Ficha 10 A   Integrais Triplos

1. $\iiint\limits_R (xy^2+z^3)\, dV \qquad R=\{(x,y,z)\in\mathbb{R}^3 \mid 0\le x\le a \wedge 0\le y\le b \wedge 0\le z\le c\}$

$$= \int_0^a\int_0^b\int_0^c (xy^2+z^3)\, dz\, dy\, dx = \int_0^a\int_0^b \left[xy^2z+\frac{z^4}{4}\right]_{z=0}^{z=c} dy\, dx =$$

$$= \int_0^a\int_0^b \left(xy^2c+\frac{c^4}{4}\right) dy\, dx = \int_0^a \left[x\frac{y^3}{3}c+\frac{c^4}{4}y\right]_{y=0}^{y=b} dx =$$

$$= \int_0^a \left(x\frac{b^3}{3}c+\frac{c^4}{4}b\right) dx = \left[\frac{x^2}{2}\frac{b^3}{3}c+\frac{c^4}{4}bx\right]_0^a =$$

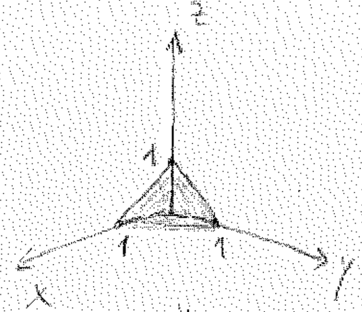
$$= \frac{a^2}{2}\frac{b^3}{3}c + ab\frac{c^4}{4} = \frac{abc}{2}\left(\frac{ab^2}{3}+\frac{c^3}{2}\right)$$

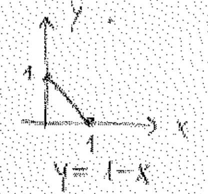
2. $\iiint\limits_R f(x,y,z)\, dx\, dy\, dz$

a) $x+y+z=1\ ,\quad x=0\ ,\quad y=0\ ,\quad z=0$

$z=1-x-y$

$$\int_0^1\int_0^{1-y}\int_0^{1-x-y} f(x,y,t)\, dz\, dx\, dy$$

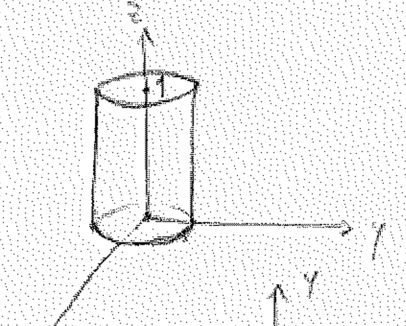


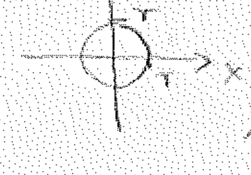


b) $x^2+y^2=r^2\ ,\quad z=0\ ,\quad z=1$

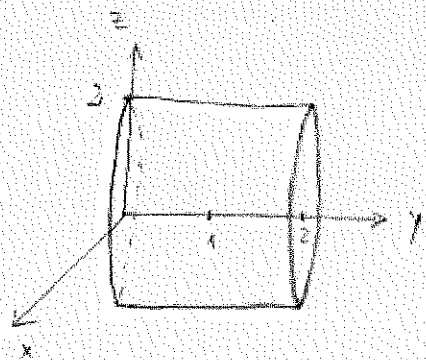
$$\int_0^1\int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f(x,y,z)\, dx\, dy\, dz$$

$$\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_0^1 \quad dx$$

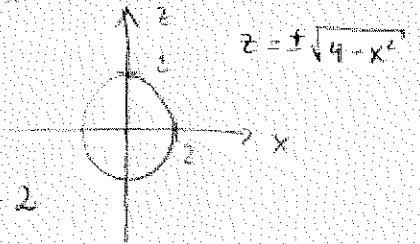




3. $z^2 + x^2 = 4 \qquad y = 0 \qquad y = 2$



$$V = \int_0^2 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \; dz\,dx\,dy$$



