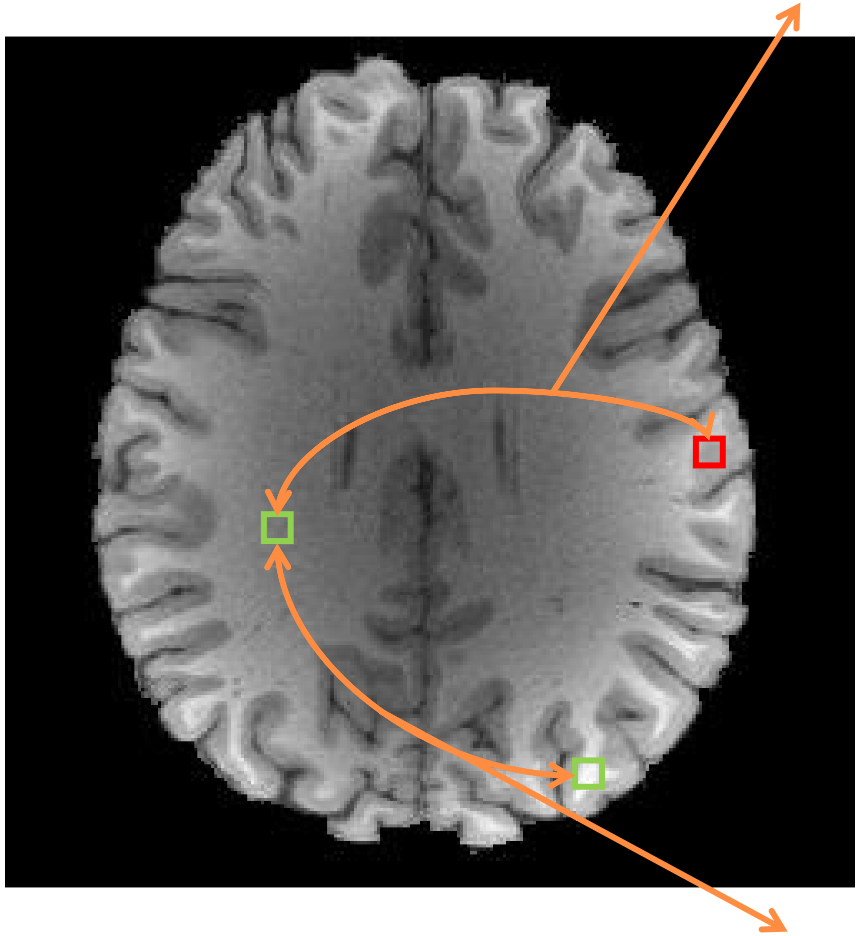
Tutorial on S2DNets

Structure and Smoothness Constrained Dual Networks for MR Bias Field Correction——*Dong Liang*

Bias field lets:

1. different tissues possess similar intensity



2. same tissues possess different intensity

Bias Field (also called Intensity Inhomogeneity, Gain Field, or Illumination Artifact) refers to a slow, spatially varying low-frequency artifact. It is created by imperfections in the imaging process, including:

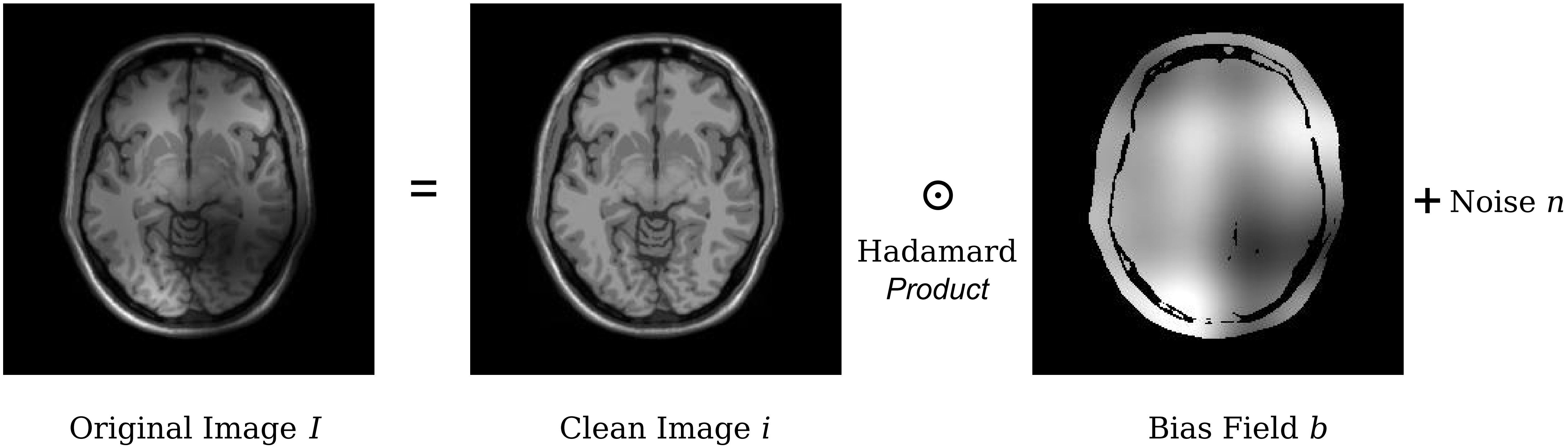
1. B_0 Radiofrequency inhomogeneity

2. B_1 Static magnetic inhomogeneity

3. Gradient nonlinearity

4. Patient-induced effects.

Bias Field Formulation

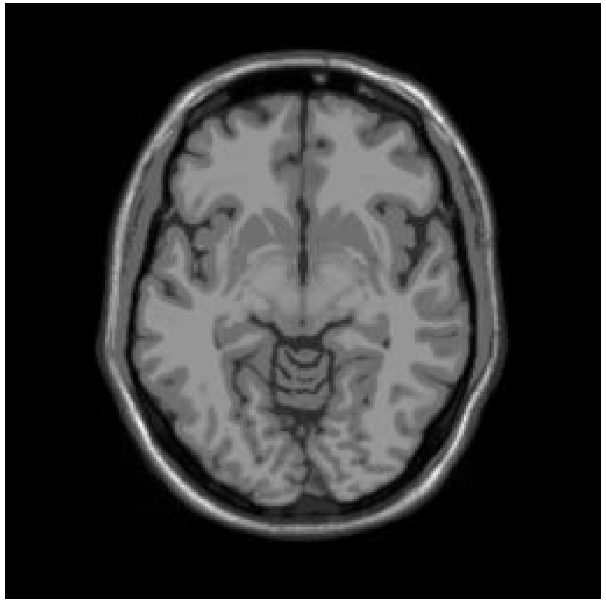


$$I(r) = i(r)b(r) + n(r), \forall r \in \Omega$$

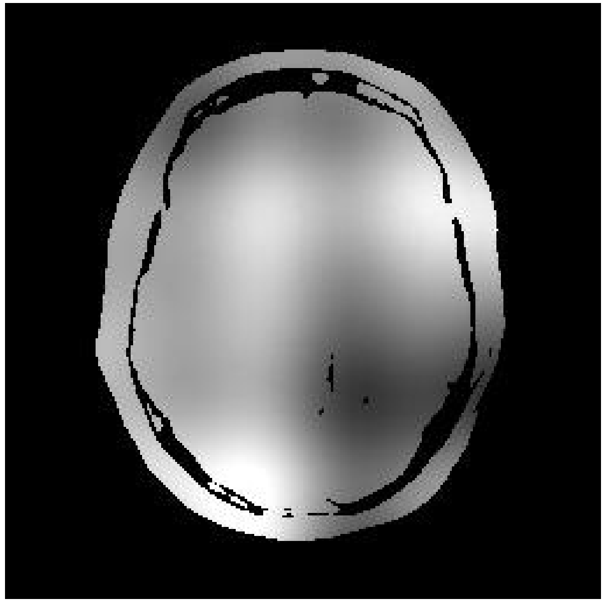
where r is position, Ω is the whole image domain.

