Ficha nº8 – Integrais triplos.

1\. a) $\iiint_P (xyz)\,dV$, $P = \underbrace{[0,1]}_{x} \times \underbrace{[-1,1]}_{y} \times \underbrace{[1,2]}_{z}$

$$\int_0^1 \int_{-1}^1 \int_1^2 (x.y.z)\,dz\,dy\,dx = \int_{-1}^1 \int_0^1 \int_1^2 xyz\,dz\,dx\,dy =$$

$$= \int_0^1 \int_1^2 \int_{-1}^1 (xyz)\,dy\,dz\,dx = \int_1^2 \int_0^1 \int_{-1}^1 xyz\,dy\,dx\,dz$$

$$= \int_{-1}^1 \int_1^2 \int_0^1 xyz\,dx\,dz\,dy = \int_1^2 \int_{-1}^1 \int_0^1 xyz\,dx\,dy\,dz \,.$$

$$\int_0^1 \int_{-1}^1 \int_1^2 (xyz)\,dz\,dy\,dx = \int_0^1 \int_{-1}^1 xy\left[\frac{z^2}{2}\right]_1^2 dy\,dx = \frac{3}{2}\int_0^1 \int_{-1}^1 xy\,dy\,dx =$$

$$= \frac{3}{2}\int_0^1 x\left[\frac{y^2}{2}\right]_{-1}^1 dx = 0$$

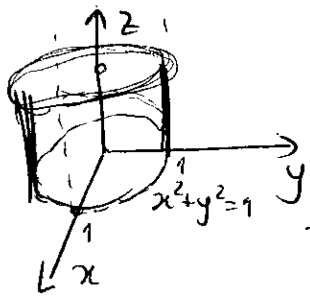
b) $\iiint_R (x^2+5y^2-z)\,dV$ $x \in [0,1]$, $y \in [1,2]$, $z \in [-1,1]$

$$\int_0^1 \int_1^2 \int_{-1}^1 (x^2+5y^2-z)\,\cancel{dV}\,dz\,dy\,dx = \int_0^1 \int_1^2 \left[x^2 z + 5y^2 z - \frac{z^2}{2}\right]_{z=-1}^{z=1} dy\,dx$$

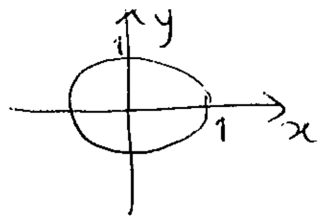
$$= \int_0^1 \int_1^2 (2x^2+10y^2)\,dy\,dx = \int_0^1 \left[2x^2 y + 10\frac{y^3}{3}\right]_{y=1}^{y=2} dx = \int_0^1 \left(2x^2+\frac{70}{3}\right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{70}{3}\cdot x\right]_0^1 = \frac{2}{3} + \frac{70}{3} = \frac{72}{3}$$

2. a) $\iiint_R f(x,y,z)\,dV$



$0 \le z \le 1$
$x^2+y^2 \le 1$
$-1 \le x \le 1$
$-\sqrt{1-y^2} \le y \le \sqrt{1-x^2}$



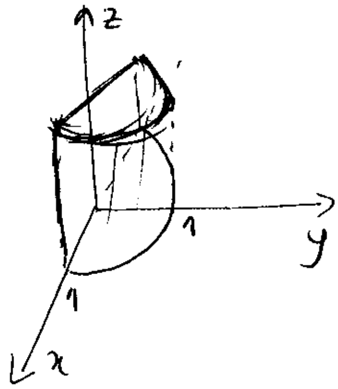
$$\int_0^1 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y,z)\,dy\,dx\,dz = \int_{-1}^1 \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 f(x,y,z)\,dz\,dy\,dx$$

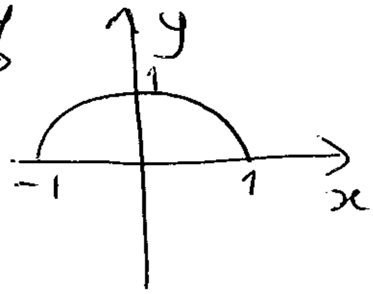
Com estes limites: $0 \le z \le 1$
$-1 \le x \le 1$
$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

tem que se integrar primeiro relativamente a y e depois relativamente a x. A ordem do integral relativo a z é indiferente.

