Ficha 7A

1. $\int_{K}^{x} f(t)\,dt = \operatorname{sen} x + \frac{1}{2}, \ \forall x \in \mathbb{R}.$

Derivando em ordem a x ambos os membros da equação dada, vem

$$f(x) = \cos x, \quad \forall x \in \mathbb{R}.$$

Por outro lado, fazendo $x = K$ na equação dada, fica

$$\int_{K}^{K} f(t)\,dt = \operatorname{sen} K + \frac{1}{2}$$

$$\Rightarrow 0 = \operatorname{sen} K + \frac{1}{2}$$

$$\Rightarrow \operatorname{sen} K = -\frac{1}{2}$$

$$\Rightarrow K = \frac{7\pi}{6} + 2n\pi \ \vee \ K = -\frac{\pi}{6} + 2n\pi, \ n \in \mathbb{Z}$$

Podemos escolher, por exemplo, $K = -\frac{\pi}{6}$.

2. a) $f(x) = \int_{1}^{x} \frac{\sqrt{1+t^4}}{t^2}\,dt, \quad x \in \mathbb{R}^+.$

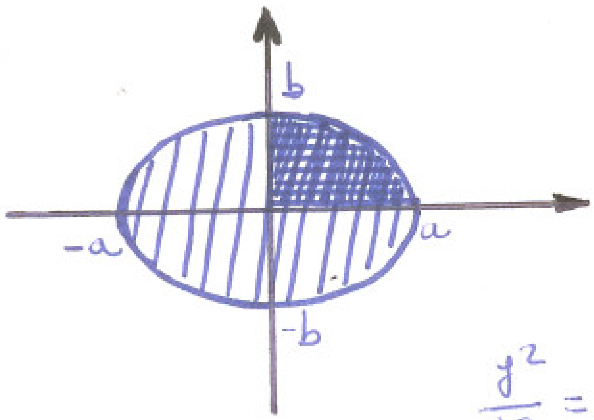
$$f'(x) = \frac{\sqrt{1+x^4}}{x^2}, \quad x \in \mathbb{R}^+.$$

b) $f(x) = \int_{1}^{\ln x} \operatorname{sen}(u + e^u)\,du, \quad x \in \mathbb{R}^+.$

$$f'(x) = \operatorname{sen}(\ln x + x) \cdot \frac{1}{x}, \quad x \in \mathbb{R}^+.$$

3. Áreas de regiões planas.

a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (a>0,\ b>0)$



→ D

→ D_1

A equação dada define uma elipse (circunferência se $a=b$)

Da equação da elipse, sai

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$\Rightarrow y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

sendo $y = \frac{b}{a}\sqrt{a^2-x^2}$ para a semi-elipse superior.

Atendendo às simetrias da figura, vem

$$\text{área } D = 4\, \text{área } D_1 = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\int_0^{\pi/2} \sqrt{a^2 - a^2\,\text{sen}^2 t}\cdot a\cos t\,dt$$

substituição $x = a\,\text{sen}\,t$

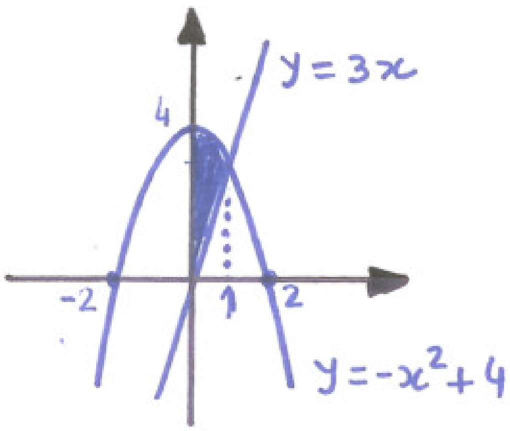
$$= 4ab\int_0^{\pi/2} \sqrt{1-\text{sen}^2 t}\,\cos t\,dt$$

$$= 4ab\int_0^{\pi/2} \cos^2 t\,dt = 4ab\int_0^{\pi/2} \frac{1+\cos 2t}{2}\,dt$$

$$= 4ab\left(\frac{1}{2}[t]_0^{\pi/2} + \frac{1}{4}[\operatorname{sen} 2t]_0^{\pi/2}\right)$$

