INSTITUTO SUPERIOR DE ENGENHARIA DE COIMBRA

DEPARTAMENTO DE FÍSICA E MATEMÁTICA

ENGENHARIA BIOMÉDICA – 1º ano /1º Semestre

Frequência-Parte I

06-Dez-2010 **Duração:2h**

Importante:
Não é permitida a consulta de quaisquer livros ou textos de apoio
A resolução completa de cada pergunta inclui a justificação do raciocínio utilizado bem como a apresentação de todos os cálculos efectuados.

1- Considere a função $f(x) = 1 + 2sen(3x - \frac{\pi}{3})$.

a) Determine o domínio e o contradomínio de f.

b) Determine os valores de x para os quais a função é não negativa.

c) Caracterize f^{-1}.

2- Considere a equação $sen(xy) + y^2x^3 = x$ que define implicitamente y como função de x.

a) Determine $y'(x)$.

b) Obtenha a equação da recta tangente à curva $y = f(x)$ no ponto $(0,1)$.

3- Determine a seguinte primitiva imediata $\int \frac{e^{x+1}}{\sqrt{4 - e^{2x}}} dx$.

4- Aplique o processo de primitivação por partes para calcular $\int \frac{x^2}{\sqrt[3]{x+1}} dx$.

5- Calcule a seguinte primitiva por substituição $\int \frac{1 - x^2}{\sqrt{4 - 9x^2}} dx$.

6- Usando a primitivação de fracções racionais determine $\int \frac{x^2 - 1}{x^4 - 3x^3 + 2x^2} dx$.

7- Calcule as seguintes primitivas:

a) $\int sen(\ln(x))dx$

b) $\int \frac{5}{(x+1)\ln^3(x+1)} dx$

c) $\int \frac{1}{\sqrt{4x} - \sqrt[4]{x}} dx$

d) $\int \frac{2}{\sqrt[3]{sen(x)sec^3x}} dx$

Cotação

1a	1b	1c	2a	2b	3	4	5	6	7a	7b	7c	7d
0,75	0,75	0,5	1	1	2	2	2	2	2	2	2	2

Resolução do teste 1 - 6.Dez.10

Eng. Biomédica

Cálculo 1

1. $f(x) = 1 + 2\,\text{sen}\left(3x - \frac{\pi}{3}\right)$

a) $Df = \mathbb{R}$

$CD = ?$

$$y = 1 + 2\,\text{sen}\left(3x - \frac{\pi}{3}\right)$$

$$y - 1 = 2\,\text{sen}\left(3x - \frac{\pi}{3}\right)$$

$$-1 \leq \text{sen}\left(3x - \frac{\pi}{3}\right) = \frac{y-1}{2} \leq 1$$

$$-2 \leq y \leq 2$$

$$1 - 2 \leq y \leq 1 + 2$$

$$y \in [-1, 3]$$

$CD = [-1, 3]$

b) $R = \{x \in \mathbb{R} : f(x) \geq 0\}$

$$f(x) \geq 0 \Leftrightarrow 1 + 2\,\text{sen}\left(3x - \frac{\pi}{3}\right) \geq 0$$

$$2\,\text{sen}\left(3x - \frac{\pi}{3}\right) \geq -1$$

$$\text{sen}\left(3x - \frac{\pi}{3}\right) \geq -\frac{1}{2}$$

$$2k\pi + \left(-\frac{\pi}{6}\right) \leq 3x - \frac{\pi}{3} \leq \pi + \frac{\pi}{6} + 2k\pi \quad , k \in \mathbb{Z}.$$

$$2k\pi - \frac{\pi}{6} \leq 3x - \frac{\pi}{3} \leq \frac{7\pi}{6} + 2k\pi$$

$$2k\pi + \frac{\pi}{6} \leq 3x \leq \frac{9\pi}{6} + 2k\pi$$

$$\frac{2k\pi}{3} + \frac{\pi}{18} \leq x \leq \frac{3.\pi}{6} + 2k\frac{\pi}{3} \quad , k \in \mathbb{Z}$$

c) f^{-1}

Restrição principal

$$D_f = \{x \in \mathbb{R} : (3x - \tfrac{\pi}{3}) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\}$$