2. b)



Projeção no plano xOy $\longrightarrow$

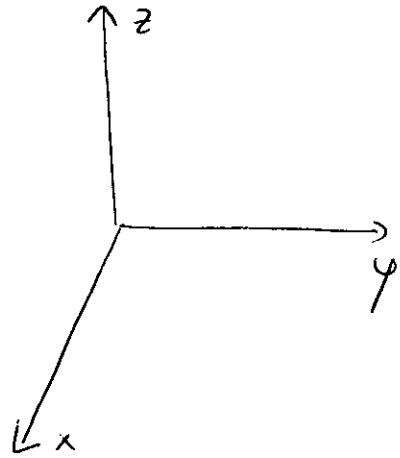


$0 \le z \le 1$
$-1 \le x \le 1$
$0 \le y \le \sqrt{1-x^2}$

Tem que se ter os mesmos cuidados que na alínea a) com a ordem de integração.

$$\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y,z)\,dy\,dx\,dz = \int_{-1}^1 \int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y,z)\,dy\,dz\,dx$$
$$= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^1 f(x,y,z)\,dz\,dy\,dx.$$

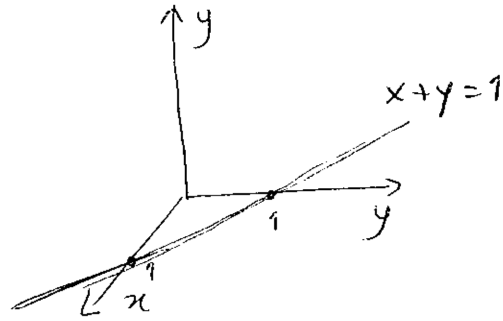
2.c) ~~R~~



A interseção do

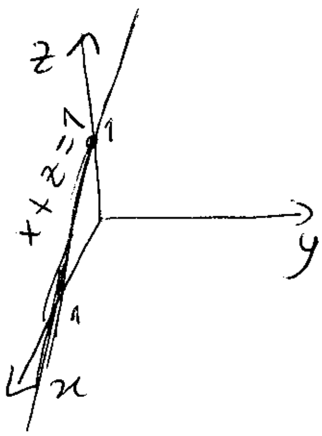
$x+y+z=1$ e plano xOy

$$\begin{cases} x+y+z=1 \\ z=0 \end{cases} \Leftrightarrow x+y=1$$



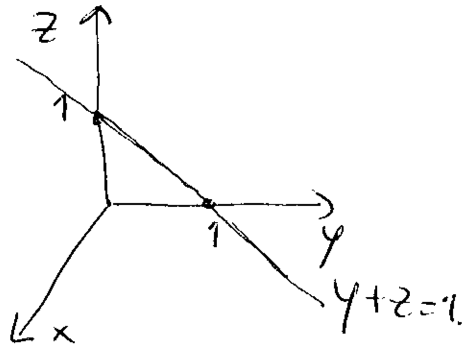
A interseção com o plano xOz

$$\begin{cases} x+y+z=1 \\ y=0 \end{cases} \Leftrightarrow x+z=1$$

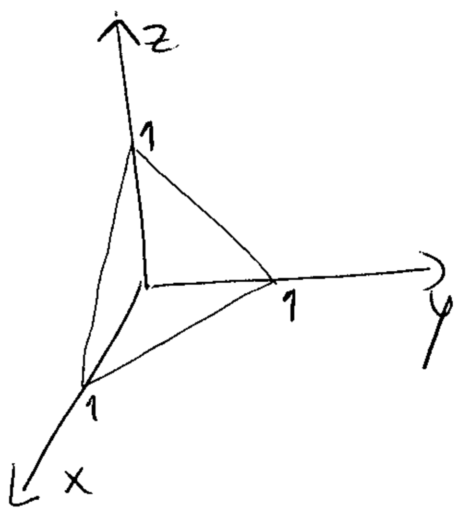


A interseção com o plano yOz

$$\begin{cases} x+y+z=1 \\ x=0 \end{cases} \Leftrightarrow y+z=1$$



A conjunção das interseções dá a localização do plano $x+y+z=1$ no espaço $xOyOz$:

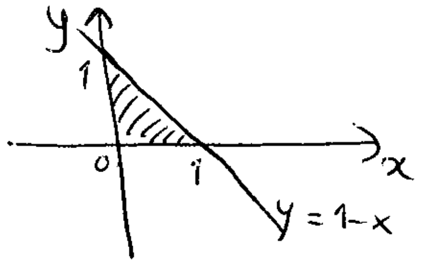


Com os outros limites $x=0$, $y=0$ e $z=0$

queremos a pirâmide triangular que fica no 1º octante

$$0 \le z \le 1-x-y$$

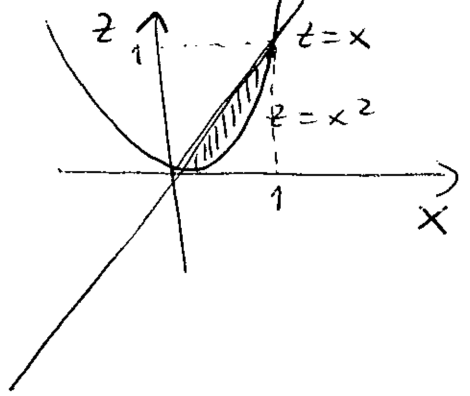
Projecção no plano xoy

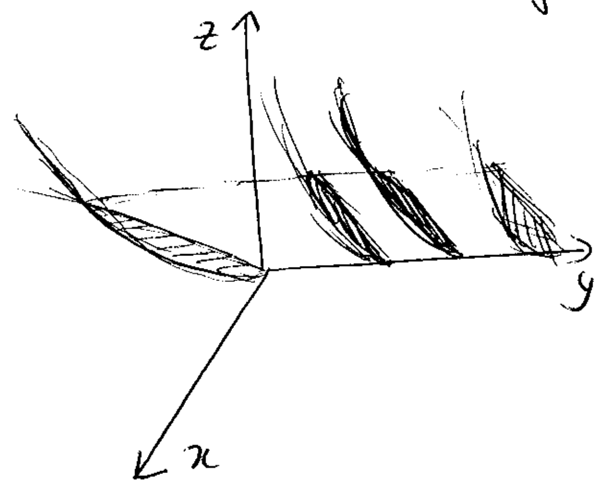


$0 \leq x \leq 1$
$0 \leq y \leq 1-x$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\, dz\, dy\, dx$$

3. As superfícies $z = x$ e $z = x^2$ são gerados pelo movimento das curvas $z = x$ e $z = x^2$ no plano xoz ~~que serão~~ ao longo do eixo Oy.





$0 \leq y \leq 2$
$x^2 \leq z \leq x$
$0 \leq x \leq 1$

$$\int_0^2 \int_0^1 \int_{x^2}^{x} 1 . dz\, dx\, dy =$$

$$= \int_0^2 \int_0^1 (x - x^2)\, dx\, dy = \int_0^2 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 dy =$$

$$= \int_0^2 \frac{1}{6}\, dy = \frac{2}{6} = \frac{1}{3}.$$

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3.b) $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\, dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx = \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx =$$

$$= \int_0^1 \frac{(1-x)^2}{2} dx = -\frac{1}{2} \left[\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{6}$$

Coordenadas cilíndricas e esféricas

1. Coordenadas cilíndricas (R, θ, z)

$$\iiint_R (x^2+y^2+z^2)\, dV =$$

Como $\begin{cases} x = R\cos\theta \\ y = R\,\text{sen}\,\theta \\ z = z \end{cases}$, tem-se

$$= \int_0^4 \int_{\pi/2}^{\pi} \int_{-1}^{1} (R^2+z^2) \cdot R\, dz\, dR\, d\theta$$

