Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 Introduction

aka chapter 1

1.1 Introduction for the Introduction

Definition 1.1 (*Differential Equations*)

Equations containing derivatives.

Definition 1.2 (*Slope Field/Direction Field*)

A buncha line segments on the plane that represent the "motion" of a diff-eq.

Direction Fields are good for studying differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

(Page 6 – How to construct a diff-eq mathematical model from a real-world situation.)
(7) Newton: Differential equations come in one of these 3 forms:

1. $\frac{dy}{dx} = f(x)$,
2. $\frac{dy}{dx} = f(y)$,
3. $\frac{dy}{dx} = f(x, y)$.

Exercise 11-16.
1.1.5 corresponds with **j**.
1.1.6 corresponds with **c**.
1.1.7 corresponds with **g**.
1.1.8 corresponds with **b**.
1.1.9 corresponds with **h**.
1.1.10 corresponds with **e**. □

Exercise 17.
(a)

$$\frac{dC}{dt} = [\text{chemicals/hour going in}] - [\text{out}] = 0.01 \cdot 300 - 300 \cdot \frac{C}{1000000}$$

where C is the number of gallons of said chemical in the pond and t is time measured in hours.

(b) After a very long time, 10000 gallons will be in the pond; this limiting amount is independent of starting conditions.

(c) Since concentration $= \frac{\text{Amount}}{\text{Volume}}$, $C = \text{volume} \cdot c = c \cdot 10^6$ where c stands for concentration. As such,

$$\frac{dc}{dt} = \frac{1}{10^6}\frac{dC}{dt} = \frac{3}{10^6} - \frac{3(c \cdot 10^6)}{10^4 \cdot 10^6}$$

So in final, $\boxed{\frac{dc}{dt} = \frac{3}{10^6} - \frac{3c}{10^4}}$. □

1.2 Introduction to Solutions

(11) - Finding the general solutions to diff-eqs of the form $\frac{dy}{dt} = ay - b$ $(a \neq 0)$;

$$\frac{dy}{dt} = ay - b \Longrightarrow y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

⟨14 - "Further Remarks on Mathematical Modeling" - essentially, the underlying assumptions we make may or may not be wrong. ⟩

Exercise 1a.
$\dfrac{dy}{dt} = -y + 5 \to \dfrac{1}{5-y} dy = dt.$

So, $\ln(5 - y(t)) = t + C$. With initial condition $y(0) = k$, we get that $\ln(5 - k) = C$, so our solution becomes $y(t) = 5 - e^{t+\ln(5-k)} = 5 - (5 - k)e^t$. (Note that $(5 - k)$ is constant.) □

Exercise 9.
9a: Since $F = ma$, $F = m\frac{dv}{dt}$. Since drag acts inversely to velocity (object falling faster has more air resistance), we should expect $\frac{dv}{dt}$ to be negative; thus, $\frac{dv}{dt} = -\frac{F}{10}$. Knowing that F is proportional to the square of the velocity, we know that $F = av^2 - b$ for constants a, b.

Now, we plug in some known values. At $v = 0$, we expect $\frac{dv}{dt} = -\frac{(-b)}{10} = -9.8$ (gravity) so $b = -98$. At $v = 49$, we reach limiting velocity which implies $\frac{dv}{dt} = 0$ so $\frac{a(49^2)-98}{10} = 0$ so $a = \frac{2}{49}$. Thus, in final, we get our differential equation as

$$\frac{dv}{dt} = \frac{2}{49 \cdot 10} v^2 - \frac{98}{10}$$

which can be re-arranged for the desired equation.

9b:

$$\frac{dv}{dt} = \frac{1}{245}\left(49^2 - v^2\right) \to 245 \frac{1}{49^2 - v^2}\, dv = dt \to 245 \int \frac{1}{49^2 - v^2}\, dv = t$$

Doing a trig sub ($v = 49 \sin\theta$, $dv = 49 \cos\theta\, d\theta$), $\dfrac{dv}{49^2 - v^2}$ becomes $\dfrac{49 \cos\theta\, d\theta}{49^2 - 49 \sin^2\theta}$ so our integral ends up turning into

$$\longrightarrow 245 \int \frac{d\theta}{49 \cos\theta} = t \Longrightarrow 5\left(\frac{1}{2} \ln\left(\frac{1 + \sin\theta}{1 - \sin\theta}\right)\right) = t + C.$$

Thus,

$$t + C = \frac{5}{2} \ln \frac{1 + v/49}{1 - v/49}.$$

Plugging in our initial condition $v(0) = 0$, we get that $C = 0$. Thus,

$$\ln\left(\frac{1 + v/49}{1 - v/49}\right) = \frac{2t}{5} \text{ so } 49 + v = (49 - v)(e^{2t/5}).$$

Simplifying, (by expanding and putting all the vs on one side of the equation) we find our final answer to be

$$v(t) = 49 \cdot \frac{e^{2t/5} - 1}{e^{2t/5} + 1} = 49 \tanh(t).$$

□

Exercise 13.
(a)
$$\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{RC} = \frac{VC - Q}{RC} \Longrightarrow RC \int \frac{dQ}{VC - Q} = t + C$$
Integrating, we get that $t + C_1 = -RC\ln(VC - Q)$. Plugging in our initial condition $Q(0) = 0$, we get that $C_1 = -RC\ln(VC)$. Thus, we can substitute and simplify as follows:
$$RC\ln(VC - Q) = RC\ln(VC) - t \to \ln(VC - Q) - \ln(VC) = -\frac{t}{RC}$$
$$\to VC - Q = VCe^{-t/RC}$$
so $Q(t) = VC(1 - e^{-t/RC})$.

(b) After a very long time ($t \sim \infty$), $Q \sim VC$ so $Q_L = VC$.

(c) From Kirchoff's voltage rule, $R\frac{dQ}{dt} + \frac{Q}{C} = 0 \longrightarrow -\frac{Q}{C} = R\frac{dQ}{dt}$. Thus, $t + C_1 = -RC\ln(Q)$. Evaluating in our initial condition, we get that $C_1 = -RC\ln(Q_L) + t_1$. As a result,
$$-(t - t_1) = RC(\ln(Q) - \ln(Q_L))$$
so
$$Q = Q_L e^{-\frac{t-t_1}{RC}}.$$
□

1.3 Classification of Diffy Qs

Definition 1.3 (*Ordinary Differential Equation*)

An Ordinary Diffy Q (ODE) is an equation where the unknown function depends on a single independent variable.
E.g. (LRC Circuit)
$$L^2\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Definition 1.4 (*Partial Differential Equation*)

A Partial Differential Eq (PDE) is when the unknown function depends on several independent variables.
E.g. (Wave Equation)
$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

(17) - If you have n unknown functions in a system of differential equations, then you gotta have at least n diffy qs to solve that system completely.

Definition 1.5 (*Order*)

The **order** of a differential equation is the highest derivative that appears in the differential equation. Thus you can have a *first-order* or *second-order* or *seventh-order* diffy q.
E.g.: $\alpha\dfrac{d^3x}{dk^3} + \beta\dfrac{d^2x}{dk^2} + \dfrac{\alpha}{\beta}x = \gamma$ is a third-order (ordinary differential) equation (when α, β, and γ are constants and x is a function of k).

Generally then, a differential equation of order n can be represented by the generic $F(t, x(t), x'(t), \dots, x^{(n)}t) = 0$ for some function $x(t)$. Replacing $y = x(t)$, we get that a general nth order differential equation is of the form
$$F(t, y, y', \dots, y^{(n)}) = 0.^{1}$$

[1] (18) Note: We assume it is always possible to solve for the highest derivative – e.g. we can rearrange to get to the form of

Definition 1.6 (***Linearity***)

A differential equation is said to be **linear** if $F(t, y, y', \ldots, y^{(n)}) = 0$ is a linear function of t, y, y', $\ldots$, $y^{(n)}$. As such, the general linear diffy q is of the form $0 = c(t) + a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)}$.

Definition 1.7 (***Linearization***)

Linearization is the process of approximating a non-linear diffy q by a linear one. Example given in the textbook is of approximating the motion of an oscillating pendulum.

(19-20) - Questions of solvability and uniqueness for general differential equations.

Exercise 1-4.
1. Order is 2, and the differential equation is linear.
2. Order is 2, and the differential equation is NOT linear (because of the term $(1 + y^2)\frac{d^2y}{dt^2}$).
3. Order is 4, and the differential equation is linear.
4. Order is 2, and the differential equation is non-linear. □

Exercise 10.
(I'm only doing this one because it looks fun)
We shall verify that $y = e^{t^2}\left(1 + \int_0^t e^{-s^2}\, ds\right)$ is a solution to the differential equation $y' - 2ty = 1$.
First, we substitute y into our equation.

$$\left[e^{t^2} + e^{t^2}\int_0^t e^{-s^2}\, ds\right]' = 1 + 2t \cdot e^{t^2}\left(1 + \int_0^t e^{-s^2}\, ds\right)$$

Next, we differentiate that left side and simplify the right.

$$\longrightarrow 2te^{t^2} + \left(2te^{t^2}\right)\left(\int_0^t e^{-s^2}\, ds\right) + \left(e^{t^2}\right)\left(e^{-t^2}\right) = 1 + 2te^{t^2} + 2te^{t^2}\left(\int_0^t e^{-k^2}\, dk\right)$$

Finally, we cancel terms and arrive at the equation

$$e^{t^2} \cdot e^{-t^2} = 1,$$

which is trivially true. Thus, we are done. □

Exercise 11-13.
Since $y = e^{rt}$, $y^{(n)} = r^n e^{rt}$. Thus, in each of problems 11-13, we're basically just solving a polynomial. To illustrate, consider problem 12:

$$y'' + y' - 6y = 0 \Longrightarrow r^2e^{rt} + re^{rt} - 6e^{rt} = e^{rt}\left(r^2 + r - 6\right) = 0,$$

which is (almost) isomorphic to solving the system $x^2 + x - 6 = 0$. Thus, we yield the solutions $r = 2, 3$ and maybe even $r = -\infty$ (which would make e^{rt} be 0).

Similar solutions follow for 11 and 13. □

Exercise 16-18.
16: 2nd order linear partial differential eq.
17: 4th order linear PDE.
18: 2nd order non-linear PDE. □

$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$.

2 First-Order Differential Equations

aka chapter 2
for chapter 2, all diffy qs will be first order.

2.1 Linear ODEs: Method of Integrating Factors

If $\frac{dy}{dt} = f(t, y)$ and f is linear (w.r.t y), then we can rewrite it in the following form (called the **first-order linear differential equation**):

$$\frac{dy}{dt} + p(t)y = g(t) \iff P(t)\frac{dy}{dt} + Q(t)y = G(t) \text{ (page 24)}$$

Definition 2.1 (*Integrating Factor*)

A **integrating factor** $\mu(t)$ is a function such that when a diffy q is multiplied by it, the equation is then immediately integratable (discovered by Leibniz). (page 25)

Exercise - Pauls Online Notes, Problem 4 (modified).
Find the general solution to the ODE

$$t\frac{dy}{dt} + 2y = t^2 - t + 1.$$

This diffy q looks hard. To start, we add on an integrating factor $\alpha(t)$ to the equation to get

$$t\alpha(t)\frac{dy}{dt} + 2\alpha(t)y = \alpha(t)(t^2 - t + 1).$$

From here, consider what happens when you take the derivative of $(t \cdot y \cdot \alpha(t))$:[a]

$$\frac{d}{dt}\left[t \cdot y \cdot \alpha(t)\right] = y\alpha(t) + t\alpha(t)\frac{dy}{dt} + ty\alpha'(t) = t\alpha(t)\frac{dy}{dt} + y(\alpha(t) + t\alpha'(t)).$$

For this equation to match the left hand side of the equation above, we then must have that $t\alpha(t) = t\alpha(t)$ and $2\alpha(t) = \alpha(t) + t\alpha'(t) \to \alpha(t) = t\alpha'(t)$. From that last equation, I recognized that the function $\alpha(t) = t$ works!
And from there, after plugging things in and integrating, I ended up with my final answer that the general solution to the given ODE is

$$y(t) = \boxed{\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}}.$$

□

[a] This is not the *actual* way to do it –Paul's math notes first divides everything by t so they only have to consider the derivative of $y\alpha(t)$.

So essentially, the process of solving diffy qs of the form $P(t)\frac{dy}{dt} + Q(t)y = G(t)$ is to first divide by $P(t)$, then find an integrating factor that "matches up" both sides of the equation.
Mathematically:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \to \frac{dy}{dt} + \frac{Q(t)}{P(t)}y = \frac{G(t)}{P(t)} \to \kappa(t)\frac{dy}{dt} + \kappa(t)\frac{Q(t)}{P(t)}y = \kappa(t)\frac{G(t)}{P(t)}$$

Since $(y\kappa(t))' = \kappa(t) \cdot y' + \kappa'(t)y$, comparing terms on the LHS, we get that we just need to find some $\kappa(t)$ such that $\kappa'(t) = \kappa(t)\frac{Q(t)}{P(t)}$. If that nasty fraction $\left(\frac{Q(t)}{P(t)}\right)$ is some constant or basic polynomial, the equation is *probably* solvable.
So assuming that some suitable $\kappa(t)$ is found, we then just kinda evaluate everything from there.

$$\longrightarrow \int \frac{d}{dt}\left[y\kappa(t)\right] = \int \kappa(t)\frac{G(t)}{P(t)} \implies y(t) = \frac{C + \int \kappa(t)\frac{G(t)}{P(t)}}{\kappa(t)}.$$

It's quite messy when written out.
(27) For equations of the form $\frac{dy}{dt}+ay=g(t)$, the right integrating factor is $\mu(t)=e^{at}$. This can be rederived pretty easily (probably).

If you want to integrate but the messy thing doesn't simplify, that's fine; put the bounds of your integral to be from some arbitrary t_0 to t, preferably in a way such that if an initial condition $y(y_0)=c_0$, $t_0=y_0$. In this way, your integral will collapse on itself when evaluated at $y=y_0$ and any other value of your function will be computed by that constant given in the problem plus the accumulated gain/loss from the function as it goes to your desired x/y value.

Exercise 1c-8c.
(I'm just going to document my answers here.)
1c: $y(t)=e^{-2t}+\frac{1}{3}\left(t-\frac{1}{3}\right)+\frac{C}{e^{3t}}$.
2c: $y(t)=\left(\frac{1}{3}t^3+C\right)e^{2t}$.
3c: $y(t)=\frac{t^2}{2e^t}+1+\frac{C}{e^t}$.
4c: $y(t)=1.5\sin(2t)+\frac{0.75}{t}\cos(2t)+\frac{C}{t}$.
5c: $y(t)=-3e^t+Ce^{2t}$.
6c: $y(t)=-te^{-t}+Ct$.
7c: $y(t)=\sin(2t)-2\cos(2t)+\frac{C}{e^t}$. Warning: This integral is hard but is very doable.
8c: $y(t)=3t^2-12t+24+\frac{C}{2e^{t/2}}$. □

Exercise 9-12.
(More answer exercise documentation. Both the general form and the specific solution to each problem will be given.)
9: $y(t)=2te^{2t}-2e^{2t}+Ce^t$. The specific case when $y(0)=1$ is given by $C=3$.
10: $y(t)=\frac{t^2/2+C}{e^{2t}}$. The specific solution when $y(1)=0$ is given by $C=-\frac{1}{2}$.
11: $y(t)=\frac{\sin t+C}{t^2}$, with $C=0$.
12: $y(t)=\frac{t-1}{t}+\frac{C}{te^t}$, $C=2$ for this particular case. □

Exercise 18.
Given the simplicity of the right hand side, we can fake solve the equation for $y(t)$; namely, from intuition, if $y(t) = \alpha t + \beta$, then $y'(t) = \alpha$ (a constant) and we can probably find values α, β that make such a solution possible.
In fact we do; $y(t) = -\frac{3}{4}t + \frac{21}{8} + Ce^{-2t/3}$, where that last term was derived from realizing that if we actually integrated this properly, our integrating factor $\mu(t)$ would be $e^{2t/3}$.
Anyways, things get a little dicey from here. Let's call the point where $y(t)$ touches (but doesn't cross) the t-axis as t_0. Then, we know that $y'(t_0) = 0$ (y must be at a local max/min as otherwise y would cross the t-axis) and $y(t_0) = 0$. From here, we can rearrange our equations as follows:

$$y'(t_0) = 0 \to 0 = -\frac{3}{4} - \frac{2}{3}Ce^{-2t_0/3} \to -\frac{9}{8} = Ce^{-2t_0/3} \text{ and}$$

$$y(t_0) = 0 \longrightarrow \frac{3}{4}t_0 - \frac{21}{8} = Ce^{-2t_0/3}$$

and we can match the LHSs of both equations to get $\frac{3}{4}t_0 - \frac{21}{8} = -\frac{9}{8}$ and find that $t_0 = 2$.
From here, we can simply plug this value of t_0 into our equations and solve for C:

$$y'(t_0) = 0 \to y'(2) = 0 \to C = -\frac{9}{8}e^{4/3}$$

Thus, $y_0 = y(0) = \frac{21}{8} + C = \frac{21}{8} - \frac{9}{8}e^{4/3} \approx \boxed{-1.64}$. □

Exercise 20.
(Note: this problem and the last problem have caused me some amount of pain because I keep misreading the problem and not sticking to the end.)
If you want $y' - y = 1 + 3\sin t$ to remain finite as $t \to \infty$, then when you get that $y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + Ce^t$, it should be pretty clear that $C = 0$. As such, $y_0 = y(0) = -1 - \frac{3}{2}(1 + 0) = \boxed{-\frac{5}{2}}$.
When doing this problem, don't doubt yourself :). □

Exercise 28 - Variation of Parameters.
(a). If $g(t) = 0\ \forall t$, then effectively $g(t) = 0$. As such, we are simply solving $\frac{dy}{dt} + p(t)y = 0$ which can be done by separating variables:

$$\frac{dy}{dt} = -p(t) \cdot y \to \frac{1}{y}dy = -p(t)dt \to \ln(y) = \int -p(t)dt + C_0 \text{ so } y(t) = C_1 \exp\left(-\int p(t)dt\right)$$

where $C_1 = e^{C_0}$. Replace C_1 with A to get the expression shown in the textbook.

(b). To show $A(t)$ must satisfy (51), we simply substitute everything in and cancel the messy equation.

$$y' + p(t)y = g(t) \Longrightarrow \left[A(t)\exp\left(-\int p(t)dt\right)\right]' + A(t)\exp\left(-\int p(t)dt\right)p(t) = g(t)$$

$$\longrightarrow \exp\left(-\int p(t)dt\right)\left([A'(t) + A(t)(-p(t))] + A(t)p(t)\right) = g(t) \text{ so } A'(t) = g(t) \cdot \exp\left(\int p(t)dt\right)$$

which is exactly the equation given by (51).

(c). This part is lowkey quite simple. Picking up from (b), we simply slap an integral sign in front of the massive equation that we derived for $A'(t)$, and after replacing $A(t)$ with the appropriate integral in an integral, it is equivalent to (33) up to a constant as the $\mu(t)$ in (33) is really the big scary integral we've been dealing with, $\int p(t)dt$. □

2.2 Separable Differential Equations

A general first-order differential equation can be written as $\frac{dy}{dx} = f(x, y)$ which can be rearranged to become $M(x, y) + N(x, y)\frac{dy}{dx} = 0$. When M is a function solely of x and N is a function solely of y, then we can rewrite the diffy q as

$$M(x)dx + N(y)dy = 0$$

which we call a **separable** equation. (A definition of a separable differential equation is a differential equation that can be written as the form above.)
To solve these equations, it's just 'simple'/'intuitive' integration. I'm not sure what exactly to write so.....
Let $A'(x) = a(x)$ and $B'(y) = b(y)$. Then,

$$a(x) + b(y)\frac{dy}{dx} = 0$$

simlifies down to

$$A(x) + B(y) = c$$

for an arbitrary constant c.

Exercise 1-4.
(Solutions only.)
1: $\frac{y^2}{2} = \frac{x^3}{3} + C$ or $y = \sqrt{\frac{2}{3}x^3 + C}$.
2: $y = \frac{1}{C - \cos x}$.
3: $\frac{1}{2}\tan(2y) = \frac{x}{2} + \frac{\sin(2x)}{4} + C$. Alternatively, the textbook gives the clever solution of $y = \frac{\pi}{2}$ (a constant) and similar solutions as $y' = 0$ and the RHS evaluates to 0. Very tricky (but nice) stuff.
4: $\ln|x| + C = \arcsin y$. Didn't fully get those inverse trig integrals right away :/.
(The rest (5-8) look trivial as you just move terms to either side of the equation and integrate.) $\square$

Exercise 17.
Literally just separate and integrate.

$$\Rightarrow \int 3y^2 - 6y\ dy = \int 1 + 3x^2\ dx + C \to y^3 - 3y^2 = x + x^3 + C$$

Plugging in our initial condition of $y(0) = 1$ (e.g. the values $x = 0$ and $y = 1$), we get that $C = 1 - 3 = -2$.
To determine the interval in which the solution is valid, we simply look at when the denominator of y' is 0 —$3y^2 - 6y = 0 \to y = 0, 2$ which correspond to the x values of 0, $\frac{-1+\sqrt{3}}{2}$, $\frac{-1-\sqrt{3}}{2}$, and $x = -1$ ($y(-1) = 2$ and y of any of the other x values is equal to 0).
I'm not sure how really to proceed from here but I think the way to actually do it is to recognize that $y(-1) = 2$, $y(1) = 0$, which given our initial condition $y(0) = 1$ means that our function y is trapped between -1 and 1 (y is not exactly a "function" at these specific points so our domain then becomes $(-1, 1)$).
It's not a great solution :/. $\square$

Exercise 19.

$$\frac{dy}{dx} = 2y^2 + xy^2 \to \int \frac{1}{y^2}\, dy = \int 2 + x\, dx \to y = \frac{1}{2x + \frac{x^2}{2} + C}.$$

Plugging our initial condition $y(0) = 1$ nets $C = -1$.
To find the minimum value of y, we simply find the derivative y' and set it to 0;

$$y' = \frac{1}{\left(2x + \frac{x^2}{2} - 1\right)} \cdot (2 + x).$$

Setting y' to 0, the only solution we get is $x = -2$ so the minimum value of our function y is $y(-2) = -\frac{1}{-4+2-1} = \boxed{\frac{1}{3}}$.
While it is true that $y(3) < y(-2)$, note that the function y is discontinuous and plotting the graph on Desmos, we see that $y(3)$ is not 'on' the particular branch of the solution we're focused on (namely, the piece of the function where $y(0) = 1$). □

Exercise 24.
Solve

$$\frac{dQ}{dt} = r(a + bQ),\ Q(0) = Q_0.$$

(Note: for some reason this exercise does not have a solution in the back of the book.)

We can rearrange to get the equation

$$\int \frac{1}{a + bQ} dQ = \int r dt \longrightarrow \frac{1}{b} \ln|a + bQ| = rt + C$$

and evidently, $C = \frac{\ln a + bQ_0}{b}$.
Solving for Q, we eventually find that

$$Q(t) = \frac{e^{rbt}(a + bQ_0) - a}{b}$$

so when $t \to \infty$, assuming that all constants are positive, $Q \to \infty$. □

Definition 2.2 (*Homogeneous Equation*)

(Note: This definition of a homogeneous diffy q will be different from other ones presented in this book.)
A homogeneous equation (for our purposes for now) is a first-order differential equation $\frac{dy}{dx} = f(x, y)$ that can be expressed as a function of the the expression $\frac{y}{x}$. In other words, $\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$ for some function g.

In particular, a differential equation is a homogeneous equation, then it is separable 'by a change of the dependent variable' by making the substitution $y = xv(x)$ (note that y is a function of x implicitly) and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. See the exercise below for more information.

Exercise 25.
25(a): Trivial. Multiply the numerator and denominator of the fraction by $\frac{1}{x}$ and simplify.

25(b):

$$y = xv \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Multiplying both sides by dx, our equation now becomes

$$dy = v\ dx + x\ dv,$$

which we can directly substitute in for dy.
As such, to conclude,

$$\frac{dy}{dx} = \frac{v\ dx + x\ dv}{dx} = \boxed{v + x\frac{dv}{dx}}.$$

25(c): Trivial. Literally just make the substitutions.

25(d + e): The key to this problem is partial fractions.

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v} \Rightarrow \ln|x| = \int \frac{1 - v}{v^2 - 4}\ dv + C$$

From here, using partial fractions, we find that the integrand is equal to $\dfrac{-0.25}{v-2} + \dfrac{-0.75}{v+2}$ which can be easily integrated to get our final answer.
As such, after integrating and substituting $v = \frac{y}{x}$ back into our equation, we get that

$$\ln|x| + C = -0.75\ln\left|\frac{y}{x} + 2\right| - 0.25\ln\left|\frac{y}{x} - 2\right|.$$

(I'm lowkey not sure how to get rid of those absolute value signs from the $\frac{1}{x}$ integral so they'll stay for now.) $\square$

2.3 Modeling with First-Order Differential Equations

To make a good mathematical model:

1. Construct the Model (in accordance with reality).
2. Analyze the model. Simplify functions/variables if needed, but always make sure the simplifications to the model make physical sense.
3. Compare the model with outside observations/experiments/data. Also take a look at long run behavior.

