Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

6 F

The above sections are spacer sections.

7 Systems of First-Order Linear Equations

7.1 Introduction

Essentially, we consider systems of first-order equations since any higher order differential equation can inevitably be transformed into multiple first order linear transformations.

Moreover, for any $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$, we cna make the substituions $x_1 = y$, $x_2 = y'$, $\dots$, $x_n = y^{(n-1)}$ and thus eventually find $x_1' = F_1(t, x_1, x_2, \dots, x_n)$, $x_2' = F_2(t, x_1, x_2, \dots, x_n)$ and so on. Thus, we have effectively converted a general differential equation into many teeny tiny first-order differential equations (that are each in their own way, granted, hard to solve).

7.2 Matrices

(note: all uppercase letters from here on out (A, B, C, $\dots$) will most likely represent matrices from here on out unless they are in function notation (e.g. $F(t)$ would be a function)).

Various matrix preliminaries are covered here. Do note that when the book talks about the **adjoint** of A, they mean the **transpose of the conjugate matrix of** A rather than the cofactor expansion matrix of A.

Integrals, derivatives, and [x] over matrices of functions are just those same operations applied to each individual operations (boring). For example, $\displaystyle\int A\;dt = \int a_{ij}\;dt$.

7.3 More Linear Algebra

(This is just a review of Math 4a......)

7.4 Basic Theory of Systems of First-Order Linear Equations

(a.k.a. A review of section 3.2 but with matrices instead of second-order linear differential equations.)
To examine a system of n first-order linear equations each of the form $x_i' = p_{i1}(t)x_1 + p_{i2}(t)x_2 + \cdots + p_{in}(t)x_n + g_i(t)$, we can rewrite everything in matrix form and obtain the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]^T$, $\mathbf{g}(t) = [g_1(t)\ g_2(t)\ \dots\ g_n(t)]^T$, and $\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{pmatrix}$.

With matrix equations, multiple solutions ($\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(k)}(t)$) for $\mathbf{x}$ may exist. Moreover, if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two solutions to a first-order homogenous matrix differential equation ($\mathbf{g} = \mathbf{0}$), then $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution to said equation for arbitrary constants c_1, c_2 (Theorem 7.4.1, Page 305).

If we make a big matrix $\mathbf{X} = [\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ \dots\ \mathbf{x}^{(n)}]$, then we can calculate its determinant; namely, $\det \mathbf{X} = W[\mathbf{x}^{(1)}\ \dots\ \mathbf{x}^{(n)}]$ and as such if $\det \mathbf{X} \neq 0$ at some particular point $t = t_0$, then the solutions $\mathbf{x}^{(1)}$, $\dots$ are all linearly independent at t_0.

Definition 7.1 (*Generalized Abel's Theorem*)

If $\mathbf{x}^{(1)}$, $\dots$, $\mathbf{x}^{(n)}$ are solutions to a homogenous first-order set of linear differential equations over some open interval I, then over I, either $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}] = 0$ or $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) \neq 0 \ \ \forall t \in I$.

Abel's theorem is super helpful as we only need to evaluate the Wronskian / determinant over one point to conclude the linear dependence/independence of our solutions. (Note: some stuff about a fundamental set of solutions is talked about here but honestly I don't really care :/.)

Similarly to when we looked at real-valued solutions to differential equations, we can turn complex-valued solutions into real solutions:

Definition 7.2 (*Theorem 7.4.5*)

If $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is a solution to the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then solely the real part $\mathbf{u}$ and solely the imaginary part $\mathbf{v}$ are also solutions to the above equation.

(I don't see much use in doing the exercises here as they are just proofs about theorems from section 3.2 in matrix form. None are like super interesting.)

7.5 Constant Coefficients and Matrices

This subsection focuses on equations of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a $n \times n$ matrix of real-valued constants.

Assuming[1] $\mathbf{x} = \xi e^{rt}$, after substituting it into the above equation, we eventually derive the equation

$$(\mathbf{A} - r\mathbf{I})\xi = 0,$$

which means solutions to $\mathbf{x}$ are given pairs of eigenvalue-eigenvector combinations (r, ξ). When $\mathbf{A}$ is specifically a 2×2 matrix, if the eigenvalues of $\mathbf{A}$ have opposite signs, then the origin is a saddle point and an unstable equilibrium. If on the other hand, the eigenvalues of $\mathbf{A}$ have the same sign, then the origin is a **node** and $\mathbf{0}$ is a stable equlibrium if the eigenvalues are negative and unstable if the eigenvalues are positive.

Returning to the more general case of when $\mathbf{A}$ is a $n \times n$ matrix, the eigenvalues of $\mathbf{A}$ (r_1, r_2, $\ldots$, r_n) can either be

1. all real and different from one another,
2. some eigenvalues are complex conjugate pairs of each other, or
3. some eigenvalues are repeated.

