① Леонов А. В

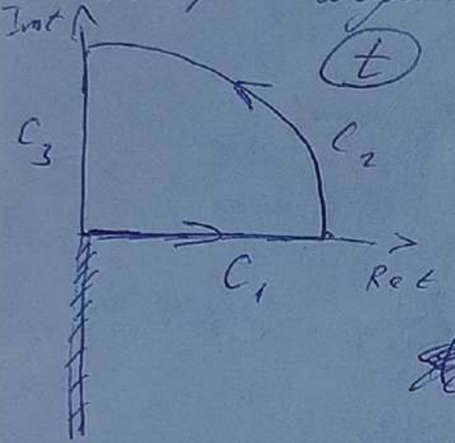
$$C = \int_0^\infty \frac{\ln t}{t} \sin t \, dt$$

$$f(a) = \int_0^\infty t^{a-1} \sin t \, dt$$

$$C = \lim_{a \to 0} f'(a)$$

$$f(a) = \operatorname{Im} F(a) \Rightarrow F(a) = \int_0^\infty t^{a-1} e^{it} dt$$

Заметим, что если подставить $t = it'$, то интеграл сведется к Гамма функции {C_2 сходится из-за больших осцилляций}



$$C_1 + C_2 + C_3 = 0$$

$$C_2 = 0 \; (\text{т.к. } C_2 \sim \int_0 \frac{\ln t}{t} \sin t = 0 \big|_{t \to \infty})$$

$$\int_0^\infty t^{a-1} e^{it} + \int_\infty^0 (i)^a \cdot t^{a-1} e^{-t} dt = 0$$

$$\int_0^\infty t^{a-1} e^{it} - \int_0^\infty e^{\frac{i\pi a}{2}} t^{a-1} e^{-t} dt = 0$$

$$F(a) - e^{\frac{i\pi a}{2}} \cancel{F(a)}\,\Gamma(a) = 0$$

$$F(a) = e^{\frac{i\pi a}{2}} \Gamma(a)$$

$$f(a) = \Gamma(a) \cdot \sin\left(\frac{\pi a}{2}\right)$$

$$C = f'(a)\big|_{a \to 0} = \left(\Gamma(a) \cdot \sin\left(\frac{\pi a}{2}\right)\right)' = \left(\left[\frac{1}{a} - \gamma\right]\left[\frac{\pi a}{2}\right]\right)'\Big|_{a \to 0} =$$

$$= \left(\left[\frac{\pi}{2} - \gamma \frac{\pi a}{2}\right]\right)'\Big|_{a \to 0} = \boxed{-\gamma \frac{\pi}{2}}$$

② $I(\nu) = \int\limits_0^{\frac{\pi}{2}} \sin^\nu x \, dx$

Сходится при $Re\, \nu > -1$

$t = \sin x$

$dt = \cos x \, dx \Rightarrow dx = \frac{dt}{\sqrt{1-t^2}}$

$I(\nu) = \int\limits_0^1 \frac{t^\nu}{\sqrt{1-t^2}} dt \,=$

$p = t^2$

$dp = 2t\,dt \Rightarrow dt = \frac{dp}{2p^{1/2}}$

$$= \int\limits_0^1 \frac{p^{\frac{1}{2}\nu}}{(1-p)^{1/2} p^{1/2} \cdot 2} dp = \int\limits_0^1 p^{1/2\nu - 1/2}(1-p)^{-1/2} dp =$$

$$= \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2 \cdot \Gamma\left(\frac{1}{2}\nu + 1\right)} = \boxed{\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\nu + 1\right)}}$$

③ $h(a) = \int\limits_0^1 \frac{t^{a-1}}{1+t} dt \;(a>0) = \cancel{\int\limits_0^1 t^{a-1}(1-t+t^2+\ldots)dt}$

$$\cancel{= \int\limits_0^1 t^{a-1} - \int\limits_0^1 t^a + \int\limits_0^1 t^{a+1} + \ldots}$$

$$= (-1)^n \sum_0^\infty \int\limits_0^1 t^{n+a-1} dt = (-1)^n \sum_0^\infty \frac{t^{a+n}}{(a+n)}\Big|_0^1 =$$

$$= \sum_0^\infty \frac{(-1)^n}{a+n} = \sum_0^\infty \left(\frac{1}{a+2n+1} - \frac{1}{a+2n}\right) = \frac{1}{2}\sum_0^\infty \left(\frac{1}{n+\frac{a+1}{2}} - \frac{1}{n+\frac{a}{2}}\right) =$$

$$= \frac{1}{2}\left(\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right)$$

