Folha 6A

1. a) $\int_1^2 e^{\pi x}\,dx = \frac{1}{\pi}\left[e^{\pi x}\right]_1^2 = \frac{1}{\pi}\left(e^{2\pi} - e^{\pi}\right).$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\operatorname{sen} x|\,dx = \int_{-\pi/2}^{0} |\operatorname{sen} x|\,dx + \int_0^{\pi/2} |\operatorname{sen} x|\,dx$

$= \int_{-\pi/2}^{0} (-\operatorname{sen} x)\,dx + \int_0^{\pi/2} \operatorname{sen} x\,dx$

$= \left[\cos x\right]_{-\pi/2}^{0} + \left[-\cos x\right]_0^{\pi/2}$

$= (1-0) + (0+1) = 2$

c) $\int_{-3}^{5} |x-1|\,dx = \int_{-3}^{1} (-x+1)\,dx + \int_1^5 (x-1)\,dx$

$= -\frac{1}{2}\left[x^2\right]_{-3}^{1} + \left[x\right]_{-3}^{1} + \frac{1}{2}\left[x^2\right]_1^5 - \left[x\right]_1^5$

$= -\frac{1}{2}(1-9) + (1+3) + \frac{1}{2}(25-1) - (5-1)$

$= 4 + 4 + 12 - 4 = 16$

d) $\int_0^2 |(x-1)(3x-2)|\,dx$

Atendendo a que

$(x-1)(3x-2) = 0 \iff x = 1 \lor x = \frac{2}{3}$

e a que

$(x-1)(3x-2) < 0 \iff x \in \left]\frac{2}{3}, 1\right[,$

vem

$$|(x-1)(3x-2)| = \begin{cases} (x-1)(3x-2), & \text{se } x \notin \left]\frac{2}{3}, 1\right[\\ -(x-1)(3x-2), & \text{se } x \left]\frac{2}{3}, 1\right[\end{cases}$$

pelo que

$$\int_0^2 |(x-1)(3x-2)|\,dx$$

$$= \int_0^{2/3} (x-1)(3x-2)\,dx + \int_{2/3}^1 \left[-(x-1)(3x-2)\right] dx$$

$$+ \int_1^2 (x-1)(3x-2)\,dx$$

$$= \int_0^{2/3} (3x^2-5x+2)\,dx - \int_{2/3}^1 (3x^2-5x+2)\,dx$$

$$+ \int_1^2 (3x^2-5x+2)\,dx$$

$$= \left[x^3\right]_0^{2/3} - \frac{5}{2}\left[x^2\right]_0^{2/3} + 2\left[x\right]_0^{2/3}$$

$$- \left[x^3\right]_{2/3}^1 + \frac{5}{2}\left[x^2\right]_{2/3}^1 - 2\left[x\right]_{2/3}^1$$

$$+ \left[x^3\right]_1^2 - \frac{5}{2}\left[x^2\right]_1^2 + 2\left[x\right]_1^2$$

$$= \left(\frac{8}{27} - 0\right) - \frac{5}{2}\left(\frac{4}{9} - 0\right) + 2\left(\frac{2}{3} - 0\right)$$

$$- \left(1 - \frac{8}{27}\right) + \frac{5}{2}\left(1 - \frac{4}{9}\right) - 2\left(1 - \frac{2}{3}\right)$$

$$+ (8-1) - \frac{5}{2}(4-1) + 2(2-1)$$

$$= \frac{8}{27} - \frac{10}{9} + \frac{4}{3} - 1 + \frac{8}{27} + \frac{5}{2} - \frac{10}{9} \cancel{-2} + \frac{4}{3} + 7 - \frac{15}{2} \cancel{+2}$$

$$= 55/27 \;//$$

2. a) $\int_0^2 f(x)dx$, $f(x) = \begin{cases} x^2 , & 0 \leq x \leq 1 \\ 3-x , & 1 < x \leq 2 . \end{cases}$

A função f possui apenas uma descontinuidade em $x=1$, logo é integrável.

