Complete derivations and formulation

The objective of finite element formulation is to derive an integral representation of the physical phenomena within an element. The fundamental assumption of FEA is that it considers a continuous mechanics for its system. As the universe is known to be unfathomably complex, the problem-solving methodology of engineers is to model variables as approximations and create assumptions to simply the problems and to derive solutions. In the 21st century, most of our solutions are a combination of analytical, numerical, and computational, and we will take the same approach with this FEA problem.

Bar element formulation

The governing equations of how stress, strain, and bending relate to each other are partial differential equations, derived from fundamental laws of physics and constitutive equations. Most of the equations can are further derivations based on constitutive laws and Newton's and Euler's Laws of motion. External forces that act on a system must be balanced by internal reaction forces. Involved in the study of mechanical systems in a frame problem are the following:

$F = ma$, Newton's second law, and potential energy $U = \int F\, dx$

$\sigma = \frac{F}{A}$ and $\varepsilon = \frac{\partial}{\partial X}(x - X)$, $\sigma = E\varepsilon$, $F_s = k(x - X)$, $\frac{d^2U}{dx^2} = k$ commonly known as Hooke's Law

These have originally been taught as the main method of analysis in rigid body mechanics, but now, they must be considered in the differential sense to explain how individual elements accelerate within the object due to the sum of internal forces experienced within the object, as compared to how the overall object moves in response to weight. For example, in the infinitesimal sense, $d\sigma = \frac{dF}{dA}$.This is especially important in the safety analysis of beams and trusses, where excessive deformation of a foundational element past its yield strength can cause catastrophic failure. Considering an infinitesimal element of an object, we take the derivation of total body force F_B over volume V, with mass m, mass density ρ,

$$F_B = \int_V b\, dV$$

where the average body force per unit volume acting on the element be b and the average acceleration and density of the element be a and ρ,

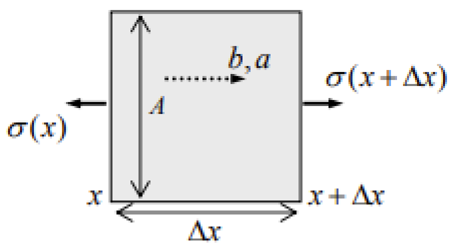


Fig 1.

In a 1D simplification, we can observe that the net surface force acting is $\sigma(x + \Delta x)A - \sigma(x)A + b\Delta x A = \rho a \Delta x A$, And since $a = \frac{d^2x}{dt^2}$, taking the limit as $\Delta x \to 0$ to obtain the differential equation,

$$\frac{d\sigma}{dx} + b = \rho a$$

Let us now define u as the displacement in the x direction. Then now $\varepsilon = \frac{du}{dx}$ and recalling that $\sigma = E\varepsilon$, we can now write $\frac{d\sigma}{dx}$ as $E\frac{d^2u}{dx^2}$. With assumptions such as no body force present, thankfully the PDE can be simplified into several forms. One of which is $EA\frac{d^2u}{dx^2} = -f_B$ in its cross-sectional area form, which can be formulated with the Galerkin method, for example, or the principle of virtual work.

Knowing that Energy is the integral of force over the displacement, what we will do in the following is to derive the strain energy formulation in conjunction with shape functions and arrive at a set of matrix equations for each element by utilizing the principle of minimum potential energy.

From the partial differential equation, we will first assume the body force is 0 to obtain

$$\frac{d\sigma}{dx} = \rho a$$

This equation is in the units of force per unit volume, or force density. Integrating both by sides by du, to obtain energy density,

$$\int \frac{d\sigma}{dx}\, du = \int \frac{dF}{dV}\, du = \frac{dU}{dV}$$

$$\int \frac{d\sigma}{dx}\, du = \int \frac{dF}{dV}\, du = \frac{dU}{dV}$$

Since $\varepsilon = \frac{du}{dx}$ implies $du = \varepsilon dx$,

$$\frac{dU}{dV} = \int \varepsilon d\sigma$$

$$\frac{d\sigma}{d\varepsilon} = E, d\sigma = E d\varepsilon$$

$$\frac{dU}{dV} = \int E\varepsilon d\varepsilon$$

$$\frac{dU}{dV} = \frac{E\varepsilon^2}{2}$$

$$\frac{dU}{dV} = \frac{\sigma\varepsilon}{2}$$

And consequently, the strain energy equation in the element,

$$U = \int \frac{\sigma\varepsilon}{2}\, dV$$

We can prove this in the generalized Hooke's Law equation as well.

