Ficha 9B

1.a) $\iint_D \exp(x+y)\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1} \exp(x+y)\,dy\,dx$

$= \int_{x=0}^{x=1} \left(e^{x+1} - e^{x}\right) dx = e^2 - 2e + 1$

1.b) $\iint_D (x^2-y^2)\,dx\,dy = \int_{x=0}^{x=\pi}\int_{y=0}^{y=\operatorname{sen} x} (x^2-y^2)\,dy\,dx =$

$= \int_{x=0}^{x=\pi} \left(x^2 \operatorname{sen} x - \frac{\operatorname{sen}^3 x}{3}\right) dx$

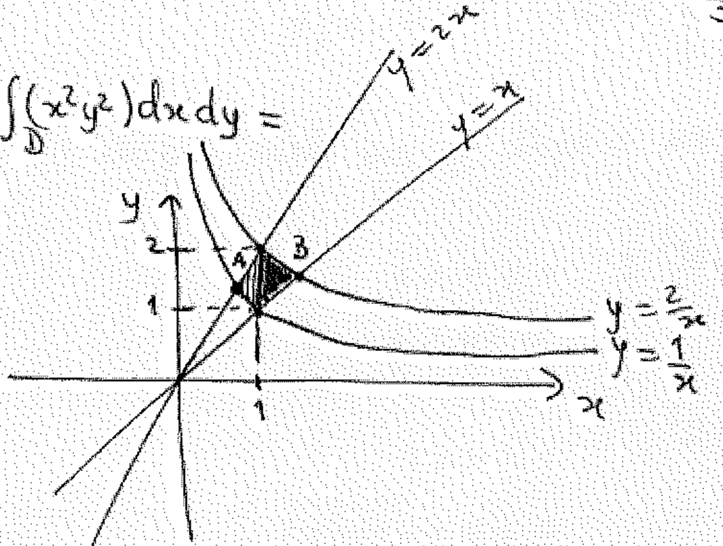
$= -\frac{16}{3} + \pi^2$

pois

$P\, x^2 \operatorname{sen} x = -x^2 \cos x + 2x \operatorname{sen} x + 2\cos x + C$

$P\, \frac{\operatorname{sen}^3 x}{3} = P \operatorname{sen} x \cdot \operatorname{sen}^2 x = -\cos x + \frac{\cos^3 x}{3}$.

c) $\iint_D (x^2 y^2)\,dx\,dy =$



Determinar coordenadas do ponto A

$\begin{cases} y = \frac{1}{x} \\ y = 2x \end{cases} \Leftrightarrow \begin{matrix} x = \frac{\sqrt{2}}{2} \\ y = \sqrt{2} \end{matrix} \qquad A \rightsquigarrow \left(\frac{\sqrt{2}}{2}, \sqrt{2}\right)$

Determinar coordenadas do ponto B

$\begin{cases} y = \frac{2}{x} \\ y = x \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{2} \\ y = \sqrt{2} \end{cases} \qquad B \rightsquigarrow \left(\sqrt{2}, \sqrt{2}\right)$

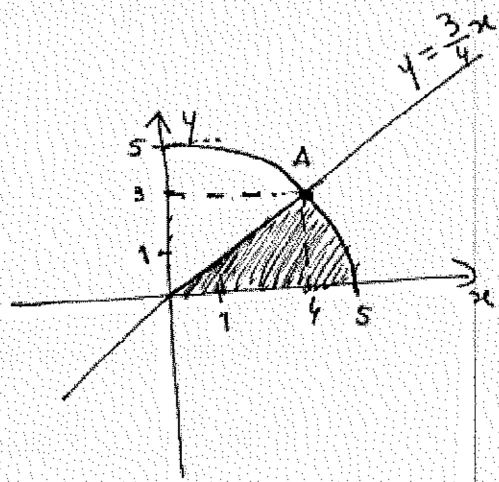
Assim,

$$\iint_D (x^2y^2)\,dx\,dy = \int_{x=\frac{\sqrt{2}}{2}}^{x=1}\int_{y=\frac{1}{x}}^{y=2x} x^2y^2\,dy\,dx + \int_{x=1}^{x=\sqrt{2}}\int_{y=x}^{y=\frac{2}{x}} x^2y^2\,dy\,dx$$

$$= \frac{1}{3}\int_{x=\frac{\sqrt{2}}{2}}^{x=1}\left(8x^5 - \frac{1}{x}\right)dx + \frac{1}{3}\int_{x=1}^{x=\sqrt{2}}\left(\frac{8}{x} - x^5\right)dx$$

$$= \frac{5}{6}\ln 2.$$

2. a) $\displaystyle\int_{y=0}^{y=3} dy \int_{x=\frac{4}{3}y}^{x=\sqrt{25-y^2}} f(x,y)\,dx$



