Ficha B

Diferenciabilidade, Diferenciais e Derivadas de função composta

(1)

a) $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{h^2 \operatorname{sen}\left(\frac{1}{\sqrt{h^2}}\right)}{h} = \lim_{h\to 0} h \operatorname{sen} \frac{1}{\sqrt{h^2}} =$ $\quad \triangle\!\!!\ \sqrt{h^2} = |h|$

$= \lim_{h\to 0} h \times \lim \operatorname{sen} \frac{1}{\sqrt{2}} = 0$

Nota: Sen u é uma função limitada e periódica cujos valores estão em $[-1,1]$ logo $\lim \operatorname{sen} \frac{1}{h^2}$ não existe e $0 \times \lim \operatorname{sen} u = 0$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x \operatorname{sen}\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 2y \operatorname{sen} \frac{1}{\sqrt{x^2+y^2}} - \frac{\cancel{x}}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

b) Se $f(x,y)$ é diferenciável em $(0,0)$ então,

$$f(h,k) - f(0,0) = \frac{\partial f}{\partial x}(0,0) \times h + \frac{\partial f}{\partial y}(0,0) \times k + \varepsilon\rho$$

com $\rho = \sqrt{h^2+k^2}$ e $\lim_{\rho\to 0} \varepsilon = 0$

Sabemos que

$$\left.\begin{array}{l} f(h,k) - f(0,0) = (h^2+k^2) \operatorname{sen} \frac{1}{\sqrt{h^2+k^2}} \\ \text{e que} \\ f(h,k) - f(0,0) = 0 \times h + 0 \times k + \varepsilon\rho \end{array}\right\} \Rightarrow \varepsilon\rho = (h^2+k^2) \operatorname{sen} \frac{1}{\sqrt{h^2+k^2}}$$

$$\frac{\varepsilon\rho}{\rho} = \varepsilon = \sqrt{h^2+k^2}\ \text{sen}\ \frac{1}{\sqrt{h^2+k^2}}$$

Como $\lim\limits_{\rho\to 0} \varepsilon = \lim\limits_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\ \text{sen}\ \frac{1}{\sqrt{h^2+h^2}} = 0$

então $f(x,y)$ é diferenciável em $(0,0)$.

c) $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(0,0)$?

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0} \frac{x}{\sqrt{x^2}} = \lim_{x\to 0} \frac{x}{|x|} \begin{cases} 1 & \text{se } x \geq 0 \\ -1 & \text{se } x < 0 \end{cases}$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

O $\lim\limits_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ e $\lim\limits_{(x,y)\to(0,0)} \cos\frac{1}{\sqrt{x^2+y^2}}$ não existem

então podemos concluir que o $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y)$ não existe.

Logo $\frac{\partial f}{\partial x}(x,y)$ não é contínua.

$\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial x}(0,0)$?

do mesmo modo

(2) a) $df(x,y) = -\dfrac{\frac{y}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\,dx + \dfrac{\frac{1}{x}}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\,dy$

b) $df(x,y) = (yz - z\,x^{z-1})\,dx + xz\,dy + (xy - x^{z}\ln x)\,dz$

③ $f(x,y,z) = xyz - x^{z}$

$f(1,2,-1) = -3$

$dx = 0{,}001 \quad ; \quad dy = -0{,}001 \quad ; \quad dz = -0{,}01$

$$f(1+0{,}001,\ 2-0{,}001,\ -1-0{,}01) = f(1,2,-1) + df(1,2,-1)$$
$$= -3 + (0{,}003 + 0{,}001 - 0{,}02)$$
$$= -3{,}016$$

④ Ao truncar um número real positivo inferior a 10 à primeira casa decimal comete-se, no máximo um erro de 0,1 (x,y ~~ou z~~).

Nota: 4,4872
Arredondado (1 casa decimal) = 4,5
Truncado (1 casa decimal) = 4,4
↳ cortado

$df(x,y,z) = yz\,dx + xz\,dy + xy\,dz$

erro máximo $= yz \times \frac{1}{10} + xz \times \frac{1}{10} + xy \times \frac{1}{10}$

xy ou xz ou yz é menor que $10 \times 10 = 10^2$

então

o erro máximo cometido será $100 \times \frac{1}{10} + 100 \times \frac{1}{10} + 100 \times \frac{1}{10} = 10 + 10 + 10 = 30$

