7) b)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{h^3+0^3}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = \lim_{h\to 0} 1 = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\frac{0^3+h^3}{0^2+h^2} - 0}{h} = 1$$

8) a) $$\frac{\partial f}{\partial x} = 3x^2y + 14x$$

$$\frac{\partial f}{\partial y} = x^3 - 6y^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2y + 14x\right) = 6xy + 14x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2y + 14x\right) = 3x^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^3 - 6y^2\right) = 3x^2$$

Teorema di Schwarz

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^3 - 6y^2\right) = -12y$$

b) $$\frac{\partial g}{\partial x} = \frac{3(7x+y) - 7(3x+y^2)}{(7x+y)^2} = \frac{21x + 3y - 21x - 7y^2}{(7x+y)^2} = \frac{3y-7y^2}{(7x+y)^2}$$

$$\frac{\partial g}{\partial y} = \frac{2y(7x+y) - 1(3x+y^2)}{(7x+y)^2} = \frac{14xy + 2y^2 - 3x - y^2}{(7x+y)^2} = \frac{y^2 + 14xy - 3x}{(7x+y)^2}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{0(7x+y)^2 - 2(7x+y)\cdot 7(3y-7y^2)}{(7x+y)^4} = \frac{-14(3y-7y^2)}{(7x+y)^3}$$

$$
\left.
\begin{aligned}
\frac{\partial^2 g}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{(3-14y)(7x+y)^2 - 2(7x+y)\times 1\,(3y-7y^2)}{(7x+y)^4} \\
&= \frac{(3-14y)(7x+y) - 2(3y-7y^2)}{(7x+y)^3} = \frac{21x - 3y - 98xy}{(7x+y)^3} \\
\frac{\partial^2 g}{\partial y \partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{(14y-3)(7x+y)^2 - 2(7x+y)\times 7\,(y^2+14xy-3x)}{(7x+y)^4} \\
&= \frac{(14y-3)(7x+y) - 14(y^2+14xy-3x)}{(7x+y)^3} = \frac{21x-3y-98xy}{(7x+y)^3}
\end{aligned}
\right\} \text{Teorema de Schwarz}
$$

8)

e) $u(x,y) = \operatorname{sen}\left(1+e^{xy}\right)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(1+e^{xy}\right)\cos\left(1+e^{xy}\right) = \frac{\partial}{\partial x}(xy)\, e^{xy}\cos\left(1+e^{xy}\right)$$

$$= y\, e^{xy}\cos\left(1+e^{xy}\right)$$

$$\frac{\partial f}{\partial y} = x\, e^{xy}\cos\left(1+e^{xy}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(y e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial x}\cos\left(1+e^{xy}\right) y e^{xy}$$

$$= y^2 e^{xy}\cos\left(1+e^{xy}\right) + \left(-y e^{xy}\operatorname{sen}\left(1+e^{xy}\right)\right) y e^{xy}$$

$$= y^2 e^{xy}\left[\cos\left(1+e^{xy}\right) - \operatorname{sen}\left(1+e^{xy}\right)\right]$$

8 e)

$$\left.\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial y}\cos\left(1+e^{xy}\right) y e^{xy} \\
&= \left(e^{xy} + xy e^{xy}\right)\cos\left(1+e^{xy}\right) - x e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right) y e^{xy} \\
&= e^{xy}\left[\left(1+xy\right)\cos\left(1+e^{xy}\right) - xy e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right)\right] \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial x}\cos\left(1+e^{xy}\right) x e^{xy} \\
&= \left(e^{xy} + xy e^{xy}\right)\cos\left(1+e^{xy}\right) - y e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right) x e^{xy} \\
&= e^{xy}\left[\left(1+xy\right)\cos\left(1+e^{xy}\right) - xy e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right)\right]
\end{aligned}\right\}\ \text{T.S.}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial y}\cos\left(1+e^{xy}\right) x e^{xy}$$

$$= x^2 e^{xy}\cos\left(1+e^{xx}\right) - x e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right) x e^{xy}$$

$$= x^2 e^{xy}\left[\cos\left(1+e^{xy}\right) - e^{xy}\,\mathrm{sen}\left(1+e^{xx}\right)\right]$$

9.

$$z = \left(x^2+y^2\right)^{1/3}$$

$$\frac{\partial z}{\partial x} = \frac{1}{3}\left(x^2+y^2\right)^{-2/3} \times 2x = \frac{2}{3} x \left(x^2+y^2\right)^{-2/3}$$

$$\frac{\partial z}{\partial y} = \frac{1}{3}\left(x^2+y^2\right)^{-2/3} \times 2y = \frac{2}{3} y \left(x^2+y^2\right)^{-2/3}$$

⑨ (cont.)