Ptº de intersecção A

$$\begin{cases} x = \frac{4}{3}y \\ x = \sqrt{25-y^2} \end{cases} \quad \begin{cases} y = \pm 3 \\ x = \pm 4 \end{cases} \quad A \rightsquigarrow (4,3)$$

É necessário dividir a região D em duas regiões D_1 e D_2

$$D_1 = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le \frac{3}{4}x \end{array}\right\}$$

$$D_2 = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{l} 4 \le x \le 5 \\ 0 \le y \le \sqrt{25-x^2} \end{array}\right\}$$

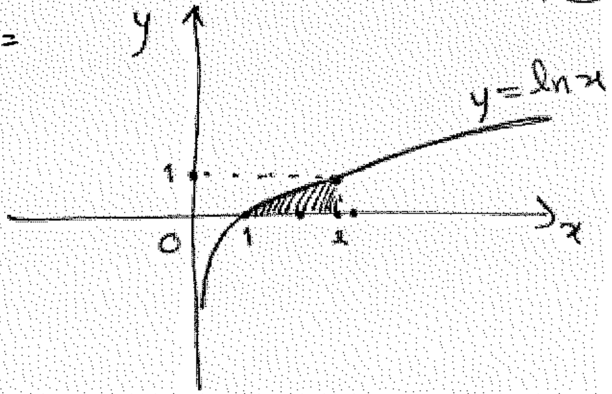
Assim

$$\int_{y=0}^{y=3}\int_{x=\frac{4}{3}y}^{x=\sqrt{25-y^2}} f(x,y)\,dx\,dy$$

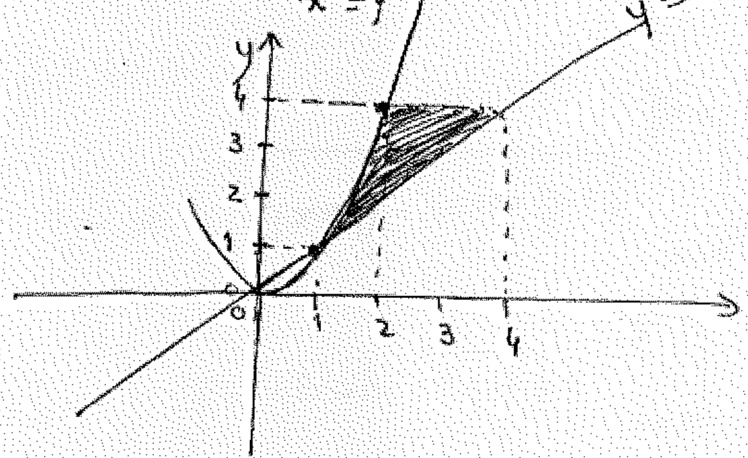
$$= \int_{x=0}^{x=4}\int_{y=0}^{y=\frac{3}{4}x} f(x,y)\,dy\,dx + \int_{x=4}^{x=5}\int_{y=0}^{y=\sqrt{25-x^2}} f(x,y)\,dy\,dx$$

b) $\displaystyle\int_{x=1}^{x=e} \int_{y=0}^{y=\ln x} f(x,y)\,dy\,dx =$

$$= \int_{y=0}^{y=1} \int_{x=e^{y}}^{x=e} f(x,y)\,dx\,dy$$

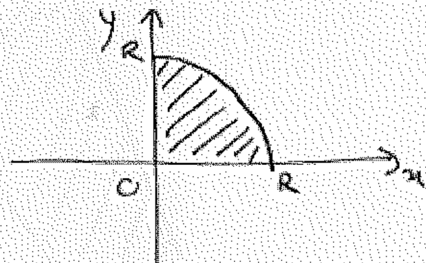


e) $\displaystyle\int_{x=1}^{x=2} \int_{y=x}^{y=x^2} f(x,y)\,dy\,dx + \int_{x=2}^{x=4} \int_{y=x}^{y=4} f(x,y)\,dy\,dx =$



