Differential Equations Notes

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Fall 2025

Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 A

2 B

3 C

4 D

5 E

6 F

7 G

8 H

The above sections are spacer sections.

9 Nonlinear Differential Equations and Stability

This chapter deals with qualitative information about a differential equation and stability/instability of given solutions.

9.1 The Phase Plane: Linear Systems

When considering the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x},$$

there are special equilibrium solutions to be aware about: namely, **critical points**, or when $\mathbf{A}\mathbf{x} = \mathbf{0}$. If we assume $\det \mathbf{A} \neq 0$, then only $\mathbf{x} = \mathbf{0}$ is the only critical point of the system.

For linear systems, we can analyze the system and its trajectories by the eigenvalues of $\mathbf{A}$; namely, if we assume $\mathbf{A}$ is a 2×2 matrix and $\mathbf{A}$ has eigenvalues r_1, r_2, then

- if $r_1 \neq r_2$ and r_1 and r_2 have the same sign, then all trajectories will approach the origin and the origin (the critical point) will either be a **nodal source** (e.g. trajectories go away from the critical point) or a **nodal sink** (trajectories end up going towards the critical point.)
- if r_1 and r_2 have differing signs (wlog $r_1 > 0 > r_2$) then as $t \to \infty$, all trajectories will converge towards the eigenvector associated with r_1. As $t \to -\infty$ however, since $e^{-\infty} = 0$, all trajectories will converge towards the eigenvector associated with r_2. As such, the critical point is known as a **saddle point** as no trajectories pass through the critical point (see textbook Page 391).
- if $r_1 = r_2$ and two *indepedent* eigenvectors can be found for $\mathbf{A}$, then all trajectories look like a line through the critical point (in this case the origin) and the critical point is called a **proper node** or **star point**.
- if conversely $r_1 = r_2$ and only one eigenvector can be found, then the critical point is called an **improper node**[1]
- If eigenvalues are complex with some real part, trajectories will look like a spiral going either towards/away the origin in which case the origin is called a spiral sink/spiral source, depending on the real part.
- If the eigenvalues are purely imaginary, trajectories will infinitely loop on themselves (since there is no real part, trajectories do not 'decay') and the origin/critical point is called a **center** of the system. For linear systems, these trajectories look like ellipses around the origin.

Essentially, all solutions either go to infinity, go to $\mathbf{0}$, or go in a spiral.

9.2 Autonomous Systems and Stability

In this section we will be concerned with Autonomous systems of two functions of the form

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y).$$

with initial condition $x(t_0) = x_0$, $y(t_0) = y_0$. This system can also be written in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(t_0) = \mathbf{x}^0$$

where $\mathbf{x} = (x, y)^T = (x(t), y(t))^T$, $\mathbf{f}(\mathbf{x}) = (F(x, y), G(x, y))^T$, and $\mathbf{x}^0 = (x_0, y_0)^T$.
For simplicity and so theorems hold, we will assume F and G are continuous and their partial derivatives are also continuous.[2]

[1] tbh idrk why; the graph just looks super weird

[2] Rigorously speaking, their partial derivatives only need to be continuous over some domain D of the xy-plane. But since most functions we work with are very simple, it might as well be (almost) the whole xy-plane over which the partial derivatives are continuous.

Definition 9.1 (*Autonomous*)

A differential equation system is said to be **autonomous** if the systems do not depend on time. In particular, the system given above ($\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$) is autonomous, and so is the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ as long as all terms in $\mathbf{A}$ do not involve the independent variable t.

The distinction between autonomous and nonautonomous systems is important as the condition of autonomy guarantees that there is only one trajectory crossing through the point (x_0, y_0) regardless of time.

9.2.1 Stability and Instability

For autonomous systems of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

Definition 9.2 (*Critical Points*)

Critical points are defined as points to the above system where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Since $\mathbf{x}' = \mathbf{0}$, critical points must be constant solutions to the system.

