MVA: Convex Optimization

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Homework 1

1- Convex Sets

1 A slab $\{x \in \mathbb{R}^n, \alpha \leq a^T x \leq \beta\}$

$$\{x \in \mathbb{R}^n, \alpha \leq a^T x \leq \beta\} = \underbrace{\{x \in \mathbb{R}^n, \alpha \leq a^T x\}}_{\text{halfspace}} \cap \underbrace{\{x \in \mathbb{R}^n, a^T x \leq \beta\}}_{\text{halfspace}}$$

A slab is the intersection of two halfspaces. We know that halspaces are convex and that the intersection of two convex set is convex.

Hence, a slab is convex.

2. A rectangle $\{x \in \mathbb{R}^n, \alpha_i \leq x_i \leq \beta_i \ \forall i \in [\![1, n]\!]\}$

$$\{x \in \mathbb{R}^n, \forall i \in [\![1, n]\!] \ \alpha_i \leq x_i \leq \beta_i\} = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n, \alpha_i \leq e_i^T x \leq \beta_i\}$$

With e_i the vector in $\mathbb{R}^n$ whose i^{th} component is 1 and all the other 0.

We found that a rectangle is an intersection of slabs. We know from the previous question that a slab is convex. An intersection of convex sets is convex, thus a rectangle is convex.

3 A wedge $\{x \in \mathbb{R}^n, a_1^T x \leq b_1, a_2^T x \leq b_2\}$

$$\{x \in \mathbb{R}^n, a_1^T x \leq b_1, a_2^T x \leq b_2\} = \underbrace{\{x \in \mathbb{R}^n, a_1^T x \leq b_1\}}_{\text{halfspace, convex}} \cap \underbrace{\{x \in \mathbb{R}^n, a_2^T x \leq b_2\}}_{\text{halfspace, convex}}$$

A wedge is the intersection of two ~~ha~~ convex sets. Thus, it is convex.

4. The set of points closer to ~~one set~~ a given point than a given set

$\{x, \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$, $S \subseteq \mathbb{R}^n$

Let $y \in S$. Show that $\{x, \|x - x_0\|_2 \leq \|x - y\|_2\}$ is a halfspace

$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff \|x - x_0\|_2^2 \leq \|x - y\|_2^2$$

$$\iff (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)$$

$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff -2 x_0^T x + \|x_0\|_2^2 \leq -2 y^T x + \|y\|_2^2$$

$\|x-x_0\|_2^2 \leq \|x-y\|_2 \iff \cancel{2(y-x_0)^T x \leq \|y\|_2^2} - \|x\|_2^2$

$\iff 2(y-x_0)^T x \leq \|y\|_2^2 - \|x\|_2^2$

With $a = 2(y-x_0)$ and $b = \|y\|_2^2 - \|x\|_2^2$, we find that

$\{x,\ \|x-x_0\|_2 \leq \|x-y\|_2\} = \{x,\ a^T x \leq b\}$

Thus, $\{x,\ \|x-x_0\|_2 \leq \|x-y\|_2\}$ is a halfspace, thus convex.

Our set can be written as $\bigcap_{y\in S} \underbrace{\{x,\ \|x-x_0\|_2 \leq \|x-y\|_2\}}_{\text{halfspace, convex}}$

Thus, $\{x,\ \|x-x_0\|_2 \leq \|x-y\|_2,\ \forall y \in S\}$ is convex (intersection of convex sets).

5. The set of points closer to one set than another $\{x,\ \mathrm{dist}(x,S) \leq \mathrm{dist}(x,T)\}$, $S, T \subseteq \mathbb{R}^n$

This set is not always convex. If we take $n=1$, $S=\{-1, 1\}$ and $T=\{0\}$.

We have $\{x,\ \mathrm{dist}(x,S) \leq \mathrm{dist}(x,T)\} = \underbrace{]-\infty, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, +\infty[}_{\text{not convex}}$

6. The set $A = \{x,\ x + S_2 \subseteq S_1\}$, $S_1, S_2 \subseteq \mathbb{R}^n$, S_1 convex

Let $A = \{x,\ x+S_2 \subseteq S_1\}$ with $S_1, S_2 \subseteq \mathbb{R}^n$ and S_1 convex.

Let $a, b \in A$ and $\lambda \in [0,1]$

Show that $(\lambda a + (1-\lambda) b) + S_2 \subseteq S_1$.

