

# Differential Equations Notes

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## Remarks

This notes thing was started on 10/19/2025.

I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).

Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

### 3 Second-Order Linear Differential Equations

Or differential equations of the form

$$y'' + p(t)y' + q(t)y = g(t).$$

#### 3.1 Homogenous Second-Order Equations

Remember, a **linear** second-order differential equation is of the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

**Nonlinear** differential equations are super hard and annoying to tackle and as such they're just not tackled in this book :/.

In second-order differential equations, a problem with an initial condition has initial condition of the form

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases}.$$

Note that there are two initial equations given - the location of  $y$  at time  $t_0$ , and the slope of  $y$  at time  $t_0$ .

##### **Definition 3.1 (Homogenous)**

A **homogenous** differential equation has no 'constant' terms (terms without  $y$ ). In the case for our second-order linear differential equations, a homogenous equation of that form can be written as

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

Anyways, it turns out if we solve the homogenous version of the differential equation  $P(t)y'' + Q(t)y' + R(t)y = G(t)$ , we can actually find an expression for  $y$  (that may or may not have an integral in it). That's pretty cool.

For this chapter (unfortunately), we will only consider the cases when  $P$ ,  $Q$ , and  $R$  are **constants**.

Thus, our differential equation becomes  $ay'' + by' + cy = 0$ . Letting  $y = e^{rt}$ , we find that our equation now becomes

$$\rightarrow ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0$$

with  $ar^2 + br + c$  called the **characteristic equation** for the general differential equation with constant coefficients shown above.

If we let  $r_1$  and  $r_2$  be two real roots that satisfy the characteristic equation above, then the **general solution** to our differential equation is  $y = c_1e^{r_1t} + c_2e^{r_2t}$  with  $c_1$  and  $c_2$  being arbitrary constants. Initial conditions can be solved for summarily.

##### **Exercise 1-4.**

1.  $y = c_1e^t + c_2e^{-3t}$ .
2.  $y = c_1e^t + c_2e^{2t}$ .
3.  $y = c_1e^{t/2} + c_2e^{-t/3}$ .
4.  $y = c_1 + c_2e^{-5t}$ .

□

##### **Exercise 13.**

If a differential equation's solution is  $c_1e^{2t} + c_2e^{-3t}$ , we have  $r_1 = 2$ ,  $r_2 = -3$  and as such our differential equation is  $y'' + y' - 6y = 0$ .

It probably can be shown that no other differential equation produces the general solution given in the problem. □

**Exercise 16.**

The characteristic equation for our differential equation is  $n^2 - n - 2 = 0$  and as such we have roots  $r_1 = 2$ ,  $r_2 = -1$ . As such, the general solution to the equation is  $y = c_1 e^{2t} + c_2 e^{-t}$ .

To make the solution approach 0 as  $t \rightarrow \infty$ , we need  $c_1 = 0$  as in any other case,  $e^{2t}$  will spiral out to infinity and our solution is unbounded. Thus, we can plug this solution into the second part of the initial value problem  $y'(0) = 2$ :

$$y'(0) = 2 \rightarrow 2 = 2 \cdot 0e^{2t} + (-1) \cdot c_2 e^{-0} \rightarrow c_2 = -2.$$

Thus, our final solution to the differential equation is  $y_{sol} = -2e^{-t}$  and  $y_{sol}(0) = \alpha = -2$ .  $\square$

## 3.2 Solutions of Linear Homogenous Equations | the Wronskian

### **Definition 3.2 (Differential Operator $L$ )**

A general differential operator *does stuff*.

For now, for continuous functions  $\alpha$  and  $\beta$  on some open interval  $I$  and for any function  $\phi$  twice differentiable on  $I$ , we define the **differential operator**  $L$  as

$$L[\phi] = \phi'' + \alpha\phi' + \beta\phi.$$

Note that the result of applying  $L$  to some function  $f$  is another function  $g$ .

In this section we will examine the equation  $L[y] = 0$ .

### **Definition 3.3 (Existence and Uniqueness Theorem)**

(Reproduced from page 110.)

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

where  $p$ ,  $q$ , and  $g$  are continuous on an open interval  $I$  with  $t_0 \in I$ . This problem has exactly one solution  $y = \phi(t)$ , and the solution exists throughout the interval  $I$ .

