Differential Equations Notes

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Fall 2025

Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 Introduction

aka chapter 1

1.1 Introduction for the Introduction

Definition (Differential Equations)**.** Equations containing derivatives.

Definition (Slope Field/Direction Field)**.** A buncha line segments on the plane that represent the "motion" of a diff-eq.

Direction Fields are good for studying differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

(Page 6 – How to construct a diff-eq mathematical model from a real-world situation.)
(7) Newton: Differential equations come in one of these 3 forms:

1. $\dfrac{dy}{dx} = f(x)$,
2. $\dfrac{dy}{dx} = f(y)$,
3. $\dfrac{dy}{dx} = f(x, y)$.

Exercise 11-16.
1.1.5 corresponds with **j**.
1.1.6 corresponds with **c**.
1.1.7 corresponds with **g**.
1.1.8 corresponds with **b**.
1.1.9 corresponds with **h**.
1.1.10 corresponds with **e**. □

Exercise 17.
(a)

$$\frac{dC}{dt} = [\text{chemicals/hour going in}] - [\text{out}] = 0.01 \cdot 300 - 300 \cdot \frac{C}{1000000}$$

where C is the number of gallons of said chemical in the pond and t is time measured in hours.

(b) After a very long time, 10000 gallons will be in the pond; this limiting amount is independent of starting conditions.

(c) Since concentration $= \dfrac{\text{Amount}}{\text{Volume}}$, $C = \text{volume} \cdot c = c \cdot 10^6$ where c stands for concentration. As such,

$$\frac{dc}{dt} = \frac{1}{10^6}\frac{dC}{dt} = \frac{3}{10^6} - \frac{3(c \cdot 10^6)}{10^4 \cdot 10^6}$$

So in final, $\boxed{\dfrac{dc}{dt} = \dfrac{3}{10^6} - \dfrac{3c}{10^4}}$. □

Exercise 18.

$$\frac{dV}{dt} = -k \cdot 4\pi r^2 = -k \cdot 4\pi \left(\frac{3}{4\pi}V\right)^{\frac{2}{3}}$$

□

Exercise 19.

$$\frac{dT}{dt} = -0.05 * (T - 70)$$

where T is the temperature of the object in Farenheit and t is time in minutes. □

1.2 Introduction to Solutions

(11) - Finding the general solutions to diff-eqs of the form $\frac{dy}{dt} = ay - b$ $(a \neq 0)$;

$$\frac{dy}{dt} = ay - b \Longrightarrow y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$$

⟨14 - "Further Remarks on Mathematical Modeling" - essentially, the underlying assumptions we make may or may not be wrong. ⟩

Exercise 1a.
$\frac{dy}{dt} = -y + 5 \to \frac{1}{5-y}dy = dt.$

So, $\ln(5 - y(t)) = t + C$. With initial condition $y(0) = k$, we get that $\ln(5 - k) = C$, so our solution becomes $y(t) = 5 - e^{t+\ln(5-k)} = 5 - (5 - k)e^t$. (Note that $(5 - k)$ is constant.) □

Exercise 9a.
Since $F = ma$, $F = m\frac{dv}{dt}$. Since drag acts inversely to velocity (object falling faster has more air resistance), we should expect $\frac{dv}{dt}$ to be negative; thus, $\frac{dv}{dt} = -\frac{F}{10}$. Knowing that F is proportional to the square of the velocity, we know that $F = av^2 - b$ for constants a, b.

Now, we plug in some known values. At $v = 0$, we expect $\frac{dv}{dt} = -\frac{(-b)}{10} = -9.8$ (gravity) so $b = -98$. At $v = 49$, we reach limiting velocity which implies $\frac{dv}{dt} = 0$ so $\frac{a(49^2)-98}{10} = 0$ so $a = \frac{2}{49}$. Thus, in final, we get our differential equation as

$$\frac{dv}{dt} = \frac{2}{49 \cdot 10}v^2 - \frac{98}{10}$$

which can be re-arranged to

$$\frac{dv}{dt} = \frac{1}{245}\left(v^2 - 49\right).$$

□

Exercise 9b.
(I'm gonna go with their equation for simplicity - it doesn't matter too much though.)

$$\frac{dv}{dt} = \frac{1}{245}\left(49^2 - v^2\right)$$
$$\to 245\frac{1}{49^2 - v^2}\ dv = dt$$
$$\to 245\int \frac{1}{49^2 - v^2}\ dv = t$$

Doing a trig sub ($v = 49\sin\theta,\ dv = 49\cos\theta\ d\theta$), $\dfrac{dv}{49^2 - v^2}$ becomes $\dfrac{49\cos\theta\ d\theta}{49^2 - 49\sin^2\theta}$ so our integral ends up turning into

$$\longrightarrow 245\int \frac{d\theta}{49\cos\theta} = t \Longrightarrow 5\left(\frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right) = t + C.$$

