1) a) $f(x,y) = \begin{cases} \dfrac{2xy}{5x^2-y^2} & \text{se } (x,y) \neq (0,0) \\ 1 & \text{se } (x,y) = (0,0) \end{cases}$

$D_f = \{(x,y) \in \mathbb{R}^2 : y \neq \pm\sqrt{5}x\} \cup \{(0,0)\}$

$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0 \Rightarrow$ o limite, se existir, é zero

para $\dfrac{2xy}{5x^2-y^2}$ nos pontos $(x,y) \neq (0,0)$ e $y \neq \pm\sqrt{5}x$: para esses pontos, a expressão de $f$ é um quociente de polinómios cujo denominador não se anula no $D_f$. O limite $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{5x^2-y^2}$ se existir é zero e $f(0,0)=1$. Logo a função não é contínua em $(0,0)$.

$\therefore$ A função é contínua no $D_f \setminus \{(0,0)\}$.

b) $f(x,y) = \begin{cases} \dfrac{xy}{x+1} & \text{se } x \neq -1 \\ 0 & \text{se } x = -1 \end{cases}$

$D_f = \mathbb{R}^2$

- para $x=-1$ temos que estudar se existe limite e se este é igual para $x=-1$, isto é,

$$\lim_{(x,y)\to(-1,y)} \frac{xy}{x+1} = f(-1,y)$$

$\lim_{x\to -1}\left(\lim_{y\to y} \dfrac{xy}{x+1}\right) = \lim_{x\to -1} \dfrac{xy}{x+1} = \dfrac{-y}{0} \to \infty$ excepto quando $y=0$.

Não existe limite.

$\therefore$ A função é contínua em todo o seu domínio excepto para $x=-1$.

② Sabemos que $\lim\limits_{u \to 0} \frac{\operatorname{sen} u}{u} = 1$.

Então $\lim\limits_{(x,y)\to(0,0)} \frac{\operatorname{Sen}(x^2+y^2)}{x^2+y^2} = 1$

$\therefore$ $g(x,y)$ é contínua em $\mathbb{R}^2$ se $K=1$

③ a) $f(x,y) = \frac{x-y}{x+y}$

$$f(2,-1) = \frac{2-(-1)}{2+(-1)} = 3$$

$$\frac{\partial f}{\partial x}(2,-1) = \lim_{h\to 0} \frac{f(2+h,-1) - f(2,1)}{h} = \lim_{h\to 0} \frac{\frac{2+h+1}{2+h-1} - 3}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{3+h}{1+h} - \frac{3(1+h)}{1+h}}{h} = \lim_{h\to 0} \frac{-2h}{h(1+h)} = \lim_{h\to 0} \frac{-2}{h+1} = -2$$

$$\frac{\partial f}{\partial y}(2,-1) = \lim_{h\to 0} \frac{f(2,h-1) - f(2,1)}{h} = \lim_{h\to 0} \frac{\frac{2-(h-1)}{2+(h-1)} - 3}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{3-h}{h+1} - 3}{h} = \lim_{h\to 0} \frac{3-h-3(h+1)}{h(h+1)} =$$

$$= \lim_{h\to 0} \frac{h-3h}{h(h+1)} = \lim_{h\to 0} \frac{-4h}{h(h+1)} = -4$$

b) $$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

$$f(0,0) = 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{h^3+0^3}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = \lim_{h\to 0} 1 = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\frac{0^3+h^3}{0^2+h^2} - 0}{h} = 1$$

(4) a) $$\frac{\partial f}{\partial x} = 3x^2y + 14x$$

$$\frac{\partial f}{\partial y} = x^3 - 6y^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2y + 14x\right) = 6xy + 14x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2y + 14x\right) = 3x^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^3 - 6y^2\right) = 3x^2$$

Teorema di Schwarz

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^3 - 6y^2\right) = -12y$$

b) $$\frac{\partial g}{\partial x} = \frac{3(7x+y) - 7(3x+y^2)}{(7x+y)^2} = \frac{21x + 3y - 21x - 7y^2}{(7x+y)^2} = \frac{3y - 7y^2}{(7x+y)^2}$$

$$\frac{\partial g}{\partial y} = \frac{2y(7x+y) - 1(3x+y^2)}{(7x+y)^2} = \frac{14xy + 2y^2 - 3x - y^2}{(7x+y)^2} = \frac{y^2 + 14xy - 3x}{(7x+y)^2}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{0(7x+y)^2 - 2(7x+y) \cdot 7(3y-7y^2)}{(7x+y)^4} = \frac{-14(3y-7y^2)}{(7x+y)^3}$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{(3-14y)(7x+y)^2 - 2(7x+y)\times 1(3y-7y^2)}{(7x+y)^4}$$

$$= \frac{(3-14y)(7x+y) - 2(3y-7y^2)}{(7x+y)^3} = \frac{21x - 3y - 98xy}{(7x+y)^3}$$

