Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

The above sections are spacer sections.

6 The Laplace Transform

6.1 Definition of the Laplace

An improper integral is an integral that has an infinity in one of its bounds or has its function be otherwise discontinuous. Examples of discontinuous integrals are

$$\int_a^\infty f(t)\ dt \text{ or } \int_0^5 \frac{1}{x-3}\ dx.$$

Sometimes these integrals converge, sometimes they diverge.

Definition 6.1 (*Piecewise Continuous*)

We call a function f **piecewise continuous** over an interval $\alpha \leq t \leq \beta$ if we can find a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ such that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$, and
2. Approaching f's endpoints of each subinterval from within the subinterval results in a finite limit.

As the book sums it up nicely, f is piecewise ocntinuous if it is "continuous except for a finite number of jump discontinuities."

The integral of a piecewise function f is just the sum of its parts:

$$\int_\alpha^\beta f(t)\ dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\ dt = \int_\alpha^{t_1} f(t)\ dt + \int_{t_1}^{t_2} f(t)\ dt + \cdots + \int_{t_{n-1}}^{\beta} f(t)\ dt.$$

With that, we get our first theorem on divergence and convergence:

Definition 6.2 (*Integral Convergence (Theorem 6.1.1)*)

If f is piecewise continuous for $t \geq \alpha$ and $|f(t)| \leq g(t)$ when $t > C$ for some constant C, if $\displaystyle\int_C^\infty g(t)\ dt$ converges, then $\displaystyle\int_a^\infty f(t)\ dt$ also converges.

Moreover, if $f(t) \geq g(t) \geq 0$ for all $t \geq C$, if $\displaystyle\int_C^\infty g(t)\ dt$ diverges, then $\displaystyle\int_a^\infty f(t)\ dt$ also diverges.

Essentially, this theorem states that if f is bounded above and the bounding function is convergent, f is similarly convergent. Also, if f is bounded below and the integral of the bounding function diverges, then f is similarly divergent.

6.1.1 The Laplace Transform

Definition 6.3 (*Integral Transform*)

An integral transform is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t) f(t)\ dt$$

where K is called the **kernel** of the transformation and limits α and β are given. F is called the **transform** of f.

In our case, the laplace transform for a given function f is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

if this improper integral converges. Thus, the kernel function used here is $K(s,t) = e^{-st}$.

In general, to solve a differential equation, we use a laplace transform on a function f to derive F, solve for F, then un-transform and recover f. In general, s may also be a complex number.

Definition 6.4 (***Laplace Transform Existence (Theorem 6.1.2)***)

If f is piecewise continuous for all $t \geq 0$ and there exist constnats $K > 0$, a, and $M > 0$ such that

$$|f(t)| \leq Ke^{at} \text{ when } t \geq M,$$

then the laplace transform $\mathcal{L}\{f\} = F$ as defined above exists for all $s > a$.

If a function f satisfies Theorem 6.1.2, then f is described as being piecewise continuous and of **exponential order** as $t \to \infty$ (e.g. The highest 'order' the function can be is $O(e^{at})$).

// (Some examples of the Laplace transformation being used out in the wild are given!!) //
Note that the Laplace transform is a **linear operator**; that is,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

Exercise 6-11.
The laplace transform of $e^{\alpha t} = \dfrac{1}{s-b}$, assuming $s > b > 0$. Thus, for (6),

$$\mathcal{L}\{\cosh(bt)\} = \mathcal{L}\left\{\frac{e^{bt}}{2} + \frac{e^{-bt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{bt}\} + \frac{1}{2}\mathcal{L}\{e^{-bt}\} = \frac{1}{2(s-b)} + \frac{1}{2(s+b)}.$$

The rest of the exercises similarly follow;

$$(7)\text{: } \mathcal{L}\{\sinh(bt)\} = \frac{1}{2}\mathcal{L}\{e^{bt}\} - \frac{1}{2}\mathcal{L}\{e^{-bt}\} = \frac{1}{2(s-b)} - \frac{1}{2(s+b)}.$$

$$(8)\text{: } \mathcal{L}\{\sin(bt)\} = \frac{1}{2i}\mathcal{L}\{e^{ibt}\} - \frac{1}{2i}\mathcal{L}\{e^{-ibt}\} = \frac{1}{2i(s-ib)} - \frac{1}{2i(s+ib)} = \frac{(s+ib)-(s-ib)}{2i(s-ib)(s+ib)} = \frac{b}{s^2+b^2}.$$

$$(9)\text{: } \mathcal{L}\{\cos(bt)\} = \frac{1}{2}\mathcal{L}\{e^{ibt}\} + \frac{1}{2}\mathcal{L}\{e^{-ibt}\} = \frac{1}{2(s-ib)} + \frac{1}{2(s+ib)}.$$

$$\begin{aligned}(10)\text{: } \mathcal{L}\{e^{at}\sin(bt)\} &= \frac{1}{2i}\mathcal{L}\{e^{t(a+ib)}\} - \frac{1}{2i}\mathcal{L}\{e^{t(a-ib)}\} = \frac{1}{2i((s-a)-ib)} - \frac{1}{2i((s-a)+ib)} \\ &= \frac{(s-a+ib)-(s-a-ib)}{2i((s-a)+ib)((s-a)-ib)} = \frac{b}{b^2+(s-a)^2}.\end{aligned}$$

