Differential Equations Notes

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 Second-Order Linear Differential Equations

Or differential equations of the form

$$y'' + p(t)y' + q(t)y = g(t).$$

3.1 Homogenous Second-Order Equations

Remember, a **linear** second-order differential equation is of the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Nonlinear differential equations are super hard and annoying to tackle and as such they're just not tackled in this book :/.

In second-order differential equations, a problem with an initial condition has initial condition of the form

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}.$$

Note that there are two initial equations given - the location of y at time t_0, and the slope of y at time t_0.

Definition 3.1 (*Homogenous*)

A **homogenous** differential equation has no 'constant' terms (terms without y). In the case for our second-order linear differential equations, a homogenous equation of that form can be written as

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

Anyways, it turns out if we solve the homogenous version of the differential equation $P(t)y'' + Q(t)y' + R(t)y = G(t)$, we can actually find an expression for y (that may or may not have an integral in it). That's pretty cool.

For this chapter (unfortunately), we will only consider the cases when $\boxed{P,\ Q,\ \text{and } R \text{ are } \textbf{constants}}$.

Thus, our differential equation becomes $ay'' + by' + cy = 0$. Letting $y = e^{rt}$, we find that our equation now becomes

$$\to ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0$$

with $ar^2 + br + c$ called the **characteristic equation** for the general differential equation with constant coefficients shown above.

If we let r_1 and r_2 be two real roots that satisfy the characteristic equation above, then the **general solution** to our differential equation is $y = c_1e^{r_1t} + c_2e^{r_2t}$ with c_1 and c_2 being arbitrary constants. Initial conditions can be solved for summarily.

Exercise 1-4.
1. $y = c_1e^t + c_2e^{-3t}$.
2. $y = c_1e^t + c_2e^{2t}$.
3. $y = c_1e^{t/2} + c_2e^{-t/3}$.
4. $y = c_1 + c_2e^{-5t}$. □

Exercise 13.
If a differential equation's solution is $c_1e^2t + c_2e^{-3t}$, we have $r_1 = 2$, $r_2 = -3$ and as such our differential equation is $y'' + y' - 6y = 0$.
It probably can be shown that no other differential equation produces the general solution given in the problem. □

Exercise 16.
The characteristic equation for our differential equation is $n^2 - n - 2 = 0$ and as such we have roots $r_1 = 2$, $r_2 = -1$. As such, the general solution to the equation is $y = c_1 e^{2t} + c_2 e^{-t}$.
To make the solution approach 0 as $t \to \infty$, we need $c_1 = 0$ as in any other case, e^{2t} will spiral out to infinity and our solution is unbounded. Thus, we can plug this solution into the second part of the initial value problem $y'(0) = 2$:

$$y'(0) = 2 \to 2 = 2 \cdot 0e^{2t} + (-1) \cdot c_2 e^{-0} \to c_2 = -2.$$

Thus, our final solution to the differential equation is $y_{sol} = -2e^{-t}$ and $y_{sol}(0) = \alpha = -2$. $\square$

3.2 Solutions of Linear Homogenous Equations — the Wronskian

Definition 3.2 (*Differential Operator L*)

A general differential operator *does stuff.*
For now, for continuous functions α and β on some open interval I and for any function ϕ twice differentiable on I, we define the **differential operator** L as

$$L[\phi] = \phi'' + \alpha\phi' + \beta\phi.$$

Note that the result of applying L to some function f is another function g.

In this section we will examine the equation $L[y] = 0$.

Definition 3.3 (*Existence and Uniqueness Theorem*)

(Reproduced from page 110.)
Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \; y'(t_0) = y_0',$$

where p, q, and g are continuous on an open interval I with $t_0 \in I$. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

This existence theorem is pretty similar to Theorem 2.4.1 but generalized to second-order linear differential equations. Note once again the guarantee and uniqueness of a solution to the given differential equation over a certain interval.

Definition 3.4 (*Principle of Superposition*)

If y_1 and y_2 are two solutions to the differential equation $L[y] = 0$, then $y_3 = c_1 y_1 + c_2 y_2$ is also a solution to the given differential equation for any $(c_1, c_2) \in \mathbb{R}^2$.

