Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

6 F

The above sections are spacer sections.

7 Systems of First-Order Linear Equations

7.1 Introduction

Essentially, we consider systems of first-order equations since any higher order differential equation can inevitably be transformed into multiple first order linear transformations.

Moreover, for any $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$, we cna make the substituions $x_1 = y$, $x_2 = y'$, $\dots$, $x_n = y^{(n-1)}$ and thus eventually find $x_1' = F_1(t, x_1, x_2, \dots, x_n)$, $x_2' = F_2(t, x_1, x_2, \dots, x_n)$ and so on. Thus, we have effectively converted a general differential equation into many teeny tiny first-order differential equations (that are each in their own way, granted, hard to solve).

7.2 Matrices

(note: all uppercase letters from here on out (A, B, C, $\dots$) will most likely represent matrices from here on out unless they are in function notation (e.g. $F(t)$ would be a function)).

Various matrix preliminaries are covered here. Do note that when the book talks about the **adjoint** of A, they mean the **transpose of the conjugate matrix of** A rather than the cofactor expansion matrix of A.

Integrals, derivatives, and [x] over matrices of functions are just those same operations applied to each individual operations (boring). For example, $\int A \, dt = \int a_{ij} \, dt$.

7.3 More Linear Algebra

(This is just a review of Math 4a......)

7.4 Basic Theory of Systems of First-Order Linear Equations

(a.k.a. A review of section 3.2 but with matrices instead of second-order linear differential equations.)
To examine a system of n first-order linear equations each of the form $x_i' = p_{i1}(t)x_1 + p_{i2}(t)x_2 + \dots + p_{in}(t)x_n + g_i(t)$, we can rewrite everything in matrix form and obtain the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T$, $\mathbf{g}(t) = [g_1(t) \ g_2(t) \ \dots \ g_n(t)]^T$, and $\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{pmatrix}$.

With matrix equations, multiple solutions ($\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(k)}(t)$) for $\mathbf{x}$ may exist. Moreover, if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two solutions to a first-order homogenous matrix differential equation ($\mathbf{g} = \mathbf{0}$), then $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution to said equation for arbitrary constants c_1, c_2 (Theorem 7.4.1, Page 305).

If we make a big matrix $\mathbf{X} = [\mathbf{x}^{(1)} \ \mathbf{x}^{(2)} \ \dots \ \mathbf{x}^{(n)}]$, then we can calculate its determinant; namely, $\det \mathbf{X} = W[\mathbf{x}^{(1)} \ \dots \ \mathbf{x}^{(n)}]$ and as such if $\det \mathbf{X} \neq 0$ at some particular point $t = t_0$, then the solutions $\mathbf{x}^{(1)}$, $\dots$ are all linearly independent at t_0.

Definition 7.1 (*Generalized Abel's Theorem*)

If $\mathbf{x}^{(1)}$, $\dots$, $\mathbf{x}^{(n)}$ are solutions to a homogenous first-order set of linear differential equations over some open interval I, then over I, either $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}] = 0$ or $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) \neq 0 \ \ \forall t \in I$.

Abel's theorem is super helpful as we only need to evaluate the Wronskian / determinant over one point to conclude the linear dependence/independence of our solutions. (Note: some stuff about a fundamental set of solutions is talked about here but honestly I don't really care :/.)

Similarly to when we looked at real-valued solutions to differential equations, we can turn complex-valued solutions into real solutions:

Definition 7.2 (*Theorem 7.4.5*)

If $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is a solution to the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then solely the real part $\mathbf{u}$ and solely the imaginary part $\mathbf{v}$ are also solutions to the above equation.

(I don't see much use in doing the exercises here as they are just proofs about theorems from section 3.2 in matrix form. None are like super interesting.)

7.5 Constant Coefficients and Matrices

This subsection focuses on equations of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a $n \times n$ matrix of real-valued constants.

Assuming[1] $\mathbf{x} = \xi e^{rt}$, after substituting it into the above equation, we eventually derive the equation

$$(\mathbf{A} - r\mathbf{I})\xi = 0,$$

which means solutions to $\mathbf{x}$ are given pairs of eigenvalue-eigenvector combinations (r, ξ). When $\mathbf{A}$ is specifically a 2×2 matrix, if the eigenvalues of $\mathbf{A}$ have opposite signs, then the origin is a saddle point and an unstable equilibrium. If on the other hand, the eigenvalues of $\mathbf{A}$ have the same sign, then the origin is a **node** and $\mathbf{0}$ is a stable equlibrium if the eigenvalues are negative and unstable if the eigenvalues are positive.