The first case is easy to take care of; if all n eigenvalues are real and different, then their corresponding eigenvectors ($\xi^{(i)}$) will all be linearly independent and as such $\mathbf{x} = c_1\xi^{(1)}e^{r_1 t} + \cdots + c_n\xi^{(n)}e^{r_n t}$. Section 7.6 deals with case two, of when some eigenvalues are complex conjugate pairs of each other. Section 7.8 deals with the case of repeated eigenvalues.

Remember: To find eigenvalues, solve the characteristic polynomial $\det(\mathbf{A} - r\mathbf{I}_n) = 0$, and to get the corresponding eigenvectors, RREF $\mathbf{A} - r\mathbf{I}_n$ for a given r.

Exercise 7-9.

7. After finding our eigenvalues to be $r = \{4, -1, 1\}$ and the associated eigenvectors to be $\xi = \{[1\ 1\ 1]^T, [1\ 0\ -1]^T, [-1\ 2\ -1]^T\}$, we conclude that $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} e^{t}$ for arbitrary constants c_1, c_2, c_3.

8. This is a nice symmetric matrix with eigenvalues. Solving, we eventually find the characteristic polynomial to be $(\lambda - 8)(\lambda + 1)(\lambda + 1) = 0$ which means the corresponding eigenvaues are $\lambda = r = 8, -1$. Solving for eigenvectors and plugging them in, we find $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$.

9. $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. □

[1] Some stuff about a phase portrait/plane is talked about here although those tools are primarily used for visualization purposes.

Exercise 20.

20a: Note that this problem operates under the assumption A and $\xi^{(1)}$ are matrices with constant coefficients. For the sake of the argument, assume $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{v} \neq \mathbf{0}$. As such, since matrix multiplication is distributive, we can rearrange the equation and get $\mathbf{A}\xi^{(1)} = r_1\mathbf{I}\xi^{(1)} + \mathbf{v} = r_1\xi^{(1)} + \mathbf{v}$ as per properties from the identity matrix.

Returning back to our original equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, since the given solution for $\mathbf{x}$ holds for any coefficients c_1 and c_2, we let $c_1 = 1$ and $c_2 = 0$ and thus a particular solution to our differential equation is $\mathbf{x} = \xi^{(1)}e^{r_1 t}$. We plug this particular solution of our differential equation into our differential equation and derive

$$\left(\xi^{(1)}e^{r_1 t}\right)' = \mathbf{A}\left(\xi^{(1)}e^{r_1 t}\right) \to r_1 e^{r_1 t}\xi^{(1)} = e^{r_1 t}\mathbf{A}\xi^{(1)} \to r_1\xi^{(1)} = \mathbf{A}\xi^{(1)}.$$

Substituting in our other expression for the RHS, we find $r_1\xi^{(1)} = r_1\xi^{(1)} + \mathbf{v}$ which implies $\mathbf{v} = \mathbf{0}$, a contradiction! Thus, our original assumption that $\mathbf{v} \neq \mathbf{0}$ is false and $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0}$. (A symmetrical argument holds for proving why $(\mathbf{A} - r_2\mathbf{I})\xi^{(2)} = \mathbf{0}$.)

20b: $(\mathbf{A} - r_2\mathbf{I})\xi^{(1)} = \mathbf{A}\xi^{(1)} - r_2\xi(1) = r_1\xi^{(1)} - r_2\xi(1)$ since $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0} \to \mathbf{A}\xi^{(1)} = r_1\xi^{(1)}$.

20c: $(\mathbf{A} - r_2\mathbf{I})(c_1\xi^{(1)} + c_2\xi^{(2)}) = c_1(\mathbf{A} - r_2\mathbf{I})\xi^{(1)} + c_2(\mathbf{A} - r_2\mathbf{I})\xi^{(2)} = c_1(r_1 - r_2)\xi^{(1)}$. However, $(\mathbf{A} - r_2\mathbf{I})(c_1\xi^{(1)} + c_2\xi^{(2)}) = (\mathbf{A} - r_2\mathbf{I})\mathbf{0} = \mathbf{0}$. Thus, $c_1(r_1 - r_2)\xi^{(1)} = \mathbf{0}$ which means either $c_1 = 0$, $r_1 = r_2$, or $\xi^{(1)} = \mathbf{0}$ (or some combination of the above).

Since we assume $r_1 \neq r_2$, the second statement must be false, and if $\xi^{(1)} = \mathbf{0}$, then the wronskian of our solution would be 0, contradicting our assumption that the general solution for $\mathbf{x}$ is a fundamental solution with respect to $\xi^{(1)}e^{r_1 t}$ and $\xi^{(2)}e^{r_2 t}$. Thus, this leads us to conclude that $c_1 = 0$ which is a contradiction, meaning our original supposition (that $\xi^{(1)}$ and $\xi^{(2)}$ are linearly dependent) is false. □

Exercise 21.

21a: We can rewrite this second order linear differential as

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2 \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

21b: The eigenvalue equation is simply the determinant of $\mathbf{A} - \lambda\mathbf{I}$ which is

$$0 = -\lambda\left(-\frac{b}{a} - \lambda\right) - (1)\left(-\frac{c}{a}\right) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \to a\lambda^2 + b\lambda + c.$$

□

Exercise 24.

