1. Sheet

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Exercise sheet 1 Leonhard Moske

1) fermionic:

$$\Pi|n_{\lambda_1}, n_{\lambda_2}, \dots n_{\lambda_N}\rangle = (a^+_{\lambda_1})^{\Pi(n_{\lambda_1})} (a^+_{\lambda_2})^{\Pi(n_{\lambda_2})} \cdots (a^+_{\lambda_N})^{\Pi(n_{\lambda_N})} (a_{\lambda_1})^{n_{\lambda_1}} (a_{\lambda_2})^{n_{\lambda_2}} \cdots (a_{\lambda_N})^{n_{\lambda_N}}$$

$$(a^+_{\lambda_1})^{n_{\lambda_1}} (a^+_{\lambda_2})^{n_{\lambda_2}} \cdots (a_{\lambda_N})^{n_{\lambda_N}} |0\rangle \quad | \quad a_{\lambda_i}|n_{\lambda_1}, n_{\lambda_2} \dots n_{\lambda_i} \dots n_{\lambda_N}\rangle = (-1)^{\sum_{j<i} n_{\lambda_j}} |n_{\lambda_1}, n_{\lambda_2} \dots n_{\lambda_i}-1 \dots n_{\lambda_N}\rangle$$

$$= (a^+_{\lambda_1})^{\Pi(n_{\lambda_1})} (a^+_{\lambda_2})^{\Pi(n_{\lambda_2})} \cdots (a^+_{\lambda_N})^{\Pi(n_{\lambda_N})} (-1)^{\sum_i \sum_{j<i} n_{\lambda_j}} |0\rangle$$

$$= (-1)^{\sum_i \sum_{j<i} n_{\lambda_j}} |\Pi(n_{\lambda_1}), \Pi(n_{\lambda_2}) \dots \Pi(n_{\lambda_N})\rangle$$

=> the wavefunction of fermions obtains a - sign if $\sum_i \sum_{j<i} n_{\lambda_j} \bmod 2 = 1$ where $\sum_{j<i} n_{\lambda_j}$ is the number of inversions, so $\sum_i \sum_{j<i} n_{\lambda_j}$ is the whole number of inversions

bosonic:

$$\Pi|n_{\lambda_1}, n_{\lambda_2}, \dots n_{\lambda_N}\rangle = (a^+_{\lambda_1})^{\Pi(n_{\lambda_1})} (a^+_{\lambda_2})^{\Pi(n_{\lambda_2})} \cdots (a^+_{\lambda_N})^{\Pi(n_{\lambda_N})} (a_{\lambda_1})^{n_{\lambda_1}} (a_{\lambda})^{n_{\lambda_2}} \cdots (a_{\lambda_N})^{n_{\lambda_N}}$$

$$(a^+_{\lambda_1})^{n_{\lambda_1}} (a^+_{\lambda_2})^{n_{\lambda_2}} \cdots (a_{\lambda_N})^{n_{\lambda_N}} |0\rangle \quad | \quad a_{\lambda_i}|n_{\lambda_1}, n_{\lambda_2} \dots n_{\lambda_i} \dots n_{\lambda_N}\rangle = \sqrt{n_{\lambda_i}} |n_{\lambda_1}, n_{\lambda_2} \dots n_{\lambda_i}-1, \dots n_{\lambda_N}\rangle$$

$$= \prod_i^N \sqrt{n_{\lambda_i}!} \, (a^+_{\lambda_1})^{\Pi(n_{\lambda_1})} (a^+_{\lambda_2})^{\Pi(n_{\lambda_2})} \cdots (a^+_{\lambda_N})^{\Pi(n_{\lambda_N})} |0\rangle$$

$$= \prod_i^N |\Pi(n_{\lambda_1}), \Pi(n_{\lambda_2}) \dots \Pi(n_{\lambda_N})\rangle$$

