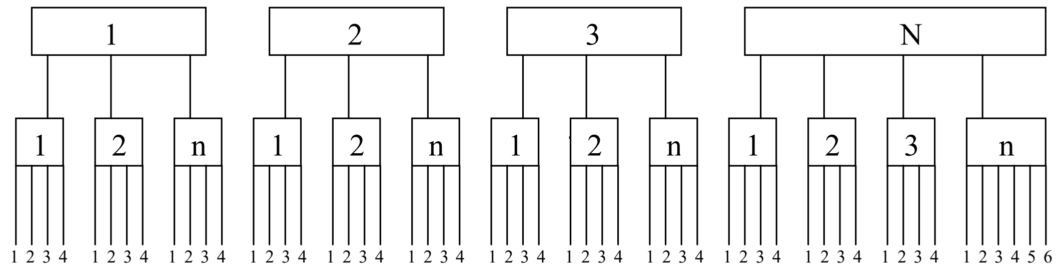
# Multilevel

# Hierarchical Analyses: The Equations

# Hierarchical analysis:

**Figure 1:** Example hierarchical data structure



| Examples: | Education | Organizational | Longitudinal |
|---|---|---|---|
| Level 3 | schools | organizations | classes |
| Level 2 | classes | departments | pupils |
| Level 1 | pupils | individuals | observations |

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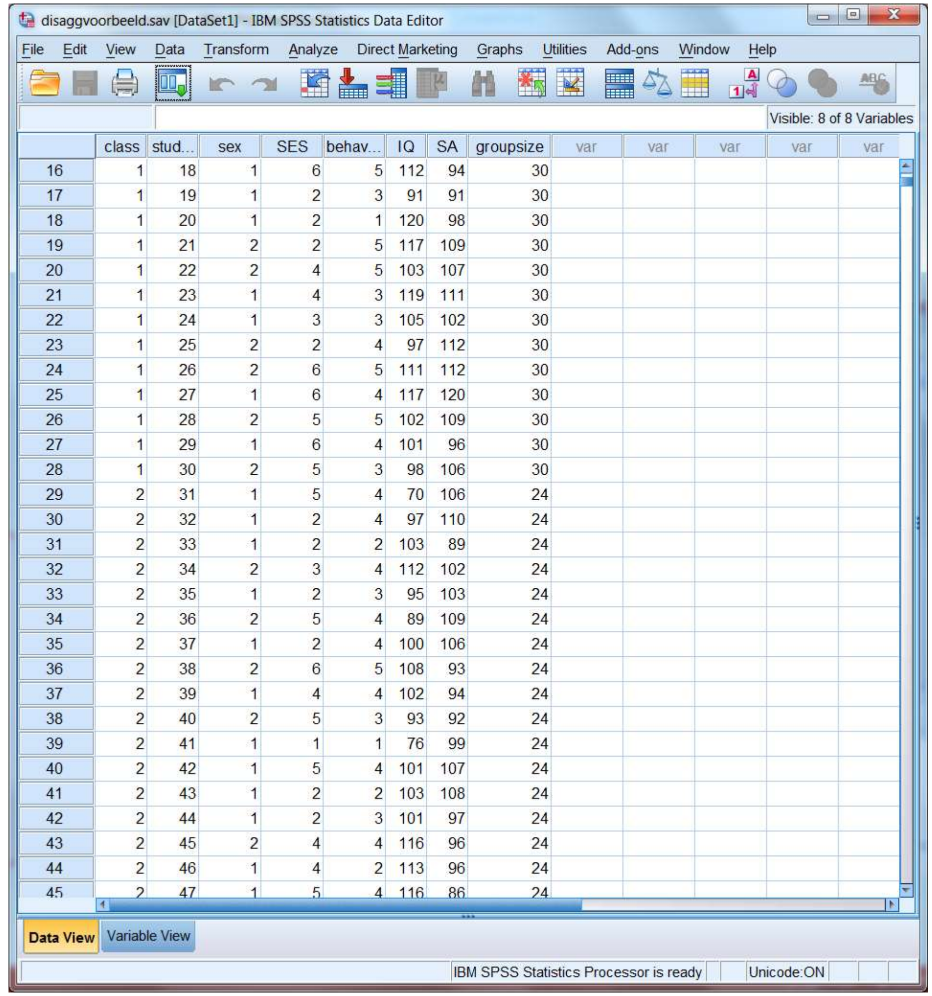
# Why Multilevel?

- Variables at different levels/Dependency.
  - Level 1: Need to correct for dependency in data.




| | class | stud... | sex | SES | behav... | IQ | SA | groupsize | var | var | var | var | var |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 1 | 18 | 1 | 6 | 5 | 112 | 94 | 30 | | | | | |
| 17 | 1 | 19 | 1 | 2 | 3 | 91 | 91 | 30 | | | | | |
| 18 | 1 | 20 | 1 | 2 | 1 | 120 | 98 | 30 | | | | | |
| 19 | 1 | 21 | 2 | 2 | 5 | 117 | 109 | 30 | | | | | |
| 20 | 1 | 22 | 2 | 4 | 5 | 103 | 107 | 30 | | | | | |
| 21 | 1 | 23 | 1 | 4 | 3 | 119 | 111 | 30 | | | | | |
| 22 | 1 | 24 | 1 | 3 | 3 | 105 | 102 | 30 | | | | | |
| 23 | 1 | 25 | 2 | 2 | 4 | 97 | 112 | 30 | | | | | |
| 24 | 1 | 26 | 2 | 6 | 5 | 111 | 112 | 30 | | | | | |
| 25 | 1 | 27 | 1 | 6 | 4 | 117 | 120 | 30 | | | | | |
| 26 | 1 | 28 | 2 | 5 | 5 | 102 | 109 | 30 | | | | | |
| 27 | 1 | 29 | 1 | 6 | 4 | 101 | 96 | 30 | | | | | |
| 28 | 1 | 30 | 2 | 5 | 3 | 98 | 106 | 30 | | | | | |
| 29 | 2 | 31 | 1 | 5 | 4 | 70 | 106 | 24 | | | | | |
| 30 | 2 | 32 | 1 | 2 | 4 | 97 | 110 | 24 | | | | | |
| 31 | 2 | 33 | 1 | 2 | 2 | 103 | 89 | 24 | | | | | |
| 32 | 2 | 34 | 2 | 3 | 4 | 112 | 102 | 24 | | | | | |
| 33 | 2 | 35 | 1 | 2 | 3 | 95 | 103 | 24 | | | | | |
| 34 | 2 | 36 | 2 | 5 | 4 | 89 | 109 | 24 | | | | | |
| 35 | 2 | 37 | 1 | 2 | 4 | 100 | 106 | 24 | | | | | |
| 36 | 2 | 38 | 2 | 6 | 5 | 108 | 93 | 24 | | | | | |
| 37 | 2 | 39 | 1 | 4 | 4 | 102 | 94 | 24 | | | | | |
| 38 | 2 | 40 | 2 | 5 | 3 | 93 | 92 | 24 | | | | | |
| 39 | 2 | 41 | 1 | 1 | 1 | 76 | 99 | 24 | | | | | |
| 40 | 2 | 42 | 1 | 5 | 4 | 101 | 107 | 24 | | | | | |
| 41 | 2 | 43 | 1 | 2 | 2 | 103 | 108 | 24 | | | | | |
| 42 | 2 | 44 | 1 | 2 | 3 | 101 | 97 | 24 | | | | | |
| 43 | 2 | 45 | 2 | 4 | 4 | 116 | 96 | 24 | | | | | |
| 44 | 2 | 46 | 1 | 4 | 2 | 113 | 96 | 24 | | | | | |
| 45 | 2 | 47 | 1 | 5 | 4 | 116 | 86 | 24 | | | | | |

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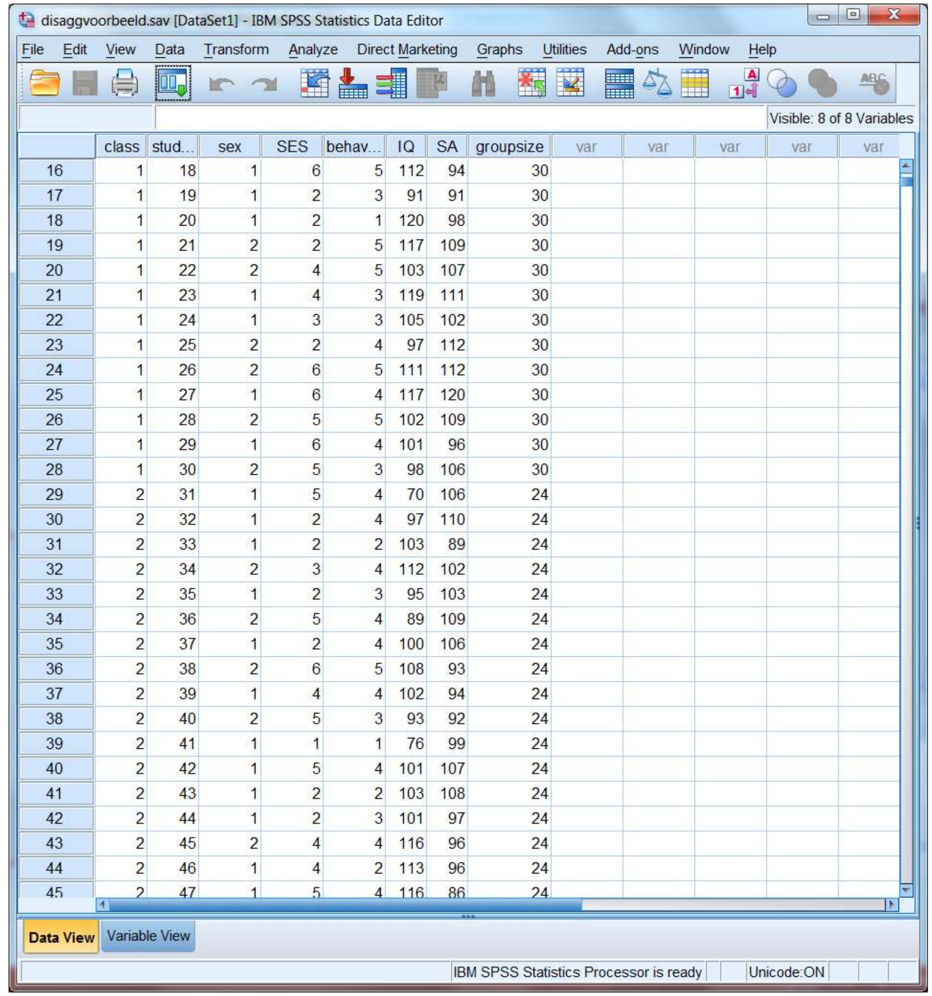


| | class | stud... | sex | SES | behav... | IQ | SA | groupsize | var | var | var | var | var |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 1 | 18 | 1 | 6 | 5 | 112 | 94 | 30 | | | | | |
| 17 | 1 | 19 | 1 | 2 | 3 | 91 | 91 | 30 | | | | | |
| 18 | 1 | 20 | 1 | 2 | 1 | 120 | 98 | 30 | | | | | |
| 19 | 1 | 21 | 2 | 2 | 5 | 117 | 109 | 30 | | | | | |
| 20 | 1 | 22 | 2 | 4 | 5 | 103 | 107 | 30 | | | | | |
| 21 | 1 | 23 | 1 | 4 | 3 | 119 | 111 | 30 | | | | | |
| 22 | 1 | 24 | 1 | 3 | 3 | 105 | 102 | 30 | | | | | |
| 23 | 1 | 25 | 2 | 2 | 4 | 97 | 112 | 30 | | | | | |
| 24 | 1 | 26 | 2 | 6 | 5 | 111 | 112 | 30 | | | | | |
| 25 | 1 | 27 | 1 | 6 | 4 | 117 | 120 | 30 | | | | | |
| 26 | 1 | 28 | 2 | 5 | 5 | 102 | 109 | 30 | | | | | |
| 27 | 1 | 29 | 1 | 6 | 4 | 101 | 96 | 30 | | | | | |
| 28 | 1 | 30 | 2 | 5 | 3 | 98 | 106 | 30 | | | | | |
| 29 | 2 | 31 | 1 | 5 | 4 | 70 | 106 | 24 | | | | | |
| 30 | 2 | 32 | 1 | 2 | 4 | 97 | 110 | 24 | | | | | |
| 31 | 2 | 33 | 1 | 2 | 2 | 103 | 89 | 24 | | | | | |
| 32 | 2 | 34 | 2 | 3 | 4 | 112 | 102 | 24 | | | | | |
| 33 | 2 | 35 | 1 | 2 | 3 | 95 | 103 | 24 | | | | | |
| 34 | 2 | 36 | 2 | 5 | 4 | 89 | 109 | 24 | | | | | |
| 35 | 2 | 37 | 1 | 2 | 4 | 100 | 106 | 24 | | | | | |
| 36 | 2 | 38 | 2 | 6 | 5 | 108 | 93 | 24 | | | | | |
| 37 | 2 | 39 | 1 | 4 | 4 | 102 | 94 | 24 | | | | | |
| 38 | 2 | 40 | 2 | 5 | 3 | 93 | 92 | 24 | | | | | |
| 39 | 2 | 41 | 1 | 1 | 1 | 76 | 99 | 24 | | | | | |
| 40 | 2 | 42 | 1 | 5 | 4 | 101 | 107 | 24 | | | | | |
| 41 | 2 | 43 | 1 | 2 | 2 | 103 | 108 | 24 | | | | | |
| 42 | 2 | 44 | 1 | 2 | 3 | 101 | 97 | 24 | | | | | |
| 43 | 2 | 45 | 2 | 4 | 4 | 116 | 96 | 24 | | | | | |
| 44 | 2 | 46 | 1 | 4 | 2 | 113 | 96 | 24 | | | | | |
| 45 | 2 | 47 | 1 | 5 | 4 | 116 | 86 | 24 | | | | | |