Usando coordenadas cilíndricas:

$$\begin{cases} x = \rho\cos\theta = x(\rho,\theta,y) \\ y = y = y(\rho,\theta,y) \\ z = \rho\sin\theta = z(\rho,\theta,y) \end{cases} \qquad \begin{array}{l} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \end{array}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial y} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & \rho\cos\theta & 0 \end{vmatrix} =$$

$$= (-1) \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = -\rho\cos^2\theta - \rho\sin^2\theta = -\rho$$

$$V = \int_0^2 \int_0^{2\pi} \int_0^2 \rho \; d\rho\,d\theta\,dy = \int_0^2 \int_0^{2\pi} \left[\frac{\rho^2}{2}\right]_0^2 d\theta\,dy = \int_0^2 \int_0^{2\pi} 2 \; d\theta\,dy$$

$$= \int_0^2 \left[2\theta\right]_0^{2\pi} dy = \int_0^2 4\pi \; dy = 4\pi \left[y\right]_0^2 = 8\pi$$

ou:

$$V = \int_0^2 \int_{-2}^{2} \left[z\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx\,dy = \int_0^2 \int_{-2}^{2} 2\sqrt{4-x^2} \; dx\,dy =$$

$$= \dots = 8\pi$$

4.

a) $x^2+y^2+z=10$ e $z-2x^2-2y^2+2=0$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2+2x^2+2y^2}^{10-x^2-y^2} 1 \, dz\, dy\, dx$$

b) $z=\sqrt{x^2+y^2}$ e $z=\sqrt{36-x^2-y^2}$

$$\int_{-\sqrt{18}}^{\sqrt{18}}\int_{-\sqrt{18-x^2}}^{\sqrt{18-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{36-x^2-y^2}} 1 \, dz\, dy\, dx$$

c) $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{64-4x^2-4y^2}}^{\sqrt{64-4x^2-4y^2}} 1 \quad dz\, dy\, dx$$

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a) $$\int_{0}^{2}\int_{0}^{2\pi}\int_{2+2\rho^2}^{10-\rho^2} \rho \, dz\, d\theta\, d\rho$$

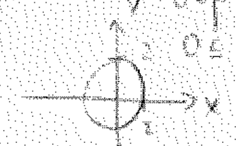
b) $$\int_{0}^{\sqrt{18}}\int_{0}^{2\pi}\int_{\rho}^{\sqrt{36-\rho^2}} \rho \, dz\, d\theta\, d\rho$$

c) $$\int_{0}^{2}\int_{0}^{2\pi}\int_{-\sqrt{64-4\rho^2}}^{\sqrt{64-4\rho^2}} \rho \, dz\, d\theta\, d\rho$$

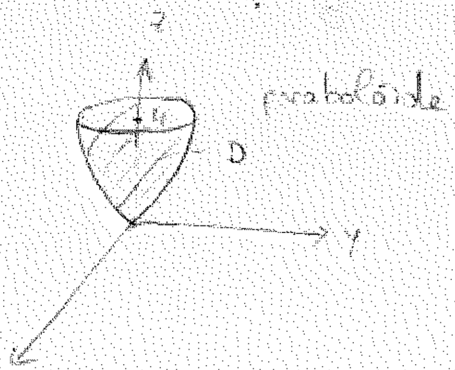
6. $y^2+x^2=z \quad , \quad z=0 \; , \; z=4$

Coordenadas cilíndricas:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases}$$



