and construct $A_0, A_1, A_2, \dots,$ by BFS.

$X = A_0 \cup A_2 \cup A_4 \cup \cdots$ and $Y = A_1 \cup A_3 \cup A_5 \cup \cdots$

We need only show that X and Y contain no edges and then all edges must join X and Y. Suppose X contains edge (a, b) where $a \in A_k$ and $b \in A_\ell$. *(we are considering X, so. | k-l | >= 2 or k=l)*

(i) If $k \neq \ell$ then $|k - \ell| \geq 2$ which contradicts (1) *(i.e. no edge can exist in the BFS construction if | k-l | >= 2)*

(ii) $k = \ell$: *if it has an edge (in X) then it must exist a cycle of odd length.*



There exist paths $(v = v_0, v_1, v_2, \ldots, v_k = a)$ and $(v = w_0, w_1, w_2, \ldots, w_k = b)$.

Let $j = \max\{t : v_t = w_t\}$.

$$(v_j, v_{j+1}, \ldots, v_k, w_k, w_{k-1}, \ldots, w_j)$$

is an odd cycle – length $2(k-j)+1$ – contradiction. $\square$

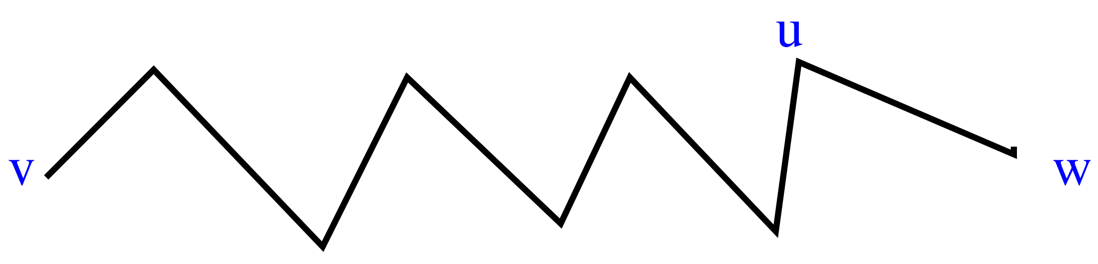
if it has an edge (in X) then it must exist a cycle of odd length.
Which is impossible because we are assuming that the graph does not have odd cycles.

Walks and powers of matrices

Theorem 2 *$A^k(v, w)$ = number of walks of length k from v to w with k edges.*

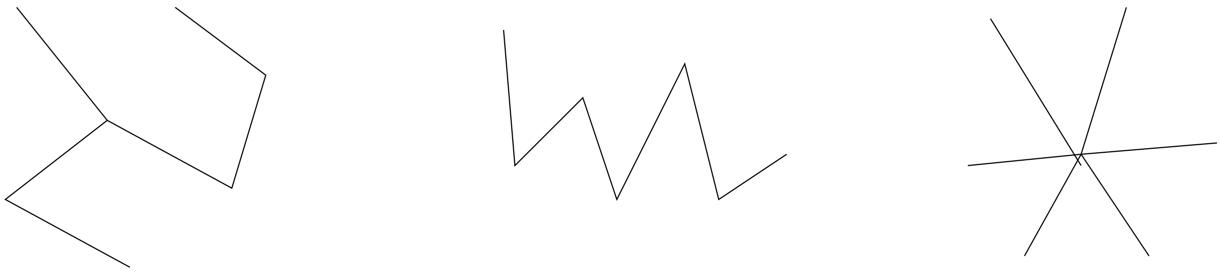
Proof By induction on k. Trivially true for $k = 1$. Assume true for some $k \geq 1$.

Let $N_t(v, w)$ be the number of walks from v to w with t edges.
Let $N_t(v, w; u)$ be the number of walks from v to w with t edges whose penultimate vertex is u.



$$
\begin{aligned}
N_{k+1}(v,w) &= \sum_{u \in V} N_{k+1}(v,w;u) \\
&= \sum_{u \in V} N_k(v,u) A(u,w) \\
&= \sum_{u \in V} A^k(v,u) A(u,w) && \text{induction} \\
&= A^{k+1}(v,w).
\end{aligned}
$$

