Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 Introduction

aka chapter 1

1.1 Introduction for the Introduction

Definition 1.1 (*Differential Equations*)

Equations containing derivatives.

Definition 1.2 (*Slope Field/Direction Field*)

A buncha line segments on the plane that represent the "motion" of a diff-eq.

Direction Fields are good for studying differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

(Page 6 – How to construct a diff-eq mathematical model from a real-world situation.)
(7) Newton: Differential equations come in one of these 3 forms:

1. $\frac{dy}{dx} = f(x)$,
2. $\frac{dy}{dx} = f(y)$,
3. $\frac{dy}{dx} = f(x, y)$.

Exercise 11-16.
1.1.5 corresponds with **j**.
1.1.6 corresponds with **c**.
1.1.7 corresponds with **g**.
1.1.8 corresponds with **b**.
1.1.9 corresponds with **h**.
1.1.10 corresponds with **e**. □

Exercise 17.
(a)

$$\frac{dC}{dt} = [\text{chemicals/hour going in}] - [\text{out}] = 0.01 \cdot 300 - 300 \cdot \frac{C}{1000000}$$

where C is the number of gallons of said chemical in the pond and t is time measured in hours.

(b) After a very long time, 10000 gallons will be in the pond; this limiting amount is independent of starting conditions.

(c) Since concentration $= \frac{\text{Amount}}{\text{Volume}}$, $C = \text{volume} \cdot c = c \cdot 10^6$ where c stands for concentration. As such,

$$\frac{dc}{dt} = \frac{1}{10^6}\frac{dC}{dt} = \frac{3}{10^6} - \frac{3(c \cdot 10^6)}{10^4 \cdot 10^6}$$

So in final, $\boxed{\frac{dc}{dt} = \frac{3}{10^6} - \frac{3c}{10^4}}$. □

1.2 Introduction to Solutions

(11) - Finding the general solutions to diff-eqs of the form $\frac{dy}{dt} = ay - b$ $(a \neq 0)$;

$$\frac{dy}{dt} = ay - b \Longrightarrow y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

⟨14 - "Further Remarks on Mathematical Modeling" - essentially, the underlying assumptions we make may or may not be wrong. ⟩

Exercise 1a.
$\dfrac{dy}{dt} = -y + 5 \to \dfrac{1}{5-y} dy = dt.$

So, $\ln(5 - y(t)) = t + C$. With initial condition $y(0) = k$, we get that $\ln(5-k) = C$, so our solution becomes $y(t) = 5 - e^{t+\ln(5-k)} = 5 - (5-k)e^t$. (Note that $(5-k)$ is constant.) □

Exercise 9.
9a: Since $F = ma$, $F = m\frac{dv}{dt}$. Since drag acts inversely to velocity (object falling faster has more air resistance), we should expect $\frac{dv}{dt}$ to be negative; thus, $\frac{dv}{dt} = -\frac{F}{10}$. Knowing that F is proportional to the square of the velocity, we know that $F = av^2 - b$ for constants a, b.

Now, we plug in some known values. At $v = 0$, we expect $\frac{dv}{dt} = -\frac{(-b)}{10} = -9.8$ (gravity) so $b = -98$. At $v = 49$, we reach limiting velocity which implies $\frac{dv}{dt} = 0$ so $\frac{a(49^2)-98}{10} = 0$ so $a = \frac{2}{49}$. Thus, in final, we get our differential equation as

$$\frac{dv}{dt} = \frac{2}{49 \cdot 10}v^2 - \frac{98}{10}$$

which can be re-arranged for the desired equation.

9b:

$$\frac{dv}{dt} = \frac{1}{245}\left(49^2 - v^2\right) \to 245\frac{1}{49^2 - v^2}\, dv = dt \to 245 \int \frac{1}{49^2 - v^2}\, dv = t$$

Doing a trig sub ($v = 49\sin\theta$, $dv = 49\cos\theta\ d\theta$), $\dfrac{dv}{49^2 - v^2}$ becomes $\dfrac{49\cos\theta\ d\theta}{49^2 - 49\sin^2\theta}$ so our integral ends up turning into

$$\longrightarrow 245 \int \frac{d\theta}{49\cos\theta} = t \Longrightarrow 5\left(\frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right) = t + C.$$

Thus,

$$t + C = \frac{5}{2}\ln\frac{1 + v/49}{1 - v/49}.$$

Plugging in our initial condition $v(0) = 0$, we get that $C = 0$. Thus,

$$\ln\left(\frac{1 + v/49}{1 - v/49}\right) = \frac{2t}{5} \text{ so } 49 + v = (49 - v)(e^{2t/5}).$$

Simplifying, (by expanding and putting all the vs on one side of the equation) we find our final answer to be

$$v(t) = 49 \cdot \frac{e^{2t/5} - 1}{e^{2t/5} + 1} = 49\tanh(t).$$

□

Exercise 13.
(a)

$$\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{RC} = \frac{VC - Q}{RC} \Longrightarrow RC\int \frac{dQ}{VC - Q} = t + C$$

Integrating, we get that $t + C_1 = -RC\ln(VC - Q)$. Plugging in our initial condition $Q(0) = 0$, we get that $C_1 = -RC\ln(VC)$. Thus, we can substitute and simplify as follows:

$$RC\ln(VC - Q) = RC\ln(VC) - t \to \ln(VC - Q) - \ln(VC) = -\frac{t}{RC}$$

$$\to VC - Q = VCe^{-t/RC}$$

so $Q(t) = VC(1 - e^{-t/RC})$.

(b) After a very long time ($t \sim \infty$), $Q \sim VC$ so $Q_L = VC$.

(c) From Kirchoff's voltage rule, $R\frac{dQ}{dt} + \frac{Q}{C} = 0 \longrightarrow -\frac{Q}{C} = R\frac{dQ}{dt}$. Thus, $t + C_1 = -RC\ln(Q)$. Evaluating in our initial condition, we get that $C_1 = -RC\ln(Q_L) + t_1$. As a result,

$$-(t - t_1) = RC(\ln(Q) - \ln(Q_L))$$

so

$$Q = Q_L e^{-\frac{t-t_1}{RC}}.$$

□

1.3 Classification of Diffy Qs

Definition 1.3 (*Ordinary Differential Equation*)

An Ordinary Diffy Q (ODE) is an equation where the unknown function depends on a single independent variable.
E.g. (LRC Circuit)

$$L^2\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Definition 1.4 (*Partial Differential Equation*)

A Partial Differential Eq (PDE) is when the unknown function depends on several independent variables.
E.g. (Wave Equation)

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

(17) - If you have n unknown functions in a system of differential equations, then you gotta have at least n diffy qs to solve that system completely.

Definition 1.5 (*Order*)

The **order** of a differential equation is the highest derivative that appears in the differential equation. Thus you can have a *first-order* or *second-order* or *seventh-order* diffy q.
E.g.: $\alpha\frac{d^3x}{dk^3} + \beta\frac{d^2x}{dk^2} + \frac{\alpha}{\beta}x = \gamma$ is a third-order (ordinary differential) equation (when α, β, and γ are constants and x is a function of k).

Generally then, a differential equation of order n can be represented by the generic $F(t, x(t), x'(t), \ldots, x^{(n)}t) = 0$ for some function $x(t)$. Replacing $y = x(t)$, we get that a general nth order differential equation is of the form

$$F(t, y, y', \ldots, y^{(n)}) = 0.^{1}$$

[1] (18) Note: We assume it is always possible to solve for the highest derivative – e.g. we can rearrange to get to the form of

Definition 1.6 (***Linearity***)

A differential equation is said to be **linear** if $F(t, y, y', \dots, y^{(n)}) = 0$ is a linear function of t, y, y', $\dots$, $y^{(n)}$. As such, the general linear diffy q is of the form $0 = c(t) + a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)}$.

Definition 1.7 (***Linearization***)

Linearization is the process of approximating a non-linear diffy q by a linear one. Example given in the textbook is of approximating the motion of an oscillating pendulum.

(19-20) - Questions of solvability and uniqueness for general differential equations.

Exercise 1-4.
1. Order is 2, and the differential equation is linear.
2. Order is 2, and the differential equation is NOT linear (because of the term $(1 + y^2)\frac{d^2y}{dt^2}$).
3. Order is 4, and the differential equation is linear.
4. Order is 2, and the differential equation is non-linear. □

Exercise 10.
(I'm only doing this one because it looks fun)
We shall verify that $y = e^{t^2}\left(1 + \int_0^t e^{-s^2}\, ds\right)$ is a solution to the differential equation $y' - 2ty = 1$.
First, we substitute y into our equation.

$$\left[e^{t^2} + e^{t^2}\int_0^t e^{-s^2}\, ds\right]' = 1 + 2t \cdot e^{t^2}\left(1 + \int_0^t e^{-s^2}\, ds\right)$$

Next, we differentiate that left side and simplify the right.

$$\longrightarrow 2te^{t^2} + \left(2te^{t^2}\right)\left(\int_0^t e^{-s^2}\, ds\right) + \left(e^{t^2}\right)\left(e^{-t^2}\right) = 1 + 2te^{t^2} + 2te^{t^2}\left(\int_0^t e^{-k^2}\, dk\right)$$

Finally, we cancel terms and arrive at the equation

$$e^{t^2} \cdot e^{-t^2} = 1,$$

which is trivially true. Thus, we are done. □

Exercise 11-13.
Since $y = e^{rt}$, $y^{(n)} = r^n e^{rt}$. Thus, in each of problems 11-13, we're basically just solving a polynomial. To illustrate, consider problem 12:

$$y'' + y' - 6y = 0 \Longrightarrow r^2e^{rt} + re^{rt} - 6e^{rt} = e^{rt}\left(r^2 + r - 6\right) = 0,$$

which is (almost) isomorphic to solving the system $x^2 + x - 6 = 0$. Thus, we yield the solutions $r = 2, 3$ and maybe even $r = -\infty$ (which would make e^{rt} be 0).