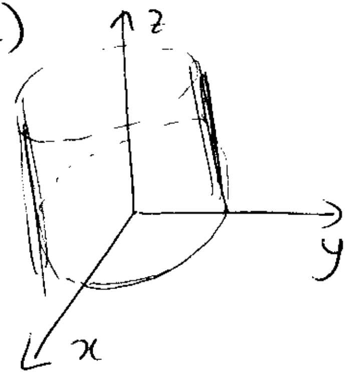
$$R = \frac{\partial(x,y,z)}{\partial(R,\theta,z)}$$

$$= \int_0^4 \int_{\pi/2}^{\pi} \left[R^3 \cdot z + R \cdot \frac{z^3}{3} \right]_{z=-1}^{z=1} dR\, d\theta =$$

$$= \int_0^4 \int_{\pi/2}^{\pi} \left(2R^3 + \frac{2R}{3} \right) dR\, d\theta = \int_0^4 \left[\frac{2R^4}{4} + \frac{\cancel{2}R^2}{3} \right]_{R=\pi/2}^{R=\pi} d\theta =$$

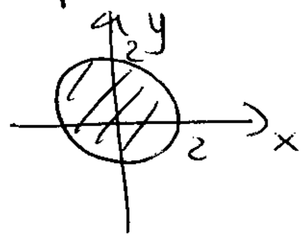
$$= \int_0^4 \frac{\pi^2}{4} \left(\frac{\pi^2 \cdot 15}{8} + 1 \right) d\theta = \pi^2 \left(\frac{\pi^2 \cdot 15}{8} + 1 \right).$$

2a)



$0 \leq z \leq 2$

Projeção no plano xoy

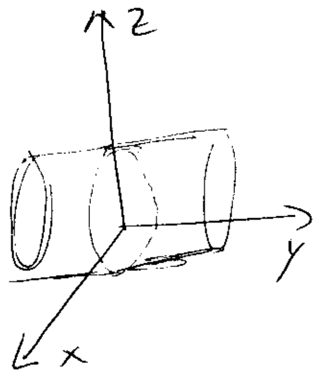


$0 \leq R \leq 2$

$0 \leq \theta < 2\pi$

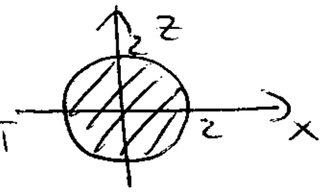
$$\int_0^2 \int_0^2 \int_0^{2\pi} R \, d\theta \, dR \, dz \qquad \frac{\partial(x,y,z)}{\partial(R,\theta,z)} = R$$

2.b)



$-1 \leq y \leq 1$

Projeção no plano xOz

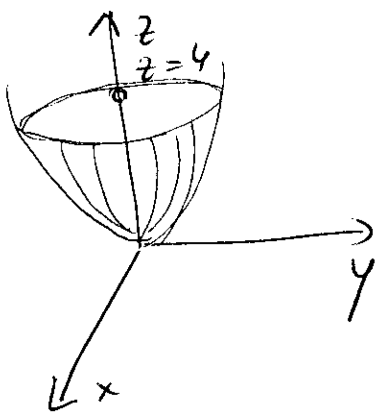


$0 \leq R \leq 2$

$0 \leq \theta \leq 2\pi$

$$\int_{-1}^1 \int_0^2 \int_0^{2\pi} R \, d\theta \, dR \, dy$$

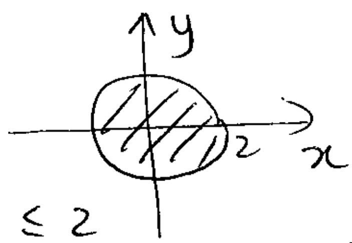
2.c)



$x^2+y^2 \leq z \leq 4$

Projeção no plano xoy é

$x^2+y^2=4$

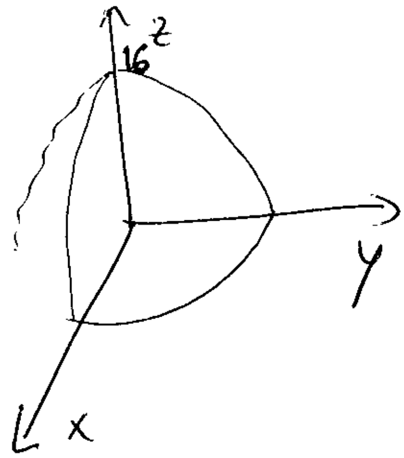


$0 \leq R \leq 2$

$0 \leq \theta \leq 2\pi$ $\Rightarrow R^2 \leq z \leq 4$

$$\int_0^2 \int_0^{2\pi} \int_{R^2}^4 R \, . \, dz \, d\theta \, dR.$$

2.d)