=> bosonic wavefunctions do not get a sign underpermutation

2) XXZ Hamiltonian

$$H = \sum_{i=1}^{L-1} J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z S_i^z S_{i+1}^z$$

$$S_i^x = 1\!\!1 \otimes 1\!\!1 \dots 1\!\!1 \otimes \frac{\sigma_x}{2} \otimes 1\!\!1 \dots$$

$$a_i^+ = \left(\prod_{k<i} \sigma_k^z\right) \sigma_i^+ \qquad a_j = \left(\prod_{k<j} \sigma_k^z\right) \sigma_j^- \qquad a_i^+ a_i = \frac{1}{2}(\sigma_i^z + 1)$$

$$\sigma_j^+ = \left(\prod_{k<j} \sigma_k^z\right)^{-1} a_i^+ = \prod_{k<j} (\sigma_k^z)^{-1} a_i^+ = \left(\prod_{k<j} \sigma_k^z\right) a_i^+ \quad \text{since}$$

$$\sigma_k^z \cdot \sigma_k^z = 1\!\!1 \Rightarrow \sigma_k^z = {\sigma_k^z}^{-1}$$

$$-i\sigma_i^+ + i\sigma_i^-$$

$$\sigma_i^\pm = \frac{1}{2}(\sigma_i^x \pm i\sigma_i^y)$$

$$\sigma_i^x = \sigma_i^+ + \sigma_i^- \qquad \sigma_i^y = -i(\sigma_i^+ - \sigma_i^-) \qquad \sigma_i^z = 2a_i^+ a_i - 1$$

$$\sigma_i^x = \left(\prod_{k<j} \sigma_k^z\right) a_i^+ + \left(\prod_{k<j} \sigma_k^z\right) a_i$$

$$= \left(\prod_{k<j} (2a_k^+ a_k - 1)\right) a_i^+ + \left(\prod_{k<j} (2a_k^+ a_k - 1)\right) a_i$$

$$\sigma_i^y = -i\left(\prod_{k<j} 2a_k^+ a_k - 1\right)(a_i^+ - a_i)$$

$A \otimes B =$ $a_{11}b_{11}\, a_{11}b_{12}\, a_{11}$ / $a_{11}b_{21}\, a_{11}b_{22}$ / a_{21} a_{21}

$+ C \otimes D =$ $a_{11}b_{11} + c_{11}d_{11}$ / $a_{11}b_{21} + c_{11}d_{21}$

$(A+C) \otimes (B+D)$

$= A \otimes B + C \otimes D$

$$a_i' = -i\left(\prod_{k<j} (2a_k^\dagger a_k - 1)\right)(a_i^\dagger - a_i)$$

$$+ C \otimes D = \begin{matrix} a_{11}b_{11} + c_{11}d_{11} \\ a_{11}b_{21} + c_{11}d_{21} \end{matrix}$$

$$(A+C)\otimes(B+D) = A\otimes B + C\otimes D + C\otimes B + C\otimes D$$

$$S_i^x = \mathbb{1}\otimes \cdots \mathbb{1}\otimes \frac{1}{2}\left(\prod_{k<i} (2a_k^\dagger a_k - 1)\right)(a_i^\dagger + a_i)\otimes \mathbb{1} \cdots$$

$$S_i^x S_{i+1}^x = \frac{1}{4}\mathbb{1}\otimes \cdots \mathbb{1}\otimes \underbrace{\left(\prod_{k<i} (2a_k^\dagger a_k - 1)\right)}_{\lambda}(a_i^\dagger + a_i)\otimes\left(\prod_{k<i+1} (2a_k^\dagger a_k - 1)\right)(a_{i+1}^\dagger + a_{i+1})\otimes \mathbb{1} \cdots$$

$$S_i^y S_{i+1}^y = -\mathbb{1}\otimes \cdots \left(\prod_{k<i} (2a_k^\dagger a_k - 1)\right)(a_i^\dagger - a_i)\otimes\left(\prod_{k<i+1} (2a_k^\dagger a_k - 1)\right)(a_{i+1}^\dagger - a_{i+1})\otimes \mathbb{1} \cdots$$

$$a_i^\dagger a_{i+1}^\dagger + a_i^\dagger a_{i+1}$$
$$a_i a_{i+1}^\dagger + a_i a_{i+1}$$
$$- a_i^\dagger a_{i+1} - a_i a_{i+1}$$
$$+ a_i^\dagger a_{i+1} + a_i a_{i+1}^\dagger$$