$u = \frac{FL}{EA}$ by algebraic manipulation of the Hooke's Law. So $U = \int F\, du\ = \int ku\, du = \frac{1}{2}ku^2 = \frac{1}{2}Fu = \frac{1}{2}\frac{F_s^2 L}{EA}$

In the derivative sense, $U = \int_0^L \frac{1}{2}\frac{F_s^2}{EA}\, dx\ =\ \int_V \frac{1}{2}\frac{F_s^2}{EA^2}\, dV =\ \int_V \frac{1}{2}\frac{\sigma F_s}{EA}\, dV$

But $\sigma = E\varepsilon = \frac{F_s}{A}$, so $F_s = EA\varepsilon$ and $\int_V \frac{1}{2}\frac{\sigma F_s}{EA}\, dV =\ \int_V \frac{1}{2}\frac{\sigma EA\varepsilon}{EA}\, dV$

$$= \int_V \frac{\sigma\varepsilon}{2}\, dV$$

To use the strain energy equation with shape functions, we must go back to the precursor

$$\frac{dU}{dV} = \frac{E\varepsilon^2}{2}$$

Since $V =\ \int_0^L A\, dx$,

$$U = \int_0^L \frac{EA}{2}\varepsilon^2 dx$$

Approximating u as a linear shape function for the axial displacement in the x direction, in Global X coordinate,

$$u(X) = a_0 + a_1 X = \frac{(X_2 - X)}{(X_2 - X_1)}\, u_1 + \frac{(X - X_1)}{(X_2 - X_1)}\, u_2$$

where $u_1 = a_0 + a_1 X_1$ and $u_2 =\ a_0 + a_1 X_2$, after solving for $a_0\ and\ a_1$

Thus, shape function for Node 1 and Node 2 are

$$N_1(X) = \frac{(X_2 - X)}{(X_2 - X_1)}$$

$$N_2(X) = \frac{(X - X_1)}{(X_2 - X_1)}$$

Applying the intrinsic coordinate system with $\xi = \frac{(2X)}{(X_2 - X_1)} - \frac{(X_2 + X_1)}{(X_2 - X_1)}$

$$u(\xi) = \frac{1}{2}(X_2 + X_1) + \frac{1}{2}(X_2 - X_1)\xi$$

$$N_1(\xi) = \frac{1 - \xi}{2}$$

$$N_2(\xi) = \frac{1 + \xi}{2}$$

In local coordinates where $x\ = X - X_1, L = X_2 - X_1$, we have
$N_1(x) = 1 - \frac{x}{L}$, $N_2(x) = \frac{x}{L}$

The final displacement along with its shape function is written in intrinsic coordinate ξ as such

$$u = N(\xi)u(\xi)$$

And displayed in matrix as

$$u = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

So $\varepsilon = \frac{du}{dx} = \begin{Bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{Bmatrix}\{u_1 \quad u_2\}^T$

Differentiating the shape functions, $\frac{dN_1}{dx} = -\frac{1}{L}, \frac{dN_2}{dx} = \frac{1}{L}$ thus $\varepsilon = [B]\,\{u_e\}$ where $[B] = [-\frac{1}{L} \quad \frac{1}{L}]$ and $\{u_e\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

ε^2 can thus be written as $\{u_e\}^T[B]^T[B]\,\{u_e\}$ and we find that $[B]^T[B] = \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}$

