Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

6 F

The above sections are spacer sections.

7 Systems of First-Order Linear Equations

7.1 Introduction

Essentially, we consider systems of first-order equations since any higher order differential equation can inevitably be transformed into multiple first order linear transformations.

Moreover, for any $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$, we cna make the substituions $x_1 = y$, $x_2 = y'$, $\dots$, $x_n = y^{(n-1)}$ and thus eventually find $x_1' = F_1(t, x_1, x_2, \dots, x_n)$, $x_2' = F_2(t, x_1, x_2, \dots, x_n)$ and so on. Thus, we have effectively converted a general differential equation into many teeny tiny first-order differential equations (that are each in their own way, granted, hard to solve).

7.2 Matrices

(note: all uppercase letters from here on out (A, B, C, $\dots$) will most likely represent matrices from here on out unless they are in function notation (e.g. $F(t)$ would be a function)).

Various matrix preliminaries are covered here. Do note that when the book talks about the **adjoint** of A, they mean the **transpose of the conjugate matrix of** A rather than the cofactor expansion matrix of A.

Integrals, derivatives, and [x] over matrices of functions are just those same operations applied to each individual operations (boring). For example, $\int A\ dt = \int a_{ij}\ dt$.

7.3 More Linear Algebra

(This is just a review of Math 4a......)

7.4 Basic Theory of Systems of First-Order Linear Equations

(a.k.a. A review of section 3.2 but with matrices instead of second-order linear differential equations.)
To examine a system of n first-order linear equations each of the form $x_i' = p_{i1}(t)x_1 + p_{i2}(t)x_2 + \cdots + p_{in}(t)x_n + g_i(t)$, we can rewrite everything in matrix form and obtain the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]^T$, $\mathbf{g}(t) = [g_1(t)\ g_2(t)\ \dots\ g_n(t)]^T$, and $\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{pmatrix}$.

With matrix equations, multiple solutions ($\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(k)}(t)$) for $\mathbf{x}$ may exist. Moreover, if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two solutions to a first-order homogenous matrix differential equation ($\mathbf{g} = \mathbf{0}$), then $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution to said equation for arbitrary constants c_1, c_2 (Theorem 7.4.1, Page 305).

If we make a big matrix $\mathbf{X} = [\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ \dots\ \mathbf{x}^{(n)}]$, then we can calculate its determinant; namely, $\det \mathbf{X} = W[\mathbf{x}^{(1)}\ \dots\ \mathbf{x}^{(n)}]$ and as such if $\det \mathbf{X} \neq 0$ at some particular point $t = t_0$, then the solutions $\mathbf{x}^{(1)}$, $\dots$ are all linearly independent at t_0.

Definition 7.1 (*Generalized Abel's Theorem*)

If $\mathbf{x}^{(1)}$, $\dots$, $\mathbf{x}^{(n)}$ are solutions to a homogenous first-order set of linear differential equations over some open interval I, then over I, either $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}] = 0$ or $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) \neq 0\ \ \forall t \in I$.

Abel's theorem is super helpful as we only need to evaluate the Wronskian / determinant over one point to conclude the linear dependence/independence of our solutions. (Note: some stuff about a fundamental set of solutions is talked about here but honestly I don't really care :/.)

Similarly to when we looked at real-valued solutions to differential equations, we can turn complex-valued solutions into real solutions:

Definition 7.2 (*Theorem 7.4.5*)

If $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is a solution to the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then solely the real part $\mathbf{u}$ and solely the imaginary part $\mathbf{v}$ are also solutions to the above equation.

(I don't see much use in doing the exercises here as they are just proofs about theorems from section 3.2 in matrix form. None are like super interesting.)

7.5 Constant Coefficients and Matrices

This subsection focuses on equations of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a $n \times n$ matrix of real-valued constants.

Assuming[1] $\mathbf{x} = \xi e^{rt}$, after substituting it into the above equation, we eventually derive the equation

$$(\mathbf{A} - r\mathbf{I})\xi = 0,$$

which means solutions to $\mathbf{x}$ are given pairs of eigenvalue-eigenvector combinations (r, ξ). When $\mathbf{A}$ is specifically a 2×2 matrix, if the eigenvalues of $\mathbf{A}$ have opposite signs, then the origin is a saddle point and an unstable equilibrium. If on the other hand, the eigenvalues of $\mathbf{A}$ have the same sign, then the origin is a **node** and $\mathbf{0}$ is a stable equlibrium if the eigenvalues are negative and unstable if the eigenvalues are positive.

Returning to the more general case of when $\mathbf{A}$ is a $n \times n$ matrix, the eigenvalues of $\mathbf{A}$ $(r_1, r_2, \ldots, r_n)$ can either be

1. all real and different from one another,
2. some eigenvalues are complex conjugate pairs of each other, or
3. some eigenvalues are repeated.

The first case is easy to take care of; if all n eigenvalues are real and different, then their corresponding eigenvectors $(\xi^{(i)})$ will all be linearly independent and as such $\mathbf{x} = c_1\xi^{(1)}e^{r_1 t} + \cdots + c_n\xi^{(n)}e^{r_n t}$. Section 7.6 deals with case two, of when some eigenvalues are complex conjugate pairs of each other. Section 7.8 deals with the case of repeated eigenvalues.

Remember: To find eigenvalues, solve the characteristic polynomial $\det(\mathbf{A} - r\mathbf{I}_n) = 0$, and to get the corresponding eigenvectors, RREF $\mathbf{A} - r\mathbf{I}_n$ for a given r.

Exercise 7-9.

7. After finding our eigenvalues to be $r = \{4, -1, 1\}$ and the associated eigenvectors to be $\xi = \{[1\ 1\ 1]^T, [1\ 0\ -1]^T, [-1\ 2\ -1]^T\}$, we conclude that $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} e^{t}$ for arbitrary constants c_1, c_2, c_3.

8. This is a nice symmetric matrix with eigenvalues. Solving, we eventually find the characteristic polynomial to be $(\lambda - 8)(\lambda + 1)(\lambda + 1) = 0$ which means the corresponding eigenvaues are $\lambda = r = 8, -1$. Solving for eigenvectors and plugging them in, we find $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$. □

[1] Some stuff about a phase portrait/plane is talked about here although those tools are primarily used for visualization purposes.