Let $s \in S_2$

$$\lambda a + (1-\lambda) b + s = \lambda \underbrace{(a+s)}_{\substack{\in x+S_2 \\ \in S_1}} + (1-\lambda) \underbrace{(b+s)}_{\substack{\in x+S_2 \\ \in S_1}} \in S_1$$

$a+s \in S_1$ and $b+s \in S_1$ because $a, b \in A$.

S_1 is convex, thus the convex combination is also in S_1.

Hence, A is convex.

7. The set $\{x,\ \|x-a\|_2 \leq \theta \|x-b\|_2\}$ with $a \neq b$ and $\theta \in [0,1]$

7 The set $\{x,\ \|x-a\|_2 \leq \theta \|x-b\|_2\}$ with $a \neq b$ and $\theta \in [0,1]$

$$\{x,\ \|x-a\|_2 \leq \theta\|x-b\|_2\} = \{x,\ \|x-a\|_2^2 \leq \theta^2 \|x-b\|_2^2\}$$
$$= \{x,\ x^Tx - 2a^Tx + a^Ta \leq \theta^2(x^Tx - 2b^Tx + b^Tb\}$$
$$= \{x,\ (1-\theta^2)x^Tx - 2(a-b\theta^2)^Tx + a^Ta - \theta^2 b^Tb \leq 0\}$$

$$\{x,\ \|x-a\|_2 \leq \theta\|x-b\|_2\} = \{x,\ (1-\theta^2)x^Tx - 2(a-b\theta^2)^Tx \leq a^Ta - \theta^2 b^Tb\}$$

If $\theta = 1$: $\{x,\ \|x-a\|_2 \leq \theta\|x-b\|_2\} = \{\underbrace{x,\ -(a-b\theta^2)^Tx \leq \tfrac{1}{2}(a^Ta - \theta^2 b^Tb)}_{\text{halfspace}}\}$

A halfspace is convex, thus, if $\theta = 1$ our set is convex.

If $\theta < 1$:

$$\{x,\ \|x-a\|_2 \leq \theta\|x-b\|_2\} = \{x,\ \|x\|_2^2 - 2\left(\frac{a-b\theta^2}{1-\theta^2}\right)^T x \leq \frac{\|a\|_2^2 - \theta^2\|b\|_2^2}{1-\theta^2}\}$$
$$= \{x,\ \|x\|_2^2 - 2\left(\frac{a-b\theta^2}{1-\theta^2}\right)^T x + \left\|\frac{a-b\theta^2}{1-\theta^2}\right\|_2^2 - \left\|\frac{a-b\theta^2}{1-\theta^2}\right\|_2^2 \leq \frac{\|a\|_2^2 - \theta^2\|b\|_2^2}{1-\theta^2}\}$$

$$\{x,\ \|x-a\|_2 \leq \theta\|x-b\|_2\} = \{x,\ (x-c)^T(x-c) \leq \frac{\|a\|_2^2 - \theta^2\|b\|_2^2}{1-\theta^2} + \|c\|_2^2\}$$

avec $c = \dfrac{a-b\theta^2}{1-\theta^2}$

We recognize a ball of center $c = \dfrac{a-b\theta^2}{1-\theta^2}$ and radius $r^2 = \dfrac{\|a\|_2^2 - \theta^2\|b\|_2^2}{1-\theta^2} + \|c\|_2^2$

Therefore our set is convex.

2 - Pointwise Maximum and Supremum

1) Let $f(x) = \max_{i \in [\![1, k]\!]} \| A^{(i)} x - b^{(i)} \|$, with $A^{(i)} \in \mathbb{R}^{m \times n}$, $b^{(i)} \in \mathbb{R}^m$ and $\|.\|$ is a norm on $\mathbb{R}^m$.

<u>Let us show that f is convex</u>

The function $x \mapsto \| A^{(i)} x - b^{(i)} \|$ is the composition of an affine function (convex) and a norm (convex). Thus $x \mapsto \| A^{(i)} x - b^{(i)} \|$ is convex for each $i \in [\![1, k]\!]$. f is the pointwise maximum of k convex functions.

Hence, <u>f is convex</u>.

2) $f(x) = \sum_{i=1}^{r} |x|_{[i]}$, with $|x|$ denotes the vector with $|x|_i = |x_i|$ and $|x|_{[i]}$ the i^{th} largest component of $|x|$.

<u>Show that f is convex</u>

~~We can say that~~ $\cancel{f(x) = \sum_{i=1}^{r} |x|_{[i]} = \max_{i_k} \sum_{i=1}^{r} |x|_{i_k}}$

We can say that $f(x) = \max \{ x_{i_1} + x_{i_2} + \dots + x_{i_{r-1}} + x_{i_r} ,\ 1 \le i_1 \le \dots \le i_r \le r \}$

Hence f is the pointwise maximum of an affine function.

