19.07

1) $e^{-z^2} \to e^{-\frac{1}{t^2}}$ $\qquad z^2 = \frac{1}{t^2}$

$$e^{-\frac{1}{t^2}} = 1 - \frac{1}{t^2} + \frac{1}{2!t^4} - \frac{1}{3!t^6} + \dots$$

$t=0$; $z=\infty$ – сущ. особая ~~точка~~ точка.

3) $\sin\frac{z_1}{z^2}$ $(a=0)$

$\lim\limits_{z\to 0}\left(\sin\frac{z}{z^2}\right) \nexists$ → сущ. особая точка.

19.15

1) $$\frac{(1+z^2)^2}{1-z^2} = \frac{(1+z^2)^2}{(1-z)(1+z)} = \left\{\begin{matrix} 1-z=\varepsilon \\ z = 1-\varepsilon \end{matrix}\right\} =$$

$$= \frac{(1+(1-\varepsilon)^2)^2}{\varepsilon(2-\varepsilon)} = \frac{1}{\varepsilon}\cdot\frac{(\varepsilon^2-2\varepsilon+2)^2}{(2-\varepsilon)}$$

1й порядок полюса, очевидно

с $(1+z)$ аналогично 1й порядок.

2) $$\operatorname{ctg} z = \frac{\cos z}{\sin z} = \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\right)}{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)} =$$

$$= \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\right)\left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right) + (\dots)^2 + \dots\right) \Rightarrow$$

$k = 0; 1; 2 \dots$

$\pm\pi k$ – полюс 1го порядка; $\frac{\pi}{2}+\pi k$ – нули

3) $z \operatorname{tg}^2 z =$

$z=0$;

$z=\pm\pi k$ – нули.

$\xi = z - \frac{\pi}{2}$

$$\left(\xi + \frac{\pi}{2}\right)\operatorname{ctg}^2\xi = \left(\xi+\frac{\pi}{2}\right)\frac{1}{\xi^2}\left(1 - \frac{\xi^2}{2!} + \frac{\xi^2}{4!} \dots\right)\left(1 + \left(\frac{\xi^2}{3!} - \frac{\xi^4}{5!} + \dots\right) + (\dots)^2 + \dots\right) = \frac{\pi}{2}\frac{1}{\xi^2}(\dots) +$$

$$+ \frac{1}{\xi}(\dots) + \dots$$

(аналогичн. пред. задаче)

$z = \frac{\pi}{2} + \pi k$ – полюс 2го порядка.

20.01

1) $\sum_{-\infty}^{+\infty} 2^{-|n|} \cdot z^n = \sum_{-\infty}^{+\infty} \frac{1}{2^{|n|}} \cdot z^n =$

$= \sum_{-\infty}^{0} \frac{z^n}{2^{-n}} + \sum_{1}^{\infty} \left(\frac{z}{2}\right)^n =$

$n = -m$

$= \sum_{0}^{\infty} \frac{1}{(2z)^m} + \sum_{1}^{\infty} \left(\frac{z}{2}\right)^n$

$\begin{cases} \left|\frac{1}{2z}\right| < 1 \\ \left|\frac{z}{2}\right| < 1 \end{cases} \Rightarrow \underline{\frac{1}{2} < |z| < 2}$

2) $\sum_{-\infty}^{\infty} \frac{z^n}{3^n+1} = \cancel{\sim \sum_{-\infty}^{\infty} \frac{z^n}{3^n}}$ ~~=> 1 < |z| < 3~~

$= \sum_{-\infty}^{0} \frac{z^n}{3^n+1} + \sum_{1}^{\infty} \frac{z^n}{3^n+1} = \sum_{0}^{\infty} \frac{1}{\left(\frac{1}{3^n}+1\right) z^n} +$ $\quad m = -n$

$+ \sum_{1}^{\infty} \frac{z^n}{3^n} = \sum_{0}^{\infty} \frac{3^m}{3^m+1} \cdot \frac{1}{z^m} + \sum_{1}^{\infty} \left(\frac{z}{3}\right)^n =$

$= \sum_{0}^{\infty} \frac{1}{z^m} + \sum_{1}^{\infty} \left(\frac{z}{3}\right)^n \quad \begin{cases} \left|\frac{z}{3}\right| < 1 \\ \left|\frac{1}{z}\right| < 1 \end{cases} \Rightarrow \underline{1 < |z| < 3}$

~~3)~~ 4) $\sum_{-\infty}^{+\infty} 2^{-n^2}(z+1)^n =$

$= \cancel{\sum_{-\infty}^{0}} \sum_{0}^{\infty} 2^{-m^2} \frac{1}{(z+1)^m} + \sum_{1}^{\infty} \frac{(z+1)^n}{2^{n^2}} =$

$\lim_{n \to \infty} \sqrt[n]{\frac{(z+1)^n}{2^{n^2}}} = \frac{z+1}{2^n} = 0 < 1$