Returning to the more general case of when $\mathbf{A}$ is a $n \times n$ matrix, the eigenvalues of $\mathbf{A}$ (r_1, r_2, $\ldots$, r_n) can either be

1. all real and different from one another,
2. some eigenvalues are complex conjugate pairs of each other, or
3. some eigenvalues are repeated.

The first case is easy to take care of; if all n eigenvalues are real and different, then their corresponding eigenvectors ($\xi^{(i)}$) will all be linearly independent and as such $\mathbf{x} = c_1\xi^{(1)}e^{r_1 t} + \cdots + c_n\xi^{(n)}e^{r_n t}$. Section 7.6 deals with case two, of when some eigenvalues are complex conjugate pairs of each other. Section 7.8 deals with the case of repeated eigenvalues.

Remember: To find eigenvalues, solve the characteristic polynomial $\det(\mathbf{A} - r\mathbf{I}_n) = 0$, and to get the corresponding eigenvectors, RREF $\mathbf{A} - r\mathbf{I}_n$ for a given r.

Exercise 7-9.
7. After finding our eigenvalues to be $r = \{4, -1, 1\}$ and the associated eigenvectors to be $\xi = \{[1\ 1\ 1]^T, [1\ 0\ -1]^T, [-1\ 2\ -1]^T\}$, we conclude that $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} e^{t}$ for arbitrary constants c_1, c_2, c_3.

8. This is a nice symmetric matrix with eigenvalues. Solving, we eventually find the characteristic polynomial to be $(\lambda - 8)(\lambda + 1)(\lambda + 1) = 0$ which means the corresponding eigenvaues are $\lambda = r = 8, -1$. Solving for eigenvectors and plugging them in, we find $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$.

9. $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. □

[1] Some stuff about a phase portrait/plane is talked about here although those tools are primarily used for visualization purposes.

Exercise 20.

20a: Note that this problem operates under the assumption A and $\xi^{(1)}$ are matrices with constant coefficients. For the sake of the argument, assume $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{v} \neq \mathbf{0}$. As such, since matrix multiplication is distributive, we can rearrange the equation and get $\mathbf{A}\xi^{(1)} = r_1\mathbf{I}\xi^{(1)} + \mathbf{v} = r_1\xi^{(1)} + \mathbf{v}$ as per properties from the identity matrix.

Returning back to our original equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, since the given solution for $\mathbf{x}$ holds for any coefficients c_1 and c_2, we let $c_1 = 1$ and $c_2 = 0$ and thus a particular solution to our differential equation is $\mathbf{x} = \xi^{(1)}e^{r_1 t}$. We plug this particular solution of our differential equation into our differential equation and derive

$$\left(\xi^{(1)}e^{r_1 t}\right)' = \mathbf{A}\left(\xi^{(1)}e^{r_1 t}\right) \to r_1 e^{r_1 t}\xi^{(1)} = e^{r_1 t}\mathbf{A}\xi^{(1)} \to r_1\xi^{(1)} = \mathbf{A}\xi^{(1)}.$$

Substituting in our other expression for the RHS, we find $r_1\xi^{(1)} = r_1\xi^{(1)} + \mathbf{v}$ which implies $\mathbf{v} = \mathbf{0}$, a contradiction! Thus, our original assumption that $\mathbf{v} \neq \mathbf{0}$ is false and $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0}$. (A symmetrical argument holds for proving why $(\mathbf{A} - r_2\mathbf{I})\xi^{(2)} = \mathbf{0}$.)

20b: $(\mathbf{A} - r_2\mathbf{I})\xi^{(1)} = \mathbf{A}\xi^{(1)} - r_2\xi(1) = r_1\xi^{(1)} - r_2\xi(1)$ since $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0} \to \mathbf{A}\xi^{(1)} = r_1\xi^{(1)}$.

20c: $(\mathbf{A} - r_2\mathbf{I})(c_1\xi^{(1)} + c_2\xi^{(2)}) = c_1(\mathbf{A} - r_2\mathbf{I})\xi^{(1)} + c_2(\mathbf{A} - r_2\mathbf{I})\xi^{(2)} = c_1(r_1 - r_2)\xi^{(1)}$. However, $(\mathbf{A} - r_2\mathbf{I})(c_1\xi^{(1)} + c_2\xi^{(2)}) = (\mathbf{A} - r_2\mathbf{I})\mathbf{0} = \mathbf{0}$. Thus, $c_1(r_1 - r_2)\xi^{(1)} = \mathbf{0}$ which means either $c_1 = 0$, $r_1 = r_2$, or $\xi^{(1)} = \mathbf{0}$ (or some combination of the above).