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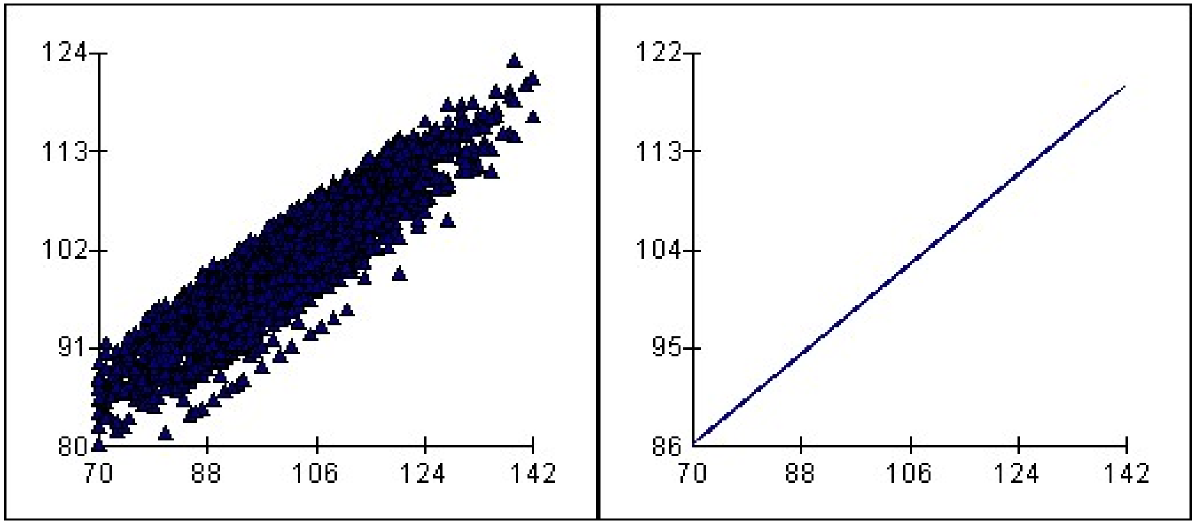
# Hierarchical analysis: Research questions




- Use multilevel because observations are likely to de dependent, however:
  - These type of questions could still be answered with normal regression.
  - Merely involves adjusting obtained s.e.

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# Hierarchical analysis: Research questions

- Could correct for this with the formula by Kish (1965):

- $v_{eff} = v(1 + (n_{clus} - 1)\rho)$, where
- $\rho = \frac{(MS_b - MS_w)}{MS_b + (n_{clus} - 1)MS_w)}$

- This obviously does not provide the nice missing data handling of multilevel

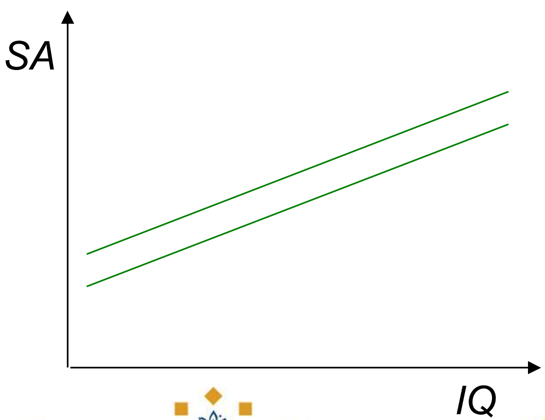
# Why Multilevel?

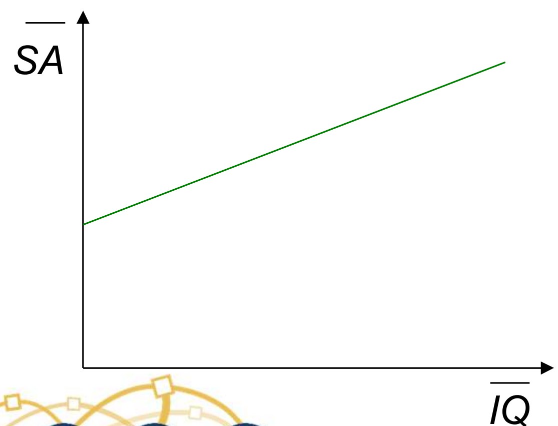
- Variables at different levels/Dependency.
  - Level 1: Need to correct for dependency in data.
  - Level 2: Disaggregation

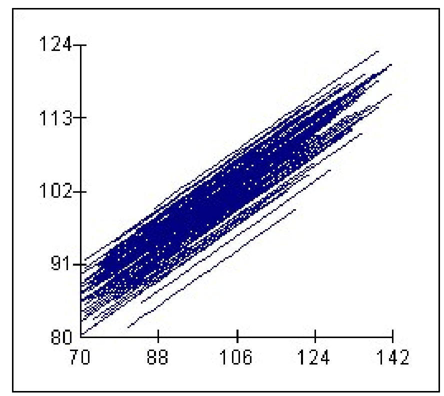
# Multilevel research questions

- Questions with respect to the influence of variables at a higher level on the dependent variable at the lowest level:
  - *mean intelligence* of a class (*MIQ*) as predictor of mean *school achievement* (*SA*); (controlling for individual *IQ*)

$$SA_{ij} = b_0 + b_1 IQ_{ij} + b_2 MIQ_j + u_j + e_{ij}$$



# Why multilevel analysis?

## **Example**: Disaggregation versus multilevel analysis

*disaggvoorbeeld.sav [DataSet1] - IBM SPSS Statistics Data Editor

1 : MIQ 106,79 — Visible: 9 of 9 Variables

| | class | stud... | sex | SES | behav... | IQ | SA | groupsize | MIQ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 6 | 3 | 106 | 97 | 30 | 106,79 |
| 2 | 1 | 3 | 1 | 3 | 4 | 109 | 104 | 30 | 106,79 |
| 3 | 1 | 4 | 2 | 2 | 4 | 104 | 109 | 30 | 106,79 |
| 4 | 1 | 5 | 1 | 6 | 3 | 113 | 128 | 30 | 106,79 |
| 5 | 1 | 6 | 1 | 5 | 2 | 93 | 103 | 30 | 106,79 |
| 6 | 1 | 7 | 2 | 5 | 5 | 115 | 125 | 30 | 106,79 |
| 7 | 1 | 8 | 1 | 2 | 1 | 114 | 62 | 30 | 106,79 |
| 8 | 1 | 9 | 2 | 2 | 4 | 101 | 107 | 30 | 106,79 |
| 9 | 1 | 11 | 1 | 5 | 1 | 103 | 91 | 30 | 106,79 |
| 10 | 1 | 12 | 2 | 4 | 4 | 103 | 113 | 30 | 106,79 |
| 11 | 1 | 13 | 2 | 6 | 5 | 93 | 114 | 30 | 106,79 |
| 12 | 1 | 14 | 2 | 4 | 5 | 118 | 121 | 30 | 106,79 |
| 13 | 1 | 15 | 2 | 3 | 5 | 106 | 104 | 30 | 106,79 |
| 14 | 1 | 16 | 1 | 6 | 4 | 96 | 98 | 30 | 106,79 |
| 15 | 1 | 17 | 1 | 5 | 5 | 123 | 133 | 30 | 106,79 |
| 16 | 1 | 18 | 1 | 6 | 5 | 112 | 94 | 30 | 106,79 |
| 17 | 1 | 19 | 1 | 2 | 3 | 91 | 91 | 30 | 106,79 |
| 18 | 1 | 20 | 1 | 2 | 1 | 120 | 98 | 30 | 106,79 |
| 19 | 1 | 21 | 2 | 2 | 5 | 117 | 109 | 30 | 106,79 |
| 20 | 1 | 22 | 2 | 4 | 5 | 103 | 107 | 30 | 106,79 |
| 21 | 1 | 23 | 1 | 4 | 3 | 119 | 111 | 30 | 106,79 |
| 22 | 1 | 24 | 1 | 3 | 3 | 105 | 102 | 30 | 106,79 |
| 23 | 1 | 25 | 2 | 2 | 4 | 97 | 112 | 30 | 106,79 |
| 24 | 1 | 26 | 2 | 6 | 5 | 111 | 112 | 30 | 106,79 |
| 25 | 1 | 27 | 1 | 6 | 4 | 117 | 120 | 30 | 106,79 |
| 26 | 1 | 28 | 2 | 5 | 5 | 102 | 109 | 30 | 106,79 |

Data View | Variable View — IBM SPSS Statistics Processor is ready | Unicode:ON | Split by class

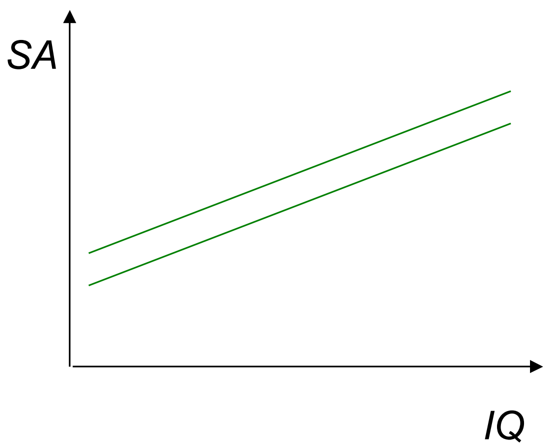
*disaggvoorbeeld.sav [DataSet1] - IBM SPSS Statistics Data Editor

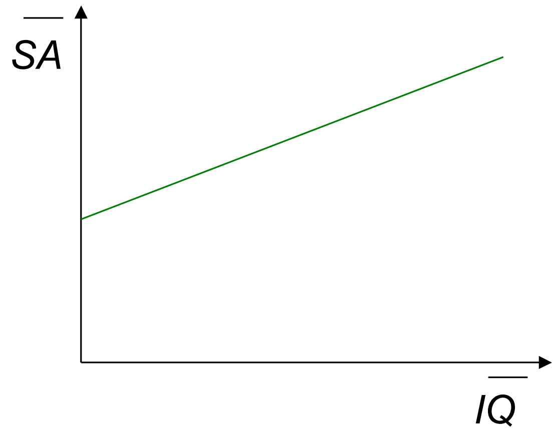
25 : — Visible: 3 of 3 Variables

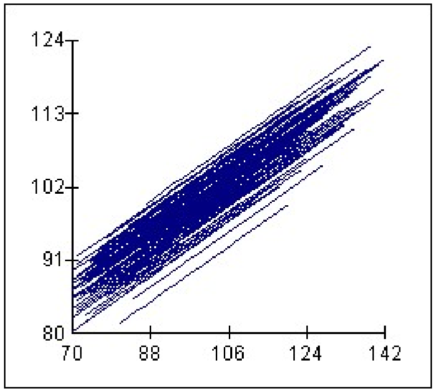
| | class | groupsize | MIQ |
|---|---|---|---|
| 1 | 1 | 30 | 106,79 |
| 2 | 2 | 24 | 102,13 |
| 3 | 3 | 31 | 108,20 |
| 4 | 4 | 29 | 102,31 |
| 5 | 5 | 24 | 100,18 |
| 6 | 6 | 22 | 95,73 |
| 7 | 7 | 21 | 86,33 |
| 8 | 8 | 33 | 106,00 |
| 9 | 9 | 19 | 81,11 |
| 10 | 10 | 23 | 106,04 |
| 11 | 11 | 31 | 9472,00 |
| 12 | 12 | 32 | 108,93 |
| 13 | 13 | 23 | 104,50 |
| 14 | 14 | 28 | 96,23 |
| 15 | 15 | 27 | 96,58 |
| 16 | 16 | 24 | 107,04 |
| 17 | 17 | 29 | 99,88 |
| 18 | 18 | 26 | 105,38 |
| 19 | 19 | 28 | 103,85 |
| 20 | 20 | 32 | 104,63 |
| 21 | 21 | 26 | 89,72 |
| 22 | 22 | 29 | 100,89 |
| 23 | 23 | 30 | 94,67 |
| 24 | 24 | 25 | 99,08 |
| 25 | 25 | 32 | 102,09 |
| 26 | 26 | 31 | 93,73 |
| 27 | 27 | 29 | 96,58 |
| 28 | 28 | 24 | 92,17 |
| 29 | 29 | 24 | 102,63 |
| 30 | 30 | 29 | 107,70 |

Data View | Variable View — IBM SPSS Statistics Processor is ready | Unicode:ON | Split by class

# Multilevel research questions




- This type of question could also be answered with normal regression by running a regression on mean scores:
  - Power obviously an issue.
  - Should means from groups of different size be treated the same?

