

The Analysis of a couple of Drinking Games

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1 Introduction

This report investigates the mathematics behind a few popular drinking games and showcases the various methods used for the investigations.

The analysis of each game involves modelling each one mathematically and extracting any useful information about them, such as:

- What is the optimal way to play the game?
- To what extent the game is fair?
- What interesting mathematical properties does the game have?

The motivation for this project is to create a fun initial research report done by the quant-finance team of the Warwick Finance Societies and increase WFS's involvement with the field of mathematics. The games investigated in this research report are 'Twenty-One' and 'Spoons'.

2 Preliminaries

2.1 Twenty-One

'Twenty-One', also known as 'Twenty-One Dares', involves a group of people arranged in a circle taking turns to collectively count up to a value, which is 21 in this case. The person who says "21" has to drink in the drinking game version of 'Twenty-One'.

The game has two possible objectives: either you aim to say "21" or avoid saying "21". The process of counting up to 21 depends on the ruleset.

This report will only consider rules that affect the counting process and the order of the turns the players take to count.

2.2 Spoons

'Spoons' involves people arranged in a circle, each with a set number of randomly dealt playing cards. Each replaces one of their cards with a card passed down by the person on their right (or left). Then, each passes their replaced card to the person on their left (or right).

The aim is to collect the maximal amount of cards of the same value.

This report will use a computer simulation of 'Spoons' to analyse it, as it has a statistically chaotic structure.

3 Twenty-One Game Analysis

3.1 Mathematical Model

The game requires n people arranged in a circle, where $n \in \mathbb{N}$. Each player is numbered 0 to $n - 1$. The first player who counts is player 0.

We model the game to be a sequence of states. We denote the game's state at period t by the vector $v_t \in (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{N}_0$, where $\mathbb{Z}/n\mathbb{Z}$ is the ring of integers modulo n . The first entry of v_t denotes the number of the player counting at period t , and the second is the number collectively counted up to in all periods before period t . At $t = 0$, the game state is $v_0 = (0, 0)$, since the game starts with player 0 counting and the count starts at 0.

Here, we generalise the counting rules as follows. At each state, the counting player can choose by how many numbers to count. Their choice is limited by the game's decision space $\{d_1, d_2, \dots, d_k\} = D \subseteq \mathbb{N}$ where each d_i is an amount by which they can count. We introduce the rule function $f : D \mapsto \mathbb{Z}/n\mathbb{Z}$, which determines how each decision from D determines the player who will count in the next turn.

Suppose the game state at period t is $v_t = (p_1, p_2)$, so player p_1 counts up after the number p_2 . If the player chooses to count by $x_t \in D$, where x_t is the decision made at period t , then the game state at period $t + 1$ will be $v_{t+1} = (p_1 + f(x_t), p_2 + x_t)$. In general:

$$v_{t+1} = \left(\sum_{i=0}^t f(x_i), \sum_{i=0}^t x_i \right) : \forall t \in \mathbb{N}_0$$

3.2 Specific Versions

There are two popular versions of 'Twenty-One'.

1. **Original 'Twenty-One'**: Each player can count up by either one, two or three. Regardless of the amount by which a player counts up, the player coming directly after them is the next player who counts. Thus, the decision space for each player is $D = \{1, 2, 3\}$, and the rule function f outputs 1 for any input element from D .
2. **Drinking game 'Twenty-One'**: Each player can count up by either one, two, three or four. Counting up by one passes the turn to the next player, and counting by two passes the turn to the previous player. Counting up by three or four would be the same as counting up by one or two, respectively, with the added twist of skipping a player. Therefore, the decision space is $D = \{1, 2, 3, 4\}$, and the rule function f has the following mapping:

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ -1 & \text{if } x = 2 \\ 2 & \text{if } x = 3 \\ -2 & \text{if } x = 4 \end{cases}$$

The game is lost (or won) by the player who counts up to 21 inclusive for both versions.

3.3 Game Theoretic Analysis Example

From here, we will assume that each player avoids counting up to some number N . This means all players are aiming to count up to $N - 1$, though it's not a necessary requirement to win, as analysis will demonstrate.

By the rules of the first version of 'Twenty-One', the main player (you) can count up by one, two or three, while your opponents can collectively count up by an amount between $n - 1$ and $3(n - 1)$ inclusive before it's your turn again.

We now look for the conditions where the main player can guarantee a win. Using backwards induction, it's evident that the main player loses if they start counting up after $N - 1$; they have to count N . However, the main player can guarantee avoiding a loss if they count up after any value from $N - 2$ to $N - 4$ for a game of at least two players, as they can always count up to $N - 1$, forcing the next player to count N and lose.

Moreover, the main player can guarantee a win if they count up after any value from $N - 2$ to $N - 2 - n$ when playing an n player game. Your $n - 1$ opponents must collectively do a minimum of $n - 1$ additional counts before it's your turn again.

