Understanding Analysis Exercises Solutions

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Overview

This document will be a compilation of all my (hopefully thorough) solutions to Stephen Abbott's *Understanding Analysis*, a classic undergraduate textbook in real analysis. I hope that all solutions in this doc will leave the reader no doubt to the solution's correctness and will (hopefully) not use words such as "trivial", "clearly", or leave things up to the reader to figure out.

This document will **only** pertain to the solutions for Abbott's exercises; namely, this means that I assume the reader is concurrently reading Abbott's analysis textbook, and at some times I may cite theorems covered in the textbook that will not be reproduced here.[1]

Credits to Ulisse Mini and Jesse Li and for putting together the [hyperlink needed] solution manual that I based this off of; it's helped me a lot in my journey and at some points where I couldn't figure out solutions, I'll be citing their solutions. You can find their solution doc/repo [hyperlink needed] here.

Also credits to Ulisse for motivating me to make this document. I'm not sure it will be helpful to anyone other than myself, but it's good to have my solutions LaTeX'd up and set in stone.

[1] Maybe. Maybe I'll find it easier to put all theorems here so it's in a sense self contained but that's to be decided.

Contents

1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

1.1.1 Notes

$\mathbb{N} \Longrightarrow 1, 2, \ldots.$
$\mathbb{Z} \Longrightarrow \ldots, -2, -1, 0, 1, 2, \ldots.$
$\mathbb{Q} \Longrightarrow \left\{ \frac{p}{q} \text{ where } p, q \in \mathbb{Z}, q \neq 0 \right\}.$

Field. A field is a set of numbers where two operations (+, × - called addition and multiplication) are well-defined (closed), commutative, associative, and distributive (in some sense). Also, additive/multiplicative inverses exist for each element, and there exists two distinct identities; the additive identity (0) and the mulitplicative identity (1).

Note that $\mathbb{Q}$ is ordered: for any $r, s \in \mathbb{Q}$, either $r > s$, $r = s$, or $r < s$. In some sense, $\mathbb{Q}$ is "dense" (kinda) since it spans the whole number line and for any $r, s \in \mathbb{Q}$ such that $r < s$, you can always find some $t \in \mathbb{Q}$ such that $r < t < s$.

1.2 Some Preliminaries

1.2.1 Notes

Preliminaries:

Sets:

- A and B are disjoint if $A \cap B = \emptyset$.
- A^c usually represents $x \in \mathbb{R}, x \notin A$.
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Functions:

- $f : A \to B$

1.2.2 Exercises

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution. (a) Essentially, we follow the proof for Theorem 1.1.1.

Assume there exists some rational $\frac{p}{q}$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$) with $\gcd(p, q) = 1$ such that $\left(\frac{p}{q}\right)^2 = 3$. Simplifying, we get that $p^2 = 3q^2$. Since the right hand side (RHS) of our equation is divisible by 3, the left hand side (LHS), p^2, must also be divisible by 3, implying p must be divisible by 3. As such, we can represent $p = 3k$ for some $k \in \mathbb{Z}$. Substituting this into our original equation, we find that $3k^2 = q^2$, which in turn implies q is divisible by 3 (see the logic above). Yet if both p and q are divisible by 3, then our original fraction $\left(\frac{p}{q}\right)$ was not in fact in its most simplified form; namely, $\gcd(p, q) = 3$! As such, our original assumption was wrong, meaning there exists **no** such rational $r = \frac{p}{q}$ such that $r^2 = 3$.

A similar argument does in fact work to prove that there exists no rational r such that $r^2 = 6$.

(b) The proof breaks down when we look at the divisibility of both sides. Namely, following the same steps at before, we arrive at the equation $p^2 = 4q^2$. Here, following the same logic as before, the 4 on the RHS implies p is even, meaning that $p = 2k$ for some k. As such, $p^2 = 4q^2 \to (2k)^2 = 4q^2 \to k^2 = q^2$. Previously, we would now be able to claim something about q (e.g. q must be divisible by 3). Yet currently, the LHS of the equation gives us no information about q. As such, there is no logical contradiction (or infinite descent argument) for us to exploit to prove the irrationality of $\sqrt{4}$. □

Exercise 1.2.2

Show that there is no rational number r satisfying $2^r = 3$.

Solution. solution to 1.2.2. □

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \subset A_2$ then x.

(b) if abc, then d.

(c) test

Solution. and their corresponding solutions. □