// Various examples of mathematical models are shown at this part in the textbook. //

Exercise 1.
The concentration of dye in the tank can be represented by $\frac{dA}{dt}$ where $A(t)$ is the concentration of dye in the tank as a function of time (t) in minutes. Specifically,

$$\frac{dA}{dt} = \text{rate in} - \text{rate out} = 0 - \frac{A}{200} \cdot 2$$

which is empirically explained as follows:
Since the water coming in is clean, rinse water, there is no dye in that water so the rate at which the dye is coming in the tank is 0. On the flip side, the rate at which the dye is going out is equal to the concentration of the dye ($\frac{\text{Amount}}{\text{Volume}} = \frac{A}{200L}$) times the amount of the water/dye mixture that flows out ($\frac{2L}{\text{min}}$).

Solving for $A(t)$, we find that

$$A(t) = e^{-\frac{t}{100}} \quad (C = 0).$$

As such, to find when the concentration of the dye reaches 1% of its original value, we simply set $A(t) = 0.01$ and solve, yielding $\boxed{t \approxeq 460.517}$(minutes) (thanks Wolfram Alpha). $\square$

Exercise 4.
4(a): More clearly, Torricelli's principle states that the outflow velocity v at the outlet is equal to the velocity of a particle in freefall from the height h **when it reaches the height of the outlet**.
From mechanics, the velocity of a particle in free fall is $v(t) = gt$ since $v_0 = 0$ in our case. Since we don't have t, we must resort to finding it by relating the height fallen (h) and time: $h = \frac{1}{2}gt^2$ so $t = \frac{2h}{g}$. As such, combining this equation with the previous, we find that $\boxed{v = \sqrt{2gh}}$ as desired.

4(b):

$$\begin{aligned}\frac{dh}{dt} &= \text{How much the height of the water drops with respect to (w.r.t) time}\\ &= \frac{\text{How much water goes out w.r.t time}}{\text{Area of the cross section of the tank at height h}} = \frac{-\alpha a v}{A(h)},\end{aligned}$$

with that last "jump" in reasoning deriving from the fact that how much water goes out of the tank is equal to the velocity of the water (v) times the area of the outflow hole (a). And apparently since water flow isn't ideal, there's a constant (α) tacked on to the ensure the equation matches with real world phenomena.

4(c): In this specific scenario, $A(h) = \pi r^2$, $r = 1$, $a = \pi(0.1)^2$, and $h(0) = 3$. Plugging in these values, we get that $\pi\frac{dh}{dt} = -\alpha\frac{\pi}{100}\sqrt{2gh}$ so $\frac{1}{\sqrt{h}}\frac{dh}{dt} = -\alpha\frac{\sqrt{2g}}{100}$. Integrating and plugging in our initial condition from here, we find that $h(t) = (\sqrt{(}3) - \frac{\alpha\sqrt{2g}}{200}t)^2$ which means that when $h(t) = 0$, $\boxed{t = \frac{200\sqrt{3}}{\alpha\sqrt{2g}} \approx 130.41}$ (seconds). $\square$

Exercise 9.
9(a): We are given that $\frac{dQ}{dt} = -rQ$ (so $Q(t) = Q_0e^{-rt}$) and that the halflife of carbon-14 is 5730 years. In other words, $Q(5730) = 0.5Q_0$ (half of the carbon-14 has decayed). As such, r can be solved for and is approximately $\boxed{0.000120968}$.

9(b): Done. See above when I did it.

9(c): In this scenario, $\frac{Q(t)}{Q_0} = 0.2$ so $e^{-rt} = 0.2$. With the r we solved for above, t can be solved for and it turns out $\boxed{t = 13304.7}$ (years). $\square$

Exercise 14ab.
14(a): If you're having trouble with this problem, that's ok, I did too. Essentially, use an integrating factor of e^{kt} and then work out the complicated mess that follows. Below is the abridged version of my work.

First, simplify and rearrange to the general form of a differential equation.

$$\frac{du}{dt} = -k(u - T(t)) \Longrightarrow \frac{du}{dt} + ku = kT(t)$$

This expression strongly suggests we use an integrating factor of e^{kt}, so let's do that!

$$\to \int \frac{d}{dt}\left[e^{kt}u(t)\right] = \int ke^{kt}T(t)\ dt$$

This integral would be terrible if $T(t)$ was weird but thankfully it's not; essentially, we can substitute the expression for $T(t)$ in and simplify the mess from there.

$$\begin{aligned}\to e^{kt}u(t) &= k\int e^{kt}T_0\ dt + kT_1\int e^{kt}\cos(\omega t)\ dt \\ &= e^{kt}T_0 + kT_1 \cdot R + C\end{aligned}$$

with R being that second integral. Let's now go simplify it! (For those who are following along at home, integrate by parts twice.)
Using integration by parts ($\int \boxdot\ d\bigstar = \boxdot\bigstar - \int \bigstar\ d\boxdot$) with $\boxdot = \cos(\omega t)$ and $\bigstar = e^{kt}$, our integral (R) can now be rewritten as

$$R = \frac{1}{k}e^{kt}\cos(\omega t) + \frac{\omega}{k}\int e^{kt}\sin(\omega t)\ dt.$$

From here, we do integration by parts again ($\boxdot = \sin(\omega t)$ and $\bigstar = e^{kt}$) and we get that

$$R = \frac{1}{k}e^{kt}\cos(\omega t) + \frac{\omega}{k}\cdot\frac{1}{k}e^{kt}\sin(\omega t) - \frac{\omega^2}{k^2}R$$

which can thus be simplified to obtain our final result,

$$R = e^{kt}\frac{\omega\sin(\omega t) + k\cos(\omega t)}{k^2+\omega^2}.$$

Returning to our original integral and problem (I told you this was messy), we thus reach our final answer for $u(t)$;

$$\boxed{u(t) = \frac{C}{e^{kt}} + T_0 + kT_1\left(\frac{\omega\sin(\omega t) + k\cos(\omega t)}{k^2+\omega^2}\right)}.$$

14(b): [Graphs not pictured] $\tau \approx 3.508$ and $R \approx 9.106$. It's interesting to note that the crossing points for $S(t)$ and $T(t)$ seem to happen exactly at the min/max points of $S(t)$ (which makes sense in the physical interpretation). $\square$

Exercise 14c.
14(c): Setting the two sides equal to each other, we have that

$$R\cos(\omega t - \omega\tau) = \frac{kT_1}{\omega^2 + k^2}(k\cos(\omega t) + \omega\sin(\omega t)).$$

Expanding that left part (and ignoring the fraction for now), we have

$$R'(\cos(\omega t)\cos(\omega\tau) + \sin(\omega t)\sin(\omega\tau)) = k\cos(\omega t) + \omega\sin(\omega t)$$

where R' is R up to a constant. Anyways, from here, we intuit that $\cos(\omega\tau)$ and $\sin(\omega\tau)$ should be constants that maintain the relative proportions of $\cos(\omega t)$ and $\sin(\omega t)$. As such, we write $\frac{w}{k} = \frac{\sin(\omega\tau)}{\cos(\omega\tau)}$ and as such find that $\tau = \frac{1}{\omega}\arcsin\left(\frac{\omega}{\sqrt{\omega^2 + k^2}}\right)$. From here, some direct simplification and term comparison leads us to figure out that $R' = \sqrt{\omega^2 + k^2}$.
Substituting this back into the original equation, we have

$$R\cos(\omega t - \omega\tau) = \frac{kT_1}{\omega^2 + k^2}R'\cos(\omega t - \omega\tau)$$

so $\boxed{R = \frac{kT_1}{\sqrt{\omega^2 + k^2}}}$. □

Exercise 19a.
19(a): The strategy to attain maximum height x_m is to first set up the differential equation, find $v(t)$, find $x(t)$, find t_m, then find x_m.
For the first part, to set up the differential equation, since the medium in our question offers a **resistance** of $k|v|$, this is equivalent to the force from the resistance always being $-kv$. As such, using the handy dandy equation $F = ma = m\frac{dv}{dt}$, we can setup our differential equation as

$$m\frac{dv}{dt} = -mg - kv.$$

Solving (and remembering the $+C$), we find that

$$v(t) = \frac{1}{k}\left((mg + kv_0)e^{-kt/m} - mg\right).$$

To find $x(t)$, integrate $v(t)$. This is made simpler by the fact that $v(t)$ consists mostly of constants. Integrating and **remembering** the $+C$ (I spent 20 minutes here since I forgor), you should find

$$x(t) = -\frac{mgt}{k} - \frac{m}{k^2}(mg + kv_0)e^{-kt/m} + (mg + kv_0)\frac{m}{k^2}.$$

To find t_m, set $v(t_m) = 0$. The physical explanation for this is that at the moment the body is at its maximum height, it has no velocity going upwards (and none going downwards). As such, find t_m by setting $v = 0$. Solving, you should find

$$t = \frac{m}{k}\ln\left(\frac{mg + kv_0}{mg}\right).$$

Putting it all together and solving the actual problem (the maximum height x_m the body reaches), we simply evaluate $x(t_m)$ to find

$$x_m = \boxed{-\frac{m^2 g}{k^2}\ln\left(\frac{mg + kv_0}{mg}\right) + \frac{mv_0}{k}}.$$

□

Exercise 19bc.
19(b): Use the taylor expansion of $\ln(1+x)$ and substitute x for $\frac{kv_0}{mg}$ in the above equations for t_m and x_m. Note that this substitution is only valid when $\frac{kv_0}{mg} < 1$ as otherwise the taylor series approximation for $\ln(1+x)$ might not converge.

19(c): The quantity mg represents a force (in Newtons). The quantity kv_0 also represents a force as resistance is a force (think of friction - friction is resistance and friction as a force). Thus the fraction $\frac{kv_0}{mg}$ is dimensionless. (There's lowkey not much to say here.) $\square$

Exercise 21.
21(a): When setting up our diffy q, we first take the downwards direction to be negative, take g to be the scalar value $9.8m/s^2$, and assume that $V_{displaced} = V_{sphere}$. From here, using similar logic to 19(a), we conclude that the resistive force $R = -6\pi\mu av$. As such, the ultimate differential equation we get is

$$m\frac{dv}{dt} = (w) + (R) + (B) = -mg - 6\pi\mu av + \rho' V_{sphere} g.$$

Note that at the limiting velocity v_L, $\frac{dv}{dt} = 0$. As such, to find v_L, we set $\frac{dv}{dt} = 0$ and solve for v in the above equation yielding $v_L = \boxed{(\rho' - \rho)\frac{2a^2 g}{9\mu}}$. Note that the difference in signage comes between my answer and the textbook comes down to the fact that in my answer I take g to be a positive scalar value rather than a negative value (for me, $g = +9.8$ not -9.8).

21(b): If the electron is held stationary, then $\frac{dv}{dt} = 0$ and $v = 0$. As such, we can directly simplify our differential equation (note the added term E pointing upwards which "holds" the electron in place):

$$m\frac{dv}{dt} = (w) + (R) + (B) + (E) = -mg - 6\pi\mu av + \rho' V_{sphere} g + Ee \Longrightarrow 0 = -\rho V_{sphere} g + \rho' V_{sphere} g + Ee$$

and as such $e = \boxed{\frac{(\rho - \rho')V_{sphere} g}{E}}$. $\square$

Exercise 24. Brachistochrone problem.
24(a): We isolate y':

$$(1 + y'^2)y = k^2 \to 1 + y'^2 = \frac{k^2}{y} \to y' = \sqrt{\frac{k^2}{y} - 1}.$$

We take the positive branch of this square root equation since generally our slope of the equation $\frac{\Delta y}{\Delta x} > 0$ and as such we expect $\frac{dy}{dx} > 0$ as well.
24(b): From more or less direct simplication:

$$y' = \sqrt{\frac{k^2}{y} - 1} \to \frac{dy}{dx} = \sqrt{\frac{k^2}{k^2 \sin^2 t} - \frac{\sin^2 t}{\sin^2 t}} \to \frac{(k^2 2 \sin t \cos t \ dt)}{dx} = \frac{\cos t}{\sin t} \to \boxed{2k^2 \sin^2 t \ dt} = dx.$$

24(c): The equation given for y is found by substituting $\sin^2 t$ for $\frac{1-\cos 2t}{2}$ in (39) when the relationship between y and t is introduced. The equation given for x is found by integrating what was found in 24(b) and is pretty direct if $\sin^2 t$ is substituted with the substitution given above.
24(d): We basically need to solve the system of equations $\begin{cases} k^2(\theta - \sin\theta) = 2 \\ k^2(1 - \cos\theta) = 4 \end{cases}$. Dividing both equations, we get that $2\theta - 2\sin\theta = 1 - \cos\theta$ which is not a very solvable equation so I used a calculator to find values for k and θ. Doing so, I find $k \approx \boxed{2.2}$. $\square$

2.4 Differences Between Linear and Nonlinear Differential Equationss

Definition 2.3 (***Existence and Uniqueness Theorem (for First-Order Linear Equations)***)

If p and q are continuous functions on an interval I: $\alpha < t < \beta$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\frac{dy}{dt} + p(t)y = q(t)$$

with initial condition $y(\gamma) = \delta$ with δ being an arbitrary intial value and $\gamma \in I$.

Note that this theorem asserts both the **existence** and **uniqueness** of a solution to a given first order linear differential equation.

Definition 2.4 (***Existence and Uniqueness Theorem for (for First-Order Nonlinear Equations)***)

Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t_0,\, y_0)$. Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ to the differential equation

$$\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0.$$

Note: The conclusion that a differential equation exists can be established based on the continuity of f alone. However, the given solution ϕ may or may not be unique.

(Example differential equations are then solved through examples to highlight example applications of the existence theorems in practice and show the necessity of the conditions.)

Note: For first order linear differential equations, its possible points of discontinuity/singularity can be found by examining the discontinuity of the coefficients p and g.

Note (page 56): General solutions may not exist for non-linear differential equations. That is, there may be functions ϕ and ψ that both satisfy the given differential equation yet not be of the same form as each other.

Exercise 1-4.
1. Rearranging, we get that the differential equation is

$$\frac{dy}{dt} + \frac{\ln t}{t-3}y = \frac{2t}{t-3}.$$

Clearly, from the $(t-3)$ in the denominator, our end function $y = \phi(t)$ will be discontinuous at $t = 3$. Similarly, since $\ln t$ is discontinuous for all $t \leq 0$[a], ϕ may be discontinuous at $t \leq 0$. As such, we know for sure that ϕ exists over the intervals $0 < t < 3$ and $3 < t < \infty$.
2. The function $\tan(t)$ is discontinuous at $\{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}$ so the solution function $y = \phi(t)$ is certain to exist over the intervals $n\pi + \frac{\pi}{2} < t < (n+1)\pi + \frac{\pi}{2}$ for all integer values of n.
3. After rewriting the differential equation, $(4 - t^2)$ appears in the denominator of p and g meaning ϕ is certain to exist over the intervals $(infty, -2)$, $(-2, 2)$, $(2, \infty)$.
4. $\cot(t)$ is discontinuous at every multiple of π and $\ln(t)$ (as mentioned before) is discontinuous for all $t \leq 0$. As such, ϕ is certain to exist over the intervals $n\pi < t < (n+1)\pi$ for all nonnegative values of n. $\square$

[a] $\ln(-1) = i\pi$

Exercise 9.
We seperate the variables and integrate to get $\frac{1}{2}y^2 = -2t^2 + C$. With initial condition $y(0) = y_0$, $C = \frac{1}{2}y_0^2$. As such, the general solution to our differential equation is

$$y = \pm\sqrt{y_0^2 - 4t^2}$$

with the $\pm$ sign indicating directionality of the answer. Notably, if $y_0 < 0$, then the negative sign of the square root is taken while if $y_0 > 0$, the positive sign of the square root is taken. Also, note that $|t|$ must be less than $|y|/2$ to make the equation work; if $|t| = |y|/2$, then $y(t_0) = 0$ for some t_0 yet then our original differential equation ($y' = -4t/y$) has no solution for y' when $y = 0$. $\square$

Exercise 10.
The solution to the given differential equation is $y = -\frac{1}{t^2 - \frac{1}{y_0}}$. Given the discontinuity in the denominator, t cannot be equal to $\frac{1}{\sqrt{y_0}}$ and as such the domain of t will be restricted to $-\frac{1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$ if $y_0 \geq 0$ (and t is unrestricted if $y_0 < 0$.) $\square$

Exercise 11.
The solution to the differential equation ends up being $t + \frac{1}{2y_0^2} = \frac{1}{2y^2}$ or $y = \pm\frac{1}{\sqrt{2(t + 1/2y_0^2)}}$. Like in exercise 9, the positive branch of the square root is taken if $y_0 > 0$ and the negative branch taken if $y_0 < 0$. Otherwise, the domain of t is restricted to $t > -\frac{1}{2y_0^2}$. $\square$

Exercise 18.
(a) Verified.
(b) $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-t + \sqrt{t^2 + 4y}}{2}\right) = \frac{1}{\sqrt{t^2 + 4y}}$ is not continuous with the initial condition $y(2) = -1$. As such, while a solution to the differential equation ϕ can be guaranteed, the solution ϕ may or may not be unique as not all requirements of Theorem 2.4.2 are satisfied.
(c) Given that we know the second solution $y_2(t)$, it's pretty easy to check no constant c plugged into the equation $y = ct + c^2$ can yield a t^2 term like in $y_2(t)$. Otherwise, fixing the initial condition $y(2) = -1$, plugging $t = 2$ and $y = -1$ **only** yields the solution $c = -1$. $\square$

Exercise 21.
By direct simplification,

$$\frac{dy}{dt} + p(t)y \longrightarrow \frac{d}{dt}[y_1(t) + y_2(t)] + p(t)(y_1(t) + y_2(t)) = y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = g(t).$$

$\square$

Discontinuous coefficients problems look daunting but really you're just solving more or less the same differential equation twice then gluing the pieces together.

Exercise 26.
Here, we solve $y' + 2y = g(t)$ by multiplying both sides by the integrating factor of e^{2t}. We then evaluate both cases ($t > 1$, $0 \le t \le 1$) separately, and glue together both functions when $t = 1$.
With the case of when $g(t) = 1$ ($0 \le t \le 1$),

$$y' + 2y = 1 \to e^{2t}\frac{dy}{dt} + 2e^{2t}y = e^{2t} \longrightarrow \int \frac{d}{dt}\left[e^{2t}y\right] = \int e^{2t}\ dt \Longrightarrow y_\alpha(t) = \frac{e^{2t} + C_\alpha}{2e^{2t}}.$$

Since we have the initial condition of $y(0) = 0$, $C_\alpha = -1$ and $y_\alpha(t) = \dfrac{e^{2t} - 1}{2e^{2t}}$.
The case when $g(t) = 0$ is simple enough to not warrant covering and the final equation obtained is $y_\beta(t) = \dfrac{C_\beta}{e^{2t}}$ ($t > 1$). To make the overall function $y(t)$ continuous ($y = y_\alpha \cup y_\beta$), we set $y_\alpha(1) = y_\beta(1)$.
Namely, $y_\alpha(1) = \dfrac{e^2 - 1}{2e^2}$ and $y_\beta(1) = \dfrac{C_\beta}{e^2}$ so $C_\beta = \dfrac{e^2 - 1}{2}$.
Overall then,

$$y(t) = \begin{cases} \frac{e^{2t}-1}{2e^{2t}} & 0 \le t \le 1 \\ \frac{e^2-1}{2e^{2t}} & t > 1 \end{cases} .$$

□

Exercise 27.
Setting up the differential equation to be separable, we obtain

$$\frac{1}{p(t)y}\ dy = -dt.$$

For the case when $p(t) = 2$, y_α is correspondingly

$$y_\alpha(t) = e^{-2t}$$

after plugging in initial condition $y(0) = y_\alpha(0) = 1$.

Similarly, when $p(t) = 1$, $y_\beta = e^{-t+C_\beta}$.
As such, 'gluing' the functions together when $t = 1$, we find that $y_\alpha(t) = e^{-2}$ and $y_\beta(t) = e^{-1+C_\beta}$ so $C_\beta = -1$ and the equation is solved. □

2.5 Autonomous Differential Equations

This section mostly examines the logistic equation modelling population growth in the context of differential equations and some goes somewhat in-depth about it.

Definition 2.5 (*Autonomous Differential Equation*)

A differential equation is **autonomous** if the independent variable does not appear explicitly. They have the form

$$\frac{dy}{dt} = f(y).$$

Note that the zeros of $f(y)$ $\{z \mid f(z) = 0\}$ are called **critical points**.

(Note: There's some definitions of **asymptotically stable** points and other terms that aren't noted down here (I don't care about those terms).)

Definition 2.6 (*Logistic (Verhulst) Equation*)

The equation of the form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

which is commonly used to represent population growth.
Notably, K is the **carrying capacity** for the population and r is called the **intrinsic growth rate**.

(65) - Logistic Growth with a Threshold:
The equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

represents a logistic growth function with a threshold (where $0 < T < K$) —graphing the solution to this function reveals that all initial values $y_0 > T$ gravitate towards K like in logistic growth yet all initial values $y_0 < T$ eventually fade to 0.

Exercise 5. Semistable Equilibrium Solutions
5a: Trivial since if $y \neq 1$, then $(1-y)^2$ will be non-zero by the trivial inequality.

5c:

$$\frac{dy}{dt} = k(1-y)^2 \to \frac{1}{(1-y)^2}\,dy = k\,dt \to \frac{1}{1-y} = kt + C.$$

With initial condition $y(0) = y_0$, $C = \frac{1}{1-y_0}$ so $y(t) = \boxed{1 - \frac{1-y_0}{kt - kty_0 + 1}}$. □

Exercise 13.
To find the inflection points of $y(t)$, we can find the min/maxes of $f(y)$ since $y'(t) = f(y)$ (kinda). As such, we find those minimum/maximum points by setting $\frac{df}{dy} = 0$ then solving.
Starting from (17), we apply a 3-part product rule to get

$$f'(y) = -r\left[\left(1 - \frac{y}{T}\right)\left(-\frac{1}{K}\right)y + \left(-\frac{1}{T}\right)\left(1 - \frac{y}{K}\right)y - \left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)\right].$$

The inside function is merely a quadratic (a very messy one) $\left[y^2\left(\frac{3}{KT}\right) - y\left(\frac{2}{K} + \frac{2}{T}\right) + 1\right]$, which can be bashed using the quadratic formula to get the desired solutions. □

Exercise 17a.
17(a): Separating, we get

$$\frac{1}{\ln K - \ln y}\frac{1}{y}\ dy = r\ dt.$$

Integrating, we have

$$rt + C = \int \frac{1}{y}\ dy\ \frac{1}{\ln K - \ln y}$$

from which we can make the substitution $\ln y = u$ (and $\frac{1}{y}\ dy = du$) so the RHS becomes $\displaystyle\int \frac{1}{\ln k - n}\ dn$. From here, integrating gets us

$$rt + C = -\ln\left|\ln\left(\frac{K}{y}\right)\right|$$

and a multitude of substitutions leads us to find $C = \ln\frac{y_0}{K}$ so

$$y(t) = K\exp\left(\ln\left(\frac{y_0}{K}\right)e^{-rt}\right).$$

□

Exercise 19.
19(a): Assuming $E < r$, to solve we set the RHS equal to 0:

$$r\left(1 - \frac{Y}{K}\right)y - Ey = 0 \to y\left(r - \left(1 - \frac{y}{K}\right) - E\right) = 0$$

meaning either $y_1 = 0$ or the inside function is 0. Simplifying that inside function reduces to the second solution, $y_2 = K\left(1 - \frac{E}{r}\right)$.

19(b): The inside function is a parabola pointed downwards. As such, function values for y values right below the first solution $y_1 = 0$ are negative and function values for y values right above 0 are positive. This translates to values drifting away from $y = 0$ as negative values become even more negative and positive values continue to get more positive making y_1 an unstable equilibrium.
On the other hand, values slightly above y_2 have a negative $\frac{dy}{dt}$ and values slightly below y_2 have a positive $\frac{dy}{dt}$ so overall those values will drift towards y_2. Thus, y_2 is an asymptotically stable solution.

19(c): Since $Y = E \cdot y_2$, $Y = EK\left(1 - \frac{E}{r}\right)$.
19(d): The maximum of a parabola lies at its vertex $-\frac{b}{2a}$. In this case, $Y(E) = -E^2\frac{K}{r} + EK$ is maximized when $E = \frac{r}{2}$ and correspondingly $Y_m = \frac{KR}{4}$. □

Exercise 27.
27(a): The limiting value of $x(t)$ as $t \to \infty$ is $x = \min\{p,\ q\}$. Slightly above this value $\frac{dx}{dt}$ is negative (meaning $x(t)$ decreases) and slightly below this value $\frac{dx}{dt}$ is positive.
We can solve the differential equation using partial fractions:

$$\frac{dx}{dt} = \alpha(p - x)(q - x) \to \int \frac{1}{(p - x)(q - x)}\ dx = \alpha t + C \to \frac{1}{q - p}\int \frac{1}{p - x} - \frac{1}{q - x}\ dx = \alpha t + C$$

so $\displaystyle\frac{1}{q - p}\left(-\ln|p - x| + \ln|q - x|\right) = \alpha t + C$ with $\displaystyle C = \frac{\ln(q/p)}{q - p}$. (The simplification afterwards for an explicit form of $x(t)$ gets really messy.)

