Differential Equations Notes

Alex Z

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Contents

Remarks

This notes thing was started on 10/19/2025.
I am to work through the WHOLE of the textbook (Elementary Differential Equations and Boundary Value Problems (11th ed)) by the end of this quarter (12/13/2025).
Hopefully, I'll also get around 20-40% of the problems in the textbook done.

1 Introduction

aka chapter 1

1.1 Introduction for the Introduction

Definition (Differential Equations)**.** Equations containing derivatives.

Definition (Slope Field/Direction Field)**.** A buncha line segments on the plane that represent the "motion" of a diff-eq.

Direction Fields are good for studying differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

(Page 6 – How to construct a diff-eq mathematical model from a real-world situation.)
(7) Newton: Differential equations come in one of these 3 forms:

1. $\dfrac{dy}{dx} = f(x)$,
2. $\dfrac{dy}{dx} = f(y)$,
3. $\dfrac{dy}{dx} = f(x, y)$.

Exercise 11-16.
1.1.5 corresponds with **j**.
1.1.6 corresponds with **c**.
1.1.7 corresponds with **g**.
1.1.8 corresponds with **b**.
1.1.9 corresponds with **h**.
1.1.10 corresponds with **e**. □

Exercise 17.
(a)

$$\frac{dC}{dt} = [\text{chemicals/hour going in}] - [\text{out}] = 0.01 \cdot 300 - 300 \cdot \frac{C}{1000000}$$

where C is the number of gallons of said chemical in the pond and t is time measured in hours.

(b) After a very long time, 10000 gallons will be in the pond; this limiting amount is independent of starting conditions.

(c) Since concentration $= \dfrac{\text{Amount}}{\text{Volume}}$, $C = \text{volume} \cdot c = c \cdot 10^6$ where c stands for concentration. As such,

$$\frac{dc}{dt} = \frac{1}{10^6}\frac{dC}{dt} = \frac{3}{10^6} - \frac{3(c \cdot 10^6)}{10^4 \cdot 10^6}$$

So in final, $\boxed{\dfrac{dc}{dt} = \dfrac{3}{10^6} - \dfrac{3c}{10^4}}$. □

Exercise 18.

$$\frac{dV}{dt} = -k \cdot 4\pi r^2 = -k \cdot 4\pi \left(\frac{3}{4\pi}V\right)^{\frac{2}{3}}$$

□

Exercise 19.

$$\frac{dT}{dt} = -0.05 * (T - 70)$$

where T is the temperature of the object in Farenheit and t is time in minutes. □

1.2 Introduction to Solutions

(11) - Finding the general solutions to diff-eqs of the form $\frac{dy}{dt} = ay - b$ ($a \neq 0$);

$$\frac{dy}{dt} = ay - b \Longrightarrow y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$$

⟨14 - "Further Remarks on Mathematical Modeling" - essentially, the underlying assumptions we make may or may not be wrong. ⟩

Exercise 1a.
$\frac{dy}{dt} = -y + 5 \rightarrow \frac{1}{5-y}dy = dt.$

So, $\ln(5 - y(t)) = t + C$. With initial condition $y(0) = k$, we get that $\ln(5 - k) = C$, so our solution becomes $y(t) = 5 - e^{t+\ln(5-k)} = 5 - (5 - k)e^t$. (Note that $(5 - k)$ is constant.) □

Exercise 9a.
Since $F = ma$, $F = m\frac{dv}{dt}$. Since drag acts inversely to velocity (object falling faster has more air resistance), we should expect $\frac{dv}{dt}$ to be negative; thus, $\frac{dv}{dt} = -\frac{F}{10}$. Knowing that F is proportional to the square of the velocity, we know that $F = av^2 - b$ for constants a, b.

Now, we plug in some known values. At $v = 0$, we expect $\frac{dv}{dt} = -\frac{(-b)}{10} = -9.8$ (gravity) so $b = -98$. At $v = 49$, we reach limiting velocity which implies $\frac{dv}{dt} = 0$ so $\frac{a(49^2)-98}{10} = 0$ so $a = \frac{2}{49}$. Thus, in final, we get our differential equation as

$$\frac{dv}{dt} = \frac{2}{49 \cdot 10}v^2 - \frac{98}{10}$$

which can be re-arranged to

$$\frac{dv}{dt} = \frac{1}{245}\left(v^2 - 49\right).$$

□

Exercise 9b.
(I'm gonna go with their equation for simplicity - it doesn't matter too much though.)

