Differential Equations Notes

Alex Z

Fall 2025

Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around ~20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

6 F

7 G

8 H

The above sections are spacer sections.

9 Nonlinear Differential Equations and Stability

This chapter deals with qualitative information about a differential equation and stability/instability of given solutions.

9.1 The Phase Plane: Linear Systems

When considering the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x},$$

there are special equilibrium solutions to be aware about: namely, **critical points**, or when $\mathbf{A}\mathbf{x} = \mathbf{0}$. If we assume $\det \mathbf{A} \neq 0$, then only $\mathbf{x} = \mathbf{0}$ is the only critical point of the system.

For linear systems, we can analyze the system and its trajectories by the eigenvalues of $\mathbf{A}$; namely, if we assume $\mathbf{A}$ is a 2×2 matrix and $\mathbf{A}$ has eigenvalues r_1, r_2, then

- if $r_1 \neq r_2$ and r_1 and r_2 have the same sign, then all trajectories will approach the origin and the origin (the critical point) will either be a **nodal source** (e.g. trajectories go away from the critical point) or a **nodal sink** (trajectories end up going towards the critical point.)
- if r_1 and r_2 have differing signs (wlog $r_1 > 0 > r_2$) then as $t \to \infty$, all trajectories will converge towards the eigenvector associated with r_1. As $t \to -\infty$ however, since $e^{-\infty} = 0$, all trajectories will converge towards the eigenvector associated with r_2. As such, the critical point is known as a **saddle point** as no trajectories pass through the critical point (see textbook Page 391).
- if $r_1 = r_2$ and two *indepedent* eigenvectors can be found for $\mathbf{A}$, then all trajectories look like a line through the critical point (in this case the origin) and the critical point is called a **proper node** or **star point**.
- if conversely $r_1 = r_2$ and only one eigenvector can be found, then the critical point is called an **improper node**[1]
- If eigenvalues are complex with some real part, trajectories will look like a spiral going either towards/away the origin in which case the origin is called a spiral sink/spiral source, depending on the real part.
- If the eigenvalues are purely imaginary, trajectories will infinitely loop on themselves (since there is no real part, trajectories do not 'decay') and the origin/critical point is called a **center** of the system. For linear systems, these trajectories look like ellipses around the origin.

Essentially, all solutions either go to infinity, go to $\mathbf{0}$, or go in a spiral.

9.2 Autonomous Systems and Stability

In this section we will be concerned with Autonomous systems of two functions of the form

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y).$$

with initial condition $x(t_0) = x_0$, $y(t_0) = y_0$. This system can also be written in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(t_0) = \mathbf{x}^0$$

where $\mathbf{x} = (x, y)^T = (x(t), y(t))^T$, $\mathbf{f}(\mathbf{x}) = (F(x, y), G(x, y))^T$, and $\mathbf{x}^0 = (x_0, y_0)^T$.
For simplicity and so theorems hold, we will assume F and G are continuous and their partial derivatives are also continuous.[2]

[1] tbh idrk why; the graph just looks super weird

[2] Rigorously speaking, their partial derivatives only need to be continuous over some domain D of the xy-plane. But since most functions we work with are very simple, it might as well be (almost) the whole xy-plane over which the partial derivatives are continuous.

Definition 9.1 (*Autonomous*)

A differential equation system is said to be **autonomous** if the systems do not depend on time. In particular, the system given above ($\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$) is autonomous, and so is the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ as long as all terms in $\mathbf{A}$ do not involve the independent variable t.

The distinction between autonomous and nonautonomous systems is important as the condition of autonomy guarantees that there is only one trajectory crossing through the point (x_0, y_0) regardless of time.

9.2.1 Stability and Instability

For autonomous systems of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

Definition 9.2 (*Critical Points*)

Critical points are defined as points to the above system where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Since $\mathbf{x}' = \mathbf{0}$, critical points must be constant solutions to the system.