# Why Multilevel?

- Variables at different levels.
  - Level 1: Need to correct for dependency in data.
  - Level 2: Disaggregation
  - Level 1&2: Cross-level interaction
    - We want to MODEL differences between level 2 units in the effects of level 1 variables.

# Hierarchical Analyses: The Steps

# Analysis Strategy

## Analysis strategy

- Multilevel model with:
  - *p* explanatory variables at the lowest level
  - *q* explanatory variables at the highest level

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + \gamma_{pq} Z_{qj} X_{pij} + u_{pj} X_{pij} + u_{0j} + e_{ij}$$

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# Steps of a multilevel analysis

1. Check whether multilevel is necessary
2. Add all level 1 main effects and interactions
3. Add all level 2 main effects and interactions
4. Check level 1 effects for random slopes
5. If random slopes are present: add cross-level interactions

# Step 1: two level intercept-only model

- Analyze a two-level model with no explanatory variables: the *intercept-only* model:

$$Y_{ij} = \beta_{0j} + e_{ij}$$

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$Y_{ij} = \gamma_{00} + u_{0j} + e_{ij}$$

- $\gamma_{00}$ : the regression intercept
- $u_{0j}$ : residual error at the second level
- $e_{ij}$ : residual error at the first level

- Decomposition of the total variance in two terms
- Should we perform a multilevel analysis?

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# Step 1: two level intercept-only model

$$Y_{ij} = \gamma_{00} + u_{0j} + e_{ij}$$

- $H_0$: individuals in the same group aren't more alike than individuals in different groups

  $H_0$: $\sigma^2_{u_0} = 0$

- $H_1$: individuals in the same group are more alike than individuals in different groups

  $H_1$: $\sigma^2_{u_0} > 0$

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# Intraclass correlation

$$\rho = \frac{\sigma_{u_0}^2}{\sigma_{u_0}^2 + \sigma_e^2}$$

- variance of the second level errors $u_{0j}$ $\sigma_{u_0}^2$
- variance of the first level errors $e_{ij}$ $\sigma_e^2$

- The ‘percentage’ variance at the second level

- The expected correlation between the observations on the dependent variable of two randomly chosen units that are in the same group.

# Step 2: first level variables

Add all first level variables (+ their interactions) with fixed effects:

$$Y_{ij} = \beta_{0j} + \beta_{pj} X_{pij} + e_{ij}$$

$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{pj} = \gamma_{p0}$$

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + u_{0j} + e_{ij}$$

# Step 2: first level variables

Add all first level variables (+ their interactions) with fixed effects:

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + u_{0j} + e_{ij}$$

- $X_{pij}$ : *p* explanatory variables at the first level (e.g., gender, extraversion, and gender*extraversion)
- $H_0$: There is no relation between the dependent variable *Y* and the explanatory variable *X*
- $H_1$: There is a relation between the dependent variable *Y* and the explanatory variable *X* (+ direction)

- Calculate the explained variance at level 1 and at level 2

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## Step 2: first level variables

- Calculate the explained variance at level 1 and level 2.

- Why?

# Step 2: first level variables

- Calculate the explained variance at level 1 and level 2.

- Variables at the lowest level can explain variance at the first level
  - Girls are more popular than boys.

- Variables at the lowest levels can explain variance at the higher levels.
  - Girls are more popular than boys.
  - In a class with a lot of girls the mean popularity is higher.

## Step 3: Second level variables

Add all second level variables (+ their interactions) with fixed effects:

$$Y_{ij} = \beta_{0j} + \beta_{pj} X_{pij} + e_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{0q} Z_{qj} + u_{0j}$$

$$\beta_{pj} = \gamma_{p0}$$

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + u_{0j} + e_{ij}$$

# Step 3: Second level variables

Add all second level variables (+ their interactions) with fixed effects:

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + u_{0j} + e_{ij}$$

- $Z_{qj}$ : *q* explanatory variables at the second level (e.g., teacher experience).
- $H_0$: there is no relation between the explanatory variable Z and the mean score of the dependent variable *Y*
- $H_1$: there is a relation between the explanatory variable *Z* and the mean score of the dependent variable *Y* (+ direction?)
- Calculate the explained variance at level 2

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# Step 4: random slopes

Assess whether any of the slopes of the explanatory variables of the first level has a significant variance between the second level units:

$$Y_{ij} = \beta_{0j} + \beta_{pj} X_{pij} + e_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{0q} Z_{qj} + u_{0j}$$

$$\beta_{pj} = \gamma_{p0} + u_{pj}$$

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + u_{pj} X_{pij} + u_{0j} + e_{ij}$$

# Step 4: random slopes

Assess whether any of the slopes of the explanatory variables of the first level has a significant variance between the second level units:

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + u_{pj} X_{pij} + u_{0j} + e_{ij}$$

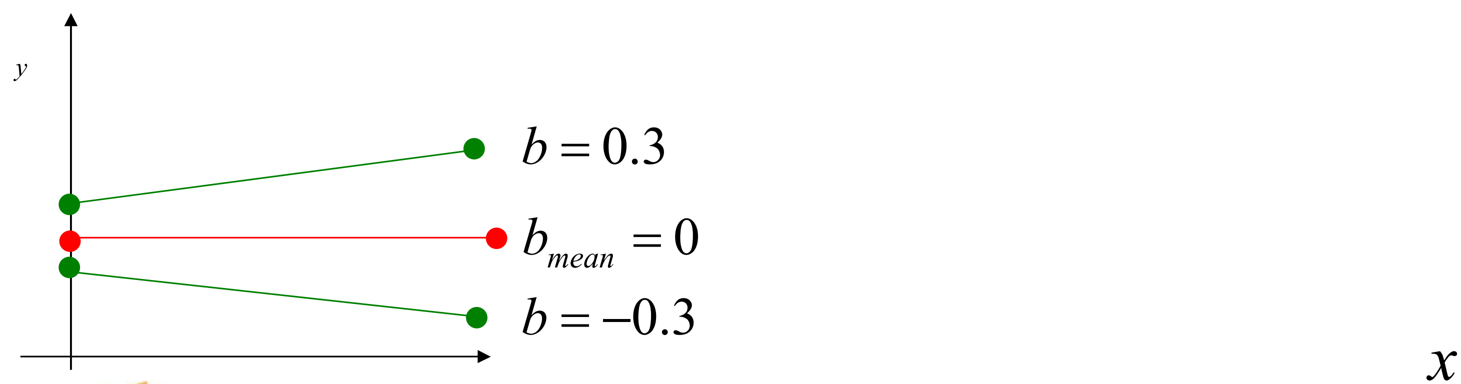
- $u_{pj}$ : second level residuals of the slopes of the first level explanatory variables $X_{pij}$

- $H_0$: the relation between the explanatory variable $X$ and the dependent variable $Y$ is the same within all level two units ($H_0$: $\sigma^2_{u_{1j}} = 0$)

- $H_1$: the relation between the explanatory variable $X$ and the dependent variable $Y$ is not the same within all level two units ($H_1$: $\sigma^2_{u_{1j}} > 0$)

# Remarks on step 4: random slopes

- Testing for random slope variation: variable by variable
- Variables that were omitted in step 2 may be analyzed again:
  - it is possible that an explanatory variable has no significant mean regression slope, but that there is slope variance
- Add all significant slopes simultaneously in a final model.



# Step 5: cross-level interactions

Add cross-level interactions between those level one explanatory variables that had significant slope variation and explanatory variables of the second level:

$$Y_{ij} = \beta_{0j} + \beta_{pj} X_{pij} + e_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{0q} Z_{qj} + u_{0j}$$

$$\beta_{pj} = \gamma_{p0} + \gamma_{pq} Z_{qj} + u_{pj}$$

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + \gamma_{pq} Z_{qj} X_{pij} + u_{pj} X_{pij} + u_{0j} + e_{ij}$$

# Step 5: cross-level interactions

Add cross-level interactions between those level one explanatory variables that had significant slope variation and explanatory variables of the second level:

$$Y_{ij} = \gamma_{00} + \gamma_{p0} X_{pij} + \gamma_{0q} Z_{qj} + \gamma_{pq} Z_{qj} X_{pij} + u_{pj} X_{pij} + u_{0j} + e_{ij}$$

- $Z_{qj} X_{pij}$ : cross-level interactions

- $H_0$: the explanatory variable *Z* can't explain the different relations between the explanatory variable *X* and the dependent variable *Y* in different level two units

- $H_1$: the explanatory variable *Z* explains (a part of) the different relations between the explanatory variable *X* and the dependent variable *Y* in different level two units

- Calculate the explained slope variance

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# Multilevel Regression

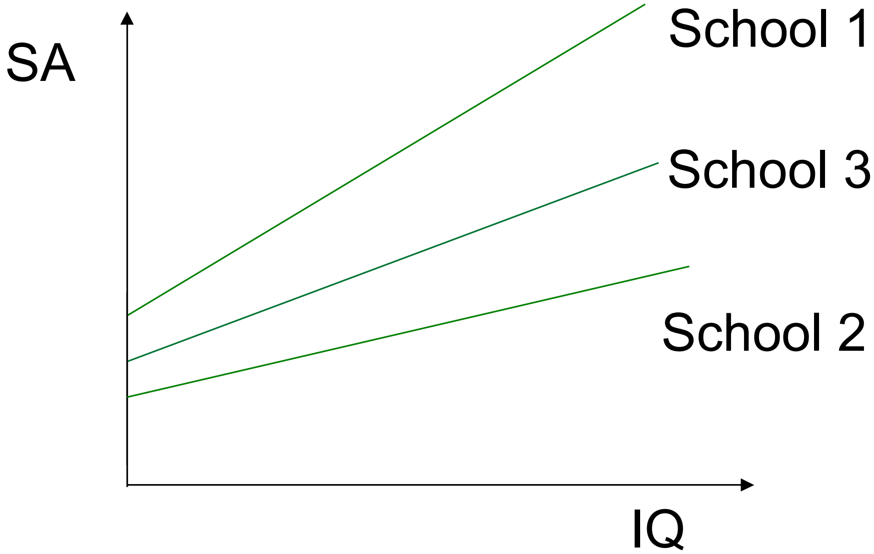
- Level 1: $SA_{ij} = b_{0j} + b_{1j}IQ_{ij} + e_{ij}$
- Level 2: $b_{0j} = \gamma_{00} + \gamma_{01}Schoolsize_j + u_{0j}$

  $b_{1j} = \gamma_{10} + u_{1j}$

Combined:

$$SA_{ij} = \gamma_{00} + \gamma_{10}IQ_{ij} + \gamma_{01}Schoolsize_j + eij + u_{0j} + u_{1j}\ IQ_{ij}$$

# Multilevel Regression




Differences in intercept (partially) explained by school size:

$$b_{0j} = \gamma_{00} + \gamma_{01} Schoolsize_j + u_{0j}$$

So far, no predictor for differences in slope:

$$b_{1j} = \gamma_{10} + u_{1j}$$

Could school size explain differences in slope as well?

$$b_{1j} = \gamma_{10} + \gamma_{11} Schoolsize_j + u_{1j}$$

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# Multilevel Regression

Level 1: $SA_{ij} = b_{0j} + b_{1j}IQ_{ij} + e_{ij}$

Level 2: $b_{0j} = \gamma_{00} + \gamma_{01}Schoolsize_j + u_{0j}$

$$\boxed{b_{1j} = \gamma_{10} + \gamma_{11}Schoolsize_j + u_{1j}}$$

Smaller School → More help → Pupils less reliant on own IQ? → Negative $\gamma_{11}$

OR

Smaller School → Less help → Pupils more reliant on own IQ?

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# Multilevel Regression

- Level 1: $SA_{ij} = b_{0j} + b_{1j}IQ_{ij} + e_{ij}$

- Level 2: $b_{0j} = \gamma_{00} + \gamma_{01}Schoolsize_j + u_{0j}$
  $b_{1j} = \gamma_{10} + \gamma_{11}Schoolsize_j + u_{1j}$

Combined:

$$SA_{ij} = \gamma_{00} + \gamma_{10}IQ_{ij} + \gamma_{01}Schoolsize_j \boxed{+ \gamma_{11}Schoolsize_j IQ_{ij}} + eij + u_{0j} + u_{1jIQij}$$

## Hierarchical Analyses: The Steps

**Update!...Don’t do the steps!!**

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# Maximum Approach

- Going through the steps is a good way to see what is going on, but…

- We always know that things will differ between people/units?
  - So why assume fixed effects at first?
  - Why not go for a full model straight away?

# Maximum Approach

- Going through the steps is a good way to see what is going on, but…
- We always know that things will differ between people/units?
  - So why assume fixed effects at first?
  - Why not go for a full model straight away? (Maximum Approach)
- What are possible advantages of staring with a full model?
- And what are disadvantages?