Therefore, <u>f is convex</u>.

3- Products and Ratios of Convex Functions

We call I the interval where the functions f and g are defined on.

Let $x, y \in I$ and $\theta \in [0,1]$.

Show that fg is convex.

Show that $fg(\theta x + (1-\theta)y) \leq fg(x)\theta + (1-\theta)fg(y)$

$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$ because f is convex

$g(\theta x + (1-\theta)y) \leq \theta g(x) + (1-\theta)g(y)$ because g is convex

Thus $fg(\theta x + (1-\theta)y) \leq (\theta f(x) + (1-\theta)f(y))(\theta g(x) + (1-\theta)g(y))$

$$\leq \theta^2 f(x)g(x) + (1-\theta)^2 f(y)g(y) + \theta(1-\theta)f(x)g(y) + \theta(1-\theta)f(y)g(x)$$

$\theta^2 = \theta(\theta+1-1)$

$(1-\theta)^2 = (1-\theta) - \theta(1-\theta)$

$$\leq \theta f(x)g(x) + \theta(\theta-1)f(x)g(x) + (1-\theta)f(y)g(y) - \theta(1-\theta)f(y)g(y)$$
$$+ \theta(1-\theta)f(x)g(y) + \theta(1-\theta)f(y)g(x)$$

$$\leq \theta f(x)g(x) + (1-\theta)f(y)g(y) + \theta(\theta-1)[f(x)g(x) - f(x)g(y)]$$
$$+ \theta(1-\theta)[-f(y)g(y) + f(y)g(x)]$$

$$\leq \theta f(x)g(x) + (1-\theta)f(y)g(y) + \theta(\theta-1)f(x)(g(x)-g(y))$$
$$+ \theta(1-\theta)f(y)(g(x)-g(y))$$

$$\leq \theta f(x)g(x) + (1-\theta)f(y)g(y) + \theta(\theta-1)(f(x)-f(y))(g(x)-g(y))$$

Since $\theta \in [0,1]$, $\theta - 1 \leq 0$ and $\theta(\theta-1) \leq 0$.

Also, since f and g or both non increasing (or non decreasing), we have:

$$(f(x)-f(y))(g(x)-g(y)) \geq 0.$$

Thus $\theta(\theta-1)(f(x)-f(y))(g(x)-g(y)) \leq 0$.

Hence $fg(\theta x + (1-\theta)y) \leq \theta fg(x) + (1-\theta)fg(y)$

Thus, fg is convex

4. Conjugate of functions

1) Max function $f(x) = \max_i x_i$ on $\mathbb{R}^n$

$$f^*(y) = \sup_{x\in\mathbb{R}^n} (y^T x - f(x)) = \sup_{x\in\mathbb{R}^n} (y^T x - \max_i x_i) \quad \forall y \in \operatorname{dom} f^*$$

Let's first look for all the y such that $x \mapsto y^T x - f(x)$ is bounded.

- Consider that $\exists k \in [\![1, n]\!]$, $y_k < 0$.

Then, if we have an x such that $x_k = -t$ and $x_i = 0$ $\forall i \neq k$:

$$y^T x - f(x) = -t y_k \xrightarrow[t\to+\infty]{} +\infty$$

It is unbounded, thus we need $y \geq 0$.

- Consider $y \geq 0$ and $\mathbb{1}^T y > 1$. If we have an x such that $x_k = t$ $\forall k \in [\![1, n]\!]$, we have:

$$\begin{aligned} y^T x - f(x) &= y^T \mathbb{1} t - t \\ &= t (\underbrace{y^T \mathbb{1} - 1}_{>0}) \xrightarrow[t\to+\infty]{} +\infty \end{aligned}$$

It is unbounded, thus we need $\mathbb{1}^T y \leq 1$.

- Consider $y \geq 0$ and $\mathbb{1}^T y < 1$. With $x = -t\mathbb{1}$ we have:

$$y^T x - f(x) = -t (\underbrace{y^T \mathbb{1} - 1}_{<0}) \xrightarrow[t\to+\infty]{} +\infty$$

It is unbounded, thus we need $\mathbb{1}^T y = 1$

- Consider $y \geq 0$ and $\mathbb{1}^T y = 1$: We have:

$$\forall x \qquad y^T x \leq \max_i x_i \Rightarrow y^T x - \max_i x_i \leq 0$$

~~Thus~~, We have the equality for $x = 0$.

