EECE562 Assignment 2

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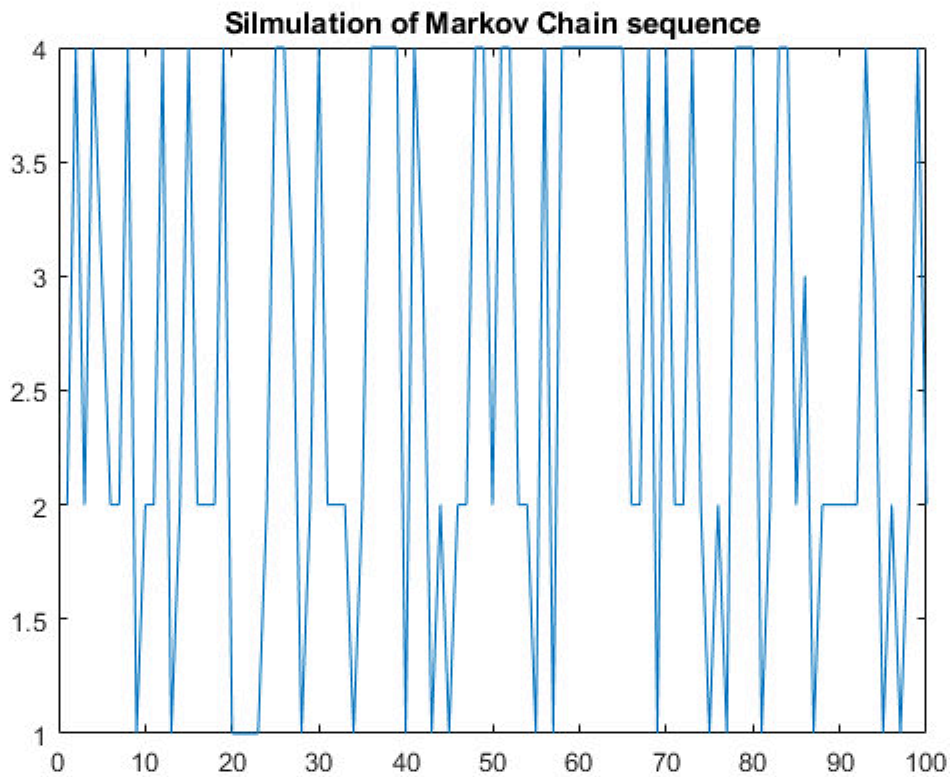
q1

```
P=[0.2 0.3 0.1 0.4;
   0.15 0.35 0.05 0.45;
   0.5 0.25 0.12 0.13;
   0.23 0.27 0.13 0.37];  %Aperiodic adn irreducible
q=1/4;
c=zeros(4,1);
for m=1:4
    c(m)=max(P(m,:))/q;
end
Y=1; % assume it starts from state 1
N=10000;
realization=zeros(1,N);
for n=1:N
    U1=rand();
    X=floor(4*U1)+1;
    U=rand();
    while U>=P(Y,X)/(c(Y)*q)
        U1=rand();
        X=floor(4*U1)+1;
        U=rand();
    end
    Y=X;
    realization(n)=Y;
end
plot(1:N,realization);
xlim([0 100]) % for better visualization, I only show part of
 realization here
title("Silmulation of Markov Chain sequence");
```