This existence theorem is pretty similar to Theorem 2.4.1 but generalized to second-order linear differential equations. Note once again the guarantee and uniqueness of a solution to the given differential equation over a certain interval.

### **Definition 3.4 (Principle of Superposition)**

If  $y_1$  and  $y_2$  are two solutions to the differential equation  $L[y] = 0$ , then  $y_3 = c_1 y_1 + c_2 y_2$  is also a solution to the given differential equation for any  $(c_1, c_2) \in \mathbb{R}^2$ .

### **Definition 3.5**

Wronskian Determinant The **Wronskian Determinant** for the system

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0, \\ c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0 \end{cases}$$

is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} = y_1(t_0)y'_2(t_0) - y'_1(t_0)y_2(t_0).$$

If  $W$  is non-zero, then there is a unique solution to the differential equation  $L[y] = 0$  with **any** given initial condition. Otherwise, there are initial conditions to the differential equation that cannot be satisfied no matter how  $c_1$  and  $c_2$  are chosen (113).

Note that if the Wronskian  $W$  is non-zero, the two solutions  $y_1$  and  $y_2$  to  $L[y] = 0$  are said to form a **fundamental set of solutions**.

(There's a lot more discussion here about uniqueness of solutions, Wronskians, and other things I frankly don't care about.)

Regarding complex valued solutions, if  $y = u(t) + iv(t)$  satisfies  $L[y] = 0$ , then  $u$  and  $v$  are also solutions to the differential equation  $L[y] = 0$  (Theorem 3.2.6, Page 117). This is important for later sections.

For another theorem in this long section, we have....

### **Definition 3.6 (Abel's Theorem)**

If  $y_1$  and  $y_2$  are solutions for the differential equation  $L[y] = 0$  (and some other general conditions are satisfied), then the Wronskian  $W[y_1, y_2](t)$  is given by

$$W[y_1, y_2](t) = c \exp \left( - \int p(t) dt \right)$$

where  $c$  is a constant dependent on  $y_1$  and  $y_2$  but not on  $t$  (Theorem 3.2.7, Page 117)

In summary (page 118), to solve  $L[y] = 0$  over some open interval  $I$ , we first find two solutions  $y_1$  and  $y_2$  then make sure that  $W[y_1, y_2](i) \neq 0$  for some  $i \in I$ . If this is achieved,  $y_1$  and  $y_2$  would then be a fundamental set of solutions to the given differential equation from which initial-value problems can be solved.

### **Exercise 12.**

We evaluate the differential equation with  $y = c\phi(t)$ :

$$y'' + p(t)y' + q(t)y = c\phi''(t) + cp(t)\phi'(t) + cq(t)\phi(t) = c(\phi''(t) + p(t)\phi'(t) + q(t)) = g(t).$$

Since we know  $\phi(t)$  is a solution to the differential equation, we thus have  $c(g(t)) = g(t)$  which cannot hold if  $c \neq 1$  and  $g(t) \neq 0$ .

This does not violate Theorem 3.2.2 (Principle of Superposition) as that principle arises from the special case of when  $g(t) = 0$ .  $\square$

### **Exercise 13.**

No.

If  $y = \sin(t^2)$  is a solution to  $L[y] = 0$ , then

$$2\cos(t^2) - 4t^2\sin(t^2) + p(t)2t\cos(t^2) + q(t)\sin(t^2) = \cos(t^2)(2 + p(t)2t) + \sin(t^2)(-4t^2 + q(t)) = 0.$$

To make  $L[\sin(t^2)]$  equal to 0, we thus have to have  $2 + p(t)2t = 0$  and  $-4t^2 + q(t) = 0$ . The latter case is easy to solve but the former implies  $p(t) = -\frac{1}{t}$ , which is a non-continuous function around the point  $t = 0$ . In any case, if we change  $q(t)$  to 'cancel' the residue  $2\cos(t^2)$  in the equation above, then in some form or another part of  $q(t)$  would contain the fraction  $\cot(t^2)$  meaning  $q$  would also be a non-continuous function around  $t = 0$ .