And substituting it back into the strain energy formula,

$$U = \int_0^L \frac{EA}{2}\{u_e\}^T[B]^T[B]\,\{u_e\}dx$$

$$U = \int_0^L \frac{EA}{2}\{u_! \quad u_2\}\begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}dx$$

$$= \frac{1}{2}\{u_! \quad u_2\}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}\int_0^L \frac{EA}{L^2}dx$$

And defining element stiffness $k = \int_0^L \frac{EA}{L^2}dx$, $U = \frac{1}{2}\{u_! \quad u_2\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

External forces can be represented as potential energy V from the external force R integrated over displacement, multiplied by -1 due to it being seen from the perspective of the element. Assuming this external force R is not a function of u, we can write $V_1 = -R_1u_1, V_2 = -R_2u_2$

Then $V_{NF} = -\{R_1 \quad R_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

External traction $T(x)$ is a distributed force with units N/m over the element length and may be a function of x. Thus $dF = T(x)dx$ and $dV_T = -dF\,u(x) = (-T(x)dx\,u(x) = -\int_0^L T(x)\,u(x)dx$

$$= -\int_0^L T(x)[N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}dx$$

$$= -\int_0^L T(x)[N_1 \quad N_2]dx\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= -\{Q_1 \quad Q_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This can be approximated with gauss quadrature sum $\int_0^L f(x)dx \approx \frac{L}{2}\sum_{j=1}^{N} w_j f(\xi\ )$

The principle of Minimum Potential Energy states that the solution of the body is the one that minimizes the total potential energy of the body, and it is at a stable equilibrium state. In other words, for total potential energy in the body

$$\Pi = U + V_T + V_{NF},\, \partial\Pi = 0, \frac{\partial\Pi}{\partial u_1} = \frac{\partial\Pi}{\partial u_2} = 0$$

For u_1, $\frac{\partial\Pi}{\partial u_1} = \frac{\partial}{\partial u_1}\left(\frac{1}{2}\{u_! \quad u_2\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}\right) + \frac{\partial}{\partial u_1}\left(-\{R_1 \quad R_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}\right) + \frac{\partial}{\partial u_1}(-\{Q_1 \quad Q_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix})$

By Leibniz rule of calculus, $\frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$, we arrive at

$$\frac{\partial\Pi}{\partial u_1} = \left(\frac{1}{2}\{1 \quad 0\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}\right) + \left(\frac{1}{2}\{u_1 \quad u_2\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\right) + \left(-\{R_1 \quad R_2\}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\right) + (-\{Q_1 \quad Q_2\}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix})$$

Since $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ is symmetric, $\{1 \quad 0\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{2}\{u_1 \quad u_2\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

$$and\ \left(\frac{1}{2}\{1 \quad 0\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}\right) + \left(\frac{1}{2}\{u_1 \quad u_2\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\right) = [1 \quad 0][K_e]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Thus, $\frac{\partial\Pi}{\partial u_1} = \{1 \quad 0\}[K_e]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - R_1 - Q_1$