# Maximum Approach

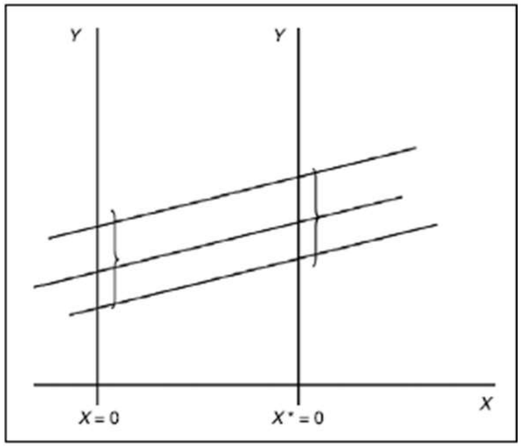



*Figure 4.1* Parallel regression lines, with shift on *X*.

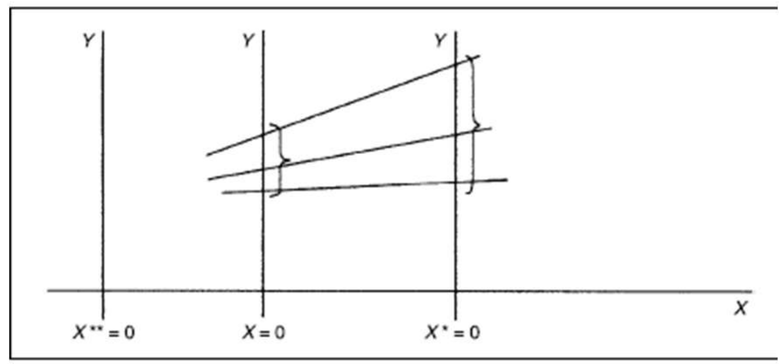
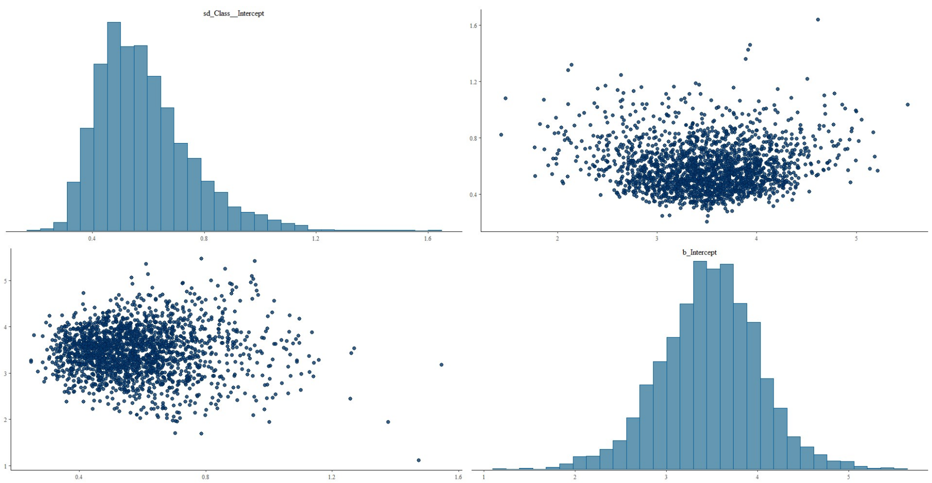
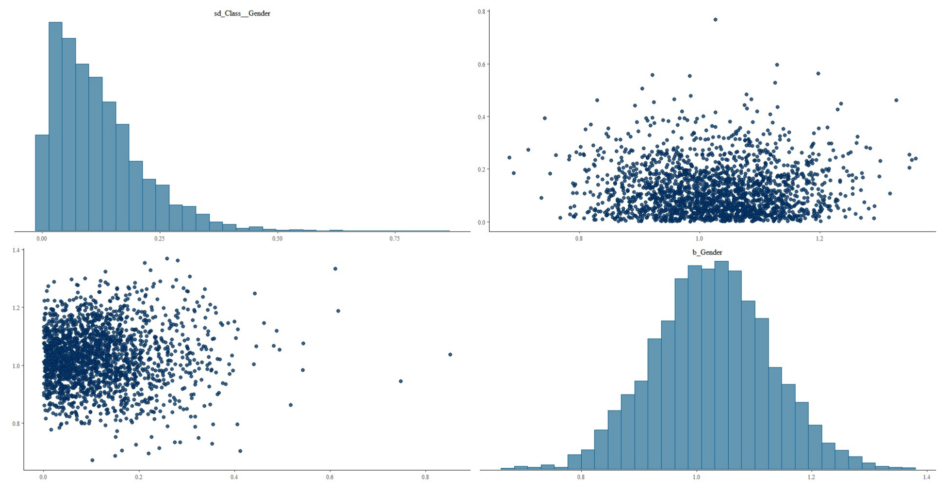
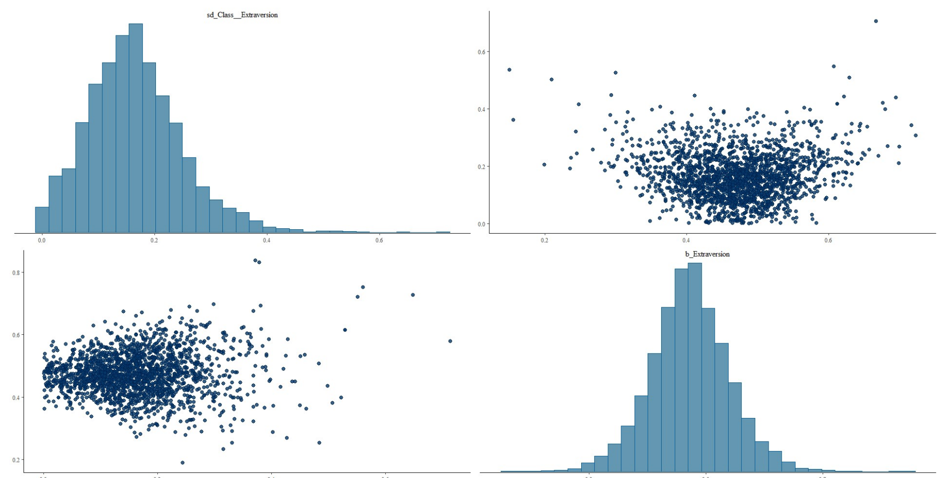



*Figure 4.2* Varying regression lines, with shifts on *X*.

# Maximum Approach

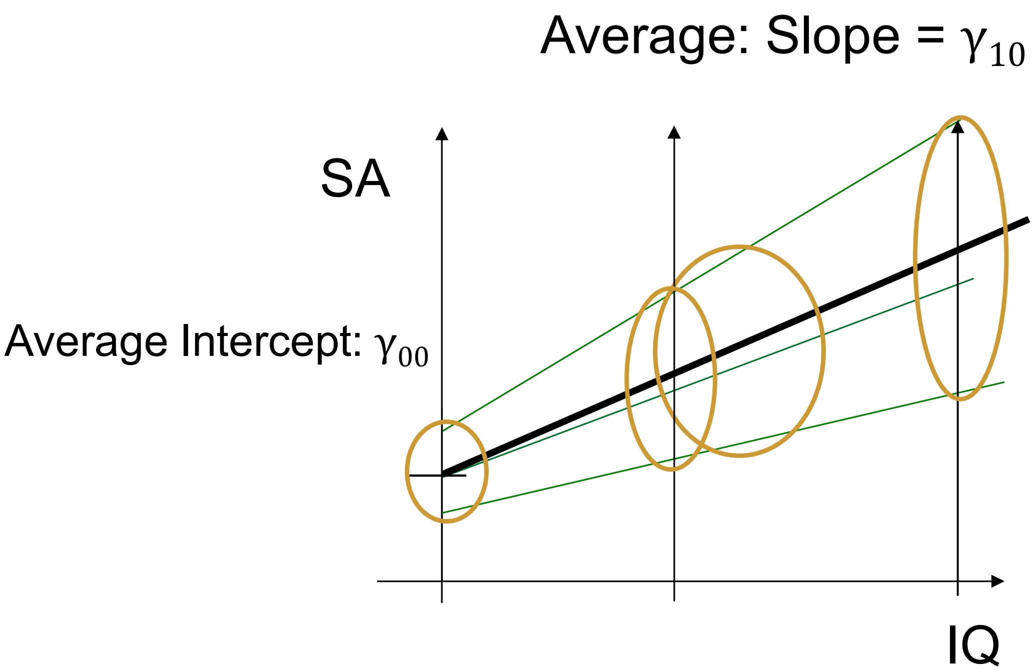


# Methodological Considerations: Centering

# Multilevel Regression



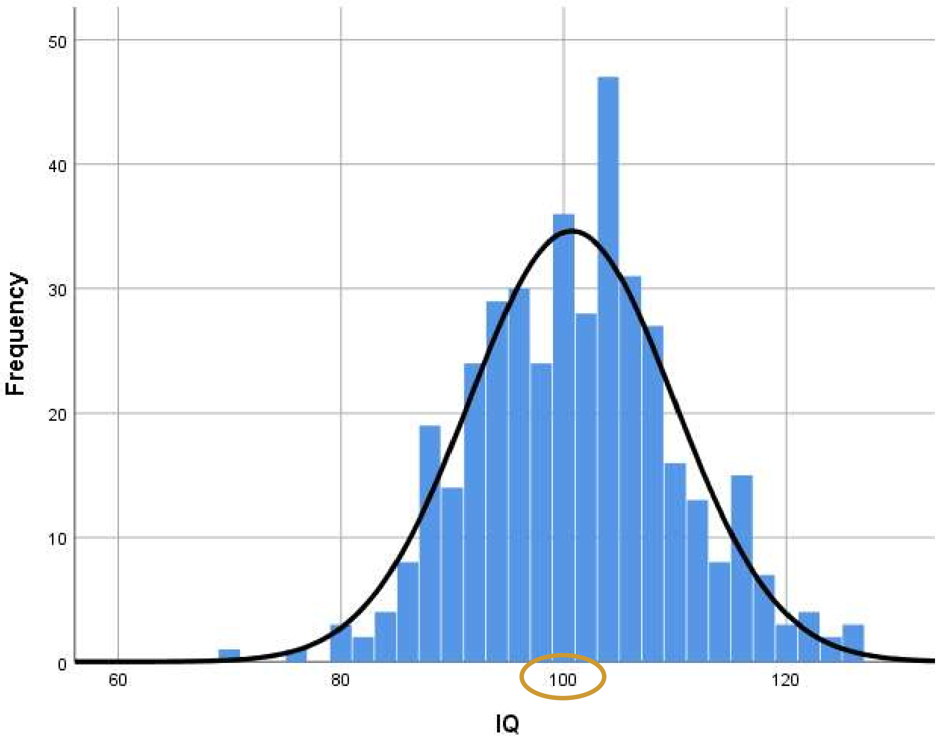

School 1: Slope = $\gamma_{10} + u_{11}$

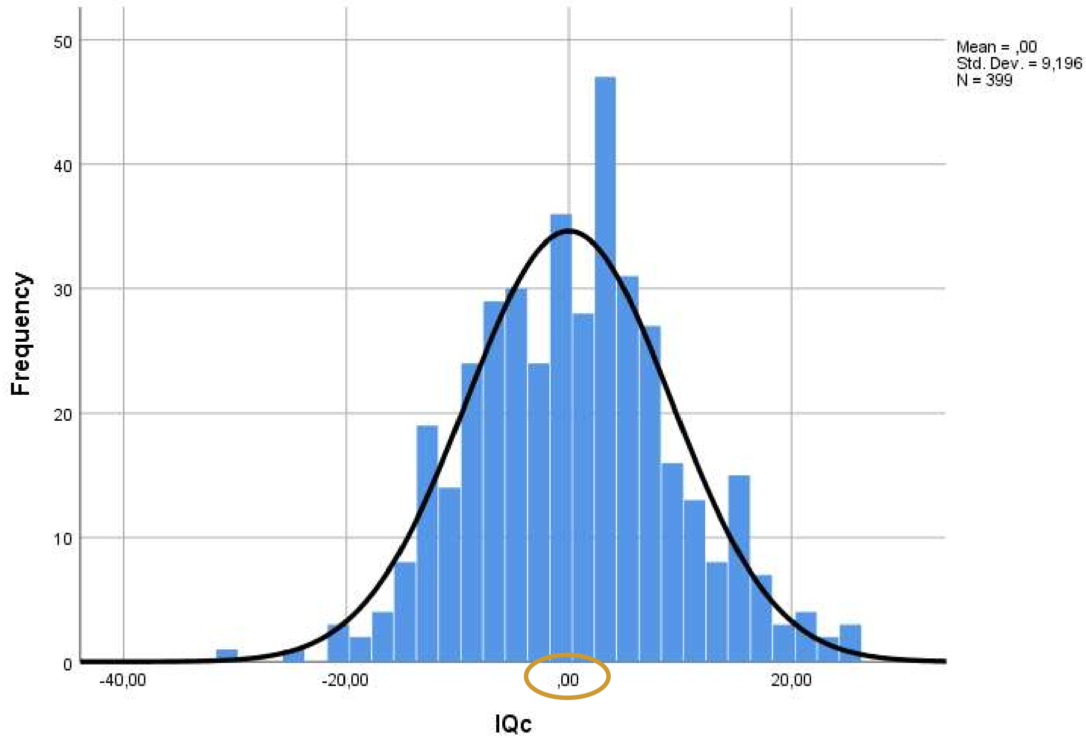
School 2: Slope = $\gamma_{10} + u_{12}$

