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COURSE TITLE: OPE 311: OPERATIONS RESEARCH

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Course Content

Meaning of operation research, techniques of operation research, model building, decision making using logic or common sense, linear programming: graphical and simplex method, sensitivity analysis,; learning curve experience, queuing theory, network analysis simulation

This Course Is Based On The Following Textbooks:

1. Hillier, Lieberman: Introduction to Operations Research, McGraw Hill, 10th edition McGraw-Hill,
2. Winston W.L. Operations Research: Applications and Algorithms, PWS-KENT Publishing Company, Boston, 1987. Additional literature
3. Anderson D. R., D. J. Sweeney, and T. A. Williams: An Introduction to Management Science, 8th ed., West, St. Paul, Mn, 2000.
4. Taha, H. Operations Research: An Introduction, 8th ed., Pearson Prentice Hall, Upper Sadle River, NJ, 2007.
5. Williams H.P.: Model Building in Mathematical Programming, 3d ed., Wiley, New York, 1990.

INTRODUCTION TO OPERATIONS RESEARCH

INTRODUCTION

-Known as Management Science, System Analysis, Quantitative Method, Decision Analysis.

-Definition: The application of the scientific methods, techniques and tools to problem involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems.

Historical Development

-Began during WW II as a separate discipline.

-England (Operational Research): British government organized teams of experts to solve perplexing strategic and tactical problems.

-USA (Operations Research): Applied OR to the deployment of merchant marine convoys to minimize losses from enemy submarines.

-Following the WW II: OR extended into industry

-Oil refineries, steel and paper mills.

-Expansion to almost all industries, services and large social and urban systems.

Benefits

-Increase the effectiveness of the decision.

-Enable evaluation of situations involving uncertainty.

-Provide a systematic and logical approach to decision making.

-Allows quick and inexpensive examination of a large number of alternatives.

Limitations

-Time-consuming.

-Can be expensive to undertake, relative to the size of the problem.

-Lack of acceptance by decision makers.

-Assessments of uncertainties are difficult to obtain.

Tools of OR

-Decision Tables	-Decision Trees
-Game Theory	-Forecasting
-Queuing Models	-Markov Analysis
-Dynamic Programming	-Linear Programming
-Integer Programming	-Inventory Models
-Transportation Models	-Simulation etc.

Mathematical Model

-Model: Simplified representation of reality.

Mathematical model: a mathematical representation of an actual situation.

-Components: sets of mathematical expressions such as equation or inequalities.

-All mathematical models are comprised of three basic components: result variables, decision variables and uncontrollable variables.

-There are two major parts.

1. The objective function

-Express the dependent variables in the model as they relate to the independent variables.

$$Z = P_1X_1 + P_2X_2$$

Z = dependent (result) variable

P_1, P_2 = uncontrollable variables

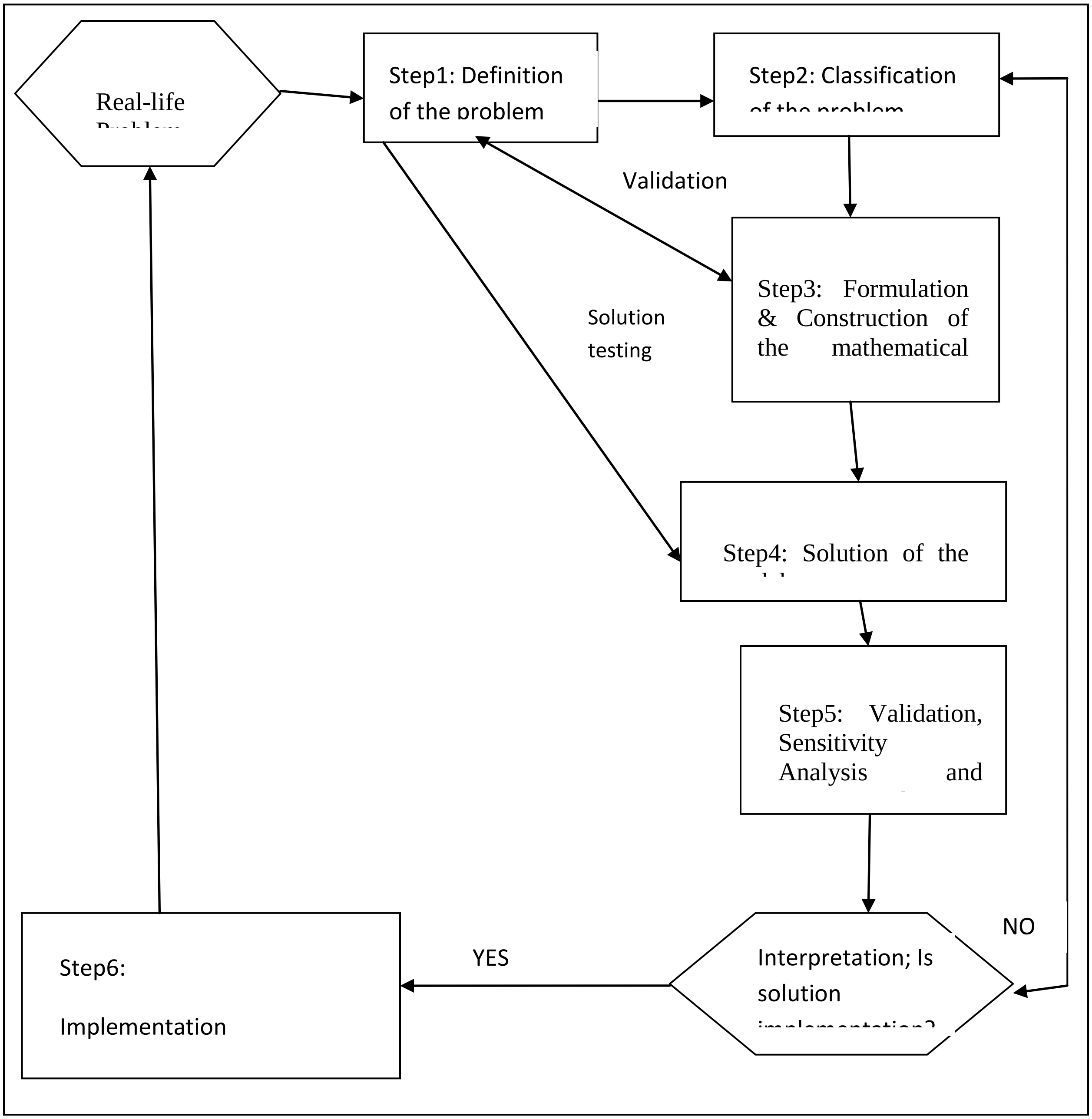
X_1, X_2 = decision variables (controllable)

2. The constraints

-Express the limitations imposed on managerial systems due to regulations, competition and technology, etc.

$$X_1 + X_2 \leq 50$$

Process of Operations Research



Linear Programming (LP)

-LP is a tool for solving optimization problem.

-All the mathematical functions in the model are linear.

-George Dantzig developed an efficient method, the simplex method, for solving LP problems.

-Deal with allocation problems: to determine an optimal allocation of an organization's limited resources.

Problems

The product-Mix Problem

-There are two or more products.

-Find out which products to include in the production plan and in what quantities.

-Mostly to maximize profit with fixed resources.

The Blending Problem

-To determine the best blend of available ingredients to form a certain quantity of a product under strict specification.

-Minimize the cost of ingredients.

Formulation

1. The decision variables

-Controllable.

-To find the value of decision variables is to find the optimal solution to the problem.

-The number of units to be produced, the quantities of the resources to be allocated.

2. The objective function

-A linear function with a single goal.

-Maximization or Minimization.

3. The constraints

-Uncontrollable restrictions or requirements.

-Expressed as linear inequalities and/or equations.

General Formulation of LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $x_1, x_2, \cdots, x_n$ maximize or minimize the objective function.

$Z = C_1x_1 + C_2x_2 + \cdots + C_nx_n$

Subject to the restrictions

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \leq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \ \leq b_m$

Data needed for a linear programming model involving the allocation of resources to activities

Resource	Resource Usage per Unit of Activity Activity 1 2 . . . n	Amount of Resource Available
1	a_{11} a_{12} . . . a_{1n}	b_1
2	a_{21} a_{22} . . . a_{2n}	b_1
⋮	. . . .	⋮
M	a_{m1} a_{m1} . . . a_{m1}	b_m
Contribution to *Z* per unit of activity	C_1 C_2 . . . C_n	

And $x_1 \geq 0$. $x_2 \geq 0$. … $x_n \geq 0$

Common terminology for the linear programming model can now be summarized.

The function being maximized, $C_1x_1 + C_2x_2 + \cdots + C_nx_n$, is called the **objective function.**

The restrictions normally are referred to as **constraints.** The first *m* constraints (those with a *function* of all the variables $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{im}x_n$ on the left-hand side) are sometimes called **functional constraints** (or *structural constraints*). Similarly, the $x_n \geq 0$ restrictions are called **nonnegativity constraints** (or *nonnegativity conditions*).

Matrix Form of LPP

The LPP can be expressed in the matrix form as follows:

$Maximaze\ or$

$Minimize\ Z = Cx \rightarrow Objective\ function$

$Subject\ to\ Ax\ (\leq = \geq) b \rightarrow Contant\ equation$

$b > 0,\ x \geq 0\ Non - negativity\ restriction$

Where $x = (x_1 \quad x_2 \quad \cdots \quad x_n)$

$c = (c_1 \quad c_2 \quad \cdots \quad c_n)$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & x_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Terminology for Solutions of the Model

You may be used to having the term *solution* mean the final answer to a problem, but the convention in linear programming (and its extensions) is quite different. Here, *any* specification of values for the decision variables $(x_1, x_2, \cdots, x_n)$ is called a **solution,** regardless of whether it is a desirable or even an allowable choice. Different types of solutions are then identified by using an appropriate adjective.

A **feasible solution** is a solution for which *all* the constraints are *satisfied.*

An **infeasible solution** is a solution for which *at least one* constraint is *violated.*

In the example, the points (2, 3) and (4, 1) in Fig. 3.2 are *feasible solutions,* while the points (1, 3) and (4, 4) are *infeasible solutions.*

The **feasible region** is the collection of all feasible solutions.

It is possible for a problem to have **no feasible solutions.**

An **optimal solution** is a feasible solution that has the *most favorable value* of the objective function.

The **most favorable value** is the *largest value* if the objective function is to be *maximized,* whereas it is the *smallest value* if the objective function is to be *minimized.*

Example 2.1:

Suppose an industry is manufacturing two types of products P_1 and P_2. The profits per Kg of the two products are 30 FRS and 40 FRS respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each machine to produce one Kg of P_1 and P_2. Formulate the problem in the form of linear programming model.

Profit/Kg	$P_1 = 30\ Frs$	$P_2 = 30Frs$	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Solution:

The procedure for linear programming problem formulation is as follows:

Introduce the decision variable as follows:

Let x_1 = amount of P_1

x_1 = amount ofP_2

In order to maximize profits, we establish the objective function as

$30x_1 + 40x_2$

Since one Kg of P_1 requires 3 hours of processing time in machine 1 while the corresponding requirement of P_2 is 2 hours. So, the first constraint can be expressed as

$3x_1 + 2x_2 \leq 600$

Similarly, corresponding to machine 2 and 3 the constraints are

$3x_1 + 5x_2 \leq 800$

$5x_1 + 6x_2 \leq 1100$

In addition to the above there is no negative production, which may be represented algebraically as $x_1 \geq 0;\ x_1 \geq 0$

Thus, the product mix problem in the linear programming model is as follows:

Maximize $30x_1 + 40x_2$

Subject to: $3x_1 + 2x_2 \leq 600$

$3x_1 + 5x_2 \leq 800$

$5x_1 + 6x_2 \leq 1100$

$x_1 \geq 0;\ x_1 \geq 0$

Example 2: A manufacturer produces two types of models M_1 and M_2. Each model of the type M_1 requires 4 hours of grinding and 2 hours of polishing; whereas each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M_1 model is Rs 3.00 and on model M_2 is Rs 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

Solution:

Decision variables: Let X_1 and X_2 be the number of units of M_1 and M_2 model.
Objective function: Since the profit on both the models is given, we have to maximize the profit, viz.

$Max\ Z = 300X_1 + 400X_2$

Constraints:
There are two constraints, one for grinding and the other for polishing. The number of hours available on each grinder for one week is 40 hrs. There are two grinders. Hence, the manufacturer does not have more than 2 × 40 = 80 hours of grinding. M_1 requires 4 hours of grinding and M_2 requires 2 hours of grinding. The grinding constraint is given by

$4x_1 + 2x_2 \leq 80$

Since there are 3 polishers, the available time for polishing in a week is given by

$3 \times 60 = 180$. M_1 requires 2 hours of polishing and M_2 requires 5 hours of polishing.

Hence, we have $2x_1 + 5x_2 \leq 180$.

Profit/Kg	$Q_1 = 300\ Frs$	$Q_2 = 400Frs$	Total available Machine hours/day
Grinder	4	2	$2 \times 40 = 80$
Polisher	2	5	$3 \times 60 = 180$

Finally we have

$Max\ Z = 300x_1 + 400x_2$

Subject to $4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

Example 2.2:

A company manufactures two products *A* and *B*. These products are processed in the same machine. It takes 10 minutes to process one unit of product *A* and 2 minutes for each unit of product *B* and the machine operates for a maximum of 35 hours in a week. Product *A* requires 1 kg. and *B* 0.5 kg. of raw material per unit the supply of which is 600 kg. per week. Market constraint on product *B* is known to be 800 units every week. Product *A* costs Rs 5 per unit and sold at Rs 10. Product *B* costs Rs 6 per unit and can be sold in the market at a unit price of Rs 8. Determine the number of units of *A* and *B* per week to maximize the profit.

Solution:

Decision variables: Let x_1 andx_2 be the number of products *A* and *B*.

Objective function: Costs of product *A* per unit is 5F and sold at 10F per unit.

$\therefore$ Profit on one unit of product $A = 10 - 5 = 5$

$\therefore x_1$ units of product *A* contributes a profit of 5F profit contribution from one unit of product

$B = 8 - 6 = 2$

$\therefore$ x_2 units of product *B* contribute a profit of 2F

$\therefore$ The objective function is given by

$Max\ Z = 5x_1 + x_2$

Constraints: Time requirement constraint is given by

$$10x_1 + 2x_2 = (35 \times 60)$$

$$10x_1 + 2x_2 = 2100$$

Raw material constraint is given by

$x_1 + 0.5x_2 \leq 600$

Market demand on product *B* is 800 units every week.

$\therefore x_2 \geq 800$

The complete LPP is

$Max\ Z = 5x_1 + x_2$

$10x_1 + 2x_2 = 2100$

$x_1 + 0.5x_2 \leq 600$

$x_2 \geq 800$

$x_1, x_2 \geq 0$

Example 2.3: A person requires 10,12, and 12 units of chemicals *A*, *B* respectively for his garden. *A* liquid product contains 5,2 and 1 units of *A*, *B* and *C* respectively per jar. *A* dry product contains 1,2 and 4 units of *A*, *B*, *C* per carton. If the liquid product sells for 3F per jar and the dry product sells for 2F per carton, how many of each should be purchased, in order to minimize the cost and meet the requirements?

Solution:

Decision variables: Let x_1 and x_2 be the number of units of liquid and dry products.

Objective function: Since the cost for the products are given we have to minimize the cost.

$Min\ Z = 3x_1 + 2x_2$

Constraints: As there are three chemicals and its requirement are given.
We have three constraints for these three chemicals

$$5x_1 + x_2 \geq 10$$

$$2x_1 + 2x_2 \geq 12$$

$$x_1 + 4x_2 \geq 12$$

Finally the complete L.PP is

$$Min\ Z = 3x_1 + 2x_2$$

Subject to $\quad 5x_1 + x_2 \geq 10$

$$2x_1 + 2x_2 \geq 12$$

$$x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Graphical Method of Solution of a Linear Programming Problem

The graphical method is applicable to solve the LPP involving two decision variables x_1, and x_2. To solve an LP, the graphical method includes two major steps.

a) The determination of the solution space that defines the feasible solution. Note that the set of values of the variable x_1, x_2, x_3,....x_n which satisfy all the constraints and also the non-negative conditions is called the feasible solution of the LP.
b) The determination of the optimal solution from the feasible region

a) To determine the feasible solution of an LP, we have the following steps.