Since we assume $r_1 \neq r_2$, the second statement must be false, and if $\xi^{(1)} = \mathbf{0}$, then the wronskian of our solution would be 0, contradicting our assumption that the general solution for $\mathbf{x}$ is a fundamental solution with respect to $\xi^{(1)}e^{r_1 t}$ and $\xi^{(2)}e^{r_2 t}$. Thus, this leads us to conclude that $c_1 = 0$ which is a contradiction, meaning our original supposition (that $\xi^{(1)}$ and $\xi^{(2)}$ are linearly dependent) is false. □

Exercise 21.

21a: We can rewrite this second order linear differential as

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2 \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

21b: The eigenvalue equation is simply the determinant of $\mathbf{A} - \lambda\mathbf{I}$ which is

$$0 = -\lambda\left(-\frac{b}{a} - \lambda\right) - (1)\left(-\frac{c}{a}\right) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \to a\lambda^2 + b\lambda + c.$$

□

Exercise 24.

24a. The general equation for eigenvalues of this matrix is $\lambda^2 + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\lambda + \left(\frac{R_1 + R_2}{CLR_2}\right) = 0$. Plugging in the specific values given, we eventually find eigenvalues of -1 and -2 and the general solution corresponding is $\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}$. Specific values that make the initial conditions hold are $c_1 = \dfrac{3I_0 - V_0}{2}$ and $c_2 = \dfrac{V_0 - I_0}{2}$.

24b. Since both terms in the above equation have e^{-t} in them, as $t \to \infty$ $\begin{pmatrix} I \\ V \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since the decay of e^{-t} is too much for the constants in the equation to handle. □

Exercise 25.
25a. $b^2 - 4ac > 0$ so $\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4\left(\frac{R_1+R_2}{CLR_2}\right) > 0$.
25b. Since all the coefficients in the characteristic polynomial are positive since R_1, R_2, C, and L cannot take on negative values (negative resistance?!), it is impossible for an eigenvalue λ to be greater than 0 as each term in the equation would thus be bigger than 0. As such, both I and V will eventually become 0 as t goes to ∞.
25c. If the eigenvalues are complex or repeated, it is possible that both I and/or V blow up to infinity (e.g. if you have negative resistance that adds energy to the circuit) or I and V settle into a stable equilibrium solution (see an LC clock circuit — no energy is being lost since there is no resistor in the circuit). $\square$

7.6 Complex Eigenvalues

Essentially, if the eigenvalues of $\mathbf{A}$ are complex[2] (in conjugate pairs), then the resulting graph of $\mathbf{x}$ will look like a spiral either convergering at the origin or diverging away from it (depending on whether the eigenvalues have positive or negative real part). If the real part is 0, then the graph will basically make a loop around the origin.

For the general linear differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $r_2 = \overline{r_1}$, then $\xi^{(2)} = \overline{\xi^{(1)}}$. Thus, a particular solution to $\mathbf{x}$ would be

$$\mathbf{x} = \mathbf{x}^{(1)} = \xi^{(1)} e^{r_1 t} \to (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} = e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) + ie^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t))$$

where we made the substitutions $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$ and $r_1 = \lambda + i\mu$. Thus, if we let $\mathbf{x}^{(1)}t = \mathbf{u}(t) + i\mathbf{v}(t)$ with $\mathbf{u}$ and $\mathbf{v}$ corresponding to the real and imaginary parts above, we have a particular solution for $\mathbf{x}$.
But since we want a real solution for $\mathbf{x}$, by that one weird theorem (Theorem 7.4.5) covered above, $\mathbf{u}$ and $\mathbf{v}$ are themselves individual solutions to $\mathbf{x}$ and we can write $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + \dots$ where the dots indicate the solutions stemming from the other roots of $\mathbf{x}$.

Exercise 5-6.
5. Our eigenvalues for this matrix are $\lambda = 1, 1 \pm 2i$. The eigenvector for $\lambda = 1$ is $\xi = [2\ \ -3\ \ 2]^T$, and the eigenvector for $\lambda = 1 + 2i$ is $\xi = [0\ 0\ 1]^T + i[0\ 1\ 0]^T$. Thus, our final solution is

$$\mathbf{x} = c_1 e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin(2t) \right) + c_2 e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos(2t) \right) + c_3 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t.$$

6. This one was hard (calculation-wise) and the answer is dicey. The eigenvalues are $\lambda = -2, -1 \pm i\sqrt{2}$ and the final solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + c_2 e^{-t} \begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}.$$

$\square$

[2] All entries in $\mathbf{A}$ must be real as otherwise complex roots may not come in conjugate pairs.

