1.

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u,$$
$$\dot{v} = -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w,$$
$$\dot{h} = v,$$

$$s\Theta(s) = -\frac{1}{T_1}\Theta(s) + U(s)$$
$$sV(s) = -\frac{1}{T_2}V(s) + \sigma\Theta(s) + \frac{1}{T_2}W(s)$$
$$sH_h(s) = V(s)$$

$$\Theta(s) = -\frac{1}{T_1 s}\Theta(s) + \frac{U(s)}{s}$$

$$\Theta(s) = \frac{\frac{U(s)}{s}}{1 + \frac{1}{T_1 s}} = \frac{U(s)T_1 s}{s^2 T_1 + s} = \frac{UT_1}{sT_1 + 1} \quad \text{--- (1)}$$

$$\left(s + \frac{1}{T_2}\right)V(s) = \sigma\Theta(s) + \frac{1}{T_2}W(s)$$

$$V(s) = \frac{\sigma\Theta + \frac{1}{T_2}W}{s + \frac{1}{T_2}} = \frac{T_2\sigma\Theta + W}{sT_2 + 1} \quad \text{plug in (1)}$$

$$= \frac{T_2\sigma \cdot \frac{UT_1}{sT_1+1} + W}{sT_2 + 1} = \frac{T_1T_2\sigma U + (sT_1+1)W}{(sT_1+1)(sT_2+1)} \quad \text{--- (2)}$$

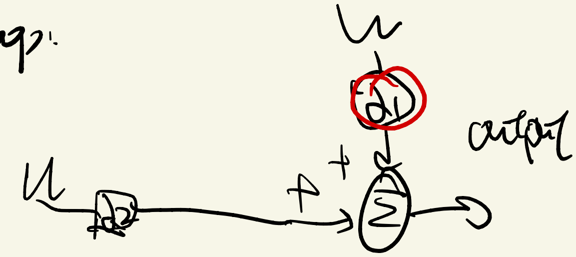
$$\therefore \underbrace{H_h(s)}_{\text{output}} = \frac{V(s)}{s} = \boxed{\frac{T_1T_2\sigma U + (sT_1+1)W}{s(sT_1+1)(sT_2+1)}} \quad \text{--- (3)}$$

(reference altitude)

open-loop output response in Laplace domain

$$H_n(s) = \frac{T_1 T_2 G U + S(T_1+1)W}{S(ST_1+1)(ST_2+1)}$$

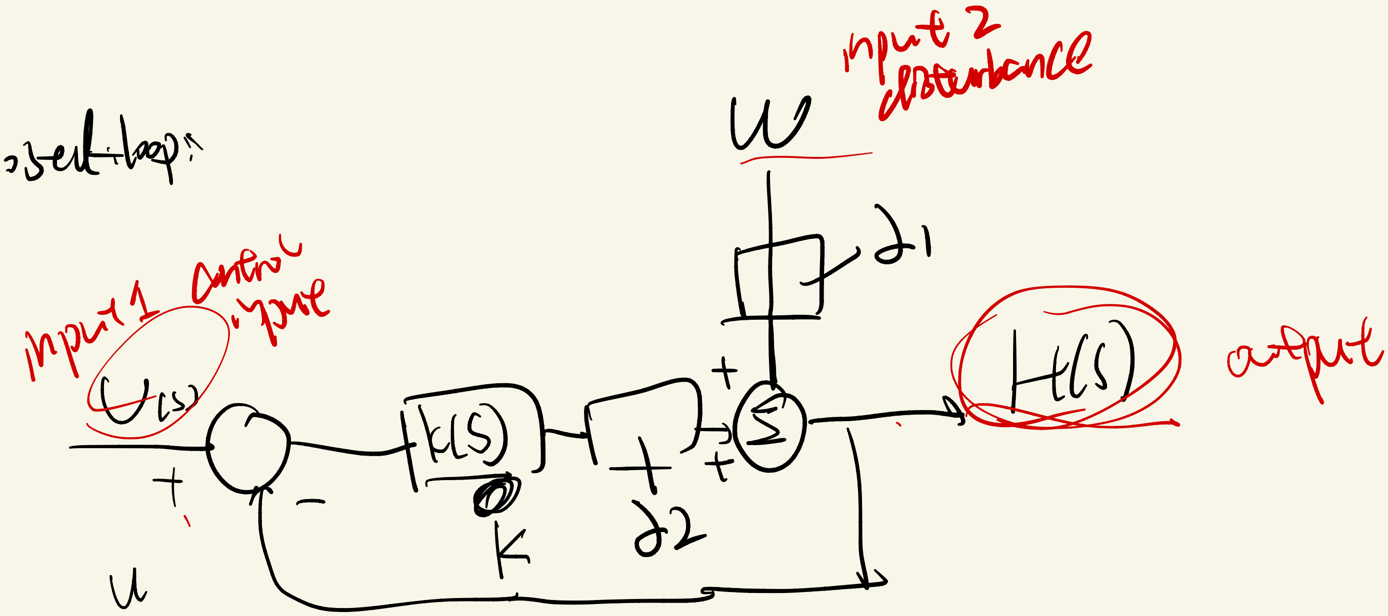
open-loop:



here our ∂_1 is $\dfrac{S T_1 + 1}{S(ST_1+1)(ST_2+1)}$

∂_2 is $\dfrac{T_1 T_2 G}{S(ST_1+1)(ST_2+1)}$

closed-loop:



$$(U - H(s))K\partial_2 + W\partial_1 = H(s)$$

$$UK\partial_2 - HK\partial_2 + W\partial_1 = H(s)$$

$$= H(\cancel{1+K\partial_2})$$

$$\frac{UK\partial_2 + W\partial_1}{1+K\partial_2}$$

All we need to choose is a K so that $1+KG_2$ is Routh-Hurwitz.

$$1+\frac{KT_1T_2G}{S(ST_1+1)(ST_2+1)} = S(ST_1+1)(ST_2+1)+KT_1T_2G$$

$$= S(S^2T_1T_2+S(T_1+T_2)+1)+KT_1T_2G$$

$$= S^3T_1T_2+S^2(T_1+T_2)+S+KT_1T_2G$$

Routh-Hurwitz:

$$\begin{bmatrix} T_1+T_2 & T_1T_2 & 0 \\ KT_1T_2G & 1 & T_1+T_2 \\ 0 & 0 & KT_1T_2G \end{bmatrix}$$

$\Delta_1 = T_1+T_2 \geq 0$ ✓.

$\Delta_2 = (T_1+T_2) - KT_1T_2G(T_1T_2)$

$= T_1+T_2 - KT_1^2T_2^2G > 0$

$K < \frac{T_1+T_2}{T_1^2T_2^2G}$

$\Delta_3 = KT_1T_2G\,\Delta_2 > 0 \Rightarrow K > 0$

$\therefore$ $\boxed{0 < K < \frac{T_1+T_2}{T_1^2T_2^2G}}$ → for system to be stable

2. $E_2(s) = U(s) - Y(s) = \frac{b}{s} - \frac{U K(s) G_2 + W G_1}{1 + K(s) G_2}$

$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s E(s)$

here our G_1 is $\frac{\cancel{s}\, T_1 + 1}{s(sT_1+1)(sT_2+1)}$

G_2 is $\frac{T_1 T_2 6}{s(sT_1+1)(sT_2+1)}$

$= \frac{b}{s} - \frac{U K(s) T_1 T_2 6 + W(\cancel{s} T_1 + 1)}{s(sT_1+1)(sT_2+1) + K(s) T_1 T_2 6}$

$U = \frac{b}{s}$

$\therefore \lim_{s\to 0} s E(s) = b - \frac{[U K(s) T_1 T_2 6 + W(sT_1+1)]\, s}{s(sT_1+1)(sT_2+1) + K T_1 T_2 6}$

If $K(s)$ is proportional controller: sub $K(s) = K$

$\lim_{s\to 0} s E(s) = b - b = 0$

If $K(s)$ is integral controller: sub $K(s) = \frac{K}{s}$

$K \in \mathbb{R}$

?