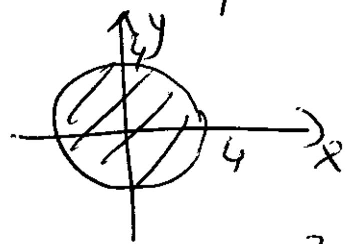


$0 \leq z \leq 16 - x^2 - y^2$

Projecção no plano xOy

$16 - x^2 - y^2 = 0$

$x^2 + y^2 = 16$



$0 \leq R \leq 4$

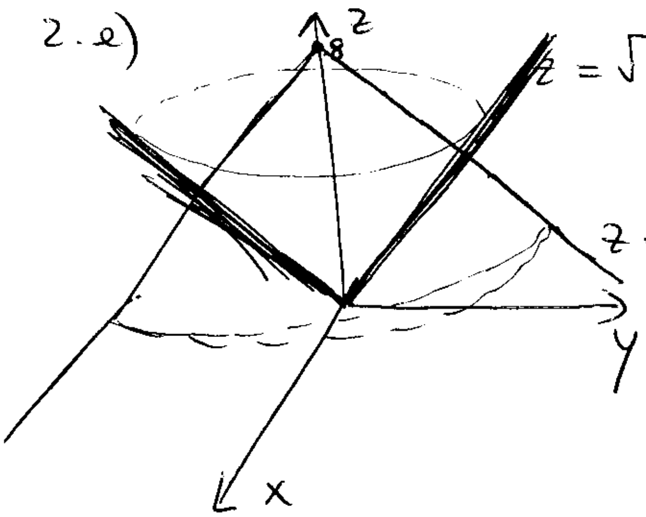
$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 16 - R^2$

$16 - x^2 - y^2 = 16 - R^2$

$$\int_0^4 \int_0^{2\pi} \int_0^{16-R^2} R \, dz \, d\theta \, dR.$$

2.e)



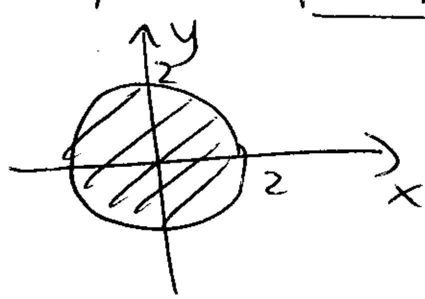
$\sqrt{x^2+y^2} \leq z \leq \sqrt{8-x^2-y^2}$

Projecção no plano xOy
da intersecção dos dois cones

$\sqrt{x^2+y^2} = \sqrt{8-x^2-y^2} \Rightarrow$

$x^2+y^2 = 8 - x^2 - y^2 \Leftrightarrow$

$2x^2+2y^2 = 8 \Leftrightarrow \boxed{x^2+y^2=4}$



$0 \leq R \leq 2$

$0 \leq \theta \leq 2\pi$

$$\sqrt{x^2+y^2} = R$$

$$\sqrt{8-x^2-y^2} = \sqrt{8-R^2}$$

$$R \leq z \leq \sqrt{8-R^2}$$

$$0 \leq R \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$\int_0^2 \int_0^{2\pi} \int_R^{\sqrt{8-R^2}} R \, dz \, d\theta \, dR$$