Exercise 20.
20a. By Kirchoff's voltage law, across the triangle formed by R_1, C, and L, we find

$$L\frac{dI}{dt} + V + R_1 I = 0 \to \frac{dI}{dt} = -\frac{R_1}{L}I - \frac{1}{L}V$$

which is the first row of our matrix equation.
Summing the current at the point joined by C, R_2, and L, we find

$$\frac{V}{R_2} + C\frac{dV}{dt} = I \to \frac{dV}{dt} = \frac{1}{C}I - \frac{1}{CR_2}V.$$

Note that the reason why voltage across resistor R_2 is V is because summing the voltage loop over the rectangle C and R_2, we must have $V + (-V_{R_2}) = 0$ so $V_{R_2} = V$. Note that there is a negative sign over V_{R_2} as when tracing the loop across the resistor, our loop goes against the current of the circuit.
Thus, turning both equations into matrix form, we have

$$\begin{cases} \frac{dI}{dt} = -\frac{R_1}{L}I - \frac{1}{L}V \\ \frac{dV}{dt} = \frac{1}{C}I - \frac{1}{CR_2}V \end{cases} \to \begin{bmatrix} I' \\ V' \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}.$$

20b. The particular eigenvalues for the matrix $\mathbf{A}$ given in the problem are $\lambda = -\frac{1}{2} \pm i\frac{1}{2}$. As such, the corresponding eigenvectors are $\xi = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. As such, the final general solution we find (after plugging everything in) is:

$$\begin{bmatrix} I \\ V \end{bmatrix} = c_1 e^{-\frac{t}{2}} \left(\begin{bmatrix} -\sin\left(\frac{t}{2}\right) \\ -4\cos\left(\frac{t}{2}\right) \end{bmatrix} \right) + c_2 e^{-\frac{t}{2}} \left(\begin{bmatrix} \cos\left(\frac{t}{2}\right) \\ -4\sin\left(\frac{t}{2}\right) \end{bmatrix} \right).$$ [a]

20c. For the particular case given above, $c_1 = -\frac{3}{4}$ and $c_2 = 2$.
20d. Since the exponential in both terms of the equation are negative, as $t \to \infty$, both I and V tend towards 0 regardless of the initial conditions given. □

[a] Note: The solution in the back of the book uses the eigenvector $\xi = [1 \quad -4i]^T$.

Exercise 21.
21a. The characteristic polynomial of this matrix is $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC}$ from which it follows that the discriminant is $\frac{1}{R^2C^2} - \frac{4}{LC} = L - 4R^2C$.
21b. $\lambda = -1 \pm i$, and a specific eigenvector that's yielded is $\xi = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. As such, our final general solution is

$$\begin{bmatrix} I \\ V \end{bmatrix} = e^{-t} \left(c_1 \begin{bmatrix} \cos(t) - \sin(t) \\ -2\cos(t) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(t) - \cos(t) \\ -2\sin(t) \end{bmatrix} \right).$$ [a]

21c. $c_1 = -\frac{1}{2}$, $c_2 = -\frac{5}{2}$.
21d. Obviously not, as $t \to \infty$, $I \to 0$ and $V \to 0$ regardless of c_1 and c_2. □

[a] I think I'm right; the fact that $(1+i)(1-i) = 2$ is making my answer not match to the one in the back of the book. Honestly atp idfk.

These problems are genuinely ragebaiting me.

7.7 Fundamental Matrices

So it turns out learning/getting a glimpse of this section (7.7) is pretty helpful for understanding the rest of the content in the chapter.

7.7.1 Fundamental Matrices

Suppose $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ..., $\mathbf{x}^{(n)}$ are a set of fundamental solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ (their Wronskian is non-zero). Then, we define the matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)}\end{pmatrix} = \begin{pmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{pmatrix}$$

to be a **fundamental matrix** for the nonhomogenous linear system described above. Note that this fundamental matrix $\mathbf{\Psi}$ has an inverse as all columns of $\mathbf{\Psi}$ are linearly independent.
As such, to solve any initial value problem $\mathbf{x}(t_0) = \mathbf{x_0}$, we can write

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} \to \mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x_0} \to \mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x_0} \;\longrightarrow\; \mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x_0}$$

where $\mathbf{c}$ is a matrix of constants.