Characteristics

Clean Image i



Bias Field b

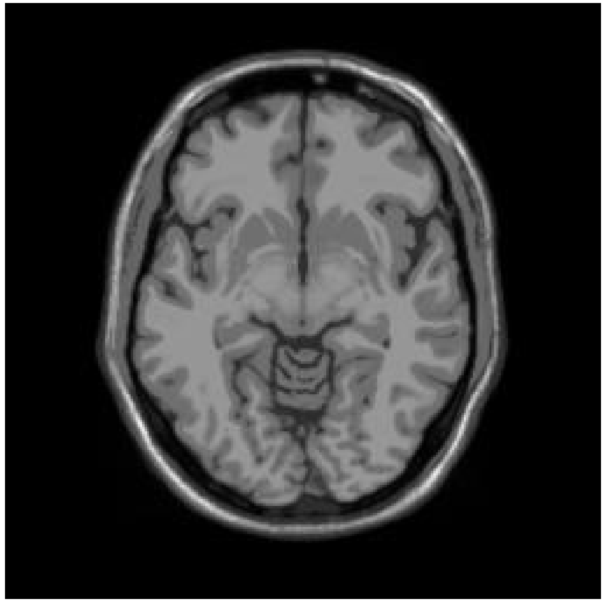


Clean image is piece-wise constant

Bias field smoothly/slowly varies across the imaged object

Characteristic 1

Clean Image i



Clean image is piece-wise constant

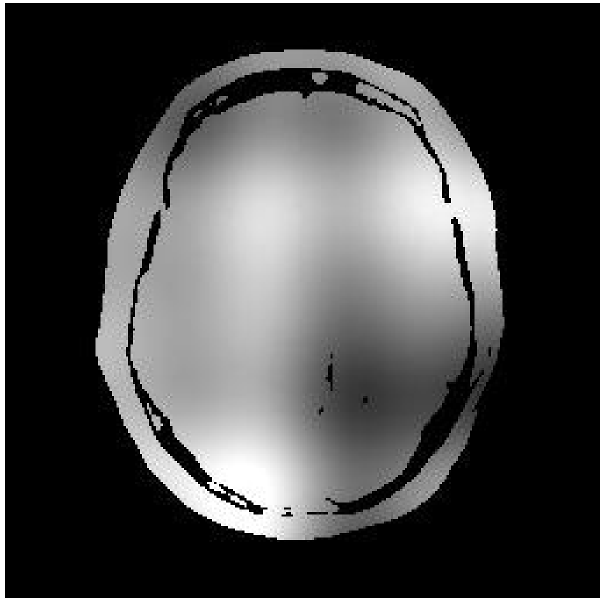
Suppose that clean image can be decomposed into N pieces with different intensity distribution, for an arbitray position r, the probability that it belongs to $i-th$ piece is represented by $u_i(r)$.

Meanwhile, the average intensity of $i-th$ piece is c_i

Thus, in clean image, the intensity on position r is computed as: $\sum_{i=1}^{N} u_i(r)\, c_i$

Characteristic 2

Bias Field b



Bias field smoothly/slowly varies across the imaged object

If two locations r and s are adjacent, there exists $b(r) \approx b(s)$.

And more adjacent two locations are, they more likely possess the same bias value.

To describe this phenomenon, we introduce Gaussian kernel:

$$K(r,s) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-s)^2}{2\sigma^2}}.$$

As this phenomenon is localized and occurs in imaged object, we introduce masked truncated Gassian kernel as:

$$K(r,s) = \begin{cases} mask(r) \cdot mask(s) \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-s)^2}{2\sigma^2}}, & if\ |r-s| < d,\ r,s \in \Omega \\ 0, & otherwise \end{cases}$$

$$mask(r) = \begin{cases} 1, \text{ if r belongs to foreground;} \\ 0, \text{ otherwise.} \end{cases}$$

$$K(r,s) = K(r,s) / \int K(r,s) ds$$

$K(r,s)$ can be regarded as the probability that locations r and s have the same bias value.