Similarly for the second equation,

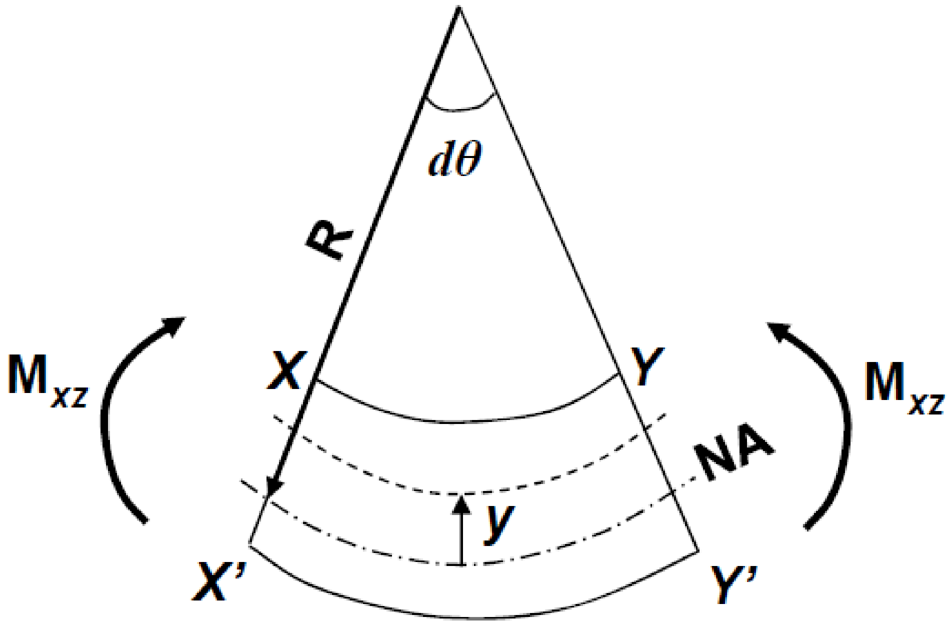
$$\frac{\partial\Pi}{\partial u_2} = \{0 \quad 1\}[K_e]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - R_2 - Q_2$$

Combining, $\frac{\partial\Pi}{\partial u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}[K_e]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} - \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ which we use as the 1D Bar equation

Beam element formulation

So far, we have looked at the formulation of a 1D Bar element. Let us now look at the case of a beam along the x direction. The Euler-Bernoulli beam theory is a simplification of the linear theory of elasticity and allows us to calculate the load-carrying and deflection characteristics in beams.

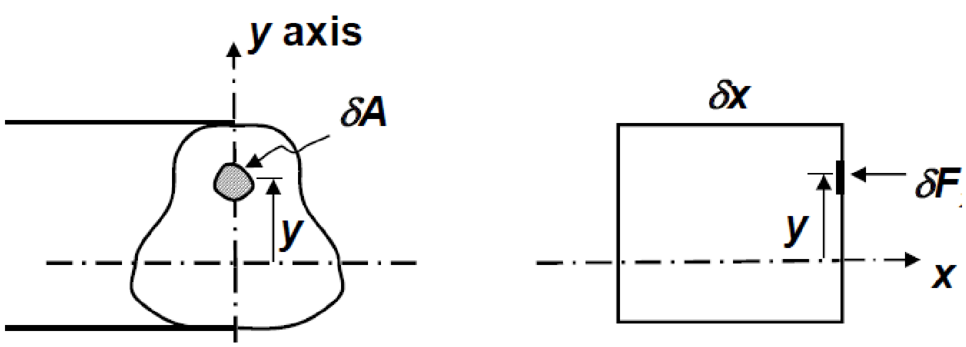
Consider and element ∂x of a bent beam that has been cut into an arc. We can define R as the radius of curvature of the neutral axis NA, and y as the vertical co-ordinate perpendicular to the neutral axis.



The length of the neutral axis is $Rd\theta$. The length of an axis at a y units above the Neutral Axis is thus $(R-y)d\theta$. Thus, the strain ε of this axis is $\frac{R-(R-y)}{Rd\theta} = \frac{y}{R}$

If we assume that the normal plane remains normal to axis after deformation, bending only occurs in the plane of the bending moment, and the plane cross-sections remain plane after bending, and there is no resultant force in the axial direction, and Young's Modulus is the same in tension and compression, we can derive certain formulas to simplify our problem.