Similar solutions follow for 11 and 13. □

Exercise 16-18.
16: 2nd order linear partial differential eq.
17: 4th order linear PDE.
18: 2nd order non-linear PDE. □

$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$.

2 First-Order Diffy Qs

aka chapter 2
for chapter 2, all diffy qs will be first order.

2.1 Linear ODEs: Method of Integrating Factors

If $\frac{dy}{dt} = f(t, y)$ and f is linear (w.r.t y), then we can rewrite it in the following form (called the **first-order linear differential equation**):

$$\frac{dy}{dt} + p(t)y = g(t) \iff P(t)\frac{dy}{dt} + Q(t)y = G(t) \text{ (page 24)}$$

Definition 2.1 (*Integrating Factor*)

A **integrating factor** $\mu(t)$ is a function such that when a diffy q is multiplied by it, the equation is then immediately integratable (discovered by Leibniz). (page 25)

Exercise - Pauls Online Notes, Problem 4 (modified).
Find the general solution to the ODE

$$t\frac{dy}{dt} + 2y = t^2 - t + 1.$$

This diffy q looks hard. To start, we add on an integrating factor $\alpha(t)$ to the equation to get

$$t\alpha(t)\frac{dy}{dt} + 2\alpha(t)y = \alpha(t)(t^2 - t + 1).$$

From here, consider what happens when you take the derivative of $(t \cdot y \cdot \alpha(t))$:[a]

$$\frac{d}{dt}\left[t \cdot y \cdot \alpha(t)\right] = y\alpha(t) + t\alpha(t)\frac{dy}{dt} + ty\alpha'(t) = t\alpha(t)\frac{dy}{dt} + y(\alpha(t) + t\alpha'(t)).$$

For this equation to match the left hand side of the equation above, we then must have that $t\alpha(t) = t\alpha(t)$ and $2\alpha(t) = \alpha(t) + t\alpha'(t) \to \alpha(t) = t\alpha'(t)$. From that last equation, I recognized that the function $\alpha(t) = t$ works!
And from there, after plugging things in and integrating, I ended up with my final answer that the general solution to the given ODE is

$$y(t) = \boxed{\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}}.$$

□

[a] This is not the *actual* way to do it –Paul's math notes first divides everything by t so they only have to consider the derivative of $y\alpha(t)$.

So essentially, the process of solving diffy qs of the form $P(t)\frac{dy}{dt} + Q(t)y = G(t)$ is to first divide by $P(t)$, then find an integrating factor that "matches up" both sides of the equation.
Mathematically:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \to \frac{dy}{dt} + \frac{Q(t)}{P(t)}y = \frac{G(t)}{P(t)} \to \kappa(t)\frac{dy}{dt} + \kappa(t)\frac{Q(t)}{P(t)}y = \kappa(t)\frac{G(t)}{P(t)}$$

Since $(y\kappa(t))' = \kappa(t) \cdot y' + \kappa'(t)y$, comparing terms on the LHS, we get that we just need to find some $\kappa(t)$ such that $\kappa'(t) = \kappa(t)\frac{Q(t)}{P(t)}$. If that nasty fraction $\left(\frac{Q(t)}{P(t)}\right)$ is some constant or basic polynomial, the equation is *probably* solvable.
So assuming that some suitable $\kappa(t)$ is found, we then just kinda evaluate everything from there.

$$\longrightarrow \int \frac{d}{dt}\left[y\kappa(t)\right] = \int \kappa(t)\frac{G(t)}{P(t)} \implies y(t) = \frac{C + \int \kappa(t)\frac{G(t)}{P(t)}}{\kappa(t)}.$$

It's quite messy when written out.
(27) For equations of the form $\frac{dy}{dt} + ay = g(t)$, the right integrating factor is $\mu(t) = e^{at}$. This can be rederived pretty easily (probably).

If you want to integrate but the messy thing doesn't simplify, that's fine; put the bounds of your integral to be from some arbitrary t_0 to t, preferably in a way such that if an initial condition $y(y_0) = c_0$, $t_0 = y_0$. In this way, your integral will collapse on itself when evaluated at $y = y_0$ and any other value of your function will be computed by that constant given in the problem plus the accumulated gain/loss from the function as it goes to your desired x/y value.

Exercise 1c-8c.
(I'm just going to document my answers here.)
1c: $y(t) = e^{-2t} + \frac{1}{3}\left(t - \frac{1}{3}\right) + \frac{C}{e^{3t}}$.
2c: $y(t) = \left(\frac{1}{3}t^3 + C\right)e^{2t}$.
3c: $y(t) = \frac{t^2}{2e^t} + 1 + \frac{C}{e^t}$.
4c: $y(t) = 1.5\sin(2t) + \frac{0.75}{t}\cos(2t) + \frac{C}{t}$.
5c: $y(t) = -3e^t + Ce^{2t}$.
6c: $y(t) = -te^{-t} + Ct$.
7c: $y(t) = \sin(2t) - 2\cos(2t) + \frac{C}{e^t}$. Warning: This integral is hard but is very doable.
8c: $y(t) = 3t^2 - 12t + 24 + \frac{C}{2e^{t/2}}$. $\square$

Exercise 9-12.
(More answer exercise documentation. Both the general form and the specific solution to each problem will be given.)
9: $y(t) = 2te^{2t} - 2e^{2t} + Ce^t$. The specific case when $y(0) = 1$ is given by $C = 3$.
10: $y(t) = \frac{t^2/2 + C}{e^{2t}}$. The specific solution when $y(1) = 0$ is given by $C = -\frac{1}{2}$.
11: $y(t) = \frac{\sin t + C}{t^2}$, with $C = 0$.
12: $y(t) = \frac{t-1}{t} + \frac{C}{te^t}$, $C = 2$ for this particular case. $\square$

Exercise 18.
Given the simplicity of the right hand side, we can fake solve the equation for $y(t)$; namely, from intuition, if $y(t) = \alpha t + \beta$, then $y'(t) = \alpha$ (a constant) and we can probably find values α, β that make such a solution possible.
In fact we do; $y(t) = -\frac{3}{4}t + \frac{21}{8} + Ce^{-2t/3}$, where that last term was derived from realizing that if we actually integrated this properly, our integrating factor $\mu(t)$ would be $e^{2t/3}$.
Anyways, things get a little dicey from here. Let's call the point where $y(t)$ touches (but doesn't cross) the t-axis as t_0. Then, we know that $y'(t_0) = 0$ (y must be at a local max/min as otherwise y would cross the t-axis) and $y(t_0) = 0$. From here, we can rearrange our equations as follows:

$$y'(t_0) = 0 \to 0 = -\frac{3}{4} - \frac{2}{3}Ce^{-2t_0/3} \to -\frac{9}{8} = Ce^{-2t_0/3} \text{ and}$$

$$y(t_0) = 0 \longrightarrow \frac{3}{4}t_0 - \frac{21}{8} = Ce^{-2t_0/3}$$

and we can match the LHSs of both equations to get $\frac{3}{4}t_0 - \frac{21}{8} = -\frac{9}{8}$ and find that $t_0 = 2$.
From here, we can simply plug this value of t_0 into our equations and solve for C:

$$y'(t_0) = 0 \to y'(2) = 0 \to C = -\frac{9}{8}e^{4/3}$$

Thus, $y_0 = y(0) = \frac{21}{8} + C = \frac{21}{8} - \frac{9}{8}e^{4/3} \approx \boxed{-1.64}$. □

Exercise 20.
(Note: this problem and the last problem have caused me some amount of pain because I keep misreading the problem and not sticking to the end.)
If you want $y' - y = 1 + 3\sin t$ to remain finite as $t \to \infty$, then when you get that $y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + Ce^t$, it should be pretty clear that $C = 0$. As such, $y_0 = y(0) = -1 - \frac{3}{2}(1 + 0) = \boxed{-\frac{5}{2}}$.
When doing this problem, don't doubt yourself :). □

Exercise 28 - Variation of Parameters.
(a). If $g(t) = 0\ \forall t$, then effectively $g(t) = 0$. As such, we are simply solving $\frac{dy}{dt} + p(t)y = 0$ which can be done by separating variables:

$$\frac{dy}{dt} = -p(t) \cdot y \to \frac{1}{y}dy = -p(t)dt \to \ln(y) = \int -p(t)dt + C_0 \text{ so } y(t) = C_1 \exp\left(-\int p(t)dt\right)$$

where $C_1 = e^{C_0}$. Replace C_1 with A to get the expression shown in the textbook.

(b). To show $A(t)$ must satisfy (51), we simply substitute everything in and cancel the messy equation.

$$y' + p(t)y = g(t) \Longrightarrow \left[A(t)\exp\left(-\int p(t)dt\right)\right]' + A(t)\exp\left(-\int p(t)dt\right)p(t) = g(t)$$

$$\longrightarrow \exp\left(-\int p(t)dt\right)\left([A'(t) + A(t)(-p(t))] + A(t)p(t)\right) = g(t) \text{ so } A'(t) = g(t) \cdot \exp\left(\int p(t)dt\right)$$

which is exactly the equation given by (51).