27(b): As seen in problem 5, this is a equation with a semistable equilibrium solution $x(t) = p$ which happens when $t \to \infty$ with initial condition $x(0) = 0$. A simple integration shows $\int \frac{1}{(p-x)^2}\ dx = \frac{1}{p-x} + C$ and thus $x(t) = p(1 - \frac{1}{p\alpha t + 1})$. □

2.6 Exact Differential Equations and Integrating Factors

Definition 2.7 (*Exact Differential Equation*)

If we can identify a function $\psi(x,y)$ such that $\frac{\partial\psi}{\partial x} = M(x,y)$ and $\frac{\partial\psi}{\partial y} = N(x,y)$ for a given differential equation $M(x,y) + N(x,y)y' = 0$, then that differential equation is **exact** and the solutions are implicitly given by $\psi(x,y) = c$ for arbitrary c.

While this itself is daunting, Clairut's Theorem simplifies this tremendously for us as under the assumption that M, N, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are continuous in some closed region, Clairut's theorem tells us

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

We can then substitute our function ψ for the generic function f and simplify:

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) \longrightarrow \frac{\partial}{\partial x}(N) = \frac{\partial}{\partial y}(M)$$

concluding that the function ψ only exists if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$.

Now with this simple test at our side, if a function ψ exists for an exact differential equation, since $M = \frac{\partial\psi}{\partial x}$, we can simply integrate M with respect to x with the constant term being replaced by an arbitrary function $C(y)$. To then solve for C, take the derivative of the integrated thing and compare terms.

Returning to the concept of integrating factors from Section 2.1, another technique to solving general differential equations is to multiply both sides by a specific integrating factor such that the resulting differential equation is exact.

In symbols, we find a function μ as an integrating factor

$$M(x,y) + N(x,y)y' = 0 \longrightarrow (\mu(x,y)\cdot M) + (\mu(x,y)\cdot N)y' = 0$$

so $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x}$ to make this new differential equation an **exact differential equation**.

Unfortunately, this is super duper difficult (there is no good way to find μ) in general. As such, we have to gimmick a little and hope that μ is a function of one variable which simplifies things considerably.

Assuming $\mu = \mu(y)$, we can use the product rule on both sides of the above equation to get

$$\mu(y)\frac{\partial M}{\partial y} + M\frac{d\mu(y)}{dy} = \mu(y)\frac{\partial N}{\partial x} + N\frac{\partial\mu(y)}{\partial x} = \mu(y)\frac{\partial N}{\partial x}$$

which can we rearrange and find $\dfrac{d\mu}{dy} = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\mu$ from which μ can be integrated and solved for.[2]

Exercise 1-8.

1. This differential equation is exact; $\psi(x,y) = x^2 + 3x + y^2 - 2y (= c)$.
2. Not exact.
3. Exact; $\psi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y$.
4. Exact (rearrange to get $(ax+by) + (bx+cy)y' = 0$); $\psi(x,y) = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2$.
5. Not exact.
6. Exact; $\psi(x,y) = e^{xy}\cos(2x) + x^2 - 3y$.
7. Exact; $\psi(x,y) = y\ln x + 3x^2 - 2y$.
8. Exact; $\psi(x,y) = -\frac{1}{\sqrt{x^2+y^2}}$ (via u-substitution). □

[2] A similar equation arises if you assume μ is a function of x.

Exercise 14.
A differential equation is exact if $M_y = N_x$. Since $M = M(x)$, $\frac{\partial M(x)}{\partial y} = 0$ (it's basically a constant) and similarly $\frac{\partial N(y)}{\partial x} = 0$ so the separable equation is exact. $\square$

Exercise 17.
From the derivation I did previously, $\dfrac{d\mu}{dy} = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\mu = \dfrac{N_x - M_y}{M}\mu$. Thus, we have $\mu' = Q(y)\mu$ so $\mu(y) = e^{\int Q(y)\ dy}$ as given. $\square$

Exercise 18-21.
18. Assuming $\mu = \mu(x)$, we find the integrating factor to be $\mu(x) = e^{3x}$. Integrating, we find $\psi(x, y) = x^2ye^{3x} + \frac{1}{3}y^3e^{3x}$.
19. The integrating factor is e^{-x} and $\psi(x, y) = ye^{-x} - e^x - e^{-x}$.
20. The integrating factor is y and $\psi(x, y) = xy + y\cos y - \sin y$.
21. The integrating factor is $\mu(y) = \frac{e^{2y}}{y}$ and integration reveals $\psi(x, y) = xe^{2y} - \ln|y|$. $\square$

2.7 Euler's Method

Euler's method, famously shoved down students' throats in Calculus BC, is just basic (first-order?) numerical approximation of a given function.
Essentially, the problem statement can be summed up by finding the value of the function y at a certain point t_1 given differential equation $\dfrac{dy}{dt} = f(t, y)$ and initial point (t_0, y_0).
Mathematically, since $y(t_1) = \displaystyle\int_{t_0}^{t_1} f(t, y)\ dt + y_0 = \sum_{t_0}^{t_1} f(t, y) \cdot dt + y_0$ (assuming y is continuous), we can find $y(t_1)$ by approximating the summation with a non-infinitesimal dt ($dt \to \Delta t$).
To Euler it up, determine how many steps n you want to take. Next, to approximate $y(t_{\text{next}})$, take a 'step' of size $\dfrac{t_1 - t_0}{n}$ and estimate $y(t_{\text{next}})$ as $y(t_{\text{now}}) + \frac{dy}{dt}\Delta t = y(t_{\text{now}}) + f(t_{\text{now}}, y)(t_{\text{next}} - t_{\text{now}})$ with $t_{\text{next}} = t_{\text{now}} + \frac{t_1 - t_0}{n}$. Iterate this until $t_{\text{next}} = t_1$, the desired endpoint.
So that's basic numerical interpolation for you. Note that the step size need not be constant though usually it is.
Most exercises below are computation-related and they're frankly boring which is why almost none are done.

Exercise 15. Convergence of Euler's Method.
15(a). Assuming $\mu = \mu(t)$ leads us to find $\mu(t) = e^{-t}$ and a semi-messy integration reveals $\psi(t, y) = e^{-t}(y - t) = c$. Rearranging for y, we find $y = t + ce^t$ and plugging in point (t_0, y_0) means $y_0 = t_0 + ce^{t_0}$ or $c = \frac{y_0 - t_0}{e^{t_0}}$ which matches the solution given in the problem.

15(b). $y_k = y_{k-1} + \dfrac{dy}{dt}h = y_{k-1} + (1 - t_{k-1} + y_{k-1})h = y_{k-1}(1 + h) + h - t_{k-1}h$.

15(c). $y_2 = (1+h)y_1 + h - ht_1 = (1+h)y_1 + t_2 - t_1 - ht_1 = (1+h)(y_1 - t_1) + t_2 \to (1+h)((1+h)(y_0 - t_0) + t_1 - t_1) + t_2$ so $y_2 = (1 + h)^2(y_0 - t_0) + t_2$. As such, by engineer's induction $y_n = (1 + h)^n(y_0 - t_0) + t_n$.

15(d): I'm not sure what conditions this problem is throwing at me but essentially:

$$y_n = (1 + h)^n(y_0 - t_0) + t_n \to (1 + (t - t_0)/n)^n(y_0 - t_0) + t_n \implies y_n \approxeq e^{t - t_0}(y_0 - t_0) + t = \phi(t)$$

as $n \to \infty$. $\square$

2.8 The Existence and Uniqueness Theorem

This long section is just kind of a proof of the existence and uniqueness of solutions of a continuous differential equation.
The proof uses a method called "method of successive approximations" which solves a differential equation by considering the solution $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ where

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\ ds$$

(f is the thing you get when you rewrite a differential equation as $\frac{dy}{dt} = f(t, y)$.)

Essentially this whole section is just dedicated to the proof of an elementary yet powerful (but also cumbersome) theorem.

Exercise 1.
$\dfrac{dy}{dt} = (t+1)^2 + (y+2)^2$. □

Exercise 5a, 6a.
5(a): A pattern clearly emerges calculating the first values of $\phi_n(t)$:

$$\phi_0(t) = t,\ \phi_1(t) = t,\ \phi_2(t) = \int_0^t t^2 + 1 = \frac{t^3}{3} + t, \phi_3(t) = \int_0^t s\left(\frac{s^3}{2} + s\right) + 1\ ds = \frac{t^5}{3\cdot 5} + \frac{t^3}{3} + t,$$

$$\phi_4(t) = \int_0^t s\left(\frac{s^5}{3\cdot 5} + \frac{s^3}{3} + s\right) + 1 = \int_0^t \frac{s^6}{3\cdot 5} + \frac{s^4}{3} + s^2 + 1 = \frac{t^7}{3\cdot 5\cdot 7} + \frac{t^5}{3\cdot 5} + \frac{t^3}{3} + t$$

from which we can draw the conclusion that $\phi_n(t) = \dfrac{t^{2n-1}}{(1)(3)\cdots(2n-1)} + \phi_{n-1}(t)$ so $\phi_n(t) = \displaystyle\sum_{i=1}^{n} \frac{t^{2i-1}}{(2i-1)!!}$.

6(a): Following the same process of value bashing:

$$\phi_0(t) = t,\ \phi_1(t) = -\frac{t^2}{2}, \phi_2(t) = \int_0^t -\frac{s^4}{2} - s\ ds = -\frac{t^5}{5\cdot 2} - \frac{t^2}{2},$$

$$\phi_3(t) = \int_0^t -\frac{s^7}{5\cdot 2} - \frac{s^4}{2} - s\ ds = -\frac{t^8}{8\cdot 5\cdot 2} - \frac{t^5}{5\cdot 2} - \frac{t^2}{2}$$

so if we do engineer's induction we arrive at the conclusion that $\phi_n(t) = -\displaystyle\sum_{i=1}^{n} \frac{t^{3i-1}}{(2)(5)\dots(3i-4)(3i-1)}$. □

Exercise 12.
12(a): We start by defining functions $f_n(x)$ and $g_n(x)$ such that $f_n(x) = 0$ and $g_n(x) = \dfrac{2nx}{1 + nx^2 + \frac{n^2x^4}{2}}$.
Next, we note that $f_n(x) \le \phi_n(x) \le g_n(x)$ over the interval $0 \le x \le 1$ when $n > 3$. The former inequality $f_n(x) \le \phi_n(x)$ is trivial to prove, and the latter inequality $\phi_n(x) \le g_n(x)$ is derived from the fact that

$$\phi_n(x) = \frac{2nx}{e^{nx^2}} = \frac{2nx}{1 + nx^2 + \frac{n^2x^4}{2} + \frac{n^3x^6}{6} + \frac{n^4x^8}{24} + \cdots} < \frac{2nx}{1 + nx^2 + \frac{n^2x^4}{2}} = g_n(x).$$

Evaluating the limits of each function, trivially $\lim_{n\to\infty} f_n(x) = 0$. For $g_n(x)$,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{2}{x + \frac{2+n^2x^4}{2nx}} \to \lim_{n\to\infty} \frac{2}{x + \frac{n^2x^4}{2nx}} = \lim_{n\to\infty} \frac{2}{x + nx^3/2} = 0$$

for fixed $x \neq 0$ ($g_n(0) = 0$ for all n). As such, by the squeeze theorem, since $f_n(x) \le \phi_n(x) \le g_n(x)$ and $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = 0$, $\boxed{\lim_{n\to\infty} \phi_n(x) = 0}$.

12(b): Recognizing that $[-nx^2]' = -2nx$,

$$\int_0^1 2nxe^{-nx^2}\, dx = \int_1^0 -2nxe^{-nx^2}\, dx = [e^{-nx^2}]_1^0 = e^0 - e^{-n} = 1 - e^{-n}.$$

□

Exercise 13.
13(a):

$$\phi(t) = \int_0^t (2s)(1+\phi(s))\, ds = \int_0^t (2s)\left(1 + s^2 + \frac{s^4}{2!} + \cdots + \frac{s^{2k}}{k!} + \cdots\right)\, ds$$

$$\int_0^t 2s + 2s^3 + \frac{2s^5}{2} + \cdots + \frac{2s^{2k+1}}{k!} + \cdots = t^2 + \frac{t^4}{2} + \frac{t^6}{2\cdot 3} + \cdots + \frac{t^{2k+2}}{(k+1)!} + \cdots$$

with that final expression being exactly $\phi(t)$.

13(b): $\phi(0) = 0$.
13(c): $\phi(t) = \sum_{k=0}^{\infty} \frac{(t^2)^k}{k!} - 1 = e^{t^2} - 1$.
13(d)/(e): (This is problem (8) by the way)

$$y' = 2t(1+y) \to 2t\, dt = \frac{1}{1+y}\, dy \to t^2 + C = \ln|y+1| \to y = e^{t^2+C} - 1$$

with the constant being $C = 0$ found by plugging in the initial condition. □

2.9 First-Order Difference Equations

Definition 2.8 (***first-order difference equation***)

An equation that takes on discrete values and is of the form $y_{n+1} = f(n, y_n)$.
Further classifications can be derived from whether or not f is linear (or non-linear) and whether or not an initial condition is provided.
A solution to the first-order difference equation is a set of values $\{y_n\}$ that satisfy the relation held above.

Assuming that $y_{n+1} = f(y_n)$ (recurrences!), we can find **equilibrium solutions** by solving $y_n = f(y_n)$.
Page (93) analyzes a more complicated model of discrete population growth similar to that of a discretized lienar differential equation.

Exercise 1-4.
1. Trivial; $y_n = -0.9^n y_0$ so the limit of y_n is 0 as $|y_{n+1}| < |y_n|$.
2. In general, y_n does not matter in the sense that since the fraction $\sqrt{\dfrac{n+3}{n+1}}$ is nearly 1, y_n does not undergo any drastic changes. In fact, $y_n = \sqrt{\dfrac{(n+2)(n+1)}{n(n-1)}}y_0$ as most square roots end up cancelling each other. Thus, asymptotically, $y_{n+1} \equiv y_n$.
3. Flip flop.
4. An equilibrium solution to this equation is $y_n = 12$. Since this equation is also linear and well-behaved, it's to be expected that any solution to this recurrence given an initial condition will converge to $y_n = 12$. □

2.10 Miscallaneous Problems

This section is dedicated to solving the end of chapter problems given on page 100 in the textbook. I'm tired of word problems so I'll just be solving the 24 miscallaneous differential equations given.

Exercise 1.
I trolled on this problem for a long time :/.
Rewriting our differential equation, we have $\frac{2}{x}y + \frac{dy}{dx} = x^2$. Multiplying by a generic integrating factor μ, our equation becomes $\frac{2}{x}\mu y + \mu\frac{dy}{dx} = x^2\mu$. Given that we want the left hand side of our equation to look something like $\frac{d}{dx}[\mu y] = \mu' y + \frac{dy}{dx}\mu$, we immediately have $\mu' = \frac{2}{x}\mu \to \mu(x) = x^2$.
As such, we can plug that back into our equation and reformat our differential equation as

$$2xy + \frac{dy}{dx}x^2 = x^4 \to \frac{d}{dx}[x^2 y] = x^4$$

and integrate both sides to get $\frac{x^5}{5} + C = x^2 y \to \boxed{y(x) = \frac{x^3}{5} + \frac{C}{x^2}}$. □

Exercise 2.

$$\frac{dy}{dx} = \frac{1+\cos x}{2-\sin y} \to 1 + \cos x \; dx = 2 - \sin y \; dy \to \boxed{x + \sin x + C = 2y + \cos y}.$$

□

Exercise 3.
We rewrite our diffy q as $(-2x-y)+(3+3y^2-x)\dfrac{dy}{dx}=0$. Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}=-1$, our differential equation is **exact** and we can simply integrate $M=-2x-y$ with respect to y and solve for the constant function to get $\psi(x,y)=-x^2-xy+3y+y^3$. With our initial condition $(0,0)$, our differential equation is implicitly solved by $-x^2-xy+3y+y^3=0$. □

Exercise 4.

$$\frac{dy}{dx}=3-6x+y-2xy\longrightarrow\frac{dy}{dx}=(3+y)(1-2x)\Rightarrow y=Ce^{x-x^2}-3.$$

□

Exercise 5.
$\rightarrow$ $(2xy+y^2+1)+(x^2+2xy)\frac{dy}{dx}$. $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}=2x+2y$ so our differential equation is exact. As such, $\psi(x,y)=\displaystyle\int x^2+2xy\;dy=\int 2xy+y^2+1\;dx=x^2y+xy^2+x\;(=C)$. □

Exercise 6.
Since there's a $-y$ on the right hand side, we have to rearrange the left hand side to account for it:

$$x\frac{dy}{dx}+xy=1-y\rightarrow\frac{dy}{dx}+\frac{x+1}{x}y=\frac{1}{x}$$

As such, we can recognize that an integrating factor of $\mu=xe^x$ is needed $\left(\mu'=\frac{x+1}{x}\mu\right)$ and as such we can integrate and get $e^x+C=xe^xy$. With initial condition, our solution thus becomes $\boxed{y=\dfrac{1}{x}-\dfrac{e}{xe^x}}$. □

Exercise 7.

$$x\frac{dy}{dx}+2y=\frac{\sin x}{x}\rightarrow x^2\frac{dy}{dx}+2xy=\sin x\rightarrow\frac{d}{dx}[x^2y]=\sin x\rightarrow y=\frac{C-\cos x}{x^2}\Longrightarrow\boxed{y=\frac{4+\cos 2-\cos x}{x^2}}.$$

□

Exercise 8.
This differential equation is exact.

$$(2xy+1)+(x^2+2y)y'=0\rightarrow\psi(x,y)=x^2y+x+y^2.$$

□

Exercise 9.
Although this differential equation looks like it can be exact, it's actually separable.

$$(x^2y + xy - y) + (x^2y - 2x^2)\frac{dy}{dx} = 0 \to (y)(x^2 + x - 1) + (x^2)(y - 2)\frac{dy}{dx} = 0$$
$$\to -\frac{y(x^2 + x - 1)}{x^2(y - 2)} = \frac{dy}{dx} \to \frac{y - 2}{y}\ dy = \frac{1 - x - x^2}{x^2}\ dx.$$

As such (break apart the fractions!), $\boxed{y - 2\ln y = -x - \ln x - \frac{1}{x} + C}$. $\square$

Exercise 10.
This time, the differential equation is exact. This makes our work super easy as we can just recognize $\psi(x, y) = \frac{x^3}{3} + xy + e^y$ which captures all solutions when paired with a level curve. $\square$

Exercise 11.
This differential equation is also exact. $\psi(x, y) = \frac{x^2}{2} + xy + y^2$ so with point $(2, 3)$, the solution is $\frac{x^2}{2} + xy + y^2 = 17$. $\square$

Exercise 12. We can rewrite our differential equation to be separable and seperate to get

$$\to \ln y = C + \int \frac{1 - e^x}{1 + e^x}.$$

Since this kind of looks u-subbable, we let $u = e^x + 1$; correspondingly, $du = e^x\ dx$ and $1 - e^x = 2 - u$. As such, our integral then becomes

$$= C + \int \frac{2 - u}{u(u - 1)}\ du = C + \int \frac{1}{u - 1} - \frac{2}{u}\ du.$$

As such, now integratable, we integrate and simplify to get $\boxed{y = Ce^x \cdot (e^x + 1)^{-2}}$. $\square$

Exercise 13.
This big scary equation is actually an exact differential equation. Solving, we get $\psi(x, y) = e^{-x}\cos y + e^{2y}\sin x$. $\square$

Exercise 14.

$$-3y + \frac{dy}{dx} = e^{2x} \to e^{-x} = -3e^{-3x}y + e^{-3x}\frac{dy}{dx} \to -e^{-x} + C = e^{-3x}y.$$

$\square$

Exercise 15. Using integrating factor e^{2x}, we get

$$e^{2x}y = \int e^{-x^2}\ dx + C$$

so $y = e^{-2x}\frac{\sqrt{\pi}}{2}\text{erf}(x) + 3e^{-2x}$. □

Exercise 16.
Again, another big scary fraction that turns super tame when we rearrange it and find it's an exact differential equation. $\psi(x, y) = xy^3 + 2xy - x^3$ is the parent solution. □

Exercise 17.

$$y' = e^{x+y} \to e^{-y}\ dy = e^x\ dx \to -y = \ln(-e^x) \to y = -x(1 + i\pi).$$

□

Exercise 18.
$\psi(x, y) = 2xy^2 + 3x^2y - 4x + y^3$. □

Exercise 19.
Note: This problem is functionally equivalent to problem 6. The integrating factor is the same (xe^x) so the problem is basically identical.

$$t\frac{dy}{dt} + (t + 1)y = e^{2t} \to e^{3t} = te^t\frac{dy}{dt} + (t + 1)e^t y \to \frac{e^{3t}}{3} + C = te^t y \to y = \frac{e^{2t}}{3t} + \frac{C}{te^t}.$$

□

Exercise 20.
This problem is very tricky and I gave up on the problem (although I was close in trying to make a substitution). Essentially, to clear things up, we set $y = nx$ and correspondingly find $\frac{dy}{dx} = n + \frac{dn}{dx}x$. We then substitute this new $\frac{dy}{dx}$ back into our differential equation and solve.
Eventually, we find $-e^{-n} = \ln x + C$ so $e^{-y/x} + \ln x = C$ is our solution. □

Exercise 21.
This problem is weird. Also, I wouldn't take the hint that the textbook gives.

Motivated by how bad the differential equation is when we first write it out $\left(\dfrac{dy}{dx} = \dfrac{x}{y} \cdot \dfrac{1}{x^2+y^2}\right)$, we are somewhat motivated in making the substitution $v = x^2 + y^2$ to clean things up. Indeed, we can rewrite the whole differential equation in terms of v and x:

$$v = x^2 + y^2 \to \frac{dv}{dx} = 2x + 2y\frac{dy}{dx} \to \frac{dy}{dx} = \frac{\frac{dv}{dx} - 2x}{2y}$$
$$\implies \frac{\frac{dv}{dx} - 2x}{2y} = \frac{x}{y} \cdot \frac{1}{v} \Rightarrow \frac{dv}{dx} = 2x\left(\frac{v+1}{v}\right)$$

from which the equation is separable and then easily integratable.

Lesson of the day then is to use clever substitutions to turn a non-linear differential equation (with term $y^3\frac{dy}{dx}$) into a linear, solvable differential equation. $\square$

Exercise 22.
Use the substitution $y = nx$ $\left(\dfrac{dy}{dx} = n + \dfrac{dn}{dx}\right)$ and rearrange:

$$\to n + \frac{dn}{dx}x = \frac{(n+1)x}{(1-n)x} \to \frac{dn}{dx} = \frac{1+n^2}{1-n} \cdot \frac{1}{x}$$

which is clearly solvable and might be integratable and might result in a solution that isn't deranged and unwritable. $\square$

Exercise 23.
I'm not even sure how this still works but once again the substitution $y = nx$ works (This time I figured it out by myself!!).
$y = nx$ so $\dfrac{dy}{dx} = n + x\dfrac{dn}{dx}$. Replacing this in our differential equation, we get:

$$3n^2x^2 + 2nx^2 - \left(n + x\frac{dn}{dx}\right)(2nx^2 + x^2) = 0 \to 3n^2 + 2n - \left[2n^2 + n + (2n+1)x\frac{dn}{dx}\right] = 0$$
$$\to n^2 + n = (2n+1)x\frac{dn}{dx} \to \frac{1}{x}\ dx = \frac{1}{n} + \frac{1}{n+1}\ dn$$

from which the equation is easily solvable :).
Lesson learned: Always substitute $y = nx$ if things look funky. $\square$

Exercise 24.
Not even going to attempt this one. I've wasted too much time on this stupid problem.
The solution is to divide the equation by y^2 and recognize that you can find an integrating factor μ for the exact differential equation by trying to find $\frac{d\mu}{dx}$. Good luck o7. $\square$

3 Second-Order Linear Differential Equations

Or differential equations of the form

$$y'' + p(t)y' + q(t)y = g(t).$$

3.1 Homogenous Second-Order Equations

Remember, a **linear** second-order differential equation is of the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Nonlinear differential equations are super hard and annoying to tackle and as such they're just not tackled in this book :/.
In second-order differential equations, a problem with an initial condition has initial condition of the form

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}.$$

Note that there are two initial equations given - the location of y at time t_0, and the slope of y at time t_0.

Definition 3.1 (*Homogenous*)

A **homogenous** differential equation has no 'constant' terms (terms without y). In the case for our second-order linear differential equations, a homogenous equation of that form can be written as

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

Anyways, it turns out if we solve the homogenous version of the differential equation $P(t)y''+Q(t)y'+R(t)y = G(t)$, we can actually find an expression for y (that may or may not have an integral in it). That's pretty cool.

For this chapter (unfortunately), we will only consider the cases when $\boxed{P,\ Q,\ \text{and } R \text{ are } \textbf{constants}}$.

Thus, our differential equation becomes $ay'' + by' + cy = 0$. Letting $y = e^{rt}$, we find that our equation now becomes

$$\to ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0$$

with $ar^2 + br + c$ called the **characteristic equation** for the general differential equation with constant coefficients shown above.

If we let r_1 and r_2 be two real roots that satisfy the characteristic equation above, then the **general solution** to our differential equation is $y = c_1e^{r_1t} + c_2e^{r_2t}$ with c_1 and c_2 being arbitrary constants. Initial conditions can be solved for summarily.