Taking an element of area ∂A within the beam,



We observe that the stress σ acts over ∂A, so $\partial F = \sigma \partial A$. Recall that $\sigma = E\varepsilon$

Substituting equations above and letting $\partial \rightarrow d$:

$$dF = -\frac{E}{R} y dA$$

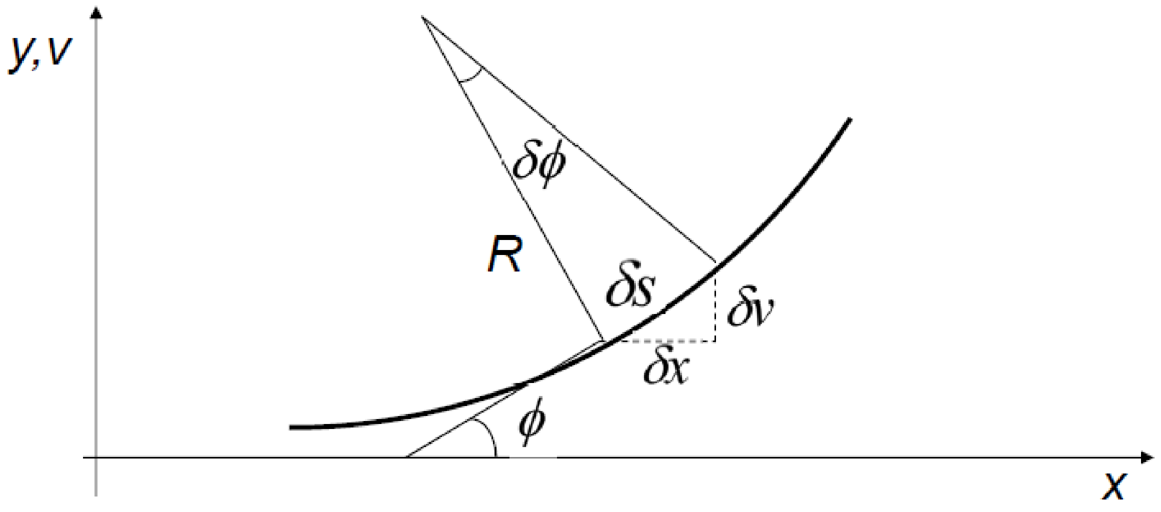
Since there is no resultant axial force, the sum of stresses across the cross-section must be zero

$$\int_A dF = -\frac{E}{R} \int_A y \, dA = 0$$

This generates a bending moment of $\partial M = y \times \partial F = \frac{E}{R} y^2 \partial A$

Integrating, $\partial M = \frac{E}{R} \int_A y^2 \, dA$, where $\int_A y^2 \, dA = I$

Let us consider an arbitrary curve in the geometrical sense on the y-x axis.



The radius of curvature is R and the curved element length is given by $\partial s = R \partial \phi$

Curvature $\kappa = \frac{1}{R} = \frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}}$

For ∂s sufficiently small enough, we can approximate it by Pythagoras theorem

$$\partial s = [(\partial x)^2 + (\partial v)^2]^{\frac{1}{2}}$$

Take $\partial \to d$, squaring both sides,

$$(ds)^2 = (dx)^2 + (dv)^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dv}{dx}\right)^2$$

$$\frac{ds}{dx} = \left(1 + \left(\frac{dv}{dx}\right)^2\right)^{\frac{1}{2}}$$

Since $\tan(\phi) = \frac{dv}{dx}$, differentiating both sides,

$$\sec^2\left(\frac{\phi d\phi}{dx}\right) = \frac{d^2v}{dx^2}$$

Then $\frac{d\phi}{dx} = \frac{\frac{d^2v}{dx^2}}{1+\tan^2\phi} = \frac{\frac{d^2v}{dx^2}}{1+\left(\frac{dv}{dx}\right)^2}$

Combining the above equations, we get $\frac{1}{R} = \frac{\frac{d^2v}{dx^2}}{\left[1+\left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$

For our beam problem, let us define the radius of curvature ρ so $\frac{1}{\rho} = \frac{\frac{d^2v}{dx^2}}{\left[1+\left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$

If we assume that the displacement is small as in the case of a deformation in a beam used for truss, $\frac{dv}{dx}$ is small so $\left[1+\left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}} \approx 1$, and so $\frac{1}{\rho} \approx \frac{d^2v}{dx^2}$

The Strain ε_{xx} of a beam element thus can be approximated from $-\frac{y}{\rho}$=- $y\frac{d^2v}{dx^2}$

where we can define 2 variables per node, v, which is the y-axis displacement, and χ,the rotation in radians. The corresponding

Cubic shape functions will be implemented to maintain continuity in displacement and rotation across elements for the beam elements.

