Ficha 8

Gradientes, Derivadas direccionais, Plano tangente, Recta Normal

① a) $\vec{\nabla} f(x,y) = 2(x,y)$

b) $\vec{\nabla} f(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$

② $\vec{\nabla} f(1,1,\pi) = (1,0,0)$

③ $f(x,y) = \ln\|\vec{R}\|$, $\vec{R} = (x,y)$, $\|\vec{R}\| = \sqrt{x^2+y^2}$

$\vec{\nabla} f(x,y) = \frac{1}{x^2+y^2}(x,y)$

$\frac{\vec{R}}{\|\vec{R}\|^2} = \frac{(x,y)}{x^2+y^2} = \frac{1}{x^2+y^2}(x,y)$

④ a) $P \to (2,1,3)$ $\quad M \to (5,5,15)$ $\quad \vec{PM} = M - P = (3,4,12)$

$D_{\vec{PM}}(2,1,3) = (4,5,3)\cdot\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right) = \frac{68}{13}$

b) $\measuredangle(\vec{u},\vec{Ox}) = 60^\circ$

$u \to (\cos 60^\circ, \sin 60^\circ) = \left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

$D_{\vec{u}} f(1,2) = (0,-9)\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = -\frac{9\sqrt{3}}{2}$

⑤ $\begin{cases} D_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)} f(a,b) = \vec{\nabla} f(a,b)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = 3\sqrt{2} \\ D_{\left(\frac{3}{5},-\frac{4}{5}\right)} f(a,b) = \vec{\nabla} f(a,b)\cdot\left(\frac{3}{5},-\frac{4}{5}\right) = 5 \end{cases}$ $\begin{cases} \frac{\partial f}{\partial x}(a,b) = 7 \\ \frac{\partial f}{\partial y}(a,b) = -1 \end{cases} \Rightarrow \vec{\nabla} f_{(a,b)} = (7,-1)$

⑥ $\vec{u} = ? = (u_1,u_2)$: $D_{\vec{u}} f(2,0) = -1$ $\quad f(x,y) = xy$

$\vec{\nabla} f(2,0)\cdot(u_1,u_2) = -1 \Leftrightarrow (u_1,u_2) = \left(u_1,-\frac{1}{2}\right)$ $\quad u_1 \in \mathbb{R}$

⑦ a) $\vec{\nabla} T(2,1) = \frac{4}{e}(1,-1)$

Nota: Ver exercício 5 da ficha 9A

b) $D_{\vec{\nabla T}} T(2,1) = \|\vec{\nabla} T(2,1)\| = \frac{4}{e}\sqrt{2}$

(8) A taxa de variação máxima no ponto (a,b,c) é $\|\vec{\nabla} f(a,b,c)\|$

A sua metade é $\frac{1}{2}\|\vec{\nabla} f(a,b,c)\| = \sqrt{a^2+b^2+c^2}$

Pretende-se determinar $\vec{u}$ tal que $D_{\vec{u}} f(a,b,c) = \sqrt{a^2+b^2+c^2} \Leftrightarrow$

$$\Leftrightarrow (2a, 2b, -2c) \cdot (u_1, u_2, u_3) = \sqrt{a^2+b^2+c^2}$$

$\vec{u}$ é tal que

$$a u_1 + b u_2 - c u_3 = \frac{\sqrt{a^2+b^2+c^2}}{2}$$

$$\vec{u} = \left(u_1, u_2, \frac{\sqrt{a^2+b^2+c^2}}{-2c} + \frac{a}{c}u_1 + \frac{b}{c}u_2\right), \quad c \neq 0$$

(9)

a) $D_{\vec{u}} f(3,4) = \vec{\nabla} f(3,4) \cdot (\cos\alpha ; \operatorname{sen}\alpha) = -6\cos\alpha - 8 \operatorname{sen}\alpha$

b) Na direcção do vector $\vec{\nabla} f(3,4)$, isto é, $(-6;-8)$

c) No ponto $(3,4,75)$, e na direcção do vector $(-6,-8)$, o valor da função aumenta

(12) $F(x,y,z) = x^2+y^2-4z$

$\vec{\nabla} f(2,-4,5) = (4,-8,-5)$

$4(x-2) - 4(y+8) - 5(z-5) = 0 \Leftrightarrow 4x - 4y - 5z - 15 = 0$ Plano Tangente

$$\frac{x-2}{4} = \frac{y+8}{-4} = \frac{z-5}{-5} \Leftrightarrow 5x - 10 = -5y - 40 = 20 - 4z$$

(13) $z=0$ é o plano tangente ao parabolóide hiperbólico.

$F(x,y,z) = x^2 - y^2 - z \quad \vec{\nabla} F(0,0,0) = (0,0,-1)$

Intersecção $\begin{cases} z=0 \\ z = x^2-y^2 \end{cases} \Rightarrow y^2 = x^2 \Leftrightarrow y = \pm x$

(14) $F(x,y,z) = x^2 - 2yz + y^3 - 4$

$G(x,y,z) = x^2 + 1 + 2y^2 - z^2$

$\vec{\nabla} F(1,-1,2) = (2,-1,2) \qquad \vec{\nabla} G(1,-1,2) = (2,-4,-4)$

$\vec{\nabla} F \cdot \vec{\nabla} G = 0 \Leftrightarrow (2,-1,2)\cdot(2,-4,-4) = 0 \Leftrightarrow$

$0 = 0$ //

10)

$$f(x,y) \approx 4 + 4(x-1) + 4(y-2) + 4(x-1)(y-2) + (y-2)^2$$

11)

a) $f(x,y) \approx 2(x-1) - (x-1)^2 + y^2 + \frac{2}{3}(x-1)^3 - 2(x-1)y^2$

b) $f(x,y) \approx x - \frac{1}{3}x^3$

c) $f(x,y) \approx 1 - (y-1) + \frac{1}{2}\left(-x^3 + 2y^2\right)$