Step 1: Since the two decision variable x and y are non-negative, consider only the first quadrant of xy-coordinate plane.

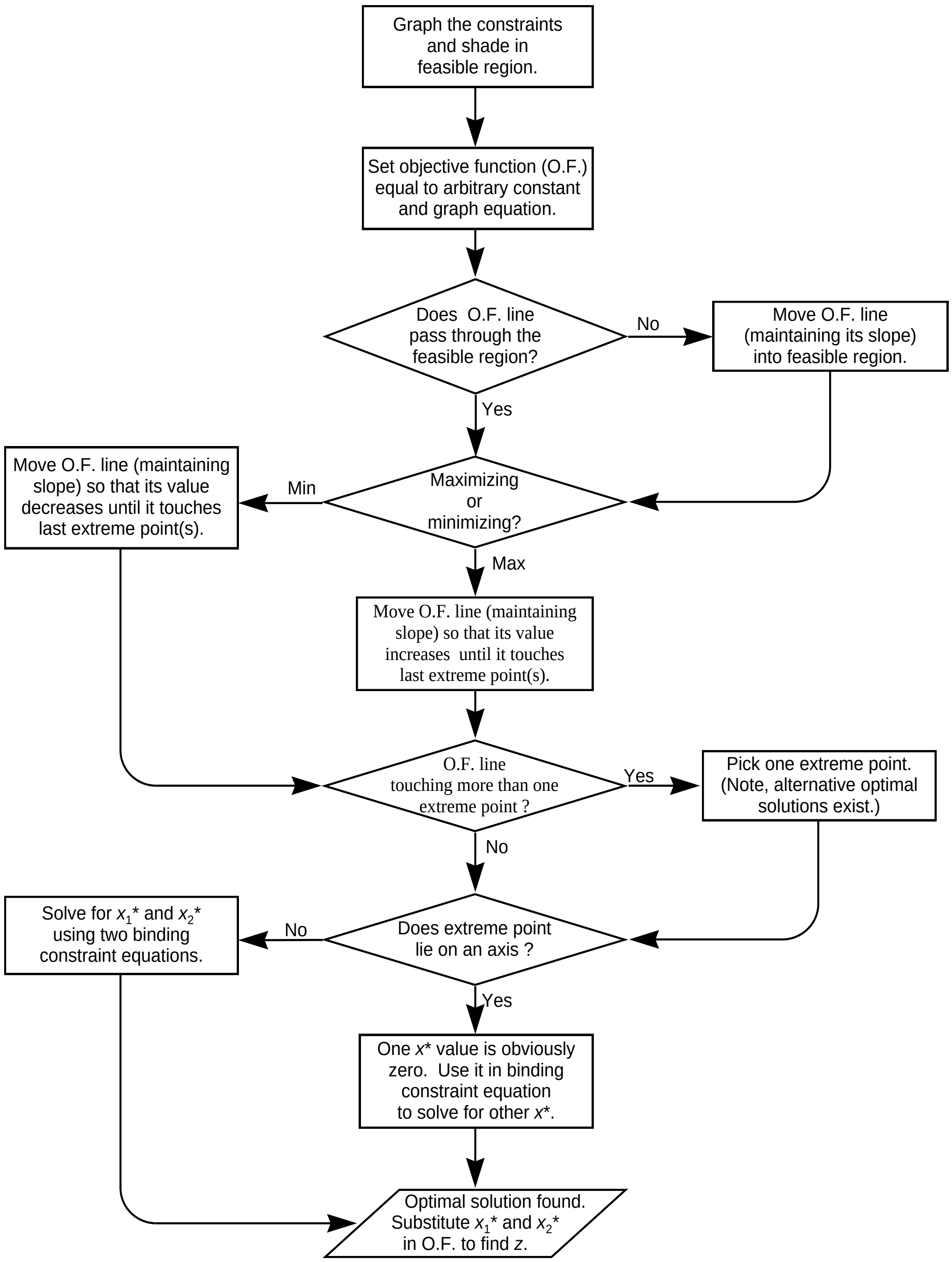
Step 2: Each constraint is of the form $ax + by \leq c\ or ax + by \geq c$

Draw the line ax + by = c

For each constraint, the line (1) divides the first quadrant in to two regions say R_1 and R_2, suppose $(x_1, 0)$ is a point in R_1. If this point satisfies the in equation $ax + by \leq c$ or $(\geq c)$, then shade the region R_1. If $(x_1, 0)$ does not satisfy the inequality, shade the region R_2.

Step 3: Corresponding to each constant, we obtain a shaded region. The intersection of all these shaded regions is the feasible region or feasible solution of the LP.

<u>FLOW CHART OFGRAPHICAL L.P. SOLUTION PROCEDURE</u>



NOTE: Plotting an initial objective function line involves little more than reversing the objective coefficients for x_1 and x_2. Consider Problem 1 below. The objective line will cross the x_1 axis at 4 (x_2's coefficient) and the x_2 axis at 3 (x_1's coefficient). If the coefficients are too large (or small) for convenient graphing, scale them down (or up) in a consistent manner by dividing (or multiplying) both by, say, 10.

GRAPHICAL SOLUTION PROCEDURE

1) Graph the constraints and shade in the feasible region, considering the feasible side of each constraint line.
2) Set the objective function equal to any arbitrary constant and graph it. If the line does not lie in the feasible region, move it (maintaining its slope) into the feasible region.
3) Move the objective function line parallel to itself in the direction that increases its value when maximizing (decreases its value when minimizing) until it touches the last point(s) of the feasible region.
4) If the optimal extreme point falls on an axis (say, x_2 axis), use the binding constraint equation to solve for the unknown x* (in this case x_2*, since x_1* is zero). Otherwise, solve the two equations (binding constraints) in two unknowns (x_1* and x_2*) that determine the optimal extreme point.
5) Find z by substituting x_1* and x_2* in the objective function.

Example. Let us find the feasible solution for the problem of a decorative item dealer whose

LPP is to maximize profit function.

Z = 50x + 18y

Subject to the constraints: $2x + y \leq 100$

$$x + y \leq 80$$

$$x \geq 0, y \geq 0,$$

Step 1: Since x ≥0, y ≥0, we consider only the first quadrant of the xy – plane

Step 2: We draw straight lines for the equation

$$2x + y \leq 100$$

$$x + y \leq 80$$

To determine two points on the straight line 2x + y = 100

Put y = 0, 2x = 100

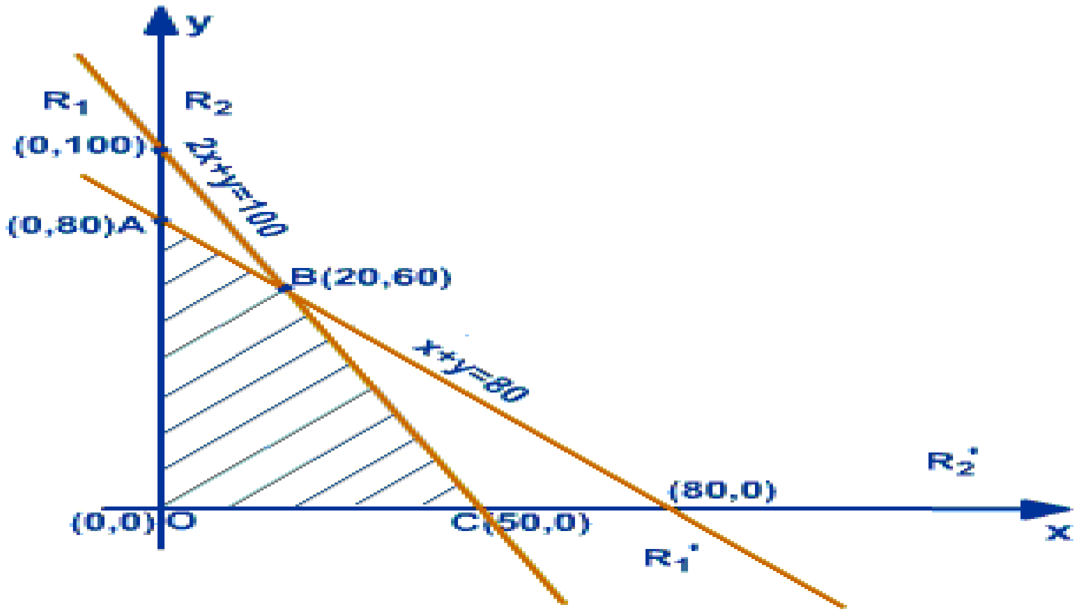
$\Rightarrow$ x = 50

$\Rightarrow$ (50, 0) is a point on the line (2)

$\Rightarrow$ put x = 0 in (2), y =100

$\Rightarrow$ $\Rightarrow$(0, 100) is the other point on the line (2)

Plotting these two points on the graph paper draw the line which represent the line 2x + y =100.



This line divides the 1st quadrant into two regions, say R_1 and R_2. Choose a point say (1, 0) in R_1. (1, 0) satisfy the inequality $2x + y \leq 100$. Therefore R_1 is the required region for the constraint

$2x + y \leq 100$.

Similarly draw the straight line x + y = 80 by joining the point (0, 80) and (80, 0). Find the required region say R_1', for the constraint $x + y \leq 80$.

The intersection of both the region R_1 and R_1' is the feasible solution of the LPP. Therefore every point in the shaded region OABC is a feasible solution of the LPP, since this point satisfies all the constraints including the non-negative constraints.

b) There are two techniques to find the optimal solution of an LPP.

Corner Point Method

The optimal solution to a LPP, if it exists, occurs at the corners of the feasible region.

The method includes the following steps

Step 1: Find the feasible region of the LLP

Step 2: Find the co-ordinates of each vertex of the feasible region.

These co-ordinates can be obtained from the graph or by solving the equation of the lines.

Step 3: At each vertex (corner point) compute the value of the objective function.

Step 4: Identify the corner point at which the value of the objective function is maximum (or minimum depending on the LP)

The co-ordinates of this vertex is the optimal solution and the value of Z is the optimal value

Example: Find the optimal solution in the above problem of decorative item dealer whose objective function is Z = 50x + 18y.

In the graph, the corners of the feasible region are

O (0, 0), A (0, 80), B(20, 60), C(50, 0)

At (0, 0) Z = 0

At (0, 80) Z = 50 (0) + 18(80) =1440

At (20, 60), Z = 50 (20) +18 (60)

= 1000 + 1080 = Rs.2080

At (50, 0) Z = 50 (50) + 18 (0) =2500

Since our object is to maximize Z and Z has maximum at (50, 0) the optimal solution is x = 50 and y = 0.

The optimal value is 2500.

If an LPP has many constraints, then it may be long and tedious to find all the corners of the feasible region. There is another alternate and more general method to find the optimal solution of an LP, known as 'ISO profit or ISO cost method'

ISO- PROFIT (OR ISO-COST)

Method of Solving Linear Programming Problems

Suppose the LPP is to

Optimize Z = ax + by subject to the constraints

$a_1x + b_1y \le (\text{or} \ge) c_1$

$a_2x + b_2y \le (\text{or} \ge) c_2$

$x \ge 0, \quad y \ge 0.$

This method of optimization involves the following method.

Step 1: Draw the half planes of all the constraints

Step 2: Shade the intersection of all the half planes which is the feasible region.

Step 3: Since the objective function is Z = ax + by, draw a dotted line for the equation ax + by = k, where k is any constant. Sometimes it is convenient to take k as the LCM of a and b.

Step 4: To maximise Z draw a line parallel to ax + by = k and farthest from the origin. This line should contain at least one point of the feasible region. Find the coordinates of this point by solving the equations of the lines on which it lies.

To minimise Z draw a line parallel to ax + by = k and nearest to the origin. This line should contain at least one point of the feasible region. Find the co-ordinates of this point by solving the equation of the line on which it lies.

Step 5: If (x_1, y_1) is the point found in step 4, then $x = x_1$, $y = y_1$, is the optimal solution of the LPP and $Z = ax_1 + by_1$ is the optimal value.

The above method of solving an LPP is more clear with the following example.

Example: Solve the following LPP graphically using ISO- profit method.

maximize Z =100 + 100y.

Subject to the constraints $10x + 5y \leq 80$

$6x + 6y \leq 66$

$4x + 8y \geq 24$

$5x + 6y \leq 90$

$x \geq 0, \quad y \geq 0$

Suggested answer:

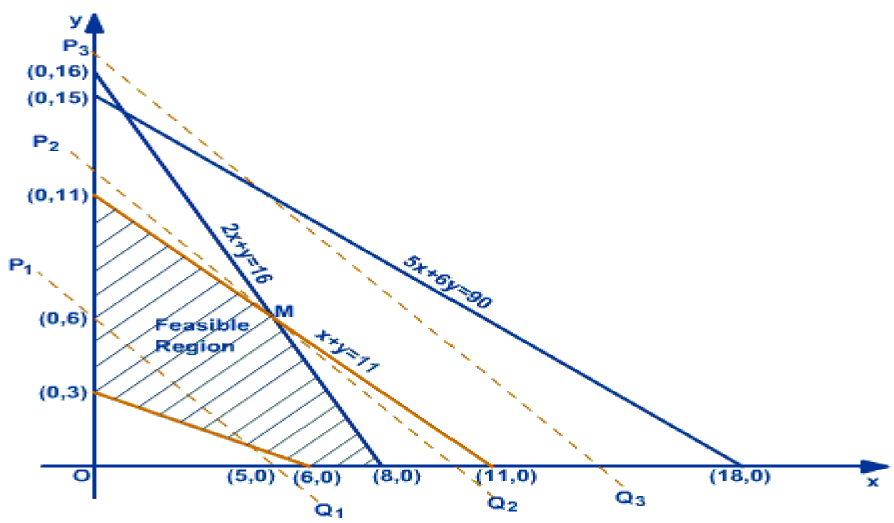
since $x \geq 0$, $y \geq 0$, consider only the first quadrant of the plane graph the following straight lines on a graph paper 10x + 5y = 80 or 2x+y =16,

6x + 6y = 66 or x +y =11

4x+ 8y = 24 or x+ 2y = 6

5x + 6y = 90

Identify all the half planes of the constraints. The intersection of all these half planes is the feasible region as shown in the figure.



Give a constant value 600 to Z in the objective function, then we have an equation of the line

$120x + 100y = 600$ (1)

or $6x + 5y = 30$ (Dividing both sides by 20)

P_1Q_1 is the line corresponding to the equation $6x + 5y = 30$.

We give a constant 1200 to Z then the P_2Q_2 represents the line.

$120x + 100y = 1200$

$6x + 5y = 60$

P_2Q_2 is a line parallel to P_1Q_1 and has one point 'M' which belongs to feasible region and farthest from the origin.

If we take any line P_3Q_3 parallel to P_2Q_2 away from the origin, it does not touch any point of the feasible region.

The co-ordinates of the point M can be obtained by solving the equation

$2x + y = 16$

$x + y = 11$ which give $x = 5$ and $y = 6$

$\Rightarrow$The optimal solution for the objective function is $x = 5$ and $y = 6$

The optimal value of Z120 (5) + 100 (6) = 600 + 600=1200

<u>PROBLEM 1</u>

Given the following linear program:

$$
\begin{array}{ll}
\text{MAX} & z = 3x_1 + 4x_2 \\
\text{s.t.} & 2x_1 + 3x_2 \leq 24 \\
& 3x_1 + x_2 \leq 21 \\
& x_1 + x_2 \leq 9 \\
& x_1, x_2 \geq 0
\end{array}
$$

a) Solve the problem graphically.

b) Write the problem in standard form.

c) Given your answer to (a), what are the optimal values of the slack variables.