Thus,

$$t + C = \frac{5}{2}\ln\frac{1+v/49}{1-v/49}.$$

Plugging in our initial condition $v(0) = 0$, we get that $C = 0$. Thus,

$$\ln\left(\frac{1+v/49}{1-v/49}\right) = \frac{2t}{5} \text{ so } 49 + v = (49 - v)(e^{2t/5}).$$

Simplifying, (by expanding and putting all the vs on one side of the equation) we find our final answer to be

$$v(t) = 49\cdot\frac{e^{2t/5}-1}{e^{2t/5}+1} = 49\tanh(t).$$

□

Exercise 13.
(a)

$$\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{RC} = \frac{VC - Q}{RC}$$

$$\Rightarrow RC \int \frac{dQ}{VC - Q} = t + C$$

Integrating, we get that $t + C_1 = -RC\ln(VC - Q)$. Plugging in our initial condition $Q(0) = 0$, we get that $C_1 = -RC\ln(VC)$. Thus, we can substitute and simplify as follows:

$$RC\ln(VC - Q) = RC\ln(VC) - t \to \ln(VC - Q) - \ln(VC) = -\frac{t}{RC}$$

$$\to VC - Q = VCe^{-t/RC}$$

so $Q(t) = VC(1 - e^{-t/RC})$.

(b) After a very long time ($t \sim \infty$), $Q \sim VC$ so $Q_L = VC$.

(c) From Kirchoff's voltage rule, $R\frac{dQ}{dt} + \frac{Q}{C} = 0 \longrightarrow -\frac{Q}{C} = R\frac{dQ}{dt}$. Thus, $t + C_1 = -RC\ln(Q)$. Evaluating in our initial condition, we get that $C_1 = -RC\ln(Q_L) + t_1$. As a result,

$$-(t - t_1) = RC(\ln(Q) - \ln(Q_L))$$

so

$$Q = Q_L e^{-\frac{t-t_1}{RC}}.$$

□

1.3 Classification of Diffy Qs

Definition (Ordinary Differential Equation)**.** An Ordinary Diffy Q (ODE) is an equation where the unknown function depends on a single independent variable.
E.g. (LRC Circuit)

$$L^2\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Definition (Partial Differential Equation)**.** A Partial Differential Eq (PDE) is when the unknown function depends on several independent variables.
E.g. (Wave Equation)

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

(17) - If you have n unknown functions in a system of differential equations, then you gotta have at least n diffy qs to solve that system completely.

Definition (Order)**.** The **order** of a differential equation is the highest derivative that appears in the differential equation. Thus you can have a *first-order* or *second-order* or *seventh-order* diffy q.
E.g.: $\alpha\dfrac{d^3x}{dk^3} + \beta\dfrac{d^2x}{dk^2} + \dfrac{\alpha}{\beta}x = \gamma$ is a third-order (ordinary differential) equation (when α, β, and γ are constants and x is a function of k).

Generally then, a differential equation of order n can be represented by the generic $F(t, x(t), x'(t), \ldots, x^{(n)}t) = 0$ for some function $x(t)$. Replacing $y = x(t)$, we get that a general nth order differential equation is of the form

$$F(t, y, y', \ldots, y^{(n)}) = 0.^{1}$$

[1] (18) Note: We assume it is always possible to solve for the highest derivative – e.g. we can rearrange to get to the form of $y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$.

Definition (Linearity)**.** A differential equation is said to be **linear** if $F(t, y, y', \ldots, y^{(n)}) = 0$ is a linear function of t, y, y', $\ldots$, $y^{(n)}$.
As such, the general linear diffy q is of the form $0 = c(t) + a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)}$.

Definition (Linearization)**.** Linearization is the process of approximating a non-linear diffy q by a linear one. Example given in the textbook is of approximating the motion of an oscillating pendulum.

(19-20) - Questions of solvability and uniqueness for general differential equations.

Exercise 1-4.
1. Order is 2, and the differential equation is linear.
2. Order is 2, and the differential equation is NOT linear (because of the term $(1+y^2)\frac{d^2y}{dt^2}$).
3. Order is 4, and the differential equation is linear.
4. Order is 2, and the differential equation is non-linear. □

Exercise 10.
(I'm only doing this one because it looks fun)
We shall verify that $y = e^{t^2}\left(1 + \int_0^t e^{-s^2}\, ds\right)$ is a solution to the differential equation $y' - 2ty = 1$.
First, we substitute y into our equation.

$$\left[e^{t^2} + e^{t^2}\int_0^t e^{-s^2}\, ds\right]' = 1 + 2t \cdot e^{t^2}\left(1 + \int_0^t e^{-s^2}\, ds\right)$$

Next, we differentiate that left side and simplify the right.

$$\longrightarrow 2te^{t^2} + \left(2te^{t^2}\right)\left(\int_0^t e^{-s^2}\, ds\right) + \left(e^{t^2}\right)\left(e^{-t^2}\right) = 1 + 2te^{t^2} + 2te^{t^2}\left(\int_0^t e^{-k^2}\, dk\right)$$