Não é necessário calcular $\frac{\partial^2 z}{\partial x^2}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{2x}{3} \times \frac{\partial}{\partial y}\left(x^2+y^2\right)^{-2/3}$$

$$= \frac{2x}{3} \times \left(-\frac{2}{3}\right)\left(x^2+y^2\right)^{-5/3} \times 2y = -\frac{8}{9} xy \left(x^2+y^2\right)^{-5/3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left[\left(\frac{2}{3}\right)\right] \times \left(x^2+y^2\right)^{-2/3} + \left[\left(-\frac{2}{3}\right)\left(x^2+y^2\right)^{-5/3} \times 2y\right] \times \frac{2}{3} y$$

$$= \frac{2}{3}\left[\left(x^2+y^2\right)^{-2/3} - \frac{4}{3} y^2 \left(x^2+y^2\right)^{-5/3}\right]$$

$$= \frac{2}{3}\left(x^2+y^2\right)^{-2/3}\left[1 - \frac{4}{3}\left(x^2+y^2\right)^{-1}\right]$$

$$3x \frac{\partial^2 z}{\partial x \partial y} + 3y \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = 0 \Leftrightarrow$$

$$\Leftrightarrow 3x\left[-\frac{8}{9} xy \left(x^2+y^2\right)^{-5/3}\right] + 3y\left[\frac{2}{3}\left(x^2+y^2\right)^{-2/3} - \frac{8}{9} y^2 \left(x^2+y^2\right)^{-5/3}\right] +$$

$$+ \frac{2}{3} y \left(x^2+y^2\right)^{-2/3} = 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{8}{3} x^2 y \left(x^2+y^2\right)^{-5/3} + 2y\left(x^2+y^2\right)^{-2/3} - \frac{8}{3} y^3 \left(x^2+y^2\right)^{-5/3} + \frac{2}{3} y \left(x^2+y^2\right)^{-2/3} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(x^2+y^2\right)^{-5/3}\left[-\frac{8}{3} x^2 y - \frac{8}{3} y^3\right] + \left(x^2+y^2\right)^{-2/3}\left[2y + \frac{2}{3} y\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(x^2+y^2\right)^{-5/3}\left[-\frac{8}{3} x^2 y - \frac{8}{3} y^3 + \left(x^2+y^2\right)\frac{8}{3} y\right] = 0 \Leftrightarrow$$

9 (cont.)

$$\Leftrightarrow (x^2+y^2)^{-5/3}\left(-\frac{8}{3}x^2y-\frac{8}{3}y^3+\frac{8}{3}x^2y+\frac{8}{3}y^3\right)=0 \Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2)^{-5/3}\times 0=0 \Leftrightarrow 0=0$$

10. $$v(x,t)=t^{-1/2}\exp\left(-\frac{x^2}{4Kt}\right)$$

$$\frac{\partial v}{\partial t}=-\frac{1}{2}t^{-3/2}\times 1\times\exp\left(-\frac{x^2}{4Kt}\right)+\frac{\partial}{\partial t}\left(-\frac{x^2}{4Kt}\right)\times\exp\left(-\frac{x^2}{4Kt}\right)\times t^{-1/2}$$

$$=\left[-\frac{1}{2}t^{-3/2}+\frac{x^2}{4Kt^2}\times t^{-1/2}\right]\exp\left(-\frac{x^2}{4Kt}\right)$$

$$=\left(-\frac{1}{2}t^{-1}+\frac{x^2}{4Kt^2}\right)t^{-1/2}\exp\left(\frac{-x^2}{4Kt}\right)$$

$$\frac{\partial v}{\partial x}=t^{-1/2}\frac{\partial}{\partial x}\left(-\frac{x^2}{4Kt}\right)\exp\left(-\frac{x^2}{4Kt}\right)=-\frac{1}{2}t^{-1/2}\frac{x}{Kt}\exp\left(-\frac{x^2}{4Kt}\right)$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=-\frac{t^{-1/2}}{2Kt}\times\frac{\partial}{\partial x}\left[x\times\exp\left(-\frac{x^2}{4Kt}\right)\right]=$$

$$=-\frac{t^{-1/2}}{2Kt}\left[1\times\exp\left(-\frac{x^2}{4Kt}\right)+\frac{\partial}{\partial x}\left(-\frac{x^2}{4Kt}\right)\exp\left(\frac{-x^2}{4Kt}\right)\times x\right]=$$

$$=-\frac{t^{-3/2}}{2K}\exp\left(-\frac{x^2}{4Kt}\right)\left[1-\frac{2x}{4Kt}x\right]=-\frac{t^{-3/2}}{2K}\left(1-\frac{2x^2}{4Kt}\right)\exp\left(-\frac{x^2}{4Kt}\right)=$$

$$=\left(-\frac{1}{2K}t^{-3/2}+\frac{x^2}{4K^2t^2}t^{-1/2}\right)\exp\left(-\frac{x^2}{4Kt}\right)=\left(-\frac{1}{2K}t^{-1}+\frac{x^2}{4K^2t^2}\right)t^{-1/2}\exp\left(-\frac{x^2}{4Kt}\right)$$