Note that we also define a special matrix $\mathbf{\Phi}$ with the property that

$$\mathbf{\Phi}(t_0) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \mathbf{I} \;\longleftrightarrow\; \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0).$$

7.7.2 Matrix Exponentiation

Since the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x} \to \mathbf{x} = \mathbf{\Phi}\mathbf{x}^0$ looks similar to the first-order differential equation $x' = ax \to x = x_0 e^{at}$, it seems tempting to exponentiate $\mathbf{A}$ and we can in fact do so by defining

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

with this definition also satisfying the property that $\frac{d}{dt}\exp \mathbf{A}t = \mathbf{A}\exp(\mathbf{A}t)$. Notably, it turns out that $\mathbf{\Phi} = \exp(\mathbf{A}t)$ as a result that they satisfy the same initial condition $\exp \mathbf{A}t|_{t=0} = \mathbf{\Phi}(0) = \mathbf{I}$.[3]

7.7.3 Diagonalizable Matrices

Solving simultaneous systems of equations is hard. What if we could solve each equation individually instead?
Suppose matrix $\mathbf{A}$ has n linearly eigenvectors $\xi^{(1)}$, $\xi^{(2)}$, ..., $\xi^{(n)}$ and associated eigenvalues λ_1, ..., λ_n. We define the matrix $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \dots & \xi_1^{(n)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \dots & \xi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \xi_n^{(2)} & \dots & \xi_n^{(n)} \end{pmatrix}$$

where $\xi_j^{(i)}$ represents the ith eigenvalue of $\mathbf{A}$ and j simply indexes the row of that eigenvalue. Since $\mathbf{A}\xi^{(k)} = \lambda_k \xi^{(k)}\ \forall k$,

$$\mathbf{AT} = \begin{pmatrix} \lambda_1\xi_1^{(1)} & \lambda_2\xi_1^{(2)} & \dots & \lambda_n\xi_1^{(n)} \\ \lambda_1\xi_2^{(1)} & \lambda_2\xi_2^{(2)} & \dots & \lambda_n\xi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\xi_n^{(1)} & \lambda_2\xi_n^{(2)} & \dots & \lambda_n\xi_n^{(n)} \end{pmatrix} = \mathbf{T}\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

[3]Assuming we let $\mathbf{\Phi}(t_0) = \mathbf{\Phi}(0) = \mathbf{I}$.

where that last matrix of eigenvalues is denoted as $\mathbf{D}$. Thus, it follows that $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{D}$ and if $\mathbf{A}$ can be transformed into $\mathbf{D}$ like this, we say $\mathbf{A}$ is **diagonalizable** and **similar** to $\mathbf{D}$.

Returning back to our original system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $\mathbf{A}$ is diagonalizable, then we can define a new matrix $\mathbf{y}$ characterized by $\mathbf{x} = \mathbf{T}\mathbf{y}$ from which it follows $\mathbf{y}' = \mathbf{D}\mathbf{y}$, which is easily solvable since $\mathbf{D}$ is a linearly independent and mostly empty matrix.
Namely, $\mathbf{\Psi} = \mathbf{T}\exp(\mathbf{D}t)$.

7.8 Repeated Eigenvalues

So what happens if there is an eigenvalues of $\mathbf{A}$ that has a multiplicity $m > 1$? Long story short, we basically use the variation of parameters method with e^{rt} and te^{rt} to find our answer.

In more detail, assume that there is an eigenvalue $\lambda = \lambda_m$ such that the multiplicity of λ_m is m (e.g. $(\lambda_m - \lambda)^m$ is in the characteristic polynomial of $\mathbf{A} - \lambda\mathbf{I}$). In this case, one trivial solution is simply $\mathbf{x} = \xi e^{\lambda_m t}$. Assuming no other linearly independent eigenvectors can be found, to find the second solution, we let $\mathbf{x} = \alpha_1 te^{\lambda_m t} + \alpha_2 e^{\lambda_m t}$, plug this particular solution into the differential equation, and simplify to find what α_1 and α_2 are. We then repeat this process (e.g. assume $\mathbf{x} = \alpha_1 \frac{t^n}{n!} e^{\lambda_m t} + \alpha_2 \frac{t^{n-1}}{(n-1)!} e^{\lambda_m t} + \cdots + \alpha_{n+1} e^{\lambda_m t}$) until we get m linearly independent solutions for $\mathbf{x}$. Note that your various solutions for $\mathbf{x}$ along the way will include previous derived solutions (e.g. $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \frac{t^2}{2} e^t + \begin{pmatrix} 100 \\ -32 \\ -101 \end{pmatrix} te^t + \begin{pmatrix} \pi \\ \zeta(3) \\ e^{e^\gamma} \end{pmatrix} e^t$).

(Some stuff about Jordan Forms and Fundamental Matrices discussed in Section 7.7 are brought up for discussion again here.)