So, by counting up by three after $N - 2 - n$, there are $n - 1$ counts left until someone counts N . Thus one of your opponents is bound to lose.

$$\overbrace{N-1}^{\text{Loss}}, \underbrace{N-2, N-3, \dots, N-2-n}_{\text{Loss avoidable}}, \overbrace{N-3-n, \dots}^{\text{Can lose}}$$

If the number of players is more than two, there are no other numbers to count up after to avoid a loss except the ones mentioned above. The two-player case is unique in that there are more values where you can prevent a loss. Loss is avoidable when counting up after any integer in the range from $N - 2 - 4k$ to $N - 4 - 4k$ for $k \in \mathbb{N}_0$. You can check it yourself.

$$\overbrace{N-1}^{\text{Loss}}, \underbrace{N-2, N-3, N-4}_{\text{Loss avoidable}}, \overbrace{N-5}^{\text{Can lose}}, \underbrace{N-6, N-7, N-8, \dots}_{\text{Loss avoidable}}$$

Setting $N = 21$, we see that, for two players, a loss is always avoidable when counting up after any number underlined in the sequence below:

$$20, \underline{19}, \underline{18}, \underline{17}, 16, \underline{15}, \underline{14}, \underline{13}, 12, \underline{11}, \underline{10}, \underline{9}, 8, \underline{7}, \underline{6}, \underline{5}, 4, \underline{3}, \underline{2}, 1, 0$$

Evidently, you can always avoid a loss by playing second.

3.4 Generating Function Analysis Example

The second version of 'Twenty-One' has trickier rules to work with, but we can try asking a different question now. What if we want the number of ways a game can start with one person and end with another given the final number they counted?

Here, we demonstrate using a matrix to create generating functions that yield our desired answers. Let matrix A have polynomial entries, where $A_{ij} = P_{ij}(x)$ is a generating function for the number of ways a single turn starts with player i and ends with player j , where the exponents of the polynomial represent the final number counted.

For simplicity, we will look at the drinking game version with three players.

There is no way for a player to end up back at themselves in one turn. So, for every player, $A_{ii} = P_{ii}(x) = 0 : \forall i \in \mathbb{Z}/3\mathbb{Z}$.

For a player to end with the next player in one turn, they can count up by one or four. Since there is only one way of counting up by either one or four, the following equations hold:

$$A_{01} = A_{12} = A_{20} = P_{01}(x) = x + x^4$$

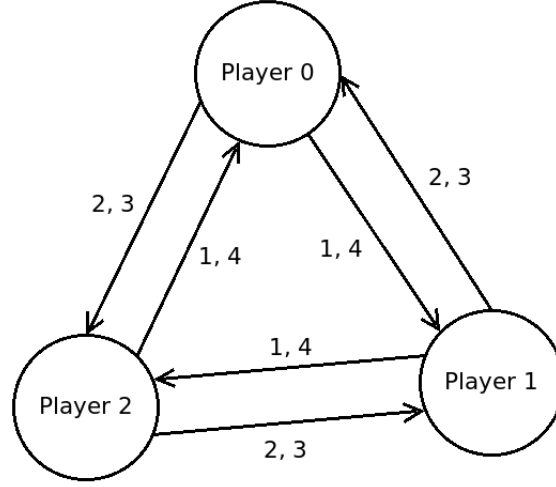


Figure 1: The graph representation of the three player drinking game version of the game 'Twenty-One'.

Lastly, a player can end with the previous player in one turn if they count up by two or three. Again, there is only one way of counting up by two or three, therefore:

$$A_{02} = A_{10} = A_{21} = P_{02}(x) = x^2 + x^3$$

The final generating function matrix A has the form:

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & x + x^4 & x^2 + x^3 \\ x^2 + x^3 & 0 & x + x^4 \\ x + x^4 & x^2 + x^3 & 0 \end{pmatrix}$$

We can now, for example, analyse the matrix A^3 to find the number of ways the game can start and end with the zeroth player in three turns, given that the final count is 6. For this, we look at the generating function formed at $(A^3)_{00}$, which is $P^3_{00}(x) = x^3 + 4x^6 + 3x^7 + 3x^8 + 4x^9 + x^{12}$. Since the final count is 6, the coefficient for the x^6 term is 4, so there are 4 ways of starting and ending with player zero in three turns and having the final count equal 6.

The generating function matrix method is a powerful combinatorial tool that also has applications in graph theory. Indeed, 'Twenty-One' can be expressed as a graph, where the nodes are the players connected with directed vertices, and each vertex is weighted by the count number required for the game to switch from one player to another.

4 Spoons Game Analysis

4.1 Mathematical Model

Consider a game of 'Spoons' with:

- t players
- k being the number of cards a player must hold at any time
- n different values the cards can take
- m suits of cards

From this, the inequality governing the values is $tk \leq nm$, since the players can't collectively hold more cards than there are.

Firstly, each player is randomly dealt k cards from the main deck. Define the vector $p_i = (p_{i1}, p_{i2}, \dots, p_{in}) \in \mathbb{N}_0^n$ to represent the number of different card values that player i is holding. So p_{ij} equals the number of cards of value j that player i is holding. Each player has to hold k cards at any time, so the following constraint holds:

$$\sum_{j=1}^n p_{ij} = \mathbf{1} \cdot p_i = k : \forall i \in \{1, 2, \dots, t\}$$

Where $\mathbf{1}$ is the unit vector, with all entries equal to one. For any player i , the aim is to have only one entry in p_i equal k , creating another restriction $k \leq m$.

