ECE 141 - Project Solutions
Spring 2023

Please carefully justify all your answers.
Submit your MATLAB scripts separately.
You can refer to the textbook for useful MATLAB commands.
If you think that there is something wrong with the questions,
just use your best interpretation and state your justification.
You are not allowed to collaborate with others.

Due: June 9, 2023 11:59 pm, 90 points in total

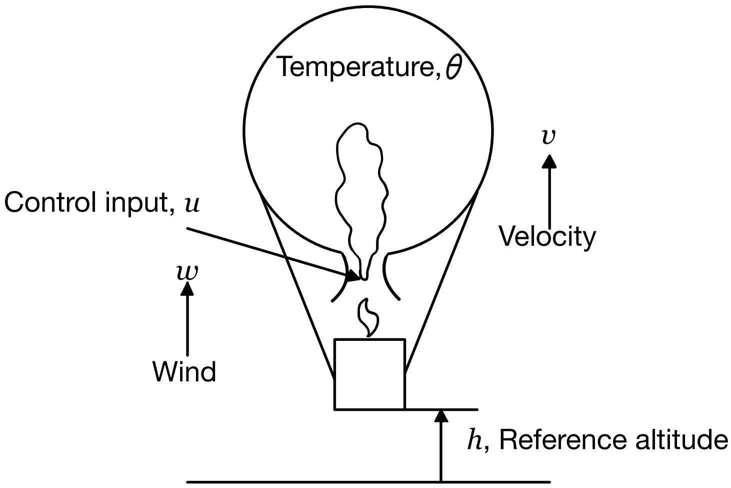

Figure 1: Hot air balloon system

In this project we will design a controller for a hot air balloon system based on an approximate differential equation describing its motion. The visual representation of the system is given in Figure 1. The equation of motion for the system is as follows

$$\begin{aligned} \dot{\theta} &= -\frac{1}{\tau_1}\theta + u, \\ \dot{v} &= -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w, \\ \dot{h} &= v, \end{aligned} \tag{1}$$

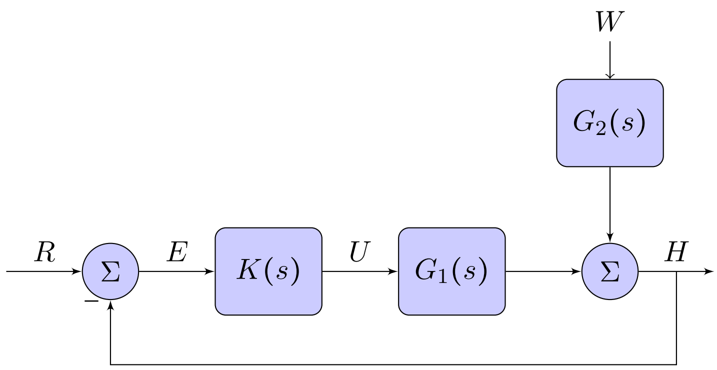

Figure 2: Block-diagram of the complete closed-loop system, where $K(s)$ is a controller

where $\theta \in \mathbb{R}$ is a temperature of air inside the balloon, $h \in \mathbb{R}$ is an altitude change from a reference point, $w \in \mathbb{R}$ is a wind disturbance, and $v \in \mathbb{R}$ is balloon's vertical velocity; $\sigma, \tau_1, \tau_2 \in \mathbb{R}$ are system parameters.

Part I, Laplace Domain

1. (15 points) Assume that $\sigma, \tau_1, \tau_2 > 0$. Find the expression for the open-loop output response of the system in Laplace domain when both the control input and the disturbance are present. Now, suppose you place the complete system in a closed-loop with a unity negative feedback. Draw a block-diagram of the complete system. Can you make the system stable using a proportional controller for the closed-loop system with a unity negative feedback?

   Solution:

$$G_1(s) = \left.\frac{h(s)}{u(s)}\right|_{w=0} = \frac{h(s)}{v(s)}\frac{v(s)}{\theta(s)}\frac{\theta(s)}{u(s)} = \frac{\sigma}{s(s+1/\tau_2)(s+1/\tau_1)} \tag{2}$$

$$G_2(s) = \left.\frac{h(s)}{w(s)}\right|_{u=0} = \frac{h(s)}{v(s)}\frac{v(s)}{w(s)} = \frac{1/\tau_2}{s(s+1/\tau_2)}. \tag{3}$$

   By superposition the output of the open-loop system is

$$H(s) = G_1(s)U(s) + G_2(s)W(s), \tag{4}$$

   where $W(s)$ is the Laplace domain representation of the disturbance. The block diagram of the complete system is given in Figure 2.