School 3: Slope = $\gamma_{10} + u_{13}$

$$b_{0j} = \gamma_{00} + \gamma_{01} Schoolsize_j + u_{0j}$$
$$b_{1j} = \gamma_{10} + u_{1j}$$

# Centering

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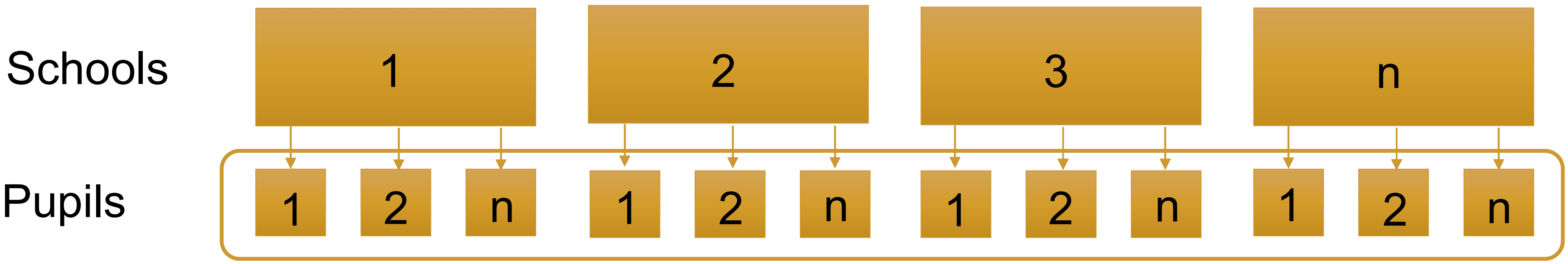
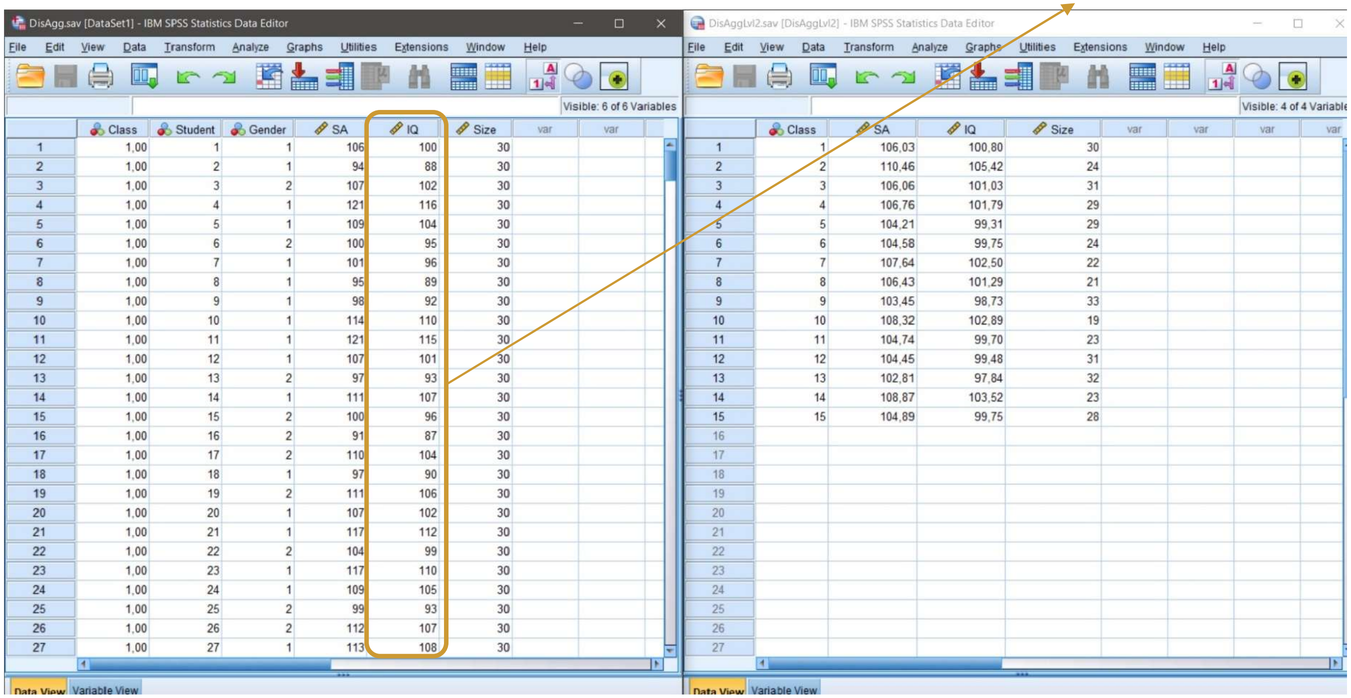
# Grand Mean Centering




| | Class | Student | Gender | SA | IQ | Size | var | var |
|---|---|---|---|---|---|---|---|---|
| 1 | 1,00 | 1 | 1 | 106 | 100 | 30 | | |
| 2 | 1,00 | 2 | 1 | 94 | 88 | 30 | | |
| 3 | 1,00 | 3 | 2 | 107 | 102 | 30 | | |
| 4 | 1,00 | 4 | 1 | 121 | 116 | 30 | | |
| 5 | 1,00 | 5 | 1 | 109 | 104 | 30 | | |
| 6 | 1,00 | 6 | 2 | 100 | 95 | 30 | | |
| 7 | 1,00 | 7 | 1 | 101 | 96 | 30 | | |
| 8 | 1,00 | 8 | 1 | 95 | 89 | 30 | | |
| 9 | 1,00 | 9 | 1 | 98 | 92 | 30 | | |
| 10 | 1,00 | 10 | 1 | 114 | 110 | 30 | | |
| 11 | 1,00 | 11 | 1 | 121 | 115 | 30 | | |
| 12 | 1,00 | 12 | 1 | 107 | 101 | 30 | | |
| 13 | 1,00 | 13 | 2 | 97 | 93 | 30 | | |
| 14 | 1,00 | 14 | 1 | 111 | 107 | 30 | | |
| 15 | 1,00 | 15 | 2 | 100 | 96 | 30 | | |
| 16 | 1,00 | 16 | 2 | 91 | 87 | 30 | | |
| 17 | 1,00 | 17 | 2 | 110 | 104 | 30 | | |
| 18 | 1,00 | 18 | 1 | 97 | 90 | 30 | | |
| 19 | 1,00 | 19 | 2 | 111 | 106 | 30 | | |
| 20 | 1,00 | 20 | 1 | 107 | 102 | 30 | | |
| 21 | 1,00 | 21 | 1 | 117 | 112 | 30 | | |
| 22 | 1,00 | 22 | 2 | 104 | 99 | 30 | | |
| 23 | 1,00 | 23 | 1 | 117 | 110 | 30 | | |
| 24 | 1,00 | 24 | 1 | 109 | 105 | 30 | | |
| 25 | 1,00 | 25 | 2 | 99 | 93 | 30 | | |
| 26 | 1,00 | 26 | 2 | 112 | 107 | 30 | | |
| 27 | 1,00 | 27 | 1 | 113 | 108 | 30 | | |

| | Class | SA | IQ | Size | var | var | var | var |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 106,03 | 100,80 | 30 | | | | |
| 2 | 2 | 110,46 | 105,42 | 24 | | | | |
| 3 | 3 | 106,06 | 101,03 | 31 | | | | |
| 4 | 4 | 106,76 | 101,79 | 29 | | | | |
| 5 | 5 | 104,21 | 99,31 | 29 | | | | |
| 6 | 6 | 104,58 | 99,75 | 24 | | | | |
| 7 | 7 | 107,64 | 102,50 | 22 | | | | |
| 8 | 8 | 106,43 | 101,29 | 21 | | | | |
| 9 | 9 | 103,45 | 98,73 | 33 | | | | |
| 10 | 10 | 108,32 | 102,89 | 19 | | | | |
| 11 | 11 | 104,74 | 99,70 | 23 | | | | |
| 12 | 12 | 104,45 | 99,48 | 31 | | | | |
| 13 | 13 | 102,81 | 97,84 | 32 | | | | |
| 14 | 14 | 108,87 | 103,52 | 23 | | | | |
| 15 | 15 | 104,89 | 99,75 | 28 | | | | |
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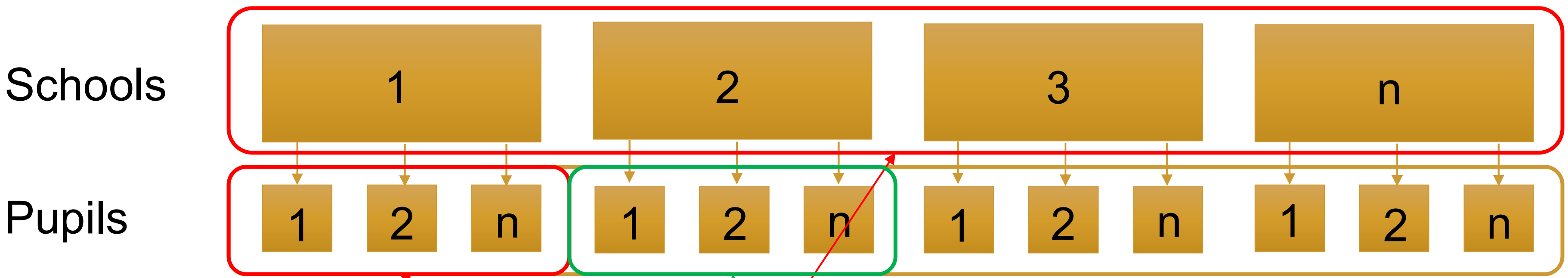
| 25 | | | | | | | | |
| 26 | | | | | | | | |
| 27 | | | | | | | | |

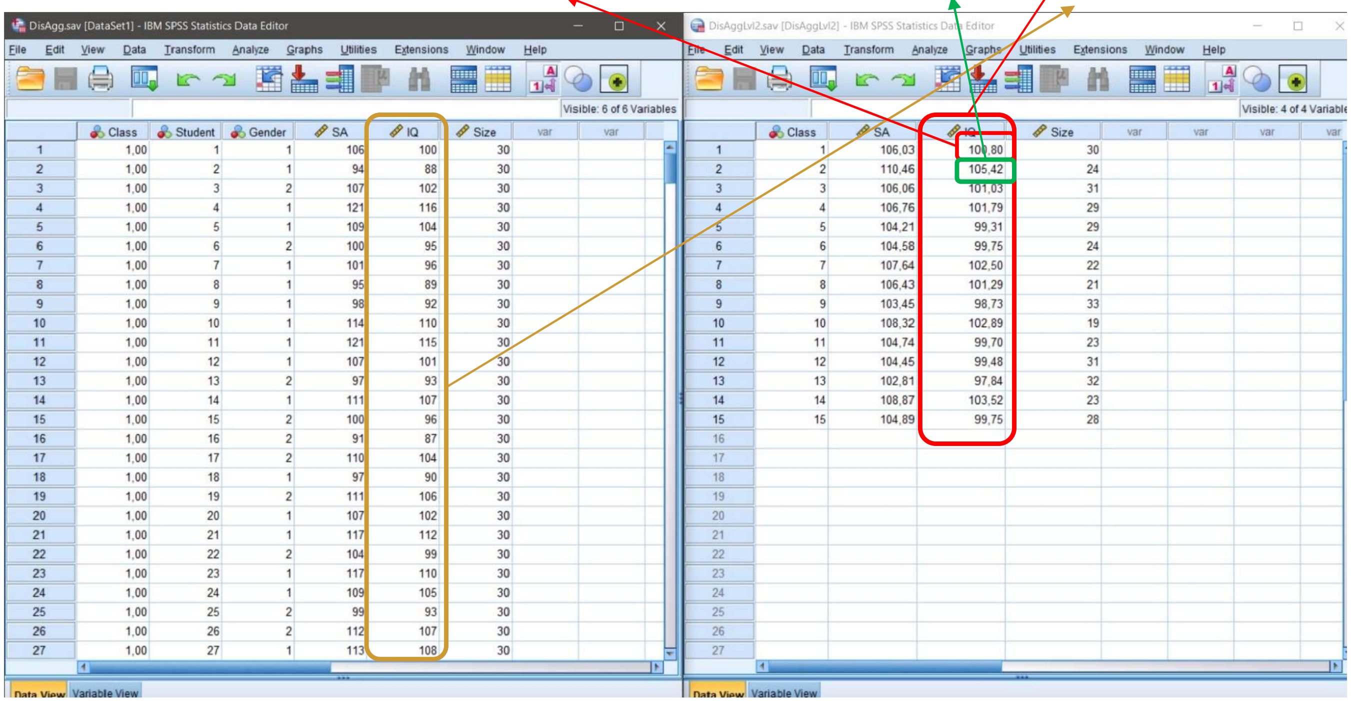
**Descriptive Statistics**

| | N | Minimum | Maximum | Mean | Std. Deviation |
|---|---|---|---|---|---|
| IQ | 399 | 70 | 126 | 100,72 | 9,196 |
| Valid N (listwise) | 399 | | | | |

$$IQc = \boxed{IQ} - \boxed{100.72}$$

# Group Mean Centering

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| 7 | 1,00 | 7 | 1 | 101 | 96 | 30 | | |
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| 13 | 1,00 | 13 | 2 | 97 | 93 | 30 | | |
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| 15 | 1,00 | 15 | 2 | 100 | 96 | 30 | | |
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| 17 | 1,00 | 17 | 2 | 110 | 104 | 30 | | |
| 18 | 1,00 | 18 | 1 | 97 | 90 | 30 | | |
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| | Class | SA | IQ | Size | var | var | var | var |
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| 27 | | | | | | | | |

**Descriptive Statistics**

| | N | Minimum | Maximum | Mean | Std. Deviation |
|---|---|---|---|---|---|
| IQ | 399 | 70 | 126 | 100,72 | 9,196 |
| Valid N (listwise) | 399 | | | | |

$$IQc=\boxed{IQ}-\boxed{100.72}$$

$$IQc\text{-}group=IQ_{ij}-\overline{IQ}_j$$

# Centering

- Group mean centering changes interpretation of variable:
  - No longer “raw” score.
  - Relative position within higher level unit.