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{(14y-3)(7x+y)^2 - 2(7x+y)\times 7(y^2+14xy-3x)}{(7x+y)^4}$$

$$= \frac{(14y-3)(7x+y) - 14(y^2+14xy-3x)}{(7x+y)^3} = \frac{21x - 3y - 98xy}{(7x+y)^3}$$

Teorema de Schwarz

e) $m(x,y) = \operatorname{sen}(1+e^{xy})$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1+e^{xy})\cos(1+e^{xy}) = \frac{\partial}{\partial x}(xy)\, e^{xy}\cos(1+e^{xy})$$

$$= y\,e^{xy}\cos(1+e^{xy})$$

$$\frac{\partial f}{\partial y} = x\,e^{xy}\cos(1+e^{xy})$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(y e^{xy})\cos(1+e^{xy}) + \frac{\partial}{\partial x}\cos(1+e^{xy})\, y e^{xy}$$

$$= y^2 e^{xy}\cos(1+e^{xy}) + \left(-y e^{xy}\operatorname{sen}(1+e^{xy})\right) y e^{xy}$$

$$= y^2 e^{xy}\left[\cos(1+e^{xy}) - \operatorname{sen}(1+e^{xy})\right]$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial y}\cos\left(1+e^{xy}\right) y e^{xy}$$

$$= \left(e^{xy} + xy e^{xy}\right)\cos\left(1+e^{xy}\right) - x e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right) y e^{xy}$$

$$= e^{xy}\left[(1+xy)\cos\left(1+e^{xy}\right) - xy e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right)\right]$$

T.S.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial x}\cos\left(1+e^{xy}\right) x e^{xy}$$

$$= \left(e^{xy} + xy e^{xy}\right)\cos\left(1+e^{xy}\right) - y e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right) x e^{xy}$$

$$= e^{xy}\left[(1+xy)\cos\left(1+e^{xy}\right) - xy e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right)\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x e^{xy}\right)\cos\left(1+e^{xy}\right) + \frac{\partial}{\partial y}\cos\left(1+e^{xy}\right) x e^{xy}$$

$$= x^2 e^{xy}\cos\left(1+e^{xy}\right) - x e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right) x e^{xy}$$

$$= x^2 e^{xy}\left[\cos\left(1+e^{xy}\right) - e^{xy}\,\mathrm{sen}\left(1+e^{xy}\right)\right]$$

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$$z = \left(x^2+y^2\right)^{1/3}$$

$$\frac{\partial z}{\partial x} = \frac{1}{3}\left(x^2+y^2\right)^{-2/3} \times 2x = \frac{2}{3}x\left(x^2+y^2\right)^{-2/3}$$

$$\frac{\partial z}{\partial y} = \frac{1}{3}\left(x^2+y^2\right)^{-2/3} \times 2y = \frac{2}{3}y\left(x^2+y^2\right)^{-2/3}$$

Não é necessário calcular $\frac{\partial^2 z}{\partial x^2}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{2x}{3} \times \frac{\partial}{\partial y}\left(x^2+y^2\right)^{-2/3}$$

$$= \frac{2x}{3} \times \left(-\frac{2}{3}\right)\left(x^2+y^2\right)^{-5/3} \times 2y = -\frac{8}{9} xy \left(x^2+y^2\right)^{-5/3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left[\left(\frac{2}{3}\right)\right] \times \left(x^2+y^2\right)^{-2/3} + \left[\left(-\frac{2}{3}\right)\left(x^2+y^2\right)^{-5/3} \times 2y\right] \times \frac{2}{3} y$$

$$= \frac{2}{3}\left[\left(x^2+y^2\right)^{-2/3} - \frac{4}{3} y^2 \left(x^2+y^2\right)^{-5/3}\right]$$

$$= \frac{2}{3}\left(x^2+y^2\right)^{-2/3}\left[1 - \frac{4}{3}\left(x^2+y^2\right)^{-1}\right]$$