(c). This part is lowkey quite simple. Picking up from (b), we simply slap an integral sign in front of the massive equation that we derived for $A'(t)$, and after replacing $A(t)$ with the appropriate integral in an integral, it is equivalent to (33) up to a constant as the $\mu(t)$ in (33) is really the big scary integral we've been dealing with, $\int p(t)dt$. □

2.2 Separable Differential Equations

A general first-order differential equation can be written as $\dfrac{dy}{dx} = f(x,y)$ which can be rearranged to become $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$. When M is a function solely of x and N is a function solely of y, then we can rewrite the diffy q as

$$M(x)dx + N(y)dy = 0$$

which we call a **separable** equation. (A definition of a separable differential equation is a differential equation that can be written as the form above.)
To solve these equations, it's just 'simple'/'intuitive' integration. I'm not sure what exactly to write so.....
Let $A'(x) = a(x)$ and $B'(y) = b(y)$. Then,

$$a(x) + b(y)\frac{dy}{dx} = 0$$

simlifies down to

$$A(x) + B(y) = c$$

for an arbitrary constant c.

Exercise 1-4.
(Solutions only.)
1: $\dfrac{y^2}{2} = \dfrac{x^3}{3} + C$ or $y = \sqrt{\dfrac{2}{3}x^3 + C}$.
2: $y = \dfrac{1}{C - \cos x}$.
3: $\dfrac{1}{2}\tan(2y) = \dfrac{x}{2} + \dfrac{\sin(2x)}{4} + C$. Alternatively, the textbook gives the clever solution of $y = \dfrac{\pi}{2}$ (a constant) and similar solutions as $y' = 0$ and the RHS evaluates to 0. Very tricky (but nice) stuff.
4: $\ln|x| + C = \arcsin y$. Didn't fully get those inverse trig integrals right away :/.
(The rest (5-8) look trivial as you just move terms to either side of the equation and integrate.) $\square$

Exercise 17.
Literally just separate and integrate.

$$\Rightarrow \int 3y^2 - 6y \; dy = \int 1 + 3x^2 \; dx + C \to y^3 - 3y^2 = x + x^3 + C$$

Plugging in our initial condition of $y(0) = 1$ (e.g. the values $x = 0$ and $y = 1$), we get that $C = 1 - 3 = -2$.
To determine the interval in which the solution is valid, we simply look at when the denominator of y' is 0 —$3y^2 - 6y = 0 \to y = 0, 2$ which correspond to the x values of 0, $\frac{-1+\sqrt{3}}{2}$, $\frac{-1-\sqrt{3}}{2}$, and $x = -1$ ($y(-1) = 2$ and y of any of the other x values is equal to 0).
I'm not sure how really to proceed from here but I think the way to actually do it is to recognize that $y(-1) = 2$, $y(1) = 0$, which given our initial condition $y(0) = 1$ means that our function y is trapped between -1 and 1 (y is not exactly a "function" at these specific points so our domain then becomes $(-1, 1)$).
It's not a great solution :/. $\square$

Exercise 19.

$$\frac{dy}{dx} = 2y^2 + xy^2 \to \int \frac{1}{y^2}\, dy = \int 2 + x\; dx \to y = \frac{1}{2x + \frac{x^2}{2} + C}.$$

Plugging our initial condition $y(0) = 1$ nets $C = -1$.
To find the minimum value of y, we simply find the derivative y' and set it to 0;

$$y' = \frac{1}{\left(2x + \frac{x^2}{2} - 1\right)} \cdot (2 + x).$$

Setting y' to 0, the only solution we get is $x = -2$ so the minimum value of our function y is $y(-2) = -\frac{1}{-4+2-1} = \boxed{\frac{1}{3}}$.
While it is true that $y(3) < y(-2)$, note that the function y is discontinuous and plotting the graph on Desmos, we see that $y(3)$ is not 'on' the particular branch of the solution we're focused on (namely, the piece of the function where $y(0) = 1$). □

Exercise 24.
Solve

$$\frac{dQ}{dt} = r(a + bQ),\ Q(0) = Q_0.$$

(Note: for some reason this exercise does not have a solution in the back of the book.)

We can rearrange to get the equation

$$\int \frac{1}{a + bQ} dQ = \int r dt \longrightarrow \frac{1}{b} \ln|a + bQ| = rt + C$$

and evidently, $C = \frac{\ln a + bQ_0}{b}$.
Solving for Q, we eventually find that

$$Q(t) = \frac{e^{rbt}(a + bQ_0) - a}{b}$$

so when $t \to \infty$, assuming that all constants are positive, $Q \to \infty$. □

Definition 2.2 (*Homogeneous Equation*)

(Note: This definition of a homogeneous diffy q will be different from other ones presented in this book.)
A homogeneous equation (for our purposes for now) is a first-order differential equation $\frac{dy}{dx} = f(x, y)$ that can be expressed as a function of the the expression $\frac{y}{x}$. In other words, $\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$ for some function g.

In particular, a differential equation is a homogeneous equation, then it is separable 'by a change of the dependent variable' by making the substitution $y = xv(x)$ (note that y is a function of x implicitly) and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. See the exercise below for more information.

Exercise 25.
25(a): Trivial. Multiply the numerator and denominator of the fraction by $\frac{1}{x}$ and simplify.

25(b):

$$y = xv \to \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Multiplying both sides by dx, our equation now becomes

$$dy = v\ dx + x\ dv,$$

which we can directly substitute in for dy.
As such, to conclude,

$$\frac{dy}{dx} = \frac{v\ dx + x\ dv}{dx} = \boxed{v + x\frac{dv}{dx}}.$$

25(c): Trivial. Literally just make the substitutions.

25(d + e): The key to this problem is partial fractions.

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v} \Rightarrow \ln|x| = \int \frac{1 - v}{v^2 - 4}\ dv + C$$

From here, using partial fractions, we find that the integrand is equal to $\frac{-0.25}{v-2} + \frac{-0.75}{v+2}$ which can be easily integrated to get our final answer.
As such, after integrating and substituting $v = \frac{y}{x}$ back into our equation, we get that

$$\ln|x| + C = -0.75 \ln\left|\frac{y}{x} + 2\right| - 0.25 \ln\left|\frac{y}{x} - 2\right|.$$

(I'm lowkey not sure how to get rid of those absolute value signs from the $\frac{1}{x}$ integral so they'll stay for now.) $\square$

2.3 Modeling with First-Order Differential Equations

To make a good mathematical model:

1. Construct the Model (in accordance with reality).
2. Analyze the model. Simplify functions/variables if needed, but always make sure the simplifications to the model make physical sense.
3. Compare the model with outside observations/experiments/data. Also take a look at long run behavior.

// Various examples of mathematical models are shown at this part in the textbook. //

Exercise 1.
The concentration of dye in the tank can be represented by $\dfrac{dA}{dt}$ where $A(t)$ is the concentration of dye in the tank as a function of time (t) in minutes. Specifically,

$$\frac{dA}{dt} = \text{rate in} - \text{rate out} = 0 - \frac{A}{200} \cdot 2$$

which is empirically explained as follows:
Since the water coming in is clean, rinse water, there is no dye in that water so the rate at which the dye is coming in the tank is 0. On the flip side, the rate at which the dye is going out is equal to the concentration of the dye ($\frac{\text{Amount}}{\text{Volume}} = \frac{A}{200L}$) times the amount of the water/dye mixture that flows out ($\frac{2L}{\text{min}}$).

Solving for $A(t)$, we find that

$$A(t) = e^{-\frac{t}{100}} \quad (C = 0).$$

As such, to find when the concentration of the dye reaches 1% of its original value, we simply set $A(t) = 0.01$ and solve, yielding $\boxed{t \approxeq 460.517}$(minutes) (thanks Wolfram Alpha). $\square$

Exercise 4.
4(a): More clearly, Torricelli's principle states that the outflow velocity v at the outlet is equal to the velocity of a particle in freefall from the height h **when it reaches the height of the outlet**.
From mechanics, the velocity of a particle in free fall is $v(t) = gt$ since $v_0 = 0$ in our case. Since we don't have t, we must resort to finding it by relating the height fallen (h) and time: $h = \frac{1}{2}gt^2$ so $t = \frac{2h}{g}$. As such, combining this equation with the previous, we find that $\boxed{v = \sqrt{2gh}}$ as desired.

4(b):

$$\begin{aligned}\frac{dh}{dt} &= \text{How much the height of the water drops with respect to (w.r.t) time}\\ &= \frac{\text{How much water goes out w.r.t time}}{\text{Area of the cross section of the tank at height h}} = \frac{-\alpha a v}{A(h)},\end{aligned}$$

with that last "jump" in reasoning deriving from the fact that how much water goes out of the tank is equal to the velocity of the water (v) times the area of the outflow hole (a). And apparently since water flow isn't ideal, there's a constant (α) tacked on to the ensure the equation matches with real world phenomena.