Definition 9.3 (*Stability*)

A critical point of the solution $\mathbf{x}^\circ$ is said to be **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every solution $\mathbf{x}$ which at $t = 0$ satisfies

$$||\mathbf{x}(0) - \mathbf{x}^\circ|| < \delta$$

exists for all $t > 0$ and satisfies

$$||\mathbf{x}(t) - \mathbf{x}^\circ|| < \varepsilon$$

for all $t \geq 0$.

Essentially, this definition codifies the notion that a given solution $\mathbf{x}^*$ should stay bounded within the critical point. Note however that this definition of stability does not require $\mathbf{x}^*$ to converge to $\mathbf{x}^\circ$; instead, it merely requires that $\mathbf{x}^*$ not leave an open disk[3] of radius ε centered at $\mathbf{x}^\circ$ for all $t \geq 0$.

Any critical points for which the condition of stability doesn't hold are said to be **unstable**.

The following definition thus distinguishes between stability and asymptotic stability:

Definition 9.4 (*Asymptotic Stability*)

A critical point $\mathbf{x}^\circ$ is said to be **asymptotically stable** if it is stable and there exists a $\delta_0 > 0$ such that if a solution $\mathbf{x} = \mathbf{x}(t)$ satisfies

$$||\mathbf{x}(0) - \mathbf{x}^\circ|| < \delta_0,$$

$$\text{then } \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^\circ.$$

In english, if a trajectory starts "sufficiently close" to $\mathbf{x}^\circ$ (within a δ_0), then it must eventually approach $\mathbf{x}^\circ$ as $t \to \infty$.

Definition 9.5 (*Basin of Attraction*)

For a two-dimensional (potentially non-linear) autonomous system with at least one asymptotically critically point, we define the **basin of attraction** for a critical points to be the set of all points P such that a trajectory passing through P eventually converges to said critical point as $t \to \infty$.
If there is a boundary to a **basin of attraction**, that trajectory which bounds the basin is called a **separatrix** as it seperates the trajectories that converge and the trajectories that don't.

If we're lucky, we can determine trajectories of a two-dimensional autonomous system by solving just a first-order differential equation. Namely, since $F(x,y)$ and $G(x,y)$ don't depend on t, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x,y)}{F(x,y)}$$

which is a first-order differential equation. In general, the differential equation arising from the quotient $\frac{G(x,y)}{F(x,y)}$ may not be solvable.

[3] If $\mathbf{x}$ is n-dimensional, then the statement should be amended to read "...not leave an open $n-1$-sphere of radius ...".

Exercise 14a-20a.

For this exercise, I recommend using this or this online plotter to plot some nice looking solutions.

14a. Following the equation above, we have $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} = \frac{8x}{2y} \to H(x,y) = y^2 - 4x^2 = c$. Solutions to this system generally look like (one-directional) hyperbolas.

15a. $\frac{dy}{dx} = \frac{-8x}{2y} \to H(x,y) = y^2 + 4x^2 = c$. As the equation suggests, solutions to this system look like an ellipse.

16a. $\frac{dy}{dx} = \frac{2x+y}{y}$. Using the substitution $y = vx$ and $dy = vdx + xdv$, we thus have

$$\frac{vdx + xdv}{dx} = v + \frac{dv}{dx}x = \frac{2+v}{v} \to \frac{dv}{dx}x = \frac{2+v-v^2}{v}$$

which with partial fraction decomposition and a bunch of tedious integration simplification reveals $H(x,y) = (x+y)(y-2x)^2 = c$.

17a. $\frac{dy}{dx} = \frac{x+y}{x-y}$. Using the substitution $y = vx$ again, we thus have $H(x,y) = \arctan\left(\frac{y}{x}\right) - \ln\left(\sqrt{\frac{y^2}{x^2}+1}\right) - \ln x = \arctan\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} = c$.

18a. While this equation looks super complicated, if you rearrange it into the form $(2xy^2 - 6xy) + (2x^2y - 3x^2 - 4y)y' = 0$, you quickly find the differential equation is exact so $H = x^2y^2 - 3x^2y - 4y^2 = c$.

19a. $\frac{dy}{dx} = \frac{-\sin x}{y} \to \frac{y^2}{2} - \cos(x) = c$.