$$= \int_{y=1}^{y=4} \int_{x=\sqrt{y}}^{x=y} f(x,y)\,dx\,dy$$

3.a) $\displaystyle\iint_D xy\,dx\,dy$



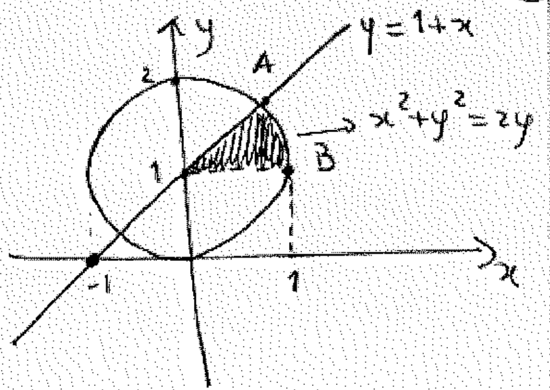
Em coordenadas polares

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases} \qquad \begin{matrix} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \rho \le R \end{matrix} \qquad e \quad dx\,dy = \rho\,d\rho\,d\theta$$

$$\int_{\rho=0}^{\rho=R} \int_{\theta=0}^{\theta=\frac{\pi}{2}} (\rho\cos\theta)(\rho\,\mathrm{sen}\,\theta)\,\rho\,d\theta\,d\rho = \int_{\rho=0}^{\rho=R} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{\rho^3}{2}\,\mathrm{sen}(2\theta)\,d\theta\,d\rho$$

$$= \int_{\rho=0}^{\rho=R} \frac{\rho^3}{2}\,d\rho = \frac{R^4}{8}.$$

b) $\iint_D \sqrt{1-x^2-(1-y)^2}\;dx\,dy$



$x^2+y^2=2y \Leftrightarrow$

$\Leftrightarrow x^2+(y-1)^2=1$

Assim, $\begin{cases} x=\rho\cos\theta \\ y-1=\rho\,\text{sen}\,\theta \end{cases}$ e a região D é descrita por

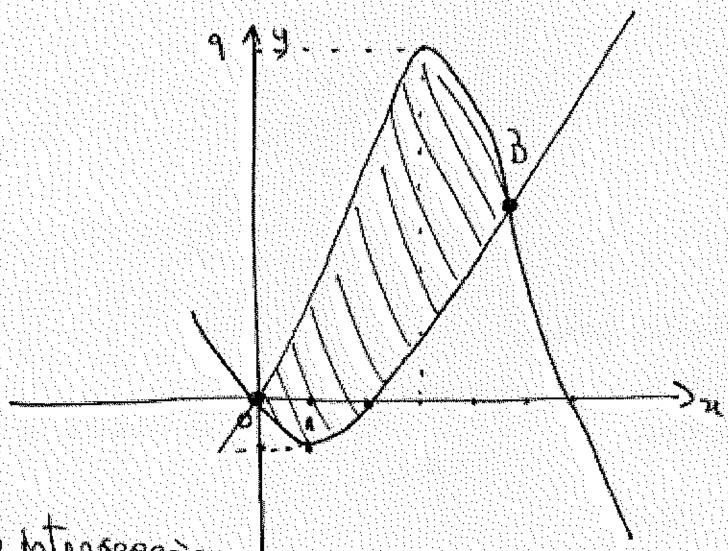
$0\le\rho\le 1$

$0\le\theta\le\frac{\pi}{4}$

$$\int_{\rho=0}^{\rho=1}\int_{\theta=0}^{\theta=\frac{\pi}{4}}\sqrt{1-(\rho\cos\theta)^2-(\rho\,\text{sen}\,\theta)^2}\;\rho\,d\theta\,d\rho$$

$$=\int_{\rho=0}^{\rho=1}\int_{\theta=0}^{\theta=\frac{\pi}{4}}\rho\sqrt{1-\rho^2}\;d\theta\,d\rho=\frac{\pi}{4}\int_0^1\rho\sqrt{1-\rho^2}\,d\rho=\frac{1}{3}$$