4(c): In this specific scenario, $A(h) = \pi r^2$, $r = 1$, $a = \pi(0.1)^2$, and $h(0) = 3$. Plugging in these values, we get that $\pi\dfrac{dh}{dt} = -\alpha\dfrac{\pi}{100}\sqrt{2gh}$ so $\dfrac{1}{\sqrt{h}}\frac{dh}{dt} = -\alpha\frac{\sqrt{2g}}{100}$. Integrating and plugging in our initial condition from here, we find that $h(t) = (\sqrt(3) - \frac{\alpha\sqrt{2g}}{200}t)^2$ which means that when $h(t) = 0$, $\boxed{t = \dfrac{200\sqrt{3}}{\alpha\sqrt{2g}} \approx 130.41}$ (seconds). $\square$

Exercise 9.
9(a): We are given that $\dfrac{dQ}{dt} = -rQ$ (so $Q(t) = Q_0 e^{-rt}$) and that the halflife of carbon-14 is 5730 years. In other words, $Q(5730) = 0.5Q_0$ (half of the carbon-14 has decayed). As such, r can be solved for and is approximately $\boxed{0.000120968}$.

9(b): Done. See above when I did it.

9(c): In this scenario, $\frac{Q(t)}{Q_0} = 0.2$ so $e^{-rt} = 0.2$. With the r we solved for above, t can be solved for and it turns out $\boxed{t = 13304.7}$ (years). $\square$

Exercise 14ab.
14(a): If you're having trouble with this problem, that's ok, I did too. Essentially, use an integrating factor of e^{kt} and then work out the complicated mess that follows. Below is the abridged version of my work.

First, simplify and rearrange to the general form of a differential equation.

$$\frac{du}{dt} = -k(u - T(t)) \Longrightarrow \frac{du}{dt} + ku = kT(t)$$

This expression strongly suggests we use an integrating factor of e^{kt}, so let's do that!

$$\to \int \frac{d}{dt}\left[e^{kt}u(t)\right] = \int ke^{kt}T(t)\ dt$$

This integral would be terrible if $T(t)$ was weird but thankfully it's not; essentially, we can substitute the expression for $T(t)$ in and simplify the mess from there.

$$\begin{aligned} \to e^{kt}u(t) &= k\int e^{kt}T_0\ dt + kT_1\int e^{kt}\cos(\omega t)\ dt \\ &= e^{kt}T_0 + kT_1 \cdot R + C \end{aligned}$$

with R being that second integral. Let's now go simplify it! (For those who are following along at home, integrate by parts twice.)
Using integration by parts ($\int \boxdot\ d\bigstar = \boxdot\bigstar - \int \bigstar\ d\boxdot$) with $\boxdot = \cos(\omega t)$ and $\bigstar = e^{kt}$, our integral (R) can now be rewritten as

$$R = \frac{1}{k}e^{kt}\cos(\omega t) + \frac{\omega}{k}\int e^{kt}\sin(\omega t)\ dt.$$

From here, we do integration by parts again ($\boxdot = \sin(\omega t)$ and $\bigstar = e^{kt}$) and we get that

$$R = \frac{1}{k}e^{kt}\cos(\omega t) + \frac{\omega}{k}\cdot\frac{1}{k}e^{kt}\sin(\omega t) - \frac{\omega^2}{k^2}R$$

which can thus be simplified to obtain our final result,

$$R = e^{kt}\frac{\omega\sin(\omega t) + k\cos(\omega t)}{k^2+\omega^2}.$$

Returning to our original integral and problem (I told you this was messy), we thus reach our final answer for $u(t)$;

$$\boxed{u(t) = \frac{C}{e^{kt}} + T_0 + kT_1\left(\frac{\omega\sin(\omega t) + k\cos(\omega t)}{k^2+\omega^2}\right)}.$$

14(b): [Graphs not pictured] $\tau \approx 3.508$ and $R \approx 9.106$. It's interesting to note that the crossing points for $S(t)$ and $T(t)$ seem to happen exactly at the min/max points of $S(t)$ (which makes sense in the physical interpretation). $\square$

Exercise 14c.
14(c): Setting the two sides equal to each other, we have that

$$R\cos(\omega t - \omega\tau) = \frac{kT_1}{\omega^2 + k^2}(k\cos(\omega t) + \omega\sin(\omega t)).$$

Expanding that left part (and ignoring the fraction for now), we have

$$R'(\cos(\omega t)\cos(\omega\tau) + \sin(\omega t)\sin(\omega\tau)) = k\cos(\omega t) + \omega\sin(\omega t)$$

where R' is R up to a constant. Anyways, from here, we intuit that $\cos(\omega\tau)$ and $\sin(\omega\tau)$ should be constants that maintain the relative proportions of $\cos(\omega t)$ and $\sin(\omega t)$. As such, we write $\frac{w}{k} = \frac{\sin(\omega\tau)}{\cos(\omega\tau)}$ and as such find that $\tau = \frac{1}{\omega}\arcsin\left(\frac{\omega}{\sqrt{\omega^2 + k^2}}\right)$. From here, some direct simplification and term comparison leads us to figure out that $R' = \sqrt{\omega^2 + k^2}$.
Substituting this back into the original equation, we have

$$R\cos(\omega t - \omega\tau) = \frac{kT_1}{\omega^2 + k^2}R'\cos(\omega t - \omega\tau)$$

so $\boxed{R = \frac{kT_1}{\sqrt{\omega^2 + k^2}}}$. □

Exercise 19a.
19(a): The strategy to attain maximum height x_m is to first set up the differential equation, find $v(t)$, find $x(t)$, find t_m, then find x_m.
For the first part, to set up the differential equation, since the medium in our question offers a **resistance** of $k|v|$, this is equivalent to the force from the resistance always being $-kv$. As such, using the handy dandy equation $F = ma = m\frac{dv}{dt}$, we can setup our differential equation as

$$m\frac{dv}{dt} = -mg - kv.$$

Solving (and remembering the $+C$), we find that

$$v(t) = \frac{1}{k}\left((mg + kv_0)e^{-kt/m} - mg\right).$$

To find $x(t)$, integrate $v(t)$. This is made simpler by the fact that $v(t)$ consists mostly of constants. Integrating and **remembering** the $+C$ (I spent 20 minutes here since I forgor), you should find

$$x(t) = -\frac{mgt}{k} - \frac{m}{k^2}(mg + kv_0)e^{-kt/m} + (mg + kv_0)\frac{m}{k^2}.$$

To find t_m, set $v(t_m) = 0$. The physical explanation for this is that at the moment the body is at its maximum height, it has no velocity going upwards (and none going downwards). As such, find t_m by setting $v = 0$. Solving, you should find

$$t = \frac{m}{k}\ln\left(\frac{mg + kv_0}{mg}\right).$$

Putting it all together and solving the actual problem (the maximum height x_m the body reaches), we simply evaluate $x(t_m)$ to find

$$x_m = \boxed{-\frac{m^2 g}{k^2}\ln\left(\frac{mg + kv_0}{mg}\right) + \frac{mv_0}{k}}.$$

□

Exercise 19bc.
19(b): Use the taylor expansion of $\ln(1+x)$ and substitute x for $\frac{kv_0}{mg}$ in the above equations for t_m and x_m. Note that this substitution is only valid when $\frac{kv_0}{mg} < 1$ as otherwise the taylor series approximation for $\ln(1+x)$ might not converge.

19(c): The quantity mg represents a force (in Newtons). The quantity kv_0 also represents a force as resistance is a force (think of friction - friction is resistance and friction as a force). Thus the fraction $\frac{kv_0}{mg}$ is dimensionless. (There's lowkey not much to say here.) □

Exercise 21.
21(a): When setting up our diffy q, we first take the downwards direction to be negative, take g to be the scalar value $9.8m/s^2$, and assume that $V_{displaced} = V_{sphere}$. From here, using similar logic to 19(a), we conclude that the resistive force $R = -6\pi\mu av$. As such, the ultimate differential equation we get is

$$m\frac{dv}{dt} = (w) + (R) + (B) = -mg - 6\pi\mu av + \rho' V_{sphere}g.$$

Note that at the limiting velocity v_L, $\frac{dv}{dt} = 0$. As such, to find v_L, we set $\frac{dv}{dt} = 0$ and solve for v in the above equation yielding $v_L = \boxed{(\rho' - \rho)\frac{2a^2g}{9\mu}}$. Note that the difference in signage comes between my answer and the textbook comes down to the fact that in my answer I take g to be a positive scalar value rather than a negative value (for me, $g = +9.8$ not -9.8).

21(b): If the electron is held stationary, then $\frac{dv}{dt} = 0$ and $v = 0$. As such, we can directly simplify our differential equation (note the added term E pointing upwards which "holds" the electron in place):

$$m\frac{dv}{dt} = (w) + (R) + (B) + (E) = -mg - 6\pi\mu av + \rho' V_{sphere}g + Ee \implies 0 = -\rho V_{sphere}g + \rho' V_{sphere}g + Ee$$

and as such $e = \boxed{\frac{(\rho - \rho')V_{sphere}g}{E}}$. □

Exercise 24. Brachistochrone problem.
24(a): We isolate y':

$$(1 + y'^2)y = k^2 \to 1 + y'^2 = \frac{k^2}{y} \to y' = \sqrt{\frac{k^2}{y} - 1}.$$

We take the positive branch of this square root equation since generally our slope of the equation $\frac{\Delta y}{\Delta x} > 0$ and as such we expect $\frac{dy}{dx} > 0$ as well.
24(b): From more or less direct simplication:

$$y' = \sqrt{\frac{k^2}{y} - 1} \to \frac{dy}{dx} = \sqrt{\frac{k^2}{k^2\sin^2 t} - \frac{\sin^2 t}{\sin^2 t}} \to \frac{(k^2 2\sin t\cos t\ dt)}{dx} = \frac{\cos t}{\sin t} \to \boxed{2k^2\sin^2 t\ dt} = dx.$$

24(c): The equation given for y is found by substituting $\sin^2 t$ for $\frac{1-\cos 2t}{2}$ in (39) when the relationship between y and t is introduced. The equation given for x is found by integrating what was found in 24(b) and is pretty direct if $\sin^2 t$ is substituted with the substitution given above.
24(d): We basically need to solve the system of equations $\begin{cases} k^2(\theta - \sin\theta) = 2 \\ k^2(1 - \cos\theta) = 4 \end{cases}$. Dividing both equations, we get that $2\theta - 2\sin\theta = 1 - \cos\theta$ which is not a very solvable equation so I used a calculator to find values for k and θ. Doing so, I find $k \approx \boxed{2.2}$. □

2.4 Differences Between Linear and Nonlinear Differential Equationss

Definition 2.3 (*Existence and Uniqueness Theorem (for First-Order Linear Equations)*)

If p and q are continuous functions on an interval I: $\alpha < t < \beta$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\frac{dy}{dt} + p(t)y = q(t)$$

with initial condition $y(\gamma) = \delta$ with δ being an arbitrary intial value and $\gamma \in I$.