⑤ $f(x,y) = e^{xy}$

$$d^2f = d(df) = \frac{\partial^2 f}{\partial x^2}\,dx^2 + 2\,\frac{\partial^2 f}{\partial x \partial y}\,dx\,dy + \frac{\partial^2 f}{\partial y^2}\,dy^2$$

$$d^2f(x,y) = e^{xy}y^2\,dx^2 + 2\left(e^{xy} + xy\,e^{xy}\right)dx\,dy + e^{xy}x^2\,dy^2$$

⑥ $v = x + y^2$

$x = -\text{sen}\, t$

$y = \arccos u + \text{sen}\, t$

$$\begin{tikzcd}[row sep=small] & x \arrow[r, dash] & t \\ v \arrow[ur, dash] \arrow[dr, dash] & & u \\ & y \arrow[ur, dash] \arrow[r, dash] & t \end{tikzcd}$$

$(\arccos u)' = -\dfrac{u'}{\sqrt{1-u^2}}$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} = \cos t\left(-1 + 2\arccos u + \text{sen}\, t\right)$$

$$\frac{\partial v}{\partial u} = \frac{-2}{\sqrt{1-u^2}}\left(\arccos u + \text{sen}\, t\right)$$

⑦ Com $\underline{\underline{x}} = \rho\cos\varphi\cos\theta$

Seja $x^2+y^2+z^2 = v$

então

$u = \phi(v)$

$v = x^2+y^2+z^2$

$x = \rho\cos\varphi\cos\theta$

$y = \rho\cos\varphi\,\text{sen}\,\theta$

$z = \rho\,\text{sen}\,\varphi$

$$\begin{tikzcd}[row sep=tiny] & & & \rho \\ & & x \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \varphi \\ & & & \theta \\ & & & \rho \\ u \arrow[r, dash] & v \arrow[uuur, dash] \arrow[r, dash] \arrow[dddr, dash] & y \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \varphi \\ & & & \theta \\ & & & \rho \\ & & z \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \varphi \\ & & & \theta \end{tikzcd}$$

$$\frac{\partial u}{\partial \rho} = \frac{du}{dv}\frac{\partial v}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{du}{dv}\frac{dv}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{du}{dv}\frac{\partial v}{\partial z}\frac{\partial z}{\partial \varphi}$$

$$= \phi'\left[2x(-\rho\,\text{sen}\,\rho\cos\theta) + 2y(-\rho\,\text{sen}\,\rho\,\text{sen}\,\theta) + 2z(\rho\cos\varphi)\right] =$$

$$= 2\phi'\rho\left(-\rho\,\text{sen}\,\varphi\cos\varphi + \rho\,\text{sen}\,\varphi\cos\varphi\right) = 0$$

$$\frac{\partial u}{\partial \theta} = 2\phi'\rho\cos\varphi\left(-\rho\cos\varphi\cos\theta\,\text{sen}\,\theta + \rho\cos\varphi\,\text{sen}\,\theta\cos\theta\right) = 0$$

⑧ Seja $F = u\varphi(u^2, v^2 + w)$

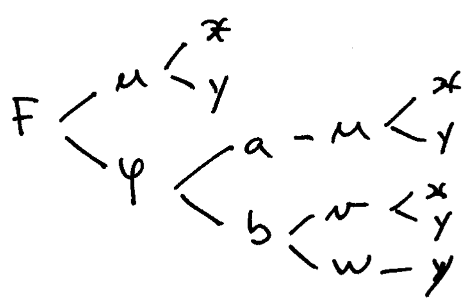
$u = \operatorname{sen}(x+y)$

$v = \cos(x+y)$

$w = y$

Fazendo $a = u^2$ e $b = v^2 + w$

então $F = u\varphi(a, b)$



$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial \varphi}\frac{\partial \varphi}{\partial a}\frac{da}{du}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial \varphi}\frac{\partial \varphi}{\partial b}\frac{\partial b}{\partial v}\frac{\partial v}{\partial x}$$

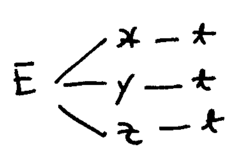
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial \varphi}\frac{\partial \varphi}{\partial a}\frac{da}{du}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial \varphi}\frac{\partial \varphi}{\partial b}\frac{\partial b}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial \varphi}\frac{\partial \varphi}{\partial b}\frac{\partial b}{\partial w}\frac{dw}{dy}$$

⑨ $E = f(x, y, z)$

$x = \operatorname{sen} t$

$y = \cos t$

$z = t$



$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial z}\frac{dz}{dt}$$

$$\frac{dE}{dt} = \cos t\,\frac{\partial E}{\partial x} - \operatorname{sen} t\,\frac{\partial E}{\partial y} + \frac{\partial E}{\partial z}$$