As such, it is impossible to find continuous  $p$  and  $q$  satisfying  $L[\sin(t^2)] = 0$  over an open interval  $I$  containing the point  $t = 0$ .  $\square$

### **Exercise 15.**

$$\begin{aligned} W[f + 3g, g - g] &= (f + 3g)'(f - g) - (f + 3g)(f - g)' = f'(f - g) + 3g'(f - g) - f'(f + 3g) + g'(f + 3g) \\ &= ff' - f'g + 3fg' - 3gg' - ff' - 3f'g + f'g + 3gg' = -4(f'g - fg') = 4\sin t - 4t\cos t. \end{aligned}$$

$\square$

**Exercise 17.**

Two solutions  $y_1, y_2$  to this differential equation are  $ce^t$  and  $ce^{-2t}$  for any  $c \in \mathbb{R}$ . To construct the fundamental set of solutions, we need to reshape our solutions such that  $y_a(0) = 1$  and  $y'_a(0) = 0$  and also  $y_b(0) = 0$  and  $y'_b(0) = 1$ .

Since our two solutions  $y_1, y_2$  seem pretty dissimilar, we first assume that  $y_a = c_1y_1 + c_2y_2$ . From here, we just solve for the properties we need; since  $y_a = 1$ ,  $c_1 + c_2 = 1$ . Similarly, since  $y'_a(0) = 0$ ,  $c_1 - 2c_2 = 0$  so  $(c_1, c_2) = (2/3, 1/3)$ .

Doing something similar for  $y_b$ , we find that the corresponding  $(c_1, c_2) = (1/3, -1/3)$ . As such,

$$\begin{cases} y_a = \frac{2}{3}e^t + \frac{1}{3}e^{-2t} \\ y_b = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \end{cases}.$$

□

**Exercise 23.**

$$W = c \exp \left( - \int p(t) dt \right) = c \exp \left( - \int \frac{-t(t+2)}{t^2} dt \right) = c \exp \left( \int 1 + \frac{2}{t} dt \right) = ce^{t+2\ln t} = ct^2 e^t.$$

□

**Exercise 25.**

$$W = c \exp \left( - \int p(x) dx \right) = c \exp \left( - \int \frac{-2x}{1-x^2} dx \right) = c \exp \left( \int -\frac{1}{u} du \right) = ce^{\ln(1/u)} = \frac{c}{1-x^2}.$$

□

**Exercise 31. Exact Equations.**

Expanding the given expression, we get

$$P'(x)y' + P(x)y'' + f'(x)y + f(x)y' = P(x)y'' + y'(P'(x) + f(x)) + f'(x)y = 0.$$

Equating the coefficients to the general form of a differential equation, we thus have  $P'(x) + f(x) = Q(x)$  and  $f'(x) = R(x)$ .

Taking the derivative of that first equation, we thus have  $P''(x) + f'(x) = Q'(x)$  or  $P''(x) - Q'(x) + R(x) = 0$  which is exactly the equation that was desired. □

**Exercise 32.**

32.  $P''(x) - Q'(x) + R(x) = 0 - 1 + 1 = 0$  so the equation is exact. Namely,  $f(x) = Q(x) - P'(x) = x$  so the problem can be restated as  $(y')' + (xy)' = 0 \rightarrow y' + xy = c$ . This equation is solvable with integrating factor  $e^{x^2/2}$  but then the error function pops out so I'm not going to finish this integral. □

**Exercise 34.**

Since  $2 - 1 + (-1) = 0$ , we can find  $f(x) = -x$ . The differential equation then becomes  $(x^2y')' + (-xy)' = 0 \rightarrow x^2y' - xy = c$ .

Solving, we find  $y = -\frac{c_1}{3x} + c_2x$ . □

### 3.3 Complex Roots of the Characteristic Equation

What happens when the roots of the characteristic equation  $ar^2 + br + c = 0$  for a general differential equation  $ay'' + by' + cy = 0$  are imaginary?

Let the roots  $r_1$  and  $r_2$  of the characteristic equation be  $r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$  for real  $\alpha, \beta$ . Then, the corresponding solutions to the differential equation are

$$\begin{cases} y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t} \cos(\beta t) + ie^{\alpha t} \sin(\beta t) \text{ and} \\ y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t} \cos(\beta t) - ie^{\alpha t} \sin(\beta t) \end{cases}.$$

In Section 3.2 (Theorem 3.2.6), it was mentioned that the real and imaginary parts of any solution to a given differential equation are each solutions to the given differential equation. In our case thus,  $y_3 = e^{\alpha t} \cos(\beta t)$  and  $y_4 = e^{\alpha t} \sin(\beta t)$  are also solutions to  $ay'' + by' + cy = 0$ , with  $W[y_3, y_4] = \beta e^{2\alpha t} \neq 0$ .

