广义相对论：第一次作业

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题目 1 对于电磁场，我们已知其动量-能量-应力张量

$$T^{\mu\nu} = \frac{1}{\mu_0}\Big(F^{\nu\sigma}F_\sigma{}^\mu - \tfrac{1}{4}\,\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) \tag{1}$$

证明它是无迹的，即 $\eta^{\mu\nu}T_{\mu\nu} = 0$。

解答 将 $T^{\mu\nu}$ 代入得

$$\begin{aligned}\eta_{\mu\nu}T^{\mu\nu} &= \frac{1}{\mu_0}\left(\eta_{\mu\nu}F^{\nu\sigma}F_\sigma{}^\mu - \frac{1}{4}\,\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\\ &= \frac{1}{\mu_0}\left(F^{\sigma\nu}F_{\sigma\nu} - \frac{1}{4}\times 4\,F_{\rho\sigma}F^{\rho\sigma}\right) = 0\end{aligned} \tag{2}$$

其中 $\mathrm{tr}(\eta^{\mu\nu}) = 4$，因此 $T^\mu{}_\mu = 0$。

题目 2 利用电磁场的能量-动量-应力张量 $T^{\mu\nu}$ 和 Maxwell 方程，证明:

$$\partial_\mu T^{\mu\nu} = -F^\nu{}_\sigma J^\sigma \tag{3}$$

实际上本题反映了电磁场的能动量守恒，请简要说明。

解答 张量形式的 Maxwell 方程为：

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu \tag{4}$$

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0 \tag{5}$$

先对能动张量一部分求微分：

$$\begin{aligned}\partial_\mu F^\mu{}_\sigma F^{\nu\sigma} &= \partial^\mu F_{\mu\sigma}F^{\nu\sigma}\\ &= F_{\mu\sigma}\partial^\mu F^{\nu\sigma} + F^{\nu\sigma}\partial^\mu F_{\mu\sigma}\\ &= F_{\mu\sigma}\partial^\mu F^{\nu\sigma} - F^{\nu\sigma}\mu_0 J_\sigma\end{aligned} \tag{6}$$

这里第二项除以 μ_0 就等于须证式 (3) 的右边，只需证剩下部分为 0。略去 μ_0, 其余部分的和为：

$$
\begin{aligned}
F_{\mu\sigma}\partial^\mu F^{\nu\sigma} - \frac{1}{4}\eta^{\mu\nu}\partial_\mu\left(F_{\rho\sigma}F^{\rho\sigma}\right) &= F_{\mu\sigma}\partial^\mu F^{\nu\sigma} - \frac{1}{2}F_{\rho\sigma}\partial^\nu F^{\rho\sigma} \\
&= \frac{1}{2}\left(F_{\mu\sigma}\partial^\mu F^{\nu\sigma} + F_{\mu\sigma}\partial^\nu F^{\nu\sigma} - F_{\mu\sigma}\partial^\nu F^{\mu\sigma}\right) \\
&= \frac{1}{2}\left(F_{\sigma\mu}\partial^\sigma F^{\nu\mu} - F_{\mu\sigma}\partial^\mu F^{\sigma\nu} - F_{\mu\sigma}\partial^\nu F^{\mu\sigma}\right) \\
&= -\frac{1}{2}F_{\mu\sigma}\left(\partial^\sigma F^{\nu\mu} + \partial^\mu F^{\sigma\nu} + \partial^\nu F^{\mu\sigma}\right) \\
&= 0
\end{aligned}
\tag{7}
$$

其中，第三行第一项交换了哑指标字母，最后一步利用了 Maxwell 方程的无源方程部分。因此式 (3) 得证。

$$
\partial_\mu T^{\mu\nu} = -F^\nu{}_\sigma J^\sigma \tag{8}
$$

实际上是能动量守恒的一个具体形式，取 $\nu = 0$ 和 $\nu = 1, 2, 3$ 分别展开，这个方程的矢量形式展开后是：

$$
\frac{\partial T^{00}}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J} \tag{9}
$$

$$
\frac{\partial \mathbf{P}}{\partial t} + \nabla\cdot\mathbf{T} = \mathbf{F} \tag{10}
$$

其中，T^{00} 表示电磁场的能量密度，$\mathbf{S}$ 是 Poynting 向量，其中，$\mathbf{P}$ 是电磁场的动量密度，$\mathbf{T}$ 是电磁场的动量流密度，$\mathbf{F}$ 是电磁力密度。从这个第一个可以看出，电磁场的能量密度会随着 Poynting 矢量在空间中传播，也会因为与物质的相互作用而产生能量的变化，从而满足整体的能量守恒。第二个方程与能量守恒方程类似，电磁场动量由于动量流的散度以及电磁力对物质的作用会发生变化，从而满足整体的动量守恒。