$$\frac{dv}{dt} = \frac{1}{245}\left(49^2 - v^2\right)$$
$$\to 245 \frac{1}{49^2 - v^2}\ dv = dt$$
$$\to 245 \int \frac{1}{49^2 - v^2}\ dv = t$$

Doing a trig sub ($v = 49\sin\theta$, $dv = 49\cos\theta\ d\theta$), $\dfrac{dv}{49^2 - v^2}$ becomes $\dfrac{49\cos\theta\ d\theta}{49^2 - 49\sin^2\theta}$ so our integral ends up turning into

$$\longrightarrow 245 \int \frac{d\theta}{49\cos\theta} = t \Longrightarrow 5\left(\frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right) = t + C.$$

Thus,

$$t + C = \frac{5}{2}\ln\frac{1+v/49}{1-v/49}.$$

Plugging in our initial condition $v(0) = 0$, we get that $C = 0$. Thus,

$$\ln\left(\frac{1+v/49}{1-v/49}\right) = \frac{2t}{5} \text{ so } 49 + v = (49 - v)(e^{2t/5}).$$

Simplifying, (by expanding and putting all the vs on one side of the equation) we find our final answer to be

$$v(t) = 49 \cdot \frac{e^{2t/5} - 1}{e^{2t/5} + 1} = 49\tanh(t).$$

□

Exercise 13.
(a)

$$\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{RC} = \frac{VC - Q}{RC}$$

$$\Rightarrow RC \int \frac{dQ}{VC - Q} = t + C$$

Integrating, we get that $t + C_1 = -RC\ln(VC - Q)$. Plugging in our initial condition $Q(0) = 0$, we get that $C_1 = -RC\ln(VC)$. Thus, we can substitute and simplify as follows:

$$RC\ln(VC - Q) = RC\ln(VC) - t \to \ln(VC - Q) - \ln(VC) = -\frac{t}{RC}$$

$$\to VC - Q = VCe^{-t/RC}$$

so $Q(t) = VC(1 - e^{-t/RC})$.

(b) After a very long time ($t \sim \infty$), $Q \sim VC$ so $Q_L = VC$.

(c) From Kirchoff's voltage rule, $R\frac{dQ}{dt} + \frac{Q}{C} = 0 \longrightarrow -\frac{Q}{C} = R\frac{dQ}{dt}$. Thus, $t + C_1 = -RC\ln(Q)$. Evaluating in our initial condition, we get that $C_1 = -RC\ln(Q_L) + t_1$. As a result,

$$-(t - t_1) = RC(\ln(Q) - \ln(Q_L))$$

so

$$Q = Q_L e^{-\frac{t-t_1}{RC}}.$$

$\square$

1.3 Classification of Diffy Qs

Definition (Ordinary Differential Equation)**.** An Ordinary Diffy Q (ODE) is an equation where the unknown function depends on a single independent variable.
E.g. (LRC Circuit)

$$L^2\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Definition (Partial Differential Equation)**.** A Partial Differential Eq (PDE) is when the unknown function depends on several independent variables.
E.g. (Wave Equation)

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

(17) - If you have n unknown functions in a system of differential equations, then you gotta have at least n diffy qs to solve that system completely.

Definition (Order)**.** The **order** of a differential equation is the highest derivative that appears in the differential equation. Thus you can have a *first-order* or *second-order* or *seventh-order* diffy q.
E.g.: $\alpha\frac{d^3x}{dk^3} + \beta\frac{d^2x}{dk^2} + \frac{\alpha}{\beta}x = \gamma$ is a third-order (ordinary differential) equation (when α, β, and γ are constants and x is a function of k).

Generally then, a differential equation of order n can be represented by the generic $F(t, x(t), x'(t), \ldots, x^{(n)}t) = 0$ for some function $x(t)$. Replacing $y = x(t)$, we get that a general nth order differential equation is of the form

$$F(t, y, y', \ldots, y^{(n)}) = 0.^{1}$$

[1] (18) Note: We assume it is always possible to solve for the highest derivative – e.g. we can rearrange to get to the form of $y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$.

Definition (Linearity)**.** A differential equation is said to be **linear** if $F(t, y, y', \ldots, y^{(n)}) = 0$ is a linear function of t, y, y', $\ldots$, $y^{(n)}$.
As such, the general linear diffy q is of the form $0 = c(t) + a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)}$.