We combine two charateristics together and construct a variational energy function to transform the bias field estimation problem into an energy minimization problem.

We estimate the bias field distribution to satisfy that:

1) estimated bias field is smoothly varying;

2) corrected image is piece-wise constant;

3) the estimated bias field is multiplied with the clean image point-to-point to reconstruct the original image.

As above mentioned, in clean image, the intensity on location r is computed as: $\mathrm{i(r)} = \sum_{i=1}^{N} u_i(r)\, c_i$.

Thus, in original image, the intensity on location r is computed as: $\boxed{\mathrm{I(r)} = \mathrm{b(r)} \bullet \sum_{i=1}^{N} u_i(r)\, c_i \qquad (1).}$

Based on (1), we can construct an energy function that: $E(b, c, u) = \int |I(r) - \mathrm{b(r)} \bullet \sum_{i=1}^{N} u_i(r)\, c_i|^2 dr \quad (2)$

(2) can be rewritten as: $E(b, c, u) = \sum_{i=1}^{N} \int |I(r) - \mathrm{b(r)} c_i|^2 u_i(r) dr$

energy function : $E(b, c, u) = \sum_{i=1}^{N} \int |I(r) - \mathrm{b(r)} c_i|^2 u_i(r) dr$

Introduce piece-wise constant constraint on clean image, but neglect smoothness constraint on bias field.

What's the problem?

Original image	100	95	50
Clean image	100	100	50
Bias Field	1	0.95	1

Real bias field is smoothly varying.

Original image	100	95	50
Estimated Clean image	100	100	100
Estimated Bias Field	1	0.95	0.5

Estimated bias field is unsmooth, for lacking of smoothness constraints, which conversely recovers incorrect clean image.

Thus, it is very important to introduce smoothness constraints into energy formulation.

energy function : $E(b,c,u) = \sum_{i=1}^{N} \int |I(r) - \mathrm{b(r)} c_i|^2 u_i(r) dr$

$$K(r,s) = \begin{cases} mask(r) \cdot mask(s) \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-s)^2}{2\sigma^2}}, & if\ |r-s| < d,\ r,s \in \Omega \\ 0, & otherwise \end{cases}$$

For an arbitrary location r in image domain, position s is adjacent to r, $b(s)$ and $b(r)$ are equal with probability $K(r,s)$. Equivalently, the bias at location s is entirely substituted by that at location r with probability $K(r,s)$. There exists:

$$E(b,c,u) = \int \sum_{i=1}^{N} \int K(r,s) |I(s) - \mathrm{b(r)} c_i|^2 u_i(s) ds dr$$

Here we introduce fuzzy clustering to describe piecewise constant characteritics of clean image, as:

$$E(b,c,u) = \int \sum_{i=1}^{N} \int K(r,s) |I(s) - \mathrm{b(r)} c_i|^2 u_i^p(s) ds dr$$

where p is fuzziness factor.

Why fuzzy clustering? (from Qwen)

Fuzzy clustering is particularly useful in MR image processing because it allows for partial membership of pixels (or voxels, in 3D imaging) to multiple tissue classes. Unlike hard clustering methods such as k-means — where each pixel is assigned exclusively to one class — fuzzy clustering assigns a degree of membership to each class, which is especially important in MR images due to the following reasons:

1. Modeling Partial Volume Effects
In MR brain imaging, especially with low spatial resolution, a single voxel may contain a mixture of different tissue types (e.g., gray matter, white matter, and cerebrospinal fluid).
Fuzzy clustering can capture this uncertainty by assigning fractional memberships to each tissue type, leading to more accurate segmentation.
2. Improved Robustness to Noise and Intensity Inhomogeneity
MR images often suffer from intensity inhomogeneities (bias fields) and noise.
Fuzzy clustering can be enhanced with spatial constraints or regularization terms to reduce the impact of these artifacts and improve segmentation accuracy.
3. Better Representation of Tissue Overlap and Boundaries
Anatomical structures in the brain often have fuzzy or gradual transitions rather than sharp boundaries.
Fuzzy clustering better models these transitional regions by allowing intermediate membership values, which is crucial for precise anatomical analysis.
4. Foundation for Further Processing
The membership functions obtained from fuzzy clustering can be used for subsequent tasks such as bias field correction, tissue classification, and even machine learning-based diagnosis.

Solution on energy minimization

$$\min_{u,b,c} J = \int (\sum_{i=1}^{N} \int K(r,s) \|I(s) - b(r)c_i\|^2 u_i^p(s)ds)dr\ ,\ subject\ to\ \sum_{i=1}^{N} u_i(s) = 1$$

lagrange multiplier

$$E = \int (\sum_{i=1}^{N} \int K(r,s) \|I(s) - b(r)c_i\|^2 u_i^p(s)ds)dr - \lambda(1 - \sum_{i=1}^{N} u_i(s))$$

$$\frac{\partial E}{\partial b(r)} = \sum_{i=1}^{N} \int K(r,s)2(I(s) - b(r)c_i)c_i\, u_i^p(s)ds = 0$$

$$b(r) = \frac{\sum_{i=1}^{N} \int K(r,s)I(s)c_i u_i^p(s)ds}{\sum_{i=1}^{N} \int K(r,s)c_i^2 u_i^p\ (s)ds}$$

$$E = \int (\sum_{i=1}^{N} \int K(r,s) \| I(s) - b(r) c_i \|^2 u_i^p(s) ds) dr - \lambda (1 - \sum_{i=1}^{N} u_i(s))$$

$$\frac{\partial E}{\partial c_i} = \int \int K(r,s) \| I(s) - b(r) c_i \| u_i^p(s) b(r) ds dr = 0$$

$$c_i = \frac{\int \int K(r,s) I(s) b(r) u_i^p(s) ds\, dr}{\int \int K(r,s) b^2(r) u_i^p(s) ds dr}$$

$$\Downarrow$$

$$c_i = \frac{\int I(s) u_i^p(s) [\int K(r,s) b(r) dr]\, ds}{\int u_i^p(s) [\int K(r,s) b^2(r) dr] ds}$$

$$E = \int \Big(\sum_{i=1}^{N} \int K(r,s) \| I(s) - b(r) c_i \|^2 u_i^p(s) ds \Big) dr - \lambda \Big(1 - \sum_{i=1}^{N} u_i(s)\Big)$$

$$\frac{\partial E}{\partial u_i(s)} = \int K(r,s) \| I(s) - b(r) c_i \|^2 p u_i^{p-1}(s) dr - \lambda = 0$$

$$u_i(s) = \left(\frac{\lambda}{p \int K(s,r) \| I(s) - b(r) c_i \|^2 dr} \right)^{\frac{1}{p-1}}$$

$$\sum_{j=1}^{N} u_j(s) = 1 \Rightarrow \sum_{j=1}^{N} \left(\frac{\lambda}{p \int K(s,r) \| I(s) - b(r) c_j \|^2 dr} \right)^{\frac{1}{p-1}} = 1$$

$$\left(\frac{\lambda}{p}\right)^{\frac{1}{p-1}} \sum_{j=1}^{N} \left(\frac{1}{\int K(s,r) \| I(s) - b(r) c_j \|^2 dr} \right)^{\frac{1}{p-1}} = 1$$

$$\lambda = p \left(\frac{1}{\sum_{j=1}^{N} \left(\frac{1}{\int K(s,r) \| I(s) - b(r) c_j \|^2 dr} \right)^{\frac{1}{p-1}}} \right)^{p-1}$$