Definition 3.5

Wronskian Determinant The **Wronskian Determinant** for the system

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{cases}$$

is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

If W is non-zero, then there is a unique solution to the differential equation $L[y] = 0$ with **any** given initial condition. Otherwise, there are initial conditions to the differential equation that cannot be satisfied no matter how c_1 and c_2 are chosen (113).

Note that if the Wronskian W is non-zero, the two solutions y_1 and y_2 to $L[y] = 0$ are said to form a **fundamental set of solutions**.
(There's a lot more discussion here about uniqueness of solutions, Wronksians, and other things I frankly don't care about.)

Regarding complex valued solutions, if $y = u(t) + iv(t)$ satisfies $L[y] = 0$, then u and v are also solutions to the differential equation $L[y] = 0$ (Theorem 3.2.6, Page 117). This is important for later sections.

For another theorem in this long section, we have....

Definition 3.6 (*Abel's Theorem*)

If y_1 and y_2 are solutions for the differential equation $L[y] = 0$ (and some other general conditions are satisfied), then the Wronksian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\ dt\right)$$

where c is a constant dependent on y_1 and y_2 but not on t (Theorem 3.2.7, Page 117)

In summary (page 118), to solve $L[y] = 0$ over some open interval I, we first find two solutions y_1 and y_2 then make sure that $W[y_1, y_2](i) \neq 0$ for some $i \in I$. If this is achieved, y_1 and y_2 would then be a fundamental set of solutions to the given differential equation from which initial-value problems can be solved.

Exercise 12.
We evaluate the differential equation with $y = c\phi(t)$:

$$y'' + p(t)y' + q(t)y = c\phi''(t) + cp(t)\phi'(t) + cq(t)\phi(t) = c(\phi''(t) + p(t)\phi'(t) + q(t)) = g(t).$$

Since we know $\phi(t)$ is a solution to the differential equation, we thus have $c(g(t)) = g(t)$ which cannot hold if $c \neq 1$ and $g(t) \neq 0$.
This does not violate Theorem 3.2.2 (Principle of Superposition) as that principle arises from the special case of when $g(t) = 0$. $\square$

Exercise 13.
No.
If $y = \sin\left(t^2\right)$ is a solution to $L[y] = 0$, then

$$2\cos\left(t^2\right) - 4t^2\sin\left(t^2\right) + p(t)2t\cos\left(t^2\right) + q(t)\sin\left(t^2\right) = \cos\left(t^2\right)(2 + p(t)2t) + \sin\left(t^2\right)\left(-4t^2 + q(t)\right) = 0.$$

To make $L[\sin\left(t^2\right)]$ equal to 0, we thus have to have $2 + p(t)2t = 0$ and $-4t^2 + q(t) = 0$. The latter case is easy to solve but the former implies $p(t) = -\frac{1}{t}$, which is a non-continuous function around the point $t = 0$.
In any case, if we change $q(t)$ to 'cancel' the residue $2\cos\left(t^2\right)$ in the equation above, then in some form or another part of $q(t)$ would contain the fraction $\cot\left(t^2\right)$ meaning q would also be a non-continous function around $t = 0$.
As such, it is impossible to find continuous p and q satisfying $L[\sin\left(t^2\right)] = 0$ over an open interval I containing the point $t = 0$. $\square$

Exercise 15.

$$W[f + 3g, g - g] = (f + 3g)'(f - g) - (f + 3g)(f - g)' = f'(f - g) + 3g'(f - g) - f'(f + 3g) + g'(f + 3g)$$

$$= ff' - f'g + 3fg' - 3gg' - ff' - 3f'g + f'g + 3gg' = -4(f'g - fg') = 4\sin t - 4t\cos t.$$

$\square$

Exercise 17.
Two solutions y_1, y_2 to this differential equation are ce^t and ce^{-2t} for any $c \in \mathbb{R}$. To construct the fundamental set of solutions, we need to reshape our solutions such that $y_a(0) = 1$ and $y_a'(0) = 0$ and also $y_b(0) = 0$ and $y_b'(0) = 1$.