Finally, introduce the vector $p_0 = m\mathbf{1} - \sum_{i=1}^t p_i$, which denotes the number of cards of different values left in the main deck.

4.2 The Optimal Strategy

A single player can treat the entire situation as though they are fed random cards from a deck of $nm - k$ cards since they can receive cards from the main deck or the other $t - 1$ players. This assumption is a substantial simplification, but the game would be significantly harder to model by accounting for all players' possible combinations of cards.

It may be obvious to prioritise keeping the cards that increase the most number of cards held of the same value and replace cards that would decrease the least number of cards held of the same value, but is there a way to show that it's the best strategy?

For player i , under the previous assumptions, the probability of getting a card of value j is $(m - p_{ij})/(nm - k)$ when they are already holding p_{ij} cards of that value.

Therefore, the probability of getting $k - p_{ij}$ cards of value j in the next $k - p_{ij}$ turns is:

$$\mathbb{P} = \prod_{r=1}^{k-p_{ij}} \frac{m - p_{ij} - r + 1}{nm - k} = \frac{(k - p_{ij})!}{(nm - k)^{k-p_{ij}}} \binom{m-p_{ij}}{m-k}$$

Since \mathbb{P} gets smaller as p_{ij} reduces, it is better to keep cards that increase the greatest value in p_i and replace cards that reduce the smallest value in p_i to increase your chances of winning the game earlier.

4.3 Small Modifications

The optimal strategy discussed previously still has nuances to resolve.

For an even valued k and $k = m$, a player could end up having half of their cards of one value and the other half of another. The problem occurs when another player attains the same combination of cards, meaning both players will never win if they play the optimal strategy. Despite being a rare occurrence, it can make the game endless if it happens, but only when the number of players t is even.

Another detail to consider is the bias towards discarding certain cards according to their value. If players prefer holding higher value cards over lower value ones, the lack of diversity of card values in circulation could lead to longer games. A player could be holding back certain cards beneficial only to another player, thus reducing the probability that their opponents could get certain cards, meaning they could also fall victim to this issue.

We resolve both issues by adding randomisations in the players' behaviours regarding what cards they discard. If a player has an equal amount of two different valued cards, they will replace one of their cards with the one passed down to them. If a player has multiple categories of card values with the smallest number of cards, they will randomly choose the category from which they will discard.

4.4 Analysis Through Simulation

By running 100000 simulations of 'Spoons' using R code and the mathematical model described above with the modified optimal strategy, it becomes evident that each player does not have the same chance of achieving k identically valued cards before everyone else. The simulations were run with the main parameters set to $t = 4$, $n = 13$, $m = 4$ and $k = 4$.

The following is a frequency bar chart for the counts of games won by each player arranged in order. The black line on the graph shows the required height of the bars for the distribution of wins to be uniform. Player 4, the last player to discard their card, has a slightly higher frequency of attaining k same valued

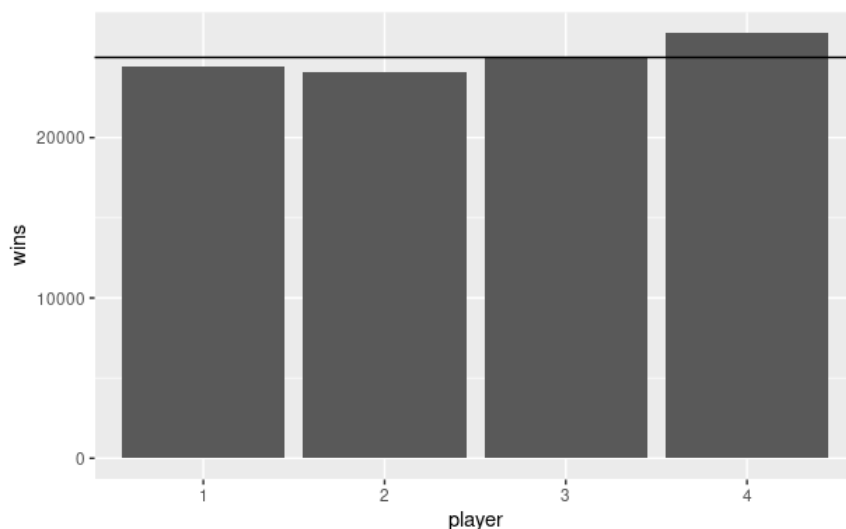


Figure 2: A bar chart for the number of wins for each player over 100000 simulations of 'Spoons'.

cards than other players. Therefore, to an extent, it's somewhat better to be the last person in the sequence of players than any other position. However, this conclusion is valid only in the specific conditions tested for in the simulation.

5 Conclusion

While this report does not thoroughly study both 'Spoons', 'Twenty-One' and their variations, it should at least explain how one can approach games from an analytical standpoint. Hopefully, this report can inspire you (the reader) to look deeper into the optimal winning strategies for any game you may encounter, especially if you want to drink the least in your next drinking game or circling session. Good luck! :D

6 Bibliography

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