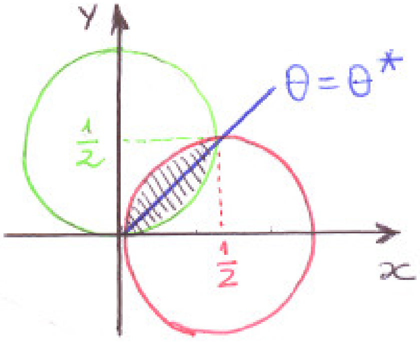
Folha 8A L1

1- $A = \{(x,y) \in \mathbb{R}^2 : (x-\frac{1}{2})^2 + y^2 \leq \frac{1}{4} \wedge x^2 + (y-\frac{1}{2})^2 \leq \frac{1}{4}\}$



$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4} \longrightarrow$ Circunf., $C = (\frac{1}{2}, 0)$; $R = \frac{1}{2}$

$x^2 + (y-\frac{1}{2})^2 = \frac{1}{4} \longrightarrow$ Circunf., $C = (0, \frac{1}{2})$; $R = \frac{1}{2}$

Equações polares das circunferências:

$$\begin{aligned}
(x-\tfrac{1}{2})^2 + y^2 = \tfrac{1}{4} &\Leftrightarrow x^2 - x + \tfrac{1}{4} + y^2 = \tfrac{1}{4} \\
&\Leftrightarrow x^2 + y^2 - x = 0 \\
&\Leftrightarrow \rho^2 - \rho\cos\theta = 0 \\
&\Leftrightarrow \rho(\rho - \cos\theta) = 0 \\
&\Leftrightarrow \rho = 0 \ \vee \ \boxed{\rho = \cos\theta}
\end{aligned}$$

$$x^2 + (y-\tfrac{1}{2})^2 = \tfrac{1}{4} \Leftrightarrow \cdots \Leftrightarrow \boxed{\rho = \operatorname{sen}\theta}$$

Para determinar θ^* (intersecção das curvas)

$$\begin{cases} \rho = \cos\theta \\ \rho = \operatorname{sen}\theta \end{cases} \Rightarrow \operatorname{sen}\theta = \cos\theta \Rightarrow \theta^* = \frac{\pi}{4} \quad (1^{\circ}\text{ quadrante})$$

Para a área,

$$\begin{aligned}
\text{área } A &= \text{área } A_1 + \text{área } A_2 \\
&= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta + \frac{1}{2}\int_0^{\pi/4} \operatorname{sen}^2\theta \, d\theta
\end{aligned}$$

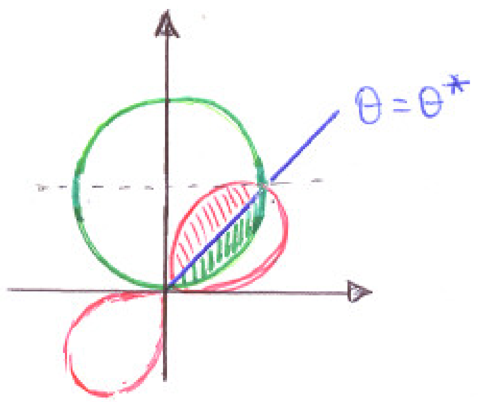
$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta + \frac{1}{2}\int_0^{\pi/4} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left([\theta]_{\pi/4}^{\pi/2} + \frac{1}{2}[\operatorname{sen}2\theta]_{\pi/4}^{\pi/2} + [\theta]_{0}^{\pi/4} - \frac{1}{2}[\operatorname{sen}2\theta]_{0}^{\pi/4} \right)$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi}{8} - \frac{1}{4}$$

2. Circunferência $\rho = \sqrt{2}\operatorname{sen}\theta$. $\theta \in [0, \frac{\pi}{2}]$

Lemniscata $\rho^2 = \operatorname{sen}2\theta$. $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$



Para θ^* (intersecção das curvas)

$$\begin{cases} \rho = \sqrt{2}\operatorname{sen}\theta \\ \rho^2 = \operatorname{sen}2\theta \end{cases} \Rightarrow (\sqrt{2}\operatorname{sen}\theta)^2 = \operatorname{sen}2\theta$$

$$\Rightarrow 2\operatorname{sen}^2\theta = \operatorname{sen}2\theta$$

$$\Rightarrow 2\operatorname{sen}\theta\operatorname{sen}\theta = 2\operatorname{sen}\theta\cos\theta$$

$$\Rightarrow \operatorname{sen}\theta = \cos\theta \underset{(IQ)}{\Rightarrow} \theta = \theta^* = \frac{\pi}{4}$$