$0 \leq \rho \leq 2$
$0 \leq \theta \leq 2\pi$



$|J| = \rho$

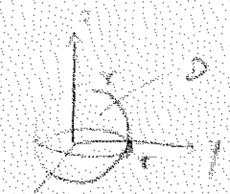
$D = \{(x,y,z) \in \mathbb{R}^3 \mid y^2+x^2=z \; , \; 0 \leq z \leq 4\}$

$$V = \iiint_D 1 \, dV = \int_0^{2\pi}\int_0^{2}\int_{\rho^2}^{4} \rho \, dz \, d\rho \, d\theta =$$

$$= \int_0^{2\pi}\int_0^{2} \rho \left[z \right]_{z=\rho^2}^{z=4} d\rho \, d\theta = \int_0^{2\pi}\int_0^{2} \rho\,(4-\rho^2)\, d\rho\, d\theta = \int_0^{2\pi}\int_0^{2} (4\rho-\rho^3)\, d\rho\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4\rho^2}{2} - \frac{\rho^4}{4} \right]_{\rho=0}^{\rho=2} d\theta = \int_0^{2\pi} (8-4)\, d\theta = 4\left[\theta\right]_0^{2\pi} = 8\pi$$

7. $$x^2+y^2+z^2 = r^2$$



$$V = \iiint_D 1 \; dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta \quad \boxed{\begin{matrix} 0 \leq \rho \leq r \\ 0 \leq \varphi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{matrix}}$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\varphi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=r} d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\varphi \, \frac{r^3}{3}\, d\varphi\, d\theta =$$

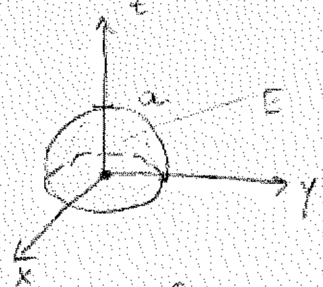
$$= \int_0^{2\pi} \left[-\cos\varphi \, \frac{r^3}{3}\right]_{\varphi=0}^{\varphi=\pi} d\theta = \int_0^{2\pi} \left(\frac{r^3}{3}+\frac{r^3}{3}\right) d\theta = \frac{2r^3}{3}\left[\theta\right]_0^{2\pi} =$$

$$= \frac{2r^3}{3}\, 2\pi = \frac{4}{3}\pi r^3$$

8 $\delta = K(2a - r)$

raio a , $K \in \mathbb{R}$
r - distância de cada ponto ao centro

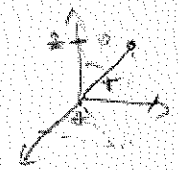
$$m = \iiint_E \delta(x,y,z)\, dV$$



$0 \leq r \leq a$
$x^2 + y^2 + z^2 = a^2$

Coordenadas esféricas:

$x = r \sin\varphi \cos\theta$
$y = r \sin\varphi \sin\theta$
$z = r \cos\varphi$



$0 \leq \theta \leq \pi$
$0 \leq \varphi \leq \pi$
$0 \leq r \leq a$

$$m = \int_0^{\pi}\int_0^{\pi}\int_0^{a} K(2a - r)\, r^2 \sin\varphi \; dr\, d\varphi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi}\int_0^{a} \left(2Kar^2\sin\varphi - Kr^3\sin\varphi\right) dr\, d\varphi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \left[2Ka\frac{r^3}{3}\sin\varphi - K\frac{r^4}{4}\sin\varphi\right]_{r=0}^{r=a} d\varphi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \left(\frac{2}{3}Ka^4\sin\varphi - \frac{Ka^4}{4}\sin\varphi\right) d\varphi\, d\theta$$

$$= \int_0^{\pi} \left[-\frac{2}{3}Ka^4\cos\varphi + \frac{Ka^4}{4}\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} d\theta$$

$$= \int_0^{\pi} \frac{2}{3}Ka^4 - \frac{Ka^4}{4} - \left(-\frac{2}{3}Ka^4 + \frac{Ka^4}{4}\right) d\theta$$

$$= \int_0^{\pi} \left(\frac{4}{3}Ka^4 - \frac{Ka^4}{2}\right) d\theta = \left(\frac{4}{3}Ka^4 - \frac{Ka^4}{2}\right)\pi$$

$$= \frac{5}{6}Ka^4\pi$$