EECE 562 HW1

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Q1. (1) $X_0 = 2X + Y$

Since X, Y are independent Gaussian,

$X \sim N(0,1) \quad Y \sim N(0,1)$

$\therefore X_0 \sim N(2\times 0 + 0,\ 2^2 \cdot 1^2 + 1) = N(0, 5)$

$\therefore$ pdf of $X_0 = \boxed{\frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{5}}\right)^2}}$

(2) $X_1 = \min(X, Y)$

$f_{X_1}(x) = P(X = x, Y > X) + P(Y = x, X > Y)$

$= f_X(x)[1 - F_Y(x)] + f_Y(x)[1 - F_X(x)]$

$\because X, Y$ are independent Gaussian distribution

$\therefore f_X(x) = f_Y(x) = f(x), \quad F_Y(x) = F_X(x) = F(x)$ where $f(x), F(x)$ are pdf, cdf of $N(0,1)$

$\therefore f_{X_1}(x) = \boxed{2 \cdot f(x) \cdot [1 - F(x)]}$

$= 2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2} \cdot \left[\frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$

(3) $X_2 = \max(X, Y)$

$f_{X_2}(x) = P(X = x, Y < X) + P(Y = x, X > Y)$

$\because X, Y \sim N(0,1)$

$\therefore f_{X_2}(x) = f_X(x) \cdot F_Y(x) + f_Y(x) \cdot F_X(x)$

$\boxed{= 2f(x) \cdot F(x)}$ where $f(x), F(x)$ are pdf, cdf of $N(0,1)$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot [1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$

Q2. (1) $X = -\ln(1-U)$

$$F_X(x_0) = Pr(X < x_0) = Pr(-\ln(1-U) < x_0) = Pr(1-u > e^{-x_0})$$
$$= Pr(u < 1 - e^{-x_0})$$
$$= F_u(1 - e^{x_0})$$

$\because$ U is an uniformal distribution

$\therefore$ $F_u = \begin{cases} 0 & u < 0 \\ u & u \in [0,1] \\ 1 & u > 1 \end{cases}$ $\quad \therefore$ $\boxed{F_X(X) = \begin{cases} 0 & X > 0 \\ 1 - e^x & X \leq 0 \end{cases}}$

(2). $Y = U^k$ for $k \geq 1$

$Y = g(u) = u^k$, Lets $u_1, u_2 \cdots u_k$ be the roots:

i.e. $Y = u_1^k = u_2^k = \cdots = u_k^k$

With formula given in lecture:

$$f_Y(y) = \frac{f_u(u_1)}{|g'(u_1)|} + \frac{f_u(u_2)}{|g'(u_2)|} \cdots$$

$\because$ $f_u(u) = \begin{cases} 1 & u \in [0,1] \\ 0 & \text{otherwise} \end{cases}$ $\quad |g'(u)| = |k \cdot u^{k-1}| = k \cdot u^{k-1}$

if k is odd: There is one root $u_1 = \sqrt[k]{Y}$

$$f_Y(y) = \begin{cases} \frac{1}{k \cdot (\sqrt[k]{y})^{k-1}} = \frac{1}{k \cdot y^{\frac{k-1}{k}}} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

if k is even: There are two roots $u_1 = -\sqrt[k]{Y}$ $\quad u_2 = \sqrt[k]{Y}$

$$\therefore f_Y(y) = \begin{cases} \frac{0}{-k y^{\frac{k-1}{k}}} + \frac{1}{k \cdot y^{\frac{k-1}{k}}} = \frac{1}{k \cdot y^{\frac{k-1}{k}}} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Combine odd and even situations:

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{k \cdot y^{\frac{k-1}{k}}} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

Q3. $f(n) = \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-n+2}{m} \times \frac{n-1}{m}$

$= \frac{n-1}{m^{n-1}} \cdot (m-1)(m-2) \cdots (m-n+2) = \boxed{\frac{(n-1)\cdot(m-1)!}{m^{n-1}\cdot(m-n+1)!}}$

Q4

$\text{mean}(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{20} x f(x)\,dx + \int_{20}^{+\infty} x f(x)\,dx$

$\Rightarrow \quad 10 = \int_0^{20} x_1 f(x_1)\,dx_1 + \int_{20}^{+\infty} x_2 f(x_2)\,dx_2$