For the Cubic polynomial in terms of local coordinates *x* for *v*, giving four arbitrary constants.

$$v(x) = a_0 + a_1x + \mathrm{a_2x^2} + 3a_3\mathrm{x^2}$$

Rotation is given by

$$\chi(x) = \frac{dv}{dx} = \mathrm{a_1} + 2\mathrm{a_2x} + 3a_3\mathrm{x^2}$$

Solving for $\mathrm{a_1, a_2, a_3}$ *and* $\mathrm{a_4}$ with x = 0 and x = L

$$v_1 = v|_{x=0} = a0$$

$$\chi_1 = \chi|_{x=0} = a1$$

$$v_2 = v|_{x=L} = a_0 + a_1L + \mathrm{a_2L^2} + 3a_3\mathrm{L^2}$$

$$\chi_2 = \chi|_{x=L} = \mathrm{a_1} + 2\mathrm{a_2L} + 3a_3\mathrm{L^2}$$

Substituting $\mathrm{a_1, a_2, a_3}$ *and* $\mathrm{a_4}$ into $v(x)$ *and* $\chi(x)$ to get the Shape Functions of the nodes, with application of intrinsic coordinate $\xi = \frac{2x}{L} - 1$ *or* $x = \frac{L(\xi+1)}{2}$

$$N_1(x) = 1 - \frac{\mathrm{3x^2}}{\mathrm{L^2}} + \frac{\mathrm{2x^3}}{\mathrm{L^3}} \text{ and } N_1(\xi) = \frac{1}{4}(2 - 3\xi + \xi^3)$$

$$N_2(x) = x - \frac{2x^2}{L^2} + \frac{x^3}{L^3} \; and \; N_2(\xi) = \frac{L}{8}(1 - \xi - \xi^2 + \xi^3)$$

$$N_3(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \; and \; N_3(\xi) = \frac{1}{4}(2 + 3\xi - \xi^3)$$

$$N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2} = \; and \; N_4(\xi) = \frac{L}{8}(-1 - \xi + \xi^2 + \xi^3)$$

For the second derivatives of the Shape Functions, we derive as such

$$N_{1xx}(\xi) = N_{1\xi\xi}(\xi)\left(\frac{d\xi}{dx}\right)^2 \; and \; N_{1\xi\xi}(\xi) = \frac{3}{2}\xi$$

$$N_{2xx}(\xi) = N_{2\xi\xi}(\xi)\left(\frac{d\xi}{dx}\right)^2 \; and \; N_{2\xi\xi}(\xi) = \frac{L}{4}(-1 + 3\xi)$$

$$N_{3xx}(\xi) = N_{3\xi\xi}(\xi)\left(\frac{d\xi}{dx}\right)^2 \; and \; N_{3\xi\xi}(\xi) = -\frac{3}{2}\xi$$

$$N_{4xx}(\xi) = N_{4\xi\xi}(\xi)\left(\frac{d\xi}{dx}\right)^2 \; and \; N_{4\xi\xi}(\xi) = \frac{L}{4}(1 + 3\xi)$$

$$where \; \frac{d\xi}{dx} = \frac{2}{L}$$

$$and \; v(\xi) = N(\xi)\{d_e\} \; where \; d_e = \begin{Bmatrix} v_1 \\ \chi_1 \\ v_2 \\ \chi_2 \end{Bmatrix}$$

$$-y\frac{d^2v}{dx^2} = -y\frac{d^2}{dx^2}\,N(\xi)\{v_e\} = -y\,N_{xx}(\xi)\{v_e\}$$

$$-y\,N_{xx}(\xi) \; can \; be \; simplified \; as \; [B] \; giving \; \varepsilon_{xx} = [B]\,\{v_e\}$$