SOLUTION 1

a) (1) **Graph the constraints.**

Constraint 1: When $x_1 = 0$, then $x_2 = 8$; when $x_2 = 0$, then $x_1 = 12$.

Connect (12,0) and (0,8). The "<" side is below the line.

Constraint 2: When $x_1 = 0$, then $x_2 = 21$; when $x_2 = 0$, then $x_1 = 7$.

Connect (7,0) and (0,21). The "<" side is below the line.

Constraint 3: When $x_1 = 0$, then $x_2 = 9$; when $x_2 = 0$, then $x_1 = 9$.

Connect (9,0) and (0,9). The "<" side is below the line.

Shade in the feasible region.

(2) Graph the objective function by setting the objective function equal to any arbitrary value (say 12) and graphing it.

For $3x_1 + 4x_2 = 12$, when $x_2 = 0$, x1 = 4; when $x_1 = 0$, $x_2 = 3$. Connect (4,0) and (0,3), the thick graphed line.

(3) Move the objective function line parallel to itself in the direction that increases its value (upward) until it touches the last point of the feasible region. It is at the intersection of the first and third constraint lines.

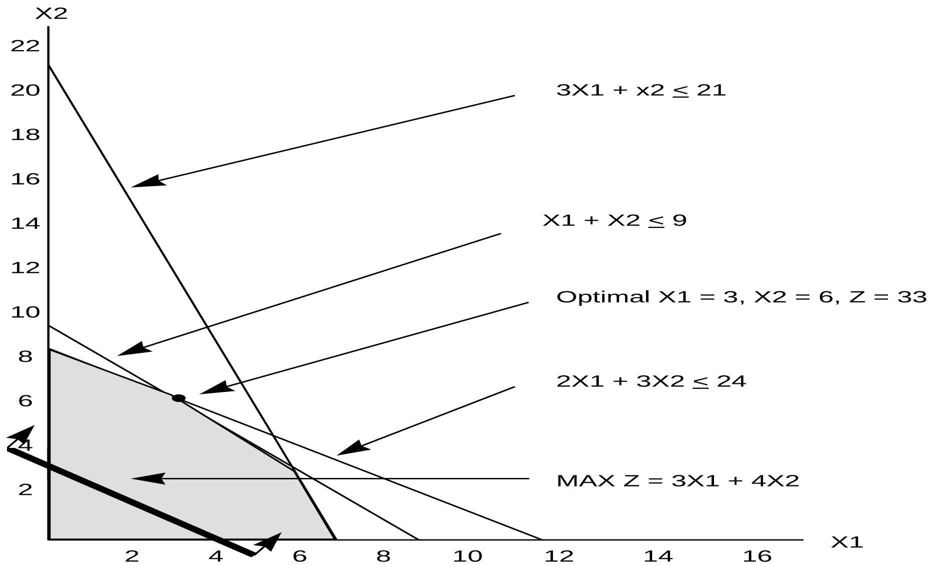
(4) Solve these two equations in two unknowns:

$$\begin{array}{rcl} 2x_1 + 3x_2 = 24 & \longrightarrow & 2x_1 + 3x_2 = 24 \\ x_1 + x_2 = 9 & \longrightarrow & 2x_1 + 2x_2 = 18 \\ & & x_2 = 6 \end{array}$$

Substituting into $x_1 + x_2 = 9$, then $x_1 = 3$.

(5) Solve for z: $z = 3x_1 + 4x_2 = 3(3) + 4(6) = 33$. Thus the optimal solution is

$x_1 = 3$, $x_2 = 6$, $z = 33$.



b) To write the problem in standard form, since each constraint is a "$\leq$" constraint, add a slack variable to each constraint.

$$
\begin{aligned}
\text{MAX } z &= 3x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t. } \quad & 2x_1 + 3x_2 + s_1 = 24 \\
& 3x_1 + x_2 + s_2 = 21 \\
& x_1 + x_2 + s_3 = 9 \\
& x_1, x_2, s_1, s_2 \geq 0
\end{aligned}
$$

c) Since the optimal solution was $x_1 = 3$, $x_2 = 6$, then substituting these values into the above equations gives:

$$s_1 = 24 - 2(3) - 3(6) = 0$$

$$s_2 = 21 - 3(3) - 1(6) = 6$$

$$s_3 = 9 - 1(3) - 1(6) = 0$$

Problem 2

Given the following linear program:

$$
\begin{aligned}
\text{MIN } z &= 5x_1 + 2x_2 \\
\text{s.t. } \quad & 2x_1 + 5x_2 \geq 10
\end{aligned}
$$

$$4x_1 - x_2 \geq 12$$

$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

a. Solve graphically for the optimal solution.

b. How does one know that although $x_1 = 5$, $x_2 = 3$ is a feasible solution for the constraints, it will never be the optimal solution no matter what objective function is imposed?

c. Solve for the optimal solution using a spreadsheet.

SOLUTION 2

a) (1) **Graph the constraints.**

Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$.

Connect (5,0) and (0,2). The ">" side is above this line.

Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting x_1 to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.

Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$.

Connect (4,0) and (0,4). The ">" side is above this line.

Shade in the feasible region.

(2) Graph the objective function by setting the objective function equal to an arbitrary constant (say 20) and graphing it.

For $5x_1 + 2x_2 = 20$, when $x_1 = 0$, then $x_2 = 10$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,10).

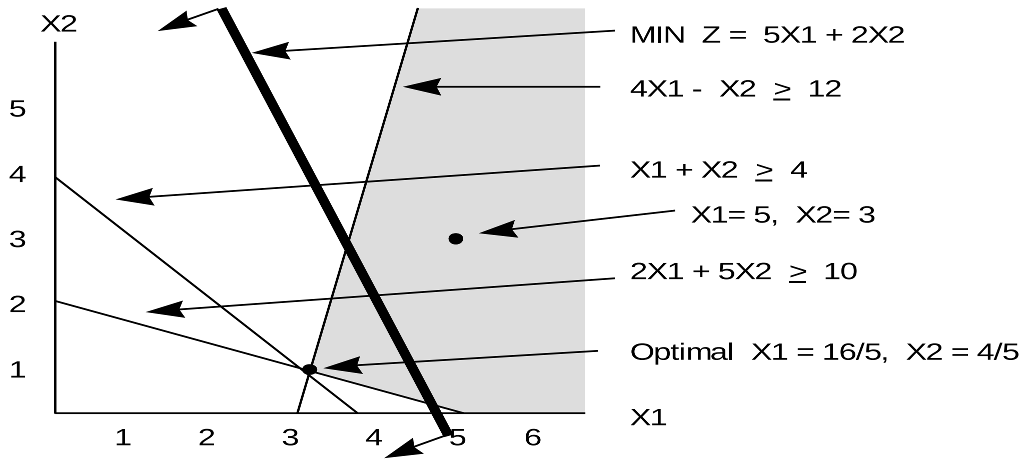
(3) Move the objective function line in the direction which lowers its value until it touches the last point of the feasible region, determined by the last two constraints.

(4) Solve these two equations in two unknowns.

$4x_1 - x_2 = 12$ and $x_1 + x_2 = 4$ Adding these two equations gives: $5x_1 = 16$ or $x_1 = 16/5$. Substituting this into $x_1 + x_2 = 4$ gives: $x_2 = 4/5$.

(5) Solve for $z = 5x_1 + 2x_2 = 5(16/5) + 2(4/5) = 88/5$. Thus the optimal solution is $x_1 = 16/5$; $x_2 = 4/5$;

$z = 88/5$.



c) (5,3) lies in the feasible region, but it is not an extreme point and can never be optimal.

PROBLEM 1

Given the following linear program:

$$
\begin{aligned}
\text{MAX} \quad z &= 3x_1 + 4x_2 \\
\text{s.t.} \quad & 2x_1 + 3x_2 \leq 24 \\
& 3x_1 + x_2 \leq 21 \\
& x_1 + x_2 \leq 9 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

a) Solve the problem graphically.

b) Write the problem in standard form.

c) Given your answer to (a), what are the optimal values of the slack variables.

SOLUTION 1

a) (1) **Graph the constraints.**

Constraint 1: When $x_1 = 0$, then $x_2 = 8$; when $x_2 = 0$, then $x_1 = 12$.

Connect (12,0) and (0,8). The "<" side is below the line.

Constraint 2: When $x_1 = 0$, then $x_2 = 21$; when $x_2 = 0$, then $x_1 = 7$.

Connect (7,0) and (0,21). The "<" side is below the line.

Constraint 3: When $x_1 = 0$, then $x_2 = 9$; when $x_2 = 0$, then $x_1 = 9$.

Connect (9,0) and (0,9). The "<" side is below the line.

Shade in the feasible region.

(2) Graph the objective function by setting the objective function equal to any arbitrary value (say 12) and graphing it.

For $3x_1 + 4x_2 = 12$, when $x_2 = 0$, x1 = 4; when $x_1 = 0$, $x_2 = 3$. Connect (4,0) and (0,3), the thick graphed line.

(3) Move the objective function line parallel to itself in the direction that increases its value (upward) until it touches the last point of the feasible region. It is at the intersection of the first and third constraint lines.

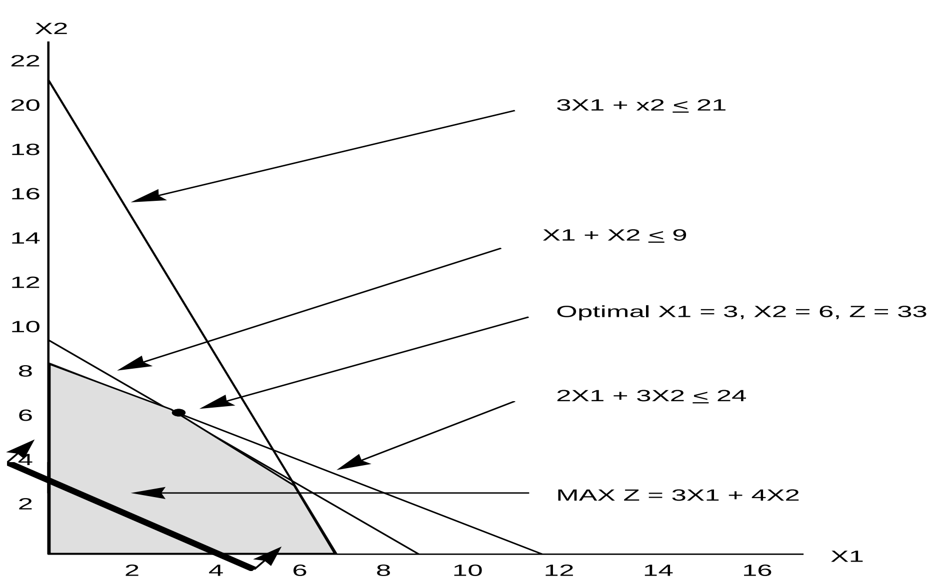
(4) Solve these two equations in two unknowns:

$$\begin{aligned} 2x_1 + 3x_2 &= 24 & \longrightarrow \quad 2x_1 + 3x_2 &= 24 \\ x_1 + x_2 &= 9 & \longrightarrow \quad 2x_1 + 2x_2 &= 18 \\ & & x_2 &= 6 \end{aligned}$$

Substituting into $x_1 + x_2 = 9$, then $x_1 = 3$.

(5) Solve for z: $z = 3x_1 + 4x_2 = 3(3) + 4(6) = 33$. Thus the optimal solution is

$x_1 = 3$, $x_2 = 6$, $z = 33$.



b) To write the problem in standard form, since each constraint is a "$\leq$" constraint, add a slack variable to each constraint.

$$\begin{aligned}
\text{MAX } z &= 3x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t. } \quad & 2x_1 + 3x_2 + s_1 = 24 \\
& 3x_1 + x_2 + s_2 = 21 \\
& x_1 + x_2 + s_3 = 9 \\
& x_1, x_2, s_1, s_2 \geq 0
\end{aligned}$$

c) Since the optimal solution was $x_1 = 3$, $x_2 = 6$, then substituting these values into the above equations gives:

$$s_1 = 24 - 2(3) - 3(6) = 0$$

$$s_2 = 21 - 3(3) - 1(6) = 6$$

$$s_3 = 9 - 1(3) - 1(6) = 0$$

Problem 2

Given the following linear program:

$$\begin{aligned}
\text{MIN } z &= 5x_1 + 2x_2 \\
\text{s.t. } \quad & 2x_1 + 5x_2 \geq 10 \\
& 4x_1 - x_2 \geq 12 \\
& x_1 + x_2 \geq 4 \\
& x_1, x_2 \geq 0
\end{aligned}$$

a. Solve graphically for the optimal solution.
b. How does one know that although $x_1 = 5$, $x_2 = 3$ is a feasible solution for the constraints, it will never be the optimal solution no matter what objective function is imposed?
c. Solve for the optimal solution using a spreadsheet.

<u>SOLUTION 2</u>

a) (1) **Graph the constraints.**

Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$.

Connect (5,0) and (0,2). The ">" side is above this line.

Constraint 2: When x_2 = 0, then x_1 = 3. But setting x_1 to 0 will yield x_2 = -12, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when x_1 = 5, then x_2 = 8. Connect (3,0) and (5,8). The ">" side is to the right.

Constraint 3: When x_1 = 0, then x_2 = 4; when x_2 = 0, then x_1 = 4.

Connect (4,0) and (0,4). The ">" side is above this line.

Shade in the feasible region.

(2) Graph the objective function by setting the objective function equal to an arbitrary constant (say 20) and graphing it.

For $5x_1 + 2x_2 = 20$, when x_1 = 0, then x_2 = 10; when x_2= 0, then x_1 = 4. Connect (4,0) and (0,10).

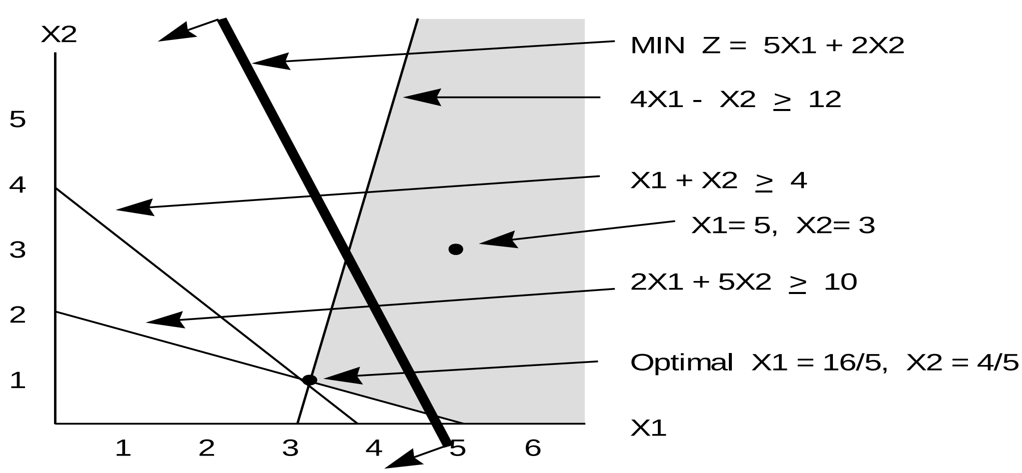
(3) <u>Move the objective function line</u> in the direction which lowers its value until it touches the last point of the feasible region, determined by the last two constraints.

(4) <u>Solve these two equations in two unknowns</u>.

$4x_1 - x_2 = 12$ and $x_1 + x_2 = 4$ Adding these two equations gives: $5x_1 = 16$ or $x_1 = 16/5$. Substituting this into $x_1 + x_2 = 4$ gives: $x_2 = 4/5$.

(5) <u>Solve for z</u> $= 5x_1 + 2x_2 = 5(16/5) + 2(4/5) = 88/5$. Thus the optimal solution is $x_1 = 16/5$; $x_2 = 4/5$;

$z = 88/5$.



b) (5,3) lies in the feasible region, but it is not an extreme point and can never be optimal.