#### Exercise 6-8.

6. The quadratic yields roots  $r_1, r_2 = 1 \pm i\sqrt{5}$  so the corresponding general solution is  $c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t)$ .
7.  $y = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$ .
8.  $y = c_1 e^{-3t} \cos(2t) - c_2 e^{-3t} \sin(2t)$ . □

#### Exercise 25.

$$\begin{aligned} \text{(a): } \frac{dy}{dt} &= \frac{dy}{dx} \frac{dx}{dt} = \frac{1}{t} \frac{dy}{dx}, \\ \frac{d^2y}{dt^2} &= \frac{d}{dt} \left( \frac{1}{t} \cdot \frac{dy}{dx} \right) = -\frac{1}{t^2} \frac{dy}{dx} + \frac{dx}{dt} \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{1}{t^2} \left( \frac{d^2y}{dx^2} - \frac{dy}{dx} \right). \end{aligned}$$

- (b): Simplify substitute everything we just derived in into the equation ... □

#### Exercise 26-29.

As seen from question 25, we can transform the coefficients of the differential equation  $(t^2, \alpha t, \beta)$  into  $(1, \alpha - 1, \beta)$ . In this case,  $\alpha = 1$  and  $\beta = 1$  so our new differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

which has solutions  $y_1 = \cos(x)$ ,  $y_2 = \sin(x)$ . As such, since  $x = \ln t$ ,  $y_1 = \cos(\ln t)$  and  $y_2 = \sin(\ln t)$  are a set of solutions to the differential equation in terms of  $t$ .

27:  $\alpha = 4$  and  $\beta = 2$  so  $y_1 = e^{-x} = \frac{1}{t}$  and  $y_2 = e^{-2x} = \frac{1}{t^2}$ .

28:  $y_1 = \frac{1}{t}$  and  $y_2 = t^6$ .

29:  $y_1 = t^2$  and  $y_2 = t^3$ . □

### 3.4 Repeated Roots | Reduction of Order

In the characteristic equation  $ar^2 + br + c$ , if the discriminant  $\Delta = b^2 - 4ac > 0$ , then we are bound to find two real roots  $r_1$  and  $r_2$  and from there derive a general solution to the differential equation (Section 3.1). If in fact  $b^2 - 4ac < 0$ , then we will have two complex roots which, as shown in Section 3.3, correspondingly lead to a real solution to the given differential equation. But what happens when  $b^2 - 4ac = 0$ ?

Assume that  $r_1 = r_2 = -\frac{b}{2a}$ . Like before, we conclude that one solution to the differential equation  $ay'' + by' + cy = 0$  would be  $y_1 = e^{-bt/2a}$ . But what about the second solution? It turns out  $y_2 = te^{-bt/2a}$  is the second solution we need<sup>1</sup>.

So to summarize, when solving the equation  $ay'' + by' + cy = 0$ , the solutions are

$$\begin{cases} y_{1,2} = e^{r_1 t} & \text{if } b^2 - 4ac > 0, \\ y_1 = e^{\lambda t} \cos(\mu t), y_2 = e^{\lambda t} \sin(\mu t) & \text{if } b^2 - 4ac < 0, \\ y_1 = e^{r_1 t}, y_2 = te^{r_1 t} & \text{(when } b^2 - 4ac = 0\text{.)} \end{cases}$$


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#### 3.4.1 Reduction of Order

D'Alembert's Method of finding 'extra' (more) solutions is to assume the new solution is of the form  $y_2(t) = v(t)y_1(t)$  and solve from there. Namely, in second order differential equations, if we know a solution  $y_1$  to  $L[y] = 0$ , we let

$$y_2 = v(t)y_1 \text{ so } y'_2 = v'(t)y_1 + v(t)y'_1 \text{ and } y''_2 = v''(t)y_1 + 2v'(t)y'_1 + v(t)y''_1.$$