$$\begin{aligned}(11)\text{: } \mathcal{L}\{e^{at}\cos(bt)\} &= \frac{1}{2}\mathcal{L}\{e^{t(a+ib)}\} + \frac{1}{2}\mathcal{L}\{e^{t(a-ib)}\} = \frac{1}{2(s-a-ib)} + \frac{1}{2(s-a+ib)} \\ &= \frac{s-a+ib+s-a-ib}{2(s-a-ib)(s-a+ib)} = \frac{s-a}{(s-a)^2+b^2}.\end{aligned}$$

$\square$

Exercise 4.
Recall from above that $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f\, dt$. Now, to find $\mathcal{L}\{t^n\}$, assume we know $\mathcal{L}\{t^{n-1}\} = g(s)$ $(t > 1)$. Then, by integration by parts,

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt = \left[\frac{t^n}{s}e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}.$$

The second expression simply simplifies to $\dfrac{n}{s}g(s)$ by definition of the laplace transform, and the first expression is 0 as when plugging in the upper bound $t = 0$, $t^n = 0$, and when plugging the lower bound $t = \infty$, e^{st} outgrows t^n (assuming $s > 0$) so $\dfrac{1}{s} \cdot \dfrac{t^n}{e^{st}} = 0$ as $t \to \infty$. $\square$

Exercise 5.

$$\mathcal{L}\{\cos(at)\} = I = \int_0^\infty e^{-st}\cos(at)\ dt \Rightarrow \left[\frac{1}{s}e^{-st}\cos(at)\right]_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\ dt$$

$$= \frac{1}{s} - \frac{a}{s}\left(\left[\frac{1}{s}e^{-st}\sin(at)\right]_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\right) = \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s}I\right) \to \boxed{I = \frac{1}{s\left(1+\frac{a^2}{s^2}\right)}}.$$

□

Exercise 16-18.

16. $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f\ dt = \int_0^\pi e^{-st}\ dt = \frac{1}{s}(1-e^{-s\pi})$.

17. $\mathcal{L}\{f\} = \int_0^1 e^{-st} t\ dt + \int_1^\infty e^{-st}\ dt = \left[-\frac{1}{s}e^{-st}\right]_1^\infty + \left[\frac{t}{s}e^{-st}\right]_1^0 + \frac{1}{s}\int_0^1 e^{-st}\ dt = \frac{1-e^{-s}}{s^2}$.

18.

$$\mathcal{L}\{f\} = \int_0^1 e^{-st} t\ dt + \int_1^2 e^{-st}(2-t)\ dt = \left(-\frac{e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}\right) + 2\int_1^2 e^{-st}\ dt - \int_1^2 e^{-st} t\ dt$$

$$= \left(-\frac{e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}\right) + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s} - \left(\left[-\frac{t}{s}e^{-st}\right]_1^2 + \frac{1}{s}\int_1^2 e^{-st}\ dt\right)$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2} + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} + \frac{2}{s}e^{-2s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \boxed{\frac{1-2e^{-s}+e^{-2s}}{s^2}}.$$

□

Exercise 24.

24(ab): Making the substitution $x = st$ and $dx = s\ dt$ (and assuming $s > 0$ else the integral bounds must be switched),

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^p\ \frac{dx}{s} = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\ dx$$

which is exactly what needs to be shown. By definition, the integral we have in the simplified form of $\mathcal{L}\{t^p\}$ evaluates to $\Gamma(p+1)$[a], so

$$\mathcal{L}\{t^p\} = \frac{\Gamma p + 1}{s^{p+1}} = \frac{n!}{s^{n+1}}$$

if $n \in \mathbb{N}$.

24(c): We essentially make the substitution $x = y^2$ ($\sqrt{x} = y$, $dx = 2y\ dy$):

$$\mathcal{L}\{t^{-1/2}\} = \frac{1}{\sqrt{s}} = \int_0^\infty e^{-x}\frac{1}{\sqrt{x}}\ dx \ \longrightarrow\ \frac{1}{\sqrt{s}}\int_{\sqrt{0}=0}^{\sqrt{\infty}=\infty} e^{-y^2}\frac{1}{y}\ 2y\ dy = \frac{2}{\sqrt{s}}\int_0^\infty e^{-y^2}\ dy$$

which is exactly what's asked for.

24(d):

$$\mathcal{L}\{t^{1/2}\} = \frac{1}{s\sqrt{s}}\int_0^\infty e^{-x}\sqrt{x}\ dx \to \frac{1}{s\sqrt{s}}\int_0^\infty (y)\left(2ye^{-y^2}\right)\ dy = \frac{1}{s\sqrt{s}}\left(\left[-ye^{-y^2}\right]_0^\infty + \int_0^\infty e^{-y^2}\ dy\right) = \boxed{\frac{\sqrt{\pi}}{2s\sqrt{s}}}.$$

□

[a] the textbook question for this part of the question is wrong :/