№2

$$I(\alpha)=\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\int_0^\infty x^\alpha(1+x)^{2\alpha}(2+x)^{3\alpha}e^{-x}\,dx$$

$\alpha=-1$

$$\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin\pi x}\Rightarrow\Gamma\left(\frac{\alpha+1}{2}\right)\approx\frac{2}{\alpha+1}\quad(\text{при } \alpha=-1)$$

Посмотрим поведение в $\varepsilon\to0$ окрестн.

$$I(-1)=\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\int_0^\varepsilon x^\alpha(1+x)^{2\alpha}(2+x)^{3\alpha}e^{-x}\,dx=\frac{\alpha+1}{2}\int_0^\varepsilon x^\alpha\cdot2^{3\alpha}\,dx=$$

$$=\frac{1}{16}(\alpha+1)\int_0^\varepsilon x^\alpha\,dx=\frac{1}{16}\varepsilon^{\alpha+1}=\frac{1}{16}$$

Остальная часть интеграла очень быстро затухает из-за $\Gamma(0)$.

$$I(-1)=\frac{1}{16}$$

$$I_{con}(\alpha)=\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\int_0^\infty x^\alpha\cdot2^{3\alpha}\cdot e^{-x}\,dx=\frac{8^\alpha}{\Gamma\left(\frac{\alpha+1}{2}\right)}\cdot\Gamma(\alpha+1)$$

$$I_{con}(-2)=\frac{1}{64\,\Gamma\left(-\frac{1}{2}\right)}\cdot\Gamma(-1)$$

$$\operatorname{Res}I_{con}(-2)=\operatorname{Res}\left(\frac{1}{64\,\Gamma\left(-\frac{1}{2}\right)}\Gamma(-1)\right)=-\frac{1}{128\sqrt{\pi}}\operatorname{Res}(\Gamma(-1))=$$

$$=\frac{1}{128\sqrt{\pi}}$$

$$I(\alpha)=\frac{8^\alpha}{\Gamma\left(\frac{\alpha+1}{2}\right)}\Gamma(\alpha)\cdot\alpha$$

$$\operatorname{Res}I(-2)=-\frac{1}{64\sqrt{\pi}}\operatorname{Res}(\Gamma(-2))=$$

$$=\frac{1}{128\sqrt{\pi}}$$

N3

$$G(in) = \sum_{k=1}^{\infty}\left(\frac{1}{-a+ik+in} - \frac{1}{-a-ik+in} + \frac{2i}{k}\right) =$$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{i(k+n+ia)} + \frac{1}{i(k+n-ia)} + \frac{2i}{k}\right) =$$

$$= i\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{(k+n+ia)} + \frac{1}{k} - \frac{1}{k-n-ia}\right) =$$

$$= i\sum_{k=}\left(\psi(1+n+ia) + \psi(1-n-ia) + 2\gamma\right) \ominus$$

$$\psi(z+1) = -\gamma + \sum\left(\frac{1}{n} - \frac{1}{n+z}\right)$$

$$\psi(1-z) = \psi(z) + \pi\cot(\pi z)$$

Чтобы избавиться от полюсов перепишем 2-ю дигамма ф. через зерк. тожд. и замены $(n = -iz)$

$$\ominus\ i\sum_{k=1}^{\infty}\left(\psi'(1-iz+ia) + \psi(ia-iz) + 2\gamma\right)$$

N4.

$$L(z) = \frac{1}{\Gamma(z)}\int_0^{\infty}\frac{x^{z-1}e^{-x}}{1+e^{-2x}}\,dx$$

$$e^x = t$$

$$e^x dx = dt$$

$$L(1) = \int_1^{\infty}\frac{t^{-2}}{1+t^{-2}}dt = \int_1^{\infty}\frac{1}{1+t^2}dt = \frac{\pi}{4}$$