As such, plugging this back into our differential equation,  $L[y_2] = 0$  becomes

$$\begin{aligned} y''_2 + P(t)y'_2 + Q(t)y_2 &= 0 \implies v''(t)y_1 + 2v'(t)y'_1 + v(t)y''_1 + P(t)(v'(t)y_1 + v(t)y'_1) + Q(t)v(t)y_1 \\ &= y_1v''(t) + (2y'_1 + P(t)y_1)v'(t) + (y''_1 + P(t)y'_1 + Q(t)y_1)v(t) = 0. \end{aligned}$$

Since  $y_1$  is a solution and thus  $L[y_1] = 0$ , that right most term is actually 0 so our new differential equation is now

$$y_1v'' + (2y'_1 + P(t)y_1)v' = 0$$

which is a first order differential equation with respect to  $v'$ . This is known as **reduction of order** since our differential equation went from being a second-order to a first-order differential equation.

#### Exercise 1-8.

(Note: The general solution  $y$  can be expressed as  $y = c_1y_1 + c_2y_2$  for arbitrary  $c_1, c_2$ . Below, I only find  $y_1$  and  $y_2$ .)

1. Since  $b^2 - 4ac = 0$ ,  $y_1 = e^{-bt/2a} = e^t$ , and  $y_2 = te^t$ .
2. Since  $b^2 - 4ac = 0$ ,  $y_1 = e^{-bt/2a} = e^{-t/3}$  and  $y_2 = te^{-t/3}$ .
3. Since  $b^2 - 4ac = 16 + 4(4)(3) = 64 > 0$ , we can simplify find the roots of the equation and derive a general solution that way. The roots to  $4n^2 - 4n - 3 = 0$  are  $n = \frac{3}{2}, -\frac{1}{2}$  so the two solutions are  $y_1 = e^{3t/2}$ ,  $y_2 = e^{-t/2}$ .
4.  $b^2 - 4ac = -36$  so the roots to this quadratic equation are  $1 \pm 3i$  (quadratic formula). As such,  $y_1 = e^t \cos(3t)$  and  $y_2 = e^t \sin(3t)$ .
5. Since  $b^2 = 4ac$ ,  $y_1 = e^{3t}$  and  $y_2 = te^{3t}$ .
6.  $y_1 = e^{-4t}$ ,  $y_2 = e^{-t/4}$ .
7.  $y_1 = e^{-3t/4}$ ,  $y_2 = ty_1$ .
8.  $y_1 = e^{-1/2} \cos(t/2)$ ,  $y_2 = e^{-1/2} \sin(t/2)$ . □

<sup>1</sup>Consult Example 1 in Section 3.4 in the textbook for a proof

**Exercise 14.**

Let  $r_1, r_2$  be the roots of the characteristic equation to the differential equation  $ay'' + by' + cy = 0$ . As discussed above, the general solution to this equation is  $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$  for arbitrary constants  $c_1$  and  $c_2$ . If we let  $y = 0$ , we can rearrange and show

$$-\frac{c_1}{c_2} = e^{(r_2 - r_1)t}.$$

Notably, the left hand side of the equation is constant for a given solution ( $c_1$  and  $c_2$  are chosen after all) and  $r_2 - r_1$  is also constant. As such, since the exponential function is a bijective function for all real inputs, there is only one  $t$  value that makes the above equation true which means if the differential equation has a non-trivial solution (e.g. not  $c_1 = c_2 = 0 \rightarrow y(t) = 0$ ), there is only one  $t$  value that makes a given solution to the differential equation ( $y$ ) 0.  $\square$

**Exercise 18-22.**

18. Reducting the order, you eventually get  $v''t^4 = 0 \rightarrow v'' = 0 \rightarrow v = c_1 t + c_2$  so  $y_2 = t^3$  as the constant  $c_1$  is arbitrarily chosen (in this case I take  $c_1 = 1$ ) and the latter term  $c_2 t^2$  is a multiple of  $y_1$  and thus not worth mentioning.

19. The post-reduction equation ends up being  $tv'' + 4v' = 0$  which yields the solution  $y = \frac{C_1}{t^3}$  which means the final second solution is  $y_2 = \frac{1}{t^2}$ . (Note: I'm ignoring the second  $+C$  at the end as its inclusion is not necessary; the final solution ends up being  $y_2 = \frac{C_1}{t^2} + C_2 t = \frac{C_1}{t^2} + C_2 y$  which means the latter term has no meaning and can be ignored.)

