① a) Função contínua em $\mathbb{R}^2$ $\left(\delta = \frac{\varepsilon}{7}\right)$

b) para $x^2+y^2<1 \Rightarrow f(x,y) = x^2+y^2$, a função é contínua é um polinómio.

para $x^2+y^2>1 \Rightarrow f(x,y) = 0$, a função é contínua, trata-se de uma constante

para $x^2+y^2=1$ $\begin{cases} f(x,y) = 1 \\ \lim\limits_{x^2+y^2 \to 1} f(x,y) = \begin{cases} 0, & \text{por valores superiores a 1 } (x^2+y^2>1) \\ 1, & \text{por valores inferiores a 1 } (x^2+y^2<1) \end{cases} \end{cases}$

$\therefore$ A função é contínua em $\mathbb{R}^2$ excepto para $(x,y): x^2+y^2=1$

c) A função é contínua em $\mathbb{R}^2$ $\boxed{\lim\limits_{u \to 0} \frac{\operatorname{sen} u}{u} = 1}$

② $\frac{\partial f}{\partial x}(0,y) = \lim\limits_{h \to 0} \frac{f(h,y) - f(0,y)}{h} = \lim\limits_{h \to 0} \frac{hy(h^2-y^2)}{h(h^2+y^2)} = -y$

$\frac{\partial f}{\partial y}(x,0) = x$

③ a) $\frac{\partial f}{\partial x} = \frac{(12x^3+15y^2)(3x-y) - 3(3x^4+5xy^3)}{(3x-y)^2} = \frac{25x^4 - 12x^3y - 5y^4}{(3x-y)^2}$

$\frac{\partial f}{\partial y} = \frac{45x^2y^2 - 10xy^3 + 3x^4}{(3x-y)^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{(100x^3 - 36x^2y)(3x-y)^2 - (18x-6y)(25x^4 - 12x^3y - 5y^4)}{(3x-y)^4}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(100x^3-36x^2y)(3x-y)-6(25x^4-12x^3y-5y^4)}{(3x-y)^3} = \frac{150x^4+30y^4-136x^3y+36x^2y^2}{(3x-y)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{(-12x^3-20y^3)(3x-y)+2(25x^4-12x^3y-5y^4)}{(3x-y)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{14x^4+10y^4-12x^3y-60xy^3}{(3x-y)^3} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{T. Schwarz}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{270x^3y-90x^2y^2+10xy^3+6x^4}{(3x-y)^3}$$

b) $h(x,y) = e^x f(x+y) + e^{-x} g(x-y)$

$(f \circ g)(x) = f[g(x)]$

$(f \circ g)'(x) = f'[g(x)] \times g'(x)$

$$\frac{\partial h}{\partial x} = e^x f(x,y) + e^x \frac{\partial f}{\partial x}(x+y) - e^{-x} g(x-y) + \frac{\partial g}{\partial x}(x-y) \times e^{-x}$$

$$\frac{\partial h}{\partial y} = e^x \frac{\partial f}{\partial y}(x+y) + e^{-x} \frac{\partial g}{\partial y}(x-y) \frac{\partial}{\partial y}(x-y) =$$

$$= e^x \frac{\partial f}{\partial y}(x+y) - e^{-x} \frac{\partial g}{\partial y}(x-y)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) =$$

$$= e^x f(x+y) + \frac{\partial f}{\partial x}(x+y) e^x + e^x \frac{\partial f}{\partial x}(x+y) + \frac{\partial^2 f}{\partial x^2}(x+y) e^x + e^{-x} g(x-y) +$$

$$+ \frac{\partial g}{\partial x}(x-y) e^{-x} + \frac{\partial^2 g}{\partial x^2}(x-y) e^{-x} - e^{-x} \frac{\partial g}{\partial x}(x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) =$$

$$= e^x \frac{\partial^2 f}{\partial x \partial y}(x+y) + e^{-x}\frac{\partial g}{\partial y}(x-y) - \frac{\partial^2 g}{\partial x \partial y}(x-y)\, e^{-x}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}(x+y)e^x + e^{-x}\frac{\partial^2 g}{\partial y^2}(x-y)$$