Trees



A *tree* is a graph which is

(a) Connected and

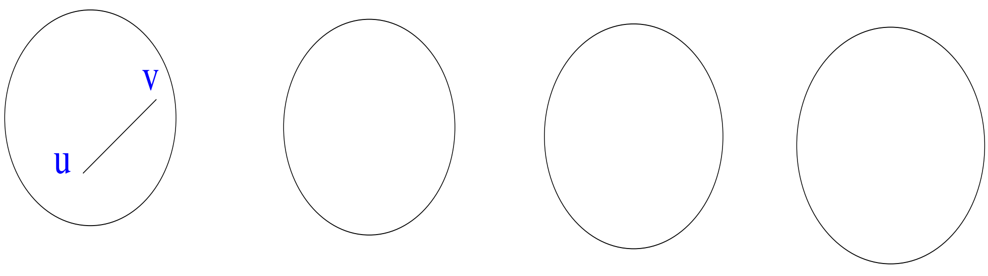
(b) has no cycles (*acyclic*).

Lemma 1 *Let the components of* G *be* $C_1, C_2, \ldots, C_r$, *Suppose* $e = (u, v) \notin E$, $u \in C_i$, $v \in C_j$.

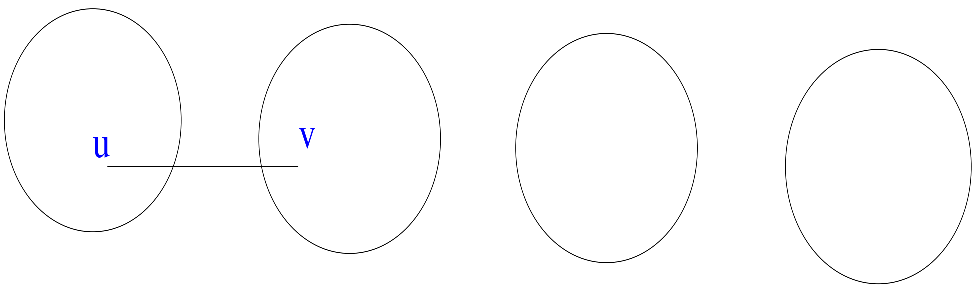
(a) $i = j \Rightarrow \omega(G + e) = \omega(G)$.

(b) $i \neq j \Rightarrow \omega(G + e) = \omega(G) - 1$.

(a)



(b)



Proof Every path P in $G + e$ which is not in G must contain e. Also,

$$\omega(G + e) \leq \omega(G).$$

Suppose

$$(x = u_0, u_1, \ldots, u_k = u, u_{k+1} = v, \ldots, u_\ell = y)$$

is a path in $G + e$ that uses e. Then clearly $x \in C_i$ and $y \in C_j$.

(a) follows as now no new relations $x \sim y$ are added.

(b) Only possible new relations $x \sim y$ are for $x \in C_i$ and $y \in C_j$. But $u \sim v$ in $G + e$ and so $C_i \cup C_j$ becomes (only) new component. $\square$

Lemma 2 $G = (V, E)$ *is acyclic (*forest*) with (tree) components* $C_1, C_2, \ldots, C_k$. $|V| = n$. $e = (u, v) \notin E$, $u \in C_i$, $v \in C_j$.

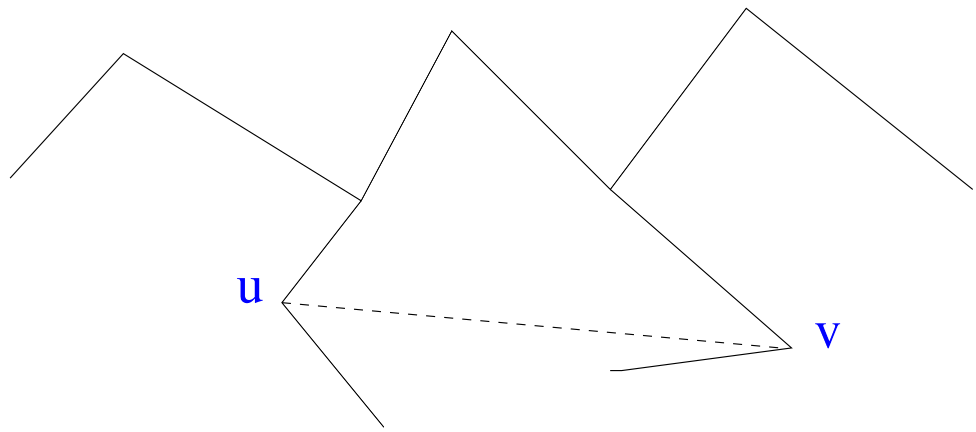
(a) $i = j \Rightarrow G + e$ *contains a cycle.*