$$S_i^x S_{i+1}^x + S_i^y S_{i+1}^y = \frac{1}{2}\mathbb{1}\cdots \mathbb{1}\otimes \lambda a_i^\dagger \otimes \lambda a_{i+1}\otimes \cdots$$
$$+ \frac{1}{2}\mathbb{1}\cdots \mathbb{1}\otimes \lambda a_i \otimes \lambda a_{i+1}^\dagger \otimes \cdots$$

$$S_i^z S_{i+1}^z = \frac{1}{4}\mathbb{1}\otimes \cdots \mathbb{1}\otimes (2a_i^\dagger a_i - 1)\otimes(2a_{i+1}^\dagger a_{i+1} - 1)\otimes \mathbb{1} \cdots$$

$$H = \sum_{i=1}^{L-1} J\left(\frac{1}{2}\lambda_i \lambda_{i+1} a_i^\dagger a_{i+1} + \frac{1}{2}\lambda_i \lambda_{i+1} a_i a_{i+1}^\dagger\right) + \frac{J_z}{4}(2a_i^\dagger a_i - 1)\cdot(2a_{i+1}^\dagger a_{i+1} - 1)$$

3) hermitian => real EV

hermitian matrix $M = M^\dagger$

let $\vec{v}$ be an eigenvector of M with eigenvalue λ

$\Rightarrow M\vec{v} = \lambda\vec{v}$

$\Leftrightarrow M\vec{v} - \lambda\mathbb{1}\vec{v} = \vec{0}$

$\Leftrightarrow (M - \mathbb{1}\lambda)\vec{v} = \vec{0}$

then $M - \mathbb{1}\lambda$ has the kernel $\vec{v}$

↳ if the kernel is not just $\vec{0}$ the determinant is 0

$\Rightarrow \det(M - \mathbb{1}\lambda) = 0$

since M is hermitian

$M\vec{v} = \lambda\vec{v} \Leftrightarrow M^\dagger\vec{v} = \lambda\vec{v}$

$\Leftrightarrow (M^\dagger\vec{v})^\dagger = (\lambda\vec{v})^\dagger$

$\Leftrightarrow \vec{v}^\dagger M = \lambda^* \vec{v}^\dagger$

$\Leftrightarrow \vec{v}^\dagger(M - \lambda^*\mathbb{1}) = 0$

$\Leftrightarrow \det(M - \lambda^*\mathbb{1}) = 0 \mid \det(M - \mathbb{1}\lambda) = 0$

$\Rightarrow \lambda^* = \lambda$ thus

unitary $M^{-1} = M^\dagger \Rightarrow |\lambda| = 1$ (λ is on complex unit circle)

M has eigenvalue λ with eigenvector $\vec{v}$

$M\vec{v} = \lambda\vec{v}$

$\Leftrightarrow \vec{v}^\dagger M^\dagger = \lambda^* \vec{v}^\dagger$

$\Leftrightarrow \vec{v}^\dagger M^{-1} = \lambda^* \vec{v}^\dagger$

$\Leftrightarrow \vec{v}^\dagger \mathbb{1} = \lambda^* \vec{v}^\dagger M$

$\Leftrightarrow \vec{v}^\dagger\left(\frac{1}{\lambda^*}\mathbb{1} - M\right) = \vec{0}$

$\Leftrightarrow \det\left(\frac{1}{\lambda^*}\mathbb{1} - M\right) = 0$

$\Rightarrow \frac{1}{\lambda^*} = \lambda \Rightarrow 1 = \lambda \cdot \lambda^* = |\lambda|^2$

$\Rightarrow 1 = |\lambda|$

4)

n fermions/bosons in N possible modes

fermions:

begin with $a_i^+|0\ldots0\rangle$

there are N possible modes

$\rightarrow |0\ldots1\ldots0\rangle$ is possible N times

$a_i^+|0\ldots1\ldots0\rangle$ is $N-1$ times possible (since one mode is full)

$\vdots$

$|0\ldots1\ldots0\ldots\rangle$ in $N-(n-1)$ times possible

the number of possibilities then is $\frac{N!}{(N-n)!}$

but since the state $|0,1,0,\ldots1\ldots0\rangle$ is generated like this $n!$ times and we can not distinguish the fermions we must devide by $n!$

$$\Rightarrow \frac{N!}{n!(N-n)!}$$

Bosons

every boson can appear in each mode

so N^n