	A	B	C	D
1		LHS Coefficients		
2	Constraints	X1	X2	RHS
3	#1	2	5	10
4	#2	4	-1	12
5	#3	1	1	4
6	Obj.Func.Coeff.	5	2	
7				
8		Decision Variables		
9		X1	X2	
10	Dec.Var.Values			
11				
12	Minimized Objective Function		=B6*B10+C6*C10	
13				
14	Constraints	Amount Used		Amount Avail.
15	#1	=B3*B10+C3*C10	>=	=D3
16	#2	=B4*B10+C4*C10	>=	=D4
17	#3	=B5*B10+C5*C10	>=	=D5

c) Spreadsheet showing data and formulas

Steps in Using Excel Solver:

Step 1: Select the <u>Tools</u> pull-down Menu.

Step 2: Select the <u>Solver</u> option.

Step 3: When the <u>Solver Parameters</u> dialog box appears:

Enter C12 in the <u>Set Target Cell</u> box.

Select the <u>Min</u> option.

Enter B10:C10 in <u>By Changing Cells</u> box.

Choose <u>Add</u>.

Step 4: When <u>Add Constraint</u> dialog box appears:

Enter B15:B17 in <u>Cell Reference</u> box.

Select >=.

Enter D15:D17 in <u>Constraint</u> box.

Choose <u>OK</u>.

Step 5: When the <u>Solver Parameters</u> dialog box appears:

Choose <u>Options</u>.

Step 6: When the <u>Solver Options</u> dialog box appears:

Select <u>Assume Linear Model</u>.

Select Assume Non-Negative.

Choose OK.

Step 7: When Solver Parameters dialog box appears:

Choose Solve.

Step 8: When Solver Results dialog box appears:

Select Keep Solver Solution.

Choose OK to produce optimal solution output.

	A	B	C	D
8		Decision Variables		
9		X1	X2	
10	Dec.Var.Values	3.20	0.800	
11				
12	Minimized Objective Function		17.600	
13				
14	Constraints	Amount Used		Amount Avail.
15	#1	10.4	>=	10
16	#2	12	>=	12
17	#3	4	>=	4

PROBLEM 3

Given the following linear program:

$$\text{MAX } z = 4x_1 + 5x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 3x_2 &\leq 22 \\ -x_1 + x_2 &\leq 4 \\ x_2 &\leq 6 \\ 2x_1 - 5x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

NOTE: If a constraint's righthand-side value is 0, the constraint line will pass through the origin ($x_1 = 0$, $x_2 = 0$). This is the case with the fourth constraint above.

a) Solve the problem by the graphical method.

b) What would be the optimal solution if the second constraint were $-x_1 + x_2 = 4$?

c) What would be the optimal solution if the first constraint were $x_1 + 3x_2 \geq 22$?

SOLUTION 3

a) (1) <u>Graph</u> <u>the</u> <u>constraints</u>.

Constraint 1: When $x_1 = 0$, $x_2 = 22/3$; when $x_2 = 0$, then $x_1 = 22$. Connect (22,0) and (0,22/3). The "<" side is below this line.

Constraint 2: When $x_1 = 0$, then $x_2 = 4$. Setting x_2 to 0 would give $x_1 = -4$, which is outside the graph. Set x_2 to a number greater than 4 and solve for x_1. When $x_2 = 6$, then $x_1 = 2$. Connect (0,4) and (2,6). (0,0) is on the "<" side.

Constraint 3: This is a horizontal line through $x_2 = 6$.

Constraint 4: When $x_2 = 0$, then $x_1 = 0$; Set x_1 to any positive constant and solve for x_2. When $x_1 = 5$, then $x_2 = 2$. Connect the points (0,0) and (5,2).

To determine the "<" side select any arbitrary point on one side of the line and substitute into the inequality. Arbitrarily choosing (0,5), this gives 2(0) - 5(5) = -25. Thus the side containing (0,5) is the "<" side.

<u>Shade in the feasible region</u>.

(2) <u>Graph</u> <u>the</u> <u>objective</u> <u>function</u> by setting it to an arbitrary value, say 20.

For $4x_1 + 5x_2 = 20$, when $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 5$. Connect with a broken line the points (5,0) and (0,4).

(3) <u>Move the objective function line parallel to itself</u> in the direction which increases its value until it touches the last point of the feasible region. This is at the intersection of the first and fourth constraints.

(4) <u>Solve these two equations in two unknowns</u>:

$$x_1 + 3x_2 = 22 \longrightarrow 2x_1 + 6x_2 = 44$$
$$2x_1 - 5x_2 = 0 \longrightarrow 2x_1 - 5x_2 = 0$$

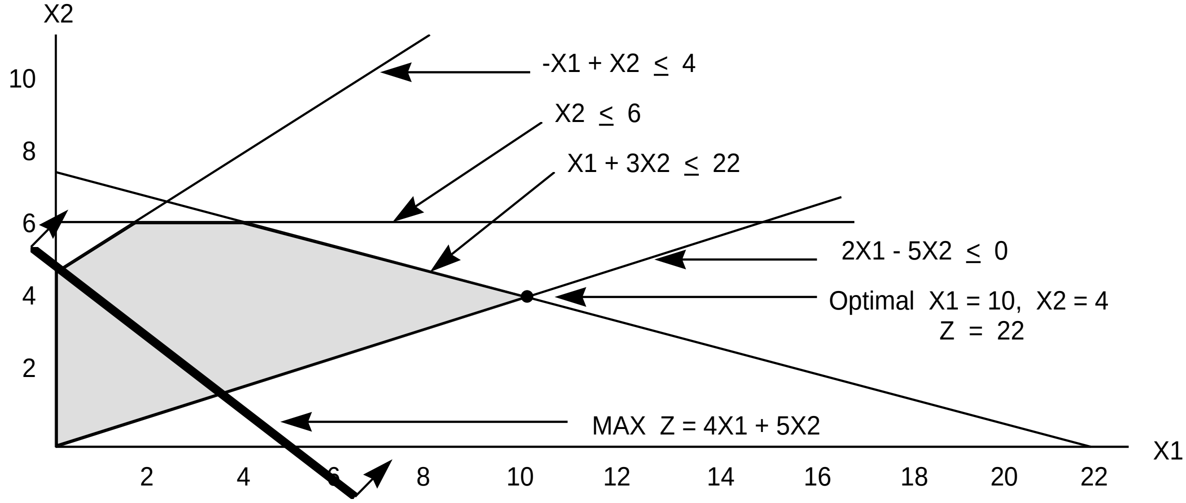
Subtracting the second equation from the first yields: $11x_2 = 44$ or $x_2 = 4$.

Substituting $x_2 = 4$ into the first equation gives $x_1 = 10$.

(5) <u>Substitute for z</u> $= 4x_1 + 5x_2 = 4(10) + 5(4) = 60$. Thus the optimal solution is $x_1 = 10$; $x_2 = 4$; $z = 60$.

b) The feasible region is now the line segment of $-x_1 + x_2 = 4$ between (0,4) and (2,6). (2,6) now gives the optimal solution.

c) The feasible region is now the triangular section between (4,6), (15,6), and (10,4). (15,6) is now the optimal solution.



PROBLEM 4

Show graphically why the following two linear programs do not have optimal solutions and explain the difference between the two.

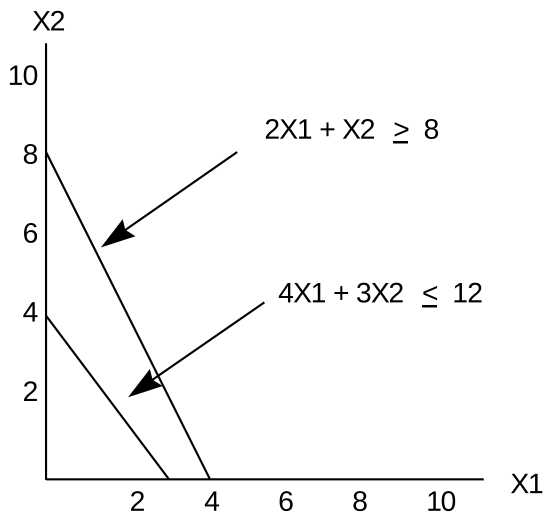
(a) MAX $z = 2x_1 + 6x_2$	(b) MAX $z = 3x_1 + 4x_2$
s.t. $4x_1 + 3x_2 \leq 12$	s.t. $x_1 + x_2 \geq 5$
$2x_1 + x_2 \geq 8$	$3x_1 + x_2 \geq 8$
$x_1, x_2 \geq 0$	$x_1, x_2 \geq 0$

SOLUTION 4

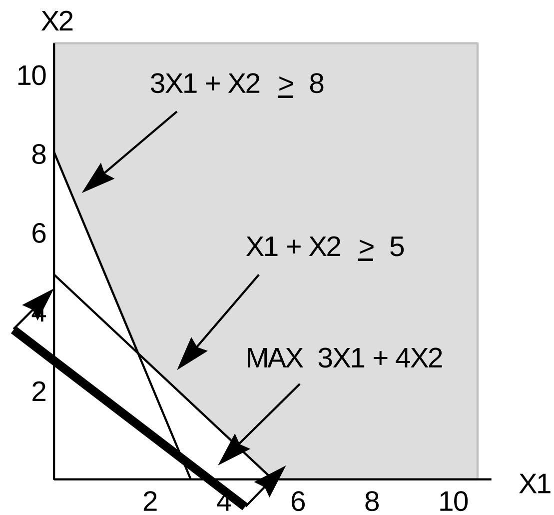
Refer to the graphs on the next page. Note that (a) has no points that satisfy both constraints, hence has no feasible region, and no optimal solution. (a) is infeasible.

Note that in (b) the feasible region is unbounded and the objective function line can be moved parallel to itself without bound so that z can be increased infinitely. (b) is unbounded.

(a)



(b)



<u>PROBLEM 5</u>

Given the following linear program:

$$
\begin{aligned}
\text{MIN } z &= 150x_1 + 210x_2 \\
\text{s.t.} \quad 3.8x_1 + 1.2x_2 &\geq 22.8 \\
x_2 &\geq 6 \\
x_2 &\leq 15 \\
45x_1 + 30x_2 &= 630 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

a) Solve the problem graphically. How many extreme points exist for this problem?

b) What would be the optimal solution if the "=" in the fourth constraint was changed to "$\leq$"?

c) If the "=" in the fourth constraint was changed to "$\geq$", how would the problem be affected?

<u>SOLUTION 5</u>

a) (1) <u>Graph the constraints</u>.

Constraint 1: When $x_1 = 0$, $x_2 = 19$; when $x_2 = 0$, then $x_1 = 6$. Connect (6,0) and (0,9). The ">" side is to the right of this line.

Constraint 2: This is a horizontal line through $x_2 = 6$. The ">" side is above this line.

Constraint 3: This is a horizontal line through $x_2 = 15$. The "<" side is above this line.

Constraint 4: When $x_1 = 0$, $x_2 = 21$; when $x_2 = 0$, then $x_1 = 14$. Connect (14,0) and (0,21).

Shade in the feasible region.

NOTE: The feasible region in this problem is limited to a segment of the line representing the "equal to" constraint. Only two extreme points exist.

(2) Graph the objective function by setting the objective function equal to an arbitrary constant as previously demonstrated or by using the following approach. Scale down the objective coefficients c_1 and c_2 (say, by dividing both by 10 to get 8 and 13, respectively). Now, use x_1's coefficient as a value to plot on the x_2 axis and use x_2's coefficient as a value to plot on the x_1 axis. Connect points (0,15) and (21,0).

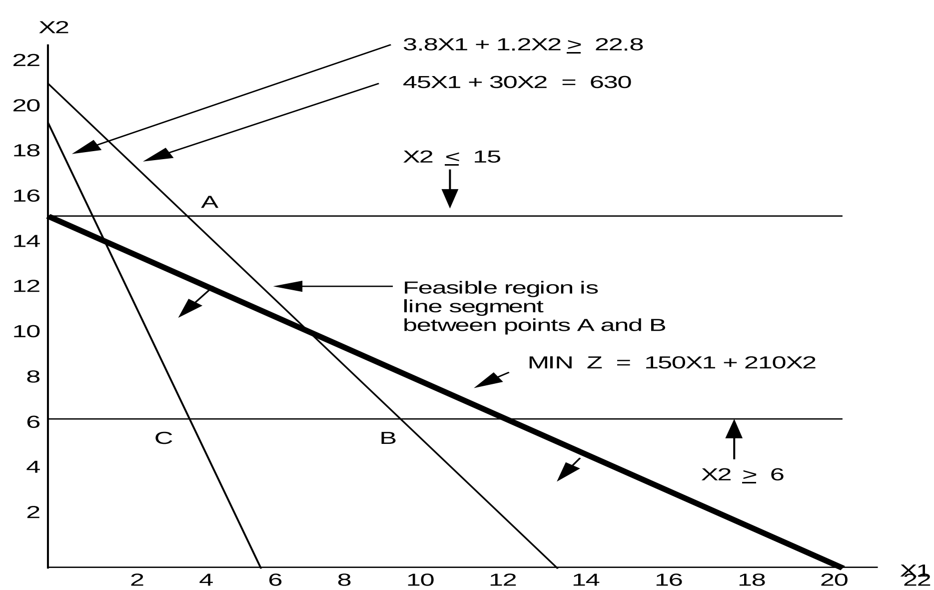
(3) Move the objective function line in the direction that lowers its value until it touches the last point of the feasible region. The point is determined by the second and fourth constraints.

(4) Solve for the unknown x by substituting $x_2 = 6$ into $45x_1 + 30x_2 = 630$, yielding $x_1 = 10$.

(5) Solve for $z = 150x_1 + 210x_2 = 150(10) + 210(6) = 2760$. Thus the optimal solution is $x_1 = 10$, $x_2 = 6$, and $z = 2760$. [See the graph on the next page.]

b) The feasible region is now shaped by all four constraints. The optimal extreme point is determined by the first and second constraints. Solving these two equations in two unknowns, the optimal solution is (4.105,6), point C on the graph.

c) The optimal solution is now (10,6), point B on the graph, and the first constraint is now redundant.



PROBLEM 6

Given the following linear program:

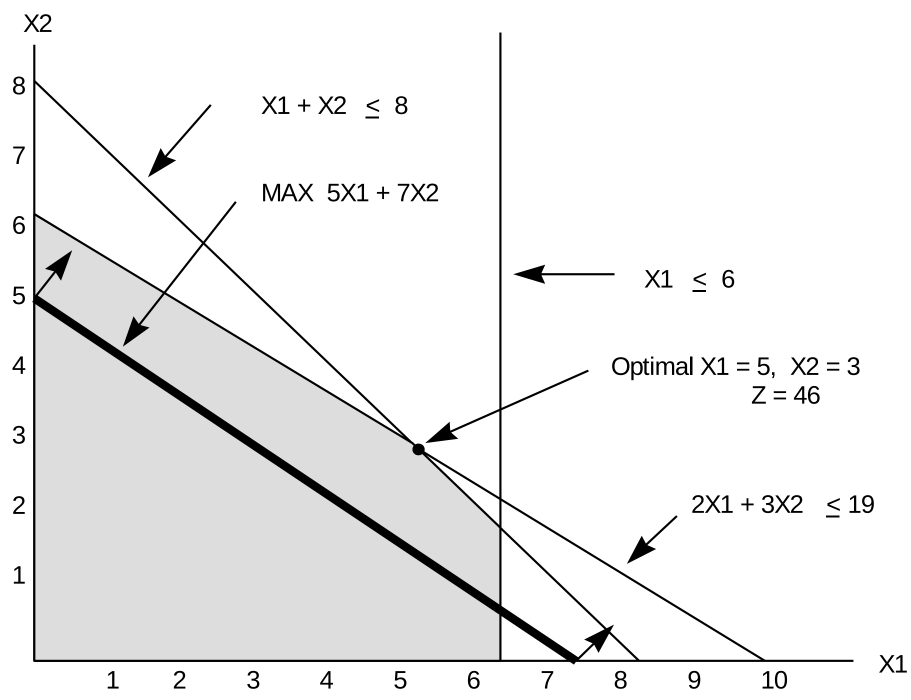
$$\text{MAX } z = 5x_1 + 7x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 &\le 6 \\ 2x_1 + 3x_2 &\le 19 \\ x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solve the problem graphically.