c) $h(x,y) = e^x \ln(y^2+3x)$

$$\frac{\partial h}{\partial x} = e^x \ln(y^2+3x) + \frac{3e^x}{y^2+3x}$$

$$\frac{\partial h}{\partial y} = \frac{2ye^x}{y^2+3x}$$

$$\frac{\partial^2 h}{\partial x^2} = e^x \ln(y^2+3x) + \frac{3}{y^2+3x} e^x + \frac{3e^x(y^2+3x-3)}{(y^2+3x)^2}$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{2y}{y^2+3x}e^x - 6\frac{y}{(y^2+3x)^2}e^x$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{6e^x y^2 x}{(y^2+3x)^2}$$

d) $\frac{\partial f}{\partial x} = \frac{1}{1+x+y^2+z^3} = (1+x+y^2+z^3)^{-1}$

$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

$(u^n)' = n u^{n-1} \times u'$

$$\frac{\partial f}{\partial y} = 2y(1+x+y^2+z^3)^{-1}$$

$$\frac{\partial f}{\partial z} = 3z^2 \left(1+x+y^2+z^3\right)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -1\left(1+x+y^2+z^3\right)^{2}$$

$$\circledast_1 \quad \frac{\partial^2 f}{\partial x \partial y} = -2y\left(1+x+y^2+z^3\right)^{-2}$$

$$\circledast_1 \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

<u>T. Schwarz $\circledast$</u>

$$\circledast_2 \quad \frac{\partial^2 f}{\partial x \partial z} = -3z^2\left(1+x+y^2+z^3\right)^{-2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2}{1+x+y^2+z^3} - \frac{4y^2}{\left(1+x+y^2+z^3\right)^2}$$

$$\circledast_3 \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{-6z^2 y}{\left(1+x+y^2+z^3\right)^2}$$

$$\circledast_2 \quad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}$$

$$\circledast_3 \quad \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{6z}{1+x+y^2+z^3} - \frac{9z^4}{\left(1+x+y^2+z^3\right)^2}$$

e) $u(x,y) = \operatorname{arctg}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

f) $p(x,y,z) = \int_0^{y \operatorname{sen} z} x \times 4^{2t}\, dt$ $\quad\left\|\ \frac{d}{dx}\int_{\varphi(t)}^{\phi(t)} f(t)\,dt = \phi'(x) f[\phi(x)] - \varphi'(x) f[\varphi(x)]\right.$

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}(y \operatorname{sen} z)\, x\, 4^{2y \operatorname{sen} z} = x \operatorname{sen} z\, 4^{2y \operatorname{sen} z}$$

$$\frac{\partial p}{\partial z} = xy \cos z\, 4^{2y \operatorname{sen} z}$$

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial x \partial z} = \frac{\partial^2 p}{\partial x^2} = 0 \qquad \frac{\partial^2 p}{\partial y \partial x} = \operatorname{sen} z\, 4^{2y \operatorname{sen} z}$$

$$\frac{\partial^2 p}{\partial y^2} = 2x \operatorname{sen}^2 z \; 4^{y \operatorname{sen} z} \ln 4 \qquad \| \, (a^u)' = u' a^u \ln a$$

$$\frac{\partial^2 p}{\partial y \partial z} = 4^{2y \operatorname{sen} z} \left[x \cos z + xy \operatorname{sen}(2z) \ln 4 \right]$$

$$\frac{\partial^2 p}{\partial z \partial x} = y \cos z \; 4^{2y \operatorname{sen} z} \left[1 + 2y \operatorname{sen} z \ln 4 \right]$$

$$\frac{\partial^2 p}{\partial z^2} = xy \, 4^{2y \operatorname{sen} z} \left[- \operatorname{sen} z + 2y \cos z \ln 4 \right]$$

5 $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \iff 2 + 2\alpha = 0 \iff \alpha = -1$$