Exercise 4.

4. The characteristic equation is $(\lambda - 2)^2(\lambda + 1) = 0$ which leads to the eigenvectors $\xi = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ with the latter ξ coming from the eigenvalue $\lambda = 2$.
To find the extra solution then, we let $\mathbf{x} = \alpha te^{2t} + \beta e^{2t}$ and find

$$2\beta e^{2t} + \alpha e^{2t} + 2\alpha te^{2t} = \mathbf{A}(\alpha te^{2t} + \beta e^{2t}).$$

Comparing coefficients, we come to the conclusion that $2\alpha = \mathbf{A}\alpha$ and $2\beta + \alpha = \mathbf{A}\beta$. The first equation ($(\mathbf{A} - 2I)\alpha = \mathbf{0}$) is already solved for us as that was the eigenvalue we found before, so we then solve $(\mathbf{A} - 2I)\beta = \alpha$, finding the general solution to be $\beta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - a \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

As such, our final solution is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - a \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

since the third term in the solution is already covered by our general solution. As such, the general solution for the system of equations is

$$\mathbf{x} = c_1 \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 e^{2t} \begin{bmatrix} 1 \\ t \\ 1 - t \end{bmatrix}.$$

□

Exercise 5.
Following the same process as in problem 4, we find the characteristic polynomial to be $(\lambda-2)(\lambda+1)^2=0$, and the eigenvectors to be $\begin{bmatrix}1\\1\\1\end{bmatrix}$ for $\lambda=2$ and $\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ to be the eigenvectors for $\lambda=-1$. Since the eigenvectors for the repeated eigenvalue we have found are linearly independent, we can proceed directly to the solution and conclude that

$$\mathbf{x}=c_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^{2t}+c_2\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{-t}+c_3\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-t}.$$

□

Exercise 6a-10a.
6a. $\mathbf{x}=2\begin{bmatrix}1\\1\end{bmatrix}e^{-3t}+4\begin{bmatrix}\frac{1}{4}+t\\t\end{bmatrix}e^{-3t}=\begin{bmatrix}3+4t\\2+4t\end{bmatrix}e^{-3t}.$
7a. $\mathbf{x}=-\begin{bmatrix}1\\1\end{bmatrix}e^{-t}-6\begin{bmatrix}-\frac{2}{3}+t\\t\end{bmatrix}e^{-t}=\begin{bmatrix}3+6t\\-1+6t\end{bmatrix}e^{-t}.$
8a. $\mathbf{x}=4\begin{bmatrix}-3\\1\end{bmatrix}-14\begin{bmatrix}-1-3t\\t\end{bmatrix}=\begin{bmatrix}2+42t\\4-14t\end{bmatrix}.$
9a. $\mathbf{x}=3\begin{bmatrix}0\\0\\1\end{bmatrix}e^{2t}+2\begin{bmatrix}0\\1\\-6\end{bmatrix}e^{t}+4\begin{bmatrix}-\frac{1}{4}\\t\\-\frac{21}{4}-6t\end{bmatrix}e^{t}=\begin{bmatrix}0\\0\\3\end{bmatrix}e^{2t}+\begin{bmatrix}-1\\2+4t\\-33-24t\end{bmatrix}e^{t}.$
10a. $\mathbf{x}=\frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}e^{-\frac{t}{2}}-\frac{5}{3}\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{-\frac{7t}{2}}+\frac{7}{3}\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-\frac{7t}{2}}.$

□

Exercise 14.
14a. Trivial; the discriminant is literally $L-4R^2C$ so the result follows.
14b. The repeated root is $\lambda=-\frac{1}{2}$, and the solution is

$$\begin{bmatrix}I\\V\end{bmatrix}=-\begin{bmatrix}1\\-2\end{bmatrix}e^{-\frac{t}{2}}+\begin{bmatrix}2+t\\-2t\end{bmatrix}e^{-\frac{t}{2}}=\begin{bmatrix}1+t\\2-2t\end{bmatrix}e^{-\frac{t}{2}}.$$

□

Exercise 15.
15a. $(\mathbf{A}-2\mathbf{I})((\mathbf{A}-2\mathbf{I})\eta)=\mathbf{0}\to(\mathbf{A}-2\mathbf{I})^2\eta=\mathbf{0}$.
15b. In this case, we can manually verify that $\begin{pmatrix}-1&-1\\1&1\end{pmatrix}^2=\mathbf{0}$.
15c. Simple matrix multiplication shows $\xi=(1,-1)^T$.
15d. More matrix multiplication shows $\xi=(-1,1)^T$.
15e. As long as $k_1\neq-k_2$, ξ and η will be independent; $\xi=(-k_1-k_2,k_1+k_2)^T$. ξ in this case will be a multiple of $\xi^{(1)}$. □

Exercise 17a-d.
17a. The characteristic polynomial ends up being $-\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3 = 0$ which yields an eigenvalue 2 of multiplicity 3. Going eigenvector hunting reveals that indeed, $\xi = (0, 1, -1)^T$ is the only eigenvector.
17b. $\mathbf{x}^{(1)} = \xi e^{2t}$.

