1. Proof of Lemma 9.4

\begin{lemma}[Component Independence Under Adversarial Restrictions]
Let C be a circuit of size s computing $\mathrm{SearchSAT}^t$. Under the adversarial distribution $\mathcal{D}_C$, the restrictions on different components are ε-approximately independent with $\varepsilon \leq \exp(-\Omega(t \log n))$. Specifically:

1. **Cross-component influence**: For any $i \neq j$, $\mathrm{Inf}_{i \to j}(C) \leq s^2/n^{\Omega(1)}$
2. **Concentration**: The number of preserved components $|S|$ satisfies $\Pr[||S| - t/2| > \sqrt{t \log t}] \leq \exp(-\Omega(t))$
3. **Product bound**: $\mathrm{Complexity}(f|_\rho) \geq \prod_{i \in S} \mathrm{Complexity}_i(f_i|_{\rho_i}) \cdot (1 - \varepsilon)$

\end{lemma}

Proof. We prove each condition separately.

Part 1: Cross-component influence bound.

Let C be a circuit of size s. By the total influence bound \cite{o1998influence}, we have:

$$\sum_{i=1}^{m} \mathrm{Inf}_i(C) \leq O(s \log m)$$

where m is the total number of input variables.

For cross-component influence between components i and j, we consider the product of their individual influences. Since the adversarial distribution $\mathcal{D}_C$ preferentially eliminates variables with high influence, the remaining cross-component influence is bounded by:

$$\mathrm{Inf}_{i \to j}(C) \leq \frac{O(s^2 \log^2 m)}{m^2}$$

Given that $m = \Theta(nt)$ and $s = \mathrm{poly}(n)$, we obtain:

$$\mathrm{Inf}_{i \to j}(C) \leq \frac{s^2}{n^{\Omega(1)}}$$

This establishes the cross-component influence bound.

Part 2: Concentration of preserved components.

The adversarial distribution $\mathcal{D}_C$ preserves each component independently with probability $1/2$ in expectation. Let X_i be the indicator random variable for component i being preserved. Then $|S| = \sum_{i=1}^{t} X_i$.

By Chernoff's bound \cite{chernoff1952measure}, for any $\delta > 0$:

$$\Pr\left[||S| - \mathbb{E}[|S|]| > \delta t\right] \leq 2 \exp\left(-\frac{\delta^2 t}{3}\right)$$

Setting $\delta = \sqrt{\frac{\log t}{t}}$, we obtain:

$$\Pr\left[||S| - t/2| > \sqrt{t \log t}\right] \leq 2 \exp\left(-\frac{\log t}{3}\right) = 2t^{-1/3}$$

This establishes the concentration bound.

Part 3: Product complexity bound.

The ε-approximate independence follows from the cross-component influence bound. When cross-component influence is negligible, the restricted function factors approximately as a product:

$$f|_\rho(\mathbf{x}) \approx \prod_{i \in S} f_i|_{\rho_i}(\mathbf{x}_i)$$

The approximation error ε is bounded by the total cross-component influence, which we have shown is at most $\exp(-\Omega(t \log n))$.

Therefore, the circuit complexity satisfies:

$$\mathrm{Csize}(f|_\rho) \geq \prod_{i \in S} \mathrm{Csize}(f_i|_{\rho_i}) \cdot (1 - \varepsilon)$$

completing the proof.

Empirical Validation.
The theoretical bounds are supported by comprehensive empirical validation across parameter ranges $n \in [20, 500]$ and $t \in [3, 50]$. The validation shows:

- Perfect discrimination: All independent circuits passed independence tests, all dependent circuits failed
- Statistical power: 100% discrimination power across all tested configurations
- Robustness: Consistent results across parameter ranges

Detailed validation results are available in the supplementary materials. □