Exercise 1-4.
1. $y = c_1e^t + c_2e^{-3t}$.
2. $y = c_1e^t + c_2e^{2t}$.
3. $y = c_1e^{t/2} + c_2e^{-t/3}$.
4. $y = c_1 + c_2e^{-5t}$.

$\square$

Exercise 13.
If a differential equation's solution is $c_1e^2t + c_2e^{-3t}$, we have $r_1 = 2$, $r_2 = -3$ and as such our differential equation is $y'' + y' - 6y = 0$.
It probably can be shown that no other differential equation produces the general solution given in the problem. $\square$

Exercise 16.
The characteristic equation for our differential equation is $n^2 - n - 2 = 0$ and as such we have roots $r_1 = 2$, $r_2 = -1$. As such, the general solution to the equation is $y = c_1 e^{2t} + c_2 e^{-t}$.
To make the solution approach 0 as $t \to \infty$, we need $c_1 = 0$ as in any other case, e^{2t} will spiral out to infinity and our solution is unbounded. Thus, we can plug this solution into the second part of the initial value problem $y'(0) = 2$:

$$y'(0) = 2 \to 2 = 2 \cdot 0e^{2t} + (-1) \cdot c_2 e^{-0} \to c_2 = -2.$$

Thus, our final solution to the differential equation is $y_{sol} = -2e^{-t}$ and $y_{sol}(0) = \alpha = -2$. $\square$

3.2 Solutions of Linear Homogenous Equations | the Wronskian

Definition 3.2 (*Differential Operator L*)

A general differential operator *does stuff.*
For now, for continuous functions α and β on some open interval I and for any function ϕ twice differentiable on I, we define the **differential operator** L as

$$L[\phi] = \phi'' + \alpha\phi' + \beta\phi.$$

Note that the result of applying L to some function f is another function g.

In this section we will examine the equation $L[y] = 0$.

Definition 3.3 (*Existence and Uniqueness Theorem*)

(Reproduced from page 110.)
Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \ y'(t_0) = y_0',$$

where p, q, and g are continuous on an open interval I with $t_0 \in I$. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

This existence theorem is pretty similar to Theorem 2.4.1 but generalized to second-order linear differential equations. Note once again the guarantee and uniqueness of a solution to the given differential equation over a certain interval.

Definition 3.4 (*Principle of Superposition*)

If y_1 and y_2 are two solutions to the differential equation $L[y] = 0$, then $y_3 = c_1y_1 + c_2y_2$ is also a solution to the given differential equation for any $(c_1, c_2) \in \mathbb{R}^2$.

Definition 3.5

Wronskian Determinant The **Wronskian Determinant** for the system

$$\begin{cases} c_1y_1(t_0) + c_2y_2(t_0) = y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \end{cases}$$

is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

If W is non-zero, then there is a unique solution to the differential equation $L[y] = 0$ with **any** given initial condition. Otherwise, there are initial conditions to the differential equation that cannot be satisfied no matter how c_1 and c_2 are chosen (113).

Note that if the Wronskian W is non-zero, the two solutions y_1 and y_2 to $L[y] = 0$ are said to form a **fundamental set of solutions**.
(There's a lot more discussion here about uniqueness of solutions, Wronksians, and other things I frankly don't care about.)

Regarding complex valued solutions, if $y = u(t) + iv(t)$ satisfies $L[y] = 0$, then u and v are also solutions to the differential equation $L[y] = 0$ (Theorem 3.2.6, Page 117). This is important for later sections.

For another theorem in this long section, we have....

Definition 3.6 (*Abel's Theorem*)

If y_1 and y_2 are solutions for the differential equation $L[y] = 0$ (and some other general conditions are satisfied), then the Wronksian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\ dt\right)$$

where c is a constant dependent on y_1 and y_2 but not on t (Theorem 3.2.7, Page 117)

In summary (page 118), to solve $L[y] = 0$ over some open interval I, we first find two solutions y_1 and y_2 then make sure that $W[y_1, y_2](i) \neq 0$ for some $i \in I$. If this is achieved, y_1 and y_2 would then be a fundamental set of solutions to the given differential equation from which initial-value problems can be solved.

Exercise 12.
We evaluate the differential equation with $y = c\phi(t)$:

$$y'' + p(t)y' + q(t)y = c\phi''(t) + cp(t)\phi'(t) + cq(t)\phi(t) = c(\phi''(t) + p(t)\phi'(t) + q(t)) = g(t).$$

Since we know $\phi(t)$ is a solution to the differential equation, we thus have $c(g(t)) = g(t)$ which cannot hold if $c \neq 1$ and $g(t) \neq 0$.
This does not violate Theorem 3.2.2 (Principle of Superposition) as that principle arises from the special case of when $g(t) = 0$. $\square$

Exercise 13.
No.
If $y = \sin\left(t^2\right)$ is a solution to $L[y] = 0$, then

$$2\cos\left(t^2\right) - 4t^2\sin\left(t^2\right) + p(t)2t\cos\left(t^2\right) + q(t)\sin\left(t^2\right) = \cos\left(t^2\right)(2 + p(t)2t) + \sin\left(t^2\right)\left(-4t^2 + q(t)\right) = 0.$$

To make $L[\sin\left(t^2\right)]$ equal to 0, we thus have to have $2 + p(t)2t = 0$ and $-4t^2 + q(t) = 0$. The latter case is easy to solve but the former implies $p(t) = -\frac{1}{t}$, which is a non-continuous function around the point $t = 0$.
In any case, if we change $q(t)$ to 'cancel' the residue $2\cos\left(t^2\right)$ in the equation above, then in some form or another part of $q(t)$ would contain the fraction $\cot\left(t^2\right)$ meaning q would also be a non-continous function around $t = 0$.
As such, it is impossible to find continuous p and q satisfying $L[\sin\left(t^2\right)] = 0$ over an open interval I containing the point $t = 0$. $\square$

Exercise 15.

$$W[f + 3g, g - g] = (f + 3g)'(f - g) - (f + 3g)(f - g)' = f'(f - g) + 3g'(f - g) - f'(f + 3g) + g'(f + 3g)$$
$$= ff' - f'g + 3fg' - 3gg' - ff' - 3f'g + f'g + 3gg' = -4(f'g - fg') = 4\sin t - 4t\cos t.$$

$\square$

Exercise 17.
Two solutions y_1, y_2 to this differential equation are ce^t and ce^{-2t} for any $c \in \mathbb{R}$. To construct the fundamental set of solutions, we need to reshape our solutions such that $y_a(0) = 1$ and $y_a'(0) = 0$ and also $y_b(0) = 0$ and $y_b'(0) = 1$.

Since our two solutions y_1, y_2 seem pretty dissimilar, we first assume that $y_a = c_1y_1 + c_2y_2$. From here, we just solve for the properties we need; since $y_a = 1$, $c_1 + c_2 = 1$. Similarly, since $y_a'(0) = 0$, $c_1 - 2c_2 = 0$ so $(c_1, c_2) = (2/3,\ 1/3)$.
Doing something similar for y_b, we find that the corresponding $(c_1, c_2) = (1/3,\ -1/3)$. As such,

$$\begin{cases} y_a = \frac{2}{3}e^t + \frac{1}{3}e^{-2t} \\ y_b = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \end{cases}.$$

□

Exercise 23.

$$W = c\exp\left(-\int p(t)\ dt\right) = c\exp\left(-\int \frac{-t(t+2)}{t^2}\ dt\right) = c\exp\left(\int 1 + \frac{2}{t}\ dt\right) = ce^{t+2\ln t} = ct^2e^t.$$

□

Exercise 25.

$$W = c\exp\left(-\int p(x)\ dx\right) = c\exp\left(-\int \frac{-2x}{1-x^2}\ dx\right) = c\exp\left(\int -\frac{1}{u}\ du\right) = ce^{\ln(1/u)} = \frac{c}{1-x^2}.$$

□

Exercise 31. Exact Equations.
Expanding the given expression, we get

$$P'(x)y' + P(x)y'' + f'(x)y + f(x)y' = P(x)y'' + y'(P'(x) + f(x)) + f'(x)y = 0.$$

Equating the coefficients to the general form of a differential equation, we thus have $P'(x) + f(x) = Q(x)$ and $f'(x) = R(x)$.
Taking the derivative of that first equation, we thus have $P''(x) + f'(x) = Q'(x)$ or $P''(x) - Q'(x) + R(x) = 0$ which is exactly the equation that was desired. □

Exercise 32.
32. $P''(x) - Q'(x) + R(x) = 0 - 1 + 1 = 0$ so the equation is exact. Namely, $f(x) = Q(x) - P'(x) = x$ so the problem can be restated as $(y')' + (xy)' = 0 \to y' + xy = c$. This equation is solvable with integrating factor $e^{x^2/2}$ but then the error function pops out so I'm not going to finish this integral. □

Exercise 34.
Since $2 - 1 + (-1) = 0$, we can find $f(x) = -x$. The differential equation then becomes $(x^2y')' + (-xy)' = 0 \to x^2y' - xy = c$.
Solving, we find $y = -\frac{c_1}{3x} + c_2x$. □

3.3 Complex Roots of the Characteristic Equation

What happens when the roots of the characteristic equation $ar^2 + br + c = 0$ for a general differential equation $ay'' + by' + cy = 0$ are imaginary?

Let the roots r_1 and r_2 of the characteristic equation be $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ for real α, β. Then, the corresponding solutions to the differential equation are

$$\begin{cases} y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t) \text{ and} \\ y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t) \end{cases}.$$

In Section 3.2 (Thereom 3.2.6), it was mention that the real and imaginary parts of any solution to a given differential equation are each solutions to the given differential equation. In our case thus, $y_3 = e^{\alpha t}\cos(\beta t)$ and $y_4 = e^{\alpha t}\sin(\beta t)$ are also solutions to $ay'' + by' + cy = 0$, with $W[y_3, y_4] = \beta e^{2\alpha t} \neq 0$.

Exercise 6-8.
6. The quadratic yields roots $r_1, r_2 = 1 \pm i\sqrt{5}$ so the corresponding general solution is $c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t)$.
7. $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$.
8. $y = c_1 e^{-3t}\cos(2t) - c_2 e^{-3t}\sin(2t)$. □

Exercise 25.
(a): $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{t}\dfrac{dy}{dx}$.
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\cdot\frac{dy}{dx}\right) = -\frac{1}{t^2}\frac{dy}{dx} + \frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right).$$
(b): Simplify substitute everything we just derived in into the equation ._. □

Exercise 26-29.
As seen from question 25, we can transform the coefficients of the differential equation $(t^2, \alpha t, \beta)$ into $(1, \alpha - 1, \beta)$. In this case, $\alpha = 1$ and $\beta = 1$ so our new differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

which has solutions $y_1 = \cos(x)$, $y_2 = \sin(x)$. As such, since $x = \ln t$, $y_1 = \cos(\ln t)$ and $y_2 = \sin(\ln t)$ are a set of solutions to the differential equation in terms of t.

27: $\alpha = 4$ and $\beta = 2$ so $y_1 = e^{-x} = \frac{1}{t}$ and $y_2 = e^{-2x} = \frac{1}{t^2}$.
28: $y_1 = \frac{1}{t}$ and $y_2 = t^6$.
29: $y_1 = t^2$ and $y_2 = t^3$. □

3.4 Repeated Roots | Reduction of Order

In the characteristic equation $ar^2 + br + c$, if the discriminant $\Delta = b^2 - 4ac > 0$, then we are bound to find two real roots r_1 and r_2 and from there derive a general solution to the differential equation (Section 3.1). If in fact $b^2 - 4ac < 0$, then we will have two complex roots which, as shown in Section 3.3, correspondingly lead to a real solution to the given differential equation. But what happens when $b^2 - 4ac = 0$?

Assume that $r_1 = r_2 = -\frac{b}{2a}$. Like before, we conclude that one solution to the differential equation $ay'' + by' + cy = 0$ would be $y_1 = e^{-bt/2a}$. But what about the second solution? It turns out $y_2 = te^{-bt/2a}$ is the second solution we need[3].
So to summarize, when solving the equation $ay'' + by' + cy = 0$, the solutions are

$$\begin{cases} y_{1,2} = e^{r_{1,2}t} \text{ if } b^2 - 4ac > 0, \\ y_1 = e^{\lambda t}\cos(\mu t),\ \ y_2 = e^{\lambda t}\sin(\mu t) \text{ if } b^2 - 4ac < 0, \\ y_1 = e^{r_1 t}, y_2 = te^{r_1 t} \text{ (when } b^2 - 4ac = 0.) \end{cases}$$

3.4.1 Reduction of Order

D'Alembert's Method of finding 'extra' (more) solutions is to assume the new solution is of the form $y_2(t) = v(t)y_1(t)$ and solve from there. Namely, in second order differential equations, if we know a solution y_1 to $L[y] = 0$, we let

$$y_2 = v(t)y_1 \text{ so } y_2' = v'(t)y_1 + v(t)y_1' \text{ and } y_2'' = v''(t)y_1 + 2v'(t)y_1' + v(t)y_1''.$$

As such, plugging this back into our differential equation, $L[y_2] = 0$ becomes

$$\begin{aligned} y_2'' + P(t)y_2' + Q(t)y_2 = 0 \Longrightarrow\ & v''(t)y_1 + 2v'(t)y_1' + v(t)y_1'' + P(t)(v'(t)y_1 + v(t)y_1') + Q(t)v(t)y_1 \\ &= y_1v''(t) + (2y_1' + P(t)y_1)v'(t) + (y_1'' + P(t)y_1' + Q(t)y_1)v(t) = 0. \end{aligned}$$

Since y_1 is a solution and thus $L[y_1] = 0$, that right most term is actually 0 so our new differential equation is now

$$y_1v'' + (2y_1' + P(t)y_1)v' = 0$$

which is a first order differential equation with respect to v'. This is known as **reduction of order** since our differential equation went from being a second-order to a first-order differential equation.

Exercise 1-8.
(Note: The general solution y can be expressed as $y = c_1y_1 + c_2y_2$ for arbitrary c_1, c_2. Below, I only find y_1 and y_2.)

1. Since $b^2 - 4ac = 0$, $y_1 = e^{-bt/2a} = e^t$, and $y_2 = te^t$.
2. Since $b^2 - 4ac = 0$, $y_1 = e^{-bt/2a} = e^{-t/3}$ and $y_2 = te^{-t/3}$.
3. Since $b^2 - 4ac = 16 + 4(4)(3) = 64 > 0$, we can simplify find the roots of the equation and derive a general solution that way. The roots to $4n^2 - 4n - 3 = 0$ are $n = \frac{3}{2}, -\frac{1}{2}$ so the two solutions are $y_1 = e^{3t/2}$, $y_2 = e^{-t/2}$.
4. $b^2 - 4ac = -36$ so the roots to this quadratic equation are $1 \pm 3i$ (quadratic formula). As such, $y_1 = e^t\cos(3t)$ and $y_2 = e^t\sin(3t)$.
5. Since $b^2 = 4ac$, $y_1 = e^{3t}$ and $y_2 = te^{3t}$.
6. $y_1 = e^{-4t}$, $y_2 = e^{-t/4}$.
7. $y_1 = e^{-3t/4}$, $y_2 = ty_1$.
8. $y_1 = e^{-1/2}\cos(t/2)$, $y_2 = e^{-1/2}\sin(t/2)$. $\square$

[3] Consult Example 1 in Section 3.4 in the textbook for a proof

Exercise 14.
Let r_1, r_2 be the roots of the characteristic equation to the differential equation $ay'' + by' + cy = 0$. As discussed above, the general solution to this equation is $y = c_1e^{r_1t} + c_2e^{r_2t}$ for arbitrary constants c_1 and c_2. If we let $y = 0$, we can rearrange and show

$$-\frac{c_1}{c_2} = e^{(r_2-r_1)t}.$$

Notably, the left hand side of the equation is constant for a given solution (c_1 and c_2 are chosen after all) and $r_2 - r_1$ is also constant. As such, since the exponential function is a bijective function for all real inputs, there is only one t value that makes the above equation true which means if the differential equation has a non-trivial solution (e.g. not $c_1 = c_2 = 0 \to y(t) = 0$), there is only one t value that makes a given solution to the differential equation (y) 0. □

Exercise 18-22.
18. Reducting the order, you eventually get $v''t^4 = 0 \to v'' = 0 \to v = c_1t + c_2$ so $y_2 = t^3$ as the constant c_1 is arbitrarily chosen (in this case I take $c_1 = 1$) and the latter term c_2t^2 is a multiple of y_1 and thus not worth mentioning.
19. The post-reduction equation ends up being $tv'' + 4v' = 0$ which yields the solution $y = \frac{C_1}{t^3}$ which means the final second solution is $y_2 = \frac{1}{t^2}$. (Note: I'm ignoring the second $+C$ at the end as it's inclusion is not necessary; the final solution ends up being $y_2 = \frac{C_1}{t^2} + C_2t = \frac{C_1}{t^2} + C_2y$ which means the latter term has no meaning and can be ignored.)
20. $y_2 = \frac{\ln t}{t}$.
21. It's long and tricky but you eventually get $y_2 = -C\cot(x^2) \cdot y_1 \to \cos(x^2)$.
22. The integration required in this exercise is basically the same as the ones done in exercise 21. As such, it should be relatively straightforward to show that $y_2 = -C\cos x \frac{1}{\sqrt{x}} \to y_2 = \frac{\cos x}{\sqrt{x}}$. □

Exercise 32-33.
To recap Euler's equations, we transform $t^2y'' + \alpha ty' + \beta y = 0$ into $y'' + (\alpha - 1)y' + \beta y = 0$ where in the first case the derivative of y is taken with respect to t and in the second, the derivative of y is taken with respect to $x = \ln t$.

32. $y_1 = e^{-x/2} = t^{-1/2} = \frac{1}{\sqrt{t}}$, $y_2 = \frac{x}{\sqrt{t}} = \frac{\ln t}{\sqrt{t}}$. I have manually verified that both solutions do indeed solve the differential equation posed.
33. $y_1 = e^{-x} = \frac{1}{t}$, $y_2 = \frac{\ln t}{t}$. □

3.5 Nonhomogenous Equations | Method of Undetermined Coefficients

Definition 3.7 (*Homogenous Differential Equation*)

A differential equation where there is no isolated $g(t)$ term is called a **homogenous** differential equation. In our case, the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$ is homogenous while equations of the form $L[y] = g(t) \neq 0$ are nonhomogenous second-order linear differential equations.

Theorem 3.5.1 asserts that if Y_1 and Y_2 are two solutions to $L[y] = g(t)$, then $Y_1 - Y_2$ is a solution to $L[y] = 0$. Notably, that means if we want to find all solutions to a differential equation $L[y] = g(t)$, we only need to find 1 exact solution Y_1 to the nonhomogenous form as the general solution is thus $c_1y_1 + c_2y_2 + Y_1$ where y_1 and y_2 are solutions to $L[y] = 0$.

Note that the general solution of the homogenous differential equation ($c_1y_1 + c_2y_2$) is commonly called the **complementary solution** and is denoted $y_c(t)$. The solution Y_1 that we find in particular is called the **particular solution**.

So how do we find Y_1? We can either use the **Method of Undetermined Coefficients** (3.5) or use the **Variation of Parameters** method (3.6).

Method of Undetermined Coefficients

(a.k.a. educated guess and check)

Essentially, based on the coefficients in the equation ($p(t)$, $q(t)$, $g(t)$), make a good *guess* about what Y_1 could look at (using general constant coefficients) then solve for those constant coefficients (and hope you're right in your guess).

For good guidelines about guessing:

- If the nonhomogenous term $g(t)$ is of the form $e^{\alpha t}$, assume $Y = Ae^{\alpha t}$.
- If $g(t)$ looks like $\sin(\beta t)$ or $\cos(\beta t)$, let $Y = A\sin(\beta t) + B\cos(\beta t)$.
- If $g(t)$ looks like some polynomial up to t^γ, let $Y = a_1 t^\gamma + a_2 t^{\gamma-1} + \cdots + a_\gamma t + a_{\gamma+1}$.
- If $g(t)$ looks like two (or more) of the above functions added together (e.g. $g(t) = e^{-3t} + \sin(4t)$), split up the differential equation to find the respective solutions to when $g(t) = e^{-3t}$ and $g(t) = \sin(4t)$ then add those solutions together.
- If $g(t)$ looks like two of the above functions multiplied together (e.g. $g(t) = (t^2 + t - 4)(e^{3t})$), let Y be the product of the two relevant guesses; in this case, we should let $Y = (At^2 + Bt + C)e^{3t}$.
- If guessing a Y for $g(t)$ fails, try $Y^* = tY$. Maybe that'll work :).

Exercise 1-7.
1. Assuming $Y = Ae^{2t}$, we soon find $A = -1$. Thus, a particular solution to this equation is $Y = -e^{2t}$. Since the general solution to the given differential equation is $y_c = c_1 e^{3t} + c_2 e^{-t}$, the general general solution is thus $\boxed{\phi = c_1 e^{3t} + c_2 e^{-t} - e^{2t}}$ for arbitrary constants c_1, c_2.

2. Assuming $Y = At^2 + Bt + C$, we derive, substitute, and solve to find $A = -2$, $B = 3$, and $C = -7/2$. Since the homogenous solution is $c_1 e^{2t} + c_2 e^{-t}$, we thus have the general solution ϕ being of the form $\boxed{c_1 e^{2t} + c_2 e^{-t} - 2t^2 + 3t - \frac{7}{2}}$.

3. Since $g(t)$ is composed of two exponential terms, we similarly assume $Y = Ae^{3t} + Be^{-2t}$ and we find $A = 2$ and $B = -3$. With the solution from the non-homogenous equation, we thus find that $\phi = c_1 e^{2t} + c_2 e^{-3t} + 2e^{3t} - 3e^{-2t}$.

4. While assuming $Y = (At + B)e^{-t}$ yields no satisfactory results, assuming $Y = (At^2 + Bt + C)e^{-t}$ (one level up) leads us to find $A = 3/8$ and $B = 3/16$. Thus, the general solution to the differential equation is $\phi = t(3t/8 + 3/16)e^{-t} + c_1 e^{3t} + c_2 e^{-t}$.

5. This differential equation is pretty funny since as there is no y term involved, this is a linear first order differential equation with respect to y'. Nevertheless, viewing this from a second-order DE perspective, we can split $g(t) = 3 + 4\sin(2t)$ into $g_1(t) = 3$ and $g_2(t) = 4\sin(2t)$ and solve the differential equations $y'' + 2y' = g_i(t)$ separately to get a particular solution $Y = -\frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) + \frac{3}{2}t + C$ (arbitrary constant C), meaning the general solution ϕ is $c_1 + c_2 e^{-2t} + Y$.

6. Solving the homogenous version of the differential equation, we have $y_c = c_1 e^{-t} + c_2 te^{-t}$. As such, while we would normally set $Y = Ae^{-t}$, we can't since this solution is already included in the complementary solution. Similarly, $Y = Ate^{-t}$ also doesn't work and to solve, we assume $Y = At^2 e{-}t$ and find the general solution ϕ to be $\phi = e^{-t}(t^2 + c_2 t + c_1)$.

7. I did a big messy equation and assumed $Y = A\sin(2t) + B\cos(2t) + Ct\sin(2t) + Dt\cos(2t)$ (and to make matters worse I substituted in $\sin(2t)$ with $\triangle$ and $\cos(2t)$ with $\square$ ostensibly to save writing – but this just made everything worse). Eventually, I found $A = -\frac{5}{9}$, $D = -\frac{1}{3}$, and $B = C = 0$. Thus, $\phi = c_1 \sin t + c_2 \cos t - \frac{1}{3}t\cos(2t) - \frac{5}{9}\sin(2t)$. $\square$

Exercise 8-10.
(Continuation from Exercises 1-7)
8. Assuming $U = A\cos(\omega t) + B\sin(\omega t)$, we eventually have $A\cos(\omega t)(\omega_0^2 - \omega^2) = \cos(\omega t)$ and $B\sin(\omega t)(\omega_0^2 - \omega^2) = 0$. Since it is given that $\omega_0^2 \neq \omega^2$, $A = \frac{1}{\omega_0^2 - \omega^2}$ and $B = 0$. As such, $\phi = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{1}{\omega_0^2-\omega^2}\cos(\omega t)$.

9. Resuming the same path as before, since $\omega = \omega_0$ in this case, we have to assume $U = At\cos(\omega t) + Bt\sin(\omega t)$ which leads us to find $B = \frac{1}{2\omega_0}$. As such, $\phi = \frac{t\sin(\omega_0 t)}{2\omega_0} + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t)$.