20.  $y_2 = \frac{\ln t}{t}$ .

21. It's long and tricky but you eventually get  $y_2 = -C \cot(x^2) \cdot y_1 \rightarrow \cos(x^2)$ .

22. The integration required in this exercise is basically the same as the ones done in exercise 21. As such, it should be relatively straightforward to show that  $y_2 = -C \cos x \frac{1}{\sqrt{x}} \rightarrow y_2 = \frac{\cos x}{\sqrt{x}}$ .  $\square$

**Exercise 32-33.**

To recap Euler's equations, we transform  $t^2 y'' + \alpha t y' + \beta y = 0$  into  $y'' + (\alpha - 1)y' + \beta y = 0$  where in the first case the derivative of  $y$  is taken with respect to  $t$  and in the second, the derivative of  $y$  is taken with respect to  $x = \ln t$ .

32.  $y_1 = e^{-x/2} = t^{-1/2} = \frac{1}{\sqrt{t}}$ ,  $y_2 = \frac{x}{\sqrt{t}} = \frac{\ln t}{\sqrt{t}}$ . I have manually verified that both solutions do indeed solve the differential equation posed.

33.  $y_1 = e^{-x} = \frac{1}{t}$ ,  $y_2 = \frac{\ln t}{t}$ .  $\square$

### 3.5 Nonhomogenous Equations | Method of Undetermined Coefficients

#### Definition 3.7 (Homogenous Differential Equation)

A differential equation where there is no isolated  $g(t)$  term is called a **homogenous** differential equation. In our case, the differential equation  $L[y] = y'' + p(t)y' + q(t)y = 0$  is homogenous while equations of the form  $L[y] = g(t) \neq 0$  are nonhomogenous second-order linear differential equations.

Theorem 3.5.1 asserts that if  $Y_1$  and  $Y_2$  are two solutions to  $L[y] = g(t)$ , then  $Y_1 - Y_2$  is a solution to  $L[y] = 0$ . Notably, that means if we want to find all solutions to a differential equation  $L[y] = g(t)$ , we only need to find 1 exact solution  $Y_1$  to the nonhomogenous form as the general solution is thus  $c_1 y_1 + c_2 y_2 + Y_1$  where  $y_1$  and  $y_2$  are solutions to  $L[y] = 0$ .

Note that the general solution of the homogenous differential equation ( $c_1 y_1 + c_2 y_2$ ) is commonly called the **complementary solution** and is denoted  $y_c(t)$ . The solution  $Y_1$  that we find in particular is called the **particular solution**.

So how do we find  $Y_1$ ? We can either use the **Method of Undetermined Coefficients** (3.5) or use the **Variation of Parameters** method (3.6).

## Method of Undetermined Coefficients

(a.k.a. educated guess and check)

Essentially, based on the coefficients in the equation  $(p(t), q(t), g(t))$ , make a good *guess* about what  $Y_1$  could look at (using general constant coefficients) then solve for those constant coefficients (and hope you're right in your guess).

For good guidelines about guessing:

- If the nonhomogenous term  $g(t)$  is of the form  $e^{\alpha t}$ , assume  $Y = Ae^{\alpha t}$ .
- If  $g(t)$  looks like  $\sin(\beta t)$  or  $\cos(\beta t)$ , let  $Y = A \sin(\beta t) + B \cos(\beta t)$ .
- If  $g(t)$  looks like some polynomial up to  $t^\gamma$ , let  $Y = a_1 t^\gamma + a_2 t^{\gamma-1} + \cdots + a_\gamma t + a_{\gamma+1}$ .
- If  $g(t)$  looks like two (or more) of the above functions added together (e.g.  $g(t) = e^{-3t} + \sin(4t)$ ), split up the differential equation to find the respective solutions to when  $g(t) = e^{-3t}$  and  $g(t) = \sin(4t)$  then add those solutions together.
- If  $g(t)$  looks like two of the above functions multiplied together (e.g.  $g(t) = (t^2 + t - 4)(e^{3t})$ ), let  $Y$  be the product of the two relevant guesses; in this case, we should let  $Y = (At^2 + Bt + C)e^{3t}$ .
- If guessing a  $Y$  for  $g(t)$  fails, try  $Y^* = tY$ . Maybe that'll work :).