Tem-se

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$

$$= \int_0^1 x^2 dx + \underbrace{\int_1^2 (3-x)\, dx}_{}$$

$f(x) \neq 3-x$ apenas para $x=1$, o que não muda o valor do integral.

$$= \frac{1}{3}\left[x^3\right]_0^1 + 3\left[x\right]_1^2 - \frac{1}{2}\left[x^2\right]_1^2$$

$$= \frac{1}{3}(1-0) + 3(2-1) - \frac{1}{2}(4-1)$$

$$= \frac{1}{3} + 3 - \frac{3}{2}$$

$$= \frac{11}{6} //$$

2. b) $\int_{-5}^0 2x\sqrt{4-x}\, dx$

Consultando a tabela de substituições verificamos que

$$4-x = t^2$$

é uma substituição de sucesso para o integral proposto.

Então temos

$$x = g(t) \quad \text{com} \quad g(t) = 4 - t^2$$

donde

$$g'(t) = -2t.$$

Quanto aos limites de integração, temos

$$\begin{cases} x = 4-t^2 \\ x = -5 \end{cases} \Rightarrow 4-t^2 = -5 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3$$

$$\begin{cases} x = 4-t^2 \\ x = 0 \end{cases} \Rightarrow 4-t^2 = 0 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$$

e a escolha mais simples é $t_1 = 3$ e $t_2 = 2$.

Resulta

$$\int_{-5}^{0} 2x\sqrt{4-x}\, dx = \int_{3}^{2} 2(4-t^2)\sqrt{t^2}\,(-2t)\,dt$$

$$= -4\int_3^2 (4-t^2)t^2\,dt = 4\int_2^3 (4t^2 - t^4)\,dt$$

$$\int_b^a \cdots = -\int_a^b \cdots$$

$$= \frac{16}{3}\left[t^3\right]_2^3 - \frac{4}{5}\left[t^5\right]_2^3$$

$$= \frac{16}{3}\left(3^3 - 2^3\right) - \frac{4}{5}\left(3^5 - 2^5\right)$$

$$= \frac{16}{3}(27-8) - \frac{4}{5}(243-32)$$

$$= \frac{16 \times 19}{3} - \frac{4 \times 211}{5}$$

$$= \frac{204}{3} - \frac{844}{5} = \frac{204 \times 5 - 844 \times 3}{15}$$

$$= \frac{1020 - 2532}{15} = -\frac{1512}{15} = \frac{504}{5}$$

2 c) Há duas versões para este enunciado.

(i) $\displaystyle\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{x^2+1}}\,dx$

Uma substituição de sucesso é $x = sh\,t$.
Então

$$x = g(t), \text{ com } g(t) = sh\,t.$$

donde

$$g'(t) = ch\,t.$$

Para os limites de integração, vem

$$\begin{cases} x = sh\,t \\ x = 3/4 \end{cases} \Rightarrow \frac{3}{4} = sh\,t \Rightarrow t = \arg sh\,\frac{3}{4}$$

$$\begin{cases} x = sh\,t \\ x = 4/3 \end{cases} \Rightarrow \frac{4}{3} = sh\,t \Rightarrow t = \arg sh\,\frac{4}{3}$$

Resulta

$$\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{x^2+1}}\,dx = \int_{\arg sh\,3/4}^{\arg sh\,4/3} \frac{1}{sh^2 t\sqrt{sh^2 t+1}}\,ch\,t\,dt$$

$$= \int_{\arg sh\,3/4}^{\arg sh\,4/3} \frac{1}{sh^2 t\cdot ch\,t}\,ch\,t\,dt = \int_{\arg sh\,3/4}^{\arg sh\,4/3} \frac{1}{sh^2 t}\,dt$$

$sh^2 t + 1 = ch^2 t$

$$= -\Big[\coth t\Big]_{\arg sh\,3/4}^{\arg sh\,4/3} = -\coth\big(\underbrace{\arg sh\,\tfrac{4}{3}}_{A}\big) + \coth\big(\underbrace{\arg sh\,\tfrac{3}{4}}_{B}\big)$$

$$A = \operatorname{argsh}\frac{4}{3} \Leftrightarrow \operatorname{sh}A = \frac{4}{3} \Rightarrow \operatorname{ch}^2 A = 1 + \left(\frac{4}{3}\right)^2$$

$$\Rightarrow \operatorname{ch}^2 A = \frac{25}{9} \Rightarrow \operatorname{ch} A = \frac{5}{3}$$