Note that this theorem asserts both the **existence** and **uniqueness** of a solution to a given first order linear differential equation.

Definition 2.4 (*Existence and Uniqueness Theorem for (for First-Order Nonlinear Equations)*)

Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t_0,\ y_0)$. Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ to the differential equation

$$\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0.$$

Note: The conclusion that a differential equation exists can be established based on the continuity of f alone. However, the given solution ϕ may or may not be unique.

(Example differential equations are then solved through examples to highlight example applications of the existence theorems in practice and show the necessity of the conditions.)

Note: For first order linear differential equations, its possible points of discontinuity/singularity can be found by examining the discontinuity of the coefficients p and g.

Note (page 56): General solutions may not exist for non-linear differential equations. That is, there may be functions ϕ and ψ that both satisfy the given differential equation yet not be of the same form as each other.

Exercise 1-4.
1. Rearranging, we get that the differential equation is

$$\frac{dy}{dt} + \frac{\ln t}{t-3}y = \frac{2t}{t-3}.$$

Clearly, from the $(t-3)$ in the denominator, our end function $y = \phi(t)$ will be discontinuous at $t = 3$. Similarly, since $\ln t$ is discontinuous for all $t \leq 0$[a], ϕ may be discontinuous at $t \leq 0$. As such, we know for sure that ϕ exists over the intervals $0 < t < 3$ and $3 < t < \infty$.
2. The function $\tan(t)$ is discontinuous at $\{\dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots\}$ so the solution function $y = \phi(t)$ is certain to exist over the intervals $n\pi + \frac{\pi}{2} < t < (n+1)\pi + \frac{\pi}{2}$ for all integer values of n.
3. After rewriting the differential equation, $(4 - t^2)$ appears in the denominator of p and g meaning ϕ is certain to exist over the intervals $(infty, -2)$, $(-2, 2)$, $(2, \infty)$.
4. $\cot(t)$ is discontinuous at every multiple of π and $\ln(t)$ (as mentioned before) is discontinuous for all $t \leq 0$. As such, ϕ is certain to exist over the intervals $n\pi < t < (n+1)\pi$ for all nonnegative values of n. □

[a] $\ln(-1) = i\pi$

Exercise 9.
We seperate the variables and integrate to get $\frac{1}{2}y^2 = -2t^2 + C$. With initial condition $y(0) = y_0$, $C = \frac{1}{2}y_0^2$. As such, the general solution to our differential equation is

$$y = \pm\sqrt{y_0^2 - 4t^2}$$

with the $\pm$ sign indicating directionality of the answer. Notably, if $y_0 < 0$, then the negative sign of the square root is taken while if $y_0 > 0$, the positive sign of the square root is taken. Also, note that $|t|$ must be less than $|y|/2$ to make the equation work; if $|t| = |y|/2$, then $y(t_0) = 0$ for some t_0 yet then our original differential equation ($y' = -4t/y$) has no solution for y' when $y = 0$. $\square$

Exercise 10.
The solution to the given differential equation is $y = -\frac{1}{t^2 - \frac{1}{y_0}}$. Given the discontinuity in the denominator, t cannot be equal to $\frac{1}{\sqrt{y_0}}$ and as such the domain of t will be restricted to $-\frac{1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$ if $y_0 \geq 0$ (and t is unrestricted if $y_0 < 0$.) $\square$

Exercise 11.
The solution to the differential equation ends up being $t + \frac{1}{2y_0^2} = \frac{1}{2y^2}$ or $y = \pm\frac{1}{\sqrt{2(t + 1/2y_0^2)}}$. Like in exercise 9, the positive branch of the square root is taken if $y_0 > 0$ and the negative branch taken if $y_0 < 0$. Otherwise, the domain of t is restricted to $t > -\frac{1}{2y_0^2}$. $\square$

Exercise 18.
(a) Verified.
(b) $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-t + \sqrt{t^2 + 4y}}{2}\right) = \frac{1}{\sqrt{t^2 + 4y}}$ is not continuous with the initial condition $y(2) = -1$. As such, while a solution to the differential equation ϕ can be guaranteed, the solution ϕ may or may not be unique as not all requirements of Theorem 2.4.2 are satisfied.
(c) Given that we know the second solution $y_2(t)$, it's pretty easy to check no constant c plugged into the equation $y = ct + c^2$ can yield a t^2 term like in $y_2(t)$. Otherwise, fixing the initial condition $y(2) = -1$, plugging $t = 2$ and $y = -1$ **only** yields the solution $c = -1$. $\square$

Exercise 21.
By direct simplification,

$$\frac{dy}{dt} + p(t)y \longrightarrow \frac{d}{dt}[y_1(t) + y_2(t)] + p(t)(y_1(t) + y_2(t)) = y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = g(t).$$

$\square$

Discontinuous coefficients problems look daunting but really you're just solving more or less the same differential equation twice then gluing the pieces together.

Exercise 26.
Here, we solve $y' + 2y = g(t)$ by multiplying both sides by the integrating factor of e^{2t}. We then evaluate both cases ($t > 1$, $0 \leq t \leq 1$) separately, and glue together both functions when $t = 1$.
With the case of when $g(t) = 1$ ($0 \leq t \leq 1$),

$$y' + 2y = 1 \to e^{2t}\frac{dy}{dt} + 2e^{2t}y = e^{2t} \longrightarrow \int \frac{d}{dt}\left[e^{2t}y\right] = \int e^{2t}\ dt \Longrightarrow y_\alpha(t) = \frac{e^{2t} + C_\alpha}{2e^{2t}}.$$

Since we have the initial condition of $y(0) = 0$, $C_\alpha = -1$ and $y_\alpha(t) = \dfrac{e^{2t} - 1}{2e^{2t}}$.
The case when $g(t) = 0$ is simple enough to not warrant covering and the final equation obtained is $y_\beta(t) = \dfrac{C_\beta}{e^{2t}}$ ($t > 1$). To make the overall function $y(t)$ continuous ($y = y_\alpha \cup y_\beta$), we set $y_\alpha(1) = y_\beta(1)$. Namely, $y_\alpha(1) = \dfrac{e^2 - 1}{2e^2}$ and $y_\beta(1) = \dfrac{C_\beta}{e^2}$ so $C_\beta = \dfrac{e^2 - 1}{2}$.
Overall then,

$$y(t) = \begin{cases} \frac{e^{2t}-1}{2e^{2t}} & 0 \leq t \leq 1 \\ \frac{e^2-1}{2e^{2t}} & t > 1 \end{cases}.$$

$\square$

Exercise 27.
Setting up the differential equation to be separable, we obtain

$$\frac{1}{p(t)y}\ dy = -dt.$$

For the case when $p(t) = 2$, y_α is correspondingly

$$y_\alpha(t) = e^{-2t}$$

after plugging in initial condition $y(0) = y_\alpha(0) = 1$.

Similarly, when $p(t) = 1$, $y_\beta = e^{-t+C_\beta}$.
As such, 'gluing' the functions together when $t = 1$, we find that $y_\alpha(t) = e^{-2}$ and $y_\beta(t) = e^{-1+C_\beta}$ so $C_\beta = -1$ and the equation is solved. $\square$

2.5 Autonomous Differential Equations

This section mostly examines the logistic equation modelling population growth in the context of differential equations and some goes somewhat in-depth about it.

Definition 2.5 (*Autonomous Differential Equation*)

A differential equation is **autonomous** if the independent variable does not appear explicitly. They have the form

$$\frac{dy}{dt} = f(y).$$

Note that the zeros of $f(y)$ $\{z \mid f(z) = 0\}$ are called **critical points**.

(Note: There's some definitions of **asymptotically stable** points and other terms that aren't noted down here (I don't care about those terms).)

Definition 2.6 (*Logistic (Verhulst) Equation*)

The equation of the form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

which is commonly used to represent population growth.
Notably, K is the **carrying capacity** for the population and r is called the **intrinsic growth rate**.

(65) - Logistic Growth with a Threshold:
The equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

represents a logistic growth function with a threshold (where $0 < T < K$) —graphing the solution to this function reveals that all initial values $y_0 > T$ gravitate towards K like in logistic growth yet all initial values $y_0 < T$ eventually fade to 0.

Exercise 5. Semistable Equilibrium Solutions
5a: Trivial since if $y \neq 1$, then $(1-y)^2$ will be non-zero by the trivial inequality.