Definition 9.3 (*Stability*)

A critical point of the solution $\mathbf{x}^\circ$ is said to be **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every solution $\mathbf{x}$ which at $t = 0$ satisfies

$$||\mathbf{x}(0) - \mathbf{x}^\circ|| < \delta$$

exists for all $t > 0$ and satisfies

$$||\mathbf{x}(t) - \mathbf{x}^\circ|| < \varepsilon$$

for all $t \geq 0$.

Essentially, this definition codifies the notion that a given solution $\mathbf{x}^*$ should stay bounded within the critical point. Note however that this definition of stability does not require $\mathbf{x}^*$ to converge to $\mathbf{x}^\circ$; instead, it merely requires that $\mathbf{x}^*$ not leave an open disk[3] of radius ε centered at $\mathbf{x}^\circ$ for all $t \geq 0$.

Any critical points for which the condition of stability doesn't hold are said to be **unstable**.

The following definition thus distinguishes between stability and asymptotic stability:

Definition 9.4 (*Asymptotic Stability*)

A critical point $\mathbf{x}^\circ$ is said to be **asymptotically stable** if it is stable and there exists a $\delta_0 > 0$ such that if a solution $\mathbf{x} = \mathbf{x}(t)$ satisfies

$$||\mathbf{x}(0) - \mathbf{x}^\circ|| < \delta_0,$$

$$\text{then } \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^\circ.$$

In english, if a trajectory starts "sufficiently close" to $\mathbf{x}^\circ$ (within a δ_0), then it must eventually approach $\mathbf{x}^\circ$ as $t \to \infty$.

Definition 9.5 (*Basin of Attraction*)

For a two-dimensional (potentially non-linear) autonomous system with at least one asymptotically critically point, we define the **basin of attraction** for a critical points to be the set of all points P such that a trajectory passing through P eventually converges to said critical point as $t \to \infty$.
If there is a boundary to a **basin of attraction**, that trajectory which bounds the basin is called a **separatrix** as it seperates the trajectories that converge and the trajectories that don't.

If we're lucky, we can determine trajectories of a two-dimensional autonomous system by solving just a first-order differential equation. Namely, since $F(x, y)$ and $G(x, y)$ don't depend on t, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x, y)}{F(x, y)}$$

which is a first-order differential equation. In general, the differential equation arising from the quotient $\frac{G(x,y)}{F(x,y)}$ may not be solvable.

[3] If $\mathbf{x}$ is n-dimensional, then the statement should be amended to read "... not leave an open $n-1$-sphere of radius ...".

Exercise 14a-20a.
For this exercise, I recommend using this online plotter to plot some nice looking solutions.
14a. Following the equation above, we have $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} = \frac{8x}{2y} \to H(x,y) = y^2 - 4x^2 = c$. Solutions to this system generally look like (one-directional) hyperbolas.
15a. $\frac{dy}{dx} = \frac{-8x}{2y} \to H(x,y) = y^2 + 4x^2 = c$. As the equation suggests, solutions to this system look like an ellipse.

16a. $\frac{dy}{dx} = \frac{2x+y}{y}$. Using the substitution $y = vx$ and $dy = vdx + xdv$, we thus have

$$\frac{vdx + xdv}{dx} = v + \frac{dv}{dx}x = \frac{2+v}{v} \to \frac{dv}{dx}x = \frac{2+v-v^2}{v}$$

which with partial fraction decomposition and a bunch of tedious integration simplification reveals $H(x,y) = (x+y)(y-2x)^2 = c$.

17a. $\frac{dy}{dx} = \frac{x+y}{x-y}$. Using the substitution $y = vx$ again, we thus have $H(x,y) = \arctan\left(\frac{y}{x}\right) - \ln\left(\sqrt{\frac{y^2}{x^2}+1}\right) - \ln x = \arctan\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} = c$.
18a. While this equation looks super complicated, if you rearrange it into the form $(2xy^2 - 6xy) + (2x^2y - 3x^2 - 4y)y' = 0$, you quickly find the differential equation is exact so $H = x^2y^2 - 3x^2y - 4y^2 = c$.
19a. $\frac{dy}{dx} = \frac{-\sin x}{y} \to \frac{y^2}{2} - \cos(x) = c$.
20a. $\frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{24} = c$. □