### Exercise 1-7.

1. Assuming  $Y = Ae^{2t}$ , we soon find  $A = -1$ . Thus, a particular solution to this equation is  $Y = -e^{2t}$ . Since the general solution to the given differential equation is  $y_c = c_1 e^{3t} + c_2 e^{-t}$ , the general general solution is thus  $\boxed{\phi = c_1 e^{3t} + c_2 e^{-t} - e^{2t}}$  for arbitrary constants  $c_1, c_2$ .

2. Assuming  $Y = At^2 + Bt + C$ , we derive, substitute, and solve to find  $A = -2$ ,  $B = 3$ , and  $C = -7/2$ . Since the homogenous solution is  $c_1 e^{2t} + c_2 e^{-t}$ , we thus have the general solution  $\phi$  being of the form

$$\boxed{c_1 e^{2t} + c_2 e^{-t} - 2t^2 + 3t - \frac{7}{2}}.$$

3. Since  $g(t)$  is composed of two exponential terms, we similarly assume  $Y = Ae^{3t} + Be^{-2t}$  and we find  $A = 2$  and  $B = -3$ . With the solution from the non-homogenous equation, we thus find that  $\phi = c_1 e^{2t} + c_2 e^{-3t} + 2e^{3t} - 3e^{-2t}$ .

4. While assuming  $Y = (At + B)e^{-t}$  yields no satisfactory results, assuming  $Y = (At^2 + Bt + C)e^{-t}$  (one level up) leads us to find  $A = 3/8$  and  $B = 3/16$ . Thus, the general solution to the differential equation is  $\phi = t(3t/8 + 3/16)e^{-t} + c_1 e^{3t} + c_2 e^{-t}$ .

5. This differential equation is pretty funny since as there is no  $y$  term involved, this is a linear first order differential equation with respect to  $y'$ . Nevertheless, viewing this from a second-order DE perspective, we can split  $g(t) = 3 + 4 \sin(2t)$  into  $g_1(t) = 3$  and  $g_2(t) = 4 \sin(2t)$  and solve the differential equations  $y'' + 2y' = g_i(t)$  separately to get a particular solution  $Y = -\frac{1}{2} \cos(2t) - \frac{1}{2} \sin(2t) + \frac{3}{2}t + C$  (arbitrary constant  $C$ ), meaning the general solution  $\phi$  is  $c_1 + c_2 e^{-2t} + Y$ .

6. Solving the homogenous version of the differential equation, we have  $y_c = c_1 e^{-t} + c_2 t e^{-t}$ . As such, while we would normally set  $Y = Ae^{-t}$ , we can't since this solution is already included in the complementary solution. Similarly,  $Y = Ate^{-t}$  also doesn't work and to solve, we assume  $Y = At^2 e^{-t}$  and find the general solution  $\phi$  to be  $\phi = e^{-t}(t^2 + c_2 t + c_1)$ .

7. I did a big messy equation and assumed  $Y = A \sin(2t) + B \cos(2t) + C t \sin(2t) + D t \cos(2t)$  (and to make matters worse I substituted in  $\sin(2t)$  with  $\Delta$  and  $\cos(2t)$  with  $\square$  ostensibly to save writing – but this just made everything worse). Eventually, I found  $A = -\frac{5}{9}$ ,  $D = -\frac{1}{3}$ , and  $B = C = 0$ . Thus,  $\phi = c_1 \sin t + c_2 \cos t - \frac{1}{3}t \cos(2t) - \frac{5}{9} \sin(2t)$ .  $\square$

### Exercise 8-10.

(Continuation from Exercises 1-7)

8. Assuming  $U = A \cos(\omega t) + B \sin(\omega t)$ , we eventually have  $A \cos(\omega t)(\omega_0^2 - \omega^2) = \cos(\omega t)$  and  $B \sin(\omega t)(\omega_0^2 - \omega^2) = 0$ . Since it is given that  $\omega_0^2 \neq \omega^2$ ,  $A = \frac{1}{\omega_0^2 - \omega^2}$  and  $B = 0$ . As such,  $\phi = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2} \cos(\omega t)$ .