20a. $\frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{24} = c$. $\square$

9.3 Locally Linear Systems

The following table/theorem (theorem 9.3.1 in the book) recaps the stability properties of the origin for the two dimensional linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Definition 9.6 (*Stability of* 0)

Let the eigenvalues for $\mathbf{A}$ be r_1 and r_2. Then, the critical point $\mathbf{x} = \mathbf{0}$ is

1. asymptotically stable if r_1 and r_2 have negative real part,
2. stable if r_1 and r_2 have 0 real part (e.g. r_1 and r_2 are pure imaginary eigenvalues),
3. unstable if either r_1 and r_2 have any sort of positive real part.

From the table we can conclude that small pertrubations in the roots r_1 and r_2 only really matter when r_1 and r_2 are pure imaginary eigenvalues, as any addition of a real part (either positive or negative) will cause the system to spiral inwards towards $\mathbf{0}$ or outwards to infinity. Thus, while $\mathbf{0}$ is stable if r_1 and r_2 have 0 real part, this stability is itself potentially unstable.

9.3.1 Linear Approximations

Definition 9.7 (*Isolated critical point*)

We say that a critical point $\mathbf{x}^\circ$ is an **isolated critical point** of the system if there is some disk around $\mathbf{x}^\circ$ with radius $r > 0$ such that there exists no other critical poitns in that disk.

Considering the linearization of trajectories around the origin ($\mathbf{x} = \mathbf{0}$) for the non-linear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$, we first assume $\mathbf{0}$ is an isolated critical point of the system. We also assume that around the critical point (in this case $\mathbf{x} = \mathbf{0}$) that $\mathbf{g}$ is small, or in rigorous terms,

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{|\mathbf{g}|}{|\mathbf{x}|} = 0$$

assuming[4] $\mathbf{g}$ has continuous first partial derivatives. If the above condition is satisfied, the system we described above can then be called a **locally linear system**.

Anyways, here's a cool theorem we can use to determine local linearity:

Definition 9.8 (*Theorem 9.3.2 (p. 410)*)

The system described by

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$$

is locally linear in the neighborhood of a critical point $\mathbf{x}^\circ$ whenever F and G have continuous partial derivatives up to order two (e.g. F and G are twice differentiable).

If the above condition holds, then the nonlinear system near $\mathbf{x}^\circ = (x^\circ, y^\circ)$ can be approximated by the linear system

$$\frac{d}{dt}\begin{pmatrix} x - x^\circ \\ y - y^\circ \end{pmatrix} = \begin{pmatrix} F_x(x^\circ, y^\circ) & F_y(x^\circ, y^\circ) \\ G_x(x^\circ, y^\circ) & G_y(x^\circ, y^\circ) \end{pmatrix} \begin{pmatrix} x - x^\circ \\ y - y^\circ \end{pmatrix} \rightarrow \frac{d\mathbf{u}}{dt} = \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x}^\circ)\,\mathbf{u}$$

where $\mathbf{u} = \begin{pmatrix} x - x^\circ \\ y - y^\circ \end{pmatrix}$.

The general coefficient matrix in the above equation

$$\mathbf{J} = \mathbf{J}[F,G](x,y) = \begin{pmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{pmatrix}$$

is called the **Jacobian**[5] of F and G with repsect to x and y. For the linear approximation system above, we need to assume $\det \mathbf{J}(\mathbf{x}^\circ) \neq 0$ so that $\mathbf{x}^\circ$ is an isolated critical point in our linear approximation system.

Anyways, to relate the properties of stability of linear and locally linear systems, here's a theorem and a table from the textbook:

Theorem 9.3.3

Let r_1 and r_2 be the eigenvalues of the linear system (1), $\mathbf{x}' = \mathbf{A}\mathbf{x}$, corresponding to the locally linear system (4), $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. Then the type and stability of the critical point (0, 0) of the linear system (1) and the locally linear system (4) are as shown in Table 9.3.1.