5c:

$$\frac{dy}{dt} = k(1-y)^2 \to \frac{1}{(1-y)^2}\,dy = k\,dt \to \frac{1}{1-y} = kt + C.$$

With initial condition $y(0) = y_0$, $C = \frac{1}{1-y_0}$ so $y(t) = \boxed{1 - \frac{1-y_0}{kt - kty_0 + 1}}$. □

Exercise 13.
To find the inflection points of $y(t)$, we can find the min/maxes of $f(y)$ since $y'(t) = f(y)$ (kinda). As such, we find those minimum/maximum points by setting $\frac{df}{dy} = 0$ then solving.
Starting from (17), we apply a 3-part product rule to get

$$f'(y) = -r\left[\left(1 - \frac{y}{T}\right)\left(-\frac{1}{K}\right)y + \left(-\frac{1}{T}\right)\left(1 - \frac{y}{K}\right)y - \left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)\right].$$

The inside function is merely a quadratic (a very messy one) $\left[y^2\left(\frac{3}{KT}\right) - y\left(\frac{2}{K} + \frac{2}{T}\right) + 1\right]$, which can be bashed using the quadratic formula to get the desired solutions. □

Exercise 17a.
17(a): Separating, we get

$$\frac{1}{\ln K - \ln y} \frac{1}{y}\ dy = r\ dt.$$

Integrating, we have

$$rt + C = \int \frac{1}{y}\ dy\ \frac{1}{\ln K - \ln y}$$

from which we can make the substitution $\ln y = u$ (and $\frac{1}{y}\ dy = du$) so the RHS becomes $\displaystyle\int \frac{1}{\ln k - n}\ dn$. From here, integrating gets us

$$rt + C = -\ln\left|\ln\left(\frac{K}{y}\right)\right|$$

and a multitude of substitutions leads us to find $C = \ln \frac{y_0}{K}$ so

$$y(t) = K\exp\left(\ln\left(\frac{y_0}{K}\right)e^{-rt}\right).$$

□

Exercise 19.
19(a): Assuming $E < r$, to solve we set the RHS equal to 0:

$$r\left(1 - \frac{Y}{K}\right)y - Ey = 0 \to y\left(r - \left(1 - \frac{y}{K}\right) - E\right) = 0$$

meaning either $y_1 = 0$ or the inside function is 0. Simplifying that inside function reduces to the second solution, $y_2 = K\left(1 - \frac{E}{r}\right)$.

19(b): The inside function is a parabola pointed downwards. As such, function values for y values right below the first solution $y_1 = 0$ are negative and function values for y values right above 0 are positive. This translates to values drifting away from $y = 0$ as negative values become even more negative and positive values continue to get more positive making y_1 an unstable equilibrium.
On the other hand, values slightly above y_2 have a negative $\frac{dy}{dt}$ and values slightly below y_2 have a positive $\frac{dy}{dt}$ so overall those values will drift towards y_2. Thus, y_2 is an asymptotically stable solution.

19(c): Since $Y = E \cdot y_2$, $Y = EK\left(1 - \frac{E}{r}\right)$.
19(d): The maximum of a parabola lies at its vertex $-\frac{b}{2a}$. In this case, $Y(E) = -E^2\frac{K}{r} + EK$ is maximized when $E = \frac{r}{2}$ and correspondingly $Y_m = \frac{KR}{4}$. □

Exercise 27.
27(a): The limiting value of $x(t)$ as $t \to \infty$ is $x = \min\{p,\ q\}$. Slightly above this value $\frac{dx}{dt}$ is negative (meaning $x(t)$ decreases) and slightly below this value $\frac{dx}{dt}$ is positive.
We can solve the differential equation using partial fractions:

$$\frac{dx}{dt} = \alpha(p - x)(q - x) \to \int \frac{1}{(p - x)(q - x)}\ dx = \alpha t + C \to \frac{1}{q - p}\int \frac{1}{p - x} - \frac{1}{q - x}\ dx = \alpha t + C$$

so $\displaystyle\frac{1}{q-p}(-\ln|p - x| + \ln|q - x|) = \alpha t + C$ with $\displaystyle C = \frac{\ln(q/p)}{q - p}$. (The simplification afterwards for an explicit form of $x(t)$ gets really messy.)

27(b): As seen in problem 5, this is a equation with a semistable equilibrium solution $x(t) = p$ which happens when $t \to \infty$ with initial condition $x(0) = 0$. A simple integration shows $\int \frac{1}{(p-x)^2}\ dx = \frac{1}{p-x} + C$ and thus $x(t) = p(1 - \frac{1}{p\alpha t + 1})$. □

2.6 Exact Differential Equations and Integrating Factors

Definition 2.7 (*Exact Differential Equation*)

If we can identify a function $\psi(x,y)$ such that $\frac{\partial\psi}{\partial x} = M(x,y)$ and $\frac{\partial\psi}{\partial y} = N(x,y)$ for a given differential equation $M(x,y)+N(x,y)y' = 0$, then that differential equation is **exact** and the solutions are implicitly given by $\psi(x,y) = c$ for arbitrary c.

While this itself is daunting, Clairut's Theorem simplifies this tremendously for us as under the assumption that M, N, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are continuous in some closed region, Clairut's theorem tells us

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

We can then substitute our function ψ for the generic function f and simplify:

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) \longrightarrow \frac{\partial}{\partial x}(N) = \frac{\partial}{\partial y}(M)$$

concluding that the function ψ only exists if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$.

Now with this simple test at our side, if a function ψ exists for an exact differential equation, since $M = \frac{\partial\psi}{\partial x}$, we can simply integrate M with respect to x with the constant term being replaced by an arbitrary function $C(y)$. To then solve for C, take the derivative of the integrated thing and compare terms.

Returning to the concept of integrating factors from Section 2.1, another technique to solving general differential equations is to multiply both sides by a specific integrating factor such that the resulting differential equation is exact.
In symbols, we find a function μ as an integrating factor

$$M(x,y) + N(x,y)y' = 0 \longrightarrow (\mu(x,y)\cdot M) + (\mu(x,y)\cdot N)y' = 0$$

so $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x}$ to make this new differential equation an **exact differential equation**.

Unfortunately, this is super duper difficult (there is no good way to find μ) in general. As such, we have to gimmick a little and hope that μ is a function of one variable which simplifies things considerably.
Assuming $\mu = \mu(y)$, we can use the product rule on both sides of the above equation to get

$$\mu(y)\frac{\partial M}{\partial y} + M\frac{d\mu(y)}{dy} = \mu(y)\frac{\partial N}{\partial x} + N\frac{\partial\mu(y)}{\partial x} = \mu(y)\frac{\partial N}{\partial x}$$

which can we rearrange and find $\dfrac{d\mu}{dy} = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\mu$ from which μ can be integrated and solved for.[2]

Exercise 1-8.
1. This differential equation is exact; $\psi(x,y) = x^2 + 3x + y^2 - 2y(=c)$.
2. Not exact.
3. Exact; $\psi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y$.
4. Exact (rearrange to get $(ax+by)+(bx+cy)y' = 0$); $\psi(x,y) = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2$.
5. Not exact.
6. Exact; $\psi(x,y) = e^{xy}\cos(2x) + x^2 - 3y$.
7. Exact; $\psi(x,y) = y\ln x + 3x^2 - 2y$.
8. Exact; $\psi(x,y) = -\frac{1}{\sqrt{x^2+y^2}}$ (via u-substitution). □

[2] A similar equation arises if you assume μ is a function of x.

Exercise 14.
A differential equation is exact if $M_y = N_x$. Since $M = M(x)$, $\frac{\partial M(x)}{\partial y} = 0$ (it's basically a constant) and similarly $\frac{\partial N(y)}{\partial x} = 0$ so the separable equation is exact. $\square$

Exercise 17.
From the derivation I did previously, $\dfrac{d\mu}{dy} = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\mu = \dfrac{N_x - M_y}{M}\mu$. Thus, we have $\mu' = Q(y)\mu$ so $\mu(y) = e^{\int Q(y)\ dy}$ as given. $\square$

Exercise 18-21.
18. Assuming $\mu = \mu(x)$, we find the integrating factor to be $\mu(x) = e^{3x}$. Integrating, we find $\psi(x, y) = x^2ye^{3x} + \frac{1}{3}y^3e^{3x}$.
19. The integrating factor is e^{-x} and $\psi(x, y) = ye^{-x} - e^x - e^{-x}$.
20. The integrating factor is y and $\psi(x, y) = xy + y \cos y - \sin y$.
21. The integrating factor is $\mu(y) = \frac{e^{2y}}{y}$ and integration reveals $\psi(x, y) = xe^{2y} - \ln |y|$. $\square$

2.7 Euler's Method

Euler's method, famously shoved down students' throats in Calculus BC, is just basic (first-order?) numerical approximation of a given function.
Essentially, the problem statement can be summed up by finding the value of the function y at a certain point t_1 given differential equation $\dfrac{dy}{dt} = f(t, y)$ and initial point (t_0, y_0).
Mathematically, since $y(t_1) = \displaystyle\int_{t_0}^{t_1} f(t, y)\ dt + y_0 = \sum_{t_0}^{t_1} f(t, y) \cdot dt + y_0$ (assuming y is continuous), we can find $y(t_1)$ by approximating the summation with a non-infinitesimal dt ($dt \to \Delta t$).
To Euler it up, determine how many steps n you want to take. Next, to approximate $y(t_{\text{next}})$, take a 'step' of size $\dfrac{t_1 - t_0}{n}$ and estimate $y(t_{\text{next}})$ as $y(t_{\text{now}}) + \frac{dy}{dt}\Delta t = y(t_{\text{now}}) + f(t_{\text{now}}, y)(t_{\text{next}} - t_{\text{now}})$ with $t_{\text{next}} = t_{\text{now}} + \frac{t_1 - t_0}{n}$. Iterate this until $t_{\text{next}} = t_1$, the desired endpoint.
So that's basic numerical interpolation for you. Note that the step size need not be constant though usually it is.
Most exercises below are computation-related and they're frankly boring which is why almost none are done.