   The output of the closed-loop system with a unity negative feedback and a proportional controller is

$$H(s) = \frac{KG_1(s)}{1+KG_1(s)}R(s) + \frac{G_2(s)}{1+KG_1(s)}W(s). \tag{5}$$

$$1 + KG_1(s) = 0 \implies s^3 + (1/\tau_1 + 1/\tau_2)s^2 + s/\tau_1\tau_2 + K\sigma = 0 \tag{6}$$

$$M_H = \begin{pmatrix} 1/\tau_1 + 1/\tau_2 & 1 & 0 \\ K\sigma & 1/\tau_1\tau_2 & 1/\tau_1 + 1/\tau_2 \\ 0 & 0 & K\sigma \end{pmatrix}$$

Routh-Hurwitz determinants:

$$\begin{aligned} \Delta_1 &= 1/\tau_1 + 1/\tau_2; \\ \Delta_2 &= (\tau_1 + \tau_2)/\tau_1^2\tau_2^2 - K\sigma; \\ \Delta_3 &= K\sigma\Delta_2. \end{aligned}$$

Now $\Delta_1 > 0$ for $\tau_1, \tau_2 > 0$;, $\Delta_2 > 0$ for $K < (\tau_1 + \tau_2)/\tau_1^2\tau_2^2\sigma$, and $\Delta_3 > 0$ for $K > 0$. Hence, stability is preserved for $0 < K < (\tau_1 + \tau_2)/\tau_1^2\tau_2^2\sigma$.

2. (30 points) Assume that w is equal to some constant value for all $t \geq 0$. Find a controller such that (1) the system tracks constant reference signals with zero steady-state error; and (2) the error signal is no less than -0.2 for all $t \geq 0$ with convergence to zero for $\sigma = 5$, $\tau_1 = 0.1$, $\tau_2 = 0.1$. Take both constants for the reference and disturbance signals as 1 for (2). Provide a plot of the error signal.

   Solution: The constant disturbance is $W(s) = d/s$ where $d \in \mathbb{R}$. The tracking error signal between the constant reference input signal $R(s) = b/s$ and the closed-loop system output $H(s)$ from (5) is

$$E(s) = R(s) - H(s) = \frac{1}{s}\left(\frac{(s + 1/\tau_1)(bs(s + 1/\tau_2) - d/\tau_2)}{s(s + 1/\tau_1)(s + 1/\tau_2) + K\sigma}\right). \tag{7}$$

   By the final value theorem

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{(s + 1/\tau_1)(bs(s + 1/\tau_2) - d)}{s(s + 1/\tau_1)(s + 1/\tau_2) + K\sigma} = \frac{-d}{K\sigma\tau_1\tau_2}. \tag{8}$$

   Hence, the proportional controller can no longer achieve zero steady-state feedback.

   We try an integral controller $K(s) = K_i/s$. The new characteristic equation becomes $1 + K_i H(s)/s$.

$$1 + K_i G_1(s) = 0 \implies s^4 + (1/\tau_1 + 1/\tau_2)s^3 + s^2/\tau_1\tau_2 + K_i\sigma = 0 \tag{9}$$

$$M_H = \begin{pmatrix} 1/\tau_1 + 1/\tau_2 & 1 & 0 & 0 \\ 0 & 1/\tau_1\tau_2 & 1/\tau_1 + 1/\tau_2 & 1 \\ 0 & K_i\sigma & 0 & 1/\tau_1\tau_2 \\ 0 & 0 & 0 & K_i\sigma \end{pmatrix}$$

   Routh-Hurwitz determinants for $\sigma, \tau_1, \tau_2 > 0$:

$$\begin{aligned} \Delta_1 &= 1/\tau_1 + 1/\tau_2 > 0; \\ \Delta_2 &= (\tau_1 + \tau_2)/\tau_1^2\tau_2^2 > 0; \\ \Delta_3 &= -K_i\sigma(1/\tau_1 + 1/\tau_2)^2 > 0 \implies K_i < 0; \\ \Delta_4 &= K_i\sigma\Delta_3 > 0 \implies K_i > 0. \end{aligned}$$

Hence, $K_i = \emptyset$ for stability, so an integral controller alone cannot do achieve stability and satisfy design specifications.