- Used for so-called frog-pond effects:
  - Someone’s actual level of an attribute doesn’t matter, what matters is how one compares to the rest of their group (or pond).

# Conclusion

- With random slopes important to think about the intercept, and therefore the meaning of a 0-score on predictors.

- Centering advised, but…
  - 2 methods: Grand mean centering and group mean centering.
  - Group mean centering complicates interpretation of model.

# Practical 3

# Methodological Considerations: Small Level 2 N

# Fixed Effects Models

- Multilevel is great, but we are modeling on level 2 as well!

- Would you run a regression on N=4?

- Would you calculate a variance on N=2?

- Same with multilevel, if N on level 2 is small (let’s say less than 10), modeling distributions there is probably not a great idea.

- Fortunately, there is a solution! ☺.

# Fixed Effects Models

- Hint, the solution is not cluster robust se’s as is sometimes suggested
  - Guess what, that correction also need to be estimated ;).

- How have you corrected for things in regression in the past?

# Fixed Effects Models

- Hint, the solution is not cluster robust se's as is sometimes suggested
  - Guess what, that correction also need to be estimated ;).

- How have you corrected for things in regression in the past?

$$SA_{ij} = b_{01}Class1 + b_{02}Class2 + \ \dots + b_{0x}ClassX + b_1 IQ_{ij} + e_{ij}$$

- Add dummies! One for each class (and remove the intercept)

# Fixed Effects Models

$$SA_{ij} = b_{01}Class1 + b_{02}Class2 + \ \dots + b_{0x}ClassX + b_1 IQ_{ij} + e_{ij}$$

- This is called a Fixed Effects model and is used often in economics.
- It works reaaaaaly well, as the dummies take care of all the level 2 differences.
- Estimates of level 1 predictors unbiased.
- Also deal with “unmodeled” level 2 influences, so is very robust.

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# Fixed Effects Models

$$SA_{ij} = b_{01}Class1 + b_{02}Class2 + \ \dots + b_{0x}ClassX + b_1 IQ_{ij} + e_{ij}$$

- But! No free lunch.

- You can’t model level 2 variables!
  - Since all level 2 variance is in the dummies they are perfectly colinear with level 2 predictors.
  - Not a big problem, if level 2 N is small what do you hope to find there anyway?

- Can model interactions between level 1 and level 2 predictors though.

# Three-level Models

# Multilevel Regression

Level 1: $$Pop_{ij} = b_{0j} + b_{1j}Ext_{ij} + e_{ij}$$

Level 2: $$b_{0j} = \gamma_{00} + \gamma_{01}Texp_j + u_{0j}$$

$$b_{1j} = \gamma_{10} + u_{1j}$$

Combined:

$$\begin{aligned} Pop_{ij} &= b_{0j} + b_{1j}Ext_{ij} + eij \\ &= \gamma_{00} + \gamma_{10}Ext_{ij} + \gamma_{01}Texp_j + e_{ij} + u_{0j} + u_{1j}Ext_{ij} \end{aligned}$$

# Multilevel Regression

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ij}$

Level 2:

$$b_{0jk} = \boxed{\gamma_{00k}} + \gamma_{01k}Texp_{jk} + u_{0jk}$$
$$b_{1jk} = \boxed{\gamma_{10k}} + u_{1jk}$$

Level 3:

$$\boxed{\gamma_{00k}} = \gamma_{000} + v_{ok}$$
$$\boxed{\gamma_{10k}} = \gamma_{100} + v_{1k}$$

Combined:

$$\begin{aligned} Pop_{ijk} &= b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk} \\ &= \gamma_{000} + \gamma_{100}Ext_{ijk} + \gamma_{01k}Texp_{jk} + e_{ij} \ + u_{0jk} + u_{1jk}Ext_{ijk} + v_{ok} \ + v_{1k}Ext_{ijk} \end{aligned}$$

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# Multilevel Regression

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ij}$

Level 2:
$$\begin{aligned} b_{0jk} &= \gamma_{00k} + \gamma_{01k}Texp_{jk} + u_{0jk} \\ b_{1jk} &= \gamma_{10k} + u_{1jk} \end{aligned}$$

Level 3:
$$\begin{aligned} \gamma_{00k} &= \gamma_{000} + v_{ok} \\ \gamma_{10k} &= \gamma_{100} + v_{1k} \end{aligned}$$

Combined:

$$\begin{aligned} Pop_{ijk} &= b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk} \\ &= \gamma_{000} + \gamma_{100}Ext_{ijk} + \gamma_{01k}Texp_{jk} + e_{ijk} + u_{0jk} + u_{1jk}Ext_{ijk} + v_{ok} + v_{1k}Ext_{ijk} \end{aligned}$$

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# Multilevel Regression

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ij}$

Level 2:

$$b_{0jk} = \boxed{\gamma_{00k}} + \gamma_{01k}Texp_{jk} + u_{0jk}$$
$$b_{1jk} = \boxed{\gamma_{10k}} + u_{1jk}$$

Level 3:

$$\boxed{\gamma_{00k}} = \boxed{\gamma_{000}} + \boxed{v_{ok}}$$
$$\boxed{\gamma_{10k}} = \boxed{\gamma_{100}} + \boxed{v_{1k}}$$

Combined:

$$\begin{aligned} Pop_{ijk} &= b_{0jk} + b_{1jk}Ext_{ijk} + e_{ij} \\ &= \gamma_{000} + \gamma_{100}Ext_{ijk} + \gamma_{01k}Texp_{jk} + e_{ijk} + u_{0jk} + \boxed{u_{1jk}Ext_{ijk}} + v_{ok} + \boxed{v_{1k}Ext_{ijk}} \end{aligned}$$

# Multilevel Regression

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk}$

Level 2: $b_{0jk} = \gamma_{00k} + \gamma_{01k}Texp_{jk} + u_{0jk}$

$b_{1jk} = \gamma_{10k} + u_{1jk}$

Level 3: $\gamma_{00k} = \gamma_{000} + v_{ok}$

$\gamma_{10k} = \gamma_{100} + v_{1k}$

Combined:

$$\begin{aligned} Pop_{ijk} &= b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk} \\ &= \gamma_{000} + \gamma_{100}Ext_{ijk} + \gamma_{01k}Texp_{jk} + e_{ijk} + u_{0jk} + u_{1jk}Ext_{ijk} + v_{ok} + v_{1k}Ext_{ijk} \end{aligned}$$

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# Multilevel Regression

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ij}$

Level 2: $b_{0jk} = \gamma_{00k} + \boxed{\gamma_{01k}} Texp_{jk} + u_{0jk}$

$b_{1jk} = \gamma_{10k} + u_{1jk}$

Level 3: $\gamma_{00k} = \gamma_{000} + v_{ok}$

$\gamma_{10k} = \gamma_{100} + v_{1k}$

Combined:

$$Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk}$$
$$= \gamma_{000} + \gamma_{100}Ext_{ijk} + \gamma_{01k}Texp_{jk} + e_{ij} + u_{0jk} + u_{1jk}Ext_{ijk} + v_{ok} + v_{1k}Ext_{ijk}$$

# Multilevel Regression

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk}$

Level 2: $b_{0jk} = \gamma_{00k} + \boxed{\gamma_{01k}}Texp_{jk} + u_{0jk}$

$b_{1jk} = \gamma_{10k} + u_{1jk}$

Level 3: $\gamma_{00k} = \gamma_{000} + v_{ok}$

$\gamma_{10k} = \gamma_{100} + v_{1k}$

$\boxed{\gamma_{01k}} = \gamma_{010} + v_{2k}$

Combined:

$$Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ijk} + e_{ij}$$

$$= \gamma_{000} + \gamma_{100}Ext_{ijk} + \boxed{\gamma_{010}}Texp_{jk} + e_{ij} + u_{0jk} + u_{1jk}Ext_{ijk} + v_{ok} + v_{1k}Ext_{ijk} + \boxed{v_{2k}Texp_{jk}}$$

# Intraclass correlation

- $\sigma^2_{v_0}$ variance of the third level errors $v_{0j}$
- $\sigma^2_{u_0}$ variance of the second level errors $u_{0j}$
- $\sigma^2_e$ variance of the first level errors $e_{ij}$

$$\rho_{class} = \frac{\sigma^2_{u_0}}{\sigma^2_{v_0} + \sigma^2_{u_0} + \sigma^2_e} \qquad \rho_{school} = \frac{\sigma^2_{v_0}}{\sigma^2_{v_0} + \sigma^2_{u_0} + \sigma^2_e}$$

- The ‘percentage’ variance at the second and third level

$$\rho_{class} = \frac{\sigma^2_{v_0} + \sigma^2_{u_0}}{\sigma^2_{v_0} + \sigma^2_{u_0} + \sigma^2_e}$$

- The expected correlation between the observations on the dependent variable of two randomly chosen units that are in the same class (and thus in the same school).

# Extra difficulty with 3 level models

- What do you do when there is a significant amount of variance on level 3, but not level 2?

# Extra difficulty with 3 level models

- What do you do when there is a significant amount of variance on level 3, but not level 2?

- The “higher you go” the smaller your sample size.
  - What do you do if the number of level 3 units is very small?

# Fixed Effect Models

Level 1: $Pop_{ijk} = b_{0j} + b_{1jk} Ext_{ij} + e_{ij}$

Level 2: $b_{0j} = \gamma_{01} School1 + \gamma_{02} School2 + \ \dots + \gamma_{0x} SchoolX + \gamma_{0,x+1} Texp_{jk} + u_{0j}$

$b_{1j} = \gamma_{11} School1 + \gamma_{12} School2 + \ \dots + \gamma_{1x} SchoolX + u_{1j}$

Combined:

$$
\begin{aligned}
Pop_{ijk} &= b_{0jk} + b_{1jk} Ext_{ijk} + e_{ij} \\
&= \gamma_{01} School1 + \gamma_{02} School2 + \ \dots + \gamma_{0x} SchoolX + \gamma_{0,x+1} Texp_{jk} \\
&\quad \gamma_{11} School1 * Ext_{ij} + \gamma_{12} School2 * Ext_{ij} + \ \dots + \gamma_{1x} SchoolX * Ext_{ij} + \\
&\quad e_{ij} + u_{0j} + u_{1j} Ext_{ij}
\end{aligned}
$$

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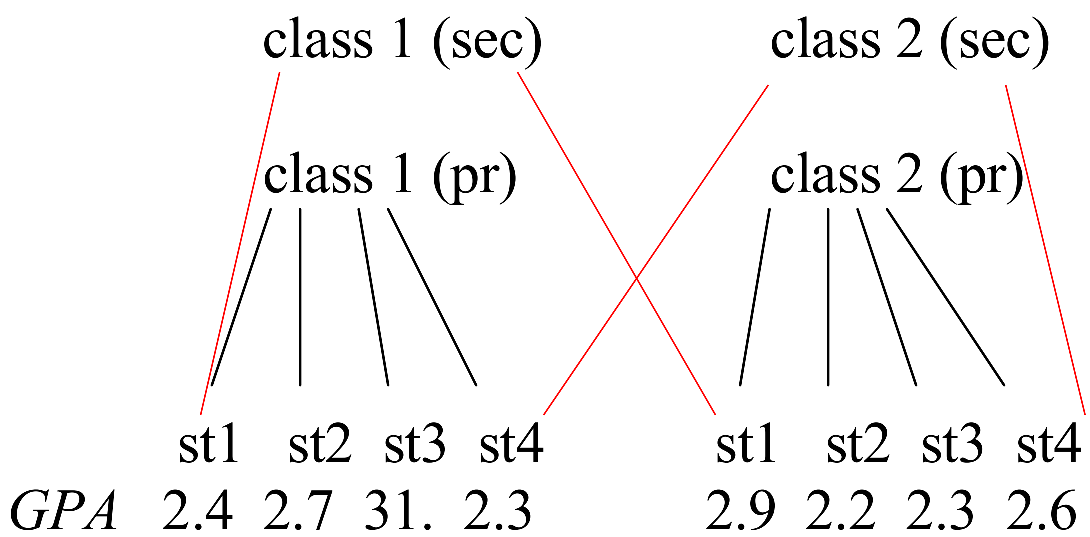
# Practical 4

# Cross-Nested Data

# Cross-Classified Models

- Sometimes the nesting structure is not constant.
  - You follow students as the transition from primary school to secondary school.
  - Some students that attend the same primary school will attend different secondary schools.
  - So students nested in a cross-classification of primary and secondary schools.