$\lim \sqrt[n]{\frac{1}{2^{n^2} \cdot (z+1)^n}} = 0 < 1$

$0 < |z+1| < \infty$

20.09 (1, 5, 7, 8)

1) $\frac{1}{z(z-3)^2}$ $\quad (a=1;\ D: 1<|z-1|<2)$ (ε)

$\varepsilon = z-1$

$$\frac{1}{(\varepsilon+1)(\varepsilon-2)^2} = \frac{A}{(\varepsilon+1)} + \frac{B}{\varepsilon-2} + \frac{C}{(\varepsilon-2)^2} =$$

$$= \frac{1}{9}\cdot\frac{1}{(\varepsilon+1)} + \frac{1}{9(\varepsilon-2)} + \frac{1}{3(\varepsilon-2)^2} =$$

$$= \frac{1}{9\varepsilon(1+\frac{1}{\varepsilon})} + \frac{1}{18(1-\frac{\varepsilon}{2})} + \frac{1}{12(1-\frac{\varepsilon}{2})^2} =$$

$$= \frac{1}{9\varepsilon}\sum_0^\infty\left(\frac{1}{\varepsilon}\right)^n(-1)^n + \frac{1}{18}\sum_0^\infty\left(\frac{\varepsilon}{2}\right)^n + \frac{1}{12}\left(\sum_{n=0}^\infty\left(\frac{\varepsilon}{2}\right)^n\right)^2 \quad (\varepsilon = z-1)$$

$\left(-\frac{3}{2} < D\right)$

5) $$\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{1}{2(z-2)} - \frac{1}{(z-1)} =$$

$$= \frac{1}{2z} - \frac{1}{4}\sum_0^\infty\left(\frac{1}{1-\frac{z}{2}}\right) - \frac{1}{z(1-\frac{1}{z})} =$$

$$= \frac{1}{2z} - \frac{1}{4}\sum_0^\infty\left(\frac{z}{2}\right)^n - \sum_0^\infty\left(\frac{1}{z}\right)^{n+1}$$

7) $$z^3\left(\frac{1}{3(z-2)} - \frac{1}{3(z+1)}\right) = \left\{\begin{matrix} t = z+1 \\ 0<|t|<3\end{matrix}\right| = \frac{(t-1)^3}{t(t-1)} =$$

$$= \frac{t^3-3t^2+3t-1}{t(t-3)} = \frac{t^2}{(t-3)} - \frac{3t}{t-3} + \frac{3}{t-3} - \frac{1}{t(t-3)}$$

$$\frac{t^2}{t-3} = -\frac{t^2}{3(1-\frac{t}{3})} = -\frac{1}{3}\sum_0^\infty\frac{t^{n+2}}{3^n} = -\sum_0^\infty\frac{t^{n+2}}{3^{n+1}}$$

$$-\frac{3t}{t-3} = \frac{t}{1-\frac{t}{3}} = \sum_0^\infty\frac{t^{n+1}}{3^n}$$

$$\frac{3}{t-3} = -\frac{1}{1-\frac{t}{3}} = -\sum_0^\infty\left(\frac{t}{3}\right)^n$$

$$-\frac{1}{t(t-3)} \quad \frac{1}{3t(1-\frac{t}{3})} = \sum_{0}^{\infty} \frac{t^{n-1}}{3^{n+1}}$$

$$\frac{(t-1)^3}{t(t-3)} = \frac{1}{3t} + \frac{1}{9} + \frac{t}{27} + \sum_{3}^{\infty} \frac{t^{n-1}}{3^{n+1}} - \sum_{0}^{\infty} \frac{t^{n+2}}{3^{n+1}} - 1$$

$$+ \sum_{1}^{\infty} \frac{t^{n+1}}{3^n} + t - \sum_{2}^{\infty} \frac{t^n}{3^n} - 1 - \frac{t}{3} =$$

$$= \frac{1}{3t} - \frac{8}{9} + \frac{19}{27} t + \sum_{2}^{\infty} \left(\frac{t^n}{3^{n+2}} - \left(\frac{t}{3}\right)^n + \frac{t^n}{3^{n-1}} - \right.$$

$$\left. - \frac{t^n}{3^{n-1}} \right). \qquad (t = z+1)$$

8)
$$\frac{1}{(z^2-1)(z^2+4)} = \frac{1}{5}\left(\frac{1}{z^2-1} - \frac{1}{z^2+4} \right)$$

$$\frac{1}{z^2-1} = \frac{1}{z^2} \frac{1}{1-\frac{1}{z^2}} = \sum_{0}^{\infty} \frac{1}{z^{2n+2}}$$

$$\frac{1}{z^2+4} = \frac{1}{z^2(1+\frac{4}{z^2})} = \sum_{0}^{\infty} (-1)^n \frac{4^n}{z^{2n+2}}$$

$$\frac{1}{(z^2-1)(z^2+4)} = \frac{1}{5} \sum_{0}^{\infty} \frac{(-1)^n 4^n + 1}{z^{2n+2}}$$