Exercise 15. Convergence of Euler's Method.
15(a). Assuming $\mu = \mu(t)$ leads us to find $\mu(t) = e^{-t}$ and a semi-messy integration reveals $\psi(t, y) = e^{-t}(y - t) = c$. Rearranging for y, we find $y = t + ce^t$ and plugging in point (t_0, y_0) means $y_0 = t_0 + ce^{t_0}$ or $c = \frac{y_0 - t_0}{e^{t_0}}$ which matches the solution given in the problem.

15(b). $y_k = y_{k-1} + \dfrac{dy}{dt}h = y_{k-1} + (1 - t_{k-1} + y_{k-1})h = y_{k-1}(1 + h) + h - t_{k-1}h$.

15(c). $y_2 = (1+h)y_1 + h - ht_1 = (1+h)y_1 + t_2 - t_1 - ht_1 = (1+h)(y_1 - t_1) + t_2 \to (1+h)((1+h)(y_0 - t_0) + t_1 - t_1) + t_2$ so $y_2 = (1 + h)^2(y_0 - t_0) + t_2$. As such, by engineer's induction $y_n = (1 + h)^n(y_0 - t_0) + t_n$.

15(d): I'm not sure what conditions this problem is throwing at me but essentially:

$$y_n = (1 + h)^n(y_0 - t_0) + t_n \to (1 + (t - t_0)/n)^n(y_0 - t_0) + t_n \implies y_n \approxeq e^{t - t_0}(y_0 - t_0) + t = \phi(t)$$

as $n \to \infty$. $\square$

2.8 The Existence and Uniqueness Theorem

This long section is just kind of a proof of the existence and uniqueness of solutions of a continuous differential equation.
The proof uses a method called "method of successive approximations" which solves a differential equation by considering the solution $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ where

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\ ds$$

(f is the thing you get when you rewrite a differential equation as $\frac{dy}{dt} = f(t, y)$.)

Essentially this whole section is just dedicated to the proof of an elementary yet powerful (but also cumbersome) theorem.

Exercise 1.
$\dfrac{dy}{dt} = (t+1)^2 + (y+2)^2$. □

Exercise 5a, 6a.
5(a): A pattern clearly emerges calculating the first values of $\phi_n(t)$:

$$\phi_0(t) = t,\ \phi_1(t) = t,\ \phi_2(t) = \int_0^t t^2 + 1 = \frac{t^3}{3} + t, \phi_3(t) = \int_0^t s\left(\frac{s^3}{2} + s\right) + 1\ ds = \frac{t^5}{3\cdot 5} + \frac{t^3}{3} + t,$$

$$\phi_4(t) = \int_0^t s\left(\frac{s^5}{3\cdot 5} + \frac{s^3}{3} + s\right) + 1 = \int_0^t \frac{s^6}{3\cdot 5} + \frac{s^4}{3} + s^2 + 1 = \frac{t^7}{3\cdot 5\cdot 7} + \frac{t^5}{3\cdot 5} + \frac{t^3}{3} + t$$

from which we can draw the conclusion that $\phi_n(t) = \dfrac{t^{2n-1}}{(1)(3)\cdots(2n-1)} + \phi_{n-1}(t)$ so $\phi_n(t) = \displaystyle\sum_{i=1}^n \frac{t^{2i-1}}{(2i-1)!!}$.

6(a): Following the same process of value bashing:

$$\phi_0(t) = t,\ \phi_1(t) = -\frac{t^2}{2}, \phi_2(t) = \int_0^t -\frac{s^4}{2} - s\ ds = -\frac{t^5}{5\cdot 2} - \frac{t^2}{2},$$

$$\phi_3(t) = \int_0^t -\frac{s^7}{5\cdot 2} - \frac{s^4}{2} - s\ ds = -\frac{t^8}{8\cdot 5\cdot 2} - \frac{t^5}{5\cdot 2} - \frac{t^2}{2}$$

so if we do engineer's induction we arrive at the conclusion that $\phi_n(t) = -\displaystyle\sum_{i=1}^n \frac{t^{3i-1}}{(2)(5)\ldots(3i-4)(3i-1)}$. □

Exercise 12.
12(a): We start by defining functions $f_n(x)$ and $g_n(x)$ such that $f_n(x) = 0$ and $g_n(x) = \dfrac{2nx}{1 + nx^2 + \frac{n^2x^4}{2}}$.
Next, we note that $f_n(x) \leq \phi_n(x) \leq g_n(x)$ over the interval $0 \leq x \leq 1$ when $n > 3$. The former inequality $f_n(x) \leq \phi_n(x)$ is trivial to prove, and the latter inequality $\phi_n(x) \leq g_n(x)$ is derived from the fact that

$$\phi_n(x) = \frac{2nx}{e^{nx^2}} = \frac{2nx}{1 + nx^2 + \frac{n^2x^4}{2} + \frac{n^3x^6}{6} + \frac{n^4x^8}{24} + \cdots} < \frac{2nx}{1 + nx^2 + \frac{n^2x^4}{2}} = g_n(x).$$

Evaluating the limits of each function, trivially $\lim_{n\to\infty} f_n(x) = 0$. For $g_n(x)$,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{2}{x + \frac{2+n^2x^4}{2nx}} \to \lim_{n\to\infty} \frac{2}{x + \frac{n^2x^4}{2nx}} = \lim_{n\to\infty} \frac{2}{x + nx^3/2} = 0$$

for fixed $x \neq 0$ ($g_n(0) = 0$ for all n). As such, by the squeeze theorem, since $f_n(x) \leq \phi_n(x) \leq g_n(x)$ and $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = 0$, $\boxed{\lim_{n\to\infty} \phi_n(x) = 0}$.

12(b): Recognizing that $[-nx^2]' = -2nx$,

$$\int_0^1 2nxe^{-nx^2}\, dx = \int_1^0 -2nxe^{-nx^2}\, dx = [e^{-nx^2}]_1^0 = e^0 - e^{-n} = 1 - e^{-n}.$$

$\square$

Exercise 13.
13(a):

$$\phi(t) = \int_0^t (2s)(1 + \phi(s))\, ds = \int_0^t (2s)\left(1 + s^2 + \frac{s^4}{2!} + \cdots + \frac{s^{2k}}{k!} + \cdots\right)\, ds$$

$$\int_0^t 2s + 2s^3 + \frac{2s^5}{2} + \cdots + \frac{2s^{2k+1}}{k!} + \cdots = t^2 + \frac{t^4}{2} + \frac{t^6}{2 \cdot 3} + \cdots + \frac{t^{2k+2}}{(k+1)!} + \cdots$$

with that final expression being exactly $\phi(t)$.

13(b): $\phi(0) = 0$.
13(c): $\phi(t) = \sum_{k=0}^{\infty} \frac{(t^2)^k}{k!} - 1 = e^{t^2} - 1$.
13(d)/(e): (This is problem (8) by the way)

$$y' = 2t(1 + y) \to 2t\, dt = \frac{1}{1+y}\, dy \to t^2 + C = \ln|y + 1| \to y = e^{t^2+C} - 1$$

with the constant being $C = 0$ found by plugging in the initial condition. $\square$

2.9 First-Order Difference Equations

Definition 2.8 (*first-order difference equation*)

An equation that takes on discrete values and is of the form $y_{n+1} = f(n, y_n)$.
Further classifications can be derived from whether or not f is linear (or non-linear) and whether or not an initial condition is provided.
A solution to the first-order difference equation is a set of values $\{y_n\}$ that satisfy the relation held above.

Assuming that $y_{n+1} = f(y_n)$ (recurrences!), we can find **equilibrium solutions** by solving $y_n = f(y_n)$.
Page (93) analyzes a more complicated model of discrete population growth similar to that of a discretized lienar differential equation.

Exercise 1-4.
1. Trivial; $y_n = -0.9^n y_0$ so the limit of y_n is 0 as $|y_{n+1}| < |y_n|$.
2. In general, y_n does not matter in the sense that since the fraction $\sqrt{\dfrac{n+3}{n+1}}$ is nearly 1, y_n does not undergo any drastic changes. In fact, $y_n = \sqrt{\dfrac{(n+2)(n+1)}{n(n-1)}}y_0$ as most square roots end up cancelling each other. Thus, asymptotically, $y_{n+1} \equiv y_n$.
3. Flip flop.
4. An equilibrium solution to this equation is $y_n = 12$. Since this equation is also linear and well-behaved, it's to be expected that any solution to this recurrence given an initial condition will converge to $y_n = 12$. $\square$

2.10 Miscallaneous Problems

This section is dedicated to solving the end of chapter problems given on page 100 in the textbook. I'm tired of word problems so I'll just be solving the 24 miscallaneous differential equations given.

