EWM Formulas

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Introduction

Let $\{X_i\}_{i\in\Omega}$ be i.i.d random variables and I be a Ω valued random variable.

Moving Average

Theorem 1. *Let* $\mu_t := \mathbb{E}_I[X_I]$ *and* $\mu_{t-1} := \mathbb{E}_I[X_I|I \neq i]$*, then:*

$$\mu_t = \mu_{t-1} \cdot (1-p) + X_i \cdot p$$

Proof.

$$\begin{aligned}
\mu_t &= \mathbb{E}_I[X_I] \\
&= \mathbb{E}_I[X_I; I \neq i] + \mathbb{E}_I[X_I; I = i] \\
&= \mathbb{E}_I[X_I|I \neq i]\mathbb{P}(I \neq i) + \mathbb{E}_I[X_I|I = i]\mathbb{P}(I = i) \\
&= \mu_{t-1} \cdot (1-p) + X_i \cdot p
\end{aligned}$$

Where: $p := \mathbb{P}(I = i)$ □

Moving Variance

Theorem 2. *Let* $v_t := \mathbb{E}_I[(X_I - \mu_t)^2]$ *and* $v_{t-1} := \mathbb{E}_I[(X_I - \mu_{t-1})^2|I \neq i]$*, then:*

$$v_t = (1-p) \cdot (v_{t-1} + p \cdot (X_i - \mu_{t-1})^2)$$

Proof.

$$\begin{aligned}
v_t &= \mathbb{E}_I[(X_I - \mu_t)^2] \\
&= \mathbb{E}_I[(X_I - \mu_{t-1})^2] - (X_i - \mu_{t-1})^2 \cdot p^2 \\
&= v_{t-1} \cdot (1-p) + \mathbb{E}_I[(X_I - \mu_{t-1})^2|I = i] \cdot p - (X_i - \mu_{t-1})^2 \cdot p^2 \\
&= (1-p) \cdot (v_{t-1} + p \cdot (X_i - \mu_{t-1})^2)
\end{aligned}$$

Where: $p := \mathbb{P}(I = i)$ □

Unbiased Variance Estimate

Theorem 3. *Let* $v_t := \mathbb{E}_I[(X_I - \mu_t)^2]$ *and* $q_t := \mathbb{P}(I \neq J)$ *where* $I \perp J$ *and* $I \overset{d}{=} J$*, then:*

$$\mathbb{E}_X[v_t] = v \cdot q_t$$

Proof.

$$\begin{aligned}
\mathbb{E}_X[v_t] &= \mathbb{E}_X[\mathbb{E}_I[(X_I - \mu_t)^2]] \\
&= \mathbb{E}_I[\mathbb{E}_X[(X_I - \mu_t)^2]]
\end{aligned}$$

Evaluating the expression inside:

$$\mathbb{E}_X[(X_I - \mu_t)^2] = \mathbb{E}_X[X_I^2] - 2\mathbb{E}_X[X_I \cdot \mu_t] + \mathbb{E}_X[\mu_t^2]$$

Evaluating the sub-expressions:

$$\mathbb{E}_X[X_I^2] = v + \mu^2$$

The middle part:

$$\begin{aligned}
\mathbb{E}_X[X_I \cdot \mu_t] &= \mathbb{E}_X[X_I \cdot \mathbb{E}_J[X_J]] \\
&= \mathbb{E}_J[\mathbb{E}_X[X_I \cdot X_J]] \\
&= \mathbb{E}_J[\mathbb{E}_X[X_I \cdot X_J] | I \neq J] \cdot q_t + \mathbb{E}_J[\mathbb{E}_X[X_J^2]] \cdot (1 - q_t) \\
&= \mu^2 \cdot q_t + (v + \mu^2) \cdot (1 - q_t) \\
&= v \cdot (1 - q_t) + \mu^2
\end{aligned}$$

The last part:

$$\begin{aligned}
\mathbb{E}_X[\mu_t^2] &= \mathbb{E}_X[\mathbb{E}_I[X_I] \cdot \mathbb{E}_J[X_J]] \\
&= \mathbb{E}_{IJ}[\mathbb{E}_X[X_I \cdot X_J]] \\
&= \mathbb{E}_{IJ}[\mathbb{E}_X[X_I \cdot X_J] | I \neq J] \cdot q_t + \mathbb{E}_J[\mathbb{E}_X[X_J^2]] \cdot (1 - q_t) \\
&= v \cdot (1 - q_t) + \mu^2
\end{aligned}$$

Therefore:

$$\begin{aligned}
\mathbb{E}_X[(X_I - \mu_t)^2] &= (v + \mu^2) - 2(v \cdot (1 - q_t) + \mu^2) + (v \cdot (1 - q_t) + \mu^2) \\
&= v \cdot q_t
\end{aligned}$$

Which is independent of I. □

Theorem 4. *Let $q_{t-1} := \mathbb{P}(I \neq J | I, J \neq i)$, then:*

$$q_t = q_{t-1} \cdot (1 - p)^2 + 2p(1 - p)$$

Proof.

$$\begin{aligned}
v \cdot q_t &= \mathbb{E}_X[v_t] \\
&= \mathbb{E}_X[(1 - p) \cdot (v_{t-1} + p \cdot (X_i - \mu_{t-1})^2)] \\
&= (1 - p) \cdot (\mathbb{E}_X[v_{t-1}] + p \cdot \mathbb{E}_X[(X_i - \mu_{t-1})^2]) \\
&= (1 - p) \cdot (v \cdot q_{t-1} + p \cdot \mathbb{E}_X[(X_i - \mu_{t-1})^2])
\end{aligned}$$

Evaluating the expectation:

$$\begin{aligned}
\mathbb{E}_X[(X_i - \mu_{t-1})^2] &= \mathbb{E}_X[X_i^2] - 2\mathbb{E}_X[X_i \cdot \mu_{t-1}] + \mathbb{E}_X[\mu_{t-1}^2] \\
&= v + \mu^2 - 2\mu^2 + v \cdot (1 - q_{t-1}) + \mu^2 \\
&= v \cdot (2 - q_{t-1})
\end{aligned}$$

Therefore:

$$\begin{aligned}
q_t &= (1 - p) \cdot (q_{t-1} + p \cdot (2 - q_{t-1})) \\
&= (1 - p) \cdot (q_{t-1} \cdot (1 - p) + 2p) \\
&= q_{t-1} \cdot (1 - p)^2 + 2p(1 - p)
\end{aligned}$$

□

There is a simpler proof of Theorem 4 using conditional probability.