Since our two solutions y_1, y_2 seem pretty dissimilar, we first assume that $y_a = c_1 y_1 + c_2 y_2$. From here, we just solve for the properties we need; since $y_a = 1$, $c_1 + c_2 = 1$. Similarly, since $y_a'(0) = 0$, $c_1 - 2c_2 = 0$ so $(c_1, c_2) = (2/3,\ 1/3)$.
Doing something similar for y_b, we find that the corresponding $(c_1, c_2) = (1/3,\ -1/3)$. As such,

$$\begin{cases} y_a = \frac{2}{3}e^t + \frac{1}{3}e^{-2t} \\ y_b = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \end{cases}.$$

$\square$

Exercise 23.

$$W = c\exp\left(-\int p(t)\ dt\right) = c\exp\left(-\int \frac{-t(t+2)}{t^2}\ dt\right) = c\exp\left(\int 1 + \frac{2}{t}\ dt\right) = ce^{t+2\ln t} = ct^2e^t.$$

$\square$

Exercise 25.

$$W = c\exp\left(-\int p(x)\ dx\right) = c\exp\left(-\int \frac{-2x}{1-x^2}\ dx\right) = c\exp\left(\int -\frac{1}{u}\ du\right) = ce^{\ln(1/u)} = \frac{c}{1-x^2}.$$

$\square$

Exercise 31. Exact Equations.
Expanding the given expression, we get

$$P'(x)y' + P(x)y'' + f'(x)y + f(x)y' = P(x)y'' + y'(P'(x) + f(x)) + f'(x)y = 0.$$

Equating the coefficients to the general form of a differential equation, we thus have $P'(x) + f(x) = Q(x)$ and $f'(x) = R(x)$.
Taking the derivative of that first equation, we thus have $P''(x) + f'(x) = Q'(x)$ or $P''(x) - Q'(x) + R(x) = 0$ which is exactly the equation that was desired. $\square$

Exercise 32.
32. $P''(x) - Q'(x) + R(x) = 0 - 1 + 1 = 0$ so the equation is exact. Namely, $f(x) = Q(x) - P'(x) = x$ so the problem can be restated as $(y')' + (xy)' = 0 \to y' + xy = c$. This equation is solvable with integrating factor $e^{x^2/2}$ but then the error function pops out so I'm not going to finish this integral. $\square$

Exercise 34.
Since $2 - 1 + (-1) = 0$, we can find $f(x) = -x$. The differential equation then becomes $(x^2y')' + (-xy)' = 0 \to x^2y' - xy = c$.
Solving, we find $y = -\frac{c_1}{3x} + c_2 x$. $\square$

3.3 Complex Roots of the Characteristic Equation

What happens when the roots of the characteristic equation $ar^2 + br + c = 0$ for a general differential equation $ay'' + by' + cy = 0$ are imaginary?

Let the roots r_1 and r_2 of the characteristic equation be $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ for real α, β. Then, the corresponding solutions to the differential equation are

$$\begin{cases} y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t) \text{ and} \\ y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t) \end{cases}.$$

In Section 3.2 (Thereom 3.2.6), it was mention that the real and imaginary parts of any solution to a given differential equation are each solutions to the given differential equation. In our case thus, $y_3 = e^{\alpha t}\cos(\beta t)$ and $y_4 = e^{\alpha t}\sin(\beta t)$ are also solutions to $ay'' + by' + cy = 0$, with $W[y_3, y_4] = \beta e^{2\alpha t} \neq 0$.

Exercise 6-8.
6. The quadratic yields roots $r_1, r_2 = 1 \pm i\sqrt{5}$ so the corresponding general solution is $c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t)$.
7. $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$.
8. $y = c_1 e^{-3t}\cos(2t) - c_2 e^{-3t}\sin(2t)$. □

Exercise 25.
(a): $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{t}\dfrac{dy}{dx}$.
$\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{1}{t}\cdot\dfrac{dy}{dx}\right) = -\dfrac{1}{t^2}\dfrac{dy}{dx} + \dfrac{dx}{dt}\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{t^2}\left(\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\right)$.