# Cross-classified data



# Cross-Classified Models

- $Y_{i(jk)}$ Score of student i within cross-classification of primary school j and secondary school k

$$Y_{i(jk)} = \beta_{0(jk)} + e_{i(jk)}$$

- $\beta_{0(jk)}$ is the intercept/overall mean

- $e_{i(jk)}$ is the residual error term

- the subscript *(jk)* is written in parentheses to indicate that they are conceptually at the same level

# Cross-Classified Models

- The intercept $\beta_{0(jk)}$ varies independently across both primary and secondary schools, so

$$Y_{i(jk)} = \beta_{0(jk)} + e_{i(jk)}$$

$$\beta_{0(jk)} = \gamma_{00} + u_{0j} + v_{ok}$$

- $u_{0j}$ error for primary schools

- $v_{ok}$ error for secondary schools

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# Cross-Classified Models

- Individual-level explanatory variables can be added to the equation.
  - Their regression slopes may be allowed to vary across primary and/or secondary schools.

$$Y_{i(jk)} = \beta_{0(jk)} + \beta_{1(jk)} X_{i(jk)} + e_{i(jk)}$$

- School-level variables can also be added,
  - Used to explain variation in the slopes of individual-level variables across schools
  - Similar to ordinary multilevel regression models.

# Cross-Classified Models

- How do you specify these things?

- Do they sort of remind you of a model we saw earlier when looking at the equations?

$$Y_{i(jk)} = \beta_{0(jk)} + e_{i(jk)}$$

$$\beta_{0(jk)} = \gamma_{00} + u_{0j} + v_{ok}$$

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# Cross-Classified Models

- How do you specify these things?

- We can start by ignoring the secondary school level, specify the individual and primary school level as usual.
  - individuals at the first level and primary schools at the second level.

- But now what?
  - We need to add another level, without actually adding another level.
  - Have we seen this before?

# Fixed Effect Models

Level 1: $Pop_{ijk} = b_{0j} + b_{1jk}Ext_{ij} + e_{ij}$

Level 2: $b_{0j} = \gamma_{01}School1 + \gamma_{02}School2 + \ \dots + \gamma_{0x}SchoolX + \gamma_{0,x+1}Texp_{jk} + u_{0j}$

$b_{1j} = \gamma_{11}School1 + \gamma_{12}School2 + \ \dots + \gamma_{1x}SchoolX + u_{1j}$

Combined:

$$
\begin{aligned}
Pop_{ijk} &= b_{0jk} + b_{1jk}Ext_{ijk} + e_{ijk} \\
&= \gamma_{01}School1 + \gamma_{02}School2 + \ \dots + \gamma_{0x}SchoolX + \gamma_{0,x+1}Texp_{jk} \\
&\quad \gamma_{11}School1 * Ext_{ij} + \gamma_{12}School2 * Ext_{ij} + \ \dots + \gamma_{1x}SchoolX * Ext_{ij} + \\
&\quad e_{ij} + u_{0j} + u_{1j}Ext_{ij}
\end{aligned}
$$

# Cross-Classified Models

- We can start by ignoring the secondary school level, specify the individual and primary school level as usual.
  - individuals at the first level and primary schools at the second level.

- To create a place to specify the crossed effects of the secondary school level, we introduce a third ‘dummy’ level

- At the pupil level: specify a full set of dummy variables to indicate all of the secondary schools.

# Cross-Classified Models

- At the pupil level: specify a full set of dummy variables to indicate all of the secondary schools.
- But this has one problem left! Which one?

# Multilevel Regression

Level 1: $Pop_{ij} = b_{0j} + b_{1j}Ext_{ij} + e_{ij}$

Level 2: $b_{0j} = \gamma_{00} + \gamma_{01}Texp_j + \boxed{u_{0j}}$

$b_{1j} = \gamma_{01} + u_{1j}$

Combined:

$$
\begin{aligned}
Pop_{ij} &= b_{0j} + b_{1j}Ext_{ij} + eij \\
&= \gamma_{00} + \gamma_{01}Ext_{ij} + \gamma_{01}Texp_j + e_{ij} + u_{0j} + u_{1j}Ext_{ij}
\end{aligned}
$$

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# Fixed Effect Models/Cross-Classified Models

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ij} + b_{2jk}School1 + b_{3jk}School2 + \ \dots + b_{xjk}SchoolX + e_{ijk}$

Level 2:

$$b_{0jk} = \gamma_{00k} + \gamma_{01k}Texp_{jk} + u_{0jk}$$
$$b_{1jk} = \gamma_{01k} + u_{1jk}$$
$$b_{2jk} = \gamma_{02k} + u_{2jk}$$
$$\vdots$$
$$b_{xjk} = \gamma_{0xk} + u_{xjk}$$

"Level" 3:

$$\gamma_{00k} = \gamma_{000} + v_{000}$$
$$\gamma_{01k} = \gamma_{001} + v_{001}$$
$$\gamma_{02k} = \gamma_{002} + v_{002}$$
$$\vdots$$
$$\gamma_{0xk} = \gamma_{00x} + v_{00x}$$

# Cross-Classified Models

- The fixed regression coefficients of these dummies are excluded from the model,
  - their slopes are allowed to vary at the third, ‘dummy’, level.
  - covariances between dummies all constrained to be zero,
  - variances all constrained to be equal.
- Thus, in the end we estimate one variance component for the secondary schools
- By putting the secondary schools in a separate “level” we assure that there are no covariances between residuals for the primary and the secondary schools.

# Fixed Effect Models/Cross-Classified Models

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ij} + b_{2jk}School1 + b_{3jk}School2 + \ \dots + b_{xjk}SchoolX + e_{ijk}$

Level 2:
$$b_{0jk} = \gamma_{00k} + \gamma_{01k}Texp_{jk} + u_{0jk}$$
$$b_{1jk} = \gamma_{01k} + u_{1jk}$$
$$b_{2jk} = \gamma_{02k} + u_{2jk}$$
$$\vdots$$
$$b_{xjk} = \gamma_{0xk} + u_{xjk}$$

"Level" 3:
$$\gamma_{00k} = \gamma_{000} + v_{000}$$
$$\gamma_{01k} = \gamma_{001} + v_{001}$$
$$\gamma_{02k} = \gamma_{002} + v_{002}$$
$$\vdots$$
$$\gamma_{0xk} = \gamma_{00x} + v_{00x}$$

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# Fixed Effect Models/Cross-Classified Models

Level 1: $$Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ij} + b_{2jk}School1 + b_{3jk}School2 + \ \dots + b_{xjk}SchoolX + e_{ij}$$

Level 2:
$$b_{0jk} = \gamma_{00k} + \gamma_{01k}Texp_{jk} + u_{0jk}$$
$$b_{1jk} = \gamma_{01k} + u_{1jk}$$
$$b_{2jk} = \gamma_{02k}$$
$$\vdots$$
$$b_{xjk} = \gamma_{0xk}$$

"Level" 3:
$$\gamma_{00k} = \gamma_{000} + v_{000}$$
$$\gamma_{01k} = \gamma_{001} + v_{001}$$
$$\gamma_{02k} = \gamma_{002} + v_{002}$$
$$\vdots$$
$$\gamma_{0xk} = \gamma_{00x} + v_{00x}$$

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# Fixed Effect Models/Cross-Classified Models

Level 1: $Pop_{ijk} = b_{0jk} + b_{1jk}Ext_{ij} + b_{2jkS}chool1 + b_{3jk}School2 + \ \dots + b_{xjk}SchoolX + e_{ijk}$

Level 2:
$$b_{0jk} = \gamma_{00k} + \gamma_{01k}Texp_{jk} + u_{0jk}$$
$$b_{1jk} = \gamma_{01k} + u_{1jk}$$
$$b_{2jk} = \gamma_{02k}$$
$$\vdots$$
$$b_{xjk} = \gamma_{0xk}$$

“Level” 3:
$$\gamma_{00k} = \gamma_{000} + v_{000}$$
$$\gamma_{01k} = \gamma_{001} + v_{001}$$
$$\gamma_{02} \quad = 0 + v_{002}$$
$$\vdots$$
$$\gamma_{0x} \quad = 0 + v_{00x}$$

$$\sigma^2_{v002} = \sigma^2_{v003} = \cdots = \sigma^2_{v0}$$

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# Fixed Effect Models/Cross-Classified Models

$$\text{achiev}_{ijk} \sim \text{N}(XB, \Omega)$$

$$\begin{aligned}
\text{achiev}_{ijk} = \beta_{30ij}\text{const} &+ v_{0k}\text{c101}_{ijk} + v_{1k}\text{c102}_{ijk} + v_{2k}\text{c103}_{ijk} + v_{3k}\text{c104}_{ijk} + v_{4k}\text{c105}_{ijk} \\
&+ v_{5k}\text{c106}_{ijk} + v_{6k}\text{c107}_{ijk} + v_{7k}\text{c108}_{ijk} + v_{8k}\text{c109}_{ijk} + v_{9k}\text{c110}_{ijk} \\
&+ v_{10k}\text{c111}_{ijk} + v_{11k}\text{c112}_{ijk} + v_{12k}\text{c113}_{ijk} + v_{13k}\text{c114}_{ijk} + v_{14k}\text{c115}_{ijk} \\
&+ v_{15k}\text{c116}_{ijk} + v_{16k}\text{c117}_{ijk} + v_{17k}\text{c118}_{ijk} + v_{18k}\text{c119}_{ijk} + v_{19k}\text{c120}_{ijk} \\
&+ v_{20k}\text{c121}_{ijk} + v_{21k}\text{c122}_{ijk} + v_{22k}\text{c123}_{ijk} + v_{23k}\text{c124}_{ijk} + v_{24k}\text{c125}_{ijk} \\
&+ v_{25k}\text{c126}_{ijk} + v_{26k}\text{c127}_{ijk} + v_{27k}\text{c128}_{ijk} + v_{28k}\text{c129}_{ijk} + v_{29k}\text{c130}_{ijk}
\end{aligned}$$

$$\beta_{30ij} = 6.349(0.078) + u_{30jk} + e_{30ijk}$$

$$\begin{bmatrix} v_{0k} \\ v_{1k} \\ v_{2k} \\ v_{3k} \\ v_{4k} \end{bmatrix} \qquad \begin{bmatrix} 0.065(.022) & & & \\ 0 & 0.065(.022) & & \\ 0 & 0 & 0.065(.022) & \\ 0 & 0 & 0 & 0.065(.022) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Cross-Classified Models

| Model | Intercept-only Coeff. (s.e.) | + pupil vars Coeff. (s.e.) | + school vars Coeff. (s.e.) | + *ses* random Coeff. (s.e.) |
|---|---|---|---|---|
| **Fixed part** | | | | |
| Intercept | 6.35 (.08) | 5.76 (.11) | 5.52 (.19) | 5.53 (.14) |
| Pupil gender | | 0.26 (.05) | 0.26 (.05) | 0.25 (.05) |
| Pupil ses | | 0.11 (.02) | 0.11 (.02) | 0.11 (.02) |
| Primary denom | | | 0.20 (.12) | 0.20 (.12) |
| Secondary denom | | | 0.18 (.10) | 0.17 (.09) |
| **Random part** | | | | |
| $\sigma^2_{\text{int/pupil}}$ | 0.51 (.02) | 0.47 (.02) | 0.47 (.02) | 0.46 (.02) |
| $\sigma^2_{\text{int/primary}}$ | 0.17 (.04) | 0.17 (.04) | 0.16 (.04) | 0.14 (.08) |
| $\sigma^2_{\text{int/secondary}}$ | 0.07 (.02) | 0.06 (.02) | 0.06 (.02) | 0.05 (.02) |
| $\sigma^2_{\text{ses/primary}}$ | | | | 0.008 (.004) |
| Deviance | 2317.8 | 2243.5 | 2237.5 | 2224.5 |
| AIC | 2325.8 | 2255.5 | 2253.5 | 2244.5 |

# Cross-Classified Models

- Alternatively you might have network data.