Definition (Linearization)**.** Linearization is the process of approximating a non-linear diffy q by a linear one. Example given in the textbook is of approximating the motion of an oscillating pendulum.

(19-20) - Questions of solvability and uniqueness for general differential equations.

Exercise 1-4.
1. Order is 2, and the differential equation is linear.
2. Order is 2, and the differential equation is NOT linear (because of the term $(1+y^2)\frac{d^2y}{dt^2}$).
3. Order is 4, and the differential equation is linear.
4. Order is 2, and the differential equation is non-linear. □

Exercise 10.
(I'm only doing this one because it looks fun)
We shall verify that $y = e^{t^2}\left(1 + \int_0^t e^{-s^2}\,ds\right)$ is a solution to the differential equation $y' - 2ty = 1$.
First, we substitute y into our equation.

$$\left[e^{t^2} + e^{t^2}\int_0^t e^{-s^2}\,ds\right]' = 1 + 2t \cdot e^{t^2}\left(1 + \int_0^t e^{-s^2}\,ds\right)$$

Next, we differentiate that left side and simplify the right.

$$\longrightarrow 2te^{t^2} + \left(2te^{t^2}\right)\left(\int_0^t e^{-s^2}\,ds\right) + \left(e^{t^2}\right)\left(e^{-t^2}\right) = 1 + 2te^{t^2} + 2te^{t^2}\left(\int_0^t e^{-k^2}\,dk\right)$$

Finally, we cancel terms and arrive at the equation

$$e^{t^2} \cdot e^{-t^2} = 1,$$

which is trivially true. Thus, we are done. □

Exercise 11-13.
Since $y = e^{rt}$, $y^{(n)} = r^n e^{rt}$. Thus, in each of problems 11-13, we're basically just solving a polynomial. To illustrate, consider problem 12:

$$y'' + y' - 6y = 0 \Longrightarrow r^2e^{rt} + re^{rt} - 6e^{rt} = e^{rt}\left(r^2 + r - 6\right) = 0,$$

which is (almost) isomorphic to solving the system $x^2 + x - 6 = 0$. Thus, we yield the solutions $r = 2, 3$ and maybe even $r = -\infty$ (which would make e^{rt} be 0).

Similar solutions follow for 11 and 13. □

Exercise 16-18.
16: 2nd order linear partial differential eq.
17: 4th order linear PDE.
18: 2nd order non-linear PDE. □

2 First-Order Diffy Qs

aka chapter 2
for chapter 2, all diffy qs will be first order.

2.1 Linear ODEs: Method of Integrating Factors

If $\frac{dy}{dt} = f(t, y)$ and f is linear (w.r.t y), then we can rewrite it in the following form (called the **first-order linear differential equation**):

$$\frac{dy}{dt} + p(t)y = g(t) \iff P(t)\frac{dy}{dt} + Q(t)y = G(t) \text{ (page 24)}$$

Definition (Integrating Factor)**.** A **integrating factor** $\mu(t)$ is a function such that when a diffy q is multiplied by it, the equation is then immediately integratable (discovered by Leibniz). (page 25)

Exercise - Pauls Online Notes, Problem 4 (modified).
Find the general solution to the ODE

$$t\frac{dy}{dt} + 2y = t^2 - t + 1.$$

This diffy q looks hard. To start, we add on an integrating factor $\alpha(t)$ to the equation to get

$$t\alpha(t)\frac{dy}{dt} + 2\alpha(t)y = \alpha(t)(t^2 - t + 1).$$

From here, consider what happens when you take the derivative of $(t \cdot y \cdot \alpha(t))$:[a]

$$\frac{d}{dt}\left[t \cdot y \cdot \alpha(t)\right] = y\alpha(t) + t\alpha(t)\frac{dy}{dt} + ty\alpha'(t) = t\alpha(t)\frac{dy}{dt} + y(\alpha(t) + t\alpha'(t)).$$

For this equation to match the left hand side of the equation above, we then must have that $t\alpha(t) = t\alpha(t)$ and $2\alpha(t) = \alpha(t) + t\alpha'(t) \to \alpha(t) = t\alpha'(t)$. From that last equation, I recognized that the function $\alpha(t) = t$ works!
And from there, after plugging things in and integrating, I ended up with my final answer that the general solution to the given ODE is

$$y(t) = \boxed{\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}}.$$

□

[a] This is not the *actual* way to do it –Paul's math notes first divides everything by t so they only have to consider the derivative of $y\alpha(t)$.