Now, let's try proportional plus integral controller $K(s) = K_p + K_i/s$. The new characteristic equation becomes $1 + (K_p + K_i/s)H(s)$.

$$1+(K_p+K_i/s)G_1(s) = 0 \implies s^4+(1/\tau_1+1/\tau_2)s^3+s^2/\tau_1\tau_2+K_p\sigma s+K_i\sigma = 0 \quad (10)$$

$$M_H = \begin{pmatrix} 1/\tau_1 + 1/\tau_2 & 1 & 0 & 0 \\ K_p\sigma & 1/\tau_1\tau_2 & 1/\tau_1 + 1/\tau_2 & 1 \\ 0 & K_i\sigma & K_p\sigma & 1/\tau_1\tau_2 \\ 0 & 0 & 0 & K_i\sigma \end{pmatrix}$$

Routh-Hurwitz determinants for $\sigma, \tau_1, \tau_2 > 0$:

$$\begin{aligned}
&\Delta_1 = 1/\tau_1 + 1/\tau_2 > 0; \\
&\Delta_2 = (\tau_1 + \tau_2)/\tau_1^2\tau_2^2 - K_p\sigma > 0 \implies K_p < (\tau_1 + \tau_2)/\tau_1^2\tau_2^2\sigma; \\
&\Delta_3 = -K_i\sigma(1/\tau_1 + 1/\tau_2)^2 + K_p\sigma((\tau_1 + \tau_2)/\tau_1^2\tau_2^2 - K_p\sigma) > 0; \\
&\implies K_p\sigma((\tau_1 + \tau_2)/\tau_1^2\tau_2^2 - K_p\sigma) > K_i\sigma(1/\tau_1 + 1/\tau_2)^2; \\
&\Delta_4 = K_i\sigma\Delta_3 > 0 \implies K_i > 0.
\end{aligned}$$

We can always pick K_i and K_p to satisfy all three conditions. Hence, stability can be satisfied. Now we replace K in (7) with $K_p + K_i/s$ to get

$$E(s) = \frac{1}{s}\left(\frac{s(s + 1/\tau_1)(bs(s + 1/\tau_2) - d/\tau_2)}{s^2(s + 1/\tau_1)(s + 1/\tau_2) + K_p\sigma s + K_i\sigma}\right). \quad (11)$$

Then, by the final value theorem

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(s + 1/\tau_1)(bs(s + 1/\tau_2) - d)}{s^2(s + 1/\tau_1)(s + 1/\tau_2) + K_p\sigma s + K_i\sigma} = \frac{0}{K_i\sigma} = 0. \quad (12)$$

Hence, a PI controller can reject the disturbance. Now to satisfy (2), we simulate the system in MATLAB and try different combinations of K_i and K_p to approximately arrive at 0.5 and 17 respectively (any other choice that satisfies the requirement is accepted). The plot of the error signal is given in

Part II, State-Space

1. (30 points) Assume the disturbance, w, is zero for all t in (1) for the remainder of the project. Write the system in the state-space matrix form. Compute the controllability matrix of the system using the appropriate matrices found in part 1. Find the determinant of the controllability matrix found in part 2. What condition should be satisfied for the system to be controllable? Suppose now that the condition you found in part 3 is violated, what part of the state can you still control?

Solution:

$$\begin{bmatrix} \dot\theta \\ \dot v \\ \dot h \end{bmatrix} = \underbrace{\begin{bmatrix} -1/\tau_1 & 0 & 0 \\ \sigma & -1/\tau_2 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{:=A} \begin{bmatrix} \theta \\ v \\ h \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{:=B} u. \quad (13)$$