20.16

1) $$z^3 e^{\frac{1}{z}} = z^3 + z^2 + \frac{1}{2} z + \frac{1}{6} + \sum_{1}^{\infty} \frac{z^{-n}}{(n+3)!}$$

2) $$z^2 \sin\left(\pi \frac{z+1}{z}\right) = z^2 \sin\left(\pi + \frac{\pi}{z}\right) = -z^2 \sin\left(\frac{\pi}{z}\right) =$$

$$= -z^2 \left(\frac{\pi}{z} - \left(\frac{\pi}{z}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{z}\right)^5 \frac{1}{5!} + \dots \right)$$

20.21

1) $\frac{z}{(z+2)^2} = \frac{z+2-2}{(z+2)^2} \quad \frac{1}{(z+2)} - \frac{2}{(z+2)^2}$

2) $\frac{e^z+1}{e^z-1} = \frac{e^{z+2\pi i k}+1}{e^{z+2\pi i k}-1} = \frac{e^{\frac{z+2\pi i k}{2}}\left(e^{\frac{z+2\pi i k}{2}}+e^{-\frac{z+2\pi i k}{2}}\right)}{e^{(z+2\pi i k)/2}\left(e^{\frac{z+2\pi i k}{2}}-e^{-\frac{z+2\pi i k}{2}}\right)} =$

$= \frac{2\cos\left(\frac{iz+2\pi k}{2}\right)}{-2i\sin\left(\frac{iz+2\pi k}{2}\right)} = \frac{1}{-i\left(\frac{iz}{2}+\pi k\right)}$

(с точн. до 1го члена) $= \frac{2}{z-2\pi i k}$

3) $\frac{z-1}{\sin^2 z} = \frac{z-1}{z^2} \approx -\frac{1}{z^2} + \frac{1}{z}$

4) $\frac{e^{iz}}{z^2+b^2} = \frac{e^{iz}}{(z+ib)(z-ib)} \left\{ z-ib=\varepsilon \right. =$

$= \frac{e^{i\varepsilon}\cdot e^{-b}}{(\varepsilon+2ib)\varepsilon} = \frac{1}{\varepsilon}\frac{1\cdot e^{-b}}{\varepsilon+2ib} \approx \frac{1}{\varepsilon}\frac{e^{-b}}{2ib} = \frac{1}{z-ib}\frac{e^{-b}}{2ib}$

5) $\frac{(z^2+1)^2}{z^2+b^2} = \left\{ \varepsilon = \frac{1}{z} \right\} = \frac{\left(\left(\frac{1}{\varepsilon}\right)^2+1\right)^2}{\left(\frac{1}{\varepsilon}\right)^2+b^2} = =$

$= \frac{(z+i)^2(z-i)^2}{(z+ib)(z-ib)} = \left\{ \varepsilon = \frac{1}{z} \right\} = \frac{\left(\frac{1}{\varepsilon}+i\right)^2\left(\frac{1}{\varepsilon}-i\right)^2}{\left(\frac{1}{\varepsilon}+ib\right)\left(\frac{1}{\varepsilon}-ib\right)} =$

$= \frac{\frac{(1+\varepsilon i)^2\cdot(1-\varepsilon i)^2}{\varepsilon^4}}{\frac{(1+i\varepsilon b)(1-i\varepsilon b)}{\varepsilon^2}} = \frac{1(1+\varepsilon i)^2(1-\varepsilon i)^2}{\varepsilon^2(1+i\varepsilon b)(1-i\varepsilon b)} \approx$

$\approx \frac{1}{\varepsilon^2} \approx z^2$

6) $$\frac{ze^{iz}}{(z^2+b^2)^2} = \frac{ze^{iz}}{(z-ib)^2(z+ib)^2} = \frac{(z-ib)e^{iz} + ibe^{iz}}{(z-ib)^2(z+ib)^2} = \frac{ibe^{iz}}{(z-ib)^2(z+ib)^2} + \frac{e^{iz}}{(z-ib)(z+ib)} \simeq$$

$$\simeq -\frac{ie^{-b}}{4b(z-ib)^2} - \frac{e^{-b}}{4b^2}\frac{1}{z-ib} = -\frac{(ib+z-ib)e^{-b}}{4b^2(z-ib)^2} =$$

$$= -\frac{ie^{-b}}{4b}\frac{1}{(z-ib)^2}$$

7) $$\frac{z}{(z^2+b^2)^2} = \frac{1}{(z-ib)(z+ib)^2} + \frac{ib}{(z-ib)^2(z+ib)^2} \simeq$$

$$\simeq -\frac{1}{4b^2(z-ib)} - \frac{i}{4b(z-ib)^2} = -\frac{i}{4b}\cdot\frac{1}{(z-ib)^2}$$