№4

$$\frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}e^{-x}}{1+e^{-2x}}dx = \frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}}{e^{x}+e^{-x}}dx = \frac{1}{\Gamma(z)}\left(\frac{x^z}{z}\frac{1}{e^x+e^{-x}}\Big|_0^\infty - \int_0^\infty \frac{x^z(e^x-e^{-x})}{z(e^x+e^{-x})^2}\right) =$$

$$= -\frac{1}{\Gamma(z)z}\int_0^\infty \frac{x^z(e^{x}-e^{-x})}{(e^x+e^{-x})^2}dx$$

Устремим z к 0^+ и получим:

$$\Gamma(z)\big|_{z\to 0^+} \sim \frac{1}{z}\ ;\ x^z \sim 1$$

$$L(0^+) = -\int_0^\infty \frac{e^x-e^{-x}}{(e^x+e^{-x})^2}dx = -\int_0^\infty \frac{d}{dx}\left(\frac{1}{e^x+e^{-x}}\right)dx \qquad L(0)=\frac{1}{2}$$

без Гамма функции: $L(0^+) \sim -\frac{1}{z}\int_0^\infty \left(\frac{1}{e^x+e^{-x}}\right)' dx$

Теперь выведем ряд для $L(z)$

$$L(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}e^{-x}}{1+e^{-2x}}dx = \frac{1}{\Gamma(z)}\int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-2xn-x}x^{z-1}dx =$$

$$= \frac{\sum(-1)^n}{\Gamma(z)}\int_0^\infty e^{-x(2n+1)}x^{z-1}dx = \frac{1}{\Gamma(z)}\left(\int_0^\infty e^{-x}x^{z-1}dx - \int_0^\infty e^{-3x}x^{z-1}dx + \dots\right) =$$

$$= \frac{1}{\Gamma(z)}\left(\Gamma(z) - \Gamma(z)\cdot\left(\frac{1}{3}\right)^z + \Gamma(z)\cdot\left(\frac{1}{5}\right)^z + \dots\right) =$$

$$= 1 - \frac{1}{3^z} + \frac{1}{5^z} + \dots$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4} \quad \left(\operatorname{arctg} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$$

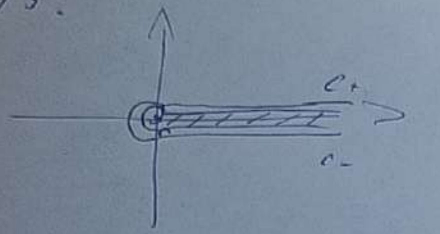
$$F_1 = \ln \prod_{n=0}^{N-1}\frac{4n+3}{4n+5} = \ln\frac{\prod_{n=0}^{N}(4n+3)}{\prod_{n=0}^{N}(4n+5)} = \ln\frac{4^N\,\Gamma(N+\frac{3}{4})\,\Gamma(\frac{5}{4})}{4^N\,\Gamma(N+\frac{5}{4})\,\Gamma(\frac{3}{4})}$$

$$\Gamma\left(N+\frac{3}{4}\right) \approx \sqrt{\frac{2\pi}{N+\frac{3}{4}}}\left(\frac{N+\frac{3}{4}}{e}\right)^{N+\frac{3}{4}}$$

$$F_1 = \ln\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}\cdot\frac{(N+\frac{3}{4})^{1/2}}{(N+\frac{5}{4})^{1/2}}\cdot\frac{(N+\frac{3}{4})^{N+\frac{3}{4}}}{(N+\frac{5}{4})^{N+\frac{5}{4}}}\cdot e^{1/2}\ \{N\to\infty\} \approx \ln\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}N^{-1/2}$$

$$L'(0) = \ln\frac{2\,\Gamma(5/4)}{\Gamma(3/4)}$$

№5.