$\xrightarrow[x_2 \text{ always } \geq 20]{x_1 \text{ always } \geq 0} \quad 10 \geq 0 \cdot \int_0^{20} f(x_1)\,dx_1 + 20 \cdot \int_{20}^{+\infty} f(x_2)\,dx_2$

$\Rightarrow \quad 10 \geq 20 \cdot \int_{20}^{+\infty} f(x)\,dx = 20 \cdot P(x>20)$

$\therefore P(x>20) \leq \frac{1}{2}$

Q5. $\because$ X, Y are jointly Guassian,

means $X = aU + bV$, $Y = cU + dV$ where U, V are independent Guassian varible

$\therefore$ pdf $f_U(u)$, $f_V(v)$ are Guassian

$\therefore$ Then linear combination X, Y of U, V are Guassian

$\therefore$ $f(x)$, $f(y)$ are guassian.

Q6. when $E\{X\} \le 1$: $E\{|X|^2\} \ge 0$ $\therefore E\{X\} \le E\{|X|^2\}+1$

when $E\{X\} > 1$: $\because E\{|X|^2\} = (E\{X\})^2 - Var(X)$

$Var(X) \ge 0$

$\therefore E\{X\} < (E\{X\})^2 \le E\{|X|^2\} < E\{|X|^2\}+1$

$\therefore$ Combine two situations: $E\{X\} \le E\{|X|^2\}+1$

Q7 If $X: \begin{cases} x = 5 \ 5 \ 5 \ 5 \\ Pr = \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \end{cases}$

$Y: \begin{cases} y = 0 \ 0 \ 0 \ 100 \\ Pr = \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \end{cases}$

$\therefore Pr(X>Y) = \frac{3}{4} > Pr(Y>X) = \frac{1}{4}$

$E\{X\} = 5 < E\{Y\} = \frac{100}{4} = 20$

$\therefore$ The statement is false.

Q8: $P(U<X) = \int_0^1 P(U<X_0 \mid X=X_0)\, dX_0$

$= \int_0^1 P_u(U<X_0) \cdot P_X(X=X_0)\, dX_0$

$= \int_0^1 F_u(X_0) \cdot f_X(X_0)\, dX_0 = \int_0^1 X_0 \cdot f_X(X_0)\, dX_0 = E\{X\}$

Q9. $P = \begin{bmatrix} 1 & b & 0 & 0 & \cdots & 0 \\ 1-b & 0 & b & & & \vdots \\ 0 & 1-b & 0 & & & \vdots \\ \vdots & 0 & 1-b & \ddots & & \\ & & & & & 1 \\ 0 & \cdots & & & & \end{bmatrix}$

$E\{N\} = E\{N_1\} + E\{N_2\}$ where $N_1: \{n \ge 0 : X_n = 1\}$

$N_2: \{n \ge 0 : X_n = M\}$

$$E\{N_1\} = \sum_{i=1}^{\infty} n_i \cdot P_i$$ where $n_i \to n_i$ steps arrive $X_n = 1$
$P_i \to$ Probability of each path

$$= \underbrace{1 \times b}_{i=1} + \underbrace{3 \times (1-b) \cdot b^2}_{i=2} + \underbrace{5 \times (1-b)^2 \times b^3 \cdot 2}_{i=3} + \underbrace{7 \times (1-b)^3 b^4 \cdot 3!}_{i=4} \cdots$$

$$= \sum_{i=1}^{\infty} (2i-1) \cdot (i-1)! \cdot (1-b)^{i-1} \cdot b^{i}$$

$$E\{N_2\} = \sum_{j=1}^{\infty} n_j \cdot P_j$$ where $n_j \to n_j$ steps arrive $X_n = M$

$$= \underbrace{(M-2) \cdot (1-b)^{M-2}}_{j=1} + \underbrace{M \cdot (1-b)^{M-1} b \cdot (M-3)}_{j=2} + \underbrace{(M+2) \cdot (M-3)^2 \cdot (1-b)^{M} \cdot b^2}_{j=3} \cdots$$

$$= \sum_{j=1}^{\infty} (M-4+2j)\, (M-3)^{j-1} \cdot (1-b)^{M-3+j} \cdot b^{j-1}$$

$$E\{N\} = \sum_{i=1}^{\infty} (2i-1) \cdot (i-1)! \cdot (1-b)^{i-1} \cdot b^{i} + \sum_{j=1}^{\infty} (M-4+2j) \cdot (M-3)^{j-1} \cdot (1-b)^{M-3+j} \cdot b^{j-1}$$