Exercise 1.
I trolled on this problem for a long time :/.
Rewriting our differential equation, we have $\dfrac{2}{x}y + \dfrac{dy}{dx} = x^2$. Multiplying by a generic integrating factor μ, our equation becomes $\dfrac{2}{x}\mu y + \mu\dfrac{dy}{dx} = x^2\mu$. Given that we want the left hand side of our equation to look something like $\dfrac{d}{dx}[\mu y] = \mu' y + \dfrac{dy}{dx}\mu$, we immediately have $\mu' = \dfrac{2}{x}\mu \to \mu(x) = x^2$.
As such, we can plug that back into our equation and reformat our differential equation as

$$2xy + \frac{dy}{dx}x^2 = x^4 \to \frac{d}{dx}[x^2 y] = x^4$$

and integrate both sides to get $\frac{x^5}{5} + C = x^2 y \to \boxed{y(x) = \dfrac{x^3}{5} + \dfrac{C}{x^2}}$. $\square$

Exercise 2.

$$\frac{dy}{dx} = \frac{1+\cos x}{2-\sin y} \to 1 + \cos x \; dx = 2 - \sin y \; dy \to \boxed{x + \sin x + C = 2y + \cos y}.$$

$\square$

Exercise 3.
We rewrite our diffy q as $(-2x-y)+(3+3y^2-x)\frac{dy}{dx}=0$. Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-1$, our differential equation is **exact** and we can simply integrate $M=-2x-y$ with respect to y and solve for the constant function to get $\psi(x,y)=-x^2-xy+3y+y^3$. With our initial condition $(0,0)$, our differential equation is implicitly solved by $-x^2-xy+3y+y^3=0$. □

Exercise 4.

$$\frac{dy}{dx}=3-6x+y-2xy \longrightarrow \frac{dy}{dx}=(3+y)(1-2x) \Rightarrow y=Ce^{x-x^2}-3.$$

□

Exercise 5.
$\to (2xy+y^2+1)+(x^2+2xy)\frac{dy}{dx}$. $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=2x+2y$ so our differential equation is exact. As such, $\psi(x,y)=\int x^2+2xy\;dy=\int 2xy+y^2+1\;dx=x^2y+xy^2+x\;(=C)$. □

Exercise 6.
Since there's a $-y$ on the right hand side, we have to rearrange the left hand side to account for it:

$$x\frac{dy}{dx}+xy=1-y \to \frac{dy}{dx}+\frac{x+1}{x}y=\frac{1}{x}$$

As such, we can recognize that an integrating factor of $\mu=xe^x$ is needed $\left(\mu'=\frac{x+1}{x}\mu\right)$ and as such we can integrate and get $e^x+C=xe^xy$. With initial condition, our solution thus becomes $\boxed{y=\frac{1}{x}-\frac{e}{xe^x}}$. □

Exercise 7.

$$x\frac{dy}{dx}+2y=\frac{\sin x}{x} \to x^2\frac{dy}{dx}+2xy=\sin x \to \frac{d}{dx}[x^2y]=\sin x \to y=\frac{C-\cos x}{x^2} \Longrightarrow \boxed{y=\frac{4+\cos 2-\cos x}{x^2}}.$$

□

Exercise 8.
This differential equation is exact.

$$(2xy+1)+(x^2+2y)y'=0 \to \psi(x,y)=x^2y+x+y^2.$$

□

Exercise 9.
Although this differential equation looks like it can be exact, it's actually separable.

$$(x^2y + xy - y) + (x^2y - 2x^2)\frac{dy}{dx} = 0 \to (y)(x^2 + x - 1) + (x^2)(y - 2)\frac{dy}{dx} = 0$$
$$\to -\frac{y(x^2 + x - 1)}{x^2(y - 2)} = \frac{dy}{dx} \to \frac{y - 2}{y}\ dy = \frac{1 - x - x^2}{x^2}\ dx.$$

As such (break apart the fractions!), $\boxed{y - 2\ln y = -x - \ln x - \frac{1}{x} + C}$. $\square$

Exercise 10.
This time, the differential equation is exact. This makes our work super easy as we can just recognize $\psi(x, y) = \frac{x^3}{3} + xy + e^y$ which captures all solutions when paired with a level curve. $\square$

Exercise 11.
This differential equation is also exact. $\psi(x, y) = \frac{x^2}{2} + xy + y^2$ so with point $(2, 3)$, the solution is $\dfrac{x^2}{2} + xy + y^2 = 17$. $\square$

Exercise 12. We can rewrite our differential equation to be separable and seperate to get

$$\to \ln y = C + \int \frac{1 - e^x}{1 + e^x}.$$

Since this kind of looks u-subbable, we let $u = e^x + 1$; correspondingly, $du = e^x\ dx$ and $1 - e^x = 2 - u$. As such, our integral then becomes

$$= C + \int \frac{2 - u}{u(u - 1)}\ du = C + \int \frac{1}{u - 1} - \frac{2}{u}\ du.$$

As such, now integratable, we integrate and simplify to get $\boxed{y = Ce^x \cdot (e^x + 1)^{-2}}$. $\square$

Exercise 13.
This big scary equation is actually an exact differential equation. Solving, we get $\psi(x, y) = e^{-x}\cos y + e^{2y}\sin x$. $\square$

Exercise 14.

$$-3y + \frac{dy}{dx} = e^{2x} \to e^{-x} = -3e^{-3x}y + e^{-3x}\frac{dy}{dx} \to -e^{-x} + C = e^{-3x}y.$$

$\square$

Exercise 15. Using integrating factor e^{2x}, we get

$$e^{2x}y = \int e^{-x^2}\ dx + C$$

so $y = e^{-2x}\dfrac{\sqrt{\pi}}{2}\text{erf}(x) + 3e^{-2x}$. □

Exercise 16.
Again, another big scary fraction that turns super tame when we rearrange it and find it's an exact differential equation. $\psi(x, y) = xy^3 + 2xy - x^3$ is the parent solution. □

Exercise 17.

$$y' = e^{x+y} \to e^{-y}\ dy = e^x\ dx \to -y = \ln(-e^x) \to y = -x(1 + i\pi).$$

□

Exercise 18.
$\psi(x, y) = 2xy^2 + 3x^2y - 4x + y^3$. □

Exercise 19.
Note: This problem is functionally equivalent to problem 6. The integrating factor is the same (xe^x) so the problem is basically identical.

$$t\frac{dy}{dt} + (t+1)y = e^{2t} \to e^{3t} = te^t\frac{dy}{dt} + (t+1)e^t y \to \frac{e^{3t}}{3} + C = te^t y \to y = \frac{e^{2t}}{3t} + \frac{C}{te^t}.$$

□

Exercise 20.
This problem is very tricky and I gave up on the problem (although I was close in trying to make a substitution). Essentially, to clear things up, we set $y = nx$ and correspondingly find $\dfrac{dy}{dx} = n + \dfrac{dn}{dx}x$. We then substitute this new $\frac{dy}{dx}$ back into our differential equation and solve.
Eventually, we find $-e^{-n} = \ln x + C$ so $e^{-y/x} + \ln x = C$ is our solution. □

Exercise 21.
This problem is weird. Also, I wouldn't take the hint that the textbook gives.

Motivated by how bad the differential equation is when we first write it out $\left(\frac{dy}{dx} = \frac{x}{y} \cdot \frac{1}{x^2+y^2}\right)$, we are somewhat motivated in making the substitution $v = x^2 + y^2$ to clean things up. Indeed, we can rewrite the whole differential equation in terms of v and x:

$$v = x^2 + y^2 \to \frac{dv}{dx} = 2x + 2y\frac{dy}{dx} \to \frac{dy}{dx} = \frac{\frac{dv}{dx} - 2x}{2y}$$

$$\implies \frac{\frac{dv}{dx} - 2x}{2y} = \frac{x}{y} \cdot \frac{1}{v} \Rightarrow \frac{dv}{dx} = 2x\left(\frac{v+1}{v}\right)$$

from which the equation is separable and then easily integratable.

Lesson of the day then is to use clever substitutions to turn a non-linear differential equation (with term $y^3\frac{dy}{dx}$) into a linear, solvable differential equation. □

Exercise 22.
Use the substitution $y = nx$ $\left(\frac{dy}{dx} = n + \frac{dn}{dx}\right)$ and rearrange:

$$\to n + \frac{dn}{dx}x = \frac{(n+1)x}{(1-n)x} \to \frac{dn}{dx} = \frac{1+n^2}{1-n} \cdot \frac{1}{x}$$

which is clearly solvable and might be integratable and might result in a solution that isn't deranged and unwritable. □

Exercise 23.
I'm not even sure how this still works but once again the substitution $y = nx$ works (This time I figured it out by myself!!).
$y = nx$ so $\frac{dy}{dx} = n + x\frac{dn}{dx}$. Replacing this in our differential equation, we get:

$$3n^2x^2 + 2nx^2 - \left(n + x\frac{dn}{dx}\right)(2nx^2 + x^2) = 0 \to 3n^2 + 2n - \left[2n^2 + n + (2n+1)x\frac{dn}{dx}\right] = 0$$

$$\to n^2 + n = (2n+1)x\frac{dn}{dx} \to \frac{1}{x}\ dx = \frac{1}{n} + \frac{1}{n+1}\ dn$$

from which the equation is easily solvable :).
Lesson learned: Always substitute $y = nx$ if things look funky. □

Exercise 24.
Not even going to attempt this one. I've wasted too much time on this stupid problem.
The solution is to divide the equation by y^2 and recognize that you can find an integrating factor μ for the exact differential equation by trying to find $\frac{d\mu}{dx}$. Good luck o7. □