Problem Sheet 2 Deadline: Monday Week 5

This problem sheet consists of two questions. Each question contains three parts.

- Parts a), worth 40% of the marks, in a question, test your knowledge of the core material, and you should aim to provide good answers to this part of all questions.
- Parts b), worth 30% of the marks, involve taking the concepts you have been taught, but applying them in ways you may not have directly been shown. You should attempt all the parts b), but you can still get a good mark without completing all of them.
- Parts c), worth 30% of the marks, are difficult questions which will test your understanding of the concepts taught in unfamiliar situations.

Question 1 (50%):

a) For each of these functions, find $f'(x)$. Show your working.

i) $f(x) = x^3 + 2x^2 + 4$

ii) $f(x) = \frac{x^2+5}{x-2}$

iii) $f(x) = \sin(2x)\cos(4x)$

iv) $f(x) = \cos(e^{4x})$

v) $f(x) = \frac{e^{x^2}-1}{\ln x}$

For each of these functions, find $\int f(x)dx$. Show your working.

vi) $f(x) = x^2 - 4x + 3$

vii) $f(x) = \sin(4x + \pi)$

viii) $f(x) = e^{3x+4}$

ix) $f(x) = 4x\cos(3x)$

x) $f(x) = x^3 \ln x$

b) In some situations, we would like to differentiate a function more than once. We call this *higher-order differentiation*, and we can define it in the following way:

$$f^{(0)}(x) = f(x)$$
$$f^{(n+1)}(x) = \left(f^{(n)}\right)'(x)$$

i) Let $f(x) = 2x^5 + 3x^4 + 4$. Calculate $f^{(3)}(x)$. Show your working.

ii) Now let $f(x)$ be a polynomial of order n, i.e. one whose highest non-zero coefficient is c_n. For example, the polynomial above is of order 5, since it has highest non-zero coefficient $c_5 = 2$. Show that $f^{(n+1)}(x) = 0$

iii) Now let $f(x) = \sin x$. Calculate $f^{(5)}(x)$. Show your working.

iv) Fix some $a \in \mathbb{R}$. With $f(x) = \sin(x)$ as before, we define a new function $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = f^{(n)}(a)$. Write an alternative definition of $g(n)$ that does not involve differentiation.

c) Below is an alternative but equivalent definition of the definite integral of $f : \mathbb{R} \to \mathbb{R}$ between a and b, given $x_k = a + k\left(\frac{b-a}{n}\right)$ for $0 \leq k \leq n$:

$$I_n(f, a, b) = \left(\frac{b-a}{n}\right)\sum_{k=0}^{n-1} f(x_k)$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} I_n(f, a, b)$$

i) For $f(x) = x^2$, calculate $I_5(f, -2, 8)$. Show your working.

ii) We now define

$$J_n(f, a, b) = \left(\frac{b-a}{n}\right)\sum_{k=1}^{n} f(x_k)$$

For $f(x) = x^2$, calculate $J_5(f, -2, 8)$. Show your working.

iii) Calculate $\int_{-2}^{8} x^2 dx$

iv) We call a function $f : \mathbb{R} \to \mathbb{R}$ *monotone increasing* if $x > y$ implies $f(x) > f(y)$. Show that, if f is monotone increasing, then, for any $a, b \in \mathbb{R}, a < b$, and $n \in \mathbb{N}$

$$I_n(f, a, b) < J_n(f, a, b)$$

v) Without formally proving it, explain why:

$$\lim_{n\to\infty} J_n(f, a, b) = \int_a^b f(x)dx = \lim_{n\to\infty} I_n(f, a, b)$$

Question 2 (50%):

a) We define the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \quad C = \begin{pmatrix} -3 & 2 \\ -4 & 3 \\ 1 & 4 \end{pmatrix}$$

i) All but one of the following matrix operations are well-defined. Indicate the one which is not, and calculate the rest. Show your working.

- $4C$
- AB

- B^2
- CA
- $\det(A)$
- A^{-1}

ii) Find the eigenvalues and corresponding eigenvectors for A. Show your working.

b) Consider the following system of linear equations:

$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= -1 \\ 3x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + 3x_2 - 4x_3 &= 2 \end{aligned}$$

i) Express this system of linear equations with a single function $f : \mathbb{R}^3 \to \mathbb{R}^3$, without using a matrix.

ii) Using the fact that

$$\begin{pmatrix} 2 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ -11 & \frac{-7}{2} & \frac{13}{2} \\ -7 & -2 & 4 \end{pmatrix}$$

calculate the solution to the system of linear equations above. Show your working.

iii) Using the fact that

$$\det\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix} = 0$$

explain why we can't use the same method as before to find solutions to the system of linear equations below:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 3x_1 - x_2 + 2x_3 &= 0 \\ 5x_1 + 3x_2 - 4x_3 &= 0 \end{aligned}$$

iv) For the purposes of this question, we will call a system of linear equations *uniform* when all the constants are zero. For example, the system in the previous part is uniform. Explain why we can always find at least one solution for any uniform system of linear equations.

c) Let A be a 2×2 matrix.

i) Show that if $\det A = 0$, then A has an eigenvalue of 0.

ii) Show that if A has an eigenvalue of 0, then $\det A = 0$.

iii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to the 2×2 matrix A. Suppose A has an eigenvalue of 0, corresponding to the eigenvector $\binom{a}{b}$. Geometrically describe the effect of this transformation on areas in 2-dimensional Euclidean space.