SOLUTION 6

From the graph below we see that the optimal solution occurs at $x_1 = 5$, $x_2 = 3$, and $z = 46$.



PROBLEM 7

A manager of a small fabrication plant must decide on a production schedule of two_new products for the automobile industry. The profit on product 1 is $1(thousand) and on product 2 is $3(thousand).

The manufacture of these products depends largely on the availability of certain subassemblies the plant receives daily from a local distributor. It takes three of these subassemblies for each unit of product 1 and two for each unit of product 2. Twelve such subassemblies are delivered daily.

Further, it takes two hours to make a unit of product 1 and six hours to make a unit of product 2. The plant has assigned only three workers working 8-hour shifts for these new products. Due to limited demand, the manager does not want more than seven units of product 2 produced daily.

a) Formulate this problem as a linear program.

b) Solve graphically for the optimal solution. Describe the set of all optimal solutions. Identify any redundant constraints.

c) Give an optimal daily production schedule that manufactures exactly one unit of product 1.

d) Discuss the applicability of linear programming for this problem.

SOLUTION 7

a) (1) Define variables: x_1 and x_2 = the amount of product 1 and 2 produced daily.

(2) Define objective:

Maximize total daily profits:

MAX $1x_1 + 3x_2$ (in thousands of dollars).

(3) Define constraints:

Subassemblies: Number used daily $\leq$ number available

$$3x_1 + 2x_2 \leq 12$$

Labor: Number of hours used daily $\leq$ (3 men)x(8 hrs./day)

$$2x_1 + 6x_2 \leq 24$$

Product 2: Quantity produced daily $\leq$ specified limit

$$x_2 \leq 7$$

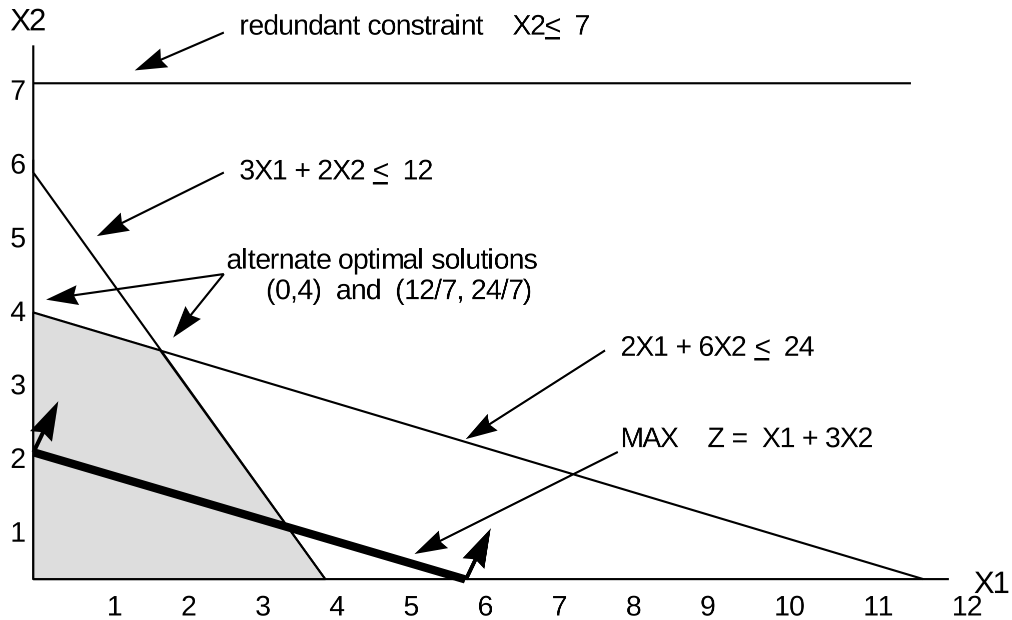
Non-negativity of variables: $x_1, x_2 \geq 0$

Summarizing,

$$\begin{aligned} \text{MAX } z &= 1x_1 + 3x_2 \\ \text{s.t. } 3x_1 + 2x_2 &\leq 12 \\ 2x_1 + 6x_2 &\leq 24 \\ x_2 &\leq 7 \end{aligned}$$

$x_1, x_2 \geq 0$

b) Graphically,



The optimal solution occurs at $x_1 = 0$, $x_2 = 4$ and at $x_1 = 12/7$, $x_2 = 24/7$, and at all points in between on the line $2x_1 + 6x_2 = 24$. At any point on this line, $z = 12$. The $x_2 \leq 7$ constraint does not help shape the feasible region and thus is redundant.

c) On the optimal solution line, $2x_1 + 6x_2 = 24$, when $x_1 = 1$, then $x_2 = 11/3$. Still, $z = 1(1) + 3(11/3) = 12$ (thousand).

d) One must consider whether these variables can be allowed to assume values which are not integers. For continuous production, frequently a fractional value can be considered as "work in progress"; products not finished on one day are simply completed the next day. Thus, LP appears to be appropriate for this problem.

PROBLEM 8

A small company will be introducing a new line of lightweight bicycle frames to be made from special aluminum and steel alloys. The frames will be produced in two models, deluxe and professional, with anticipated unit profits of $10 and $15, respectively.

The number of pounds of aluminum alloy and steel alloy needed per deluxe frame is 2 and 3, respectively. The number of pounds of aluminum alloy and steel alloy needed per professional frame is 4 and 2, respectively. A supplier delivers 100 pounds of the aluminum alloy and 80 pounds of the steel alloy weekly. What is the optimal weekly production schedule?

SOLUTION 8

Let x_1 and x_2 equal the number of deluxe and professional frames produced week

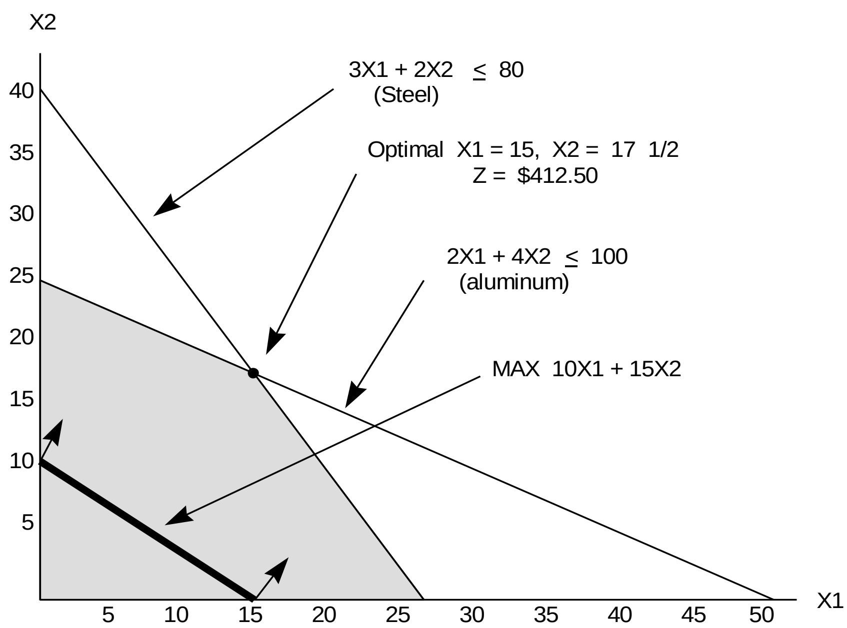
$$\text{MAX } z = 10x_1 + 15x_2$$

$$\text{s.t.} \quad 2x_1 + 4x_2 \leq 100$$

$$3x_1 + 2x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

Solving graphically, the optimal production schedule is to produce x_1 = 15 deluxe frames weekly and x_2 = 17.5 professional frames weekly for an optimal weekly profit of $412.50.



Example

Christabel runs a small creamery shop, named ENENA, in which she gets a shipment of 240 pints (30 gallons) of cream every day. Each container of ice cream requires one-half pint of cream and sells at a profit of 75 F. Each container of sherbet requires one-quarter pint of cream and sells at a profit of 50 F.

a) Define the variables.

b) Set up a mixture chart.

c) Write the resource constraints for cream and minimal constraints and the profit formula that applies to Christabel ENENA's creamery shop.

d) Draw the feasible region for Christabel ENENA's creamery shop.

e) Find the Christabel ENENA's creamery shop optimal production policy.

f) Find the point of intersection between $x = 3$ and $3x + 2y = 171$.

g) Dan has made agreements with his customers that obligate him to produce at least 100 containers of ice cream and 80 containers of sherbet. Find the Dippy Dan's creamery shop optimal production policy.

h) Suppose a competitor drives down ENENA's price of sherbet to the point that Dan only makes a profit of 25F per container. What is ENENA's optimal production policy?

i) Find the point of intersection between $5x + 2y = 17$ and $x + 3y = 6$.

j) Dan's creamery decides to produce raspberry versions of both its ice cream and sherbet lines. Dan is limited to, at most, 600 pounds of raspberries, and he adds one pound of raspberries to each container of ice cream or sherbet. Find the Christabel ENENA's creamery shop optimal production policy. (Assume he still only makes a profit of \$0.25 on a sherbet container and has non-zero minimums.)

Solution

a) Let x be the number of containers of ice cream and y be the number of containers of sherbet.

b)

	Cream (240 pints)	Minimums	Profit
Ice cream, x containers	$\frac{1}{2}$	0	0.75F
Sherbet, y containers	$\frac{1}{4}$	0	0.50F

c) Constraints: $x \geq 0$ and $y \geq 0$ (minimums); $\frac{1}{2}x + \frac{1}{4}y \leq 240(\text{creme})$

Profit formula: $P = \$0.75x + \$0.50y$

d) The minimum constraints $x \geq 0$ and $y \geq 0$ imply we are restricted to quadrant I.

We need to first graph $\frac{1}{2}x+\frac{1}{4}y \le 240.$

The *y*-intercept of $\frac{1}{2}x+\frac{1}{4}y=240$ can be found by substituting $x=0.$

$$\begin{aligned} \tfrac{1}{2}(0)+\tfrac{1}{4}y &= 240 \\ 0+\tfrac{1}{4}y &= 240 \\ \tfrac{1}{4}y &= 240 \Rightarrow y=960 \end{aligned}$$

The *y*-intercept is $(0,960).$

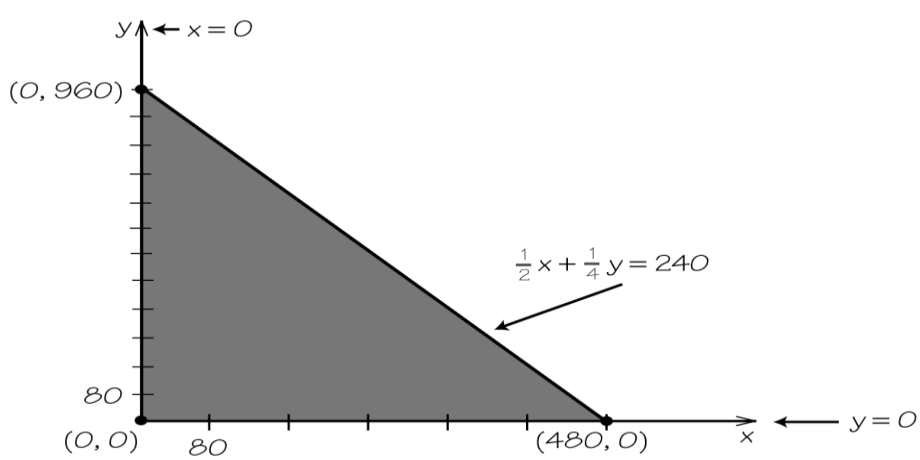
The *x*-intercept of $\frac{1}{2}x+\frac{1}{4}y=240$ can be found by substituting $y=0.$

$$\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{4}(0) &= 240 \\ \tfrac{1}{2}x+0 &= 240 \\ \tfrac{1}{2}x &= 240 \Rightarrow x=480 \end{aligned}$$

The *x*-intercept is $(480,0).$

We draw a line connecting these points. Testing the point $(0,0),$ we have the statement $\frac{1}{2}(0)+\frac{1}{4}(0)\le 240$ or $0\le 240.$

This is a true statement, thus we shade the half-plane containing our test point, the down side of the line in quadrant I.



e) We wish to maximize $0.75xF + 0.50yF.$

Corner Point	Value of the Profit Formula: $0.75xF + 0.50yF.$

(0, 0) (0, 960) (480, 0)	$0.75(0)F + 0.5(0)F = 00F + 00F = 00F$ $0.75(0)F + 0.5(960)F = 00F + 480F = 480F$ $0.75(480)F + 0.5(0)F = 480F + 0F = 480F$

ENENA's optimal production policy: Make 0 containers of ice cream and 960 containers of sherbet for a profit of 480 F.

f)

	Cream (240 pints)	Minimums	Profit
Ice cream, x containers	$\frac{1}{2}$	100	0.75F
Sherbet, y containers	$\frac{1}{4}$	80	0.50F

Constraints: $x \geq 100$ and $y \geq 80$ (minimums); $\frac{1}{2}x + \frac{1}{4}y \leq 240$ (cream)

Profit formula: $P = 0.75xF + 0.50yF.$ $P = \$0.75x + \$0.50y$

The point of intersection between $x = 100$ and $y = 80$ is $(100, 80)$.

The point of intersection between $x = 100$ and $\frac{1}{2}x + \frac{1}{4}y = 240$ can be found by substituting $x = 100$ into $\frac{1}{2}x + \frac{1}{4}y = 240$.

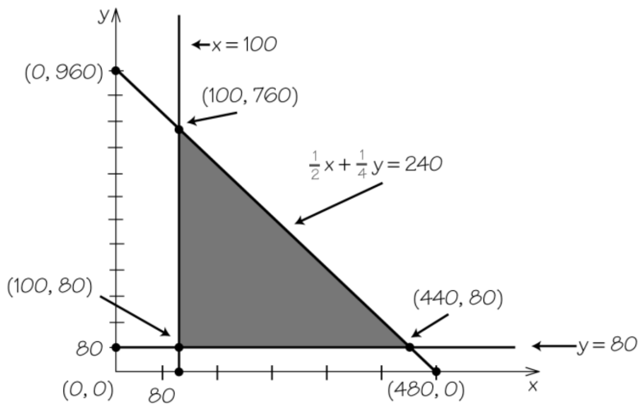
We have $\frac{1}{2}(100) + \frac{1}{4}y = 240 \Rightarrow 50 + \frac{1}{4}y = 240 \Rightarrow \frac{1}{4}y = 190 \Rightarrow y = 760$.

Thus, the point of intersection is $(100, 760)$.

The point of intersection between $y = 80$ and $\frac{1}{2}x + \frac{1}{4}y = 240$ can be found by substituting $y = 80$ into $\frac{1}{2}x + \frac{1}{4}y = 240$.

We have $\frac{1}{2}x + \frac{1}{4}(80) = 240 \Rightarrow \frac{1}{2}x + 20 = 240 \Rightarrow \frac{1}{2}x = 220 \Rightarrow x = 440$.

Thus, the point of intersection is $(440, 80)$.



We wish to maximize $0.75xF + 0.25yF$.

Corner Point	Value of the Profit Formula: $0.75xF + 0.25yF$.
(100, 80)	$0.75(100)F + 0.5(80)F = 75F + 40F = 115F$
(100, 760)	$0.75(100)F + 0.5(760)F = 75F + 380F = 455F$
(440, 80)	$0.75(100)F + 0.5(760)F = 75F + 380F = 455F$

ENENA's optimal production policy: Make 100 containers of ice cream and 760 containers of sherbet for a profit of 455F.

g) The feasible region and corner points will not change. However, we need to now maximize $0.75xF + 0.25yF$.

