Differential Equations Notes

Alex Z

Fall 2025

Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 Second-Order Linear Differential Equations

Or differential equations of the form
$$y'' + p(t)y' + q(t)y = g(t).$$

3.1 Homogenous Second-Order Equations

Remember, a **linear** second-order differential equation is of the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Nonlinear differential equations are super hard and annoying to tackle and as such they're just not tackled in this book :/.
In second-order differential equations, a problem with an initial condition has initial condition of the form

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases}.$$

Note that there are two initial equations given - the location of y at time t_0, and the slope of y at time t_0.

Definition 3.1 (*Homogenous*)

A **homogenous** differential equation has no 'constant' terms (terms without y). In the case for our second-order linear differential equations, a homogenous equation of that form can be written as

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

Anyways, it turns out if we solve the homogenous version of the differential equation $P(t)y''+Q(t)y'+R(t)y = G(t)$, we can actually find an expression for y (that may or may not have an integral in it). That's pretty cool.

For this chapter (unfortunately), we will only consider the cases when $\boxed{P,\ Q,\ \text{and } R \text{ are } \textbf{constants}}$.

Thus, our differential equation becomes $ay'' + by' + cy = 0$. Letting $y = e^{rt}$, we find that our equation now becomes

$$\to ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0$$

with $ar^2 + br + c$ called the **characteristic equation** for the general differential equation with constant coefficients shown above.

If we let r_1 and r_2 be two real roots that satisfy the characteristic equation above, then the **general solution** to our differential equation is $y = c_1e^{r_1t} + c_2e^{r_2t}$ with c_1 and c_2 being arbitrary constants. Initial conditions can be solved for summarily.

Exercise 1-4.
1. $y = c_1e^t + c_2e^{-3t}$.
2. $y = c_1e^t + c_2e^{2t}$.
3. $y = c_1e^{t/2} + c_2e^{-t/3}$.
4. $y = c_1 + c_2e^{-5t}$.

□

Exercise 13.
If a differential equation's solution is $c_1e^2t + c_2e^{-3t}$, we have $r_1 = 2$, $r_2 = -3$ and as such our differential equation is $y'' + y' - 6y = 0$.
It probably can be shown that no other differential equation produces the general solution given in the problem. □

Exercise 16.
The characteristic equation for our differential equation is $n^2 - n - 2 = 0$ and as such we have roots $r_1 = 2$, $r_2 = -1$. As such, the general solution to the equation is $y = c_1 e^{2t} + c_2 e^{-t}$.
To make the solution approach 0 as $t \to \infty$, we need $c_1 = 0$ as in any other case, e^{2t} will spiral out to infinity and our solution is unbounded. Thus, we can plug this solution into the second part of the initial value problem $y'(0) = 2$:

$$y'(0) = 2 \to 2 = 2 \cdot 0e^{2t} + (-1) \cdot c_2 e^{-0} \to c_2 = -2.$$

Thus, our final solution to the differential equation is $y_{sol} = -2e^{-t}$ and $y_{sol}(0) = \alpha = -2$. $\square$

3.2 Solutions of Linear Homogenous Equations — the Wronskian

Definition 3.2 (*Differential Operator L*)

A general differential operator *does stuff.*
For now, for continuous functions α and β on some open interval I and for any function ϕ twice differentiable on I, we define the **differential operator** L as

$$L[\phi] = \phi'' + \alpha\phi' + \beta\phi.$$

Note that the result of applying L to some function f is another function g.

In this section we will examine the equation $L[y] = 0$.

Definition 3.3 (*Existence and Uniqueness Theorem*)

(Reproduced from page 110.)
Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \; y'(t_0) = y_0',$$

where p, q, and g are continuous on an open interval I with $t_0 \in I$. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

This existence theorem is pretty similar to Theorem 2.4.1 but generalized to second-order linear differential equations. Note once again the guarantee and uniqueness of a solution to the given differential equation over a certain interval.

Definition 3.4 (*Principle of Superposition*)

If y_1 and y_2 are two solutions to the differential equation $L[y] = 0$, then $y_3 = c_1y_1 + c_2y_2$ is also a solution to the given differential equation for any $(c_1, c_2) \in \mathbb{R}^2$.

Definition 3.5

Wronskian Determinant The **Wronskian Determinant** for the system

$$\begin{cases} c_1y_1(t_0) + c_2y_2(t_0) = y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \end{cases}$$

is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

If W is non-zero, then there is a unique solution to the differential equation $L[y] = 0$ with **any** given initial condition. Otherwise, there are initial conditions to the differential equation that cannot be satisfied no matter how c_1 and c_2 are chosen (113).