$$= 2ab\,\frac{\pi}{2} + 0 = \pi ab$$

b) $x=0,\ x=1,\ y=3x,\ y=-x^2+4.$



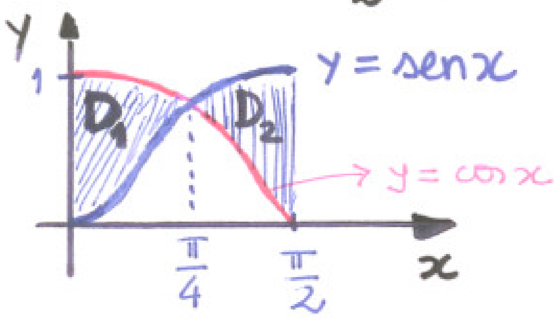
$$\text{área } D = \int_0^1 (-x^2+4-3x)\,dx$$

$$= -\frac{1}{3}[x^3]_0^1 + 4[x]_0^1 - \frac{3}{2}[x^2$$

$$= -\frac{1}{3} + 4 - \frac{3}{2}$$

$$= \frac{13}{12}$$

c) $x=0,\ x=\frac{\pi}{2},\ y=\operatorname{sen}x,\ y=\cos x.$



$$\text{área } D = \text{área } D_1 + \text{área } D_2$$

$$= 2\,\text{área } D_1$$

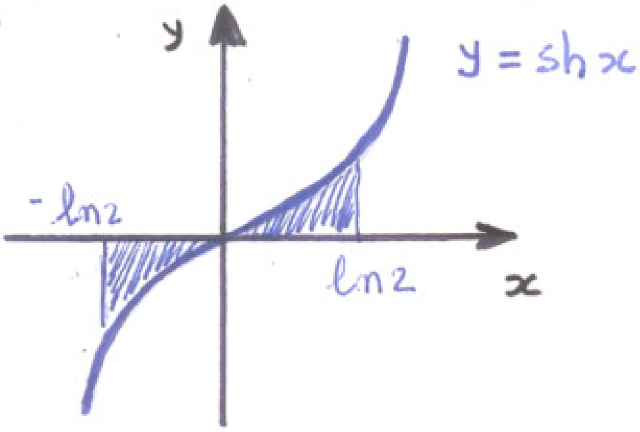
simetria

$$= 2\int_0^{\pi/4} (\cos x - \operatorname{sen} x)\,dx$$

$$= 2\left[\operatorname{sen}x + \cos x\right]_0^{\pi/4}$$

$$= 2\left(\sqrt{2} - 1\right)$$

d) $y=0,\ x=-\ln 2,\ x=\ln 2,\ y=\operatorname{sh} x.$



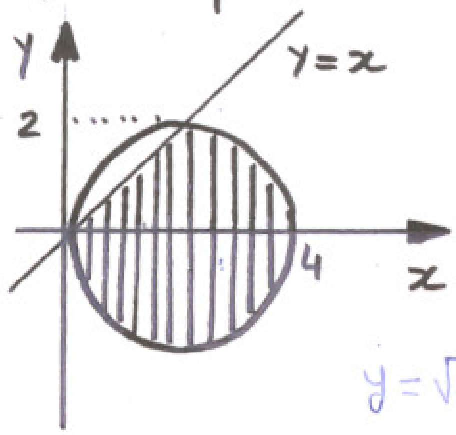
Pela simetria da região,

$$\text{área } D = 2\int_0^{\ln 2} \operatorname{sh} x \, dx$$

$$= 2\left[\operatorname{ch} x\right]_0^{\ln 2} = \left[e^{x}+e^{-x}\right]_0^{\ln 2} = 2+\frac{1}{2}-2 = \frac{1}{2}$$

4. Estabelecer o integral (ou soma de integrais) que dá a área.

a) $A=\{(x,y)\in\mathbb{R}^2 : (x-2)^2+y^2\le 4 \ \wedge \ y\le x\}$



Versão 1

$(x-2)^2+y^2=4 \Leftrightarrow y=\pm\sqrt{4-(x-2)^2}$

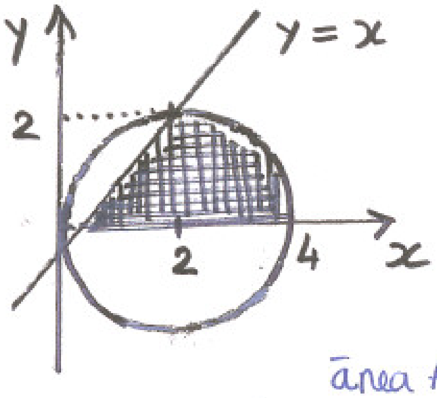
$y=\sqrt{4-(x-2)^2}$ para ⌒, $y=-\sqrt{4-(x-2)^2}$ para ◡

$$\text{Área}(A)=\int_0^2\left(x+\sqrt{4-(x-2)^2}\right)dx + 2\int_2^4 \sqrt{4-(x-2)^2}\,dx$$

simetria do semi-círculo "leste"