Corner Point	Value of the Profit Formula: $0.75xF + 0.25yF$.
(100, 80)	$0.75(100)F + 0.25(80)F = 75F + 20F = 95F$
(100, 760)	$0.75(100)F + 0.25(760)F = 75F + 190F = 455F$
(440, 80)	$0.75(100)F + 0.25(760)F = 75F + 190F = 455F$

ENENA's optimal production policy: Make 440 containers of ice cream and 80 containers of sherbet for a profit of 350F.

h) We can find this by multiplying both sides of $x+3y=6$ by $-5,$ and adding the result to

$$5x+2y=17.$$

$$\begin{array}{r} -5x-15y=-30 \\ 5x+\ 2y=17 \\ \hline -13y=-13 \Rightarrow y=\frac{-13}{-13}=1 \end{array}$$

Substitute $y=1$ into $x+3y=6$ and solve to x. We have $x+3(1)=6 \Rightarrow x+3=6 \Rightarrow x=3.$

Thus the point of intersection is therefore $(3,1).$

i) The mixture chart is now as follows.

	Cream (240 pints)	Raspberries (600 lb)	Minimums	Profit
Ice cream, x containers	$\frac{1}{2}$	1	100	0.75F
Sherbet, y containers	$\frac{1}{4}$	1	80	0.25F

Constraints: $x \geq 100$ and $y \geq 80$ $(\text{minimums}); \frac{1}{2}x+\frac{1}{4}y \leq 240(\text{cream}); x+y \leq 600(\text{raspberries})$

Profit formula: $P=\$0.75x+\$0.25y$

The y-intercept of $x+y=600$ is $(0,600),$ and the x-intercept of $x+y=600$ is $(600,0).$

The point of intersection between $x=100$ and $x+y=600$ can be found by substituting $x=100$ into $x+y=600.$ We have $100+y=600 \Rightarrow y=500.$

Thus, the point of intersection is $(100,500).$

The final new corner point is the point of intersection between $x+y=600$ and $\frac{1}{2}x+\frac{1}{4}y=240.$

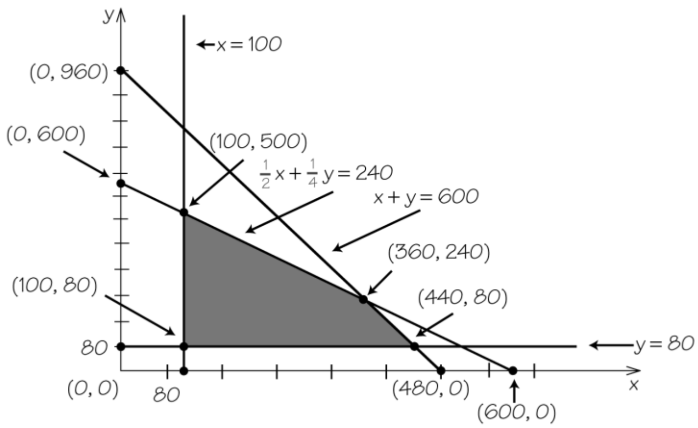
We can find this by multiplying both sides of $\frac{1}{2}x+\frac{1}{4}y=240$ by $-4,$ and adding the result to

$$x+y=600.$$

$$\begin{array}{l} -2x-y=-960 \\ \ \ x+y=600 \\ \hline -x \quad\ \ =-360 \Rightarrow x=360 \end{array}$$

j) Substitute $x=360$ into $x+y=600$ and solve for y. We have, $360+y=600 \Rightarrow y=240.$

Thus, the point of intersection is $(360,240).$



We wish to maximize $0.75xF + 0.25yF$.

Corner Point	Value of the Profit Formula: $0.75xF + 0.25yF$.
(100, 500)	$0.75(100)F + 0.25(500)F = 75F + 125F = 200F$
(360, 240)	$0.75(360)F + 0.25(240)F = 270F + 60F = 4330F$
(440, 80)	$0.75(440)F + 0.25(80)F = 330F + 20F = 350F$
(100, 80)	$0.75(100)F + 0.25(80)F = 75F + 20F = 95F$

ENENA's optimal production policy: Make 440 containers of ice cream and 80 containers of sherbet for a profit of 350F.

Note: With the additional constraint, there was no change because the optimal production policy already obeyed all constraints.

Simplex Method

-The simplex method is an iterative algorithm for efficiently solving LP problems.

-In the simplex method, the search usually starts at the origin and moves to that adjacent corner that increases (for maximization) or decreases (for minimization) the value of the objective function the most.

-The search moves to an even better corner adjacent to the new one.

-The process continues until no further improvement is possible. In each iteration, the objective function improves.

The Simplex Procedure

Step 1: Standardize the problem.

Step 2: Generate an Initial Solution.

Step 3: Test for Optimality. If the solution is optimal, go to
Step 6. Otherwise, go to Step 4.

Step 4: Identify the Incoming and Outgoing Variables.

Step 5: Generate an Improved Solution. Go to Step 3.

Step 6: Check for other Optimal Solutions.

Simplex Process

Step1. Standardize the problem

-Only corner points of feasible solution space are to be checked.

-Because these corner points are defined by the intersection of equations, convert the inequalities (constraints) in the problem statement into equations in order to find the coordinates of the point.

For less-than-or-equal-to constraints ($\leq$)

-Add a slack variable (s)

$$2x_1 + x_2 \leq 40$$

$$2x_1 + x_2 + s_1 = 40$$

-At the origin ($x_1 = 0 \text{ and } x_2 = 0$), s_1 becomes 40 (OK).

For greater-than-or-equal-to constraints ($\geq$)

-Use both a surplus variable (s) and an artificial variable (a) to form an equation.

$$x_1 + x_2 \geq 300$$

$$x_1 + x_2 - s_2 = 300$$

-When $x_1 = 0 \text{ and } x_2 = 0$, $s_2 = -300$ which is not allowable. Therefore, an artificial variable a is introduced into the equation. Therefore,

$$x_1 + x_2 - s_2 + a_2 = 300 \quad \text{where } a_2 \geq 0$$

-The objective of the artificial variable is to simply form an initial solution.

-Because the artificial variable has no meaning so its value must be zero in the final solution.

For constraints with an equality (=)

-Add an artificial variable.

$$3x_1 + x_2 = 10$$

-When $x_1 = 0 \text{ and } x_2 = 0$, 0 = 10 which is unacceptable so an artificial variable is added to the equation.

$$3x_1 + x_2 + a_3 = 10$$

-Once the constraints have been modified, the problem can be written in a standard form.

Standard Form

-All slack, surplus and artificial variables must be written in the objective function.

-The coefficient of the slack and surplus variables is zero in
the objective function because they do not generate any profit or have any costs.

-To ensure that the artificial variable is kept out of the final solution, a very large penalty (M or $-M$) is assigned as a coefficient of the artificial variable in the objective function.

-For a maximization problem, the coefficient of the artificial variable is $-M$.

-For a minimization problem, the coefficient of the artificial variable is M .

Ex:

$Max\, z = 300x_1 + 250x_2$ $\quad$ $Max\, z = 300x_1 + 250x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to: $2x_1 + x_2 \le 40$ $\quad$ Subject to: $2x_1 + x_2 + s_1 = 40$

$x_1 + 3x_2 \le 45$ $\quad$ $x_1 + 3x_2 + s_2 = 45$

$x_1 \le 12$ $\quad$ $x_1 + s_3 = 12$

$x_1, x_2 \ge 0$ $\quad$ $x_1, x_2, s_1, s_2, s_3 \ge 0$

$Min\, z = 45x_1 + 12x_2$ $\quad$ $Min\, z = 45x_1 + 12x_2 + 0s_1 + 0s_2 + Ma_1 + Ma_2$

Subject to: $x_1 + x_2 \ge 300$ $\quad$ Subject to: $x_1 + x_2 - s_1 + a_1 = 300$

$3x_1 \ge 250$ $\quad$ $3x_1 - s_2 + a_2 = 250$

$x_1, x_2 \ge 0$ $\quad$ $x_1, x_2, s_1, s_2, a_1, a_2 \ge 0$

Step2. Generate an initial solution

-Consider the standardized maximization problem.

$$Max\, z = 300x_1 + 250x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to: $2x_1 + x_2 + s_1 = 40$

$$x_1 + 3x_2 + s_2 = 45$$

$$x_1 + s_3 = 12$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

-To obtain the value of z, we solve for $x_1, x_2, s_1, s_2 \text{ and } s_3$ from all equations (constraints).

-With 5 unknown variables and 3 equations, two of five variables are set to zero so that it is possible to solve the system of linear equations in order to get an initial solution or the strating point for the simplex algorithm.

-The variables that have nonzero values are called basic variables and the ones that have values of zero are called nonbasic variables.

-From the problem considered, $x_1 \text{ and } x_2$ are set to zero as nonbasic variables with $s_1, s_2 \text{ and } s_3$ as basic variables.

Therefore, $s_1 = 40, s_2 = 45, s_3 = 12, x_1 = 0 \text{ and } x_2 = 0$ is a simple initial solution.

-The simplex method is to switch only one of the basic variables for one of the nonbasic variables at a time.

Structure of Tableau

Basis	Unit Profit	Quantity	x_1	x_2	s_1	s_2	s_3	Ratio
s_1	0	40	2	1	1	0	0	
s_2	0	45	1	3	0	1	0	
s_3	0	12	1	0	0	0	1	
c_j			300	250	0	0	0	
z_j			0	0	0	0	0	
$c_j - z_j$			300	250	0	0	0	

Example 3

Convert the linear program in Example 1 to its standard form.

Solution

For convenience, the linear program is reproduced below.

$$Max\ Z = 120x_1 + 160x_2$$

$$2x_1 \le 10$$
$$3x_2 \le 11$$
$$x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

To convert the first constraint form an inequality to equality, we introduce the first slack variable s_1 where

$$s_1 = 10 - 2x_1 \text{ or } 2x_1 + s_1 = 10.$$

Similarly after introducing s_2 and s_3, we can convert the linear program into standard form as follows:

$$
\begin{array}{llllll}
\max z = & 120x_1 & +160x_2 & & & \\
s.t. & 2x_1 & & +s_1 & & =10 \\
 & & +3x_2 & & +s_2 & =11 \\
 & x_1 & +x_2 & & +s_3 & =5 \\
 & & & x_1, x_2, s_1, s_2, s_3 & & \geq 0
\end{array}
$$

Basic and Nonbasic Variables, and Basic Feasible Solutions

If we define

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ \mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \text{ and } \mathrm{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix},
$$

the constraints of the standard form of a linear program can be simply represented by a system of simultaneous equations $A\mathrm{x} = \mathrm{b}$.

A basic solution to system of m linear equations with n unknowns is found by setting $n-m$ variables to zero and solving the m equations for the remaining m variables. The variables with zero values are referred to as the nonbasic variables and the remaining m variables are called the basic variables. Note that the choice of different nonbasic variables will lead to different solutions. If all basic variables are nonnegative, the solution is called a basic feasible solution. The optimum solution will be one of the basic feasible solutions. Let us illustrate this with an example.

Example 3

Determine a basic feasible solution for the linear program in Example 1.

Solution

The system of equation representing the constraints for this linear program is as follows:

$$
\begin{array}{llllll}
2x_1 & & +s_1 & & & =10 \\
 & +3x_2 & & +s_2 & & =11 \\
x_1 & +x_2 & & & +s_3 & =5
\end{array}
$$

where $n=5$ and $m=3$. To obtain a basic feasible solution we need to set $n-m=2$ nonbasic variables to zero and solve the remaining system of 3×3 linear equations. Let us start with setting values of x_1 and x_2 to zero. We can easily see that the solution to the system becomes

$s_1 = 10$
$s_2 = 11$
$s_3 = 5$

In this solution, all basic variables are nonnegative; therefore the solution is a basic feasible solution.

Relationship between extreme points of a feasible region and basic feasible solutions

To establish the relationship between basic feasible solutions and extreme points of the feasible region, refer to Figure 4. The above basic feasible solution corresponds to the extreme point C at the origin since in this basic feasible solution $x_1 = 0 \text{ and } x_2 = 0$. Alternatively, if we set the values of $s_3 \text{ and } x_2$ to zero, we see that we obtain the basic feasible solution where

$$x_1 = 5, s_1 = 5, \text{and } s_2 = 11$$

which corresponds to the extreme point D.

There is a special relationship between extreme points C and D arising from their adjacency that is relevant to the simplex method. For a linear program with m constraints, two basic feasible solutions are adjacent if they have $m-1$ basic variables in common. In the basic feasible solutions corresponding to adjacent points C and D, the $m-1$ common basic variables are $s_1 = 5, \text{and } s_2 = 11.$

The Simplex Algorithm

The simplex algorithm, instead of evaluating all basic feasible solutions (which can be prohibitive even for moderate-size problems), starts with a basic feasible solution and moves through other basic feasible solutions that successively improve the value of the objective function. The algorithm terminates once the optimal value is reached. Below we present a step-wise description of the simplex algorithm.

1. Convert the linear program into standard form.
2. Obtain a basic feasible solution from the standard form.
3. Determine if the basic feasible solution is optimal.

4. If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.
5. Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

Steps 1 and 2 of the algorithm have been previously discussed. Steps 3, 4 and 5 of the algorithm are best executed with the help of a tableau which is simply a table with a particular format that shows a summary of the key information regarding the linear program. For example the tableau shown in Table 1 below corresponds to the linear program described in Example 1 and the basic feasible solution in Example 3. There are several things to note about Table 1.

1. The first row of the table (also called row 0) corresponds to the objective function where all the variables are on the left-hand side following the format
2. $z - 120x_1 - 160x_2 = 0$
3. The basic feasible solution corresponds to the solution in Example 3. In addition note that variable $z = 0$ is also considered as a basic variable.
4. In this particular example, the initial tableau where the decision variables x_1 and x_2 are considered as nonbasic variables leads to a basic feasible solution due to the fact that all the right hand side variables are nonnegative.
5. The tableau is in proper form which means the solution can be read directly by looking at the tableau and the RHS values. For a tableau to be in proper form it must meet all the following requirements:

one basic variable per row

the coefficient of all basic variables are +1 and the coefficients above and below the basic variables are zero

z is the basic variable for row 0

Table 1. The initial tableau for example on in proper form

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	None
s_2	0	0	3	0	1	0	11	$11/3$
s_3	0	1	1	0	0	1	5	5

In step 3, to determine if a basic feasible solution is optimal, we need to determine if any of the nonbasic variables (who has value zero) can be increased to improve the value of the objective function. For example, in Table 1, since $z = 120x_1 + 160x_2$, increasing either one of the nonbasic variables x_1 and x_2 would increase the value of the objective function value. In the tableau, this equates to looking for negative coefficients in row 0 due to the format the objective function is written. The basic feasible solution shown in Table 1 is therefore not optimal since the coefficients of x_1 and x_2 are less than zero.

To improve the solution, we can increase the value of either x_1 or x_2. We choose to increase x_2 since the value of the objective function increases at a higher rate (160 vs. 120 per unit of increase). The nonbasic variable with the most negative coefficient (in a maximization problem) in row 0, in this case x_2, is called the entering variable and is always selected as the nonbasic variable that becomes a basic variable.

The basic variable that is replaced by the entering variable, also called the leaving variable, is determined by looking at the values in the "Ratio" column in the tableau. The values in this column are simply the ratio of the RHS values divided by the coefficient of the entering variable in that row. The leaving variable is selected to be the basic variable in the row with the smallest ratio. This is the highest value that the entering variable can have and still result in a basic feasible solution. For the tableau shown in Table 1, the leaving variable is s_2 in row 3.