Modifying the Strain Energy Equation gives

$$U = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E\varepsilon^2 b \; dy \; dx$$

$$= \frac{1}{2}\{v_e\}^T \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} E[B]^T[B] \; b \; dy \; dx \; \{v_e\}$$

$The \; K_e \; Matrix \; is \; derived \; as$

$$K_e = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} E[B]^T[B] \; b \; dy \; dx$$

$$= \left[\frac{L}{2}\int_{-1}^{1} E\left(\frac{d\xi}{dx}\right)^4 N_{i\xi\xi}(\xi)N_{j\xi\xi}(\xi) \; d\xi\right]\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 \; b \; dy\right] \; where \; \frac{d\xi}{dx} = \frac{2}{L}$$

$$= \left[\frac{8}{L^3}\int_{-1}^{1} EN_{i\xi\xi}(\xi)N_{j\xi\xi}(\xi)\ d\xi\right]\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\ b\ dy\right]$$

$$with\ the\ second\ moement\ of\ area\ I = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b\ dy$$

$The\ K_e\ Matrix\ is\ simplified\ as$

$$K_e = \frac{8}{L^3}\int_{-1}^{1} EIN_{i\xi\xi}(\xi)N_{j\xi\xi}\ d\xi$$

Assuming a Uniform Rectangular Bar, $EI = \frac{Ebh_0^3}{12}$, the integration gives

$$K_e = \frac{Ebh_0^3}{12L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} a\ symetric\ matrix$$

External distributed transverse Load can be converted to nodal shear and bending moment

$$V_q = -\int_0^L q(x)v(x)dx$$

$$= -\int_0^L q(x)\ [N\}dx\ \{v_e\} \approx -\frac{L}{2}\sum_{j=1}^{N} w_j\ N_j(\xi)q(\xi)\ \{v_e\}$$

$$= -\{Q_e\}^T\{v_e\}$$

External Concentrated forces and moments can be represented as

$$\{R_e\} = \begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix}$$

The Final FEA can be solved as $\{K_e\}\{v_e\} = \{Q_e\} + \{R_e\}$

The objective of the finite element formulation is to obtain an overall expression that relate the forces, nodal displacements, and material properties of the structure, represented as the overall equation below:

$$\{F\} = [K]\{u\}$$

Each element matrix K_e combines the stiffness matrix from bar and beam formulations. Since all elements are solid uniform rectangular bars with width b and height h, the stiffness matrices are:

$$K_{beam} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$K_{bar} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$where\ I = \frac{bh^3}{12}\ and\ A = bh$$

$$[K_e]_{xy} = \begin{bmatrix} K_{bar}^{11} & 0 & 0 & K_{bar}^{12} & 0 & 0 \\ 0 & K_{beam}^{11} & K_{beam}^{12} & 0 & K_{beam}^{13} & K_{beam}^{14} \\ 0 & K_{beam}^{21} & K_{beam}^{22} & 0 & K_{beam}^{23} & K_{beam}^{24} \\ K_{bar}^{21} & 0 & 0 & K_{bar}^{22} & 0 & 0 \\ 0 & K_{beam}^{31} & K_{beam}^{32} & 0 & K_{beam}^{33} & K_{beam}^{34} \\ 0 & K_{beam}^{41} & K_{beam}^{42} & 0 & K_{beam}^{43} & K_{beam}^{44} \end{bmatrix}$$

Since K_e is initially constructed using local coordinates, a transformation matrix T is required to represent K_e in global coordinates.

$$T_e = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} T_e & 0_{[3x3]} \\ 0_{[3x3]} & T_e \end{bmatrix}$$

where α is the angle between the local x-axis and the global x-axis.

$$[K_e]_{XY} = [T][K_e]_{xy}[T]^{-1}$$

The element stiffness matrix can be grouped into four blocks of 3x3, which is then assembled into the global matrix. i and j represent the two nodes that form the element while n is the number of nodes in the structure. This assembles each element's stiffness matrix

$$[K_e]_{XY} = \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix}$$

Into:

$$K = \begin{bmatrix} K_{11} & \cdots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \cdots & K_{nn} \end{bmatrix}$$