$$= \{x \in \mathbb{R} : -\tfrac{\pi}{2} \leq 3x - \tfrac{\pi}{3} \leq \tfrac{\pi}{2}\}$$

$$= \{x \in \mathbb{R} : -\tfrac{\pi}{6} \leq 3x \leq \tfrac{5\pi}{6}\}$$

$$= \{x \in \mathbb{R} : -\tfrac{\pi}{18} \leq x \leq \tfrac{5\pi}{18}\}$$

$$\frac{y-1}{2} = \operatorname{sen}(3x - \tfrac{\pi}{3})$$

$$3x - \pi = \operatorname{arcsen}\left(\frac{y-1}{2}\right)$$

$$3x = \frac{\pi}{3} + \operatorname{arcsen}\left(\frac{y-1}{2}\right)$$

$$x = \frac{\pi}{9} + \frac{1}{3}\operatorname{arcsen}\left(\frac{y-1}{2}\right)$$

$$f^{-1} : [-1, 3] \longrightarrow [\tfrac{\pi}{18}, \tfrac{5\pi}{18}]$$

$$x \longmapsto \frac{\pi}{9} + \frac{1}{3}\operatorname{arcsen}\left(\frac{y-1}{2}\right)$$

2. $\operatorname{sen}(xy) + y^2x^3 = x$

a) $y' = ?$

$$(xy)'\cos(xy) + (y^2x^3)' = x'$$

$$(xy' + y)\cos(xy) + 2yy'x^3 + 3x^2y^2 = 1$$

$$xy'\cos(xy) + y\cos(xy) + 2yy'x^3 + 3x^2y^2 = 1$$

$$y'(x\cos(xy) + 2yx^3) = 1 - y\cos(xy) - 3x^2y^2$$

$$y' = \frac{1 - y\cos(xy) - 3x^2y^2}{x\cos(xy) + 2yx^3}$$

b) $y - y_0 = m(x - x_0)$ com $m = y'(x_0)$

$$m = y'(0) = \frac{1 - \cos 0 - 0}{\cos 0 + 0} = \frac{0}{0}$$ Indeterminação!

(Enunciado ERRADO!)

CORRECÇÃO:

$$\boxed{\operatorname{sen}(xy) + y^2x^3 = -x}$$

3. $$\int \frac{e^{x+1}}{\sqrt{4 - e^{2x}}}\,dx = \int \frac{e^x \cdot e}{\sqrt{4\left(1 - \frac{e^{2x}}{4}\right)}}\,dx = e \int \frac{e^x}{2\sqrt{1 - \left(\frac{e^x}{2}\right)^2}}\,dx$$

$$= \frac{e}{2} \times 2 \int \frac{\frac{1}{2}e^x}{\sqrt{1 - \left(\frac{e^x}{2}\right)^2}}\,dx = e \operatorname{arcsen} \frac{e^x}{2} + C, \quad C \in \mathbb{R}$$

4. $$\int \frac{x^2}{\sqrt[3]{x+1}}\,dx = \int \underbrace{x^2}_{D} \underbrace{(x+1)^{-1/3}}_{P}\,dx =$$

$$= \int (x+1)^{-1/3}dx \; x^2 - \int \left[\int (x+1)^{-1/3}dx \; (x^2)'\right] dx$$

$$= \frac{(x+1)^{2/3}}{2/3} x^2 - \int \left[\frac{(x+1)^{2/3}}{2/3} \cdot 2x\right] dx$$

$$= \frac{3}{2}(x+1)^{2/3} \cdot x^2 - \frac{3}{2} \cdot 2 \int \underbrace{(x+1)^{2/3}}_{P} \cdot \underbrace{x}_{D}\,dx$$

$$= \frac{3}{2}(x+1)^{2/3} \cdot x^2 - 3\left[\frac{(x+1)^{5/3}}{5/3} \cdot x - \int \left[\frac{(x+1)^{5/3}}{5/3} \cdot 1\right]\right] dx$$

$$= \frac{3}{2}(x+1)^{2/3} \cdot x^2 - \frac{9}{5}(x+1)^{5/3} \cdot x + \frac{9}{5}\,\frac{(x+1)^{8/3}}{8/3} + C, \quad C \in \mathbb{R}$$