Once the entering and leaving variables are determined, we use elementary row operations (add link?) (EROs) to make the entering variable a basic variable in the row of the leaving variable by making its coefficient 1 in that row and 0 in all other rows. For example, for Table 1 where the entering and leaving variables are x_2 and s_2 respectively, after the EROs, the tableau is shown in Table 2. The tableau shows a new basic feasible solution (note that all RHS are nonnegative) where

$$s_1 = 10$$
$$x_2 = 11/3$$
$$s_3 = 4/3$$

This basic feasible solution corresponds to the adjacent extreme point A in Figure 4 with coordinates $x_1 = 0$ and $x_2 = 11/3$ and objective function value $1760/3$.

Table 2. The tableau for the basic feasible solution corresponding to extreme point A in proper form.

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	0	0	$160/3$	0	$1760/3$	
s_1	0	2	0	1	0	0	10	5
x_2	0	0	1	0	$1/3$	0	$11/3$	None
s_3	0	1	0	0	$-1/3$	1	$4/3$	$4/3$

After a new basic feasible solution is obtained, the algorithm returns to Step 3 to check if the new basic feasible solution is optimal. This cycle continues until the objective function value cannot be increased by increasing the value of any of the nonbasic variables. In other words, in a maximization problem, this is the same as having no negative valued coefficients in row 0.

A note about minimization problems:

It is important to note that the optimality condition of no negative valued coefficients in row 0 is only applicable in maximization problems. In a minimization problem, the optimality condition exists when none of the coefficients in row 0 are positive. Furthermore, in minimization problems, the entering variable is chosen to be the nonbasic variable with the highest positive coefficient in row 0.

Let us illustrate the simplex algorithm by solving the problem presented in Example 1.

Example 4

Solve the linear program in Example 1using the simplex algorithm.

Solution

Step 1:

Convert the linear program into standard form.

The linear program in standard form is

$$
\begin{array}{llllllll}
\max z = & 120x_1 & +160x_2 & & & & \\
s.t. & 2x_1 & & +s_1 & & & =10 \\
 & & +3x_2 & & +s_2 & & =11 \\
 & x_1 & +x_2 & & & +s_3 & =5 \\
 & & & & & x_1, x_2, s_1, s_2, s_3 & \geq 0
\end{array}
$$

Step 2:

Obtain a basic feasible solution from the standard form.

Previously we have shown that the solution where $x_1 = 0$ and $x_2 = 0$ is a basic feasible solution so we will start the algorithm here.

Step 3:

Determine if the basic feasible solution is optimal.

At this step we create the tableau for this basic feasible solution which was initially shown in Table 1. For convenience the table is reproduced as Table 3.

Table 3. The initial tableau in proper form

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	None
s_2	0	0	3	0	1	0	11	$11/3$
s_3	0	1	1	0	0	1	5	5

Step 4:

If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.

The current solution is not optimal. There are negative coefficients in row 0. Since x_2 has the most negative coefficient in row 0 and s_2 has the lowest ratio, the entering and the leaving variables are x_2 and s_2, respectively.

Step 5:

Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

The new basic feasible solution is shown in Table 4, which is the same as Table 2.

Table 4. The tableau for the new basic feasible solution in the first iteration

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	0	0	$160/3$	0	$1760/3$	
s_1	0	2	0	1	0	0	10	5
x_2	0	0	1	0	$1/3$	0	$11/3$	None

s_3	0	1	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{4}{3}$

Step 3:

Determine if the basic feasible solution is optimal.

The basic solution in Table 4 is still not optimal as the objective function value can be increased by increasing the value of x_1.

Step 4:

If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.

In the second iteration, since x_1 has the most (and only) negative coefficient in row 0 and s_3 has the lowest ratio, the entering and leaving variables are x_1 and s_3, respectively.

Step 5:

Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

The new basic feasible solution is shown in Table 5.

Table 5. The tableau for the basic feasible solution in the second iteration .

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	0	0	0	$\frac{40}{3}$	120	$\frac{2240}{3}$	
s_1	0	0	0	1	$\frac{2}{3}$	-2	$\frac{22}{3}$	
x_2	0	0	1	0	$\frac{1}{3}$	0	$\frac{11}{3}$	

x_1	0	1	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$

Step 3:

Determine if the basic feasible solution is optimal.

Since there are no negative coefficients in row 0, we have reached the optimal solution where the objective function value is $2240/3$ and

$$
\begin{aligned}
s_1 &= 22/3 \\
x_2 &= 11/3 \\
x_1 &= 4/3 \\
s_2 &= s_3 = 0
\end{aligned}
$$

Note that all these values can be read from the tableau shown in Table 5. This solution also corresponds to the extreme point B in Figure 4 which was also determined to be optimal using the graphical solution approach.

Finally, the woodworker should build $4/3$ Type I boxes and $11/3$ Type II boxes to maximize his revenue to $746.67.

Basis Column

-Identify the initial solution variable or basic variables.

Unit Profit/Cost Column

-Display coefficients of basic variables in the objective function.

Quantity Column

-Indicate values of basic variables in the solution.

-For the proposed initial solution, this column displays the right hand side quantities of the constraints.

Columns of Decision, Slack and Surplus Variables

-In the initial tableau, values under each of these columns are coefficients of the corresponding variable in each constraint.

-Each value under these columns shows the rate of utilization of the corresponding resources. In other words, it implies how much the value of the relevant basic variable (in the same row) will decrease if the value of the corresponding nonbasic variable increase by 1.

Ratio Column

-This column is used to identify the basic variable that should be removed from the solution (to become a nonbasic variable) without violating any constraint.

-Each value under this column in each row is the ratio of the Quantity to the coefficient of the variable considered in the same row.

Cj Row

-Show the coefficients of all variables in the objective function.

Zj Row

-Show the amount of profit/cost the objective function will be reduced by when one unit of the variable, in each column, is brought into the basis.

Cj-Zj Row

-Display the net impact on the value of the objective function of bringing one unit of each of the column variable into the basis.

-C_j indicates how much will be gained while Z_j shows how much will be simultaneously be lost.

Relationship between extreme points of a feasible region and basic feasible solutions

To establish the relationship between basic feasible solutions and extreme points of the feasible region, refer to Figure 4. The above basic feasible solution corresponds to the extreme point C at the origin since in this basic feasible solution $x_1 = 0 \text{ and } x_2 = 0$. Alternatively, if we set the values of $s_3 \text{ and } x_2$ to zero, we see that we obtain the basic feasible solution where $x_1 = 5, s_1 = 5, \text{ and } s_2 = 11$ which corresponds to the extreme point D in Figure 4.

There is a special relationship between extreme points C and D arising from their adjacency that is relevant to the simplex method. For a linear program with m constraints, two basic feasible solutions are adjacent if they have $m-1$ basic variables in common. In the basic feasible solutions corresponding to adjacent points C and D, the $m-1$ common basic variables are $s_1 = 5, \text{and} s_2 = 11.$

The Simplex Algorithm

The simplex algorithm, instead of evaluating all basic feasible solutions (which can be prohibitive even for moderate-size problems), starts with a basic feasible solution and moves through other basic feasible solutions that successively improve the value of the objective function. The algorithm terminates once the optimal value is reached. Below we present a step-wise description of the simplex algorithm.

6. Convert the linear program into standard form.
7. Obtain a basic feasible solution from the standard form.
8. Determine if the basic feasible solution is optimal.
9. If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.
10. Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

Steps 1 and 2 of the algorithm have been previously discussed. Steps 3, 4 and 5 of the algorithm are best executed with the help of a tableau which is simply a table with a particular format that shows a summary of the key information regarding the linear program. For example the tableau shown in Table 1 below corresponds to the linear program described in Example 1 and the basic feasible solution in Example 3. There are several things to note about Table 1.

6. The first row of the table (also called row 0) corresponds to the objective function where all the variables are on the left-hand side following the format
7. $z - 120x_1 - 160x_2 = 0$

8. The basic feasible solution corresponds to the solution in Example 3. In addition note that variable $z = 0$ is also considered as a basic variable.
9. In this particular example, the initial tableau where the decision variables x_1 and x_2 are considered as nonbasic variables leads to a basic feasible solution due to the fact that all the right hand side variables are nonnegative.
10. The tableau is in proper form which means the solution can be read directly by looking at the tableau and the RHS values. For a tableau to be in proper form it must meet all the following requirements:

one basic variable per row

the coefficient of all basic variables are +1 and the coefficients above and below the basic variables are zero

z is the basic variable for row 0

Table 1. The initial tableau for example on in proper form

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	None
s_2	0	0	3	0	1	0	11	$11/3$
s_3	0	1	1	0	0	1	5	5

In step 3, to determine if a basic feasible solution is optimal, we need to determine if any of the nonbasic variables (who has value zero) can be increased to improve the value of the objective function. For example, in Table 1, since $z = 120x_1 + 160x_2$, increasing either one of the nonbasic variables x_1 and x_2 would increase the value of the objective function value. In the tableau, this equates to looking for negative coefficients in row 0 due to the format the objective function is written. The basic feasible solution shown in Table 1 is therefore not optimal since the coefficients of x_1 and x_2 are less than zero.

To improve the solution, we can increase the value of either x_1 or x_2. We choose to increase x_2 since the value of the objective function increases at a higher rate (160 vs. 120 per unit of increase). The nonbasic variable with the most negative coefficient (in a maximization problem) in row 0, in this case x_2, is called the entering variable and is always selected as the nonbasic variable that becomes a basic variable.

The basic variable that is replaced by the entering variable, also called the leaving variable, is determined by looking at the values in the "Ratio" column in the tableau. The values in this column are simply the ratio of the RHS values divided by the coefficient of the entering variable in that row. The leaving variable is selected to be the basic variable in the row with the smallest ratio. This is the highest value that the entering variable can have and still result in a basic feasible solution. For the tableau shown in Table 1, the leaving variable is s_2 in row 3.

Once the entering and leaving variables are determined, we use elementary row operations (add link?) (EROs) to make the entering variable a basic variable in the row of the leaving variable by making its coefficient 1 in that row and 0 in all other rows. For example, for Table 1 where the entering and leaving variables are x_2 and s_2 respectively, after the EROs, the tableau is shown in Table 2. The tableau shows a new basic feasible solution (note that all RHS are nonnegative) where

$$s_1 = 10$$
$$x_2 = 11/3$$
$$s_3 = 4/3$$

This basic feasible solution corresponds to the adjacent extreme point A in Figure 4 with coordinates $x_1 = 0$ and $x_2 = 11/3$ and objective function value $1760/3$.

Table 2. The tableau for the basic feasible solution corresponding to extreme point A in proper form.

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	0	0	$160/3$	0	$1760/3$	
s_1	0	2	0	1	0	0	10	5
x_2	0	0	1	0	$1/3$	0	$11/3$	None
s_3	0	1	0	0	$-1/3$	1	$4/3$	$4/3$

After a new basic feasible solution is obtained, the algorithm returns to Step 3 to check if the new basic feasible solution is optimal. This cycle continues until the objective function value cannot be increased by increasing the value of any of the nonbasic variables. In other words, in a maximization problem, this is the same as having no negative valued coefficients in row 0.

A note about minimization problems:

It is important to note that the optimality condition of no negative valued coefficients in row 0 is only applicable in maximization problems. In a minimization problem, the optimality condition exists when none of the coefficients in row 0 are positive. Furthermore, in minimization problems, the entering variable is chosen to be the nonbasic variable with the highest positive coefficient in row 0.

Let us illustrate the simplex algorithm by solving the problem presented in Example 1.

Example 4

Solve the linear program in Example 1using the simplex algorithm.

Solution

Step 1:

Convert the linear program into standard form.

The linear program in standard form is

$$
\begin{array}{llllllll}
\max z = & 120x_1 & +160x_2 & & & & \\
s.t. & 2x_1 & & +s_1 & & & =10 \\
 & & +3x_2 & & +s_2 & & =11 \\
 & x_1 & +x_2 & & & +s_3 & =5 \\
 & & & & & x_1, x_2, s_1, s_2, s_3 & \geq 0
\end{array}
$$

Step 2:
Obtain a basic feasible solution from the standard form.

Previously we have shown that the solution where $x_1 = 0$ and $x_2 = 0$ is a basic feasible solution so we will start the algorithm here.

Step 3:
Determine if the basic feasible solution is optimal.

At this step we create the tableau for this basic feasible solution which was initially shown in Table 1. For convenience the table is reproduced as Table 3.

Table 3. The initial tableau in proper form

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	None
s_2	0	0	3	0	1	0	11	$11/3$
s_3	0	1	1	0	0	1	5	5

Step 4:

If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.

The current solution is not optimal. There are negative coefficients in row 0. Since x_2 has the most negative coefficient in row 0 and s_2 has the lowest ratio, the entering and the leaving variables are x_2 and s_2, respectively.

Step 5:

Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

The new basic feasible solution is shown in Table 4, which is the same as Table 2.

Table 4. The tableau for the new basic feasible solution in the first iteration

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	0	0	$160/3$	0	$1760/3$	
s_1	0	2	0	1	0	0	10	5
x_2	0	0	1	0	$1/3$	0	$11/3$	None
s_3	0	1	0	0	$-1/3$	1	$4/3$	$4/3$

Step 3:

Determine if the basic feasible solution is optimal.

The basic solution in Table 4 is still not optimal as the objective function value can be increased by increasing the value of x_1.

Step 4:

If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.

In the second iteration, since x_1 has the most (and only) negative coefficient in row 0 and s_3 has the lowest ratio, the entering and leaving variables are x_1 and s_3, respectively.

Step 5:

Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

The new basic feasible solution is shown in Table 5.

Table 5. The tableau for the basic feasible solution in the second iteration .

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	0	0	0	$40/3$	120	$2240/3$	
s_1	0	0	0	1	$2/3$	-2	$22/3$	
x_2	0	0	1	0	$1/3$	0	$11/3$	
x_1	0	1	0	0	$-1/3$	1	$4/3$	

Step 3:

Determine if the basic feasible solution is optimal.

Since there are no negative coefficients in row 0, we have reached the optimal solution where the objective function value is $2240/3$ and

$$s_1 = 22/3$$
$$x_2 = 11/3$$
$$x_1 = 4/3$$
$$s_2 = s_3 = 0$$

Note that all these values can be read from the tableau shown in Table 5. This solution also corresponds to the extreme point B in Figure 4 which was also determined to be optimal using the graphical solution approach.

Finally, the woodworker should build $4/3$ Type I boxes and $11/3$ Type II boxes to maximize his revenue to $746.67.

Basis Column

-Identify the initial solution variable or basic variables.

Unit Profit/Cost Column

-Display coefficients of basic variables in the objective function.

Quantity Column

-Indicate values of basic variables in the solution.

-For the proposed initial solution, this column displays the right hand side quantities of the constraints.

Columns of Decision, Slack and Surplus Variables

-In the initial tableau, values under each of these columns are coefficients of the corresponding variable in each constraint.

-Each value under these columns shows the rate of utilization of the corresponding resources. In other words, it implies how much the value of the relevant basic variable (in the same row) will decrease if the value of the corresponding nonbasic variable increase by 1.

Ratio Column

-This column is used to identify the basic variable that should be removed from the solution (to become a nonbasic variable) without violating any constraint.

-Each value under this column in each row is the ratio of the Quantity to the coefficient of the variable considered in the same row.

Cj Row

-Show the coefficients of all variables in the objective function.

Zj Row

-Show the amount of profit/cost the objective function will be reduced by when one unit of the variable, in each column, is brought into the basis.

Cj-Zj Row

-Display the net impact on the value of the objective function of bringing one unit of each of the column variable into the basis.

-C_j indicates how much will be gained while Z_j shows how much will be simultaneously be lost.

-If the value of $C_j - Z_j$ is positive, the value of the objective function can be increased by introducing the variable in that column into the solution.

-In the maximization problems, if one or more of the $C_j - Z_j$ values is positive, then the solution is not optimal and it can be improved.

- In the minimization problems, if one or more of the $C_j - Z_j$ values is negative, then the solution is not optimal and it can be improved.