$\lim_{s\to 0} s E(s) = b - \frac{s \cdot \cancel{s}\, U \frac{k}{\cancel{s}} T_1 T_2 6 + W s^2 (sT_1+1)}{s^2(sT_1+1)(sT_2+1) + \frac{k}{\cancel{s}} T_1 T_2 6}$

$= b - b = 0$

Also, check stability: $1 + \frac{K}{s} G_2 = 1 + \frac{K T_1 T_2 6}{s^2 (sT_1+1)(sT_2+1)}$

$= s^2 (sT_1+1)(sT_2+1) + K T_1 T_2 6$

$=$

$$= s^2(s^2T_1T_2 + s(T_1+T_2)+1) + kT_1T_2 6$$

$$= s^4(T_1T_2) + s^3(T_1+T_2) + s^2 + 0s + kT_1T_2 6.$$

Routh-Hurwitz:

$$\begin{bmatrix} T_1+T_2 & T_1T_2 & 0 & 0 \\ 0 & 1 & T_1+T_2 & T_1T_2 \\ 0 & kT_1T_26 & 0 & 1 \\ 0 & 0 & 0 & kT_1T_26 \end{bmatrix}$$

$\Delta_1 > 0$

$\Delta_2 = T_1+T_2 > 0$

$\Delta_3 = -kT_1T_26\,(T_1+T_2)^2 < 0$

not stable!

Combination of proportional and integral controller:

$$K(s) = k + \frac{k}{s}$$

check stability: $1 + (k+\frac{k}{s})G_2 = 1 + k(1+\frac{1}{s})\left(\frac{T_1T_26}{s(sT_1+1)(sT_2+1)}\right)$

$$\Rightarrow s(s^2T_1T_2 + s(T_1+T_2)+1) + k(1+\tfrac{1}{s})(T_1T_26)$$

$$= s^2(s^2T_1T_2 + s(T_1+T_2)+1) + k(s+1)(T_1T_26)$$

$$= s^4T_1T_2 + s^3(T_1+T_2) + s^2 + s(kT_1T_26) + kT_1T_26$$

Routh-Hurwitz:

$$\begin{bmatrix} T_1+T_2 & T_1T_2 & 0 & 0 \\ kT_1T_26 & 1 & T_1+T_2 & T_1T_2 \\ 0 & kT_1T_26 & kT_1T_26 & 1 \\ 0 & 0 & 0 & kT_1T_26 \end{bmatrix}$$

Δ_1 Δ_2 Δ_3 Δ_4

$\Delta_1 > 0$

$$\Delta_2 = T_1+T_2 - kT_1^2T_2^26 > 0 \Rightarrow T_1+T_2 > kT_1^2T_2^26$$

$$k < \frac{T_1+T_2}{T_1^2T_2^26}$$

$$\Delta_3 = -kT_1T_26(T_1+T_2)^2 + kT_1T_26\Delta_2$$

$$= kT_1T_26(T_1+T_2 - kT_1^2T_2^26 - (T_1+T_2)^2) > 0$$

$$T_1+T_2 > kT_1^2T_2^26 + (T_1+T_2)^2$$

$$k < \frac{T_1+T_2-(T_1+T_2)^2}{T_1^2T_2^26}$$

$\Delta_4 = kT_1T_26\Delta_3 > 0$ So system could be stable!

for $\boxed{0 < k < \frac{T_1+T_2-(T_1+T_2)^2}{T_1^2T_2^26}}$

$$T_2(s) = R(s) - Y(s) = \frac{b}{s} - \frac{uK(s)T_1T_2G + W(sT_1+1)}{s(sT_1+1)(sT_2+1) + K(s)T_1T_2G}$$

replace $K(s)$ with $K + \frac{K}{s}$

$$T_2(s) = \frac{b}{s} - \frac{u(sK+K)T_1T_2G + W(sT_1+1)s}{s^2(sT_1+1)(sT_2+1) + (sK+K)T_1T_2G}$$

$$\lim_{s\to 0} sT_2(s) = b - \frac{su(sK+K)T_1T_2G + W(sT_1+1)s^2}{s^2(sT_1+1)(sT_2+1) + (Ks+K)T_1T_2G}$$

