

# Differential Equations Notes

Alex Z

Fall 2025

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	Introduction for the Introduction . . . . .	4
1.2	Introduction to Solutions . . . . .	5

## Remarks

This notes thing was started on 10/19/2025.

I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).

Hopefully, I'll also get around 20-40% of the problems in the textbook done.

# 1 Introduction

aka chapter 1

## 1.1 Introduction for the Introduction

**Definition** (Differential Equations). Equations containing derivatives.

**Definition** (Slope Field/Direction Field). A buncha line segments on the plane that represent the “motion” of a diff-eq.

Direction Fields are good for studying differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

(Page 6 – How to construct a diff-eq mathematical model from a real-world situation.)

(7) Newton: Differential equations come in one of these 3 forms:

1.  $\frac{dy}{dx} = f(x),$
2.  $\frac{dy}{dx} = f(y),$
3.  $\frac{dy}{dx} = f(x, y).$

---

### Exercise 11-16.

- 1.1.5 corresponds with **j**.
- 1.1.6 corresponds with **c**.
- 1.1.7 corresponds with **g**.
- 1.1.8 corresponds with **b**.
- 1.1.9 corresponds with **h**.
- 1.1.10 corresponds with **e**.

□

### Exercise 17.

(a)

$$\frac{dC}{dt} = [\text{chemicals/hour going in}] - [\text{out}] = 0.01 \cdot 300 - 300 \cdot \frac{C}{1000000}$$

where  $C$  is the number of gallons of said chemical in the pond and  $t$  is time measured in hours.

(b) After a very long time, 10000 gallons will be in the pond; this limiting amount is independent of starting conditions.

(c) Since concentration =  $\frac{\text{Amount}}{\text{Volume}}$ ,  $C = \text{volume} \cdot c = c \cdot 10^6$  where  $c$  stands for concentration. As such,

$$\frac{dc}{dt} = \frac{1}{10^6} \frac{dC}{dt} = \frac{3}{10^6} - \frac{3(c \cdot 10^6)}{10^4 \cdot 10^6}$$

So in final,  $\boxed{\frac{dc}{dt} = \frac{3}{10^6} - \frac{3c}{10^4}}.$

□

**Exercise 18.**

$$\frac{dV}{dt} = -k \cdot 4\pi r^2 = -k \cdot 4\pi \left(\frac{3}{4\pi} V\right)^{\frac{2}{3}}$$

□

**Exercise 19.**

$$\frac{dT}{dt} = -0.05 * (T - 70)$$

where  $T$  is the temperature of the object in Fahrenheit and  $t$  is time in minutes.

□

---

## 1.2 Introduction to Solutions

(11) - Finding the general solutions to diff-eqs of the form  $\frac{dy}{dt} = ay - b$  ( $a \neq 0$ );

$$\frac{dy}{dt} = ay - b \implies y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$$

(14 - "Further Remarks on Mathematical Modeling" - essentially, the underlying assumptions we make may or may not be wrong. )

---

**Exercise 1a.**

$$\frac{dy}{dt} = -y + 5 \rightarrow \frac{1}{5-y} dy = dt.$$

So,  $\ln(5 - y(t)) = t + C$ . With initial condition  $y(0) = k$ , we get that  $\ln(5 - k) = C$ , so our solution becomes  $y(t) = 5 - e^{t+\ln(5-k)} = 5 - (5 - k)e^t$ . (Note that  $(5 - k)$  is constant.)

□

**Exercise 9a.**

Since  $F = ma$ ,  $F = m \frac{dv}{dt}$ . Since drag acts inversely to velocity (object falling faster has more air resistance), we should expect  $\frac{dv}{dt}$  to be negative; thus,  $\frac{dv}{dt} = -\frac{F}{10}$ . Knowing that  $F$  is proportional to the square of the velocity, we know that  $F = av^2 - b$  for constants  $a, b$ .

Now, we plug in some known values. At  $v = 0$ , we expect  $\frac{dv}{dt} = -\frac{(-b)}{10} = -9.8$  (gravity) so  $b = -98$ . At  $v = 49$ , we reach limiting velocity which implies  $\frac{dv}{dt} = 0$  so  $\frac{a(49^2)-98}{10} = 0$  so  $a = \frac{2}{49}$ . Thus, in final, we get our differential equation as

$$\frac{dv}{dt} = \frac{2}{49 \cdot 10} v^2 - \frac{98}{10}$$

which can be re-arranged to

$$\frac{dv}{dt} = \frac{1}{245} (v^2 - 49).$$

□

**Exercise 9b.**

(I'm gonna go with their equation for simplicity - it doesn't matter too much though.)

$$\begin{aligned}\frac{dv}{dt} &= \frac{1}{245} (49^2 - v^2) \\ \rightarrow 245 \frac{1}{49^2 - v^2} dv &= dt \\ \rightarrow 245 \int \frac{1}{49^2 - v^2} dv &= t\end{aligned}$$

Doing a trig sub ( $v = 49 \sin \theta$ ,  $dv = 49 \cos \theta d\theta$ ),  $\frac{dv}{49^2 - v^2}$  becomes  $\frac{49 \cos \theta d\theta}{49^2 - 49 \sin^2 \theta}$  so our integral ends up turning into

$$\rightarrow 245 \int \frac{d\theta}{49 \cos \theta} = t \implies 5 \left( \frac{1}{2} \ln \left( \frac{1 + \sin \theta}{1 - \sin \theta} \right) \right) = t + C.$$

Thus,

$$t + C = \frac{5}{2} \ln \frac{1 + v/49}{1 - v/49}.$$

Plugging in our initial condition  $v(0) = 0$ , we get that  $C = 0$ . Thus,

$$\ln \left( \frac{1 + v/49}{1 - v/49} \right) = \frac{2t}{5} \text{ so } 49 + v = (49 - v)(e^{2t/5}).$$

Simplifying, (by expanding and putting all the  $v$ s on one side of the equation) we find our final answer to be

$$v(t) = 49 \cdot \frac{e^{2t/5} - 1}{e^{2t/5} + 1} = 49 \tanh(t).$$

□

**Exercise 13.**

(a)

$$\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{RC} = \frac{VC - Q}{RC}$$

$$\Rightarrow RC \int \frac{dQ}{VC - Q} = t + C$$

Integrating, we get that  $t + C_1 = -RC \ln(VC - Q)$ . Plugging in our initial condition  $Q(0) = 0$ , we get that  $C_1 = -RC \ln(VC)$ . Thus, we can substitute and simplify as follows:

$$RC \ln(VC - Q) = RC \ln(VC) - t \rightarrow \ln(VC - Q) - \ln(VC) = -\frac{t}{RC}$$

$$\rightarrow VC - Q = VCe^{-t/RC}$$

so  $Q(t) = VC(1 - e^{-t/RC})$ .

(b) After a very long time ( $t \sim \infty$ ),  $Q \sim VC$  so  $Q_L = VC$ .

(c) Effectively,  $-\frac{Q}{C} = R \frac{dQ}{dt}$  since after the battery is removed, charge flows in the opposite direction.<sup>a</sup> Thus,  $t + C_1 = -RC \ln(Q)$ . Evaluating in our initial condition, we get that  $C_1 = -RC \ln(Q_L) + t_1$ . As a result,

$$-(t - t_1) = RC(\ln(Q) - \ln(Q_L))$$

so

$$Q = Q_L e^{-\frac{t-t_1}{RC}}.$$

□

---

<sup>a</sup>lowkey don't understand this fully - this just makes it work.