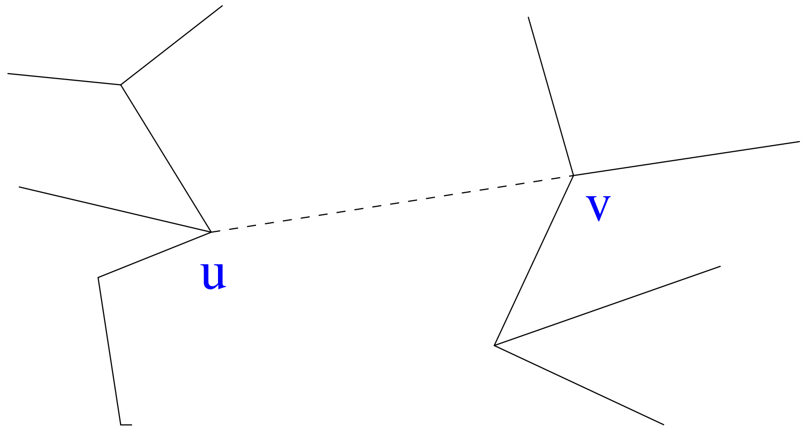
(b) $i \neq j \Rightarrow G + e$ *is acyclic and has one less component.*

(c) G *has* $n - k$ *edges.*

(a) $u, v \in C_i$ implies there exists a path $(u = u_0, u_1, \ldots, u_\ell = v)$ in G.

So $G + e$ contains the cycle $u_0, u_1, \ldots, u_\ell, u_0$.

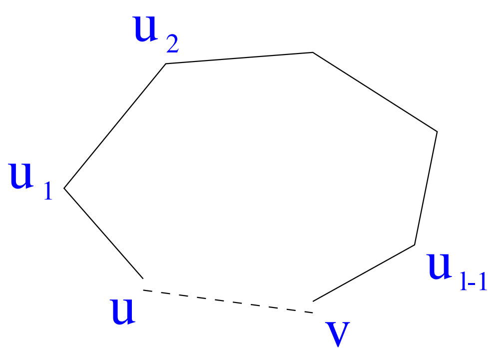




(b) Suppose $G + e$ contains the cycle C. $e \in C$ else C is a cycle of G.

$$C = (u = u_0, u_1, \ldots , u_\ell = v, u_0).$$

But then G contains the path $(u_0, u_1, \ldots , u_\ell)$ from u to v – contradiction.



The drop in the number of components follows from Lemma 1.

The rest follows from
(c) Suppose $E = \{e_1, e_2, \dots, e_r\}$ and $G_i = (V, \{e_1, e_2, \dots, e_i\})$ for $0 \le i \le r$.

Claim: G_i has $n - i$ components.
Induction on i.
$i = 0$: G_0 has no edges.
$i > 0$: G_{i-1} is acyclic and so is G_i. It follows from part (a) that e_i joins vertices in distinct components of G_{i-1}. It follows from (b) that G_i has one less component than G_{i-1}.
End of proof of claim

Thus $r = n - k$ (we assumed G had k components).

$\square$

Corollary 1 *If a tree T has n vertices then*

(a) *It has $n - 1$ edges.*

(b) *It has at least 2 vertices of degree 1, ($n \geq 2$).*

Proof (a) is part (c) of previous lemma. $k = 1$ since T is connnected.

(b) Let s be the number of vertices of degree 1 in T. There are no vertices of degree 0 – these would form separate components. Thus

$$2n - 2 = \sum_{v \in V} d_T(v) \geq 2(n - s) + s.$$

So $s \geq 2$. $\square$

Theorem 1 *Suppose $|V| = n$ and $|E| = n - 1$. The following three statements become equivalent.*

(a) G *is connected.*

(b) G *is acyclic.*

(c) G *is a tree.*

Let $E = \{e_1, e_2, \ldots, e_{n-1}\}$ and
$G_i = (V, \{e_1, e_2, \ldots, e_i\})$ for $0 \leq i \leq n - 1$.