题目 3　商定理：设在某坐标系中有张量关系

$$
A^\mu(x) = B^\mu{}_\alpha(x)\, C^\alpha(x), \tag{11}
$$

其中 A^μ 为 $(1,0)$ 张量、$B^\mu{}_\alpha$ 为 $(1,1)$ 张量。证明 C^α 为 $(1,0)$ 张量，即在坐标变换 $x \mapsto x'$ 下满足

$$
C'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\beta} C^\beta(x) \tag{12}
$$

解答　张量分量在坐标变换下的变换律为

$$
A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\, A^\nu(x), \tag{13}
$$

$$
B'^\mu{}_\alpha(x') = \frac{\partial x'^\mu}{\partial x^\nu}\,\frac{\partial x^\beta}{\partial x'^\alpha}\, B^\nu{}_\beta(x) \tag{14}
$$

由给定关系 $A^\mu = B^\mu{}_\alpha C^\alpha$ 得

$$\begin{aligned}
A'^\mu(x') &= \frac{\partial x'^\mu}{\partial x^\nu}\, B^\nu{}_\beta(x)\, C^\beta(x) \\
&= \left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\beta}{\partial x'^\alpha} B^\nu{}_\beta(x)\right)\left(\frac{\partial x'^\alpha}{\partial x^\gamma} C^\gamma(x)\right) \\
&= B'^\mu{}_\alpha(x')\left(\frac{\partial x'^\alpha}{\partial x^\gamma} C^\gamma(x)\right) \qquad (15)
\end{aligned}$$

而在 x' 系同样有 $A'^\mu = B'^\mu{}_\alpha C'^\alpha$。若 $B^\mu{}_\alpha$ 在该点可逆，则可得

$$C'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\gamma} C^\gamma(x) \qquad (16)$$

题目 4 设 $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}(x)$ 为 (p,q) 张量。证明：将其中一上指标与一下指标缩并得到的

$$S^{\mu_1\cdots\widehat{\mu_r}\cdots\mu_p}{}_{\nu_1\cdots\widehat{\nu_s}\cdots\nu_q}(x) := T^{\mu_1\cdots\alpha\cdots\mu_p}{}_{\nu_1\cdots\alpha\cdots\nu_q}(x) \qquad (17)$$

是一个 $(p-1,q-1)$ 张量。

解答 张量分量在坐标变换 $x \mapsto x'$ 下满足

$$T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}}\,\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_q}}{\partial x'^{\nu_q}}\, T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(x) \qquad (18)$$

把第 r 个上指标与第 s 个下指标缩并，有

$$\begin{aligned}
S'^{\mu_1\cdots\widehat{\mu_r}\cdots\mu_p}{}_{\nu_1\cdots\widehat{\nu_s}\cdots\nu_q}(x') &= T'^{\mu_1\cdots\alpha\cdots\mu_p}{}_{\nu_1\cdots\alpha\cdots\nu_q}(x') \\
&= \left(\prod_{\substack{i=1\\ i\neq r}}^{p}\frac{\partial x'^{\mu_i}}{\partial x^{\alpha_i}}\right)\frac{\partial x'^\alpha}{\partial x^\rho}\,\left(\prod_{\substack{j=1\\ j\neq s}}^{q}\frac{\partial x^{\beta_j}}{\partial x'^{\nu_j}}\right)\frac{\partial x^\sigma}{\partial x'^\alpha}\, T^{\alpha_1\cdots\rho\cdots\alpha_p}{}_{\beta_1\cdots\sigma\cdots\beta_q}(x) \qquad (19)
\end{aligned}$$

利用链式法则得到 Kronecker 符号

$$\frac{\partial x'^\alpha}{\partial x^\rho}\,\frac{\partial x^\sigma}{\partial x'^\alpha} = \delta^\sigma_\rho, \qquad (20)$$

于是

$$S'^{\mu_1\cdots\widehat{\mu_r}\cdots\mu_p}{}_{\nu_1\cdots\widehat{\nu_s}\cdots\nu_q}(x') = \left(\prod_{\substack{i=1\\ i\neq r}}^{p}\frac{\partial x'^{\mu_i}}{\partial x^{\alpha_i}}\right)\left(\prod_{\substack{j=1\\ j\neq s}}^{q}\frac{\partial x^{\beta_j}}{\partial x'^{\nu_j}}\right) T^{\alpha_1\cdots\lambda\cdots\alpha_p}{}_{\beta_1\cdots\lambda\cdots\beta_q}(x) \qquad (21)$$

其中把哑指标 ρ,σ 重命名为 λ。这正是 $(p-1,q-1)$ 张量的变换律，故缩并后仍为张量。

题目 5 证明：逆变矢量的平行输运在坐标变换下仍保持逆变矢量的性质。即若沿位移 dx^ν 从点 p 到 $q = p + dx$ 的一阶平行输运写为

$$B^\gamma(p\to q) = B^\gamma(p) - \Gamma^\gamma_{\mu\nu}(p)\, B^\mu(p)\, dx^\nu, \qquad (22)$$