$$3x \frac{\partial^2 z}{\partial x \partial y} + 3y \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = 0 \Leftrightarrow$$

$$\Leftrightarrow 3x\left[-\frac{8}{9} xy \left(x^2+y^2\right)^{-5/3}\right] + 3y\left[\frac{2}{3}\left(x^2+y^2\right)^{-2/3} - \frac{8}{9} y^2 \left(x^2+y^2\right)^{-5/3}\right] +$$

$$+ \frac{2}{3} y \left(x^2+y^2\right)^{-2/3} = 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{8}{3} x^2 y \left(x^2+y^2\right)^{-5/3} + 2y\left(x^2+y^2\right)^{-2/3} - \frac{8}{3} y^3 \left(x^2+y^2\right)^{-5/3} + \frac{2}{3} y \left(x^2+y^2\right)^{-2/3} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(x^2+y^2\right)^{-5/3}\left[-\frac{8}{3} x^2 y - \frac{8}{3} y^3\right] + \left(x^2+y^2\right)^{-2/3}\left[2y + \frac{2}{3} y\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(x^2+y^2\right)^{-5/3}\left[-\frac{8}{3} x^2 y - \frac{8}{3} y^3 + \left(x^2+y^2\right)\frac{8}{3} y\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2)^{-5/3}\left(-\frac{8}{3}x^2y-\frac{8}{3}y^3+\frac{8}{3}x^2y+\frac{8}{3}y^3\right)=0 \Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2)^{-5/3}\times 0=0 \Leftrightarrow 0=0$$

(6) $v(x,t)=t^{-1/2}\exp\left(-\frac{x^2}{4Kt}\right)$

$$\frac{\partial v}{\partial t}=-\frac{1}{2}t^{-3/2}\times 1\times\exp\left(-\frac{x^2}{4Kt}\right)+\frac{\partial}{\partial t}\left(-\frac{x^2}{4Kt}\right)\times\exp\left(-\frac{x^2}{4Kt}\right)\times t^{-1/2}$$

$$=\left[-\frac{1}{2}t^{-3/2}+\frac{x^2}{4Kt^2}\times t^{-1/2}\right]\exp\left(-\frac{x^2}{4Kt}\right)$$

$$=\left(-\frac{1}{2}t^{-1}+\frac{x^2}{4Kt^2}\right)t^{-1/2}\exp\left(\frac{-x^2}{4Kt}\right)$$

$$\frac{\partial v}{\partial x}=t^{-1/2}\frac{\partial}{\partial x}\left(-\frac{x^2}{4Kt}\right)\exp\left(-\frac{x^2}{4Kt}\right)=-\frac{1}{2}t^{-1/2}\frac{x}{Kt}\exp\left(-\frac{x^2}{4Kt}\right)$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=-\frac{t^{-1/2}}{2Kt}\times\frac{\partial}{\partial x}\left[x\times\exp\left(-\frac{x^2}{4Kt}\right)\right]=$$

$$=-\frac{t^{-1/2}}{2Kt}\left[1\times\exp\left(-\frac{x^2}{4Kt}\right)+\frac{\partial}{\partial x}\left(-\frac{x^2}{4Kt}\right)\exp\left(\frac{-x^2}{4Kt}\right)\times x\right]=$$

$$=-\frac{t^{-3/2}}{2K}\exp\left(-\frac{x^2}{4Kt}\right)\left[1-\frac{2x}{4Kt}x\right]=-\frac{t^{-3/2}}{2K}\left(1-\frac{2x^2}{4Kt}\right)\exp\left(-\frac{x^2}{4Kt}\right)=$$

$$=\left(-\frac{1}{2K}t^{-3/2}+\frac{x^2}{4K^2t^2}t^{-1/2}\right)\exp\left(-\frac{x^2}{4Kt}\right)=\left(-\frac{1}{2K}t^{-1}+\frac{x^2}{4K^2t^2}\right)t^{-1/2}\exp\left(-\frac{x^2}{4Kt}\right)$$

$$K\frac{\partial^2 v}{\partial x^2} = K\left(-\frac{1}{2K}t^{-1} + \frac{x^2}{4K^2t^2}\right)t^{-1/2}\exp\left(-\frac{x^2}{4Kt}\right) =$$

$$= \left(-\frac{1}{2}t^{-1} + \frac{x^2}{4Kt^2}\right)t^{-1/2}\exp\left(-\frac{x^2}{4Kt}\right) =$$

$$= \frac{\partial v}{\partial t}$$