9. Resuming the same path as before, since  $\omega = \omega_0$  in this case, we have to assume  $U = At \cos(\omega t) + Bt \sin(\omega t)$  which leads us to find  $B = \frac{1}{2\omega_0}$ . As such,  $\phi = \frac{t \sin(\omega_0 t)}{2\omega_0} + c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t)$ .

10. The particular solution is quite easy to find in this case; although  $\sinh t$  is scary, it is easily mitigated by letting  $Y = Ae^t + Be^{-t} \rightarrow A = \frac{1}{6}$ ,  $B = -\frac{1}{4}$ . In contrast, the roots of the characteristic equation are  $-\frac{1}{2} \pm \frac{i\sqrt{15}}{2}$  so the general solution is  $\phi = c_1 e^{-t/2} \sin\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$ .  $\square$

### Exercise 23.

From many problems before, it can be intuited that  $y_c = c_1 \cos(\lambda t) + c_2 \sin(\lambda t)$ . For the particular solution, we consider the differential equation of an arbitrary term in the summation  $a_k \sin(k\pi t)$ :

$$y'' + \lambda^2 y = a_k \sin(k\pi t).$$

Some uncomplicated guessing and checking ( $Y = A \sin(k\pi t) + B \cos(k\pi t)$ ) leads us to find that for this arbitrary case, a particular solution is  $Y_k = \frac{a_k}{\lambda^2 - k^2\pi^2} \sin(k\pi t)$  which lets us reasonably conclude that

$$\phi = c_1 \cos(\lambda t) + c_2 \sin(\lambda t) + \sum_{m=1}^N \frac{a_m}{\lambda^2 - m^2\pi^2} \sin(m\pi t)$$

is the general solution to this scary-looking differential equation.  $\square$

### Exercise 28-30.

Note: Everything in exercise 27 (which this problem is based off of) is true. I'm not sure how to verify it because each step seems somewhat trivially obvious.

28.  $y'' - 3y - 4y = 3e^{2t} = y(D-4)(D+1)$ . As such, letting  $u = (D+1)y$ , our aim is to find  $y$  by first finding  $u$  by solving the differential equation  $(D-4)u = g(t) \rightarrow u' - 4u = 3e^{2t}$ . This first differential equation can be done with an integrating factor  $\mu = e^{-4t}$  which leads us to find  $u = -\frac{3}{2}e^{2t}$  (screw the constant). Having found  $u$ , we can now find  $y$  with the equation  $(D - r_2)y = u \rightarrow y' + y = -\frac{3}{2}e^{2t}$ . With integrating factor  $\mu = e^t$ , we easily find  $y = -\frac{1}{2}e^{2t}$  as a particular solution to the given differential equation, and solve the problem correspondingly.

29. Our two first-order equations are  $(D+1)u = 2e^{-t}$  and  $(D+1)y = u$ . Solving the first, we have  $u' + u = 2e^{-t}$  so with integrating factor  $e^t$  we find  $u = 2te^{-t}$ . Next, we solve  $y' + y = 2te^{-t}$  and find  $y = t^2e^{-t}$  (same simple integrating factor, same process) which is indeed a particular solution to the given differential equation.

30. Our two equations are  $(D+2)u = 3 + 4 \sin(2t)$  and  $Dy = u$  (root order doesn't matter mathematically). Solving the first differential equation, with integrating factor  $\mu = e^{2t}$ , we find<sup>a</sup>

$$(e^{2t}u) = \frac{3}{2}e^{2t} + 4 \int e^{2t} \sin(2t) dt \rightarrow u = \frac{3}{2} + \sin(2t) - \cos(2t).$$

The second differential equation is simply  $y' = u$  or  $y = \int u$  so a particular solution  $y$  that we find is  $y = \frac{3}{2}t - \frac{1}{2} \cos(2t) - \frac{1}{2} \sin(2t)$ .  $\square$

<sup>a</sup>The complicated  $\int e^t \sin t$  integral is solved cleverly using integration by parts.

<sup>b</sup>Remark: In this case, the strategem of solving two first order DEs to find a particular solution works much faster (and cleaner) than the method of undetermined coefficients. It also feels a lot more straightforward.