(b): Simplify substitute everything we just derived in into the equation ._. □

Exercise 26-29.
As seen from question 25, we can transform the coefficients of the differential equation $(t^2, \alpha t, \beta)$ into $(1, \alpha - 1, \beta)$. In this case, $\alpha = 1$ and $\beta = 1$ so our new differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

which has solutions $y_1 = \cos(x)$, $y_2 = \sin(x)$. As such, since $x = \ln t$, $y_1 = \cos(\ln t)$ and $y_2 = \sin(\ln t)$ are a set of solutions to the differential equation in terms of t.

27: $\alpha = 4$ and $\beta = 2$ so $y_1 = e^{-x} = \frac{1}{t}$ and $y_2 = e^{-2x} = \frac{1}{t^2}$.
28: $y_1 = \frac{1}{t}$ and $y_2 = t^6$.
29: $y_1 = t^2$ and $y_2 = t^3$. □

3.4 Repeated Roots | Reduction of Order

In the characteristic equation $ar^2 + br + c$, if the discriminant $\Delta = b^2 - 4ac > 0$, then we are bound to find two real roots r_1 and r_2 and from there derive a general solution to the differential equation (Section 3.1). If in fact $b^2 - 4ac < 0$, then we will have two complex roots which, as shown in Section 3.3, correspondingly lead to a real solution to the given differential equation. But what happens when $b^2 - 4ac = 0$?

Assume that $r_1 = r_2 = -\frac{b}{2a}$. Like before, we conclude that one solution to the differential equation $ay'' + by' + cy = 0$ would be $y_1 = e^{-bt/2a}$. But what about the second solution? It turns out $y_2 = te^{-bt/2a}$ is the second solution we need[1].
So to summarize, when solving the equation $ay'' + by' + cy = 0$, the solutions are

$$\begin{cases} y_{1,2} = e^{r_{1,2}t} \text{ if } b^2 - 4ac > 0, \\ y_1 = e^{\lambda t}\cos(\mu t),\ y_2 = e^{\lambda t}\sin(\mu t) \text{ if } b^2 - 4ac < 0, \\ y_1 = e^{r_1 t}, y_2 = te^{r_1 t} \text{ (when } b^2 - 4ac = 0.) \end{cases}$$

3.4.1 Reduction of Order

D'Alembert's Method of finding 'extra' (more) solutions is to assume the new solution is of the form $y_2(t) = v(t)y_1(t)$ and solve from there. Namely, in second order differential equations, if we know a solution y_1 to $L[y] = 0$, we let

$$y_2 = v(t)y_1 \text{ so } y_2' = v'(t)y_1 + v(t)y_1' \text{ and } y_2'' = v''(t)y_1 + 2v'(t)y_1' + v(t)y_1''.$$

As such, plugging this back into our differential equation, $L[y_2] = 0$ becomes

$$\begin{aligned} y_2'' + P(t)y_2' + Q(t)y_2 = 0 \Longrightarrow v''(t)y_1 + 2v'(t)y_1' + v(t)y_1'' + P(t)(v'(t)y_1 + v(t)y_1') + Q(t)v(t)y_1 \\ = y_1v''(t) + (2y_1' + P(t)y_1)v'(t) + (y_1'' + P(t)y_1' + Q(t)y_1)v(t) = 0. \end{aligned}$$

Since y_1 is a solution and thus $L[y_1] = 0$, that right most term is actually 0 so our new differential equation is now

$$y_1v'' + (2y_1' + P(t)y_1)v' = 0$$

which is a first order differential equation with respect to v'. This is known as **reduction of order** since our differential equation went from being a second-order to a first-order differential equation.

Exercise 1-8.
(Note: The general solution y can be expressed as $y = c_1y_1 + c_2y_2$ for arbitrary c_1, c_2. Below, I only find y_1 and y_2.)