The Value of the Objective Function

-Computed by multiplying each unit profit by its corresponding quantity and then totaling the results.

$$Z = 0(40)+0(45)+0(12) = 0$$

Step3: Test for Optimality

-Examine the $C_j - Z_j$ row in the tableau.

-For the maximization problems, the solution is optimal if every value in the $C_j - Z_j$ row is nonpositive (zero or negative).

- For the minimization problems, the solution is optimal if all values in the $C_j - Z_j$ row are nonnegative (zero or positive).

Step4: Identify the Incoming and Outgoing Variables

Incoming Variable

-An incoming variable is currently a nonbasic variable (the current value is zero) and will be changed to a basic variable (introduced into the solution).

-For the maximization problems, the incoming variable is the variable with the largest positive value(coefficient) in the $C_j - Z_j$ row.

- For the minimization problems, the incoming variable is the variable with the largest negative value in the $C_j - Z_j$ row.

Outgoing Variable

-An outgoing variable is currently a basic variable that is first reduced to zero when increasing the value of the incoming variable and will be changed to a nonbasic variable (removed from the solution).

-To determine the outgoing variable, compute the ratio of the Quantity to the coefficient of the incoming variable for each basis row.

-For both the maximization and minimization problems, the outgoing variable is the basic variable with the smallest ratio.

-The coefficient of the incoming variable in the outgoing row is called the pivot element.

Step5: Generate an Improved Solution

-The solution is improved by introducing the incoming variable into the basis and removing the outgoing variable.

-Transform the row of the outgoing variable, then the other basis rows and finally transform the Z_j and $C_j - Z_j$ rows.

-The C_j row does not change.

Transforming of the outgoing row

-Divide the elements(coefficients) and the Quantity of the outgoing row by the pivot element.

-Replace the unit profit/cost(coefficient in the objective function) of the outgoing variable with the one of the incoming variable and also replace the outgoing variable with the incoming variable.

Transforming of other rows

-Use an elementary row operation (ERO) to make other elements in the column of the incoming variable become zero.

Transforming of the zj row

-For the new Z_j of each column, multiply the Unit Profits

/ Costs in each row by the column coefficients and sum

the results.

Transforming of the $C_j - Z_j$ row

-Subtract Z_j from C_j.

Step3: Test for Optimality (repeat)

Step4: Identify the Incoming and Outgoing Variables

(repeat)

Step5: Generate an Improved Solution (repeat)

Step3: Test for Optimality (repeat)

Step6: Check for Other Optimal Solutions

-If the coefficient of one of the nonbasic variable in the final $C_j - Z_j$ row is zero, then multiple optimal solutions exist.

-The nonbasic variable with the zero value of $C_j - Z_j$ can enter the basis without changing the value of the objective function (creating another feasible solution).

-If the value of $C_j - Z_j$ is positive, the value of the objective function can be increased by introducing the variable in that column into the solution.

-In the maximization problems, if one or more of the $C_j - Z_j$ values is positive, then the solution is not optimal and it can be improved.

- In the minimization problems, if one or more of the $C_j - Z_j$ values is negative, then the solution is not optimal and it can be improved.

The Value of the Objective Function

-Computed by multiplying each unit profit by its corresponding quantity and then totaling the results.

$$Z = 0(40)+0(45)+0(12) = 0$$

Step3: Test for Optimality

-Examine the $C_j - Z_j$ row in the tableau.

-For the maximization problems, the solution is optimal if every value in the $C_j - Z_j$ row is nonpositive (zero or negative).

- For the minimization problems, the solution is optimal if all values in the $C_j - Z_j$ row are nonnegative (zero or positive).

Step4: Identify the Incoming and Outgoing Variables

Incoming Variable

-An incoming variable is currently a nonbasic variable (the current value is zero) and will be changed to a basic variable (introduced into the solution).

-For the maximization problems, the incoming variable is the variable with the largest positive value(coefficient) in the $C_j - Z_j$ row.

- For the minimization problems, the incoming variable is the variable with the largest negative value in the $C_j - Z_j$ row.

Outgoing Variable

-An outgoing variable is currently a basic variable that is first reduced to zero when increasing the value of the incoming variable and will be changed to a nonbasic variable (removed from the solution).

-To determine the outgoing variable, compute the ratio of the Quantity to the coefficient of the incoming variable for each basis row.

-For both the maximization and minimization problems, the outgoing variable is the basic variable with the smallest ratio.

-The coefficient of the incoming variable in the outgoing row is called the pivot element.

Step5: Generate an Improved Solution

-The solution is improved by introducing the incoming variable into the basis and removing the outgoing variable.

-Transform the row of the outgoing variable, then the other basis rows and finally transform the Z_j and $C_j - Z_j$ rows.

-The C_j row does not change.

Transforming of the outgoing row

-Divide the elements(coefficients) and the Quantity of the outgoing row by the pivot element.

-Replace the unit profit/cost(coefficient in the objective function) of the outgoing variable with the one of the incoming variable and also replace the outgoing variable with the incoming variable.

Transforming of other rows

-Use an elementary row operation (ERO) to make other elements in the column of the incoming variable become zero.

Transforming of the zj row

-For the new Z_j of each column, multiply the Unit Profits

/ Costs in each row by the column coefficients and sum
the results.

Transforming of the $C_j - Z_j$ row

-Subtract Z_j from C_j.

Step3: Test for Optimality (repeat)

Step4: Identify the Incoming and Outgoing Variables (Repeat)

Step5: Generate an Improved Solution (repeat)

Step3: Test for Optimality (repeat)

Step6: Check for Other Optimal Solutions

-If the coefficient of one of the nonbasic variable in the final $C_j - Z_j$ row is zero, then multiple optimal solutions exist.

-The nonbasic variable with the zero value of $C_j - Z_j$ can enter the basis without changing the value of the objective function (creating another feasible solution).

Transportation Model

Introduction

If the basic feasible solution of a transportation problem with m origins and n destinations has fewer than m + n – 1 positive x_{ij} (occupied cells), the problem is said to be a degenerate transportation problem.

Degeneracy can occur at two stages:

1. At the initial solution
2. During the testing of the optimal solution

To resolve degeneracy, we make use of an artificial quantity (*d*). The quantity *d* is assigned to that unoccupied cell, which has the minimum transportation cost.

The use of *d* is illustrated in the following example.

Example

Factory	**Dealer**				**Supply**
	1	**2**	**3**	**4**	
A	2	2	2	4	1000
B	4	6	4	3	700
C	3	2	1	0	900
Requirement	900	800	500	400	

Solution.

An initial basic feasible solution is obtained by Matrix Minimum Method.

Table 1

Factory	**Dealer**				**Supply**
	1	**2**	**3**	**4**	
A	2 (900)	2 (100)	2	4	1000
B	4	6 (700)	4	3	700
C	3	2	1 (500)	0 (400)	900
Requirement	900	800	500	400	

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6

Since number of basic variables is less than 6, therefore, it is a degenerate transportation problem.

To resolve degeneracy, we make use of an artificial quantity(d). The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost.

In the above table, there is a tie in selecting the smallest unoccupied cell. In this situation, you can choose any cell arbitrarily. We select the cell C2 as shown in the following table.

Table 2

Factory	**Dealer**				**Supply**
	1	**2**	**3**	**4**	
A	2 (900)	2 (100)	2	4	1000
B	4	6 (700)	4	3	700
C	3	2 (d)	1 (500)	0 (400)	900 + d
Requirement	900	800 + d	500	400	2600 + d

Now, we use the stepping stone method to find an optimal solution.

Calculating opportunity cost

Unoccupied cells	**Increase in cost per unit of reallocation**	**Remarks**
A3	+2 – 2 + 2 – 1 = 1	Cost Increases
A4	+4 – 2 + 2 – 0 = 4	Cost Increases
B1	+4 – 6 + 2 – 2 = –2	Cost Decreases
B3	+4 – 6 + 2 – 1 = –1	Cost Decreases
B4	+3 – 6 + 2 – 0 = –1	Cost Decreases

C1	+3 – 2 + 2 – 2 = 1	Cost Increases

The cell B1 is having the maximum improvement potential, which is equal to -2. The maximum amount that can be allocated to B1 is 700 and this will make the current basic variable corresponding to cell B2 non basic. The improved solution is shown in the following table.

Table 3

Factory	**Dealer**				**Supply**
	1	**2**	**3**	**4**	
A	2 200	2 800	2	4	1000
B	4 700	6	4	3	700
C	3	2 d	1 500	0 400	900
Requirement	900	800	500	400	2600

The optimal solution is

2 X 200 + 2 X 800 + 4 X 700 + 2 X d + 1 X 500 + 0 X 400 = 5300 + 2d.

Notice that *d* is a very small quantity so it can be neglected in the optimal solution. Thus, the net transportation cost is Rs. 5300

Unbalanced Transportation Problem

So far we have assumed that the total supply at the origins is equal to the total requirement at the destinations. Specifically,

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

But in certain situations, the total supply is not equal to the total demand. Thus, the transportation problem with unequal supply and demand is said to be unbalanced transportation problem.

If the total supply is more than the total demand, we introduce an additional column, which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply, an additional row is introduced in the table, which represents unsatisfied demand with transportation cost zero. The balancing of an unbalanced transportation problem is illustrated in the following example.

Example

Plant	**Warehouse**			**Supply**
	W1	**W2**	**W3**	
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Solution:

The total demand is 1000, whereas the total supply is 800.

$$\sum_{i=1}^{m} S_i < \sum_{j=1}^{n} D_j$$

Total supply < total demand.

To solve the problem, we introduce an additional row with transportation cost zero indicating the unsatisfied demand.

Plant	**Warehouse**			**Supply**
	W1	**W2**	**W3**	
A	28	17	26	500
B	19	12	16	300
Unsatisfied demand	0	0	0	200
Demand	250	250	500	1000

Using matrix minimum method, we get the following allocations.

Plant	**Warehouse**	**Supply**

	W1	W2	W3	
A	50 28	17	450 26	~~500~~
B	19	250 12	50 16	~~300~~
Unsatisfied demand	200 0	0	0	~~200~~
Demand	~~250~~	~~250~~	~~500~~	1000

Initial basic feasible solution

50 X 28 + 450 X 26 + 250 X 12 + 50 X 16 + 200 X 0 = 16900.

Maximization In A Transportation Problem

There are certain types of transportation problems where the objective function is to be maximized instead of being minimized. These problems can be solved by converting the maximization problem into a minimization problem.

Example

Surya Roshni Ltd. has three factories - X, Y, and Z. It supplies goods to four dealers spread all over the country. The production capacities of these factories are 200, 500 and 300 per month respectively.

Factory	Dealer				Capacity
	A	**B**	**C**	**D**	
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand	180	320	100	400	

Determine a suitable allocation to maximize the total net return.

Solution.

Maximization transportation problem can be converted into minimization transportation problem by subtracting each transportation cost from maximum transportation cost.

Here, the maximum transportation cost is 25. So subtract each value from 25. The revised transportation problem is shown below.

Table 1

Factory	**Dealer**				**Capacity**
	A	**B**	**C**	**D**	
X	13	7	19	0	200
Y	17	18	15	7	500
Z	11	22	14	5	300
Demand	180	320	100	400	

An initial basic feasible solution is obtained by matrix-minimum method and is shown in the final table.

Final table

Factory	**Dealer**				**Capacity**
	A	**B**	**C**	**D**	
X	13	7	19	200 0	~~200~~
Y	80 17	320 18	100 15	7	~~500~~
Z	100 11	22	14	200 5	~~300~~
Demand	~~180~~	~~320~~	~~100~~	~~400~~	

The maximum net return is

25 X 200 + 8 X 80 + 7 X 320 + 10 X 100 + 14 X 100 + 20 X 200 = 14280.

<u>Prohibited Routes</u>

Sometimes there may be situations, where it is not possible to use certain routes in a transportation problem. For example, road construction, bad road conditions, strike, unexpected floods, local traffic rules, etc. We can handle such type of problems in different ways:

A very large cost represented by M or ∞ is assigned to each of such routes, which are not available.

To block the allocation to a cell with a prohibited route, we can cross out that cell.

The problem can then be solved in its usual way

Example

Consider the following transportation problem.

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16	□	12	200
F_2	14	8	18	160
F_3	26	□	16	90
Demand	180	120	150	450

Solution.

An initial solution is obtained by the matrix minimum method and is shown in the final table.

Final Table

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	50 16	□	150 12	~~200~~
F_2	40 14	120 8	18	~~160~~

F_3	90 26	□	16	~~90~~
Demand	~~180~~	~~120~~	~~150~~	450

Initial basic feasible solution

16 X 50 + 12 X 150 + 14 X 40 + 8 X 120 + 26 X 90 = 6460.

The minimum transportation cost is Rs. 6460.

Time Minimizing Problem

Succinctly, it is a transportation problem in which the objective is to minimize the time. This problem is same as the transportation problem of minimizing the cost, expect that the unit transportation cost is replaced by the time t_{ij}.

Steps

1. Determine an initial basic feasible solution using any one of the following:

   North West Corner Rule

   Matrix Minimum Method

   Vogel Approximation Method

2. Find T_k for this feasible plan and cross out all the unoccupied cells for which$t_{ij \geq T_k}$.

3. Trace a closed path for the occupied cells corresponding to T_k. If no such closed path can be formed, the solution obtained is optimum otherwise, go to step 2.

Example 1

The following matrix gives data concerning the transportation times t_{ij}

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25	30	20	40	45	37	37
O2	30	25	20	30	40	20	22
O3	40	20	40	35	45	22	32
O4	25	24	50	27	30	25	14

Demand	15	20	15	25	20	10	

Solution.

We compute an initial basic feasible solution by north west corner rule which is shown in table 1.

Table 1

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25 15	30 20	20 2	40	45	37	37
O2	30	25	20 13	30 9	40	20	22
O3	40	20	40	35 16	45 16	22	32
O4	25	24	50	27	30 4	25 10	14
Demand	15	20	15	25	20	10	

Here, $t_{11} = 25$, $t_{12} = 30$, $t_{13} = 20$, $t_{23} = 20$, $t_{24} = 30$, $t_{34} = 35$, $t_{35} = 45$, $t_{45} = 30$, $t_{46} = 25$

Choose maximum from t_{ij}, i.e., $T_1 = 45$. Now, cross out all the unoccupied cells that are □ T1. The unoccupied cell (O3D6) enters into the basis as shown in table 2.

Table 2

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25 15	30 20	20 2	40	~~45~~	37	37
O2	30	25	20 13	30 9	40	20	22
O3	40	20	40	35 16	- 45 16	+ 22	32
O4	25	24	~~50~~	27	+ 30 4	- 25 10	14
Demand	15	20	15	25	20	10	

Choose the smallest value with a negative position on the closed path, i.e., 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

Table 3

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25 15	30 20	20 2	40	~~45~~	37	37
O2	30	25	20 13	30 9	40	20	22
O3	40	20	40	35 16	45 6	22 10	32
O4	25	24	~~50~~	27	30 14	25	14
Demand	15	20	15	25	20	10	

Here, T_2 = Max(25, 30, 20, 20, 20, 35, 45, 22, 30) = 45. Now, cross out all the unoccupied cells that are □ T_2.

Table 4

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25 15	30 20	20 2	40	~~45~~	37	37
O2	30	25	20 13	- 30 9	+ 40	20	22
O3	40	20	40	+ 35 16	- 45 6	22 10	32
O4	25	24	~~50~~	27	30 14	25	14
Demand	15	20	15	25	20	10	

By following the same procedure as explained above, we get the following revised matrix.

Table 6

Destination

Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	~~40~~	~~45~~	37	37
O2	30	25	20 (13)	30 (3)	40 (6)	20	22
O3	~~40~~	20	~~40~~	35 (22)	~~45~~	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

T_3 = Max(25, 30, 20, 20, 30, 40, 35, 22, 30) = 40.

Now, cross out all the unoccupied cells that are $\geq T_3$.

Now we cannot form any other closed loop with T_3.

Hence, the solution obtained at this stage is optimal.

Thus, all the shipments can be made within 40 units.