17c. I mean they just kinda do satisfy those equations. Solving for η and neglecting the part we already found, we find $\eta = (1, 1, 0)^T$, meaning our second solution is literally $\mathbf{x}^{(2)} = \xi t e^{2t} + \eta e^{2t}$ for the ξ and η we have found.

17d. One particular solution for ζ is $\zeta = (-2, 3, 0)^T$, which means our final general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) e^{2t} + c_3 \left(\frac{t^2}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \right) e^{2t}.$$

□

7.9 Nonhomogenous Linear Systems

Finally, we return to the opening section and consider differential equation systems of the form

$$\mathbf{x}' = \mathbf{P}(t)x + \mathbf{g}(t).$$

Like in section 3.5, we can express all solutions $\mathbf{x}$ as $\mathbf{x}_c + \mathbf{x}_p$ where $\mathbf{x}_c$ is the solution to the homogenous differential equation $\mathbf{x}' = \mathbf{P}\mathbf{x}$ and $\mathbf{x}_p$ is a solution to the nonhomogenous system described above. So how do we find $\mathbf{x}_p$?

7.9.1 Diagonalization

In the differential system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$, if we assume $\mathbf{A}$ is diagonalizable, we can make the substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$ and find

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g} \ \rightarrow \ \mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$$

which is simply a set of n uncoupled (unrelated(?)) first order linear differential equations that can each be solved separately, and $\mathbf{x}$ can be recovered by left-multiplying $\mathbf{y}$ by $\mathbf{T}$.
It is also possible to solve for $\mathbf{x}$ even if $\mathbf{A}$ is not diagonalizable by reducing $\mathbf{A}$ to a jordan form $\mathbf{J}$ (I have no clue what this is) from which it's possible to solve for $\mathbf{J}$ from the last row to the top (as most rows have differential equations that are not totally uncoupled).

If that seems too hard, you can always try the method of:

7.9.2 Undetermined Coefficients (Hard ver).

(Note that this method is really only applicable when $\mathbf{P}$ is a bunch of constants and all terms in $\mathbf{g}$ look simple enough to be 'guessed'.)
Note that the difficulty level in this mode of undetermined coefficients is upped since generally, if there is a term in $\mathbf{g}$ of the form $\mathbf{u}e^{\lambda t}$, the solution must be assumed to be of the form $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$ for coefficient constant matrices $\mathbf{a}$ and $\mathbf{b}$.

7.9.3 Variation of Parameters

Assuming a (general) fundamental matrix $\mathbf{\Psi}$ has been found for the homogenous version of the differential equation $\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$, we seek solutions to the nonhomogenous system by replacing the constant vector $\mathbf{c}$ that would normally be multiplied to $\mathbf{\Psi}$ ($\mathbf{\Psi}\mathbf{c}$) by a vector of functions $\mathbf{u}$. Thus, letting $\mathbf{x} = \mathbf{\Psi}\mathbf{u}$, we have

$$\mathbf{x}' = \mathbf{\Psi}'\mathbf{u} + \mathbf{\Psi}\mathbf{u}' = \mathbf{P}\mathbf{\Psi}\mathbf{u} + \mathbf{g} \ \rightarrow \ \mathbf{\Psi}\mathbf{u}' = \mathbf{g}$$

since $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}$ since $\mathbf{\Psi}$ is after all, a solution for the homogenous version of $\mathbf{x}$.

Since the inverse of $\mathbf{\Psi}$ exists (since by definition the columns of $\mathbf{\Psi}$ ($\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$) are linearly independent),

$$\mathbf{u} = \int \mathbf{\Psi}^{-1}\mathbf{g}\, dt + \mathbf{c}$$

for arbitrary constant vector $\mathbf{c}$. Hopefully the integrals in the above equation can be evaluated because if not, a direct solution to the differential equation might not be possible 💀.[4]

7.9.4 Laplace Transform (Hard Version)

We can define a laplace transform over a vector as simply the vectors whos respective elemetns are the laplace transform of the elements in the original vector.
By an extension of the Laplace transform then,

$$\mathcal{L}\{\mathbf{x}'(t)\} = s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}(0)$$

which means we can transform the differential system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$ to $s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\{\mathbf{x}\} + \mathcal{L}\{\mathbf{g}\}$. For simplicity, if we are not solving an initial value problem, we can set $\mathbf{x}(0) = \mathbf{0}$ and do some messy calculations to find $\mathcal{L}\{\mathbf{x}\}$ from which we can do an inverse transform to find $\mathbf{L}$ (warning: messy).