So essentially, the process of solving diffy qs of the form $P(t)\frac{dy}{dt} + Q(t)y = G(t)$ is to first divide by $P(t)$, then find an integrating factor that "matches up" both sides of the equation.
Mathematically:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \to \frac{dy}{dt} + \frac{Q(t)}{P(t)}y = \frac{G(t)}{P(t)} \to \kappa(t)\frac{dy}{dt} + \kappa(t)\frac{Q(t)}{P(t)}y = \kappa(t)\frac{G(t)}{P(t)}$$

Since $(y\kappa(t))' = \kappa(t) \cdot y' + \kappa'(t)y$, comparing terms on the LHS, we get that we just need to find some $\kappa(t)$ such that $\kappa'(t) = \kappa(t)\frac{Q(t)}{P(t)}$. If that nasty fraction $\left(\frac{Q(t)}{P(t)}\right)$ is some constant or basic polynomial, the equation is *probably* solvable.
So assuming that some suitable $\kappa(t)$ is found, we then just kinda evaluate everything from there.

$$\longrightarrow \int \frac{d}{dt}\left[y\kappa(t)\right] = \int \kappa(t)\frac{G(t)}{P(t)} \implies y(t) = \frac{C + \int \kappa(t)\frac{G(t)}{P(t)}}{\kappa(t)}.$$

It's quite messy when written out.

(27) For equations of the form $\frac{dy}{dt} + ay = g(t)$, the right integrating factor is $\mu(t) = e^{at}$. This can be rederived pretty easily (probably).

If you want to integrate but the messy thing doesn't simplify, that's fine; put the bounds of your integral to be from some arbitrary t_0 to t, preferably in a way such that if an initial condition $y(y_0) = c_0$, $t_0 = y_0$. In this way, your integral will collapse on itself when evaluated at $y = y_0$ and any other value of your function will be computed by that constant given in the problem plus the accumulated gain/loss from the function as it goes to your desired x/y value.

Exercise 1c-8c.
(I'm just going to document my answers here.)
1c: $y(t) = e^{-2t} + \frac{1}{3}\left(t - \frac{1}{3}\right) + \frac{C}{e^{3t}}$.
2c: $y(t) = \left(\frac{1}{3}t^3 + C\right)e^{2t}$.
3c: $y(t) = \frac{t^2}{2e^t} + 1 + \frac{C}{e^t}$.
4c: $y(t) = 1.5\sin(2t) + \frac{0.75}{t}\cos(2t) + \frac{C}{t}$.
5c: $y(t) = -3e^t + Ce^{2t}$.
6c: $y(t) = -te^{-t} + Ct$.
7c: $y(t) = \sin(2t) - 2\cos(2t) + \frac{C}{e^t}$. Warning: This integral is hard but is very doable.
8c: $y(t) = 3t^2 - 12t + 24 + \frac{C}{2e^{t/2}}$. □

Exercise 9-12.
(More answer exercise documentation. Both the general form and the specific solution to each problem will be given.)
9: $y(t) = 2te^{2t} - 2e^{2t} + Ce^t$. The specific case when $y(0) = 1$ is given by $C = 3$.
10: $y(t) = \frac{t^2/2 + C}{e^{2t}}$. The specific solution when $y(1) = 0$ is given by $C = -\frac{1}{2}$.
11: $y(t) = \frac{\sin t + C}{t^2}$, with $C = 0$.
12: $y(t) = \frac{t-1}{t} + \frac{C}{te^t}$, $C = 2$ for this particular case. □

Exercise 18.
Given the simplicity of the right hand side, we can fake solve the equation for $y(t)$; namely, from intuition, if $y(t) = \alpha t + \beta$, then $y'(t) = \alpha$ (a constant) and we can probably find values α, β that make such a solution possible.
In fact we do; $y(t) = -\frac{3}{4}t + \frac{21}{8} + Ce^{-2t/3}$, where that last term was derived from realizing that if we actually integrated this properly, our integrating factor $\mu(t)$ would be $e^{2t/3}$.
Anyways, things get a little dicey from here. Let's call the point where $y(t)$ touches (but doesn't cross) the t-axis as t_0. Then, we know that $y'(t_0) = 0$ (y must be at a local max/min as otherwise y would cross the t-axis) and $y(t_0) = 0$. From here, we can rearrange our equations as follows:

$$y'(t_0) = 0 \to 0 = -\frac{3}{4} - \frac{2}{3}Ce^{-2t_0/3} \to -\frac{9}{8} = Ce^{-2t_0/3} \text{ and}$$