则在任意坐标变换 $x \mapsto x'$ 下有

$$B'^{\gamma}(p \to q) = \left[\frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\right]_q B^{\alpha}(p \to q) \tag{23}$$

解答 在 x 系： $B^{\alpha}(q) = B^{\alpha}(p) - \Gamma^{\alpha}{}_{\beta\delta}(p)\, B^{\beta}(p)\, dx^{\delta}$。

在 x' 系按同一定义：

$$B'^{\gamma}(p \to q) = B'^{\gamma}(p) - \Gamma'^{\gamma}{}_{\mu\nu}(p)\, B'^{\mu}(p)\, dx'^{\nu} \tag{24}$$

代入变换律并约去中间雅可比（此处量均在 p 取值）：

$$B'^{\gamma}(p \to q) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} B^{\alpha} - \frac{\partial x'^{\gamma}}{\partial x^{\rho}}\Gamma^{\rho}{}_{\beta\delta}\, B^{\beta} dx^{\delta} - \frac{\partial x'^{\gamma}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\delta}} B^{\beta} dx^{\delta}$$

对恒等式 $\dfrac{\partial x^{\rho}}{\partial x'^{\mu}}\dfrac{\partial x'^{\mu}}{\partial x^{\beta}} = \delta^{\rho}{}_{\beta}$ 对 x^{δ} 求导，得

$$\frac{\partial x'^{\gamma}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\delta}} = -\frac{\partial^2 x'^{\gamma}}{\partial x^{\beta}\partial x^{\delta}} \tag{25}$$

因此

$$B'^{\gamma}(p \to q) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} B^{\alpha} - \frac{\partial x'^{\gamma}}{\partial x^{\rho}}\Gamma^{\rho}{}_{\beta\delta}\, B^{\beta} dx^{\delta} + \frac{\partial^2 x'^{\gamma}}{\partial x^{\beta}\partial x^{\delta}} B^{\beta} dx^{\delta} \tag{26}$$

另一方面，雅可比在 $q = p + dx$ 的一阶展开：

$$\left[\frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\right]_q = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} + \frac{\partial^2 x'^{\gamma}}{\partial x^{\alpha}\partial x^{\delta}} dx^{\delta} \tag{27}$$

注意到：

$$\left[\frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\right]_q \left(B^{\alpha} - \Gamma^{\alpha}{}_{\beta\delta} B^{\beta} dx^{\delta}\right) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} B^{\alpha} - \frac{\partial x'^{\gamma}}{\partial x^{\rho}}\Gamma^{\rho}{}_{\beta\delta}\, B^{\beta} dx^{\delta} + \frac{\partial^2 x'^{\gamma}}{\partial x^{\beta}\partial x^{\delta}} B^{\beta} dx^{\delta}$$

与上式相同，故

$$B'^{\gamma}(p \to q) = \left[\frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\right]_q \left(B^{\alpha}(p) - \Gamma^{\alpha}{}_{\beta\delta}(p)\, B^{\beta}(p)\, dx^{\delta}\right) = \left[\frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\right]_q B^{\alpha}(p \to q) \tag{28}$$

即平移后的 B 在 q 点按逆变矢量变换。

题目 6 黎曼几何基本定理：在一个给定的流形上，与流形上某给定度规相关的无挠联络只有一个。

解答 展开度规的协变导数并轮换其指标

$$\nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \Gamma^{\lambda}_{\mu\rho} g_{\lambda\nu} - \Gamma^{\lambda}_{\nu\rho} g_{\mu\lambda} = 0, \tag{29}$$

$$\nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\nu\mu} g_{\lambda\rho} = 0, \tag{30}$$

$$\nabla_{\nu} g_{\mu\rho} = \partial_{\nu} g_{\mu\rho} - \Gamma^{\lambda}_{\nu\mu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = 0 \tag{31}$$

第一式减去第二式和第三式的和：

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0 \tag{32}$$

双边乘以度规张量 $g^{\rho\sigma}$，可以得到：

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) \tag{33}$$

题目 7 第一题: 曲线的弧长可以表示为:

$$S = \int_{\lambda_i}^{\lambda_f} ds = \int_{\lambda_i}^{\lambda_f} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \qquad (1)$$

根据作用量原理 $\delta S = 0$ 分别证明:(1) 当取弧长作为参量时，能够导出测地线方程:

$$\frac{dx^\mu}{ds} \nabla_\mu \frac{dx^\nu}{ds} = 0, \qquad (2)$$

其中，∇_μ 为无挠且与度规相容的协变导数，满足 $\Gamma^\mu_{[\alpha\beta]} = 0$ 以及 $\nabla_\mu g_{\alpha\beta} = 0$。(2) 请写出当采取任意参数时，测地线方程的形式。