q2

```
%%%%%%%% Estimating the transition probabilities %%%%%%%
np=zeros(4,4);
for n=1:N-1
    switch realization(n)
    % Find state = one of the cases
        case 1
            switch realization(n+1)
                % Find next state = one of the cases
                case 1
                    np(1,1)= np(1,1)+1;
                case 2
                    np(1,2)= np(1,2)+1;
                case 3
                    np(1,3)= np(1,3)+1;
                case 4
                    np(1,4)= np(1,4)+1;
            end
        case 2
            switch realization(n+1)
                case 1
                    np(2,1)= np(2,1)+1;
                case 2
                    np(2,2)= np(2,2)+1;
```

```
                case 3
                    np(2,3)= np(2,3)+1;
                case 4
                    np(2,4)= np(2,4)+1;
            end
        case 3
            switch realization(n+1)
                case 1
                    np(3,1)= np(3,1)+1;
                case 2
                    np(3,2)= np(3,2)+1;
                case 3
                    np(3,3)= np(3,3)+1;
                case 4
                    np(3,4)= np(3,4)+1;
            end
        case 4
            switch realization(n+1)
                case 1
                    np(4,1)= np(4,1)+1;
                case 2
                    np(4,2)= np(4,2)+1;
                case 3
                    np(4,3)= np(4,3)+1;
                case 4
                    np(4,4)= np(4,4)+1;
            end
    end
end
estimated_p=zeros(4,4);
% Normalize estimated_p
for m=1:4
    for n=1:4
        estimated_p(m,n)=np(m,n)/sum(np(m,:));
    end
end

%%%%%%%% Estimating the stationary distribution of the Markov Chain %%
%%%%%%
[V,D]=eig(estimated_p.');
distribution_est=zeros(4,1);
for m=1:4
     distribution_est(m)=V(m,1)/sum(V(:,1));
end
distribution_est

%%%%%%%% Estimating the stationary distribution from the transition
 probability matrix %%%%%%%%
[V,D]=eig(P.');
distribution_P=zeros(4,1);
for m=1:4
    distribution_P(m)=V(m,1)/sum(V(:,1));
end
```

```
% distribution_est is close to distribution_P, which proved that my
 realization and estimator is correct
distribution_P


distribution_est =

    0.2237
    0.3028
    0.1002
    0.3732


distribution_P =

    0.2259
    0.2987
    0.0983
    0.3771
```

q3

```
mu=[0.3 0.2 0.1 0.35 0.05];
%reconstruct
mu1=sort(mu);
N=5;
P=zeros(N,N);
muk=zeros(N,N);
muk(1,:)=mu1;
beta=zeros(N,1);
alpha=ones(N,1);
% iteratively compute beta, alpha and mu
for k=1:N-1
    beta(k+1)=1-muk(k,k);
    alpha(k+1)=1-muk(k,k)/beta(k+1);
    muk(k+1,:)=muk(k,:)/beta(k+1);
end
% compute P matrix
for m=1:N
    for n=1:N
        if m==n
            P(m,n)=0;
        end
        if m>n
            P(m,n)=prod(alpha(1:n))*(1-alpha(n+1));
        end
        if m<n
            P(m,n)=prod(alpha(1:m))*muk(m,n)/beta(n);
        end
    end
end
```

```
% Estimating the stationary distribution from the transition
 probability matrix
[V,D]=eig(P.');
distribution_P=zeros(5,1);
for m=1:5
    distribution_P(m)=V(m,1)/sum(V(:,1));
end
% Results should be close to the given mu=[0.3 0.2 0.1 0.35 0.05]
sort(mu)
distribution_P'


ans =

    0.0500    0.1000    0.2000    0.3000    0.3500


ans =

    0.0476    0.0954    0.1925    0.2967    0.3678
```

q4

Problem 4 is written by hand and attached after problem 5.

q5

(a)

```
p=[1/3 1/3 1/3];
N=10000;
x1=zeros(N,1);
for n=1:N
    i_star=randi(3);
    U1=rand();
    switch i_star
        case 1
            x1(n)=U1;
        case 2
            x1(n)=power(U1,1/3);
        case 3
            x1(n)=power(U1,1/5);
    end
end
figure
hist(x1,50)
title("Histogram of the empirical distribution function");
% compare with the actual distribution
figure
x_actual=linspace(0,1);
plot(x_actual,(1+3*x_actual.^2+5*x_actual.^4)/3)
title("Histogram of the actual distribution function");
```

```
% (b)
% Assume n=4, alpha=[1/4, 1/4, 1/4, 1/4];
p=[1/4, 1/4, 1/4, 1/4];
N=10000;
x2=zeros(N,1);
for n=1:N
    i_star=randi(4);
    U2=rand();
    switch i_star
        case 1
            x2(n)=U2;
        case 2
            x2(n)=power(U2,1/2);
        case 3
            x2(n)=power(U2,1/3);
        case 4
            x2(n)=power(U2,1/4);
    end
end
figure
hist(x2,50)
title("Histogram of the empirical distribution function");
% compare with the actual distribution
figure
x_actual=linspace(0,1);
plot(x_actual,(1+2*x_actual+3*x_actual.^2+4*x_actual.^3)/4)
title("Histogram of the actual distribution function");
```