10. The particular solution is quite easy to find in this case; although $\sinh t$ is scary, it is easily mitigated by letting $Y = Ae^t + Be^{-t} \to A = \frac{1}{6}$, $B = -\frac{1}{4}$. In contrast, the roots of the characteristic equation are $-\frac{1}{2} \pm \frac{i\sqrt{15}}{2}$ so the general solution is $\phi = c_1 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$. $\square$

Exercise 23.
From many problems before, it can be intuited that $y_c = c_1\cos(\lambda t) + c_2\sin(\lambda t)$. For the particular solution, we consider the differential equation of an arbitrary term in the summation $a_k\sin(k\pi t)$:

$$y'' + \lambda^2 y = a_k\sin(k\pi t).$$

Some uncomplicated guessing and checking ($Y = A\sin(k\pi t) + B\cos(k\pi t)$) leads us to find that for this arbitrary case, a particular solution is $Y_k = \frac{a_k}{\lambda^2 - k^2\pi^2}\sin(k\pi t)$ which lets us reasonably conclude that

$$\phi = c_1\cos(\lambda t) + c_2\sin(\lambda t) + \sum_{m=1}^{N}\frac{a_m}{\lambda^2 - m^2\pi^2}\sin(m\pi t)$$

is the general solution to this scary-looking differential equation. $\square$

Exercise 28-30.
Note: Everything in exercise 27 (which this problem is based off of) is true. I'm not sure how to verify it because each step seems somewhat trivially obvious.

28. $y'' - 3y - 4y = 3e^{2t} = y(D-4)(D+1)$. As such, leetting $u = (D+1)y$, our aim is to find y by first finding u by solving the differential equation $(D-4)u = g(t) \to u' - 4u = 3e^{2t}$. This first differential equation can be done with an integrating factor $\mu = e^{-4t}$ which leads us to find $u = -\frac{3}{2}e^{2t}$ (screw the constant).
Having found u, we can now find y with the equation $(D - r_2)y = u \to y' + y = -\frac{3}{2}e^{2t}$. With integrating factor $\mu = e^t$, we easily find $y = -\frac{1}{2}e^{2t}$ as a particular solution to the given differential equation, and solve the problem correspondingly.

29. Our two first-order equations are $(D+1)u = 2e^{-t}$ and $(D+1)y = u$. Solving the first, we have $u' + u = 2e^{-t}$ so with integrating factor e^t we find $u = 2te^{-t}$. Next, we solve $y' + y = 2te^{-t}$ and find $y = t^2e^{-t}$ (same simple integrating factor, same process) which is indeed a particular solution to the given differential equation.

30. Our two equations are $(D+2)u = 3 + 4\sin(2t)$ and $Dy = u$ (root order doesn't matter mathematically). Solving the first differential equation, with integrating factor $\mu = e^{2t}$, we find[a]

$$(e^{2t}u) = \frac{3}{2}e^{2t} + 4\int e^{2t}\sin(2t)\ dt \longrightarrow u = \frac{3}{2} + \sin(2t) - \cos(2t).$$

The second differential equation is simply $y' = u$ or $y = \int u$ so a particular solution y that we find is $y = \frac{3}{2}t - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t)$.[b] $\square$

[a]The complicated $\int e^t \sin t$ integral is solved cleverly using integration by parts.
[b]Remark: In this case, the strategem of solving two first order DEs to find a particular solution works much faster (and cleaner) than the method of undetermined coefficients. It also feels a lot more straightforward.

3.6 Variation of Parameters

Thank you Lagrange for this method.

Lagrange's idea to solving general differential equations $L[y] = g(t)$ is to replace constants with functions:
Say we have a differential equation $y'' + p(t)y' + q(t)y = g(t)$ and we know the complementary solution $y_c(t) = c_1y_1 + c_2y_2$ to the homogenous version of the differential equation. From here, the idea is to replace the constants c_1 and c_2 with functions u_1 and u_2 so $y = u_1y_1 + u_2y_2$ ends up being a particular solution to the differential equation. Assuming this, we differentiate our particular solution:

$$\longrightarrow y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Since we're not interested in solving another second-order differential equation and we have a free condition we can impose on the equation, we let $u_1'y_1 + u_2'y_2 = 0$ so that we have

$$y' = u_1y_1' + u_2y_2'.$$

As such, differentiating again, we have

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

From here, substituting in y'' and y' and y into the general differential equation, much simplifcation eventually leads us to find $u_1'y_1' + u_2'y_2' = g(t)$.
Thus, with this equation, we have a linear system from which we can solve for u_1 and u_2:

$$\begin{cases} u_1'y_1' + u_2'y_2' = g(t) \text{ (derived)} \\ u_1'y_1 + u_2'y_2 = 0 \text{ (mandated -- see above)} \end{cases}.$$

The solutions to this system ends up being

$$\begin{cases} u_1 = -\int \frac{y_2 g}{W[y_1,y_2]}\,dt + c_1 \\ u_2 = \int \frac{y_1 g}{W[y_1,y_2]}\,dt + c_2 \end{cases}$$

with $W[a,b](t) = a(t)b'(t) - a'(t)b(t)$. Thanks Lagrange :).
Note that this methodology is not a silver bullet — y_1 and y_2 may be hard to find solutions for if $p(t)$ and $q(t)$ are complicated, and the integrals solving for u_1 and u_2 may vary in nice-ness to solve.

Exercise 4-8.
4. The complementary solution to this equation is $y_c = c_1\cos t + c_2\sin t$. Calculating the respective functions u_1 and u_2, we find that $u_2 = -\cos t$ (which is useless since that's included in the complementary function) and $u_1 = \sin t - \ln|\sec t + \tan t|$. As such, the general solution to this equation would be $\boxed{\phi = c_1\cos t + c_2\sin t - (\cos t)\ln|\sec t + \tan t|}$.

5. The general solution here is $y_c = c_1\cos(3t) + c_2\sin(3t)$. Using the plug and chug formulas, we find $u_1 = -\sec(3t)$ and $u_2 = \ln|\sec(3t) + \tan(3t)|$. Thus, the general solution ϕ is of the form $\boxed{c_1\cos(3t) + c_2\sin(3t) - 1 + \sin(3t)\ln|\sec(3t) + \tan(3t)|}$.

6. $u_1 = -\ln t$, $u_2 = -\frac{1}{t}$, so the general solution is $\phi = c_1e^{-2t} + c_2te^{-2t} - \ln te^{-2t}$ (the last term can be merged in with the constant).

7. $\phi(t) = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right) + 8\cos\left(\frac{t}{2}\right)\ln\left|\cos\left(\frac{t}{2}\right)\right| + 4t\sin\left(\frac{t}{2}\right)$. Note that the last two terms can each be divided by 4 yielding the solution in the back of the book.

8. $\phi(t) = c_1e^t + c_2te^t - \frac{1}{2}e^t\ln(1+t^2) + te^t\arctan(t)$. Note that the absolute value can be removed from the natural log as it is assumed that the domain of t is $\mathbb{R}$ and as such $1 + t^2 > 0$ for all t. □

Exercise 23-25. Reduction of Order.
(Note: These problems are similar to those exercises covered in section 3.4.)
23. Plugging $v(t)y_1(t)$[a] in for y, we simplify the general differential equation:

$$(vy_1)'' + p(t)(vy_1)' + q(t)(vy_1) = g(t) \to v''y_1 + 2v'y_1' + vy_1'' + p(t)v'y_1 + p(t)vy_1' + q(t)vy_1 = g(t)$$
$$\to v''(y_1) + v'(2y_1' + p(t)y_1') + v(y_1'' + p(t)y_1' + q(t)y_1) = g(t).$$

Since the expression $y_1'' + p(t)y_1' + q(t)y_1$ simplifies to 0, the desired equation given in the textbook soon follows. Notably, as the textbook mentions, the equation above is a first-order differential equation for v'. Once $v'(t)$ is found, $v(t)$ and $v(t)y_1(t)$ soon follow.

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24. Rearranging, our DE is $y'' - \frac{2}{t}y + \frac{2}{t^2}y = 4$ which correspondingly means $p(t) = -2/t$ and $g(t) = 4$. As such, our 'formula' for v' is

$$t\frac{dv'}{dt} + (2-2)v' = 4 \to v' = 4\ln t + c_1.$$

Thus, $v(t) = 4(t\ln t - t) + c_1 t + c_2$ and our general solution is $y = y_1(t)v(t) = \boxed{4t^2\ln t - 4t^2 + c_1t^2 + c_2t}$ (that second term is redundant).

25. Our 'formula' tells us

$$\frac{1}{t}\frac{dv'}{dt} + \left(\frac{-2}{t^2} + \frac{7}{t}\frac{1}{t}\right)v' = \frac{1}{t} \longrightarrow \frac{dv'}{dt} + \frac{5}{t}v' = 1.$$

From here, a simple integration factor of $\mu = t^5$ leads us to find $v' = \frac{1}{6}t + \frac{c_1}{t^5}$ so $v = \frac{1}{12}t^2 - \frac{c_1}{5t^4} + c_2$ so $y = \phi(t) = \boxed{\frac{1}{12}t - \frac{c_1}{5t^5} + \frac{c_2}{t}}$ (with that 5 in $5t^5$ being extraneous due to the constant c_1). □

[a]Note that $y_1(t)$ need only be a solution for the homogenous second-order linear DE

3.7 Mechanical and Electrical Vibrations

// Derivation and example of a mass on a string with simple harmonic motion derived (page 151). Equations for a damped mass on a spring are also studied. //

3.7.1 Electric Circuits

(Page 156) In a classic RLC circuit (R-C-L arranged in sequential order), we know these facts:

- $I = \frac{dQ}{dt}$.
- V_r (voltage across the resistor) $= IR$.
- $V_c = \frac{Q}{C}$.
- $V_l = L\frac{dI}{dt}$.
- $V_r + V_c + V_l = V(t)$.

As such, by substituting in I for $\frac{dQ}{dt}$ in subsequent equations, we find a second-order linear differential equation with constant coefficients:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V.$$

(I'm skipping all the non-electrical problems because I'm not a physicist).

Exercise 1-2.
1. $R\cos(\omega_0 t - \delta) = (R\cos\delta)\cos\omega_0 t + (R\sin\delta)\sin\omega_0 t$. As such, in the case for this problem, $w_0 = 2$ and we have to solve $R\cos\delta = 3$ and $R\sin\delta = 4$. Squaring and summing both equations, we find $R = 5$ and thus $\delta = \arccos\left(\frac{3}{5}\right)$.
Our final equation is thus $u = 5\cos\left(2t - \arccos\frac{3}{5}\right)$.

2. $u = -\sqrt{13}\cos\left(\pi t - \arccos\frac{2}{\sqrt{13}}\right)$. □

Exercise 7.
The differential equation we set up is

$$0.2Q'' + 300Q' + 10^5 Q = V(t) = 0$$

since the problem says nothing about voltage. The general solution for this equation is $Q(t) = c_1 e^{1000t} + c_2 e^{500t}$, and to satisfy the given initial conditions, we find $c_1 = -10^{-6}$ and $c_2 = 2 \cdot 10^{-6}$. Our final equation for Q is thus $\boxed{-\frac{e^{1000t}}{10^6} + \frac{2e^{500t}}{10^6}}$. □

Exercise 12.
For motion to be critically damped, $\gamma = 2\sqrt{km}$, with the values γ, k, and m being taken from the damped differential equation $mu'' + \gamma u' + ku = 0$.
In the case for this electric circuit, our differential equation is $0.2Q'' + RQ' + 1.25 \cdot 10^6 Q = 0$. As such, our '$\gamma$' value ($R$) would be equal to $2\sqrt{0.2 \cdot 1.25 \cdot 10^6} = 1000$ (Ω). □

3.8 Forced Periodic Vibrations

Section 3.8 covers physical spring-mass systems when $g(t) \neq 0$ or when an external force constantly acts on the spring with time. I will skip this section since there are no LRC problems.

4 Higher Order Linear Differential Equations

TBD.

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5 Series Solutions for 2nd-order ODEs

TBD.

6 The Laplace Transform

6.1 Definition of the Laplace

An improper integral is an integral that has an infinity in one of its bounds or has its function be otherwise discontinuous. Examples of discontinuous integrals are

$$\int_a^\infty f(t)\ dt \text{ or } \int_0^5 \frac{1}{x-3}\ dx.$$

Sometimes these integrals converge, sometimes they diverge.

Definition 6.1 (*Piecewise Continuous*)

We call a function f **piecewise continuous** over an interval $\alpha \leq t \leq \beta$ if we can find a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ such that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$, and
2. Approaching f's endpoints of each subinterval from within the subinterval results in a finite limit.

As the book sums it up nicely, f is piecewise ocntinuous if it is "continuous except for a finite number of jump discontinuities."
The integral of a piecewise function f is just the sum of its parts:

$$\int_\alpha^\beta f(t)\ dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\ dt = \int_\alpha^{t_1} f(t)\ dt + \int_{t_1}^{t_2} f(t)\ dt + \cdots + \int_{t_{n-1}}^{\beta} f(t)\ dt.$$

With that, we get our first theorem on divergence and convergence:

Definition 6.2 (*Integral Convergence (Theorem 6.1.1)*)

If f is piecewise continuous for $t \geq \alpha$ and $|f(t)| \leq g(t)$ when $t > C$ for some constant C, if $\displaystyle\int_C^\infty g(t)\ dt$ converges, then $\displaystyle\int_a^\infty f(t)\ dt$ also converges.
Moreover, if $f(t) \geq g(t) \geq 0$ for all $t \geq C$, if $\displaystyle\int_C^\infty g(t)\ dt$ diverges, then $\displaystyle\int_a^\infty f(t)\ dt$ also diverges.

Essentially, this theorem states that if f is bounded above and the bounding function is convergent, f is similarly convergent. Also, if f is bounded below and the integral of the bounding function diverges, then f is similarly divergent.

6.1.1 The Laplace Transform

Definition 6.3 (*Integral Transform*)

An integral transform is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t) f(t)\ dt$$

where K is called the **kernel** of the transformation and limits α and β are given. F is called the **transform** of f.

In our case, the laplace transform for a given function f is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

if this improper integral converges. Thus, the kernel function used here is $K(s,t) = e^{-st}$.

In general, to solve a differential equation, we use a laplace transform on a function f to derive F, solve for F, then un-transform and recover f. In general, s may also be a complex number.

Definition 6.4 (*Laplace Transform Existence (Theorem 6.1.2)*)

If f is piecewise continuous for all $t \geq 0$ and there exist constnats $K > 0$, a, and $M > 0$ such that

$$|f(t)| \leq Ke^{at} \text{ when } t \geq M,$$

then the laplace transform $\mathcal{L}\{f\} = F$ as defined above exists for all $s > a$.

If a function f satisfies Theorem 6.1.2, then f is described as being piecewise continuous and of **exponential order** as $t \to \infty$ (e.g. The highest 'order' the function can be is $O(e^{at})$).

// (Some examples of the Laplace transformation being used out in the wild are given!!) //
Note that the Laplace transform is a **linear operator**; that is,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

Exercise 6-11.
The laplace transform of $e^{\alpha t} = \dfrac{1}{s-b}$, assuming $s > b > 0$. Thus, for (6),

$$\mathcal{L}\{\cosh(bt)\} = \mathcal{L}\left\{\frac{e^{bt}}{2} + \frac{e^{-bt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{bt}\} + \frac{1}{2}\mathcal{L}\{e^{-bt}\} = \frac{1}{2(s-b)} + \frac{1}{2(s+b)}.$$

The rest of the exercises similarly follow;

$$(7)\text{: } \mathcal{L}\{\sinh(bt)\} = \frac{1}{2}\mathcal{L}\{e^{bt}\} - \frac{1}{2}\mathcal{L}\{e^{-bt}\} = \frac{1}{2(s-b)} - \frac{1}{2(s+b)}.$$

$$(8)\text{: } \mathcal{L}\{\sin(bt)\} = \frac{1}{2i}\mathcal{L}\{e^{ibt}\} - \frac{1}{2i}\mathcal{L}\{e^{-ibt}\} = \frac{1}{2i(s-ib)} - \frac{1}{2i(s+ib)} = \frac{(s+ib)-(s-ib)}{2i(s-ib)(s+ib)} = \frac{b}{s^2+b^2}.$$

$$(9)\text{: } \mathcal{L}\{\cos(bt)\} = \frac{1}{2}\mathcal{L}\{e^{ibt}\} + \frac{1}{2}\mathcal{L}\{e^{-ibt}\} = \frac{1}{2(s-ib)} + \frac{1}{2(s+ib)}.$$

$$\begin{aligned}(10)\text{: } \mathcal{L}\{e^{at}\sin(bt)\} &= \frac{1}{2i}\mathcal{L}\{e^{t(a+ib)}\} - \frac{1}{2i}\mathcal{L}\{e^{t(a-ib)}\} = \frac{1}{2i((s-a)-ib)} - \frac{1}{2i((s-a)+ib)} \\ &= \frac{(s-a+ib)-(s-a-ib)}{2i((s-a)+ib)((s-a)-ib)} = \frac{b}{b^2+(s-a)^2}.\end{aligned}$$

$$\begin{aligned}(11)\text{: } \mathcal{L}\{e^{at}\cos(bt)\} &= \frac{1}{2}\mathcal{L}\{e^{t(a+ib)}\} + \frac{1}{2}\mathcal{L}\{e^{t(a-ib)}\} = \frac{1}{2(s-a-ib)} + \frac{1}{2(s-a+ib)} \\ &= \frac{s-a+ib+s-a-ib}{2(s-a-ib)(s-a+ib)} = \frac{s-a}{(s-a)^2+b^2}.\end{aligned}$$

$\square$

Exercise 4.
Recall from above that $\mathcal{L}\{f\} = \displaystyle\int_0^\infty e^{-st} f\, dt$. Now, to find $\mathcal{L}\{t^n\}$, assume we know $\mathcal{L}\{t^{n-1}\} = g(s)$ $(t > 1)$. Then, by integration by parts,

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n\, dt = \left[\frac{t^n}{s}e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}.$$

The second expression simply simplifies to $\dfrac{n}{s}g(s)$ by definition of the laplace transform, and the first expression is 0 as when plugging in the upper bound $t = 0$, $t^n = 0$, and when plugging the lower bound $t = \infty$, e^{st} outgrows t^n (assuming $s > 0$) so $\dfrac{1}{s} \cdot \dfrac{t^n}{e^{st}} = 0$ as $t \to \infty$. $\square$

Exercise 5.

$$\mathcal{L}\{\cos(at)\} = I = \int_0^\infty e^{-st}\cos(at)\ dt \Rightarrow \left[\frac{1}{s}e^{-st}\cos(at)\right]_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\ dt$$

$$= \frac{1}{s} - \frac{a}{s}\left(\left[\frac{1}{s}e^{-st}\sin(at)\right]_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\right) = \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s}I\right) \to \boxed{I = \frac{1}{s\left(1+\frac{a^2}{s^2}\right)}}.$$

□

Exercise 16-18.

16. $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f\ dt = \int_0^\pi e^{-st}\ dt = \frac{1}{s}(1-e^{-s\pi})$.

17. $\mathcal{L}\{f\} = \int_0^1 e^{-st} t\ dt + \int_1^\infty e^{-st}\ dt = \left[-\frac{1}{s}e^{-st}\right]_1^\infty + \left[\frac{t}{s}e^{-st}\right]_1^0 + \frac{1}{s}\int_0^1 e^{-st}\ dt = \frac{1-e^{-s}}{s^2}$.

18.

$$\mathcal{L}\{f\} = \int_0^1 e^{-st} t\ dt + \int_1^2 e^{-st}(2-t)\ dt = \left(-\frac{e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}\right) + 2\int_1^2 e^{-st}\ dt - \int_1^2 e^{-st} t\ dt$$

$$= \left(-\frac{e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}\right) + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s} - \left(\left[-\frac{t}{s}e^{-st}\right]_1^2 + \frac{1}{s}\int_1^2 e^{-st}\ dt\right)$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2} + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} + \frac{2}{s}e^{-2s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \boxed{\frac{1-2e^{-s}+e^{-2s}}{s^2}}.$$

□

Exercise 24.

24(ab): Making the substitution $x = st$ and $dx = s\ dt$ (and assuming $s > 0$ else the integral bounds must be switched),

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^p \frac{dx}{s} = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\ dx$$

which is exactly what needs to be shown. By definition, the integral we have in the simplified form of $\mathcal{L}\{t^p\}$ evaluates to $\Gamma(p+1)$[a], so

$$\mathcal{L}\{t^p\} = \frac{\Gamma p+1}{s^{p+1}} = \frac{n!}{s^{n+1}}$$

if $n \in \mathbb{N}$.

24(c): We essentially make the substitution $x = y^2$ ($\sqrt{x} = y$, $dx = 2y\ dy$):

$$\mathcal{L}\{t^{-1/2}\} = \frac{1}{\sqrt{s}} = \int_0^\infty e^{-x}\frac{1}{\sqrt{x}}\ dx \longrightarrow \frac{1}{\sqrt{s}}\int_{\sqrt{0}=0}^{\sqrt{\infty}=\infty} e^{-y^2}\frac{1}{y}\ 2y\ dy = \frac{2}{\sqrt{s}}\int_0^\infty e^{-y^2}\ dy$$

which is exactly what's asked for.

24(d):

$$\mathcal{L}\{t^{1/2}\} = \frac{1}{s\sqrt{s}}\int_0^\infty e^{-x}\sqrt{x}\ dx \to \frac{1}{s\sqrt{s}}\int_0^\infty (y)\left(2ye^{-y^2}\right)\ dy = \frac{1}{s\sqrt{s}}\left(\left[-ye^{-y^2}\right]_0^\infty + \int_0^\infty e^{-y^2}\ dy\right) = \boxed{\frac{\sqrt{\pi}}{2s\sqrt{s}}}.$$

□

[a] the textbook question for this part of the question is wrong :/

6.2 Solutions to IVPs

Definition 6.5 (*Theorem 6.2.1 (Page 248)*)

Suppose f is continuous on some interval I $(0 \le t \le A)$ and f' is also piecewise continuous over some inerval of I. Then, if $O(2^t) \ge f(t)$ (or formally, there exists constants K, a, and M such that $Ke^{at} \ge |f(t)|$ for all $t \ge M$), then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and moreover,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

As a corollary,

Definition 6.6

Corollary 6.2.2 If f, f', f'', $\ldots$, $f^{(n-1)}$ all satisfy the conditions laid out in Theorem 6.2.1, and $f^{(n)}$ is piecewise continuous, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

As long as the derivatives of f are all continuous, the theorem can then be applied successively to get the laplace transform of the nth derivative of f.

In general, while the laplace transform converts a differential equation into an algebraic equation, the challenge lies in inverting the laplace transform to find a function y such that $\mathcal{L}\{y\} = Y$.

Note by the linearity of the laplace transform, if $F = F_1 + F_2 + F_3 + \cdots$, then $\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\} + \cdots$.

Exercise 1-7.

1. $F(s) = \frac{3}{2}\frac{2}{s^2+2^2} \to \mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}\sin(2t)$.
2. $F(s) = 2\frac{2!}{(s-1)^{2+1}} \to \mathcal{L}^{-1}\{F(s)\} = 2t^2e^t$.
3. By partial fractions, $F(s) = \frac{2/5}{s-1} + \frac{2/5}{s+4}$ so $\mathcal{L}^{-1}\{F\} = \frac{2}{5}e^t - \frac{2}{5}e^{-4t}$.
4. $F(s) = 2\frac{s-(-1)}{(s-(-1))^2+2^2} \to \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos 2t$.
5. $\mathcal{L}^{-1}\{F(s)\} = \frac{e^{2t}}{4} + \frac{7e^{-2t}}{4}$.
6. While I initially decomposed $F(s)$ into $\frac{A}{s} + \frac{B}{s+2i} + \frac{C}{s-2i}$ (and got imaginary values for B and C), a much simpler way is to recognize that $F(s)$ decomposes into $\frac{3}{s} + 5\frac{s}{s^2+2^2} - 2\frac{2}{s^2+2^2}$ from which it is immediately evident that $\mathcal{L}^{-1}\{F(s)\} = 3 + 5\cos(2t) - 2\sin(2t)$.
7. $\mathcal{L}^{-1}\{F(s)\} = -2e^{-2t}\cos(t) + 5e^{-2t}\sin t$. $\square$

Exercise 8-9.

8. To solve the differential equation (with initial values given) $y'' - y' - 6y = 0$, we recall that $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$ and $\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s(s\mathcal{L}\{y\} - y(0)) - y'(0)$; thus, we can rewrite our differential equation as

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - s\mathcal{L}\{y\} + y(0) - 6s\mathcal{L}\{y\} = 0 \to \mathcal{L}\{y\}(s^2 - s - 6) - s + 1 + 1 = 0 \to \mathcal{L}\{y\} = \frac{s-2}{s^2-s-6}.$$

Un-laplacing the transform, we find $y = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$.

9. $\mathcal{L}\{y\} = \frac{s+3}{(s+1)(s+2)}$ so $y = 2e^{-2} - e^{-2t}$. $\square$

For all the problems below, note that $y'' + ay' + by = g(t)$ can be transformed into

$$L(s^2 + as + b) - (s + a)f(0) - f'(0) = \mathcal{L}\{g(t)\} \to L = \frac{\mathcal{L}\{g(t)\} + (s + a)f(0) + f'(0)}{s^2 + as + b}$$

where $L = \mathcal{L}\{y\}$ and $y = f(t)$ is the solution to the initial value problem.

Exercise 10-16.
10. $y = e^t \sin t$.
11. $y = 2e^t \cos\left(\sqrt{3}t\right) - \frac{2}{\sqrt{3}}e^t \sin\left(\sqrt{3}t\right)$.
12. $y = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$.
13. After lots of simplication, we find $\mathcal{L}\{y\} = \dfrac{s^2 - 4s + 7}{(s-1)^4}$. We can decompose piece-wise and find $y = te^t - t^2e^t + \frac{2}{3}t^3e^t$.
14. $y = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$.
15.

$$\mathcal{L}\{y\} = \frac{1}{\omega^2 - 4}\frac{s}{s^2 + 4} + \frac{\omega^2 - 5}{\omega^2 - 4}\frac{s}{s^2 + \omega^2}$$

so $y = \dfrac{1}{\omega^2 - 4}\cos(2t) + \dfrac{\omega^2 - 5}{\omega^2 - 4}\cos(\omega t)$.

