EECE 562 HW 4 $\boxed{\text{Ziqi Su}}$

P1 $x_{k+1} = x_k + v_k$

$y_k = x_k + w_k$

At iteration k: update $\hat{x}_{k|k} \triangleq E\{x_k | y_0, y_1 \cdots y_k\}$

- Measurement update: and $\Sigma_{k|k} \triangleq \mathrm{cov}(\hat{x}_{k|k})$

We have $y_k | Y_{k-1} = x_k | Y_{k-1} + v_k | Y_{k-1} = x_k | Y_{k-1} + v_k$

So: $\begin{cases} x_{k|k-1} \sim N(\hat{x}_{k|k-1}, \Sigma_{k|k-1}) \\ y_{k|k-1} \triangleq y_k | Y_{k-1} \sim N(\hat{x}_{k|k-1}, \Sigma_{k|k-1} + \Sigma_v) \end{cases}$

Since $x_{k|k-1} | y_{k|k-1} = x_k | Y_k = \hat{x}_{k|k}$

use $P(x|y) = \dfrac{P(x,y)}{P(y)}$, we can calculate $x_{k|k}$ as:

$$\hat{x}_{k|k} = x_{k|k-1} + \Sigma_{k|k-1} \cdot (\Sigma_{k|k-1} + \Sigma_v)^{-1} (y_k - \hat{x}_{k|k-1})$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1} (\Sigma_{k|k-1} + \Sigma_k)^{-1} \cdot \Sigma_{k|k-1}$$

- Time update: use $\vec{x}_{k+1} = \vec{x}_k + \vec{v}_k$

$\bar{x}_{k+1} | \bar{Y}_k = \bar{x}_k + \bar{v}_k$

So. $\begin{cases} x_{k+1|k} = x_{k|k} \\ \Sigma_{k+1|k} = \Sigma_{k|k} + \Sigma_w \end{cases}$

P 2.

$x(k)$ — $\boxed{FIR}$ — $y(k) = x(k) + h_1 x(k-1) + w(k)$ where $w(k) \sim N(0, \sigma^2)$

Assume $X(k) = \{-1, 1\}$ $P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

we want use observation $\{y_{(0)}, y_{(1)} \cdots y_{(k)}\}$ to estimate $\{x(k)\}$

Define $t(k) = X(k) + h_1 \cdot X(k)$

$\therefore t(k) \in \{1+h_1, 1-h_1, -1+h_1, -1-h_1\}$

with $P_t = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ a_{11} & 0 & a_{12} & 0 \\ 0 & a_{21} & 0 & a_{22} \\ 0 & a_{21} & 0 & a_{22} \end{bmatrix}$

With new state variable t, we have HMM filter $\begin{cases} t_k \sim MC(P_t) \\ y_k = t_k + w_k \end{cases}$

Start at π_0,

at iteration k, we have $\pi_k(j) = P\{t_k = t_j \mid Y_{1:k}\}$

$$\pi_{k+1}(j) = \frac{P(y_{k+1} \mid k_{k+1} = k_j) \cdot \sum_{i=1}^{4} P_{ij}\, \pi_k(j)}{\sum_{i=1}^{4} \left(P(y_{k+1} \mid t_{k+1} = t_i) \sum_{i=1}^{4} P_{ij}\, \pi_k(j)\right)}$$

After found π_k, we can estimate $j^* = \arg\max_i \pi_k(j)$

$t_k^* = t_{j^*}$

P3: $f(y_i) = \sum_{j=1}^{M} \pi_j f_j(y_i|\theta)$ where $f_j(y_i|\theta) = f_i(y_i|\theta_i) = \begin{cases} \lambda_j e^{-\lambda_j y_i}, & y \geq 0 \\ 0 & o.w \end{cases}$

E step : Let $z_{ij} = \begin{cases} 1 & y_i \in f_j \\ 0 & y_i \notin f_j \end{cases}$ $\sum_{j=1}^{M} z_{ij} = 1$

$$Q(\theta|\theta^{(t)}) = E\{\sum_{i=1}^{n}\sum_{j=1}^{k} z_{ij}[\ln \pi_j + \ln f_j(y_i|\lambda_j)] \mid Y=y, \pi^{(t)}, \lambda_1^{(t)} \cdots \lambda_k^{(t)}\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} w_{ij}^{(t)} \ln \pi_j + \sum_{i=1}^{n}\sum_{j=1}^{k} w_{ij}^{(t)} [\ln \lambda_j - \lambda_i y_i]$$

where $w_{ij}^{(t)} = \dfrac{\pi_j^{(t)} f_j(y_i|\lambda_j^{(t)})}{\sum_{j'=1}^{k} \pi_{j'}^{(t)} f_j(y_i|\lambda_{j'}^{(t)})}$

M step : $\dfrac{\partial Q}{\partial \theta} = 0$

$$\frac{\partial Q}{\partial \pi_j} = \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{w_{ij}^{(t)}}{\pi_j}, \quad s.t. \sum \pi_j = 1 \Rightarrow \pi_j^{(t+1)} = \frac{\sum_{i=1}^{n} w_{ij}^{(t)}}{n}$$

$$\frac{\partial Q}{\partial \lambda_j} = \sum_{i=1}^{n}\sum_{j=1}^{k} w_{ij}\left[\frac{1}{\lambda_j} - y_i\right] = 0$$

$$\lambda_j^{(t+1)} = \frac{\sum_{i=1}^{n} w_{ij}^{(t)}}{\sum_{i=1}^{n} w_{ij}^{(t)} \cdot y_i}$$

```
%% q2
N=1000;
x=zeros(N,1);
y=zeros(N,1);
P=[0.8, 0.2;
    0.2, 0.8];
Pt=[0.8 0 0.2 0;
    0.8 0 0.2 0;
    0 0.2 0 0.2;
    0 0.2 0 0.2];
h1=1;
for i=1:N
    x(i)=-1+2*round(rand);
    if i~=1
        y(i)=x(i)+h1*x(i-1)+randn(1);
    end
end



%% q3
N=1000;
ite=100000;
M=3;
pi=[0.3,0.5,0.2];
lamda=[1,2,5];
% Observation:
y=zeros(N,1);
for n=1:N
    randpi=rand();
    if randpi<=0.2
        y(n)=exprnd(1);
    elseif randpi<=0.7
            y(n)=exprnd(1/2);
    else
        y(n)=exprnd(1/5);
    end
end
%EM algorithm
pi_m=[0.25,0.55,0.2];
lamda_m=[1.2, 2.5, 4.6];
w=zeros(N,M);
for t=1:ite
    for i=1:N
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```
        %E step:
        w_denom=0;
        for jp=1:M

w_denom=w_denom+pi_m(jp)*exppdf(y(i),1/lamda_m(jp));
        end
        for j=1:M

w(i,j)=pi_m(j)*exppdf(y(i),1/lamda_m(j))/w_denom;
        end
    end
    %M step:
    w_sum=zeros(M,1);
    wy_sum=zeros(M,1);
    for j=1:M
        for i=1:N
            w_sum(j)=w_sum(j)+w(i,j);
            wy_sum(j)=wy_sum(j)+w(i,j)*y(i);
        end
        pi_m(j)=w_sum(j)/N;
        lamda_m(j)=w_sum(j)/wy_sum(j);
    end
end
pi_m
lamda_m
```

Command Window

```
>> hw4

pi_m =

    0.2150    0.5041    0.2810


lamda_m =

    1.0186    2.0333    5.0003
```