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
Eigenvalues	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	*PN or IN*	Unstable	*N or SpP*	Unstable
$r_1 = r_2 < 0$	*PN or IN*	Asymptotically stable	*N or SpP*	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\lambda = 0$	*C*	Stable	*C or SpP*	Indeterminate

Key: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

[4] It is also possible to convert this limit into polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$, $|\mathbf{x}| = r$) to make this limit easier to evaluate.

[5] Recognize this from Calc 3?

or in summary, except for two special cases ($r_1 = r_2$, $\lambda = 0$), the non-linear terms of the nonlinear system do not affect the stability of the system determined by the linear systems.
If every trajectory approaches the critical point at the origin, then the critical point $\mathbf{0}$ is said to be **globally asymptotically stable**.

/// The textbook then goes into more detail about the stability of a damped pendulum around some critical points (Page 413-415). ///

Exercise 1-3.
1. We verify $(0,0)$ is a critical point by noting that $(0,0)$ satsifies $x - y^2 = x - 2y + x^2 = 0$.
To find the locally linear version of the system, since both $F = x - y^2$ and $G = x - 2y + x^2$ have continuous second partial derivatives, we can invoke theorem 9.3.2 which tells us that our non-linear system can be approximated by the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which is verifiably linear. This linear system has two real, distinct eigenvalues $\lambda = 1$ and $\lambda = -2$ so by Table 9.3.1, both the linear and non-linear system around $(0,0)$ are unstable as in both cases, the origin is a saddle point.

2. $(0,0)$ can be verified to be a critical point, and it should be noted that it is not the only critical point of the system (see $(-2,\pi)$, $(0,2\pi)$, etc). Since the partial derivatives of both F and G up to order two are continuous, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

so $\lambda = \pm i$. As such, for the linear system, the origin is a stable center. For the nonlinear system however, the origin could be either a center or a spiral point so its stability is unknown.

3. $(0,0)$ is verifiably a critical point, and the corresponding locally linear matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which yields the eigenvalue $\lambda = 1$ with multiplicity two. Thus, the origin in both the locally linear and nonlinear systems is unstable, with the origin being an improper node[a] in the locally linear system and either a spiral point or a node in the nonlinear system. □

[a] Since the system only has one eigenvector, namely $(0,1)^T$.

(For the exercises below, this website gives some marvelous slope fields/phase diagrams.)

Exercise 4abc.

4a. Critical points to the equations occur when $\frac{dx}{dt} = 0 = \frac{dy}{dt}$. Examining the first equation, $\frac{dx}{dt}$ is 0 when $x = y$ or $x = -2$. The second equation shows $\frac{dy}{dx} = 0$ when $x = -y$ or $x = 4$. Testing out each of the 4 combination of critical equations between the points, we find that there are 3 critical points, namely $(0,0)$, $(-2,2)$, and $(4,4)$.

4b. As a straightforward application of Theorem 9.3.2, for each critical point (x°, y°) found above, the locally linear system is of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y^\circ - 2x^\circ - 2 & x^\circ + 2 \\ 4 - y^\circ - 2x^\circ & 4 - x^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 6 & 6 \end{pmatrix}, \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}.$$

4c. For the system with critical point $(0,0)$, its eigenvalues are $\lambda = 1 \pm \sqrt{17}$, which means that in the nonlinear system, the origin is an unstable saddle point.
For the system with critical point $(-2,2)$, its eigenvalues are $\lambda = 3 \pm \sqrt{33}$ which means in the nonlinear system, it is also an unstable saddle point.
For the system with critical point $(4,4)$, its eigenvalues are $\lambda = -3 \pm i\sqrt{39}$ so in the nonlinear system $(4,4)$ is an asymptotically stable spiral point. □

Exercise {5, 6, 7}abc.

5a. The critical points are $(0,0)$, $(1,0)$, $\left(0, \frac{3}{2}\right)$, and $(-1,2)$. The first three solutions can be found by solving $\frac{dx}{dt} - \frac{dy}{dt} = 0$, and the last solution can be found by solving $x - x^2 = xy = 3y - 2y^2$ as this can be broken up into two equations $1 - x = y$ and $x = 3 - 2y$.