q6

Problem 6 is attached at the end.

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Q4: (1) For $f(x) = 2e^{-2x}$ Exponential distribution $x \in [0, +\infty)$

$$F(x) = \int_0^x 2e^{-2x_0} dx_0 = -e^{-2x_0}\Big|_0^x = 1 - e^{-2x}$$

$$u = F(x_1) = 1 - e^{-2x_1}$$

$$e^{-2x_1} = 1 - u$$

$$x_1 = -\frac{1}{2} \cdot \log(1-u)$$

(2) ∴ Generate rv $u \sim U[0,1]$, the $x_1 = -\frac{1}{2}\log(1-u)$ has density as $f(x)$.

For $f(x_2) = 3e^{-3x_2}$

$$x_2 = \frac{2}{3} x_1 = -\frac{1}{3}\log(1-u)$$

(3). Assume $X_1 \sim \theta e^{-\theta x}$

for $(\theta + \Delta\theta) e^{-(\theta+\Delta\theta)x}$, $x_2 = \frac{\theta}{\theta + \Delta\theta} x_1$

$$\frac{dx}{d\theta} = \frac{x_2 - x_1}{\Delta\theta} = \frac{-\Delta\theta\, x_1}{(\theta + \Delta\theta)\cdot\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{-x_1}{\theta + \Delta\theta} = \frac{-x_1}{\theta}$$

Q6. In (a): $\begin{bmatrix} (P-I)' \\ 1' \end{bmatrix} \pi = \begin{bmatrix} 0_x \\ 1 \end{bmatrix}$, $\pi \geq 0$

∴ $M = \begin{bmatrix} P-I' \\ 1' \end{bmatrix}$, $x = \pi$, $b = \begin{bmatrix} 0_x \\ 1 \end{bmatrix}$

Q6: Given (a) $\begin{bmatrix} (P-I)' \\ 1' \end{bmatrix} \pi = \begin{bmatrix} 0_x \\ 1 \end{bmatrix}$, $\pi > 0$

where $M = \begin{bmatrix} (P-I)' \\ 1' \end{bmatrix}$, $x = \pi$, $b = \begin{bmatrix} 0_x \\ 1 \end{bmatrix}$

Plug. into (b): we want to show there is no solution that

$$[P-I \ , \ 1] \, y \geq 0 \quad \text{and} \quad [0_x \ \ 1] \, y < 0$$

Assume $y = \begin{bmatrix} y_a : N\times1 \\ y_b : 1\times1 \end{bmatrix}$

For P we know: $\sum_{i=1}^{n} P_{i,j} = 1$, $j = 1, 2, \cdots n$

$$[P-I] \cdot y_a + y_b \geq 0 \quad \text{with} \quad y_b < 0$$

$\therefore$ we need $[P-I] \cdot y_a > 0$ for every row

$$P_1 y_{a1} > y_{a1}$$
$$P_2 y_{a2} > y_{a2}$$
$$\vdots$$
$$P_n y_{an} > y_{an}$$

$\Rightarrow [P_1 + P_2 + \cdots P_n] \cdot y_a > 1' \cdot y_a$

$\Rightarrow$ $P_1 + P_2 + \cdots P_n > 1'$ for all components

Since P is transition matrix, in k row, $P_{1k} + P_{2k} + \cdots P_{nk}$ means the sum of probabilities that next state is k.