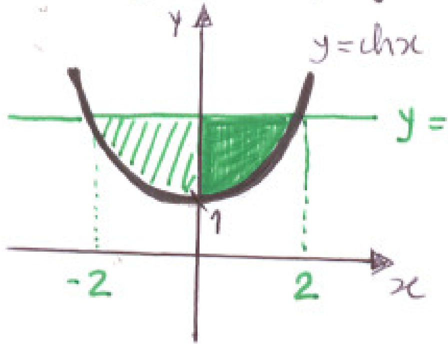
$$\text{área} = \frac{1}{2}\int_0^{\pi/4} 2\operatorname{sen}^2\theta\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \operatorname{sen}2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta - \frac{1}{4}[\cos 2\theta]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left([\theta]_0^{\pi/4} - \frac{1}{2}[\operatorname{sen}2\theta]_0^{\pi/4} \right) - \frac{1}{4}(-1-0)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2} \right) + \frac{1}{4} = \frac{\pi}{8}.$$

3. $y = \text{ch}\,x$, $y = \text{ch}\,2$



simetria

$$\text{área} = 2\int_0^2 (\text{ch}\,2 - \text{ch}\,x)\,dx$$

$$= 2\,\text{ch}\,2\,[x]_0^2 - 2\,[\text{sh}\,x]_0^2$$

$$= 4\,\text{ch}\,2 - 2\,\text{sh}\,2 .$$

Comprimento (segmento e curva sobre $y = \text{ch}\,x$).

$$\textbf{comp} = \text{comp(segm)} + \text{comp(arco)}$$

$$= 4 + 2\int_0^2 \sqrt{1+\text{sh}^2 x}\,dx$$

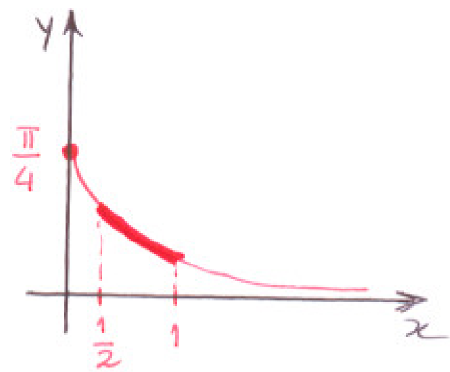
$(\text{ch}^2 x - \text{sh}^2 x = 1)$

$$= 4 + 2\int_0^2 \text{ch}\,x\,dx = 4 + 2\,[\text{sh}\,x]_0^2$$

$$= 4 + 2\,\text{sh}\,2 .$$

4.

a)



$$f(x) = \text{arcsen}(e^{-x})$$

$$f'(x) = \frac{-e^{-x}}{\sqrt{1-e^{-2x}}}$$

$$1 + [f'(x)]^2 = 1 + \frac{e^{-2x}}{1-e^{-2x}} = \frac{1-e^{-2x}+e^{-2x}}{1-e^{-2x}} = \frac{1}{1-e^{-2x}}$$

$$= \frac{1}{1-\frac{1}{e^{2x}}} = \frac{e^{2x}}{e^{2x}-1}$$

Então o comprimento pedido é dado por

$$\text{comp} = \int_{1/2}^{1} \sqrt{\frac{e^{2x}}{e^{2x}-1}}\, dx$$

$$\overset{*}{=} \int_{\sqrt{e}}^{e} \sqrt{\frac{t^2}{t^2-1}}\, \frac{1}{t}\, dt$$

(*) Substituição $e^x = t$.

$x = \ln t \Rightarrow x' = \frac{1}{t}$.

$\begin{cases} e^x = t \\ x = 1 \end{cases} \Rightarrow t = e$

$\begin{cases} e^x = t \\ x = 1/2 \end{cases} \Rightarrow t = \sqrt{e}$

$$= \int_{\sqrt{e}}^{e} \frac{1}{\sqrt{t^2-1}}\, dt = \Big[\operatorname{argch} t\Big]_{\sqrt{e}}^{e}$$

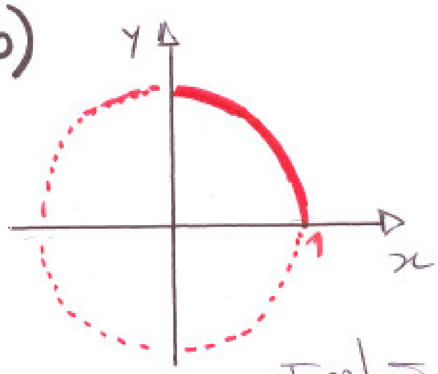
$$= \operatorname{argch} e - \operatorname{argch} \sqrt{e}$$

$$= \ln\left(e + \sqrt{e^2-1}\right) - \operatorname{argch}\left(\sqrt{e} + \sqrt{e-1}\right).$$

$\operatorname{argch} x = \ln\left(x + \sqrt{x^2-1}\right)$

$x \in [1, +\infty[$

b)



$y = \sqrt{1-x^2}$, $0 \leq x \leq 1$

$y = \sqrt{1-x^2} \Leftrightarrow y^2 = 1 - x^2 \wedge y \geq 0$

$\Leftrightarrow x^2 + y^2 = 1 \wedge y \geq 0$

Então $y = \sqrt{1-x^2}$, com $0 \leq x \leq 1$, representa o arco de circunferência $x^2+y^2=1$, $x \geq 0$, $y \geq 0$.