Finally, we cancel terms and arrive at the equation

$$e^{t^2} \cdot e^{-t^2} = 1,$$

which is trivially true. Thus, we are done. □

Exercise 11-13.
Since $y = e^{rt}$, $y^{(n)} = r^n e^{rt}$. Thus, in each of problems 11-13, we're basically just solving a polynomial. To illustrate, consider problem 12:

$$y'' + y' - 6y = 0 \Longrightarrow r^2e^{rt} + re^{rt} - 6e^{rt} = e^{rt}\left(r^2 + r - 6\right) = 0,$$

which is (almost) isomorphic to solving the system $x^2 + x - 6 = 0$. Thus, we yield the solutions $r = 2, 3$ and maybe even $r = -\infty$ (which would make e^{rt} be 0).

Similar solutions follow for 11 and 13. □

Exercise 16-18.
16: 2nd order linear partial differential eq.
17: 4th order linear PDE.
18: 2nd order non-linear PDE. □

2 First-Order Diffy Qs

aka chapter 2
for chapter 2, all diffy qs will be first order.

2.1 Linear ODEs: Method of Integrating Factors

If $\frac{dy}{dt} = f(t, y)$ and f is linear (w.r.t y), then we can rewrite it in the following form (called the **first-order linear differential equation**):

$$\frac{dy}{dt} + p(t)y = g(t) \iff P(t)\frac{dy}{dt} + Q(t)y = G(t) \text{ (page 24)}$$

Definition (Integrating Factor)**.** A **integrating factor** $\mu(t)$ is a function such that when a diffy q is multiplied by it, the equation is then immediately integratable (discovered by Leibniz). (page 25)

Exercise - Pauls Online Notes, Problem 4 (modified).
Find the general solution to the ODE

$$t\frac{dy}{dt} + 2y = t^2 - t + 1.$$

This diffy q looks hard. To start, we add on an integrating factor $\alpha(t)$ to the equation to get

$$t\alpha(t)\frac{dy}{dt} + 2\alpha(t)y = \alpha(t)(t^2 - t + 1).$$

From here, consider what happens when you take the derivative of $(t \cdot y \cdot \alpha(t))$:[a]

$$\frac{d}{dt}\left[t \cdot y \cdot \alpha(t)\right] = y\alpha(t) + t\alpha(t)\frac{dy}{dt} + ty\alpha'(t) = t\alpha(t)\frac{dy}{dt} + y(\alpha(t) + t\alpha'(t)).$$

For this equation to match the left hand side of the equation above, we then must have that $t\alpha(t) = t\alpha(t)$ and $2\alpha(t) = \alpha(t) + t\alpha'(t) \to \alpha(t) = t\alpha'(t)$. From that last equation, I recognized that the function $\alpha(t) = t$ works!
And from there, after plugging things in and integrating, I ended up with my final answer that the general solution to the given ODE is

$$y(t) = \boxed{\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}}.$$

□

[a] This is not the *actual* way to do it –Paul's math notes first divides everything by t so they only have to consider the derivative of $y\alpha(t)$.

So essentially, the process of solving diffy qs of the form $P(t)\frac{dy}{dt} + Q(t)y = G(t)$ is to first divide by $P(t)$, then find an integrating factor that "matches up" both sides of the equation.
Mathematically:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \to \frac{dy}{dt} + \frac{Q(t)}{P(t)}y = \frac{G(t)}{P(t)} \to \kappa(t)\frac{dy}{dt} + \kappa(t)\frac{Q(t)}{P(t)}y = \kappa(t)\frac{G(t)}{P(t)}$$

Since $(y\kappa(t))' = \kappa(t) \cdot y' + \kappa'(t)y$, comparing terms on the LHS, we get that we just need to find some $\kappa(t)$ such that $\kappa'(t) = \kappa(t)\frac{Q(t)}{P(t)}$. If that nasty fraction $\left(\frac{Q(t)}{P(t)}\right)$ is some constant or basic polynomial, the equation is *probably* solvable.
So assuming that some suitable $\kappa(t)$ is found, we then just kinda evaluate everything from there.

$$\longrightarrow \int \frac{d}{dt}\left[y\kappa(t)\right] = \int \kappa(t)\frac{G(t)}{P(t)} \Longrightarrow y(t) = \frac{C + \int \kappa(t)\frac{G(t)}{P(t)}}{\kappa(t)}.$$

It's quite messy when written out.

(27) For equations of the form $\frac{dy}{dt} + ay = g(t)$, the right integrating factor is $\mu(t) = e^{at}$. This can be rederived pretty easily (probably).

If you want to integrate but the messy thing doesn't simplify, that's fine; put the bounds of your integral to be from some arbitrary t_0 to t, preferably in a way such that if an initial condition $y(y_0) = c_0$, $t_0 = y_0$. In this way, your integral will collapse on itself when evaluated at $y = y_0$ and any other value of your function will be computed by that constant given in the problem plus the accumulated gain/loss from the function as it goes to your desired x/y value.