5. $\displaystyle\int \frac{1-x^2}{\sqrt{4-9x^2}}\,dx =$ $\qquad x=\frac{2}{3}\operatorname{sen} t \qquad a=2$

$\qquad\qquad dx=\frac{2}{3}\cos t\,dt \qquad b=3$

$$= \int \frac{1-\frac{4}{9}\operatorname{sen}^2 t}{\sqrt{4-9\left(\frac{4}{9}\operatorname{sen}^2 t\right)}}\cdot\frac{2}{3}\cos t\,dt$$

$$= \int \frac{1-\frac{4}{9}\operatorname{sen}^2 t}{\sqrt{4-4\operatorname{sen}^2 t}}\cdot\frac{2}{3}\cos t\,dt$$

$$= \int \frac{1-\frac{4}{9}\operatorname{sen}^2 t}{2\sqrt{1-\operatorname{sen}^2 t}}\cdot\frac{2}{3}\cos t\,dt = \frac{1}{2}\int \frac{1-\frac{4}{9}\operatorname{sen}^2 t}{\cancel{\cos t}}\,\frac{2}{3}\cancel{\cos t}\,dt$$

$$= \frac{1}{3}\int 1-\frac{4}{9}\operatorname{sen}^2 t\,dt = \frac{1}{3}\left[t-\int\frac{4}{9}\operatorname{sen}^2 t\,dt\right] =$$

$$= \frac{1}{3}t-\frac{4}{27}\int \operatorname{sen}^2 t\,dt = \frac{1}{3}t-\frac{4}{27}\int \frac{1-\cos 2t}{2}\,dt$$

$$= \frac{1}{3}t-\frac{2}{27}\int 1-\cos 2t\,dt = \frac{1}{3}t-\frac{2}{27}\left[\int 1\,dt-\int \cos 2t\,dt\right.$$

$$= \frac{1}{3}t-\frac{2}{27}\left[t-\frac{1}{2}\operatorname{sen} 2t\right]+C$$

$$= \frac{1}{3}t-\frac{2}{27}t+\frac{1}{27}\operatorname{sen} 2t+C,\quad C\in\mathbb{R}$$

Retomar x:

$$x=\frac{2}{3}\operatorname{sen} t \Rightarrow \operatorname{sen} t=\frac{3}{2}x \Rightarrow t=\operatorname{arcsen}\frac{3}{2}x$$

$$\int \frac{1-x^2}{\sqrt{4-9x^2}}\,dx = \frac{7}{27}\operatorname{arcsen}\frac{3}{2}x+\frac{1}{27}\operatorname{sen} 2\left(\operatorname{arcsen}\frac{3}{2}x\right)+C,\ C\in$$

6. $\int \frac{x^2-1}{x^4-3x^3+2x^2}\,dx$

Passo 1. Factorização do denominador

$$x^4-3x^3+2x^2 = x^2(x^2-3x+2)$$

$$x^2-3x+2=0 \Leftrightarrow x = \frac{+3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$$

$$\Leftrightarrow x=2 \quad \vee x=1$$

$$x^4-3x^3+2x^2 = x^2(x-1)(x-2)$$

Passo 2. Decomposição em elementos simples

$$\frac{x^2-1}{x^4-3x^3+2x^2} = \frac{x^2-1}{x^2(x-1)(x-2)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x-2}$$

Cálculo da constante

$$x^2-1 = A(x-1)(x-2) + Bx(x-1)(x-2) + Cx^2(x-2) + Dx^2(x$$

$$\begin{array}{l} x=0 \\ x=1 \\ x=2 \\ x=-1 \end{array} \left\{ \begin{array}{lll} -1 = A(-1)(-2) + 0 + 0 + 0 & \Rightarrow & A = -1/2 \\ 0 = 0 + 0 + C(1)(-1) + 0 & \Rightarrow & C = 0 \\ 3 = 0 + 0 + 0 + D(4)(1) & \Rightarrow & D = 3/4 \\ 0 = A(-2)(-3) + B(-1)(-2)(-3) + C\,1(-3) + D(1)(-2) & & \end{array} \right.$$

$$\left\{ \begin{array}{l} A = -1/2 \\ C = 0 \\ D = 3/4 \\ 0 = -3 + (-6B) + 0 + \left(-\frac{3}{2}\right) \Rightarrow B = -\frac{3}{4} \end{array} \right.$$