Temos

$$f(x) = \sqrt{1-x^2} \Rightarrow f'(x) = \frac{1}{2}(-2x)(1-x^2)^{-1/2} = \frac{-x}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(f'(x)\right)^2 = \frac{x^2}{1-x^2} \Rightarrow 1 + \left(f'(x)\right)^2 = \frac{1}{1-x^2}.$$

$\longrightarrow$

O comprimento pedido é dado por

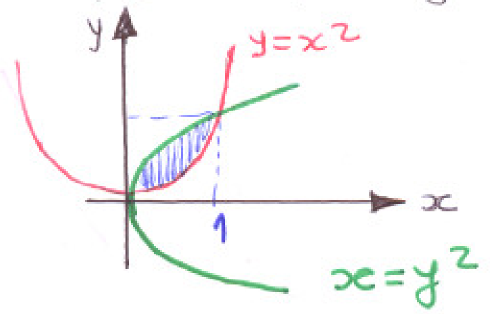
$$L = \int_0^1 \sqrt{\frac{1}{1-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \left[\operatorname{arcsen} x\right]_0^1 = \operatorname{arcsen} 1 - \operatorname{arcsen} 0 = \frac{\pi}{2}.$$

Notar que o perímetro de um circunferência de raio R é dado por $P = 2\pi R$.

5. Volumes (Rotação em torno de OX)

a) $y = x^2$, $x = y^2$.



$y=\sqrt{x}$ v $y=-\sqrt{x}$

As parábolas intersectam-se quando

$$\begin{cases} y = x^2 \\ x = y^2 \end{cases} \Rightarrow x = x^4$$

$$\Rightarrow x - x^4 = 0 \Rightarrow x(1-x^3) = 0$$

$$\Rightarrow x = 0 \lor \boxed{x = 1}$$

$$\text{Volume} = \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] dx$$

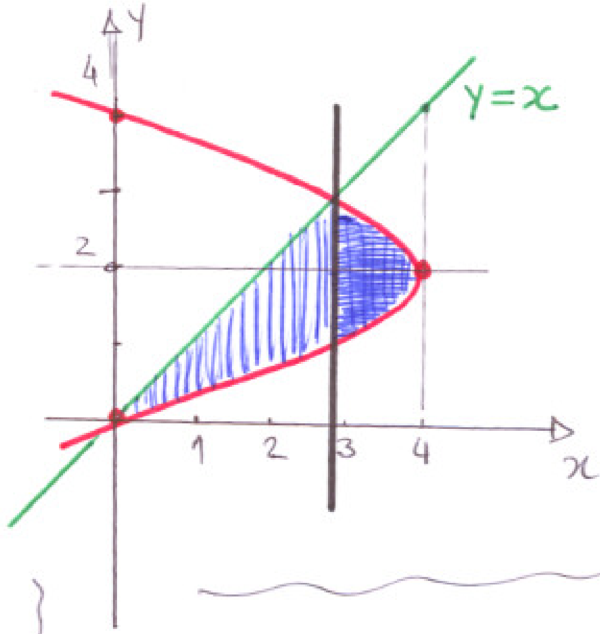
$$= \pi \int_0^1 (x - x^4)\,dx = \pi\left(\frac{1}{2}\left[x^2\right]_0^1 - \frac{1}{5}\left[x^5\right]_0^1\right)$$

$$= \frac{\pi}{2}(1-0) - \frac{\pi}{5}(1-0)$$

$$= \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}.$$

//

b) $y = x, \quad x = 4y - y^2$.



$$x = 4y - y^2 \Leftrightarrow x = -(y^2 - 4y)$$

$$\Leftrightarrow x = -(y^2 - 4y) - 4 + 4$$

$$\Leftrightarrow x = -(y^2 - 4y + 4) + 4$$

$$\Leftrightarrow x = -(y-2)^2 + 4$$

$$\Leftrightarrow \boxed{x - 4 = -(y-2)^2}$$

Parábola de vértice $(4,2)$, eixo de simetria horizontal, concavidade para a esquerda.

$$\to (y-2)^2 = 4 - x \Rightarrow \boxed{y = 2 \pm \sqrt{4-x}}$$

Intersecção

$$\begin{cases} y = x \\ x = 4y - y^2 \end{cases} \Rightarrow$$

$$\Rightarrow y^2 - 4y + y = 0$$

$$\Rightarrow y^2 - 3y = 0 \Rightarrow y(y-3) = 0 \Rightarrow y = 0 \vee y = 3.$$