Exercise 1.
Letting $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = \mathbf{A}$ for simplicity, we thus have to solve $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$. First, we find the general solution; chopping off the last term and solving a simple system with eigenvalues $\lambda = 1, -1$, we find our complementary solution to be $\mathbf{x_c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. As such, we now seek a particular solution.
Using the method of undetermined coefficients, since e^t is an eigenvalue root, we must assume that a particular solution $\mathbf{x_p}$ looks of the form $\mathbf{x_p} = \mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}$. Differentiating and plugging in both sides, we conclude that

$$\mathbf{a}te^t + (\mathbf{a}+\mathbf{b})e^t + \mathbf{c} = \mathbf{A}\mathbf{a}te^t + \mathbf{A}\mathbf{b}e^t + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$$

Matching the coefficients (te^t, e^t, t, 1) on both sides of the equation, we thus have the system

$$\begin{cases} \mathbf{A}\mathbf{a} = \mathbf{a} \\ \mathbf{A}\mathbf{b} = \mathbf{a} + \mathbf{b} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{A}\mathbf{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \mathbf{A}\mathbf{d} = \mathbf{c} \end{cases}$$

to solve. To take the simpler ones, the third equation simply tells us $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and similarly $\mathbf{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The first equation tells us $\mathbf{a}$ is of the form $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$, and plugging that in to the second equation and rearranging, we have $(\mathbf{A} - \mathbf{I})\mathbf{b} = \begin{pmatrix} \alpha - 1 \\ \alpha \end{pmatrix}$. Since a simplification of the left hand side reveals $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \mathbf{b} = \begin{pmatrix} \alpha - 1 \\ \alpha \end{pmatrix}$, it follows that $3(\alpha - 1) = \alpha$ or $\alpha = \frac{3}{2}$ and thus the general solution for $\mathbf{b}$ is $\begin{pmatrix} \frac{1}{2} + k \\ k \end{pmatrix}$. Taking $k = 0$ for simplicity, our final answer to the equation (which an online solver agrees with) is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

□

[4] Me after reading this section: ☠️.

Note: for all the exercises below, there's almost no consensus (from me, wolfram alpha, the textbook, and an online solver) on what the right answer is. Tread carefully here.

Exercise 2.
Again, we're going to let $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ for ease of writing. Solving for the complementary solution yields eigenvalues of $\lambda = \pm i$ and a particular solution of $\mathbf{x_c} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$.
Now for the fun part; the particular solution, and in this case, we'll use the method of variation of parameters. Recalling that our general matrix in this case is simply $\mathbf{\Psi} = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}$ (which has a non-zero Wronskian everywhere since $\det \mathbf{\Psi} = -1$), we can jump straight ahead to the conclusion and try to evaluate $\mathbf{u}$ by finding

$$\mathbf{u} = \int \mathbf{\Psi}^{-1}\mathbf{g}\; dt + \mathbf{c}.$$

Recalling that an inverse of a 2-by-2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, it turns out we can simplify our above expression and find

$$\mathbf{u} = \int \begin{pmatrix} -\sin t & 2\sin t + \cos t \\ \cos t & -2\cos t + \sin t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} dt + \mathbf{c} = \int \begin{pmatrix} \sin(2t) + 1 - \cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} dt + \mathbf{c}$$

Integrating each part, we find

$$\mathbf{u} = \begin{pmatrix} t - \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t) \\ \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) \end{pmatrix} + \mathbf{c}$$

and taking $\mathbf{c} = 0$ and using the equation $\mathbf{x_{(p)}} = \mathbf{\Psi}\mathbf{u}$ (and many simplifications and tricky substitutions (notably $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$) and the cosine addition formula), we find that

$$\mathbf{x_p} = \begin{pmatrix} 2t\cos(t) - t\sin(t) - \frac{3}{2}\sin(t) - \frac{1}{2}\cos(t) \\ t\cos(t) - \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{pmatrix}$$

so our final general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} + \begin{pmatrix} 2t\cos(t) - t\sin(t) - \frac{3}{2}\sin(t) - \frac{1}{2}\cos(t) \\ t\cos(t) - \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{pmatrix}.$$

□