$u = \frac{b}{s}$

$$= b - \frac{b(sK+K)T_1T_2G + W(sT_1+1)s^2}{s^2(sT_1+1)(sT_2+1) + (Ks+K)T_1T_2G}$$

$$= b - \frac{b(K)T_1T_2G + 0}{KT_1T_2G} = \boxed{0}$$

So we can either use a proportional controller or a combination of proportional & integral controller.

Try combination of proportional & integral controller:

$$\text{error (function to be plotted)} = 1 - \frac{(sk+k)(0.1)(0.1)(5) + (0.1s+1)s^2}{s^2(0.1s+1)(0.1s+1) + (ks+k)(0.1)(0.1)(5)}$$

$$= \frac{s^2(0.1s+1)^2\cancel{(0.1s+1)} - (0.1s+1)s^2}{s^2(0.1s+1)(0.1s+1) + k(s+1)(0.05)}$$

$$= \frac{s^2\left[(0.1s+1)^2 - (0.1s+1)\right]}{\sim} = \frac{s^2(0.1s+1)(0.1s+1-1)}{\sim}$$

$$= \frac{s^2(0.1s+1)(0.1s)}{s^2(0.1s+1)(0.1s+1) + k(s+1)(0.05)} = \boxed{\frac{s^2(s+10)(s)}{s^2(s+10)^2 + k(s+1)\cdot 5}}$$

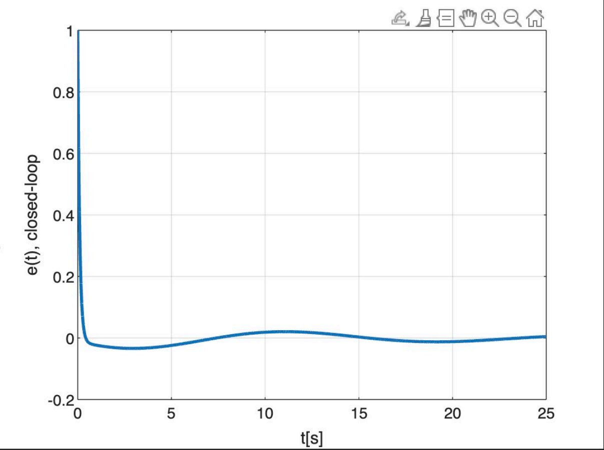
for $0 < k < 320$

rlocus.m × Finalp.m ×

/MATLAB Drive/Finalp.m

```
clc
close all
clear all
%% closed-loop combination of propoertional and integral controller
t = 0:0.001:25; % time
s = tf('s'); %laplace variable
%% plotting error signal for constant reference signal
K = 3;
sysE = (s^2*(s+10)*s)/(s^2*(s+10)^2+K*(s+1)*5)
err = step(sysE,t);
figure(1)
plot(t,err,'LineWidth', 2)
grid on
xlabel('t[s]')
ylabel('e(t), closed-loop')
```

Figure 1 ×



Command Window

New to MATLAB? See resources for Getting Started.

```
            s^4 + 10 s^3
  ----------------------------------
  s^4 + 20 s^3 + 100 s^2 + 15 s + 15

Continuous-time transfer function.
Model Properties
>>
```

Part II 1.

