Resolução da ficha 4-A

1.a) $P \dfrac{1}{2(x-1)(x+3)} = \dfrac{1}{2} P \dfrac{1}{(x-1)(x+3)}$

Decompor a fracção $\dfrac{1}{(x-1)(x+3)}$:

$$\frac{1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{A(x+3) + B(x-1)}{(x-1)(x+3)}$$

$$\frac{1}{(x-1)(x+3)} = \frac{(A+B)x + (3A-B)}{(x-1)(x+3)}$$

Desta igualdade de funções racionais, temos que

$$\begin{cases} A+B=0 \\ 3A-B=1 \end{cases} \quad \begin{cases} B=-A \\ 3A+A=1 \end{cases} \quad \begin{cases} B=-1/4 \\ A=1/4 \end{cases}.$$

Assim,

$$P \frac{1}{2(x-1)(x+3)} = \frac{1}{2}\left[P \frac{1/4}{x-1} + P \frac{-1/4}{x+3}\right] =$$

$$= \frac{1}{8}\left[P \frac{1}{x-1} - P \frac{1}{x+3}\right] =$$

$$= \frac{1}{8}\left[\ln|x-1| - \ln|x+3|\right] + C$$

$$= \ln \sqrt[8]{\left|\frac{x-1}{x+3}\right|} + C$$

1.b) $P \dfrac{x^5+x^4-8}{x^3-4x}$

Dividindo

$$
\begin{array}{rl|l}
\cancel{x^5} + x^4 - 8 & & x^3-4x \\
\hline
-\cancel{x^5} + 4x^3 & & x^2+x+4 \\
\hline
\cancel{x^4} + 4x^3 - 8 & & \\
-\cancel{x^4} + 4x^2 & & \\
\hline
\cancel{4x^3} + 4x^2 - 8 & & \\
-\cancel{4x^3} + 16x & & \\
\hline
4x^2 + 16x - 8 & &
\end{array}
$$

Assim,

$$P \frac{x^5+x^4-8}{x^3-4x} = P\left(x^2+x+4\right) + P \frac{4x^2+16x-8}{x^3-4x}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 4P \frac{x^2+4x-2}{x(x^2-4)}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 4P \frac{x^2+4x-2}{x(x-2)(x+2)}$$

Decompor a função racional em fracções elementares

$$\frac{x^2+4x-2}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$= \frac{A(x^2-4) + Bx(x+2) + Cx(x-2)}{x(x^2-4)}$$

$$= \frac{A(x^2-4) + B(x^2+2x) + C(x^2-2x)}{x(x^2-4)}$$

$$\frac{x^2+4x-2}{x(x^2-4)} = \frac{(A+B+C)x^2 + (2B-2C)x - 4A}{x(x^2-4)}$$

Da igualdade entre funções racionais, tem-se

$$\begin{cases} A+B+C=1 \\ 2B-2C=4 \\ -4A=-2 \end{cases} \quad \begin{cases} A+B+C=1 \\ B-C=2 \\ A=1/2 \end{cases} \quad \begin{cases} B+C=1/2 \\ B-C=2 \\ - \end{cases} \quad \begin{cases} C=-3/4 \\ B=5/4 \\ A=1/2 \end{cases}$$

Assim,

$$P\,\frac{x^2+4x-2}{x^3-4x} = P\,\frac{1/2}{x} + P\,\frac{5/4}{x-2} + P\,\frac{-3/4}{x+2} =$$

$$= \frac{1}{2}\ln|x| + \frac{5}{4}\ln|x-2| - \frac{3}{4}\ln|x+2| + C$$

$$= \ln\sqrt{|x|} + \ln\sqrt[4]{|x-2|^5} - \ln\sqrt[4]{|x+2|^3} + C$$

$$= \ln\left[\sqrt{|x|}\,\sqrt[4]{\frac{|x-2|^5}{|x+2|^3}}\right] + C$$

e

$$P\,\frac{x^5+x^4-8}{x^3-4x} = \frac{x^3}{3} + \frac{x^2}{2} + 4x + \ln\left[\frac{|x|^2\,|x-2|^5}{|x+2|^3}\right] + C$$

1.c) $$P\,\frac{x+1}{x(x-1)^2} = P\left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right]$$

$$\frac{x+1}{x(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

$$= \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2}$$

$$\frac{x+1}{x(x-1)^2} = \frac{(A+B)x^2 + (-2A-B+C)x + A}{x(x-1)^2}$$

$$\begin{cases} A+B=0 \\ -2A-B+C=1 \\ A=1 \end{cases} \quad \begin{cases} B=-1 \\ \text{—} \\ A=1 \end{cases} \quad \begin{cases} -2\,[\text{illegible}]+1+C=1 \end{cases} \Leftrightarrow \begin{cases} B=-1 \\ C=2 \\ A=1 \end{cases}$$