16. $y = \dfrac{1}{5}\left(e^{-t} - e^t \cos t + 7e^t \sin t\right)$. □

Exercise 22.
Since $F'(s) = \mathcal{L}\{-tf(t)\}$, letting $f(t) = -e^{at}$, we thus have $\mathcal{L}\{te^{at}\} = F'(s)$.
Since $F(s) = \displaystyle\int_0^\infty e^{-st}(-e^{at})\,dt = -\frac{1}{s-a}$,

$$\mathcal{L}\{te^{at}\} = \mathcal{L}\{-t(-e^{at})\} = F'(s) = \frac{d}{ds}\left[-\frac{1}{s-a}\right] = \frac{1}{(s-a)^2}.$$

□

Exercise 29.
29(a): Taking the book's suggestion and multiplying equation (36) by $s - r_k$, our equation becomes

$$\frac{P(s)(s - r_k)}{Q(s)} = \frac{A_1(s - r_k)}{s - r_1} + \cdots + A_k + \cdots + \frac{A_n(s - r_k)}{s - r_n}.$$

Now, taking the limit as $s \to r_k$, each term on the RHS has that has the expression $(s - r_k)$ in the numerator goes to 0 which means we have $A_k = \displaystyle\lim_{s\to r_k}\frac{P(s)(s - r_k)}{Q(s)}$, where the RHS of this limit is not simplifiable due to the fact that r_k is a root of Q and the numerator is 0 when $s = r_k$. By L'hopital, $A_k = \displaystyle\lim_{s\to r_k}\frac{P(s)(s - r_k)}{Q(s)} = \lim_{s\to r_k}\frac{P'(s)(s - r_k) + P(s)}{Q'(s)}$ and since this limit is evaluatable, we find the desired expression for A_k.
29(b): Piecewise, the inverse laplace transform of $\frac{A_i}{s - r_i}$ is simply $A_ie^{r_it}$ so when summing over all i,

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^{n} A_ie^{r_it} = \sum_{i=1}^{n}\frac{P(r_i)}{Q'(r_i)}e^{r_it}.$$

□

7 Systems of First-Order Linear Equations

7.1 Introduction

Essentially, we consider systems of first-order equations since any higher order differential equation can inevitably be transformed into multiple first order linear transformations.

Moreover, for any $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$, we cna make the substituions $x_1 = y$, $x_2 = y'$, $\dots$, $x_n = y^{(n-1)}$ and thus eventually find $x_1' = F_1(t, x_1, x_2, \dots, x_n)$, $x_2' = F_2(t, x_1, x_2, \dots, x_n)$ and so on. Thus, we have effectively converted a general differential equation into many teeny tiny first-order differential equations (that are each in their own way, granted, hard to solve).

7.2 Matrices

(note: all uppercase letters from here on out (A, B, C, $\dots$) will most likely represent matrices from here on out unless they are in function notation (e.g. $F(t)$ would be a function)).

Various matrix preliminaries are covered here. Do note that when the book talks about the **adjoint** of A, they mean the **transpose of the conjugate matrix of** A rather than the cofactor expansion matrix of A.

Integrals, derivatives, and [x] over matrices of functions are just those same operations applied to each individual operations (boring). For example, $\displaystyle\int A\,dt = \int a_{ij}\,dt$.

7.3 More Linear Algebra

(This is just a review of Math 4a......)

7.4 Basic Theory of Systems of First-Order Linear Equations

(a.k.a. A review of section 3.2 but with matrices instead of second-order linear differential equations.)
To examine a system of n first-order linear equations each of the form $x_i' = p_{i1}(t)x_1 + p_{i2}(t)x_2 + \cdots + p_{in}(t)x_n + g_i(t)$, we can rewrite everything in matrix form and obtain the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]^T$, $\mathbf{g}(t) = [g_1(t)\ g_2(t)\ \dots\ g_n(t)]^T$, and $\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{pmatrix}$.

With matrix equations, multiple solutions ($\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(k)}(t)$) for $\mathbf{x}$ may exist. Moreover, if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two solutions to a first-order homogenous matrix differential equation ($\mathbf{g} = \mathbf{0}$), then $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution to said equation for arbitrary constants c_1, c_2 (Theorem 7.4.1, Page 305).

If we make a big matrix $\mathbf{X} = [\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ \dots\ \mathbf{x}^{(n)}]$, then we can calculate its determinant; namely, $\det \mathbf{X} = W[\mathbf{x}^{(1)}\ \dots\ \mathbf{x}^{(n)}]$ and as such if $\det \mathbf{X} \neq 0$ at some particular point $t = t_0$, then the solutions $\mathbf{x}^{(1)}$, $\dots$ are all linearly independent at t_0.

Definition 7.1 (*Generalized Abel's Theorem*)

If $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are solutions to a homogenous first-order set of linear differential equations over some open interval I, then over I, either $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}] = 0$ or $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) \neq 0\ \ \forall t \in I$.

Abel's theorem is super helpful as we only need to evaluate the Wronskian / determinant over one point to conclude the linear dependence/independence of our solutions. (Note: some stuff about a fundamental set of solutions is talked about here but honestly I don't really care :/.)

Similarly to when we looked at real-valued solutions to differential equations, we can turn complex-valued solutions into real solutions:

Definition 7.2 (*Theorem 7.4.5*)

If $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is a solution to the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then solely the real part $\mathbf{u}$ and solely the imaginary part $\mathbf{v}$ are also solutions to the above equation.

(I don't see much use in doing the exercises here as they are just proofs about theorems from section 3.2 in matrix form. None are like super interesting.)

7.5 Constant Coefficients and Matrices

This subsection focuses on equations of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a $n \times n$ matrix of real-valued constants.

Assuming[4] $\mathbf{x} = \xi e^{rt}$, after substituting it into the above equation, we eventually derive the equation

$$(\mathbf{A} - r\mathbf{I})\xi = 0,$$

which means solutions to $\mathbf{x}$ are given pairs of eigenvalue-eigenvector combinations (r, ξ). When $\mathbf{A}$ is specifically a 2×2 matrix, if the eigenvalues of $\mathbf{A}$ have opposite signs, then the origin is a saddle point and an unstable equilibrium. If on the other hand, the eigenvalues of $\mathbf{A}$ have the same sign, then the origin is a **node** and $\mathbf{0}$ is a stable equlibrium if the eigenvalues are negative and unstable if the eigenvalues are positive.

Returning to the more general case of when $\mathbf{A}$ is a $n \times n$ matrix, the eigenvalues of $\mathbf{A}$ (r_1, r_2, $\ldots$, r_n) can either be

1. all real and different from one another,
2. some eigenvalues are complex conjugate pairs of each other, or
3. some eigenvalues are repeated.

The first case is easy to take care of; if all n eigenvalues are real and different, then their corresponding eigenvectors ($\xi^{(i)}$) will all be linearly independent and as such $\mathbf{x} = c_1\xi^{(1)}e^{r_1 t} + \cdots + c_n\xi^{(n)}e^{r_n t}$. Section 7.6 deals with case two, of when some eigenvalues are complex conjugate pairs of each other. Section 7.8 deals with the case of repeated eigenvalues.

Remember: To find eigenvalues, solve the characteristic polynomial $\det(\mathbf{A} - r\mathbf{I}_n) = 0$, and to get the corresponding eigenvectors, RREF $\mathbf{A} - r\mathbf{I}_n$ for a given r.

Exercise 7-9.
7. After finding our eigenvalues to be $r = \{4, -1, 1\}$ and the associated eigenvectors to be $\xi = \{[1\ 1\ 1]^T, [1\ 0\ -1]^T, [-1\ 2\ -1]^T\}$, we conclude that $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} e^{t}$ for arbitrary constants c_1, c_2, c_3.

8. This is a nice symmetric matrix with eigenvalues. Solving, we eventually find the characteristic polynomial to be $(\lambda - 8)(\lambda + 1)(\lambda + 1) = 0$ which means the corresponding eigenvaues are $\lambda = r = 8, -1$. Solving for eigenvectors and plugging them in, we find $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$.

9. $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. □

[4] Some stuff about a phase portrait/plane is talked about here although those tools are primarily used for visualization purposes.

Exercise 20.
20a: Note that this problem operates under the assumption A and $\xi^{(1)}$ are matrices with constant coefficients. For the sake of the argument, assume $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{v} \neq \mathbf{0}$. As such, since matrix multiplication is distributive, we can rearrange the equation and get $\mathbf{A}\xi^{(1)} = r_1\mathbf{I}\xi^{(1)} + \mathbf{v} = r_1\xi^{(1)} + \mathbf{v}$ as per properties from the identity matrix.
Returning back to our original equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, since the given solution for $\mathbf{x}$ holds for any coefficients c_1 and c_2, we let $c_1 = 1$ and $c_2 = 0$ and thus a particular solution to our differential equation is $\mathbf{x} = \xi^{(1)}e^{r_1 t}$. We plug this particular solution of our differential equation into our differential equation and derive

$$\left(\xi^{(1)}e^{r_1 t}\right)' = \mathbf{A}\left(\xi^{(1)}e^{r_1 t}\right) \rightarrow r_1 e^{r_1 t}\xi^{(1)} = e^{r_1 t}\mathbf{A}\xi^{(1)} \rightarrow r_1\xi^{(1)} = \mathbf{A}\xi^{(1)}.$$

Substituting in our other expression for the RHS, we find $r_1\xi^{(1)} = r_1\xi^{(1)} + \mathbf{v}$ which implies $\mathbf{v} = \mathbf{0}$, a contradiction! Thus, our original assumption that $\mathbf{v} \neq \mathbf{0}$ is false and $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0}$. (A symmetrical argument holds for proving why $(\mathbf{A} - r_2\mathbf{I})\xi^{(2)} = \mathbf{0}$.)

20b: $(\mathbf{A} - r_2\mathbf{I})\xi^{(1)} = \mathbf{A}\xi^{(1)} - r_2\xi(1) = r_1\xi^{(1)} - r_2\xi(1)$ since $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0} \rightarrow \mathbf{A}\xi^{(1)} = r_1\xi^{(1)}$.

20c: $(\mathbf{A}-r_2\mathbf{I})(c_1\xi^{(1)}+c_2\xi^{(2)}) = c_1(\mathbf{A}-r_2\mathbf{I})\xi^{(1)}+c_2(\mathbf{A}-r_2\mathbf{I})\xi^{(2)} = c_1(r_1-r_2)\xi^{(1)}$. However, $(\mathbf{A}-r_2\mathbf{I})(c_1\xi^{(1)}+c_2\xi^{(2)}) = (\mathbf{A} - r_2\mathbf{I})\mathbf{0} = \mathbf{0}$. Thus, $c_1(r_1 - r_2)\xi^{(1)} = \mathbf{0}$ which means either $c_1 = 0$, $r_1 = r_2$, or $\xi^{(1)} = \mathbf{0}$ (or some combination of the above).
Since we assume $r_1 \neq r_2$, the second statement must be false, and if $\xi^{(1)} = \mathbf{0}$, then the wronskian of our solution would be 0, contradicting our assumption that the general solution for $\mathbf{x}$ is a fundamental solution with respect to $\xi^{(1)}e^{r_1 t}$ and $\xi^{(2)}e^{r_2 t}$. Thus, this leads us to conclude that $c_1 = 0$ which is a contradiction, meaning our original supposition (that $\xi^{(1)}$ and $\xi^{(2)}$ are linearly dependent) is false. $\square$

Exercise 21.
21a: We can rewrite this second order linear differential as

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2 \end{cases} \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

21b: The eigenvalue equation is simply the determinant of $\mathbf{A} - \lambda\mathbf{I}$ which is

$$0 = -\lambda\left(-\frac{b}{a} - \lambda\right) - (1)\left(-\frac{c}{a}\right) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \rightarrow a\lambda^2 + b\lambda + c.$$

$\square$

Exercise 24.
24a. The general equation for eigenvalues of this matrix is $\lambda^2 + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\lambda + \left(\frac{R_1+R_2}{CLR_2}\right) = 0$. Plugging in the specific values given, we eventually find eigenvalues of -1 and -2 and the general solution corresponding is $\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}$. Specific values that make the initial conditions hold are $c_1 = \dfrac{3I_0 - V_0}{2}$ and $c_2 = \dfrac{V_0 - I_0}{2}$.

24b. Since both terms in the above equation have e^{-t} in them, as $t \to \infty$ $\begin{pmatrix} I \\ V \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since the decay of e^{-t} is too much for the constants in the equation to handle. $\square$

Exercise 25.
25a. $b^2 - 4ac > 0$ so $\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1+R_2}{CLR_2}\right) > 0$.
25b. Since all the coefficients in the characteristic polynomial are positive since R_1, R_2, C, and L cannot take on negative values (negative resistance?!), it is impossible for an eigenvalue λ to be greater than 0 as each term in the equation would thus be bigger than 0. As such, both I and V will eventually become 0 as t goes to ∞.
25c. If the eigenvalues are complex or repeated, it is possible that both I and/or V blow up to infinity (e.g. if you have negative resistance that adds energy to the circuit) or I and V settle into a stable equilibrium solution (see an LC clock circuit — no energy is being lost since there is no resistor in the circuit). $\square$

7.6 Complex Eigenvalues

Essentially, if the eigenvalues of $\mathbf{A}$ are complex[5] (in conjugate pairs), then the resulting graph of $\mathbf{x}$ will look like a spiral either convergering at the origin or diverging away from it (depending on whether the eigenvalues have positive or negative real part). If the real part is 0, then the graph will basically make a loop around the origin.

For the general linear differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $r_2 = \overline{r_1}$, then $\xi^{(2)} = \overline{\xi^{(1)}}$. Thus, a particular solution to $\mathbf{x}$ would be

$$\mathbf{x} = \mathbf{x}^{(1)} = \xi^{(1)} e^{r_1 t} \to (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} = e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) + ie^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t))$$

where we made the substitutions $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$ and $r_1 = \lambda + i\mu$. Thus, if we let $\mathbf{x}^{(1)}t = \mathbf{u}(t) + i\mathbf{v}(t)$ with $\mathbf{u}$ and $\mathbf{v}$ corresponding to the real and imaginary parts above, we have a particular solution for $\mathbf{x}$.
But since we want a real solution for $\mathbf{x}$, by that one weird theorem (Theorem 7.4.5) covered above, $\mathbf{u}$ and $\mathbf{v}$ are themselves individual solutions to $\mathbf{x}$ and we can write $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + \dots$ where the dots indicate the solutions stemming from the other roots of $\mathbf{x}$.

Exercise 5-6.
5. Our eigenvalues for this matrix are $\lambda = 1, 1 \pm 2i$. The eigenvector for $\lambda = 1$ is $\xi = [2\ \ -3\ \ 2]^T$, and the eigenvector for $\lambda = 1 + 2i$ is $\xi = [0\ 0\ 1]^T + i[0\ 1\ 0]^T$. Thus, our final solution is

$$\mathbf{x} = c_1 e^t \left(\begin{pmatrix}0\\0\\1\end{pmatrix}\cos(2t) - \begin{pmatrix}0\\1\\0\end{pmatrix}\sin(2t)\right) + c_2 e^t \left(\begin{pmatrix}0\\0\\1\end{pmatrix}\sin(2t) + \begin{pmatrix}0\\1\\0\end{pmatrix}\cos(2t)\right) + c_3\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t.$$

6. This one was hard (calculation-wise) and the answer is dicey. The eigenvalues are $\lambda = -2, -1 \pm i\sqrt{2}$ and the final solution is

$$\mathbf{x} = c_1\begin{pmatrix}2\\-2\\1\end{pmatrix}e^{-2t} + c_2 e^{-t}\begin{pmatrix}-\sqrt{2}\sin(\sqrt{2}t)\\ \cos(\sqrt{2}t)\\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t)\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}\sqrt{2}\cos(\sqrt{2}t)\\ \sin(\sqrt{2}t)\\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t)\end{pmatrix}.$$

$\square$

[5] All entries in $\mathbf{A}$ must be real as otherwise complex roots may not come in conjugate pairs.

Exercise 20.
20a. By Kirchoff's voltage law, across the triangle formed by R_1, C, and L, we find

$$L\frac{dI}{dt} + V + R_1 I = 0 \to \frac{dI}{dt} = -\frac{R_1}{L}I - \frac{1}{L}V$$

which is the first row of our matrix equation.
Summing the current at the point joined by C, R_2, and L, we find

$$\frac{V}{R_2} + C\frac{dV}{dt} = I \to \frac{dV}{dt} = \frac{1}{C}I - \frac{1}{CR_2}V.$$

Note that the reason why voltage across resistor R_2 is V is because summing the voltage loop over the rectangle C and R_2, we must have $V + (-V_{R_2}) = 0$ so $V_{R_2} = V$. Note that there is a negative sign over V_{R_2} as when tracing the loop across the resistor, our loop goes against the current of the circuit.
Thus, turning both equations into matrix form, we have

$$\begin{cases} \frac{dI}{dt} = -\frac{R_1}{L}I - \frac{1}{L}V \\ \frac{dV}{dt} = \frac{1}{C}I - \frac{1}{CR_2}V \end{cases} \to \begin{bmatrix} I' \\ V' \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}.$$

20b. The particular eigenvalues for the matrix $\mathbf{A}$ given in the problem are $\lambda = -\frac{1}{2} \pm i\frac{1}{2}$. As such, the corresponding eigenvectors are $\xi = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. As such, the final general solution we find (after plugging everything in) is:

$$\begin{bmatrix} I \\ V \end{bmatrix} = c_1 e^{-\frac{t}{2}} \left(\begin{bmatrix} -\sin\left(\frac{t}{2}\right) \\ -4\cos\left(\frac{t}{2}\right) \end{bmatrix} \right) + c_2 e^{-\frac{t}{2}} \left(\begin{bmatrix} \cos\left(\frac{t}{2}\right) \\ -4\sin\left(\frac{t}{2}\right) \end{bmatrix} \right).$$ [a]

20c. For the particular case given above, $c_1 = -\frac{3}{4}$ and $c_2 = 2$.
20d. Since the exponential in both terms of the equation are negative, as $t \to \infty$, both I and V tend towards 0 regardless of the initial conditions given. □

[a] Note: The solution in the back of the book uses the eigenvector $\xi = [1 \quad -4i]^T$.

Exercise 21.
21a. The characteristic polynomial of this matrix is $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC}$ from which it follows that the discriminant is $\frac{1}{R^2C^2} - \frac{4}{LC} = L - 4R^2C$.
21b. $\lambda = -1 \pm i$, and a specific eigenvector that's yielded is $\xi = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. As such, our final general solution is

$$\begin{bmatrix} I \\ V \end{bmatrix} = e^{-t} \left(c_1 \begin{bmatrix} \cos(t) - \sin(t) \\ -2\cos(t) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(t) - \cos(t) \\ -2\sin(t) \end{bmatrix} \right).$$ [a]

21c. $c_1 = -\frac{1}{2}$, $c_2 = -\frac{5}{2}$.
21d. Obviously not, as $t \to \infty$, $I \to 0$ and $V \to 0$ regardless of c_1 and c_2. □

[a] I think I'm right; the fact that $(1+i)(1-i) = 2$ is making my answer not match to the one in the back of the book. Honestly atp idfk.

These problems are genuinely ragebaiting me.

7.7 Fundamental Matrices

So it turns out learning/getting a glimpse of this section (7.7) is pretty helpful for understanding the rest of the content in the chapter.

7.7.1 Fundamental Matrices

Suppose $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ..., $\mathbf{x}^{(n)}$ are a set of fundamental solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ (their Wronskian is non-zero). Then, we define the matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)} \end{pmatrix} = \begin{pmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{pmatrix}$$

to be a **fundamental matrix** for the nonhomogenous linear system described above. Note that this fundamental matrix $\mathbf{\Psi}$ has an inverse as all columns of $\mathbf{\Psi}$ are linearly independent.
As such, to solve any initial value problem $\mathbf{x}(t_0) = \mathbf{x_0}$, we can write

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} \to \mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x_0} \to \mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x_0} \longrightarrow \mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x_0}$$

where $\mathbf{c}$ is a matrix of constants.

Note that we also define a special matrix $\mathbf{\Phi}$ with the property that

$$\mathbf{\Phi}(t_0) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \mathbf{I} \longleftrightarrow \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0).$$

7.7.2 Matrix Exponentiation

Since the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x} \to \mathbf{x} = \mathbf{\Phi}\mathbf{x}^0$ looks similar to the first-order differential equation $x' = ax \to x = x_0 e^{at}$, it seems tempting to exponentiate $\mathbf{A}$ and we can in fact do so by defining

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

with this definition also satisfying the property that $\frac{d}{dt} \exp \mathbf{A}t = \mathbf{A} \exp(\mathbf{A}t)$. Notably, it turns out that $\mathbf{\Phi} = \exp(\mathbf{A}t)$ as a result that they satisfy the same initial condition $\exp \mathbf{A}t|_{t=0} = \mathbf{\Phi}(0) = \mathbf{I}$.[6]

7.7.3 Diagonalizable Matrices

Solving simultaneous systems of equations is hard. What if we could solve each equation individually instead?
Suppose matrix $\mathbf{A}$ has n linearly eigenvectors $\xi^{(1)}$, $\xi^{(2)}$, ..., $\xi^{(n)}$ and associated eigenvalues λ_1, ..., λ_n. We define the matrix $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \dots & \xi_1^{(n)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \dots & \xi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \xi_n^{(2)} & \dots & \xi_n^{(n)} \end{pmatrix}$$

where $\xi_j^{(i)}$ represents the ith eigenvalue of $\mathbf{A}$ and j simply indexes the row of that eigenvalue. Since $\mathbf{A}\xi^{(k)} = \lambda_k \xi^{(k)} \; \forall k$,

$$\mathbf{AT} = \begin{pmatrix} \lambda_1\xi_1^{(1)} & \lambda_2\xi_1^{(2)} & \dots & \lambda_n\xi_1^{(n)} \\ \lambda_1\xi_2^{(1)} & \lambda_2\xi_2^{(2)} & \dots & \lambda_n\xi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\xi_n^{(1)} & \lambda_2\xi_n^{(2)} & \dots & \lambda_n\xi_n^{(n)} \end{pmatrix} = \mathbf{T} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

[6] Assuming we let $\mathbf{\Phi}(t_0) = \mathbf{\Phi}(0) = \mathbf{I}$.

where that last matrix of eigenvalues is denoted as $\mathbf{D}$. Thus, it follows that $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{D}$ and if $\mathbf{A}$ can be transformed into $\mathbf{D}$ like this, we say $\mathbf{A}$ is **diagonalizable** and **similar** to $\mathbf{D}$.

Returning back to our original system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $\mathbf{A}$ is diagonalizable, then we can define a new matrix $\mathbf{y}$ characterized by $\mathbf{x} = \mathbf{T}\mathbf{y}$ from which it follows $\mathbf{y}' = \mathbf{D}\mathbf{y}$, which is easily solvable since $\mathbf{D}$ is a linearly independent and mostly empty matrix.
Namely, $\mathbf{\Psi} = \mathbf{T}\exp(\mathbf{D}t)$.

7.8 Repeated Eigenvalues

So what happens if there is an eigenvalues of $\mathbf{A}$ that has a multiplicity $m > 1$? Long story short, we basically use the variation of parameters method with e^{rt} and te^{rt} to find our answer.

In more detail, assume that there is an eigenvalue $\lambda = \lambda_m$ such that the multiplicity of λ_m is m (e.g. $(\lambda_m - \lambda)^m$ is in the characteristic polynomial of $\mathbf{A} - \lambda\mathbf{I}$). In this case, one trivial solution is simply $\mathbf{x} = \xi e^{\lambda_m t}$. Assuming no other linearly independent eigenvectors can be found, to find the second solution, we let $\mathbf{x} = \alpha_1 t e^{\lambda_m t} + \alpha_2 e^{\lambda_m t}$, plug this particular solution into the differential equation, and simplify to find what α_1 and α_2 are. We then repeat this process (e.g. assume $\mathbf{x} = \alpha_1 \frac{t^n}{n!} e^{\lambda_m t} + \alpha_2 \frac{t^{n-1}}{(n-1)!} e^{\lambda_m t} + \cdots + \alpha_{n+1} e^{\lambda_m t}$) until we get m linearly independent solutions for $\mathbf{x}$. Note that your various solutions for $\mathbf{x}$ along the way will include previous derived solutions (e.g. $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \frac{t^2}{2} e^t + \begin{pmatrix} 100 \\ -32 \\ -101 \end{pmatrix} te^t + \begin{pmatrix} \pi \\ \zeta(3) \\ e^{e^\gamma} \end{pmatrix} e^t$).