define $X = \begin{bmatrix} \theta \\ v \\ h \end{bmatrix}$

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u,$$

$$\dot{v} = -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w,$$

$$\dot{h} = v,$$

$$\dot{X} = \begin{bmatrix} \dot{\theta} \\ \dot{v} \\ \dot{h} \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 \\ \sigma & -\frac{1}{\tau_2} & 0 \\ 0 & 1 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} \theta \\ v \\ h \end{bmatrix}}^{X} + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{B} u$$

$$G = \left[B \mid AB \mid A^2B \right]$$

$$AB = \begin{bmatrix} -\frac{1}{\tau_1} \\ \sigma \\ 0 \end{bmatrix} \qquad A^2B = \begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 \\ \sigma & -\frac{1}{\tau_2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\tau_1} \\ \sigma \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\tau_1^2} \\ -\sigma\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) \\ \sigma \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & -\frac{1}{\tau_1} & \frac{1}{\tau_1^2} \\ 0 & \sigma & -\sigma\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) \\ 0 & 0 & \sigma \end{bmatrix}$$

← upper triangular matrix

$\det G = 1 \cdot \sigma \sigma = \sigma^2$

proof:

$\det \mathcal{C} = 6\left[6 - 0 \cdot \frac{1}{\tau_1}\right] = 6^2$

for the controllability ~~to be controllable~~, $\det \mathcal{C} \neq 0 \Rightarrow \boxed{6 \neq 0}$

If this condition is violated $\Rightarrow 6 = 0$.

$BK = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [14\ -10\ -1] = \begin{bmatrix} 14 & -10 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$+ A = \begin{bmatrix} -10 & 0 & 0 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$(A+BK) = \begin{bmatrix} 4 & -10 & -1 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

then $\mathcal{C} = \begin{bmatrix} 1 & -\frac{1}{\tau_1} & \frac{1}{\tau_1^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\mathcal{C}X = \begin{bmatrix} 1 & -\frac{1}{\tau_1} & \frac{1}{\tau_1^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ v \\ h \end{bmatrix} = \begin{bmatrix} \theta - \frac{v}{\tau_1} + \frac{h}{\tau_1^2} \\ 0 \\ 0 \end{bmatrix}$

we can see that for arbitrary input X, $\mathcal{C}X$ has no control on v, h but we can still control θ.

2. $\dot{X} = AX + Bu$ $\quad A = \begin{bmatrix} -10 & 0 & 0 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= AX + BKX$ $\quad K = [14.\ -10.\ -1]$

$u = KX = [14.\ -10.\ -1] \begin{bmatrix} \theta \\ v \\ h \end{bmatrix} = [14\theta - 10v - h]$

$BKX = \begin{bmatrix} 14\theta - 10v - h \\ 0 \\ 0 \end{bmatrix}$ $\quad AX = \begin{bmatrix} -10\theta \\ -10v + 5\theta \\ v \end{bmatrix}$

$\dot{X} = AX + BKX$

$$= \begin{bmatrix} 4\theta - 10v - h \\ -10v + 5\theta \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -10 & -1 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} \theta \\ v \\ h \end{bmatrix}}_{X}$$

or calculate $\hat{A} = A + BK$ (result are the same)

$\dot{\theta} = -\frac{1}{10}\theta$

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u,$$
$$\dot{v} = -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w,$$
$$\dot{h} = v,$$

$$\underbrace{\begin{bmatrix} 4 & -10 & -1 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\hat{A}} \longrightarrow \det|\hat{A} - \lambda I| = 0$$

$$\det\begin{bmatrix} 4-\lambda & -10 & -1 \\ 5 & -10-\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = (4-\lambda)\begin{vmatrix} -10-\lambda & 0 \\ 1 & -\lambda \end{vmatrix} - 5\begin{vmatrix} -10 & -1 \\ 1 & -\lambda \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2 + 10\lambda) - 5(10\lambda + 1)$$

$$= -\lambda^3 + 4\lambda^2 + 40\lambda - 10\lambda^2 - 50\lambda - 5$$

$= -\lambda^3 - 6\lambda^2 - 10\lambda - 5 = 0$

$\Rightarrow \lambda^3 + 6\lambda^2 + 10\lambda + 5 = 0$ — characteristic eq

Routh-Hurwitz:

$$\begin{bmatrix} 6 & 1 & 0 \\ 5 & 10 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$D_1 = 6 > 0$

$D_2 = 60 - 5 = 55 > 0$

$D_3 = 5D_2 > 0$

$\therefore$ The system is stable!