10 (cont.)

$$K \frac{\partial^2 v}{\partial x^2} = K\left(-\frac{1}{2K} t^{-1} + \frac{x^2}{4K^2t^2}\right) t^{-1/2} \exp\left(-\frac{x^2}{4Kt}\right) =$$

$$= \left(-\frac{1}{2} t^{-1} + \frac{x^2}{4Kt^2}\right) e^{-1/2} \exp\left(-\frac{x^2}{4Kt}\right) = \frac{\partial v}{\partial t} \quad //$$

11. Considere a função real definida em $\mathbf{R}^2$ $f(x,y) = \begin{cases} x+y \text{ se } xy = 0 \\ 1 \text{ se } xy \neq 0 \end{cases}$

a) Verifique se existem as derivadas parciais de primeira ordem no ponto $(0,0)$.

b) Mostre que f não é contínua em $(0,0)$.

c) f é ou não uma função diferenciável?

12. Calcule o diferencial de f (df) da função definida do seguinte modo $f(x,y,z,t) = 3x - 2y^2 - z^3 + t$.

13. Usando diferenciais calcule um valor aproximado de $\ln\left(1.01^2 + 0.02^3\right)$.

(11)

$$f(x,y) = \begin{cases} x+y & \text{se } xy = 0 \\ 1 & \text{se } xy \neq 0 \end{cases}$$

a) $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h-0-0}{h} = \lim_{h\to 0} 1 = 1$

$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

As derivadas parciais existem e são finitas no ponto $(0,0)$.

b) $\left.\begin{array}{l} \lim_{x\to 0}\left(\lim_{y\to 0} x+y\right) = \lim_{x\to 0} x = 0 \\ \lim_{y\to 0}\left(\lim_{x\to 0} x+y\right) = \lim_{y\to 0} y = 0 \end{array}\right\}$ limites iterados

O limite se existir é zero

Vamos verificar o limite ao longo das rectas $y = mx$ com $m \neq 0$

$\lim_{\substack{x\to 0 \\ y=mx}} f(x,mx) = \lim_{x\to 0} 1 = 1$

Nota: ao longo das rectas que não os eixos coordenados, a função assume o valor 1

c) Não sendo a função contínua no ponto $(0,0)$ não é diferenciável nesse ponto

Se se verificar que $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas numa vizinhança do ponto (x_0,y_0), então f é diferenciável em

__Se__ $f(x,y)$ é diferenciável em (x_0,y_0) __então__:

- $f(x,y)$ é contínua em (x_0,y_0)
- $f(x,y)$ admite derivadas parciais de 1.ª ordem em (x_0,y_0)

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Delta f(x_0,y_0) = f(x_0+h, y_0+k) - f(x_0,y_0) \Leftrightarrow f(x_0+h, y_0+k) = \Delta f(x_0,y_0) + f(x_0,y_0)$$

$$f(x_0+h, y_0+k) \simeq df(x_0,y_0) + f(x_0,y_0)$$

(12) $f(x,y,z,t) = 3x - 2y^2 - z^3 + t$

$\frac{\partial f}{\partial x} = 3$; $\frac{\partial f}{\partial y} = -4y$; $\frac{\partial f}{\partial z} = -3z^2$; $\frac{\partial f}{\partial t} = 1$

$df(x,y,z,t) = 3dx - 4y\,dy - 3z^2dz + dt$

(13) $f(x,y) = \ln(x^2+y^3)$

$(x_0,y_0) = (1,0)$

$dx = 1{,}01 - 1 = 0{,}01 = \frac{1}{100}$ $\qquad dy = 0{,}02 - 0 = 0{,}02 = \frac{2}{100}$

$f(1+0{,}01, 0+0{,}02) \simeq df(1,0) + f(1,0)$

$f(1,0) = \ln(1^2+0^3) = \ln 1 = 0$

$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^3}$ $\qquad \frac{\partial f}{\partial x}(1,0) = \frac{2}{1+0} = 2$

$\frac{\partial f}{\partial y} = \frac{3y^2}{x^2+y^3}$ $\qquad \frac{\partial f}{\partial y}(1,0) = \frac{0}{1+0} = 0$

$df(1,0) = 2 \times \frac{1}{100} + 0 \times \frac{2}{100} = \frac{2}{100}$

$f(1{,}01; 0{,}02) \simeq \frac{2}{100} + 0 = 0{,}02$ $\qquad$ Valor Real $= 0{,}01990950$

erro inferior $0{,}0001$