Para o volume do gerado pela rotação em torno de OX da região da figura, vem

$$Vol = \pi \int_0^3 \left[x^2 - \left(2 - \sqrt{4-x}\right)^2\right] dx + \pi \int_3^4 \left[\left(2+\sqrt{4-x}\right)^2 - \left(2 - \sqrt{4-x}\right)^2\right] dx$$

$$= \pi \int_0^3 \left(x^2 - 4 + 4\sqrt{4-x} - 4 + x\right) dx$$

$$+ \pi \int_3^4 \left(\cancel{4} + 4\sqrt{4-x} + \cancel{4-x} - \cancel{4} + 4\sqrt{4-x} - \cancel{4+x}\right) dx$$

$$= \pi \int_0^3 \left(x^2 + x - 8 + 4\sqrt{4-x}\right) dx + \pi \int_3^4 8\sqrt{4-x}\, dx$$

$$= \pi \left(\frac{1}{3}\left[x^3\right]_0^3 + \frac{1}{2}\left[x^2\right]_0^3 - 8\left[x\right]_0^3 - \frac{8}{3}\left[\sqrt{(4-x)^3}\right]_0^3 \right.$$

$$\left. - \frac{16}{3}\left[\sqrt{(4-x)^3}\right]_3^4 \right)$$

$$= \pi\left(\frac{1}{3}3^3 + \frac{1}{2}3^2 - 24 - \frac{8}{3}\left(1-4\sqrt{4}\right) - \frac{16}{3}\left(0-1\right)\right)$$

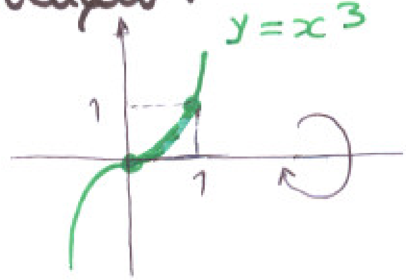
$$= \pi\left(3^2 + \frac{1}{2}3^2 - 24 - \frac{8}{3} + \frac{32}{3}\sqrt{4} + \frac{16}{3}\right)$$

$$= \pi\left(\frac{27}{2} - 24 - \frac{8}{3} + \frac{64}{3} + \frac{16}{3}\right) = \pi\left(\frac{27}{2} - 24 + \frac{72}{3}\right)$$

$$= \frac{81}{6}\pi$$

6. Áreas de superfícies de revolução.

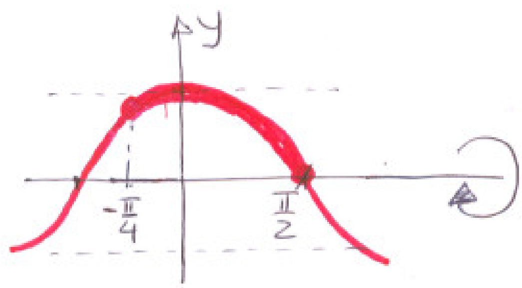
a) $y = x^3,\ x \in [0,1]$



$$\text{área sup} = 2\pi\int_0^1 x^3\sqrt{1+\left((x^3)'\right)^2}\,dx$$

$$= 2\pi\int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$$

b) $y = \cos x,\ -\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$

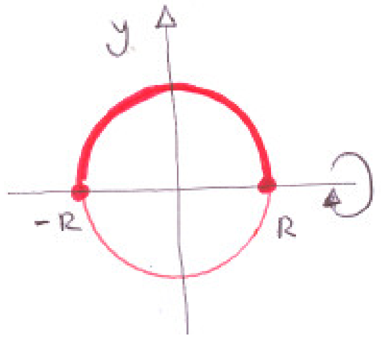


$$\text{área sup} = 2\pi\int_{-\pi/4}^{\pi/2} \cos x\sqrt{1+(-\operatorname{sen} x)^2}\,dx$$

$$= 2\pi\int_{-\pi/4}^{\pi/2} \cos x\sqrt{1+\operatorname{sen}^2 x}\,dx$$

c) $\underbrace{y = \sqrt{R^2-x^2}}_{\text{arco de circunferência}}$, $-R \le x \le R$ $(R>0)$

$y^2 = R^2 - x^2, \ y \geq 0 \iff x^2+y^2=R^2, \ y \geq 0$



$$\text{área sup} \underset{\text{simetria}}{=} 4\pi \int_0^R \sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx$$

$$y = \sqrt{R^2-x^2} = (R^2-x^2)^{1/2}$$

$$y' = \frac{1}{2}(-2x)(R^2-x^2)^{-1/2}$$

$$= -\frac{x}{\sqrt{R^2-x^2}}$$