解答 我先求解任意参数情况下的测地线方程，然后再代入弧长参数条件 $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ 得到弧长参数下的测地线方程。任意参数 λ 下的弧长为：

$$S = \int_{\lambda_i}^{\lambda_f} L \, d\lambda, \quad \text{其中} \quad L = \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

定义 $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$，则 $L = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$。根据作用量原理 $\delta S = 0$，得到欧拉-拉格朗日方程：

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0 \tag{34}$$

此处度规是计算偏导数：

$$\frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2L} \cdot 2 g_{\mu\nu} \dot{x}^\nu = \frac{g_{\mu\nu} \dot{x}^\nu}{L}, \tag{35}$$

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2L} \partial_\mu g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{36}$$

代入方程 (34) 得：

$$\frac{d}{d\lambda} \left(\frac{g_{\mu\nu} \dot{x}^\nu}{L} \right) - \frac{1}{2L} \partial_\mu g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \tag{37}$$

令 $u^\mu = \dot{x}^\mu$，则方程可写为：

$$\frac{d}{d\lambda} \left(\frac{g_{\mu\nu} u^\nu}{L} \right) = \frac{1}{2L} \partial_\mu g_{\alpha\beta} u^\alpha u^\beta \tag{38}$$

展开左边：

$$\frac{d}{d\lambda}\left(\frac{g_{\mu\nu}u^\nu}{L}\right)=\frac{1}{L}\frac{d}{d\lambda}(g_{\mu\nu}u^\nu)-\frac{g_{\mu\nu}u^\nu}{L^2}\frac{dL}{d\lambda} \tag{39}$$

其中，

$$\frac{d}{d\lambda}(g_{\mu\nu}u^\nu)=\partial_\alpha g_{\mu\nu}u^\alpha u^\nu+g_{\mu\nu}\frac{du^\nu}{d\lambda} \tag{40}$$

代入后乘以 L：

$$\partial_\alpha g_{\mu\nu}u^\alpha u^\nu+g_{\mu\nu}\frac{du^\nu}{d\lambda}-\frac{g_{\mu\nu}u^\nu}{L}\frac{dL}{d\lambda}-\frac{1}{2}\partial_\mu g_{\alpha\beta}u^\alpha u^\beta=0 \tag{41}$$

重组项：

$$g_{\mu\nu}\frac{du^\nu}{d\lambda}+\left(\partial_\alpha g_{\mu\nu}u^\alpha u^\nu-\frac{1}{2}\partial_\mu g_{\alpha\beta}u^\alpha u^\beta\right)-\frac{g_{\mu\nu}u^\nu}{L}\frac{dL}{d\lambda}=0 \tag{42}$$

对于哑指标 α、β 和 ν，通过更改指标并写为对称形式，括号内项可表示为：

$$\partial_\alpha g_{\mu\nu}u^\alpha u^\nu-\frac{1}{2}\partial_\mu g_{\alpha\beta}u^\alpha u^\beta=\frac{1}{2}\left(\partial_\beta g_{\mu\alpha}+\partial_\alpha g_{\mu\beta}-\partial_\mu g_{\alpha\beta}\right)u^\alpha u^\beta \tag{43}$$

用 $g^{\sigma\mu}$ 收缩方程 (42)，根据克里斯托费尔联络的定义可得：

$$\frac{du^\sigma}{d\lambda}+\Gamma^\sigma_{\alpha\beta}u^\alpha u^\beta-\frac{u^\sigma}{L}\frac{dL}{d\lambda}=0 \tag{44}$$

即：

$$\frac{d^2x^\sigma}{d\lambda^2}+\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}=\frac{1}{L}\frac{dL}{d\lambda}\frac{dx^\sigma}{d\lambda} \tag{45}$$

令 $f(\lambda)=\frac{d}{d\lambda}\ln L$，则任意参数下的测地线方程为：

$$\frac{d^2x^\sigma}{d\lambda^2}+\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}=f(\lambda)\frac{dx^\sigma}{d\lambda} \tag{46}$$

用协变导数形式可写为：

$$\boxed{\frac{dx^\alpha}{d\lambda}\nabla_\alpha\frac{dx^\sigma}{d\lambda}=f(\lambda)\frac{dx^\sigma}{d\lambda}} \tag{47}$$

现在代入弧长参数条件。当参数为弧长 s 时，有 $L=1$ 于是 $f(s)=0$。代入方程 (46) 得：

$$\frac{d^2x^\sigma}{ds^2}+\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0 \tag{48}$$

这就是弧长参数下的测地线方程。用协变导数表示：

$$\boxed{\frac{dx^\alpha}{ds}\nabla_\alpha\frac{dx^\sigma}{ds}=0} \tag{49}$$