$$\int_\gamma = \int_\varepsilon^{\to 0} + \int_{C_+} + \int_{C_-} = (1 - z e^{2\pi i z}) \int_{C_+}$$

$$L(z) = \frac{1}{1-e^{2\pi i z}} \frac{1}{\Gamma(z)} \int_\gamma \frac{t^{z-1}e^{-t}}{1+e^{-2t}}\,dt$$

$t = \frac{i\pi}{2} + i\pi n \quad (n \in \mathbb{Z})$ – полюса

$= e^{\frac{i\pi}{2}} \cdot \frac{\pi}{2}(1+2n)$ $\quad n > 0$ – полож. полюса

$n < 0$ – отриц. полюса.

$$\operatorname*{Res}_{n\geq 0} \frac{t^{z-1}e^{-t}}{1+e^{-2t}} = e^{-t}t^{z-1}\Big|_{t=\frac{\pi}{2}e^{\frac{i\pi}{2}}(2n+1)} = \frac{(-1)^n e^{-\frac{i\pi}{2}}}{2i}\left(\frac{\pi}{2}\right)^{z-1}(2n+1)^{z-1}\cdot e^{\frac{i\pi z}{2}}$$

$$\operatorname*{Res}_{n\leq 0} \frac{t^{z-1}e^{-t}}{1+e^{-2t}} = e^{-t}t^{z-1}\Big|_{t=\frac{\pi}{2}e^{-\frac{i\pi}{2}}(2|n|+1)} = \frac{e^{-\frac{i\pi}{2}}}{2i}(-1)^n\left(\frac{\pi}{2}\right)^{z-1}(2n+1)^{z-1}e^{-\frac{i\pi}{2}z}$$

~~N5~~

№6.

$$C = \int_0^\infty \frac{\ln x\, dx}{\cosh x}$$

$$L(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x + e^{-x}}\, dx = \frac{1}{2\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{\cosh(x)}\, dx$$

$$\frac{d}{dz}\left[L(z)\cdot\Gamma(z)\right] = \frac{1}{2}\int_0^\infty \frac{\ln x \cdot x^{z-1}}{\cosh(x)}\, dx$$

$$C = 2\frac{d}{dz}\left[L(z)\cdot\Gamma(z)\right]\Big|_{z=1} = \cancel{2L} \; 2\left[L'(z)\,\Gamma(z) + \right.$$

$$\left. + L(z)\,\Gamma'(z)\right]\Big|_{z=1} = 2\left[L'(1)\,\Gamma(1) + L(1)\,\Gamma'(1)\right]$$

~~$L(1-1) = \frac{2}{\pi}\cdot\sin\frac{\pi}{2}\,\Gamma(1)\,L(1)$ (исп. зерк. формулу)~~

~~$L(0) = \frac{2}{\pi} L(1)$~~

~~$L(1-1) = \frac{2}{\pi}\sin\frac{\pi}{2}\,\Gamma(1)\,L(1)$ (зерк. формула)~~

~~т.е. $L'(1) = \frac{\pi}{2} L'(0)$ $\quad \Gamma(1)\,L(1) = \frac{\pi}{2} L(0)$~~

~~$C = 2\left[\frac{\pi}{2} L'(0)\right]$~~

$$C = 2\frac{d}{dz}\left(L(z)\cdot\Gamma(z)\right) \,=$$

Подставим зеркальную формулу $L(1-z) = \left(\frac{2}{\pi}\right)^z \sin\frac{\pi z}{2}\,\Gamma(z)L(z)$

$$= \cancel{2L}\; 2\cdot\frac{d}{dz}\left[L(1-z)\left(\frac{\pi}{2}\right)^z\cdot\frac{1}{\sin\frac{\pi z}{2}}\right] = -2\cdot L'(1-z)\left(\frac{\pi}{2}\right)^z\cdot\frac{1}{\sin\frac{\pi z}{2}} +$$

$$+ 2\cdot L(1-z)\ln\frac{\pi}{2}\cdot\left(\frac{\pi}{2}\right)^z\cdot\frac{1}{\sin\frac{\pi z}{2}} + 0 =$$

$$= -L'(0)\cdot\pi + \cancel{2}L(0)\cdot\ln\frac{\pi}{2}\cdot\pi$$