5b. From Theorem 9.3.2, the linear matrices are of the form $\begin{pmatrix} 1 - 2x - y & -x \\ -y & 3 - x - 4y \end{pmatrix}$ so the matrices for the critical points $(0,0)$, $(1,0)$, $\left(0, \frac{3}{2}\right)$, and $(-1,2)$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}$$

respectively.

5c. The first matrix has eigenvalues $\lambda = \{1, 3\}$ so $(0,0)$ is an unstable node in the nonlinear system.
The second matrix has eigenvalues $\lambda = \{-1, 2\}$ which means $(1,0)$ is an unstable saddle point.
The third matrix has eigenvalues $\lambda = \{-\frac{1}{2}, -3\}$ so $\left(0, \frac{3}{2}\right)$ is an asymptotically stable node.
The fourth matrix has eigenvalues $\lambda = \frac{1}{2}(3 \pm \sqrt{17})$ so $(-1,2)$ is an unstable saddle point.

6. $(-1,1) \to \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix} \to \lambda = -1 \pm \sqrt{3}$ so $(-1,1)$ is an unstable saddle.
$(1,1) \to \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix} \to \lambda = -1 \pm i$ so $(1,1)$ is an asymptotically stable spiral point.

7. $(0,0) \to \begin{pmatrix} -2 & 4 \\ 2 & 4 \end{pmatrix} \to \lambda = 1 \pm \sqrt{17}$ so $(0,0)$ is an unstable saddle.
$(2,1) \to \begin{pmatrix} -3 & 6 \\ -4 & 0 \end{pmatrix} \to \lambda = \frac{1}{2}(-3 \pm i\sqrt{87})$ so $(2,1)$ is an asymptotically stable spiral point.
$(2,-2) \to \begin{pmatrix} 0 & -6 \\ 2 & 0 \end{pmatrix} \to \lambda = \pm i\sqrt{12}$ so $(2,-2)$ is either a center or a spiral point and its stability is indeterminate.
$(4,-2) \to \begin{pmatrix} 0 & -8 \\ -2 & -4 \end{pmatrix} \to \lambda = 2 \pm \sqrt{20}$ so $(4,-2)$ is an unstable saddle point. □

Exercise 23.
(Note: (26) is just (25) with $\alpha = -1$.)
23a. Trivially, $\frac{dx}{dt} = 0 + \alpha(0)(0^2 + 0^2) = \frac{dy}{dt} = 0$ so $(0,0)$ is a critical point of the system given in (25). To show that $(0,0)$ is a center, we note that the system can be rewritten as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha(x^2+y^2) & 1 \\ -1 & \alpha(x^2+y^2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \to (\lambda - \alpha(x^2+y^2))^2 + 1 = 0.$$

Since $\alpha(x^2 + y^2)$ is 0 at the point $(0,0)$, $\lambda = \pm i$ which is pure imaginary which means that $(0,0)$ is a center around the corresponding linear system.

23b. (25) is twice differentiably continuous so (25) is locally linear.
23c.

$$\frac{dr}{dt} = \frac{xx' + yy'}{r} = \frac{xy + \alpha x^2 r^2 - xy + \alpha y^2 r^2}{r} = \frac{\alpha r^2(x^2+y^2)}{r} = \alpha r^3.$$

23d. Solving the given differential equation, we have $\alpha t + C = -\frac{1}{2r^2}$ so $r = \sqrt{\frac{1}{C - 2\alpha t}}$. If $\alpha < 0$, then as $t \to \infty$, that bottom fraction will go towards infinity so r will go towards 0.

23e. If $r(0) = r_0$, then $C = \frac{1}{r_0^2}$ so $r(t) = \sqrt{\frac{1}{\frac{1}{r_0^2} - 2\alpha t}}$. Clearly, as the bottom of that denominator becomes 0 (when $t \to 1/2\alpha r_0^2$), r becomes unbounded which translates into $\sqrt{x^2+y^2}$ becoming unbounded meaning $\frac{dx}{dt}$ and $\frac{dy}{dt}$ will grow astronmically as $t \to 1/2\alpha r_0^2$ when $\alpha > 0$. $\square$