Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

6 F

The above sections are spacer sections.

7 Systems of First-Order Linear Equations

7.1 Introduction

Essentially, we consider systems of first-order equations since any higher order differential equation can inevitably be transformed into multiple first order linear transformations.

Moreover, for any $y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$, we cna make the substituions $x_1 = y$, $x_2 = y'$, $\ldots$, $x_n = y^{(n-1)}$ and thus eventually find $x_1' = F_1(t, x_1, x_2, \ldots, x_n)$, $x_2' = F_2(t, x_1, x_2, \ldots, x_n)$ and so on. Thus, we have effectively converted a general differential equation into many teeny tiny first-order differential equations (that are each in their own way, granted, hard to solve).

7.2 Matrices

(note: all uppercase letters from here on out (A, B, C, $\ldots$) will most likely represent matrices from here on out unless they are in function notation (e.g. $F(t)$ would be a function)).

Various matrix preliminaries are covered here. Do note that when the book talks about the **adjoint** of A, they mean the **transpose of the conjugate matrix of** A rather than the cofactor expansion matrix of A.

Integrals, derivatives, and [x] over matrices of functions are just those same operations applied to each individual operations (boring). For example, $\displaystyle\int A\ dt = \int a_{ij}\ dt$.

7.3 More Linear Algebra

(This is just a review of Math 4a......)

7.4 Basic Theory of Systems of First-Order Linear Equations

(a.k.a. A review of section 3.2 but with matrices instead of second-order linear differential equations.)
To examine a system of n first-order linear equations each of the form $x_i' = p_{i1}(t)x_1 + p_{i2}(t)x_2 + \cdots + p_{in}(t)x_n + g_i(t)$, we can rewrite everything in matrix form and obtain the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$, $\mathbf{g}(t) = [g_1(t)\ g_2(t)\ \ldots\ g_n(t)]^T$, and $\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \ldots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \ldots & p_{nn}(t) \end{pmatrix}$.

With matrix equations, multiple solutions ($\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(k)}(t)$) for $\mathbf{x}$ may exist. Moreover, if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two solutions to a first-order homogenous matrix differential equation ($\mathbf{g} = \mathbf{0}$), then $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution to said equation for arbitrary constants c_1, c_2 (Theorem 7.4.1, Page 305).

If we make a big matrix $\mathbf{X} = [\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ dots\ \mathbf{x}^{(n)}]$, then we can calculate its determinant; namely, $\det \mathbf{X} = W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ and as such if $\det \mathbf{X} \neq 0$ at some particular point $t = t_0$, then the solutions $\mathbf{x}^{(1)}$, $\ldots$ are all linearly independent at that point.

Definition 7.1 (*Generalized Abel's Theorem*)

If $\mathbf{x}^{(1)}$, $\ldots$, $\mathbf{x}^{(n)}$ are solutions to a homogenous first-order set of linear differential equations over some open interval I, then over I, either $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}] = 0$ or $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t) \neq 0\ \ \forall t \in I$.

Abel's theorem is super helpful as we only need to evaluate the Wronskian / determinant over one point to conclude the linear dependence/independence of our solutions. (Note: some stuff about a fundamental set of solutions is talked about here but honestly I don't really care :/.)

Similarly to when we looked at real-valued solutions to differential equations, we can turn complex-valued solutions into real solutions:

Definition 7.2 (***Theorem 7.4.5***)

If $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is a solution to the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then solely the real part $\mathbf{u}$ and solely the imaginary part $\mathbf{v}$ are also solutions to the above equation.