$$E = \int (\sum_{i=1}^{N} \int K(r,s) \| I(s) - b(r) c_i \|^2 u_i^p(s) ds) dr - \lambda (1 - \sum_{i=1}^{N} u_i(s))$$

$$\frac{\partial E}{\partial u_i(s)} = \int K(s,r) \| I(s) - b(r) c_i \|^2 p u_i^{p-1}(s) dr - \lambda = 0$$

$$u_i(s) = \left(\frac{\lambda}{p \int K(s,r) \| I(s) - b(r) c_i \|^2 dr}\right)^{\frac{1}{p-1}}$$

$$\lambda = p\left(\frac{1}{\sum_{j=1}^{N} \left(\frac{1}{\int K(s,r) \| I(s) - b(r) c_j \|^2 dr}\right)^{\frac{1}{p-1}}}\right)^{p-1}$$

$$u_i(s) = \left(\frac{p\left(\frac{1}{\sum_{j=1}^{N} \left(\frac{1}{\int K(s,r) \| I(s) - b(r) c_j \|^2 dr}\right)^{\frac{1}{p-1}}}\right)^{p-1}}{p \int K(s,r) \| I(s) - b(r) c_i \|^2 dr}\right)^{\frac{1}{p-1}}$$

$$u_i(s) = \left(\frac{\left(\frac{1}{\sum_{j=1}^{N}\left(\frac{1}{\int K(s,r)\|I(s)-b(r)c_j\|^2 ds}\right)^{\frac{1}{p-1}}}\right)^{p-1}}{\int K(s,r)\|I(s)-b(r)c_i\|^2 dr}\right)^{\frac{1}{p-1}}$$

$$\Downarrow$$

$$u_i(s) = \frac{\frac{1}{\sum_{j=1}^{N}\left(\frac{1}{\int K(s,r)\|I(s)-b(r)c_j\|^2 dr}\right)^{\frac{1}{p-1}}}}{\left(\int K(s,r)\|I(s)-b(r)c_i\|^2 ds\right)^{\frac{1}{p-1}}}$$

$$\Downarrow$$

$$u_i(s) = \frac{1}{\sum_{j=1}^{N}\left(\frac{\int K(s,r)\|I(s)-b(r)c_i\|^2 dr}{\int K(s,r)\|I(s)-b(r)c_j\|^2 dr}\right)^{\frac{1}{p-1}}}$$

$$\Downarrow$$

$$u_i(r) = \frac{1}{\sum_{j=1}^{N}\left(\frac{\int K(r,s)\|I(r)-b(s)c_i\|^2 ds}{\int K(r,s)\|I(r)-b(s)c_j\|^2 ds}\right)^{\frac{1}{p-1}}}$$

$$c_i = \frac{\int\int K(r,s) I(s) b(r) u_i^p(s) ds\, dr}{\int\int K(r,s) b^2(r) u_i^p(s) ds dr} \quad (1)$$

$$b(r) = \frac{\sum_{i=1}^{N} \int K(r,s) I(s) c_i u_i^p(s) ds}{\sum_{i=1}^{N} \int K(r,s) c_i^2 u_i^p\,(s) ds} \quad (2)$$

$$u_i(r) = \frac{1}{\sum_{j=1}^{N} \left(\frac{\int K(r,s) \|I(r) - b(s) c_i\|^2 ds}{\int K(r,s) \|I(r) - b(s) c_j\|^2 ds}\right)^{\frac{1}{p-1}}}$$

local smoothness $\leftarrow b(r) \approx \int K(r,s) b(s) ds$

also can be written as:

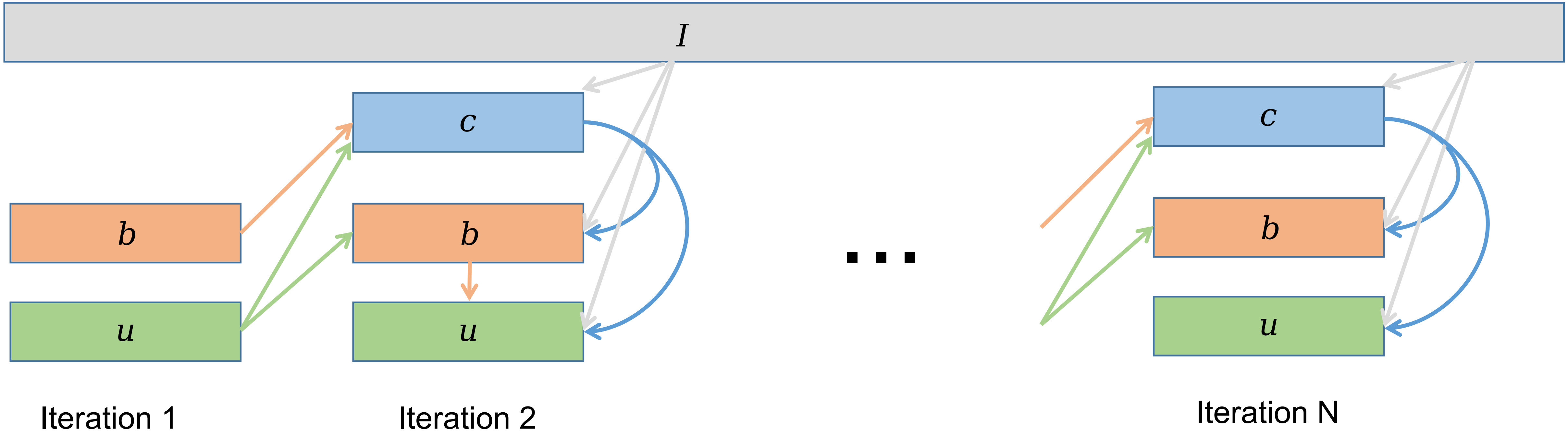
$$u_i(r) = \frac{1}{\sum_{j=1}^{N} \left(\frac{\|I(r) - c_i \int K(r,s) b(s) ds\|^2}{\|I(r) - c_j \int K(r,s) b(s) ds\|^2}\right)^{\frac{1}{p-1}}} \quad (3)$$

Optimization details:

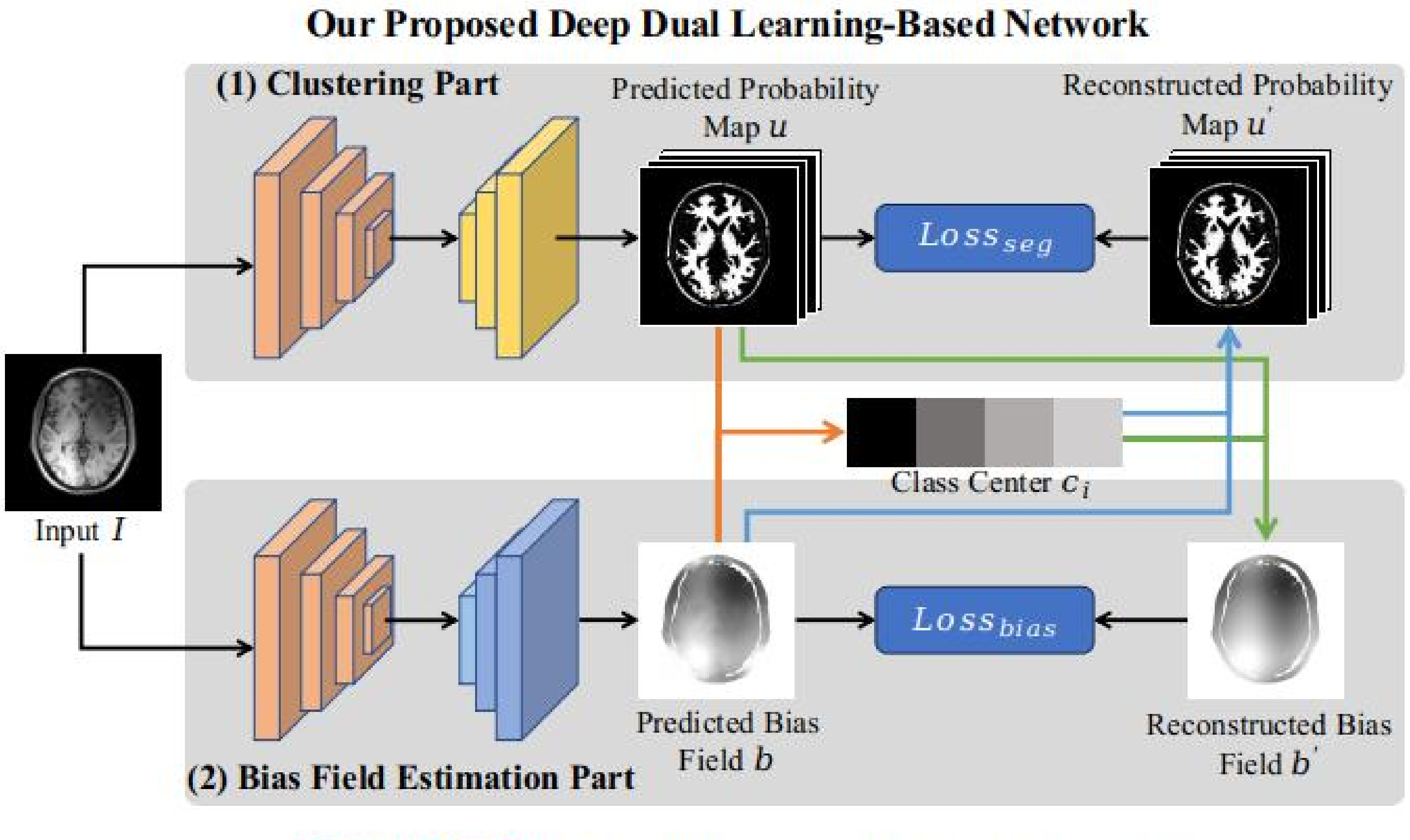
$$c_i = \frac{\int\int K(r,s) I(s) b(r) u_i^p(s) ds\, dr}{\int\int K(r,s) b^2(r) u_i^p(s) ds dr} \qquad (1)$$

$$b(r) = \frac{\sum_{i=1}^{N} \int K(r,s) I(s) c_i u_i^p(s) ds}{\sum_{i=1}^{N} \int K(r,s) c_i^2 u_i^p\ (s) ds} \qquad (2)$$

$$u_i(r) = \frac{1}{\sum_{j=1}^{N} \left(\frac{\|I(r) - c_i \int K(r,s) b(s) ds\|^2}{\|I(r) - c_j \int K(r,s) b(s) ds\|^2}\right)^{\frac{1}{p-1}}} \qquad (3)$$



Transform iterative optimization into network training:



Self-supervised Dual learning-based network for bias field estimation.

$$c_i = \frac{\int\int \mathbf{K}(\mathbf{r},\mathbf{s}) b(r) I(s) u_i^p(s) ds\, dr}{\int\int \mathbf{K}(\mathbf{r},\mathbf{s}) b^2(r) u_i^p(s) ds\, dr} \quad (1)$$

$$b'(r) = \frac{\sum_{i=1}^{N_c} \int \mathbf{K}(\mathbf{r},\mathbf{s}) I(s) c_i u_i^p(s) ds}{\sum_{i=1}^{N_c} \int \mathbf{K}(\mathbf{r},\mathbf{s}) c_i^2 u_i^p(s) ds} \quad (2)$$

$$u_i'(r) = \frac{1}{\sum_{j=1}^{N_c} \left(\frac{\| I(r) - c_i \int \mathbf{K}(\mathbf{r},\mathbf{s})\, b(s) ds \|^2}{\| I(r) - c_j \int \mathbf{K}(\mathbf{r},\mathbf{s})\, b(s) ds \|^2} \right)^{\frac{1}{p-1}}} \quad (3)$$