Note that if the Wronskian W is non-zero, the two solutions y_1 and y_2 to $L[y] = 0$ are said to form a **fundamental set of solutions**.
(There's a lot more discussion here about uniqueness of solutions, Wronksians, and other things I frankly don't care about.)

Regarding complex valued solutions, if $y = u(t) + iv(t)$ satisfies $L[y] = 0$, then u and v are also solutions to the differential equation $L[y] = 0$ (Theorem 3.2.6, Page 117). This is important for later sections.

For another theorem in this long section, we have....

Definition 3.6 (*Abel's Theorem*)

If y_1 and y_2 are solutions for the differential equation $L[y] = 0$ (and some other general conditions are satisfied), then the Wronksian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c\exp\left(-\int p(t)\ dt\right)$$

where c is a constant dependent on y_1 and y_2 but not on t (Theorem 3.2.7, Page 117)

In summary (page 118), to solve $L[y] = 0$ over some open interval I, we first find two solutions y_1 and y_2 then make sure that $W[y_1, y_2](i) \neq 0$ for some $i \in I$. If this is achieved, y_1 and y_2 would then be a fundamental set of solutions to the given differential equation from which initial-value problems can be solved.

Exercise 12.
We evaluate the differential equation with $y = c\phi(t)$:

$$y'' + p(t)y' + q(t)y = c\phi''(t) + cp(t)\phi'(t) + cq(t)\phi(t) = c(\phi''(t) + p(t)\phi'(t) + q(t)) = g(t).$$

Since we know $\phi(t)$ is a solution to the differential equation, we thus have $c(g(t)) = g(t)$ which cannot hold if $c \neq 1$ and $g(t) \neq 0$.
This does not violate Theorem 3.2.2 (Principle of Superposition) as that principle arises from the special case of when $g(t) = 0$. $\square$

Exercise 13.
No.
If $y = \sin\left(t^2\right)$ is a solution to $L[y] = 0$, then

$$2\cos\left(t^2\right) - 4t^2\sin\left(t^2\right) + p(t)2t\cos\left(t^2\right) + q(t)\sin\left(t^2\right) = \cos\left(t^2\right)(2 + p(t)2t) + \sin\left(t^2\right)\left(-4t^2 + q(t)\right) = 0.$$

To make $L[\sin\left(t^2\right)]$ equal to 0, we thus have to have $2 + p(t)2t = 0$ and $-4t^2 + q(t) = 0$. The latter case is easy to solve but the former implies $p(t) = -\frac{1}{t}$, which is a non-continuous function around the point $t = 0$.
In any case, if we change $q(t)$ to 'cancel' the residue $2\cos\left(t^2\right)$ in the equation above, then in some form or another part of $q(t)$ would contain the fraction $\cot\left(t^2\right)$ meaning q would also be a non-continous function around $t = 0$.
As such, it is impossible to find continuous p and q satisfying $L[\sin\left(t^2\right)] = 0$ over an open interval I containing the point $t = 0$. $\square$

Exercise 15.

$$W[f + 3g, g - g] = (f + 3g)'(f - g) - (f + 3g)(f - g)' = f'(f - g) + 3g'(f - g) - f'(f + 3g) + g'(f + 3g)$$

$$= ff' - f'g + 3fg' - 3gg' - ff' - 3f'g + f'g + 3gg' = -4(f'g - fg') = 4\sin t - 4t\cos t.$$

$\square$

Exercise 17.
Two solutions y_1, y_2 to this differential equation are ce^t and ce^{-2t} for any $c \in \mathbb{R}$. To construct the fundamental set of solutions, we need to reshape our solutions such that $y_a(0) = 1$ and $y_a'(0) = 0$ and also $y_b(0) = 0$ and $y_b'(0) = 1$.