1. Since $b^2 - 4ac = 0$, $y_1 = e^{-bt/2a} = e^t$, and $y_2 = te^t$.
2. Since $b^2 - 4ac = 0$, $y_1 = e^{-bt/2a} = e^{-t/3}$ and $y_2 = te^{-t/3}$.
3. Since $b^2 - 4ac = 16 + 4(4)(3) = 64 > 0$, we can simplify find the roots of the equation and derive a general solution that way. The roots to $4n^2 - 4n - 3 = 0$ are $n = \frac{3}{2}, -\frac{1}{2}$ so the two solutions are $y_1 = e^{3t/2}$, $y_2 = e^{-t/2}$.
4. $b^2 - 4ac = -36$ so the roots to this quadratic equation are $1 \pm 3i$ (quadratic formula). As such, $y_1 = e^t\cos(3t)$ and $y_2 = e^t\sin(3t)$.
5. Since $b^2 = 4ac$, $y_1 = e^{3t}$ and $y_2 = te^{3t}$.
6. $y_1 = e^{-4t}$, $y_2 = e^{-t/4}$.
7. $y_1 = e^{-3t/4}$, $y_2 = ty_1$.
8. $y_1 = e^{-1/2}\cos(t/2)$, $y_2 = e^{-1/2}\sin(t/2)$. $\square$

[1] Consult Example 1 in Section 3.4 in the textbook for a proof

Exercise 14.
Let r_1, r_2 be the roots of the characteristic equation to the differential equation $ay'' + by' + cy = 0$. As discussed above, the general solution to this equation is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for arbitrary constants c_1 and c_2. If we let $y = 0$, we can rearrange and show

$$-\frac{c_1}{c_2} = e^{(r_2 - r_1)t}.$$

Notably, the left hand side of the equation is constant for a given solution (c_1 and c_2 are chosen after all) and $r_2 - r_1$ is also constant. As such, since the exponential function is a bijective function for all real inputs, there is only one t value that makes the above equation true which means if the differential equation has a non-trivial solution (e.g. not $c_1 = c_2 = 0 \to y(t) = 0$), there is only one t value that makes a given solution to the differential equation (y) 0. □

Exercise 18-22.
18. Reducting the order, you eventually get $v''t^4 = 0 \to v'' = 0 \to v = c_1 t + c_2$ so $y_2 = t^3$ as the constant c_1 is arbitrarily chosen (in this case I take $c_1 = 1$) and the latter term $c_2 t^2$ is a multiple of y_1 and thus not worth mentioning.
19. The post-reduction equation ends up being $tv'' + 4v' = 0$ which yields the solution $y = \frac{C_1}{t^3}$ which means the final second solution is $y_2 = \frac{1}{t^2}$. (Note: I'm ignoring the second $+C$ at the end as it's inclusion is not necessary; the final solution ends up being $y_2 = \frac{C_1}{t^2} + C_2 t = \frac{C_1}{t^2} + C_2 y$ which means the latter term has no meaning and can be ignored.)
20. $y_2 = \frac{\ln t}{t}$.
21. It's long and tricky but you eventually get $y_2 = -C\cot(x^2) \cdot y_1 \to \cos(x^2)$.
22. The integration required in this exercise is basically the same as the ones done in exercise 21. As such, it should be relatively straightforward to show that $y_2 = -C\cos x \frac{1}{\sqrt{x}} \to y_2 = \frac{\cos x}{\sqrt{x}}$. □

Exercise 32-33.
To recap Euler's equations, we transform $t^2 y'' + \alpha t y' + \beta y = 0$ into $y'' + (\alpha - 1)y' + \beta y = 0$ where in the first case the derivative of y is taken with respect to t and in the second, the derivative of y is taken with respect to $x = \ln t$.

32. $y_1 = e^{-x/2} = t^{-1/2} = \frac{1}{\sqrt{t}}$, $y_2 = \frac{x}{\sqrt{t}} = \frac{\ln t}{\sqrt{t}}$. I have manually verified that both solutions do indeed solve the differential equation posed.
33. $y_1 = e^{-x} = \frac{1}{t}$, $y_2 = \frac{\ln t}{t}$. □