CLRS 16.3-4

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September 30, 2008

It were a contradiction to have an optimal tree whose frequencies and codewords were monotonically increasing in the strict sense; since, given $f(x_1) > \cdots > f(x_n) \wedge d_T(x_1) > \cdots > d_T(x_n)$, it follows that (where n is odd):

$$f(x_1)d_T(x_1) + \cdots + f(x_n)d_T(x_n) > f(x_1)d_T(x_n) + \cdots + f(x_n)d_T(x_1) \tag{1}$$

$$\begin{aligned} f(x_1)(d_T(x_1) - d_T(x_n)) + \cdots + \\ f(x_n)(d_T(x_n) - d_T(x_1)) > 0 \end{aligned} \tag{2}$$

$$\begin{aligned} f(x_1)(d_T(x_1) - d_T(x_n)) + \cdots + \\ f(x_{\lfloor \frac{n}{2} - 1 \rfloor})(d_T(x_{\lfloor \frac{n}{2} \rfloor - 1}) - d_T(x_{\lfloor \frac{n}{2} \rfloor + 1})) > \\ f(x_{\lfloor \frac{n}{2} + 1 \rfloor})(d_T(x_{\lfloor \frac{n}{2} \rfloor - 1}) - d_T(x_{\lfloor \frac{n}{2} \rfloor + 1})) + \cdots + \\ f(x_n)(d_T(x_1) - d_T(x_n)) \end{aligned} \tag{3}$$

That is, where i and j are the upper and lower median, respectively; and $c_i = d_T(x_i) - d_T(x_{n-i+1})$:

$$f(x_1)c_1 + \cdots + f(x_i)c_i > f(x_j)c_i + \cdots + f(x_n)c_1 \tag{4}$$

since

$$f(x_i) > f(x_{n-i+1}) \qquad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \tag{5}$$