- E.g. you ask all members of a group to rate all other members
  - how much they would like to share some activity with the rated person
  - Sociometric rating

# Sociometric rating

| | group | child | age | sex | grsize | rating1 | rating2 | rating3 | rating4 | rating5 | rating6 | rating7 | rating8 | rating9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 8 | 1 | 7 | . | 3 | 6 | 4 | 4 | 7 | 6 | . | . |
| 2 | 1 | 2 | 10 | 1 | 7 | 5 | . | 6 | 4 | 5 | 7 | 5 | . | . |
| 3 | 1 | 3 | 11 | 1 | 7 | 4 | 6 | . | 4 | 5 | 7 | 6 | . | . |
| 4 | 1 | 4 | 9 | 0 | 7 | 4 | 4 | 6 | . | 5 | 7 | 5 | . | . |
| 5 | 1 | 5 | 11 | 0 | 7 | 5 | 5 | 6 | 5 | . | 7 | 6 | . | . |
| 6 | 1 | 6 | 10 | 1 | 7 | 4 | 5 | 6 | 3 | 4 | . | 6 | . | . |
| 7 | 1 | 7 | 10 | 1 | 7 | 3 | 5 | 6 | 5 | 3 | 6 | . | . | . |
| 8 | 2 | 1 | 9 | 0 | 9 | . | 3 | 5 | 3 | 4 | 6 | 6 | 4 | 5 |
| 9 | 2 | 2 | 9 | 0 | 9 | 2 | . | 4 | 5 | 6 | 5 | 4 | 4 | 5 |
| 10 | 2 | 3 | 9 | 0 | 9 | 5 | 3 | . | 4 | 3 | 6 | 5 | 4 | 6 |
| 11 | 2 | 4 | 8 | 1 | 9 | 3 | 2 | 5 | . | 6 | 6 | 5 | 3 | 4 |
| 12 | 2 | 5 | 9 | 1 | 9 | 4 | 4 | 5 | 5 | . | 5 | 7 | 4 | 5 |
| 13 | 2 | 6 | 9 | 0 | 9 | 3 | 4 | 4 | 4 | 4 | . | 5 | 4 | 5 |
| 14 | 2 | 7 | 9 | 1 | 9 | 4 | 4 | 6 | 5 | 6 | 5 | . | 4 | 5 |
| 15 | 2 | 8 | 11 | 0 | 9 | 3 | 4 | 5 | 4 | 5 | 6 | 6 | . | 5 |
| 16 | 2 | 9 | 8 | 1 | 9 | 3 | 4 | 5 | 5 | 4 | 6 | 7 | 5 | . |
| 17 | 3 | 1 | 11 | 0 | 5 | . | 5 | 7 | 5 | 6 | . | . | . | . |
| 18 | 3 | 2 | 11 | 0 | 5 | 5 | . | 7 | 6 | 6 | . | . | . | . |
| 19 | 3 | 3 | 13 | 1 | 5 | 5 | 5 | . | 6 | 8 | . | . | . | . |
| 20 | 3 | 4 | 12 | 1 | 5 | 4 | 4 | 6 | . | 6 | . | . | . | . |

# Cross-Classified Models

- DV: Ratings
- Nested:

# Cross-Classified Models

- DV: Ratings
- Nested: Cross-classification of senders and receivers
- Can be further nested within groups

# Sociometric rating

| | group | sender | receiver | rating | agesend | sexsend | agerec | sexrec | grsize |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 8 | 1 | 10 | 1 | 7 |
| 2 | 1 | 1 | 3 | 6 | 8 | 1 | 11 | 1 | 7 |
| 3 | 1 | 1 | 4 | 4 | 8 | 1 | 9 | 0 | 7 |
| 4 | 1 | 1 | 5 | 4 | 8 | 1 | 11 | 0 | 7 |
| 5 | 1 | 1 | 6 | 7 | 8 | 1 | 10 | 1 | 7 |
| 6 | 1 | 1 | 7 | 6 | 8 | 1 | 10 | 1 | 7 |
| 7 | 1 | 2 | 1 | 5 | 10 | 1 | 8 | 1 | 7 |
| 8 | 1 | 2 | 3 | 6 | 10 | 1 | 11 | 1 | 7 |
| 9 | 1 | 2 | 4 | 4 | 10 | 1 | 9 | 0 | 7 |
| 10 | 1 | 2 | 5 | 5 | 10 | 1 | 11 | 0 | 7 |
| 11 | 1 | 2 | 6 | 7 | 10 | 1 | 10 | 1 | 7 |
| 12 | 1 | 2 | 7 | 5 | 10 | 1 | 10 | 1 | 7 |
| 13 | 1 | 3 | 1 | 4 | 11 | 1 | 8 | 1 | 7 |
| 14 | 1 | 3 | 2 | 6 | 11 | 1 | 10 | 1 | 7 |
| 15 | 1 | 3 | 4 | 4 | 11 | 1 | 9 | 0 | 7 |
| 16 | 1 | 3 | 5 | 5 | 11 | 1 | 11 | 0 | 7 |
| 17 | 1 | 3 | 6 | 7 | 11 | 1 | 10 | 1 | 7 |
| 18 | 1 | 3 | 7 | 6 | 11 | 1 | 10 | 1 | 7 |
| 19 | 1 | 4 | 1 | 4 | 9 | 0 | 8 | 1 | 7 |
| 20 | 1 | 4 | 2 | 4 | 9 | 0 | 10 | 1 | 7 |

# Cross-Classified Models

- $Y_{i(jk)l}$ Score of student i within cross-classification of sender j and receiver k, and within group l

$$Y_{i(jk)l} = \beta_{0(jk)l} + e_{i(jk)l}$$

- $\beta_{0(jk)l}$ is the intercept/overall mean
- $e_{i(jk)l}$ is the residual error term
- the subscript *(jk)* is written in parentheses to indicate that they are conceptually at the same level

# Cross-Classified Models

- The intercept $\beta_{0(jk)l}$ varies independently across both senders, receivers, and group, so

$$Y_{i(jk)l} = \beta_{0(jk)l} + e_{i(jk)l}$$

$$\beta_{0(jk)l} = \beta_{0l} + u_{0jl} + v_{okl}$$

$$\beta_{0l} = \gamma_{00} + f_{0l}$$

# Cross-Classified Models

$$Y_{i(jk)l} = \beta_{0(jk)l} + e_{i(jk)l}$$

$$\beta_{0(jk)l} = \beta_{0l} + u_{0jl} + v_{okl}$$

$$\beta_{0l} = \gamma_{00} + f_{0l}$$

- Score consists of:
  - Overall mean $\gamma_{00}$,
  - residual error term $f_{0l}$ for group *l*,
  - individual-level residual error terms $u_{jl}$ for sender *j* in group *l*
  - individual-level residual error $v_{kl}$ for receiver *k* in group *l*, and
  - the measurement-level error term $e_{i(jk)l}$.

# Cross-Classified Models

- The crossed effects of the receiver level are incorporated using dummies.
- At the lowest level, the ratings, we specify:
  - Dummy variables that indicate the receivers.
  - The fixed coefficients of these dummies are excluded from the model, but their slopes are allowed to vary.
  - Since the cross-classification is nested within the sociometric groups, the slopes of the dummy variables are set to vary at a third group level.
  - In addition, the covariances between the receiver dummies are constrained to be zero, and
  - the variances are constrained to be equal.

# Cross-Classified Models

- The covariances between the receiver dummies are constrained to be zero, and the variances are constrained to be equal.
  - Thus, we estimate one variance component for the receivers.

- By putting the variance term(s) for the receivers on a “separate” level we assure that there are no covariances between the residuals for the sender and the receiver level.

# Cross-Classified Models

- Both sender and receiver characteristics like *age* and *gender* and group characteristics like *group size* can be added to the model as predictors.

- Child characteristics may be allowed to have random slopes at the group level.

- The analysis proceeds along exactly the same lines as outlined for the cross-classification of primary and secondary schools.

# Cross-Classified Models: Hard in some software!

- Since the third ‘group’ level is already used to specify the random variation for the receiver dummies:
  - we must make sure that the intercept and possible slope variation at the ‘real’ group level are not correlated with the dummies.
  - This can be accomplished by adding the appropriate constraints to the model.

- When the software supports more than three levels (e.g., R/MLWin/HLM), the same result can be accomplished more conveniently:
  - Add a fourth level to the model; also for the groups
  - Used for the random part at the real group level.
  - Conceptually we still have three levels, with senders and receivers crossed at the second level

# Cross-Classified Models

- Individual-level explanatory variables can be added to the equation.
  - Their regression slopes may be allowed to vary across primary and/or secondary schools.

$$Y_{i(jk)} = \beta_{0(jk)} + \beta_{1(jk)} X_{i(jk)} + e_{i(jk)}$$

- School-level variables can also be added,
  - Used to explain variation in the slopes of individual-level variables across schools
  - Similar to ordinary multilevel regression models.

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# Non-Normal Data

# Dichotomous data and proportions

- Standard regression: continuous outcome variable with normal distribution of errors
  - What do we do again if one or more violations are violated?

- Some outcome variables *must* violate this assumption

  - Dichotomous outcome variable

  - Outcome variable is a proportion

# Nonlinear Regression models

- Classic solution: transform the dependent variable
  - Proportions: $f(p) = \text{logit}(p) = \ln((p)/(1-p))$
  - Breaks down if $p = 0,1$
    - Does not work with dichotomous variable

- Modern solution: use generalized linear model
  - i.e., Logistic regression

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# Nonlinear Regression models

- Generalized linear model:
- Don’t transform DV and then run a “normal” regression.
- Run a “normal” regression and then transform the prediction of the analysis!

# Generalized Linear Model

**Three common generalized linear models**

| (a) Linear regression | (b) Logistic regression | (c) Poisson regression |
|---|---|---|
| $\alpha + \beta x_i$ | $logistic(\alpha + \beta x_i)$ | $exp(\alpha + \beta x_i)$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $y_i \sim Normal(\mu_i, \sigma)$ | $y_i \sim Bernoulli(p_i)$ | $y_i \sim Poisson(\lambda_i)$ |

$$y = b_0 + b_1 X_1 + b_2 X_2 + e$$

$$\text{pred} = b_0 + b_1 X_1 + b_2 X_2 + e$$

$$\text{logistic(pred)} = \frac{exp^{(pred)}}{1 + exp^{(pred)}}$$

$$y \sim Bernoulli(logistic(pred))$$

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# Generalized Linear Model

**Three common generalized linear models**

| (a) Linear regression | (b) Logistic regression | (c) Poisson regression |
|---|---|---|
| $\alpha + \beta x_i$ | $logistic(\alpha + \beta x_i)$ | $exp(\alpha + \beta x_i)$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $y_i \sim Normal(\mu_i, \sigma)$ | $y_i \sim Bernoulli(p_i)$ | $y_i \sim Poisson(\lambda_i)$ |

- The output of $b_0 + b_1X_1 + b_2X_2 + e$ can be any value ranging from negative to positive infinity.
- A ‘link’ function is introduced to ensure that only sensible parameter values are allowed.
- Logistic regression: the logistic function ensures that values from negative to positive infinity are compressed into the range [0, 1].

$$\text{pred} = b_0 + b_1X_1 + b_2X_2 + e$$

$$\text{logistic(pred)} = \frac{exp^{(pred)}}{1 + exp^{(pred)}}$$

$$y \sim Bernoulli(logistic(pred))$$

# Generalized Linear Model

- Easy to do in R!!
- Just have to change the distribution we use in an analysis!
  - Just like changing from normal to t-distribution
  - But now use Bernoulli or Poisson or…
- Just need to remember that the parameter-estimates are in terms of the non-transformed “pred” part, and so we need to transform them to the “original” scale for interpretation.

# Example: Multilevel Logistic model

- Thailand Education Data
  - 8582 pupils in 411 primary schools
  - Outcome variable dichotomous
  - Repeat a class: yes (1) or no (0)

  - Pupil level predictors
  - *sex* (1 = male, 0 = female)
  - *pre-primary education* (PPED: 1 = yes, 0 = no)
  - School level predictor
  - Mean *SES*

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# Example: Multilevel Logistic model

Final estimation of fixed effects: (Unit-specific model)

| Fixed Effect | Coefficient | Standard Error | T-ratio | Approx. d.f. | P-value |
|---|---|---|---|---|---|
| For INTRCPT1, B0 | | | | | |
| INTRCPT2, G00 | -2.038493 | 0.093685 | -21.759 | 355 | 0.000 |
| For SEX slope, B1 | | | | | |
| INTRCPT2, G10 | 0.509270 | 0.073925 | 6.889 | 7513 | 0.000 |
| For PPED slope, B2 | | | | | |
| INTRCPT2, G20 | -0.609289 | 0.095220 | -6.399 | 7513 | 0.000 |

$$p_{ij} = logistic(-2.04 + 0.51 sex - 0.61 pped)$$

# Example: Multilevel Logistic model

- Interpretation: $p_{ij} = logistic(-2.04 + 0.51 sex - 0.61 pped)$

  - girls, no pped: predict: –2.04 on the underlying continuous scale.
  Transform: $g(x) = \frac{e^x}{1+e^x}$ = 0.115

  - boys, no pped: predict: -2.04 + 0.51 = -1.53
  Transformed: 0.178

  - The estimated repeat rate for the girls is 11.5% and for
  the boys 17.8% (no pped).

# Example: Multilevel Logistic model

- Interpretation: $p_{ij} = logistic(-2.04 + 0.51 sex - 0.61 pped)$
  - girls, with pped: predict: –2.65 on the underlying continuous scale
    Transformed: 0.066

  - boys, with pped: predict: -2.04 + 0.51 – 0.61 = -2.14
    Transformed: 0.105

- The estimated repeat rate for the girls with pped is 6.6% and for the boys 10.5%.

Practical 5

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