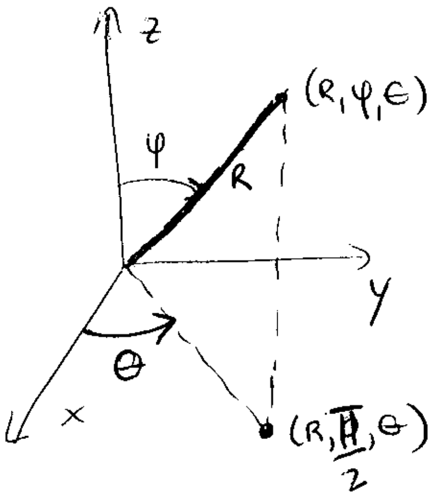
3. Coordenadas esféricas (R, φ, θ)

$$R \geq 0$$
$$0 \leq \varphi \leq \pi$$
$$0 \leq \theta < 2\pi$$

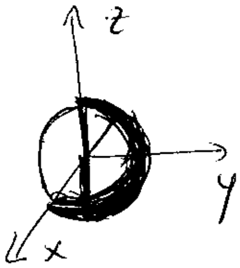
Relação entre coord. retangulares e coord. esféricas

$$\begin{cases} x = R \operatorname{sen}\varphi \cos\theta \\ y = R \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = R \cos\varphi \end{cases}$$

e $$\frac{\partial(x,y,z)}{\partial(R,\varphi,\theta)} = R^2 \operatorname{sen}\varphi$$



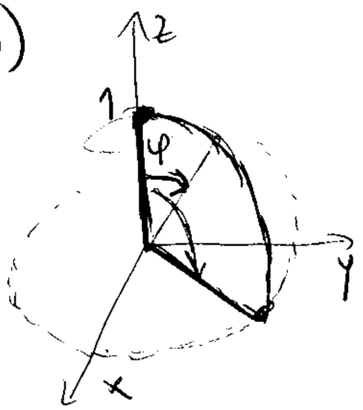
3. a) esfera de raio 2



$0 \leq R \leq 2$
$0 \leq \varphi \leq \pi$
$0 \leq \theta < 2\pi$

$$\int_0^2 \int_0^{\pi} \int_0^{2\pi} R^2 \operatorname{sen}\varphi \, d\theta \, d\varphi \, dR$$

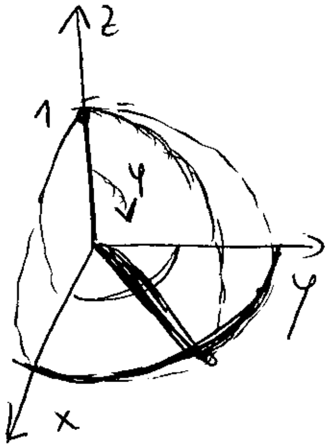
b)



$0 \leq R \leq 1$
$0 \leq \varphi \leq \frac{\pi}{2}$
$0 \leq \theta \leq 2\pi$

$$\int_0^1 \int_0^{\pi/2} \int_0^{2\pi} R^2 \operatorname{sen}\varphi \, d\theta \, d\varphi \, dR$$

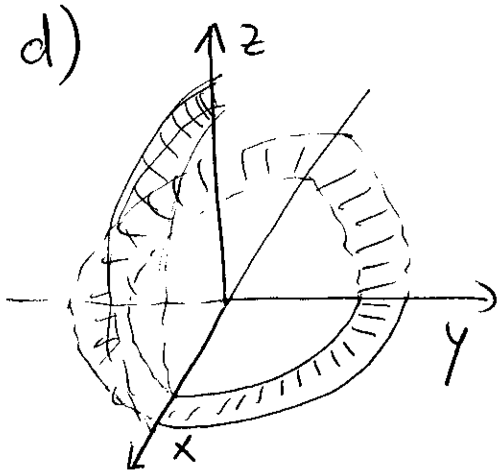
c)



$0 \leq R \leq 1$
$0 \leq \varphi \leq \frac{\pi}{2}$
$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} R^2 \operatorname{sen}\varphi \, d\theta \, d\varphi \, dR.$$

d)



$1 \leq R \leq 2$
$0 \leq \varphi \leq \frac{\pi}{2}$
$0 \leq \theta \leq 2\pi$

$$\int_1^2 \int_0^{\pi/2} \int_0^{2\pi} R^2 \operatorname{sen}\varphi \, d\theta \, d\varphi \, dR.$$