$$y(t_0) = 0 \longrightarrow \frac{3}{4}t_0 - \frac{21}{8} = Ce^{-2t_0/3}$$

and we can match the LHSs of both equations to get $\frac{3}{4}t_0 - \frac{21}{8} = -\frac{9}{8}$ and find that $t_0 = 2$.
From here, we can simply plug this value of t_0 into our equations and solve for C:

$$y'(t_0) = 0 \to y'(2) = 0 \to C = -\frac{9}{8}e^{4/3}$$

Thus, $y_0 = y(0) = \frac{21}{8} + C = \frac{21}{8} - \frac{9}{8}e^{4/3} \approx \boxed{-1.64}$. □

Exercise 20.
(Note: this problem and the last problem have caused me some amount of pain because I keep misreading the problem and not sticking to the end.)
If you want $y' - y = 1 + 3\sin t$ to remain finite as $t \to \infty$, then when you get that $y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + Ce^t$, it should be pretty clear that $C = 0$. As such, $y_0 = y(0) = -1 - \frac{3}{2}(1 + 0) = \boxed{-\frac{5}{2}}$.
When doing this problem, don't doubt yourself :). □

Exercise 28 - Variation of Parameters.
(a). If $g(t) = 0\ \forall t$, then effectively $g(t) = 0$. As such, we are simply solving $\frac{dy}{dt} + p(t)y = 0$ which can be done by separating variables:

$$\frac{dy}{dt} = -p(t) \cdot y \to \frac{1}{y}dy = -p(t)dt \to \ln(y) = \int -p(t)dt + C_0 \text{ so } y(t) = C_1 \exp\left(-\int p(t)dt\right)$$

where $C_1 = e^{C_0}$. Replace C_1 with A to get the expression shown in the textbook.

(b). To show $A(t)$ must satisfy (51), we simply substitute everything in and cancel the messy equation.

$$y' + p(t)y = g(t) \implies \left[A(t)\exp\left(-\int p(t)dt\right)\right]' + A(t)\exp\left(-\int p(t)dt\right)p(t) = g(t)$$

$$\longrightarrow \exp\left(-\int p(t)dt\right)\left([A'(t) + A(t)(-p(t))] + A(t)p(t)\right) = g(t) \text{ so } A'(t) = g(t) \cdot \exp\left(\int p(t)dt\right)$$

which is exactly the equation given by (51).

(c). This part is lowkey quite simple. Picking up from (b), we simply slap an integral sign in front of the massive equation that we derived for $A'(t)$, and after replacing $A(t)$ with the appropriate integral in an integral, it is equivalent to (33) up to a constant as the $\mu(t)$ in (33) is really the big scary integral we've been dealing with, $\int p(t)dt$. □

2.2 Separable Differential Equations

A general first-order differential equation can be written as $\dfrac{dy}{dx} = f(x, y)$ which can be rearranged to become $M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$. When M is a function solely of x and N is a function solely of y, then we can rewrite the diffy q as

$$M(x)dx + N(y)dy = 0$$

which we call a **separable** equation. (A definition of a separable differential equation is a differential equation that can be written as the form above.)
To solve these equations, it's just 'simple'/'intuitive' integration. I'm not sure what exactly to write so.....
Let $A'(x) = a(x)$ and $B'(y) = b(y)$. Then,

$$a(x) + b(y)\frac{dy}{dx} = 0$$

simlifies down to

$$A(x) + B(y) = c$$

for an arbitrary constant c.

Exercise 1-4.
(Solutions only.)
1: $\dfrac{y^2}{2} = \dfrac{x^3}{3} + C$ or $y = \sqrt{\dfrac{2}{3}x^3 + C}$.
2: $y = \dfrac{1}{C - \cos x}$.
3: $\dfrac{1}{2}\tan(2y) = \dfrac{x}{2} + \dfrac{\sin(2x)}{4} + C$. Alternatively, the textbook gives the clever solution of $y = \dfrac{\pi}{2}$ (a constant) and similar solutions as $y' = 0$ and the RHS evaluates to 0. Very tricky (but nice) stuff.
4: $\ln|x| + C = \arcsin y$. Didn't fully get those inverse trig integrals right away :/.
(The rest (5-8) look trivial as you just move terms to either side of the equation and integrate.) $\square$

Exercise 17.
Literally just separate and integrate.