4.a) $y=6x-x^2$ e $y=x^2-2x$



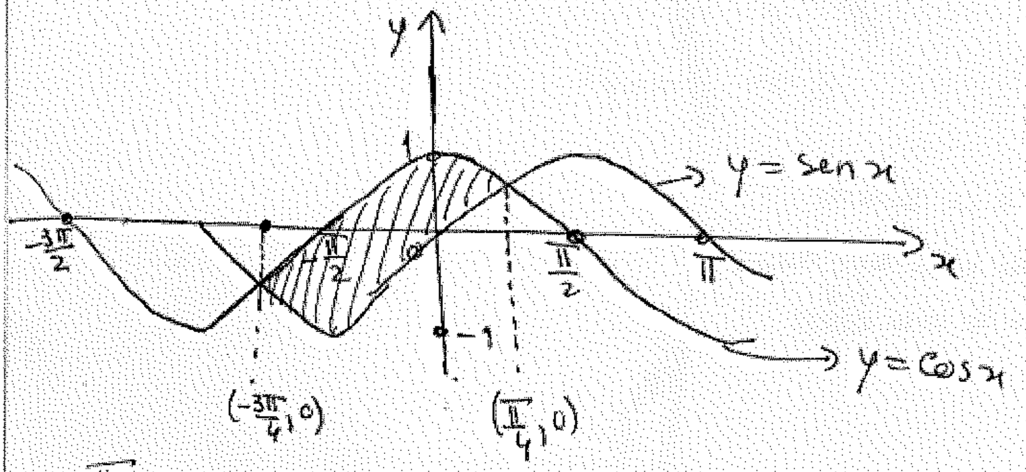
B é ptº de intersecção

$\begin{cases} y=6x-x^2 \\ y=x^2-2x \end{cases}$ $B\to(4,8)$

Área da região é dada pelo integral

$$\int_{x=0}^{x=4}\int_{y=x^2-2x}^{y=6x-x^2} 1 \, dy \, dx = \frac{4^3}{3}$$

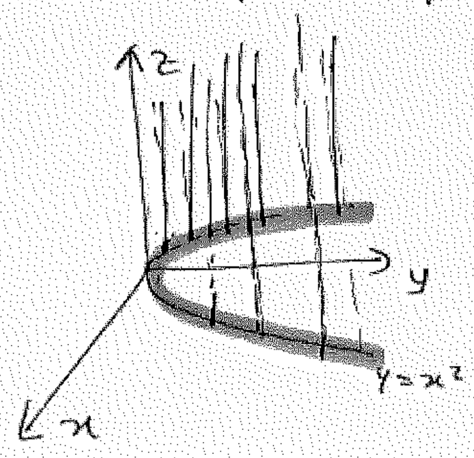
b) $y = \operatorname{sen} x$, $y = \cos x$, $-\frac{3\pi}{4} \leq x \leq \frac{\pi}{4}$



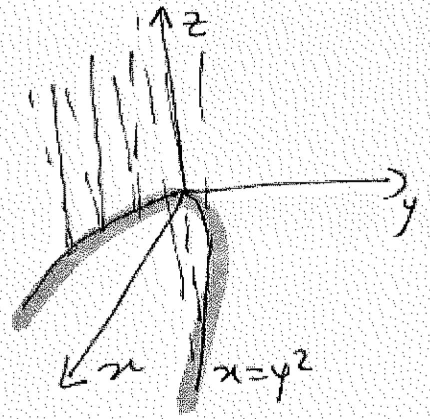
$$\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\int_{\operatorname{sen} x}^{\cos x} 1 \, dy \, dx = 2\sqrt{2} .$$

5 $S = \{(x,y,z) \in \mathbb{R}^3 : y \geq x^2 \wedge x \geq y^2 \wedge 0 \leq z \leq 3\}$

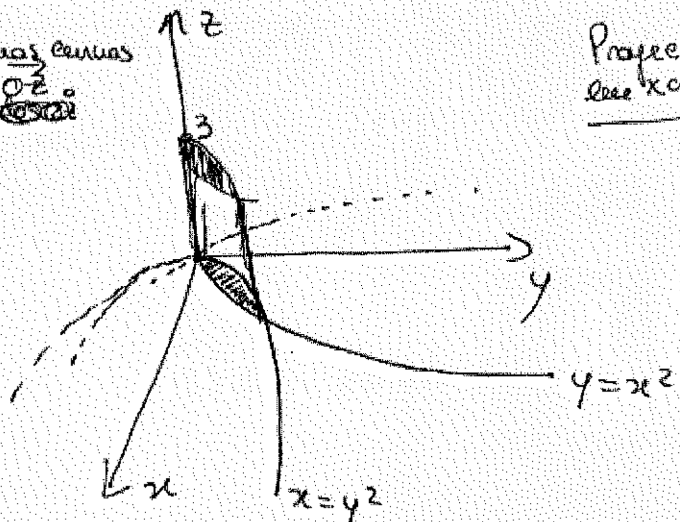
$y = x^2$ é um cilindro com geratriz paralela a $\overline{Oz}$ e directriz a parábola $y = x^2$



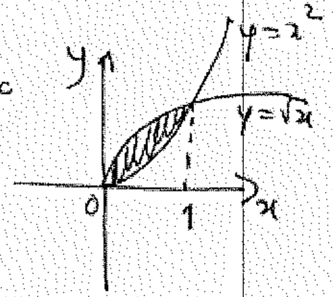
$x=y^2$ é um cilindro com geratriz paralela a $\vec{Oz}$ e directriz a parábola $x=y^2$



S é o sólido limitado pelas duas curvas ao longo do eixo $\vec{Oz}$



Projecção em xoy



$$\int_{z=0}^{z=3}\int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} 1\,dy\,dx\,dz = 1.$$