Therefore

$$\boxed{f^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \text{ and } \mathbb{1}^T y = 1 \\ +\infty & \text{otherwise} \end{cases}}$$

2) Sum of largest elements $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$

$$f^*(y) = \sup_x (y^T x - f(x)) = \sup_x \left(y^T x - \sum_{i=1}^{r} x_{[i]}\right) \quad \forall y \in \text{dom } f^*$$

Again, let's look at all the y such that $x \mapsto y^T x - f(x)$ is bounded.

- Consider that $\exists k \in [\![1, n]\!]$ $y_k < 0$

If we take x such that $x_k = -t$ and $x_i = 0$ $\forall i \neq k$, we have:

$$y^T x - f(x) = -t y_k + t \xrightarrow[t \to +\infty]{} +\infty \quad \text{unbounded}$$

Thus, we need $y \geq 0$

- Consider that $\exists k \in [\![1, n]\!]$, $y_k > 1$

If we take x such that $x_k = t$ and $x_i = 0$ $\forall i \neq k$, we have:

$$y^T x - f(x) = t y_k - t \xrightarrow[t \to +\infty]{} +\infty \quad \text{unbounded}$$

Thus, we need $y \leq 1$

- Consider that $\mathbb{1}^T y \neq r$

If we take $x = t\mathbb{1}$, we have:

$$y^T x - f(x) = y^T \mathbb{1} t - tr$$
$$= t(\underbrace{y^T \mathbb{1} - r}_{\neq 0}) \quad \text{unbounded when } t \to \pm\infty.$$

Thus, we need $0 \leq y \leq 1$ and $\mathbb{1}^T y = r$.

- Consider $0 \leq y \leq 1$ and $\mathbb{1}^T y = r$. We have:

$$x^T y \leq f(x) \quad \forall x$$

We have the equality for $x = 0$

Therefore
$$\boxed{f^*(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \text{ and } \mathbb{1}^T y = r \\ +\infty & \text{otherwise} \end{cases}}$$

3) Piecewise-linear function on $\mathbb{R}$

$f(x) = \max_{i \in [\![1, m]\!]} (a_i x + b_i)$ with $a_1 \leq \dots \leq a_m$ and no redundant function.

$$f^*(y) = \sup_{x \in \mathbb{R}} (y^T x - f(x)) = \sup_{x \in \mathbb{R}} (y^T x - \max_{i \in [\![1, n]\!]} (a_i x + b_i))$$

Let's first look for all the y such that $y \mapsto y^T x - f(x)$ is bounded.

- Consider that $y < a_1$. If we take $x = -t\mathbb{1}$, we have:

$$\cancel{y^T x - f(x)}$$

$$\begin{aligned} y^T x - f(x) &< -a_1 t - f(t) \\ &= -a_1 t + a_j t - b_j \\ &= \underbrace{t(a_1 + a_j)}_{\geq 0} - b_j \xrightarrow[t \to +\infty]{} +\infty \quad \text{unbounded.} \end{aligned}$$

With the same idea we find that $y \in [a_1, a_m]$

- Consider $y \in [a_1, a_m]$. Then, $\exists i \in [\![1, m]\!]$ such that $a_i \leq y \leq a_{i+1}$.

~~Therefore, we have~~ $\cancel{y - a_1, \dots, y - a_i \geq 0}$ ~~and~~ $\cancel{a_{i+1} - y, \dots, a_m}$

Therefore, we have $y - a_1, \dots, y - a_i \geq 0$ and $y - a_{i+1}, \dots, y - a_m \leq 0$

We have a breakpoint between the segments i and $i+1$.

$$\begin{aligned} f^*(y) &= y x_i - a_i x_i - b_i \\ &= y \frac{b_i - b_{i+1}}{a_{i+1} - a_i} - a_i \frac{b_i - b_{i+1}}{a_{i+1} - a_i} - b_i \end{aligned}$$

$$f^*(y) = (y - a_i)\left(\frac{b_i - b_{i+1}}{a_{i+1} - a_i}\right) - b_i$$

Therefore

$$\boxed{f^*(y) = \begin{cases} -b_i + (y - a_i)\left(\frac{b_i - b_{i+1}}{a_{i+1} - a_i}\right) & \text{if } y \in [a_i, a_{i+1}] \subset [a_1, a_m] \\ +\infty & \text{otherwise} \end{cases}}$$