If a matrix $\mathbf{A}$ cannot be diagonalized because of repeated eigenvalues leading to a shortage of eigenvectors, $\mathbf{A}$ can still be transformed into its **Jordan form** which is like an almost-diagonalized matrix.
To turn a matrix $\mathbf{A}$ into its jordan form, simply do $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. If we then use the substitution used in 7.7 ($\mathbf{x} = \mathbf{T}\mathbf{y}$), we get $\mathbf{y}' = \mathbf{J}\mathbf{y}$. Since the last row of $\mathbf{J}$ always contains one element, we can solve for the function y_n in the last row, then work our way upwards (solve for y_{n-1}, y_{n-2}, ...) to eventually find the whole matrix $\mathbf{y}$.

Exercise 4.

The characteristic equation is $(\lambda - 2)^2(\lambda + 1) = 0$ which leads to the eigenvectors $\xi = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ with the latter ξ coming from the eigenvalue $\lambda = 2$.
To find the extra solution then, we let $\mathbf{x} = \alpha t e^{2t} + \beta e^{2t}$ and find

$$2\beta e^{2t} + \alpha e^{2t} + 2\alpha t e^{2t} = \mathbf{A}(\alpha t e^{2t} + \beta e^{2t}).$$

Comparing coefficients, we come to the conclusion that $2\alpha = \mathbf{A}\alpha$ and $2\beta + \alpha = \mathbf{A}\beta$. The first equation ($(\mathbf{A} - 2I)\alpha = \mathbf{0}$) is already solved for us as that was the eigenvalue we found before, so we then solve $(\mathbf{A} - 2I)\beta = \alpha$, finding the general solution to be $\beta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - a \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

As such, our final solution is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - a \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

since the third term in the solution is already covered by our general solution. As such, the general solution for the system of equations is

$$\mathbf{x} = c_1 \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 e^{2t} \begin{bmatrix} 1 \\ t \\ 1 - t \end{bmatrix}.$$

□

Exercise 5.

Following the same process as in problem 4, we find the characteristic polynomial to be $(\lambda-2)(\lambda+1)^2=0$, and the eigenvectors to be $\begin{bmatrix}1\\1\\1\end{bmatrix}$ for $\lambda=2$ and $\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ to be the eigenvectors for $\lambda=-1$. Since the eigenvectors for the repeated eigenvalue we have found are linearly independent, we can proceed directly to the solution and conclude that

$$\mathbf{x}=c_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^{2t}+c_2\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{-t}+c_3\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-t}.$$

□

Exercise 6a-10a.

6a. $\mathbf{x}=2\begin{bmatrix}1\\1\end{bmatrix}e^{-3t}+4\begin{bmatrix}\frac{1}{4}+t\\t\end{bmatrix}e^{-3t}=\begin{bmatrix}3+4t\\2+4t\end{bmatrix}e^{-3t}$.

7a. $\mathbf{x}=-\begin{bmatrix}1\\1\end{bmatrix}e^{-t}-6\begin{bmatrix}-\frac{2}{3}+t\\t\end{bmatrix}e^{-t}=\begin{bmatrix}3+6t\\-1+6t\end{bmatrix}e^{-t}$.

8a. $\mathbf{x}=4\begin{bmatrix}-3\\1\end{bmatrix}-14\begin{bmatrix}-1-3t\\t\end{bmatrix}=\begin{bmatrix}2+42t\\4-14t\end{bmatrix}$.

9a. $\mathbf{x}=3\begin{bmatrix}0\\0\\1\end{bmatrix}e^{2t}+2\begin{bmatrix}0\\1\\-6\end{bmatrix}e^{t}+4\begin{bmatrix}-\frac{1}{4}\\t\\-\frac{21}{4}-6t\end{bmatrix}e^{t}=\begin{bmatrix}0\\0\\3\end{bmatrix}e^{2t}+\begin{bmatrix}-1\\2+4t\\-33-24t\end{bmatrix}e^{t}$.

10a. $\mathbf{x}=\frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}e^{-\frac{t}{2}}-\frac{5}{3}\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{-\frac{7t}{2}}+\frac{7}{3}\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-\frac{7t}{2}}$.

□

Exercise 14.

14a. Trivial; the discriminant is literally $L-4R^2C$ so the result follows.

14b. The repeated root is $\lambda=-\frac{1}{2}$, and the solution is

$$\begin{bmatrix}I\\V\end{bmatrix}=-\begin{bmatrix}1\\-2\end{bmatrix}e^{-\frac{t}{2}}+\begin{bmatrix}2+t\\-2t\end{bmatrix}e^{-\frac{t}{2}}=\begin{bmatrix}1+t\\2-2t\end{bmatrix}e^{-\frac{t}{2}}.$$

□

Exercise 15.

15a. $(\mathbf{A}-2\mathbf{I})((\mathbf{A}-2\mathbf{I})\eta)=\mathbf{0}\rightarrow(\mathbf{A}-2\mathbf{I})^2\eta=\mathbf{0}$.

15b. In this case, we can manually verify that $\begin{pmatrix}-1 & -1\\1 & 1\end{pmatrix}^2=\mathbf{0}$.

15c. Simple matrix multiplication shows $\xi=(1,-1)^T$.

15d. More matrix multiplication shows $\xi=(-1,1)^T$.

15e. As long as $k_1\neq-k_2$, ξ and η will be independent; $\xi=(-k_1-k_2,k_1+k_2)^T$. ξ in this case will be a multiple of $\xi^{(1)}$. □

Exercise 17a-d.
17a. The characteristic polynomial ends up being $-\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3 = 0$ which yields an eigenvalue 2 of multiplicity 3. Going eigenvector hunting reveals that indeed, $\xi = (0, 1, -1)^T$ is the only eigenvector.
17b. $\mathbf{x}^{(1)} = \xi e^{2t}$.

17c. I mean they just kinda do satisfy those equations. Solving for η and neglecting the part we already found, we find $\eta = (1, 1, 0)^T$, meaning our second solution is literally $\mathbf{x}^{(2)} = \xi te^{2t} + \eta e^{2t}$ for the ξ and η we have found.

17d. One particular solution for ζ is $\zeta = (-2, 3, 0)^T$, which means our final general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) e^{2t} + c_3 \left(\frac{t^2}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \right) e^{2t}.$$

□

7.9 Nonhomogenous Linear Systems

Finally, we return to the opening section and consider differential equation systems of the form

$$\mathbf{x}' = \mathbf{P}(t)x + \mathbf{g}(t).$$

Like in section 3.5, we can express all solutions $\mathbf{x}$ as $\mathbf{x}_c + \mathbf{x}_p$ where $\mathbf{x}_c$ is the solution to the homogenous differential equation $\mathbf{x}' = \mathbf{P}\mathbf{x}$ and $\mathbf{x}_p$ is a solution to the nonhomogenous system described above. So how do we find $\mathbf{x}_p$?

7.9.1 Diagonalization

In the differential system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$, if we assume $\mathbf{A}$ is diagonalizable, we can make the substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$ and find

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g} \;\rightarrow\; \mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$$

which is simply a set of n uncoupled (unrelated(?)) first order linear differential equations that can each be solved separately, and $\mathbf{x}$ can be recovered by left-multiplying $\mathbf{y}$ by $\mathbf{T}$.
It is also possible to solve for $\mathbf{x}$ even if $\mathbf{A}$ is not diagonalizable by reducing $\mathbf{A}$ to a jordan form $\mathbf{J}$ (I have no clue what this is) from which it's possible to solve for $\mathbf{J}$ from the last row to the top (as most rows have differential equations that are not totally uncoupled).

If that seems too hard, you can always try the method of:

7.9.2 Undetermined Coefficients (Hard ver).

(Note that this method is really only applicable when $\mathbf{P}$ is a bunch of constants and all terms in $\mathbf{g}$ look simple enough to be 'guessed'.)
Note that the difficulty level in this mode of undetermined coefficients is upped since generally, if there is a term in $\mathbf{g}$ of the form $\mathbf{u}e^{\lambda t}$, the solution must be assumed to be of the form $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$ for coefficient constant matrices $\mathbf{a}$ and $\mathbf{b}$.

7.9.3 Variation of Parameters

Assuming a (general) fundamental matrix $\mathbf{\Psi}$ has been found for the homogenous version of the differential equation $\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$, we seek solutions to the nonhomogenous system by replacing the constant vector $\mathbf{c}$ that would normally be multiplied to $\mathbf{\Psi}$ ($\mathbf{\Psi}\mathbf{c}$) by a vector of functions $\mathbf{u}$. Thus, letting $\mathbf{x} = \mathbf{\Psi}\mathbf{u}$, we have

$$\mathbf{x}' = \mathbf{\Psi}'\mathbf{u} + \mathbf{\Psi}\mathbf{u}' = \mathbf{P}\mathbf{\Psi}\mathbf{u} + \mathbf{g} \;\rightarrow\; \mathbf{\Psi}\mathbf{u}' = \mathbf{g}$$

since $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}$ since $\mathbf{\Psi}$ is after all, a solution for the homogenous version of $\mathbf{x}$.

Since the inverse of $\mathbf{\Psi}$ exists (since by definition the columns of $\mathbf{\Psi}$ ($\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$) are linearly independent),

$$\mathbf{u} = \int \mathbf{\Psi}^{-1}\mathbf{g}\, dt + \mathbf{c}$$

for arbitrary constant vector $\mathbf{c}$. Hopefully the integrals in the above equation can be evaluated because if not, a direct solution to the differential equation might not be possible 💀.[7]

7.9.4 Laplace Transform (Hard Version)

We can define a laplace transform over a vector as simply the vectors whos respective elemetns are the laplace transform of the elements in the original vector.
By an extension of the Laplace transform then,

$$\mathcal{L}\{\mathbf{x}'(t)\} = s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}(0)$$

which means we can transform the differential system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$ to $s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\{\mathbf{x}\} + \mathcal{L}\{\mathbf{g}\}$. For simplicity, if we are not solving an initial value problem, we can set $\mathbf{x}(0) = \mathbf{0}$ and do some messy calculations to find $\mathcal{L}\{\mathbf{x}\}$ from which we can do an inverse transform to find $\mathbf{x}$ (warning: messy).

Exercise 1.
Letting $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = \mathbf{A}$ for simplicity, we thus have to solve $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$. First, we find the general solution; chopping off the last term and solving a simple system with eigenvalues $\lambda = 1, -1$, we find our complementary solution to be $\mathbf{x_c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. As such, we now seek a particular solution.
Using the method of undetermined coefficients, since e^t is an eigenvalue root, we must assume that a particular solution $\mathbf{x_p}$ looks of the form $\mathbf{x_p} = \mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}$. Differentiating and plugging in both sides, we conclude that

$$\mathbf{a}te^t + (\mathbf{a} + \mathbf{b})e^t + \mathbf{c} = \mathbf{A}\mathbf{a}te^t + \mathbf{A}\mathbf{b}e^t + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$$

Matching the coefficients (te^t, e^t, t, 1) on both sides of the equation, we thus have the system

$$\begin{cases} \mathbf{A}\mathbf{a} = \mathbf{a} \\ \mathbf{A}\mathbf{b} = \mathbf{a} + \mathbf{b} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{A}\mathbf{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \mathbf{A}\mathbf{d} = \mathbf{c} \end{cases}$$

to solve. To take the simpler ones, the third equation simply tells us $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and similarly $\mathbf{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The first equation tells us $\mathbf{a}$ is of the form $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$, and plugging that in to the second equation and rearranging, we have $(\mathbf{A} - \mathbf{I})\mathbf{b} = \begin{pmatrix} \alpha - 1 \\ \alpha \end{pmatrix}$. Since a simplification of the left hand side reveals $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \mathbf{b} = \begin{pmatrix} \alpha - 1 \\ \alpha \end{pmatrix}$, it follows that $3(\alpha - 1) = \alpha$ or $\alpha = \frac{3}{2}$ and thus the general solution for $\mathbf{b}$ is $\begin{pmatrix} \frac{1}{2} + k \\ k \end{pmatrix}$. Taking $k = 0$ for simplicity, our final answer to the equation (which an online solver agrees with) is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

□

[7] Me after reading this section: ☠️.

Note: for all the exercises below (and above, frankly), there's almost no consensus (from me, wolfram alpha, the textbook, and an online solver) on what the right answer is. Tread carefully.

Exercise 2.
Again, we're going to let $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ for ease of writing. Solving for the complementary solution yields eigenvalues of $\lambda = \pm i$ and a particular solution of $\mathbf{x_c} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$.
Now for the fun part; the particular solution, and in this case, we'll use the method of variation of parameters. Recalling that our general matrix in this case is simply $\mathbf{\Psi} = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}$ (which has a non-zero Wronskian everywhere since $\det \mathbf{\Psi} = -1$), we can jump straight ahead to the conclusion and try to evaluate $\mathbf{u}$ by finding

$$\mathbf{u} = \int \mathbf{\Psi}^{-1} \mathbf{g} \; dt + \mathbf{c}.$$

Recalling that an inverse of a 2-by-2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, it turns out we can simplify our above expression and find

$$\mathbf{u} = \int \begin{pmatrix} -\sin t & 2\sin t + \cos t \\ \cos t & -2\cos t + \sin t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} \; dt + \mathbf{c} = \int \begin{pmatrix} \sin(2t) + 1 - \cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} \; dt + \mathbf{c}$$

Integrating each part, we find

$$\mathbf{u} = \begin{pmatrix} t - \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t) \\ \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) \end{pmatrix} + \mathbf{c}$$

and taking $\mathbf{c} = 0$ and using the equation $\mathbf{x_{(p)}} = \mathbf{\Psi u}$ (and many simplifications and tricky substitutions (notably $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$) and the cosine addition formula), we find that

$$\mathbf{x_p} = \begin{pmatrix} 2t\cos(t) - t\sin(t) - \frac{3}{2}\sin(t) - \frac{1}{2}\cos(t) \\ t\cos(t) - \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{pmatrix}$$

so our final general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} + \begin{pmatrix} 2t\cos(t) - t\sin(t) - \frac{3}{2}\sin(t) - \frac{1}{2}\cos(t) \\ t\cos(t) - \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{pmatrix}.$$

□

Exercise 3.

To solve for a solution $\mathbf{x}$, we will use the method of diagonalization. First, we let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ and note it is diagonalizable with eigenvalues $\lambda = 2, -3$. Thus, we make the substitution $\mathbf{x} = \mathbf{T}\mathbf{y} = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} \\ \xi_2^{(1)} & \xi_2^{(2)} \end{pmatrix} \mathbf{y} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \mathbf{y}$. As such, using this substitution and some properties, we derive

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{y} + \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}.$$

As such, letting $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, we can split the two differential equations and solve:

$$\mathbf{y}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{y} + \frac{1}{5} \begin{pmatrix} 4e^{-2t} - 2e^t \\ e^{-2t} + 2e^t \end{pmatrix} \rightarrow \begin{cases} y_1' = 2y_1 + \frac{1}{5}(4e^{-2t} - 2e^t) \\ y_2' = -3y_2 + \frac{1}{5}(e^{-2t} + 2e^t) \end{cases}.$$

Since each equation is a first-order thats solvable, we can solve and get $y_1 = -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_1 e^{2t}$ and $y_2 = \frac{1}{10}e^t + \frac{1}{5}e^{-2t} + c_2 e^{-3t}$. As such, we can solve for $\mathbf{x}$ given that we have $\mathbf{T}$ and $\mathbf{y}$:

$$\begin{aligned} \mathbf{x} = \mathbf{T}\mathbf{y} &= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_1 e^{2t} \\ \frac{1}{10}e^t + \frac{1}{5}e^{-2t} + c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_1 e^{2t} + \frac{1}{10}e^t + \frac{1}{5}e^{-2t} + c_2 e^{-3t} \\ -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_1 e^{2t} - \frac{2}{5}e^t - \frac{4}{5}e^{-2t} - 4c_2 e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}e^t + c_1 e^{2t} + c_2 e^{-3t} \\ -e^{-2t} + c_1 e^{2t} - 4c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}. \end{aligned}$$

□

Exercise 4.

A general solution to the homogenous version of this equation is $\mathbf{x_c} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t + \frac{1}{4} \\ 2t \end{pmatrix}$.

To find a particular solution, I will use the method of variation of parameters since $\mathbf{A}$ is not diagonalizable, $\mathbf{g}$ looks like it's made up of non-elementary functions, and a Laplace transform fails here.

First, finding our fundamental matrix $\mathbf{\Psi}$, recall that $\mathbf{\Psi} = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & t + \frac{1}{4} \\ 2 & 2t \end{pmatrix}$. Since the Wronksian (which is $\det \mathbf{\Psi}$) is non-zero everywhere, we can jump straight in and find $\mathbf{u}$ by evaluating

$$\mathbf{u} = \int \mathbf{\Psi}^{-1}\mathbf{g} \; dt + \mathbf{c}.$$

So, we simply evaluate the integral;

$$\rightarrow \int \frac{1}{-\frac{1}{2}} \begin{pmatrix} 2t & -t - \frac{1}{4} \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{pmatrix} \; dt + \mathbf{c} = -2 \int \begin{pmatrix} \frac{9}{4t^2} + \frac{1}{t} \\ -\frac{2}{t^3} - \frac{1}{t^2} \end{pmatrix} \; dt + \mathbf{c} = -2 \begin{pmatrix} -\frac{9}{4t} + \ln t \\ \frac{1}{t^2} + \frac{1}{t} \end{pmatrix} + \mathbf{c}.$$

Thus, $\mathbf{u}$ is whatever the garabge we got above. Then, since $\mathbf{x} = \mathbf{\Psi}\mathbf{u}$, we can simply do some matrix multiplication to find $\mathbf{x}$ (note: for simplicity, we let $\mathbf{c} = \mathbf{0}$):

$$\mathbf{x} = -2 \begin{pmatrix} 1 & t + \frac{1}{4} \\ 2 & 2t \end{pmatrix} \begin{pmatrix} -\frac{9}{4t} + \ln t \\ \frac{1}{t^2} + \frac{1}{t} \end{pmatrix} = \begin{pmatrix} \frac{2}{t} - 2\ln t - 2 - \frac{1}{2t^2} \\ \frac{5}{t} - 4\ln t + -4 \end{pmatrix}.$$

As such, our final solution for $\mathbf{x}$ is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t + \frac{1}{4} \\ 2t \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} \frac{1}{t} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \ln t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{2t^2}.$$

□

Exercise 5.

To use the Laplace transform on this system of differential equations, we first should find a complementary solution, which we can do easily to get $\mathbf{x_c} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$.

From here, to use the Laplace transform to solve this system, simply laplace transform the equation to get

$$s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\{\mathbf{x}\} + \mathcal{L}\{\mathbf{g}\} \ \rightarrow \ (s\mathbf{I} - \mathbf{A})\mathcal{L}\{\mathbf{x}\} = \begin{pmatrix} \frac{2}{s-1} \\ -\frac{1}{s-1} \end{pmatrix}$$

where the assumption we're making for our particular solution is that $\mathbf{x}(0) = \mathbf{0}$. From here, we can simply calculate $(s\mathbf{I} - \mathbf{A})$ and left-multiply its inverse to both sides of the equation then do an inverse laplace transform to solve for $\mathbf{x}$:

$$\rightarrow \begin{pmatrix} s-1 & -1 \\ -4 & s-1 \end{pmatrix} \mathcal{L}\{\mathbf{x}\} = \begin{pmatrix} \frac{2}{s-1} \\ -\frac{1}{s-1} \end{pmatrix} \ \rightarrow \ \mathcal{L}\{\mathbf{x}\} = \left(\frac{1}{(s-1)^2 - 4} \begin{pmatrix} s-1 & 1 \\ 4 & s-1 \end{pmatrix} \right) \begin{pmatrix} \frac{2}{s-1} \\ -\frac{1}{s-1} \end{pmatrix}$$

$$\rightarrow \mathcal{L}\{\mathbf{x}\} = \begin{pmatrix} \frac{2}{(s+1)(s-3)} - \frac{1}{(s-1)(s+1)(s-3)} \\ \frac{8}{(s-1)(s+1)(s-3)} - \frac{1}{(s+1)(s-3)} \end{pmatrix}.$$

Leveraging partial fractions (namely: $\frac{1}{(s+1)(s-3)} = \frac{1}{4(s-3)} - \frac{1}{4(s+1)}$ and $\frac{1}{(s-1)(s+1)(s-3)} = \frac{1}{8(s+1)} + \frac{1}{8(s-3)} - \frac{1}{4(s-1)}$) and the simplicity that the inverse Laplace transform of $\frac{1}{s-a}$ is e^{at}, we quickly find that a solution for $\mathbf{x}$ is

$$\mathbf{x_p} = \frac{1}{8} \begin{pmatrix} 3e^{3t} - 5e^{-t} + 2e^{t} \\ 6e^{3t} + 10e^{-t} - 16e^{t} \end{pmatrix}$$

so a general solution would be

$$\mathbf{x} = \frac{1}{4} \begin{pmatrix} 1 \\ -8 \end{pmatrix} e^{t} + c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

since the first two terms in the particular solution we found (that has the specific property that $\mathbf{x_p}(0) = \mathbf{0}$) can be eliminated by modifying c_1 and c_2. $\square$

Exercise 6.

A general solution is $\mathbf{x_c} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. Using the method of undetermined coefficients, we assume $\mathbf{x}$ is of the form $\alpha te^t + \beta e^t$. As such, we get the equation

$$\alpha e^t + \beta e^t + \alpha te^t = \mathbf{A}\alpha te^t + \alpha\beta e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

which when comparing coefficients gets us

$$\begin{cases} \mathbf{A}\alpha = \alpha \rightarrow \alpha = \begin{pmatrix} a \\ a \end{pmatrix} \\ \mathbf{A}\beta = \alpha + \beta + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{cases}.$$

Simplifying that second equation further, we have $(\mathbf{A} - \mathbf{I})\beta = \alpha + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \beta = \begin{pmatrix} a-1 \\ a+1 \end{pmatrix}$ leading us to find $a = 2$ and the simplest form for β is $\beta = (1, 0)^T$. As such, our final solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t}.$$

$\square$

Exercise 7.

A general solution to the homogenous version of this differential equation is $\mathbf{x_c} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-4t}$. Using the method of undetermined coefficients and assuming $\mathbf{x} = \alpha t e^{-t} + \beta e^{-t}$, we can once again set up equations similar to the ones above and find

$$\begin{cases} \mathbf{A}\alpha = -\alpha \\ \mathbf{A}\beta = \alpha - \beta + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{cases}.$$

We can solve the first one to find that α must be of the form $\begin{pmatrix} a \\ a\sqrt{2} \end{pmatrix}$, and a simplification like above for the second equation yields $\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \beta = \begin{pmatrix} a-1 \\ a\sqrt{2}+1 \end{pmatrix}$ leading us to find $a = \frac{1-\sqrt{2}}{3}$ and a simple form for β, namely $\beta = \left(\frac{1}{3}, -\frac{1}{3}\right)^T$. Combining, we thus find a solution for $\mathbf{x}$ is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-4t} + \frac{1}{3} \begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix} t e^{-t} + \frac{1}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.^{a}$$

□

[a] The book gives some silly/crazy answer but I'm pretty sure my answer is right; their answer is derived from a different β value.

(I skipped 8 since I don't want to spend more time in hell. Also the solution I have for 8 is super sketchy.)

Exercise 9.

9a. Recall that $\Psi(t) = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix}$. Since a general solution for the homogenous linear system is $\mathbf{x} = c_1 e^{-t/2} \begin{pmatrix} \cos(t/2) \\ 4\sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t/2) \\ -4\cos(t/2) \end{pmatrix}$, we can simply substitute these values in and find

$$\Psi = e^{-t/2} \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ 4\sin(t/2) & -4\cos(t/2) \end{pmatrix}.$$

9b. Since we want an initial condition of $\mathbf{x}(0) = \mathbf{0}$ to be satisfied, this seems like the perfect opportunity to pull out the Laplace transform!

Namely,

$$s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\{\mathbf{x}\} + \mathcal{L}\{\mathbf{g}\} \rightarrow (s\mathbf{I} - \mathbf{A})\mathcal{L}\{\mathbf{x}\} = \begin{pmatrix} \frac{1}{2s+1} \\ 0 \end{pmatrix} \rightarrow \mathcal{L}\{\mathbf{x}\} = \frac{1}{(s+\frac{1}{2})^2 + \frac{1}{4}} \begin{pmatrix} s+\frac{1}{2} & -\frac{1}{8} \\ 2 & s+\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2s+1} \\ 0 \end{pmatrix}$$

$$= \frac{1}{(s+\frac{1}{2})^2 + \frac{1}{4}} \begin{pmatrix} \frac{1}{2} \\ \frac{2}{2s+1} \end{pmatrix} = \left(\frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{1}{2}\right)^2}, \ \frac{4}{s+\frac{1}{2}} - 4\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \left(\frac{1}{2}\right)^2} \right)^T$$

$$\longrightarrow \mathbf{x} = \left(e^{-t/2}\sin(t/2), \ 4e^{-t/2} - 4e^{-t/2}\cos(t/2) \right)^T$$

so our particular solution that solves the problem and satisfies the given initial condition is

$$\mathbf{x} = e^{-\frac{t}{2}} \begin{pmatrix} \sin\left(\frac{t}{2}\right) \\ 4 - 4\cos\left(\frac{t}{2}\right) \end{pmatrix}.$$

□

8 Numerical Methods

TBD.

9 Nonlinear Differential Equations and Stability

This chapter deals with qualitative information about a differential equation and stability/instability of given solutions.