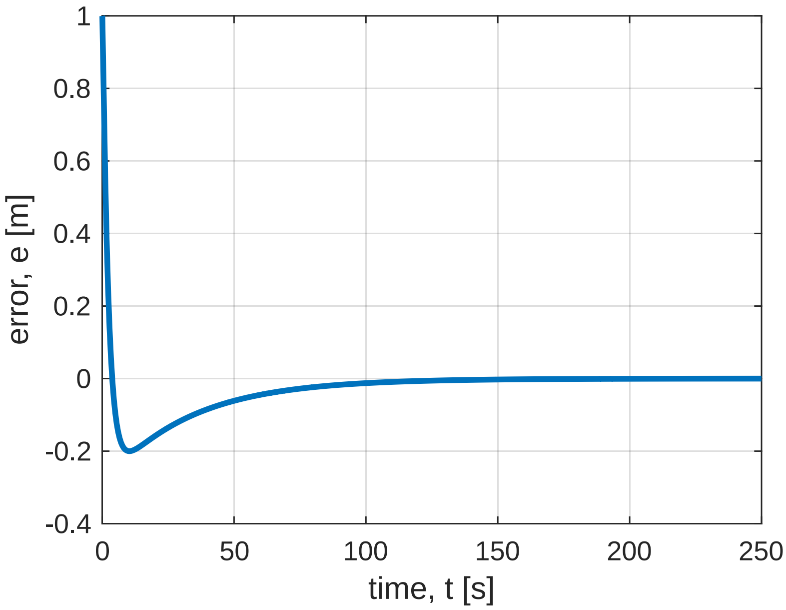

Figure 3: The error signal that satisfies design objectives (1) and (2) for question 2

We note from (13) that there are 3 states in total. Hence, the controllability matrix $\mathscr{C} = [B|AB|A^2B]$ is 3 by 3. The colmuns of $\mathscr{C}$ are computed as follows

$$AB = \begin{bmatrix} -1/\tau_1 \\ \sigma \\ 0 \end{bmatrix} \implies A^2B = A(AB) = \begin{bmatrix} 1/\tau_1^2 \\ -\sigma/\tau_1 - \sigma/\tau_2 \\ \sigma \end{bmatrix} \implies C = \begin{bmatrix} 1 & -1/\tau_1 & 1/\tau_1^2 \\ 0 & \sigma & -\sigma/\tau_1 - \sigma/\tau_2 \\ 0 & 0 & \sigma \end{bmatrix}.$$

Since $\mathscr{C}$ is an upper-triangular matrix, its determinant is equal to the product of diagonal entries. Hence,

$$\det \mathscr{C} = \sigma^2. \tag{14}$$

Now, $\det \mathscr{C} = 0 \iff \sigma = 0$. Thus, the condition to be satisfies is $\sigma \neq 0$.

If $\sigma = 0$, then we can observe from (1) once we plug in $\sigma = 0$ in the equation, that the only state influenced by the control input is θ.

2. (15 points) Take the parameters as $\sigma = 5$, $\tau_1 = 0.1$, $\tau_2 = 0.1$. Suppose your controller in state-space is given by $u = Kx$ where $K \in \mathbb{R}^{1\times 3}$ is the controller gain and $x \in \mathbb{R}^3$ is the state of the system. Take $K = \begin{bmatrix} 14 & -10 & -1 \end{bmatrix}$. Verify that your closed-loop system is stable.
Solution: Now the state space looks like this: $\dot{x} = (A + BK)x$ where we compute: $A + BK = \begin{bmatrix} -10 & 0 & 0 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 14 & -10 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -10 & -1 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix} := \widehat{A}$
Now we compute the eigenvalues of $\widehat{A}$ and verify that they all have negative

real parts.

$$\det(\widehat{A} - \lambda I) = -\lambda^3 - 6\lambda^2 - 10\lambda - 5 = 0 \tag{15}$$

$$\lambda = -1, \frac{-5+\sqrt{5}}{2}, \frac{-5-\sqrt{5}}{2} \in \mathbb{C}^- \tag{16}$$

NOTE: this part can be solved using Routh-Hurwitz as well. Perform Rough-Hurwitz test on equation (15) to show that the roots are in C^-.

MATLAB codes:

```
clc
close all
clear all

s=tf('s');
sigma_3 = 5; tau_13 = 0.1; tau_23 = 0.2;
t=0:0.0001:250;
format long
Kd=17;
Ki=0.5;
b = 1;d=1;
E_1=s*(s+1/tau_13)*(s*(s+1/tau_23)-1)/(s^2*(s+1/tau_13)*(s+1/tau_23)+Kd*s*sigma_3+Ki*sigma_3);
e_1=step(E_1,t);
figure(5)
plot(t,e_1,'LineWidth',2)
grid on
xlabel('time, t [s]');
ylabel('error, e [m]');
```