Assim,

$$P\frac{x+1}{x(x-1)^2} = P\frac{1}{x} - P\frac{1}{x-1} + 2P\frac{1}{(x-1)^2}$$

$$= \ln|x| - \ln|x-1| - \frac{2}{x-1} + C$$

$$= \ln\left|\frac{x}{x-1}\right| - \frac{2}{x-1} + C.$$

d) $P\dfrac{x}{(x^2+1)(x-1)^2}$

Decompor: $$\frac{x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{(Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$= \frac{(Ax^3-2Ax^2+Ax+Bx^2-2Bx+B) + C(x^3+x-x^2-1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$= \frac{(A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + B-C+D}{(x^2+1)(x-1)^2}$$

$$\begin{cases} A+C=0 \\ -2A+B-C+D=0 \\ A-2B+C=1 \\ B-C+D=0 \end{cases} \quad \begin{cases} C=-A \\ \text{—} \\ \text{—} \\ D=C-B \end{cases} \quad \begin{cases} C=-A \\ \\ \\ D=-A-B \end{cases} \quad \begin{cases} -2A+\cancel{B}+\cancel{A}-\cancel{A}-\cancel{B}=0 \\ \cancel{A}-2B-\cancel{A}=1 \end{cases}$$

$$\begin{cases} C=-A \\ A=0 \\ B=-1/2 \\ D=-A-B \end{cases} \qquad \begin{cases} C=0 \\ A=0 \\ B=-1/2 \\ D=1/2 \end{cases}$$

$$P \frac{x}{(x^2+1)(x-1)^2} = -1/2 \ P \frac{1}{x^2+1} + \frac{1}{2} P \frac{1}{(x-1)^2}$$

$$= -\frac{1}{2} \operatorname{arctg} x - \frac{1}{2} \cdot \frac{1}{(x-1)} + C$$

2a) $P \dfrac{x^2}{\sqrt{1-x^2}} =$ utilizando a substituição $x = \operatorname{sen} t$

$$\frac{dx}{dt} = \cos t$$

$$= P \frac{\operatorname{sen}^2 t}{\sqrt{1-\operatorname{sen}^2 t}} \cdot \cos t = P \frac{\operatorname{sen}^2 t}{\cancel{\cos t}} \cdot \cancel{\cos t} = P \operatorname{sen}^2 t = P \left(\frac{1-\cos(2t)}{2} \right)$$

$$= \frac{1}{2} P (1-\cos(2t)) = \frac{1}{2} \left(t - \frac{\operatorname{sen}(2t)}{2} \right) + C$$

onde $t = \operatorname{arcsen} x$ e $\operatorname{sen} 2t = 2 \cos t \cdot \operatorname{sen} t = 2\sqrt{1-x^2} \cdot x$

Assim

$$P \frac{x^2}{\sqrt{1-x^2}} = \frac{1}{2} \left(\operatorname{arcsen} x - x\sqrt{1-x^2} \right) + C$$

b) $P\, x(x+3)^{1/3}$

$$x+3 = t^3$$
$$x = t^3 - 3$$
$$\frac{dx}{dt} = 3t^2$$

Assim $P\, x(x+3)^{1/3} = P\, (t^3-3) \cdot t \cdot 3t^2 = 3P\left(t^6 - 3t^3\right)$

$$= 3\left[\frac{t^7}{7} - 3\frac{t^4}{4}\right] + C$$

and $t = (x+3)^{1/3}$

$$= \frac{3}{7}(x+3)^{7/3} - \frac{9}{4}(x+3)^{4/3} + C$$

e) $P \dfrac{\sqrt{x}}{x - \sqrt[3]{x}}$, $x = t^6$

$$\frac{dx}{dt} = 6t^5$$

$$P \frac{\sqrt{x}}{x-\sqrt[3]{x}} = P \frac{t^{6/2}}{t^6 - t^{6/3}} . 6t^5 = P \frac{t^3}{t^6 - t^2} . 6t^5 =$$

$$= 6P \frac{t^8}{t^6 - t^2} = 6P \frac{t^6}{t^4 - 1} = 6P\left(t^2 + \frac{t^2}{t^4 - 1}\right) =$$

Dividindo

$$\begin{array}{r|l} t^6 & t^4 - 1 \\ \hline -\cancel{t^6} + t^2 & t^2 \\ t^2 & \end{array}$$

$$6\left[\frac{t^3}{3} + P \frac{t^2}{(t^2-1)(t^2+1)}\right] = 2t^3 + 6P \frac{t^2}{(t-1)(t+1)(t^2+1)} =$$

$$= 2t^3 + 6P\left[\frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct + D}{t^2+1}\right]$$

$$= 2t^3 + 6P\left[\frac{A(t+1)(t^2+1) + B(t-1)(t^2+1) + (Ct+D)(t^2-1)}{(t^2-1)(t^2+1)}\right]$$

$$\begin{cases} A + B + C = 0 \\ A - B + D = 1 \\ A + B - C = 0 \\ A - B - D = 0 \end{cases} \quad \begin{cases} C = -B - A \\ 2A - 2B = 1 \\ 2A + 2B = 0 \\ D = A - B \end{cases} \quad \begin{cases} - \\ 2A + 2A = 1 \\ B = -A \\ - \end{cases} \quad \begin{cases} C = 0 \\ A = 1/4 \\ B = -1/4 \\ D = 1/2 \end{cases}$$

Assim,

$$2t^3 + 6\,P\left[\frac{1/4}{t-1} - \frac{1/4}{t+1} + \frac{1/2}{t^2+1}\right] =$$

$$= 2t^3 + \frac{3}{2}\ln|t-1| - \frac{3}{2}\ln|t+1| + 3\,\text{arctg}\,t + C$$

$$= 2t^3 + 3\,\text{arctg}\,t + \ln\sqrt{\left|\frac{t-1}{t+1}\right|^3} + C$$

onde $t = \sqrt[6]{x}$

$$= 2\sqrt{x} + 3\,\text{arctg}\,\sqrt[6]{x} + \ln\sqrt{\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|^3} + C .$$