24a. The general equation for eigenvalues of this matrix is $\lambda^2 + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\lambda + \left(\frac{R_1 + R_2}{CLR_2}\right) = 0$. Plugging in the specific values given, we eventually find eigenvalues of -1 and -2 and the general solution corresponding is $\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}$. Specific values that make the initial conditions hold are $c_1 = \dfrac{3I_0 - V_0}{2}$ and $c_2 = \dfrac{V_0 - I_0}{2}$.

24b. Since both terms in the above equation have e^{-t} in them, as $t \to \infty$ $\begin{pmatrix} I \\ V \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since the decay of e^{-t} is too much for the constants in the equation to handle. □

Exercise 25.
25a. $b^2 - 4ac > 0$ so $\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1+R_2}{CLR_2}\right) > 0$.
25b. Since all the coefficients in the characteristic polynomial are positive since R_1, R_2, C, and L cannot take on negative values (negative resistance?!), it is impossible for an eigenvalue λ to be greater than 0 as each term in the equation would thus be bigger than 0. As such, both I and V will eventually become 0 as t goes to ∞.
25c. If the eigenvalues are complex or repeated, it is possible that both I and/or V blow up to infinity (e.g. if you have negative resistance that adds energy to the circuit) or I and V settle into a stable equilibrium solution (see an LC clock circuit — no energy is being lost since there is no resistor in the circuit). □

7.6 Complex Eigenvalues

Essentially, if the eigenvalues of $\mathbf{A}$ are complex[2] (in conjugate pairs), then the resulting graph of $\mathbf{x}$ will look like a spiral either convergering at the origin or diverging away from it (depending on whether the eigenvalues have positive or negative real part). If the real part is 0, then the graph will basically make a loop around the origin.

For the general linear differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $r_2 = \overline{r_1}$, then $\xi^{(2)} = \overline{\xi^{(1)}}$. Thus, a particular solution to $\mathbf{x}$ would be

$$\mathbf{x} = \mathbf{x}^{(1)} = \xi^{(1)} e^{r_1 t} \to (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} = e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) + ie^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t))$$

where we made the substitutions $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$ and $r_1 = \lambda + i\mu$. Thus, if we let $\mathbf{x}^{(1)}t = \mathbf{u}(t) + i\mathbf{v}(t)$ with $\mathbf{u}$ and $\mathbf{v}$ corresponding to the real and imaginary parts above, we have a particular solution for $\mathbf{x}$.
But since we want a real solution for $\mathbf{x}$, by that one weird theorem (Theorem 7.4.5) covered above, $\mathbf{u}$ and $\mathbf{v}$ are themselves individual solutions to $\mathbf{x}$ and we can write $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + \dots$ where the dots indicate the solutions stemming from the other roots of $\mathbf{x}$.

Exercise 5-6.
5. Our eigenvalues for this matrix are $\lambda = 1, 1 \pm 2i$. The eigenvector for $\lambda = 1$ is $\xi = [2 \ \ -3 \ \ 2]^T$, and the eigenvector for $\lambda = 1 + 2i$ is $\xi = [0 \ 0 \ 1]^T + i[0 \ 1 \ 0]^T$. Thus, our final solution is

$$\mathbf{x} = c_1 e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin(2t) \right) + c_2 e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos(2t) \right) + c_3 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t.$$

6. This one was hard (calculation-wise) and the answer is dicey. The eigenvalues are $\lambda = -2, -1 \pm i\sqrt{2}$ and the final solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + c_2 e^{-t} \begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}.$$

□

[2] All entries in $\mathbf{A}$ must be real as otherwise complex roots may not come in conjugate pairs.

Exercise 20.
20a. By Kirchoff's voltage law, across the triangle formed by R_1, C, and L, we find

$$L\frac{dI}{dt} + V + R_1 I = 0 \to \frac{dI}{dt} = -\frac{R_1}{L}I - \frac{1}{L}V$$

which is the first row of our matrix equation.
Summing the current at the point joined by C, R_2, and L, we find

$$\frac{V}{R_2} + C\frac{dV}{dt} = I \to \frac{dV}{dt} = \frac{1}{C}I - \frac{1}{CR_2}V.$$

Note that the reason why voltage across resistor R_2 is V is because summing the voltage loop over the rectangle C and R_2, we must have $V + (-V_{R_2}) = 0$ so $V_{R_2} = V$. Note that there is a negative sign over V_{R_2} as when tracing the loop across the resistor, our loop goes against the current of the circuit.
Thus, turning both equations into matrix form, we have

$$\begin{cases} \frac{dI}{dt} = -\frac{R_1}{L}I - \frac{1}{L}V \\ \frac{dV}{dt} = \frac{1}{C}I - \frac{1}{CR_2}V \end{cases} \to \begin{bmatrix} I' \\ V' \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}$$

□