9.1 The Phase Plane: Linear Systems

When considering the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x},$$

there are special equilibrium solutions to be aware about: namely, **critical points**, or when $\mathbf{A}\mathbf{x} = \mathbf{0}$. If we assume $\det \mathbf{A} \neq 0$, then only $\mathbf{x} = \mathbf{0}$ is the only critical point of the system.

For linear systems, we can analyze the system and its trajectories by the eigenvalues of $\mathbf{A}$; namely, if we assume $\mathbf{A}$ is a 2×2 matrix and $\mathbf{A}$ has eigenvalues r_1, r_2, then

- if $r_1 \neq r_2$ and r_1 and r_2 have the same sign, then all trajectories will approach the origin and the origin (the critical point) will either be a **nodal source** (e.g. trajectories go away from the critical point) or a **nodal sink** (trajectories end up going towards the critical point.)
- if r_1 and r_2 have differing signs (wlog $r_1 > 0 > r_2$) then as $t \to \infty$, all trajectories will converge towards the eigenvector associated with r_1. As $t \to -\infty$ however, since $e^{-\infty} = 0$, all trajectories will converge towards the eigenvector associated with r_2. As such, the critical point is known as a **saddle point** as no trajectories pass through the critical point (see textbook Page 391).
- if $r_1 = r_2$ and two *indepedent* eigenvectors can be found for $\mathbf{A}$, then all trajectories look like a line through the critical point (in this case the origin) and the critical point is called a **proper node** or **star point**.
- if conversely $r_1 = r_2$ and only one eigenvector can be found, then the critical point is called an **improper node**[8]
- If eigenvalues are complex with some real part, trajectories will look like a spiral going either towards/away the origin in which case the origin is called a spiral sink/spiral source, depending on the real part.
- If the eigenvalues are purely imaginary, trajectories will infinitely loop on themselves (since there is no real part, trajectories do not 'decay') and the origin/critical point is called a **center** of the system. For linear systems, these trajectories look like ellipses around the origin.

Essentially, all solutions either go to infinity, go to $\mathbf{0}$, or go in a spiral.

9.2 Autonomous Systems and Stability

In this section we will be concerned with Autonomous systems of two functions of the form

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y).$$

with initial condition $x(t_0) = x_0$, $y(t_0) = y_0$. This system can also be written in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(t_0) = \mathbf{x}^0$$

where $\mathbf{x} = (x,y)^T = (x(t), y(t))^T$, $\mathbf{f}(\mathbf{x}) = (F(x,y), G(x,y))^T$, and $\mathbf{x}^0 = (x_0, y_0)^T$.
For simplicity and so theorems hold, we will assume F and G are continuous and their partial derivatives are also continuous.[9]

[8] tbh idrk why; the graph just looks super weird

[9] Rigorously speaking, their partial derivatives only need to be continuous over some domain D of the xy-plane. But since most functions we work with are very simple, it might as well be (almost) the whole xy-plane over which the partial derivatives are continuous.

Definition 9.1 (***Autonomous***)

A differential equation system is said to be **autonomous** if the systems do not depend on time. In particular, the system given above ($\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$) is autonomous, and so is the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ as long as all terms in $\mathbf{A}$ do not involve the independent variable t.

The distinction between autonomous and nonautonomous systems is important as the condition of autonomy guarantees that there is only one trajectory crossing through the point (x_0, y_0) regardless of time.

9.2.1 Stability and Instability

For autonomous systems of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

Definition 9.2 (***Critical Points***)

Critical points are defined as points to the above system where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Since $\mathbf{x}' = \mathbf{0}$, critical points must be constant solutions to the system.

Definition 9.3 (***Stability***)

A critical point of the solution $\mathbf{x}^\circ$ is said to be **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every solution $\mathbf{x}$ which at $t = 0$ satisfies

$$||\mathbf{x}(0) - \mathbf{x}^\circ|| < \delta$$

exists for all $t > 0$ and satisfies

$$||\mathbf{x}(t) - \mathbf{x}^\circ|| < \varepsilon$$

for all $t \geq 0$.

Essentially, this definition codifies the notion that a given solution $\mathbf{x}^*$ should stay bounded within the critical point. Note however that this definition of stability does not require $\mathbf{x}^*$ to converge to $\mathbf{x}^\circ$; instead, it merely requires that $\mathbf{x}^*$ not leave an open disk[10] of radius ε centered at $\mathbf{x}^\circ$ for all $t \geq 0$.

Any critical points for which the condition of stability doesn't hold are said to be **unstable**.

The following definition thus distinguishes between stability and asymptotic stability:

Definition 9.4 (***Asymptotic Stability***)

A critical point $\mathbf{x}^\circ$ is said to be **asymptotically stable** if it is stable and there exists a $\delta_0 > 0$ such that if a solution $\mathbf{x} = \mathbf{x}(t)$ satisfies

$$||\mathbf{x}(0) - \mathbf{x}^\circ|| < \delta_0,$$

$$\text{then } \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^\circ.$$

In english, if a trajectory starts "sufficiently close" to $\mathbf{x}^\circ$ (within a δ_0), then it must eventually approach $\mathbf{x}^\circ$ as $t \to \infty$.

Definition 9.5 (***Basin of Attraction***)

For a two-dimensional (potentially non-linear) autonomous system with at least one asymptotically critically point, we define the **basin of attraction** for a critical points to be the set of all points P such that a trajectory passing through P eventually converges to said critical point as $t \to \infty$.
If there is a boundary to a **basin of attraction**, that trajectory which bounds the basin is called a **separatrix** as it seperates the trajectories that converge and the trajectories that don't.

If we're lucky, we can determine trajectories of a two-dimensional autonomous system by solving just a first-order differential equation. Namely, since $F(x, y)$ and $G(x, y)$ don't depend on t, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x, y)}{F(x, y)}$$

which is a first-order differential equation. In general, the differential equation arising from the quotient $\frac{G(x,y)}{F(x,y)}$ may not be solvable.

[10] If $\mathbf{x}$ is n-dimensional, then the statement should be amended to read "…not leave an open $n-1$-sphere of radius …".

Exercise 14a-20a.

For this exercise, I recommend using this or this online plotter to plot some nice looking solutions.

14a. Following the equation above, we have $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} = \frac{8x}{2y} \to H(x,y) = y^2 - 4x^2 = c$. Solutions to this system generally look like (one-directional) hyperbolas.

15a. $\frac{dy}{dx} = \frac{-8x}{2y} \to H(x,y) = y^2 + 4x^2 = c$. As the equation suggests, solutions to this system look like an ellipse.

16a. $\frac{dy}{dx} = \frac{2x+y}{y}$. Using the substitution $y = vx$ and $dy = vdx + xdv$, we thus have

$$\frac{vdx + xdv}{dx} = v + \frac{dv}{dx}x = \frac{2+v}{v} \to \frac{dv}{dx}x = \frac{2+v-v^2}{v}$$

which with partial fraction decomposition and a bunch of tedious integration simplification reveals $H(x,y) = (x+y)(y-2x)^2 = c$.

17a. $\frac{dy}{dx} = \frac{x+y}{x-y}$. Using the substitution $y = vx$ again, we thus have $H(x,y) = \arctan\left(\frac{y}{x}\right) - \ln\left(\sqrt{\frac{y^2}{x^2}+1}\right) - \ln x = \arctan\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} = c$.

18a. While this equation looks super complicated, if you rearrange it into the form $(2xy^2 - 6xy) + (2x^2y - 3x^2 - 4y)y' = 0$, you quickly find the differential equation is exact so $H = x^2y^2 - 3x^2y - 4y^2 = c$.

19a. $\frac{dy}{dx} = \frac{-\sin x}{y} \to \frac{y^2}{2} - \cos(x) = c$.

20a. $\frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{24} = c$. □

9.3 Locally Linear Systems

The following table/theorem (theorem 9.3.1 in the book) recaps the stability properties of the origin for the two dimensional linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Definition 9.6 (*Stability of* 0)

Let the eigenvalues for $\mathbf{A}$ be r_1 and r_2. Then, the critical point $\mathbf{x} = \mathbf{0}$ is

1. asymptotically stable if r_1 and r_2 have negative real part,
2. stable if r_1 and r_2 have 0 real part (e.g. r_1 and r_2 are pure imaginary eigenvalues),
3. unstable if either r_1 and r_2 have any sort of positive real part.

From the table we can conclude that small pertrubations in the roots r_1 and r_2 only really matter when r_1 and r_2 are pure imaginary eigenvalues, as any addition of a real part (either positive or negative) will cause the system to spiral inwards towards $\mathbf{0}$ or outwards to infinity. Thus, while $\mathbf{0}$ is stable if r_1 and r_2 have 0 real part, this stability is itself potentially unstable.

9.3.1 Linear Approximations

Definition 9.7 (*Isolated critical point*)

We say that a critical point $\mathbf{x}^\circ$ is an **isolated critical point** of the system if there is some disk around $\mathbf{x}^\circ$ with radius $r > 0$ such that there exists no other critical poitns in that disk.

Considering the linearization of trajectories around the origin ($\mathbf{x} = \mathbf{0}$) for the non-linear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$, we first assume $\mathbf{0}$ is an isolated critical point of the system. We also assume that around the critical point (in this case $\mathbf{x} = \mathbf{0}$) that $\mathbf{g}$ is small, or in rigorous terms,

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{|\mathbf{g}|}{|\mathbf{x}|} = 0$$

assuming[11] $\mathbf{g}$ has continuous first partial derivatives. If the above condition is satisfied, the system we described above can then be called a **locally linear system**.

Anyways, here's a cool theorem we can use to determine local linearity:

Definition 9.8 (*Theorem 9.3.2 (p. 410)*)

The system described by

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$$

is locally linear in the neighborhood of a critical point $\mathbf{x}^\circ$ whenever F and G have continuous partial derivatives up to order two (e.g. F and G are twice differentiable).

If the above condition holds, then the nonlinear system near $\mathbf{x}^\circ = (x^\circ, y^\circ)$ can be approximated by the linear system

$$\frac{d}{dt}\begin{pmatrix} x - x^\circ \\ y - y^\circ \end{pmatrix} = \begin{pmatrix} F_x(x^\circ, y^\circ) & F_y(x^\circ, y^\circ) \\ G_x(x^\circ, y^\circ) & G_y(x^\circ, y^\circ) \end{pmatrix} \begin{pmatrix} x - x^\circ \\ y - y^\circ \end{pmatrix} \rightarrow \frac{d\mathbf{u}}{dt} = \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x}^\circ)\,\mathbf{u}$$

where $\mathbf{u} = \begin{pmatrix} x - x^\circ \\ y - y^\circ \end{pmatrix}$.

The general coefficient matrix in the above equation

$$\mathbf{J} = \mathbf{J}[F,G](x,y) = \begin{pmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{pmatrix}$$

is called the **Jacobian**[12] of F and G with repsect to x and y. For the linear approximation system above, we need to assume $\det \mathbf{J}(\mathbf{x}^\circ) \neq 0$ so that $\mathbf{x}^\circ$ is an isolated critical point in our linear approximation system.

Anyways, to relate the properties of stability of linear and locally linear systems, here's a theorem and a table from the textbook:

Theorem 9.3.3

Let r_1 and r_2 be the eigenvalues of the linear system (1), $\mathbf{x}' = \mathbf{A}\mathbf{x}$, corresponding to the locally linear system (4), $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. Then the type and stability of the critical point (0, 0) of the linear system (1) and the locally linear system (4) are shown in Table 9.3.1.

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
Eigenvalues	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	*PN or IN*	Unstable	*N or SpP*	Unstable
$r_1 = r_2 < 0$	*PN or IN*	Asymptotically stable	*N or SpP*	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\lambda = 0$	*C*	Stable	*C or SpP*	Indeterminate

Key: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

[11] It is also possible to convert this limit into polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$, $|\mathbf{x}| = r$) to make this limit easier to evaluate.

[12] Recognize this from Calc 3?

or in summary, except for two special cases ($r_1 = r_2$, $\lambda = 0$), the non-linear terms of the nonlinear system do not affect the stability of the system determined by the linear systems.
If every trajectory approaches the critical point at the origin, then the critical point $\mathbf{0}$ is said to be **globally asymptotically stable**.

/// The textbook then goes into more detail about the stability of a damped pendulum around some critical points (Page 413-415). ///

Exercise 1-3.
1. We verify $(0,0)$ is a critical point by noting that $(0,0)$ satsifies $x - y^2 = x - 2y + x^2 = 0$.
To find the locally linear version of the system, since both $F = x - y^2$ and $G = x - 2y + x^2$ have continuous second partial derivatives, we can invoke theorem 9.3.2 which tells us that our non-linear system can be approximated by the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which is verifiably linear. This linear system has two real, distinct eigenvalues $\lambda = 1$ and $\lambda = -2$ so by Table 9.3.1, both the linear and non-linear system around $(0,0)$ are unstable as in both cases, the origin is a saddle point.

2. $(0,0)$ can be verified to be a critical point, and it should be noted that it is not the only critical point of the system (see $(-2,\pi)$, $(0,2\pi)$, etc). Since the partial derivatives of both F and G up to order two are continuous, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

so $\lambda = \pm i$. As such, for the linear system, the origin is a stable center. For the nonlinear system however, the origin could be either a center or a spiral point so its stability is unknown.

3. $(0,0)$ is verifiably a critical point, and the corresponding locally linear matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which yields the eigenvalue $\lambda = 1$ with multiplicity two. Thus, the origin in both the locally linear and nonlinear systems is unstable, with the origin being an improper node[a] in the locally linear system and either a spiral point or a node in the nonlinear system. □

[a]Since the system only has one eigenvector, namely $(0,1)^T$.

Exercise 4abc.
4a. Critical points to the equations occur when $\frac{dx}{dt} = 0 = \frac{dy}{dt}$. Examining the first equation, $\frac{dx}{dt}$ is 0 when $x = y$ or $x = -2$. The second equation shows $\frac{dy}{dx} = 0$ when $x = -y$ or $x = 4$. Testing out each of the 4 combination of critical equations between the points, we find that there are 3 critical points, namely $(0,0)$, $(-2,2)$, and $(4,4)$.

4b. As a straightforward application of Theorem 9.3.2, for each critical point (x°, y°) found above, the locally linear system is of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y^\circ - 2x^\circ - 2 & x^\circ + 2 \\ 4 - y^\circ - 2x^\circ & 4 - x^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 6 & 6 \end{pmatrix}, \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}.$$

4c. For the system with critical point $(0,0)$, its eigenvalues are $\lambda = 1 \pm \sqrt{17}$, which means that in the nonlinear system, the origin is an unstable saddle point.
For the system with critical point $(-2,2)$, its eigenvalues are $\lambda = 3 \pm \sqrt{33}$ which means in the nonlinear system, it is also an unstable saddle point.
For the system with critical point $(4,4)$, its eigenvalues are $\lambda = -3 \pm i\sqrt{39}$ so in the nonlinear system $(4,4)$ is an asymptotically stable spiral point. □

(For the exercises below, this website gives some marvelous slope fields/phase diagrams.)

Exercise {5, 6, 7}abc.
5a. The critical points are $(0,0)$, $(1,0)$, $\left(0,\frac{3}{2}\right)$, and $(-1,2)$. The first three solutions can be found by solving $\frac{dx}{dt}-\frac{dy}{dt}=0$, and the last solution can be found by solving $x-x^2=xy=3y-2y^2$ as this can be broken up into two equations $1-x=y$ and $x=3-2y$.
5b. From Theorem 9.3.2, the linear matrices are of the form $\begin{pmatrix}1-2x-y & -x\\ -y & 3-x-4y\end{pmatrix}$ so the matrices for the critical points $(0,0)$, $(1,0)$, $\left(0,\frac{3}{2}\right)$, and $(-1,2)$ are

$$\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix},\begin{pmatrix}-1 & -1\\ 0 & 2\end{pmatrix},\begin{pmatrix}-\frac{1}{2} & 0\\ -\frac{3}{2} & -3\end{pmatrix},\begin{pmatrix}1 & 1\\ -2 & -4\end{pmatrix}$$

respectively.
5c. The first matrix has eigenvalues $\lambda=\{1,3\}$ so $(0,0)$ is an unstable node in the nonlinear system.
The second matrix has eigenvalues $\lambda=\{-1,2\}$ which means $(1,0)$ is an unstable saddle point.
The third matrix has eigenvalues $\lambda=\{-\frac{1}{2},-3\}$ so $\left(0,\frac{3}{2}\right)$ is an asymptotically stable node.
The fourth matrix has eigenvalues $\lambda=\frac{1}{2}(3\pm\sqrt{17})$ so $(-1,2)$ is an unstable saddle point.

6. $(-1,1)\to\begin{pmatrix}0 & -1\\ -2 & -2\end{pmatrix}\to\lambda=-1\pm\sqrt{3}$ so $(-1,1)$ is an unstable saddle.
$(1,1)\to\begin{pmatrix}0 & -1\\ 2 & -2\end{pmatrix}\to\lambda=-1\pm i$ so $(1,1)$ is an asymptotically stable spiral point.

7. $(0,0)\to\begin{pmatrix}-2 & 4\\ 2 & 4\end{pmatrix}\to\lambda=1\pm\sqrt{17}$ so $(0,0)$ is an unstable saddle.
$(2,1)\to\begin{pmatrix}-3 & 6\\ -4 & 0\end{pmatrix}\to\lambda=\frac{1}{2}(-3\pm i\sqrt{87})$ so $(2,1)$ is an asymptotically stable spiral point.
$(2,-2)\to\begin{pmatrix}0 & -6\\ 2 & 0\end{pmatrix}\to\lambda=\pm i\sqrt{12}$ so $(2,-2)$ has indeterminate stability and is either a center or a spiral point.
$(4,-2)\to\begin{pmatrix}0 & -8\\ -2 & -4\end{pmatrix}\to\lambda=2\pm\sqrt{20}$ so $(4,-2)$ is an unstable saddle point. □

Exercise 23.
(Note: (26) is just (25) with $\alpha=-1$.)
23a. Trivially, $\frac{dx}{dt}=0+\alpha(0)(0^2+0^2)=\frac{dy}{dt}=0$ so $(0,0)$ is a critical point of the system given in (25). To show that $(0,0)$ is a center, we note that the system can be rewritten as

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}\alpha(x^2+y^2) & 1\\ -1 & \alpha(x^2+y^2)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\to(\lambda-\alpha(x^2+y^2))^2+1=0.$$

Since $\alpha(x^2+y^2)$ is 0 at the point $(0,0)$, $\lambda=\pm i$ which is pure imaginary which means that $(0,0)$ is a center around the corresponding linear system.

23b. (25) is twice differentiably continuous so (25) is locally linear.
23c. $\dfrac{dr}{dt}=\dfrac{xx'+yy'}{r}=\dfrac{xy+\alpha x^2r^2-xy+\alpha y^2r^2}{r}=\dfrac{\alpha r^2(x^2+y^2)}{r}=\alpha r^3.$

23d. Solving the given differential equation, we have $\alpha t+C=-\frac{1}{2r^2}$ so $r=\sqrt{\dfrac{1}{C-2\alpha t}}$. If $\alpha<0$, then as $t\to\infty$, that bottom fraction will go towards infinity so r will go towards 0.

23e. If $r(0)=r_0$, then $C=\frac{1}{r_0^2}$ so $r(t)=\sqrt{\dfrac{1}{\frac{1}{r_0^2}-2\alpha t}}$. Clearly, as the bottom of that denominator becomes 0 (when $t\to 1/2\alpha r_0^2$), r becomes unbounded which translates into $\sqrt{x^2+y^2}$ becoming unbounded meaning $\frac{dx}{dt}$ and $\frac{dy}{dt}$ will grow astronmically as $t\to 1/2\alpha r_0^2$ when $\alpha>0$. □

9.4 Competing Species

A simple model for competing species in the same environment is

$$\begin{cases} x' = x(\alpha_1 - \sigma_1 x - \beta_1 y), \\ y' = y(\alpha_2 - \sigma_2 x - \beta_2 y) \end{cases}$$

where x and y are the populations, α_1 and α_2 are their growth rates, $\frac{\alpha_1}{\sigma_1}$ and $\frac{\alpha_2}{\sigma_2}$ their carrying capacities, and β_1 and β_2 the degree to which the other population interferes with the population of a given species. Since this system is meant to model competing populations, we mandate $x \geq 0$ and $y \geq 0$ at all times.

In general, there are 4 cases for the resulting equilibrium points in the two species depending on the **nullclines** of the system.

Definition 9.9 (*Nullcline*)

A **nullcline** of a system is a line where either $x' = 0$ or $y' = 0$. In the case of competing population dynamics, the nullclines are $\alpha_1 - \sigma_1 x - \beta_1 y = 0$ and $\alpha_2 - \sigma_2 x - \beta_2 y = 0$.

Essentially, the 4 cases boil down to two cases: in the first there exist x_0 and y_0 such that $\alpha_1 - \sigma_1 x_0 - \beta_1 y_9 = \alpha_2 - \sigma_2 x_0 - \beta_2 y_0 = 0$, or where no such point exists. In the former case, coexistence of the species is possible (although (x_0, y_0) can either be an asymptotic node or a saddle point), and in the latter, coexistence is unfortunately impossible :/.

Exercise 6.
The set of equation has 4 critical points; 3 when either $x = 0$ or $y = 0$ (or both), and the 4th is when both x and y are not equal to 0. Examining that fourth equation further, since we want $x \neq 0$ and $y \neq 0$, we thus have

$$\begin{cases} \frac{dx}{dt} = 0 \to \varepsilon_1 - \sigma_1 x - \alpha_1 y = 0 \\ \frac{dy}{dt} = 0 \to \varepsilon_2 - \sigma_2 y - \alpha_2 x = 0 \end{cases} \longrightarrow \begin{cases} \sigma_1 x + \alpha_1 y = \varepsilon_1 \\ \alpha_2 x + \sigma_2 y = \varepsilon_2 \end{cases}$$

which is an easily solvable linear system. Solving for x and y, we find

$$x = \frac{\varepsilon_2 \alpha_1 - \varepsilon_1 \sigma_2}{\alpha_1 \alpha_2 - \sigma_1 \sigma_2}, \quad y = \frac{\varepsilon_1 \alpha_2 - \varepsilon_2 \sigma_1}{\alpha_1 \alpha_2 - \sigma_1 \sigma_2}$$

which is problematic for our linear system. Since we are given $\varepsilon_2 \sigma_1 > \varepsilon_1 \alpha_2$ and $\varepsilon_2 \alpha_1 > \varepsilon_1 \sigma_2$, the numerators in the fractions for x and y have opposing signs which implies either $x < 0$ or $y < 0$. Since negative populations don't make sense, we disregard this fourth critical point and since no other critical points exist, we conclude there is no stable equilibirum where both $x > 0$ and $y > 0$ meaning as $t \to \infty$, either $x \to 0$, $y \to 0$, or both.

A similar conclusion applies if both $\varepsilon_1 \alpha_2 > \varepsilon_2 \sigma_1$ and $\varepsilon_1 \sigma_2 > \varepsilon_2 \alpha_1$ hold (problem 6b). □

Exercise 7.
7a. Trivial; from each equation, factor out either ε_1 or ε_2.
7b. Since the equilibrium point (with $x > 0$ and $y > 0$) exists at

$$\left(\frac{B - R\gamma_1}{1 - \gamma_1 \gamma_2 - 1}, \frac{R - B\gamma_2}{1 - \gamma_1 \gamma_2} \right)^{a},$$

reducing B has the effect of decreasing the x-coordinate and increasing the y-coordinate of the equlibrium thus in effect pushing the equilibrium closer to the y-axis. As such, it follows that it is indeed possible to reduce the population of bluegill to a level in which they will die out by simply moving reducing B to where $B = R\gamma_1$ at which point the equilibrium would be at the point $(0, R)$. □

[a] Since apparently $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$, $1 > \frac{\alpha_1}{\sigma_1} \frac{\alpha_2}{\sigma_2} = \gamma_1 \gamma_2$.

9.5 Predator-Prey Equations

In 9.4, we modelled populations x and y asfighting over some common natural resource (such as food). Here, we will examine situations where one population preys on the other (e.g. foxes and rabbits). Note that x will always represent the population of the species being preyed on, and y will always represent the population of the predatorial species.

In general, under some assumptions[13], we set up the equations

$$\frac{dx}{dt} = x(a - \alpha y),$$
$$\frac{dx}{dt} = y(-c - \gamma x)$$

where $a, c, \alpha, \gamma > 0$.[14] The physical interpretation of these constants is that a and c are the growth rate of the prey and the death rate of the predator, and α and γ are a measure of the effect of the interaction between the species.

A treatise examining solutions of the equations above are given in detail in the textbook, but the main takeaways are that:

- The origin $((0,0))$ will always be a saddle point.
- The "center" of the system, $\left(\frac{c}{\gamma}, \frac{a}{\alpha}\right)$, is a critical point upon which all predator-prey trajectories oscillate/circle around.
- In particular, x and y can be solved for:

  $$x = \frac{c}{\gamma}(1 + K\cos(\sqrt{ac}t + \phi)),\ y = \frac{a}{\alpha}\left(1 + \sqrt{\frac{c}{a}}K\sin(\sqrt{ac}t + \phi)\right)$$

  for constants K, ϕ.

[13]See page 428.

[14]These equations are also known as the **Lotka - Volterra equations**.