$\operatorname{ch} x \geq 1, \forall x \in \mathbb{R}$

$$B = \operatorname{argsh}\frac{3}{4} \Leftrightarrow \operatorname{sh}B = \frac{3}{4} \Rightarrow \operatorname{ch}^2 B = 1 + \left(\frac{3}{4}\right)^2$$

$$\Rightarrow \operatorname{ch}^2 B = \frac{25}{16} \Rightarrow \operatorname{ch} B = \frac{5}{4}$$

Então,

$$\coth A = \frac{\operatorname{ch} A}{\operatorname{sh} A} = \frac{\frac{5}{3}}{\frac{4}{3}} = \frac{5}{4}$$

$$\coth B = \frac{\operatorname{ch} B}{\operatorname{sh} B} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$$

Finalmente, para o integral proposto, vem

$$\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{x^2+1}}\,dx = -\frac{5}{4} + \frac{5}{3} = \frac{5}{12} \,//\,.$$

(ii) $\displaystyle\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{1}{x^2\sqrt{x^2+1}}\,dx$

Vamos fazer agora $x = \operatorname{tg} t$.
Então

$$x = g(t), \text{ com } g(t) = \operatorname{tg} t,$$

donde

$$g'(t) = \frac{1}{\cos^2 t}.$$

Para os limites de integração, vem

$$\begin{cases} x = \operatorname{tg} t \\ x = \sqrt{3}/3 \end{cases} \Rightarrow \operatorname{tg} t = \frac{\sqrt{3}}{3} \Rightarrow t = \frac{\pi}{6} + k\pi, \ k \in \mathbb{Z}$$

$$\begin{cases} x = \operatorname{tg} t \\ x = \sqrt{3} \end{cases} \Rightarrow \operatorname{tg} t = \sqrt{3} \Rightarrow t = \frac{\pi}{3} + k\pi, \ k \in \mathbb{Z}.$$

A escolha mais simples é $t_1 = \frac{\pi}{6}$ e $t_2 = \frac{\pi}{3}$.

Resulta

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{1}{x^2\sqrt{x^2+1}}\, dx = \int_{\pi/6}^{\pi/3} \frac{1}{\operatorname{tg}^2 t \sqrt{\operatorname{tg}^2 t + 1}} \frac{1}{\cos^2 t}\, dt$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{\frac{\operatorname{sen}^2 t}{\cos^2 t}\sqrt{\frac{1}{\cos^2 t}}} \cdot \frac{1}{\cos^2 t}\, dt$$

$$= \int_{\pi/6}^{\pi/3} \frac{\cos t}{\operatorname{sen}^2 t}\, dt = \int_{\pi/6}^{\pi/3} \cos t\, (\operatorname{sen} t)^{-2}\, dt$$

$$= \left[-\frac{1}{\operatorname{sen} t}\right]_{\pi/6}^{\pi/3} = -\frac{1}{\operatorname{sen}\frac{\pi}{3}} + \frac{1}{\operatorname{sen}\frac{\pi}{6}}$$

$$= -\frac{1}{\frac{\sqrt{3}}{2}} + \frac{1}{\frac{1}{2}} = -\frac{2}{\sqrt{3}} + 2 = \frac{-2\sqrt{3}+6}{3}$$

d) $\int_0^2 x^3 e^{x^2} dx = \int_0^2 \underbrace{(x e^{x^2})}_{\uparrow} x^2 \, dx$

$= \left[\frac{1}{2} e^{x^2} \cdot x^2\right]_0^2 - \int_0^2 \frac{1}{2} e^{x^2} \cdot 2x \, dx$

$= \frac{1}{2}\left(4e^4 - 0\right) - \frac{1}{2}\left[e^{x^2}\right]_0^2$

$= 2e^4 - \frac{1}{2}(e^4 - 1) = \frac{3}{2} e^4 + \frac{1}{2}$ //