Since our two solutions y_1, y_2 seem pretty dissimilar, we first assume that $y_a = c_1y_1 + c_2y_2$. From here, we just solve for the properties we need; since $y_a = 1$, $c_1 + c_2 = 1$. Similarly, since $y_a'(0) = 0$, $c_1 - 2c_2 = 0$ so $(c_1, c_2) = (2/3,\ 1/3)$.
Doing something similar for y_b, we find that the corresponding $(c_1, c_2) = (1/3,\ -1/3)$. As such,

$$\begin{cases} y_a = \frac{2}{3}e^t + \frac{1}{3}e^{-2t} \\ y_b = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \end{cases}.$$

$\square$

Exercise 23.

$$W = c\exp\left(-\int p(t)\ dt\right) = c\exp\left(-\int \frac{-t(t+2)}{t^2}\ dt\right) = c\exp\left(\int 1 + \frac{2}{t}\ dt\right) = ce^{t+2\ln t} = ct^2e^t.$$

$\square$

Exercise 25.

$$W = c\exp\left(-\int p(x)\ dx\right) = c\exp\left(-\int \frac{-2x}{1-x^2}\ dx\right) = c\exp\left(\int -\frac{1}{u}\ du\right) = ce^{\ln(1/u)} = \frac{c}{1-x^2}.$$

$\square$

Exercise 31. Exact Equations.
Expanding the given expression, we get

$$P'(x)y' + P(x)y'' + f'(x)y + f(x)y' = P(x)y'' + y'(P'(x) + f(x)) + f'(x)y = 0.$$

Equating the coefficients to the general form of a differential equation, we thus have $P'(x) + f(x) = Q(x)$ and $f'(x) = R(x)$.
Taking the derivative of that first equation, we thus have $P''(x) + f'(x) = Q'(x)$ or $P''(x) - Q'(x) + R(x) = 0$ which is exactly the equation that was desired. $\square$

Exercise 32.
32. $P''(x) - Q'(x) + R(x) = 0 - 1 + 1 = 0$ so the equation is exact. Namely, $f(x) = Q(x) - P'(x) = x$ so the problem can be restated as $(y')' + (xy)' = 0 \to y' + xy = c$. This equation is solvable with integrating factor $e^{x^2/2}$ but then the error function pops out so I'm not going to finish this integral. $\square$

Exercise 34.
Since $2 - 1 + (-1) = 0$, we can find $f(x) = -x$. The differential equation then becomes $(x^2y')' + (-xy)' = 0 \to x^2y' - xy = c$.
Solving, we find $y = -\frac{c_1}{3x} + c_2x$. $\square$

3.3 Complex Roots of the Characteristic Equation

What happens when the roots of the characteristic equation $ar^2 + br + c = 0$ for a general differential equation $ay'' + by' + cy = 0$ are imaginary?

Let the roots r_1 and r_2 of the characteristic equation be $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ for real α, β. Then, the corresponding solutions to the differential equation are

$$\begin{cases} y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t) \text{ and} \\ y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t) \end{cases}.$$

In Section 3.2 (Thereom 3.2.6), it was mention that the real and imaginary parts of any solution to a given differential equation are each solutions to the given differential equation. In our case thus, $y_3 = e^{\alpha t}\cos(\beta t)$ and $y_4 = e^{\alpha t}\sin(\beta t)$ are also solutions to $ay'' + by' + cy = 0$, with $W[y_3, y_4] = \beta e^{2\alpha t} \neq 0$.

Exercise 6-8.
6. The quadratic yields roots $r_1, r_2 = 1 \pm i\sqrt{5}$ so the corresponding general solution is $c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t)$.
7. $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$.
8. $y = c_1 e^{-3t}\cos(2t) - c_2 e^{-3t}\sin(2t)$. □

Exercise 25.
(a): $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{t}\dfrac{dy}{dx}$.
$\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{1}{t}\cdot\dfrac{dy}{dx}\right) = -\dfrac{1}{t^2}\dfrac{dy}{dx} + \dfrac{dx}{dt}\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{t^2}\left(\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\right)$.

(b): Simplify substitute everything we just derived in into the equation ... □

Exercise 26-29.
As seen from question 25, we can transform the coefficients of the differential equation $(t^2, \alpha t, \beta)$ into $(1, \alpha - 1, \beta)$. In this case, $\alpha = 1$ and $\beta = 1$ so our new differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

which has solutions $y_1 = \cos(x)$, $y_2 = \sin(x)$. As such, since $x = \ln t$, $y_1 = \cos(\ln t)$ and $y_2 = \sin(\ln t)$ are a set of solutions to the differential equation in terms of t.

27: $\alpha = 4$ and $\beta = 2$ so $y_1 = e^{-x} = \frac{1}{t}$ and $y_2 = e^{-2x} = \frac{1}{t^2}$.
28: $y_1 = \frac{1}{t}$ and $y_2 = t^6$.
29: $y_1 = t^2$ and $y_2 = t^3$. □