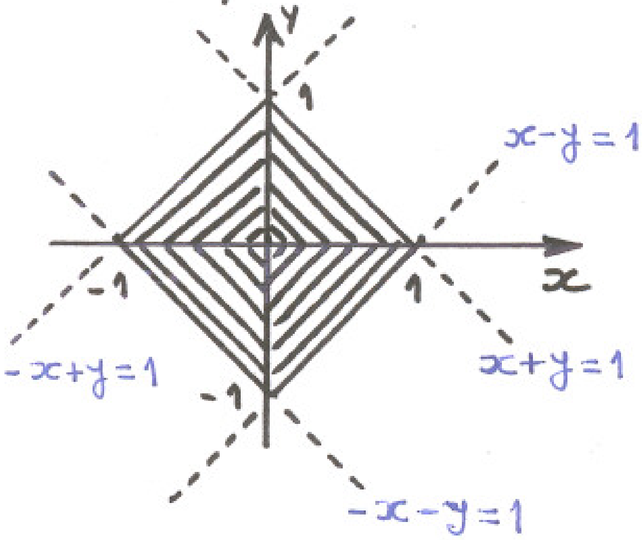
$$A = \{ (x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 \leq 4 \wedge 0 \leq y \leq x \}$$

Versão 2



$$\text{área } A = \int_0^2 x \, dx + \int_2^4 \sqrt{4-(x-2)^2} \, dx$$

b) $B = \{ (x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1 \}$



• Notar que

$$|x|+|y| = 1 \Leftrightarrow \begin{cases} x+y=1, & x \geq 0, y \geq 0 \\ x-y=1, & x \geq 0, y<0 \\ -x+y=1, & x<0, y \geq 0 \\ -x-y=1, & x<0, y<0 \end{cases}$$

Pela simetria da região em causa,

$$\text{área } B = 4 \int_0^1 (1-x) \, dx$$

c) $C = \{(x,y) \in \mathbb{R}^2 : x \leq 3 \wedge y \geq x^2 - 4x + 3 \wedge y \leq -x^2 + 5x - 4$

$y = x^2 - 4x + 3$

$y - 3 = (x-2)^2 - 4$

$\boxed{y + 1 = (x-2)^2}$

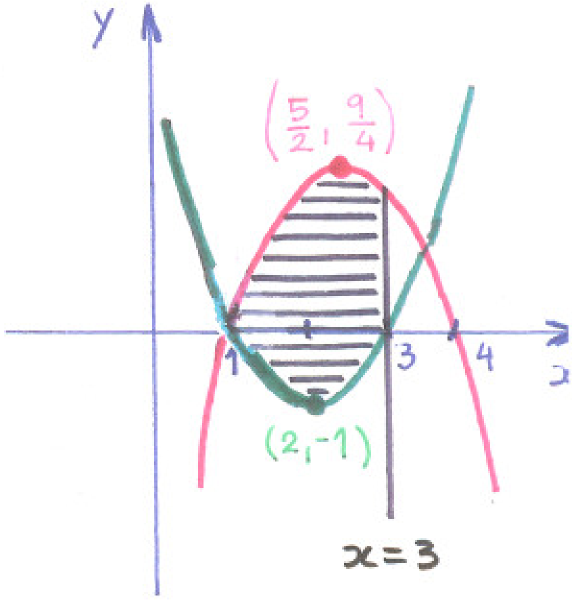
$y = -x^2 + 5x - 4$

$y + 4 = -(x^2 - 5x)$

$y + 4 = -\left(x - \frac{5}{2}\right)^2 + \frac{25}{4}$

$y + 4 - \frac{25}{4} = -\left(x - \frac{5}{2}\right)^2$

$\boxed{y - \frac{9}{4} = -\left(x - \frac{5}{2}\right)^2}$



$$\text{área } C = \int_1^3 \left[\left(-x^2 + 5x - 4\right) - \left(x^2 - 4x + 3\right)\right] dx$$

$$= \int_1^3 \left(-2x^2 + 9x - 7\right) dx$$