Passo 3. $\displaystyle\int \frac{x^2-1}{x^4-3x^3+2x^2}\,dx = \int \frac{-1/2}{x^2} + \frac{-3/4}{x} + \frac{3/4}{x-2}\,dx$

$\displaystyle = -\frac{1}{2}\frac{x^{-1}}{-1} - \frac{3}{4}\ln|x| + \frac{3}{4}\ln|x-2| + C,\ C\in\mathbb{R}$

7\. a) $\displaystyle \underbrace{\boxed{\int \sin(\ln(x))\,dx}}^{A} = \int 1\cdot \sin(\ln x)\,dx =$

$\displaystyle = x\sin(\ln(x)) - \int x\cos(\ln x)\cdot\frac{1}{x}\,dx$

$\displaystyle = x\sin(\ln(x)) - \int \cos(\ln x)\,dx$

$\displaystyle = x\sin(\ln(x)) - \left[x\cos(\ln x) + \int \cancel{x}\,\mathrm{sen}(\ln x)\frac{1}{\cancel{x}}\,dx\right]$

$\displaystyle = x\sin(\ln(x)) - x\cos(\ln x) - \underbrace{\boxed{\int \mathrm{sen}(\ln x)\,dx}}_{A}$

$\Rightarrow$

$\displaystyle A = \frac{1}{2}\Big(x\sin(\ln(x)) - x\cos(\ln(x))\Big) + C,\ C\in\mathbb{R}$

b) $\displaystyle \int \frac{5}{(x+1)\ln^3(x+1)}\,dx = 5\int \frac{1}{x+1}\ln^{-3}(x+1)\,dx$

$\displaystyle (\ln(x+1))' = \frac{1}{x+1}$

$\displaystyle = 5\,\frac{\ln^{-2}(x+1)}{-2} + C,\ C\in\mathbb{R}$

i) $\int \frac{1}{2t^2-t} \cdot 4t^3\, dt = \int \frac{4t^3}{t(2t-1)}\, dt$

$x = t^4 \qquad 4 = m.m.c(2,4)$

$dx = 4t^3\, dt$

$= \int \frac{1}{2t-1} \cdot 4t^2\, dt = 4\int \frac{t^2}{2t-1}\, dt$

$= 4\int \frac{1}{2}t + \frac{1}{4} + \frac{\frac{1}{4}}{2t-1}\, dt$

$$\begin{array}{l|l} t^2 & 2t-1 \\ \hline -t^2 + \frac{1}{2}t & \frac{1}{2}t + \frac{1}{4} \\ \frac{1}{2}t & \\ -\frac{1}{2}t + \frac{1}{4} & \\ \frac{1}{4} & \end{array}$$

$= 2\frac{t^2}{2} + t + \frac{1}{2}\ln|2t-1| + C,\ C\in\mathbb{R}$

Retoma x:

$x = t^4$

$t = \sqrt[4]{x}$

$= 2\frac{\sqrt[4]{x^2}}{2} + \sqrt[4]{x} + \frac{1}{2}\ln|2\sqrt[4]{x} - 1| + C,\ C\in\mathbb{R}$

l) $\int \frac{2}{\sqrt[3]{\sin(x)\sec^3(x)}}\, dx = 2\int (\sin x)^{-1/3}\cos^3(x)\, dx =$

$= 2\int (\sin x)^{-1/3}\cos x\,(\cos^2 x)\, dx = 2\int (\sin x)^{-1/3}\cos x\,(1-\sen^2 x)\, dx$

$= 2\int (\sin x)^{-1/3}\cos x - (\sin x)^{+5/3}\cos x\, dx = 2\frac{(\sin x)^{2/3}}{2/3} - \frac{2(\cos x)^{8/3}}{8/3} + C$

$C\in\mathbb{R}$

$= 3(\sin x)^{2/3} - \frac{3}{4}(\cos x)^{8/3} + C,\ C\in\mathbb{R}$