$$\Rightarrow \int 3y^2 - 6y\ dy = \int 1 + 3x^2\ dx + C \to y^3 - 3y^2 = x + x^3 + C$$

Plugging in our initial condition of $y(0) = 1$ (e.g. the values $x = 0$ and $y = 1$), we get that $C = 1 - 3 = -2$.
To determine the interval in which the solution is valid, we simply look at when the denominator of y' is 0 —$3y^2 - 6y = 0 \to y = 0, 2$ which correspond to the x values of 0, $\frac{-1+\sqrt{3}}{2}$, $\frac{-1-\sqrt{3}}{2}$, and $x = -1$ ($y(-1) = 2$ and y of any of the other x values is equal to 0).
I'm not sure how really to proceed from here but I think the way to actually do it is to recognize that $y(-1) = 2$, $y(1) = 0$, which given our initial condition $y(0) = 1$ means that our function y is trapped between -1 and 1 (y is not exactly a "function" at these specific points so our domain then becomes $(-1, 1)$).
It's not a great solution :/. $\square$

Exercise 19.

$$\frac{dy}{dx} = 2y^2 + xy^2 \to \int \frac{1}{y^2}\,dy = \int 2 + x\,dx \to y = \frac{1}{2x + \frac{x^2}{2} + C}.$$

Plugging our initial condition $y(0) = 1$ nets $C = -1$.
To find the minimum value of y, we simply find the derivative y' and set it to 0;

$$y' = \frac{1}{\left(2x + \frac{x^2}{2} - 1\right)} \cdot (2 + x).$$

Setting y' to 0, the only solution we get is $x = -2$ so the minimum value of our function y is $y(-2) = -\frac{1}{-4+2-1} = \boxed{\frac{1}{3}}$.
While it is true that $y(3) < y(-2)$, note that the function y is discontinuous and plotting the graph on Desmos, we see that $y(3)$ is not 'on' the particular branch of the solution we're focused on (namely, the piece of the function where $y(0) = 1$). □

Exercise 24.
Solve

$$\frac{dQ}{dt} = r(a + bQ),\ Q(0) = Q_0.$$

(Note: for some reason this exercise does not have a solution in the back of the book.)

We can rearrange to get the equation

$$\int \frac{1}{a + bQ} dQ = \int r dt \longrightarrow \frac{1}{b} \ln|a + bQ| = rt + C$$

and evidently, $C = \frac{\ln a + bQ_0}{b}$.
Solving for Q, we eventually find that

$$Q(t) = \frac{e^{rbt}(a + bQ_0) - a}{b}$$

so when $t \to \infty$, assuming that all constants are positive, $Q \to \infty$. □

Definition (Homogeneous Equation)**.** (Note: This definition of a homogeneous diffy q will be different from other ones presented in this book.)
A homogeneous equation (for our purposes for now) is a first-order differential equation $\frac{dy}{dx} = f(x, y)$ that can be expressed as a function of the the expression $\frac{y}{x}$. In other words, $\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$ for some function g.

In particular, a differential equation is a homogeneous equation, then it is separable 'by a change of the dependent variable' by making the substitution $y = xv(x)$ (note that y is a function of x implicitly) and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. See the exercise below for more information.

Exercise 25.
25(a): Trivial. Multiply the numerator and denominator of the fraction by $\frac{1}{x}$ and simplify.

25(b):

$$y = xv \to \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Multiplying both sides by dx, our equation now becomes

$$dy = v\ dx + x\ dv,$$

which we can directly substitute in for dy.
As such, to conclude,

$$\frac{dy}{dx} = \frac{v\ dx + x\ dv}{dx} = \boxed{v + x\frac{dv}{dx}}.$$

25(c): Trivial. Literally just make the substitutions.

25(d + e): The key to this problem is partial fractions.

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v} \Rightarrow \ln|x| = \int \frac{1 - v}{v^2 - 4}\ dv + C$$

From here, using partial fractions, we find that the integrand is equal to $\dfrac{-0.25}{v-2} + \dfrac{-0.75}{v+2}$ which can be easily integrated to get our final answer.
As such, after integrating and substituting $v = \frac{y}{x}$ back into our equation, we get that

$$\ln|x| + C = -0.75\ln\left|\frac{y}{x} + 2\right| - 0.25\ln\left|\frac{y}{x} - 2\right|.$$

(I'm lowkey not sure how to get rid of those absolute value signs from the $\frac{1}{x}$ integral so they'll stay for now.) □