Considering 2D analysis, each node has 3 degree-of-freedom: X-axis translation (x), Y-axis translation (y) and Z-axis rotation (θ). The node displacement u_{node} is assembled into the global displacement matrix u. In vector form, we assemble each nodal vector

$$u_{node_i} = \begin{bmatrix} u_i \\ v_i \\ \theta_i \end{bmatrix}$$

Into:

$$u = \begin{bmatrix} u_{node_1} \\ \vdots \\ u_{node_n} \end{bmatrix}$$

The load acting on each node can also be separated into three components: X-axis force (F_x), Y-axis force (F_y) and Z-axis moment (M). The nodal forces F_e is also assembled into the global force matrix F.

In the same manner, we assemble

$$F_{node_i} = \begin{bmatrix} F_{X_i} \\ F_{Y_i} \\ M_i \end{bmatrix}$$

Into:

$$F = \begin{bmatrix} F_{node_1} \\ \vdots \\ F_{nodse_n} \end{bmatrix}$$

There are two types of loading conditions: nodal forces and traction forces along the element. Nodal forces can be added directly into the global force matrix, while forces acting on the nodes due to traction need to be calculated. Since the traction forces only arise from the elements' self-weight w, we only need to consider forces in the y direction. The formulation for the traction forces acting on the nodes is below:

Consider an element connected by nodes i and j.

Calculating self-weight:

$$w = (b \times h \times L) \times \rho \times g$$

where ρ is the density of the material and g is 9.81ms^{-2} and L is the length of the element.

Since the element is uniform, the load is shared equally between the two nodes.

$$F_{Y_i} = F_Y = {}^{w}\!/_{2}$$

The self-weight can be considered as a point load acting at the center of the element. The x-coordinate of the point is:

$$\bar{X} = \frac{X_i - X_j}{2}$$

Calculating moment at the nodes:

$$M_i = w \times (X_i - \bar{X})$$

$$M_j = w \times (X_j - \bar{X})$$

To determine the stresses in the element, we need to add the magnitudes of the axial and bending stresses. Formulation of the axial and bending stresses are below.

$$\sigma_{axial} = E \times \varepsilon = E \times \frac{\Delta L}{L}$$

$$where\ \Delta L = L_{deformed} - L$$

$$\sigma_{bending} = \frac{Mc}{I}, where\ M = EI\kappa$$

Since the cross-section is a solid rectangle, the expression can be simplified to:

$$\sigma_{bending} = \frac{Eh\kappa}{2}$$

Let the deformed coordinates of the nodes be:

$$\begin{bmatrix} P_i \\ Q_i \\ \vartheta_i \end{bmatrix} = \begin{bmatrix} X_i \\ Y_i \\ \alpha_i \end{bmatrix} + \begin{bmatrix} u_i \\ v_i \\ \theta_i \end{bmatrix} \text{ and } \begin{bmatrix} P_j \\ Q_j \\ \vartheta_j \end{bmatrix} = \begin{bmatrix} X_j \\ Y_j \\ \alpha_j \end{bmatrix} + \begin{bmatrix} u_j \\ v_j \\ \theta_j \end{bmatrix}$$

The deformed curve of the beam can be represented using a cubic equation and modelled using Ferguson's curve model, governed by the nodal coordinates and the tangents at the coordinates.

$$Y(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} Q_i \\ Q_j \\ \vartheta_i \\ \vartheta_j \end{bmatrix}, t \in [0,1]$$

$$X(t) = t \times L_{deformed} + P_i, t \in [0,1]$$

$$\kappa = \frac{{d^2Y}/{dX^2}}{\left(1 + \left({dY}/{dX}\right)^2\right)^{1.5}}$$

Hence, the total stress within an element is:

$$\sigma_{elem} = |\sigma_{axial}| + |max(\sigma_{bending})|$$