Soluções da Ficha 6B - Integral definido

1. Propriedades do integral.

(a) $\displaystyle\int_1^4 f(t)\,dt = \int_1^4 f(x)\,dx = 3.$

(b) $\displaystyle\int_4^2 f(t)\,dt = -\int_{-2}^4 f(t)\,dt = -\int_{-2}^4 f(x)\,dx = -5.$

(c) $\displaystyle\int_1^2 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^2 f(x)\,dx = \int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx = 3-5=-2.$

(d) $\displaystyle\int_{1/2}^2 f(2x)\,dx \underset{\text{substituição } x=\frac{1}{2}t}{=} \int_1^4 f(t)\frac{1}{2}\,dt = \frac{1}{2}\int_1^4 f(t)\,dt = \frac{3}{2}.$

2. Integral e área de uma região plana.

Na figura podemos identificar 5 regiões planas limitadas pelo gráfico de f, pelo eixo OX e por rectas verticais. São elas 4 regiões triangulares e uma região quadrangular, digamos T_1, T_2, Q, T_3 e T_4.

(a) $\displaystyle\int_0^1 f(x)dx = \text{área } (T_1) = \frac{1}{2}.$

(b) $\displaystyle\int_1^2 f(x)dx = -\text{área } (T_2) = -\frac{1}{2}.$

(c) $\displaystyle\int_0^5 f(x)dx = \text{área } (T_1) - \text{área}(T_2) - \text{área}(Q) - \text{área}(T_3) + \text{área } (T_4) = -1.$

3. Exemplos.

(a) $f\colon [0,2] \longrightarrow \mathbb{R}, \quad \displaystyle\int_0^2 f(x)\,dx = 0, \quad f(x) \neq 0, \forall x \in [0,2].$

Por exemplo $f(x) = \begin{cases} -1 & \text{se} \quad 0 \le x \le 1, \\ 1 & \text{se} \quad 1 < x \le 2. \end{cases}$

(b) $f, g\colon [0,2] \longrightarrow \mathbb{R}, \quad \displaystyle\int_0^2 f(x)\,dx = \int_0^2 g(x)\,dx, \quad f(x) \neq g(x), \forall x \in [0,2].$

Por exemplo $f(x) = \begin{cases} 1 & \text{se} \quad 0 \le x \le 1, \\ 3 & \text{se} \quad 1 < x \le 2, \end{cases}$ e $g(x) = 2, \ \forall x \in [0,2].$

Os dois integrai são iguais a 4.

4. Cálculo de integrais definidos.

(a) $\displaystyle\int_0^3 \sqrt{9-x^2}\; dx;$ Substituição $x = 3\,\mathrm{sen}\, t$.

Obter $\displaystyle\int_0^3 \sqrt{9-x^2}\; dx = 9\int_0^{\pi/2} \cos^2 t\, dt = \frac{9}{2}\left[t + \frac{1}{2}\,\mathrm{sen}\, 2t\right]_0^{\pi/2} = \frac{9\pi}{4}.$

(b) $\displaystyle\int_0^1 \ln(x^2+1)dx = \ln 2 - 2 + \frac{\pi}{2}.$ (Partes)

(c) $\displaystyle\int_0^\pi x\,\mathrm{sen}\, x\; dx = \pi.$ (Partes)

(d) $\displaystyle\int_0^{\sqrt{2}/2} \mathrm{arcsen}\, x\; dx = \frac{\sqrt{2}}{2}\left(\frac{\pi}{4}+1\right) - 1.$ (Partes)

(e) $\displaystyle\int_{-3}^2 \sqrt{|x|}\; dx = \int_{-3}^0 \sqrt{-x}\, dx + \int_0^2 \sqrt{x}\, dx = 2\sqrt{3} + \frac{4}{3}\sqrt{2}.$

(f) $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}\; dx = \frac{\pi}{2}.$

(g) $\displaystyle\int_3^4 \frac{1-4x^3}{x-x^4}\; dx = \left[\ln\left|x-x^4\right|\,\right]_3^4 = \ln\frac{42}{13}.$

(h) $\displaystyle\int_0^{\frac{\pi}{2}} \mathrm{sen}\, 2x \cos 5x\; dx = -\frac{2}{21}.$ (Partes $2\times$, reaparece a primitiva inicial)

(i) $\displaystyle\int_0^1 x\,\mathrm{arctg}\, x^2\; dx = \frac{\pi}{8} - \frac{1}{4}\ln 2.$ (Partes)

(j) $\displaystyle\int_0^3 2-|x|\; dx = \frac{3}{2}.$

(k) $\displaystyle\int_0^2 \frac{2x-1}{(x-3)(x+1)}\; dx = \frac{5}{4}\int_0^2 \frac{1}{x-3}\, dx + \frac{3}{4}\int_0^2 \frac{1}{x+1}\, dx = -\frac{1}{2}\ln 3.$

(l) $\displaystyle\int_{-3}^2 \sqrt{|x|}\; dx$ Feito na alínea (e).

(m) $\displaystyle\int_e^{e^2} \frac{\ln(\ln x^2)}{x}\; dx = \Big[\ln x\, \ln(\ln x^2)\Big]_e^{e^2} - \int_e^{e^2} \frac{1}{x}\, dx = 3\ln 2 - 1.$

(n) $\displaystyle\int_0^8 \frac{\sqrt[3]{x}}{\sqrt[3]{x}^2+1}\; dx$ Substituição $x = t^3$.

Obter $\displaystyle\int_0^8 \frac{\sqrt[3]{x}}{\sqrt[3]{x}^2+1}\; dx = \int_0^2 \frac{t}{t^2+1}\, 3t^2\, dt = 6 - \frac{3}{2}\ln 5.$

(o) $\displaystyle\int_0^{2\pi} |\cos x|\ dx = 4\int_0^{\pi/2} \cos x\, dx = 4.$

apelando à noção de área

(p) $\displaystyle\int_{-1}^{2} x|x|\ dx = \int_{-1}^{0}(-x^2)dx + \int_0^2 x^2 dx = \frac{7}{3}.$

(q) $\displaystyle\int_0^1 g(x)\ dx, \quad \text{com} \ \ g(x) = \begin{cases} x & \text{se} \quad 0 \le x \le \frac{1}{2}, \\ -x & \text{se} \quad \frac{1}{2} < x \le 1; \end{cases}$

$$\int_0^1 g(x)\ dx = \int_0^{1/2} x\, dx - \int_{1/2} x\, dx = -\frac{1}{4}.$$

(r) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\ dx$, utilizando a mudança de variável definida por $x = \operatorname{tg} t$.

Obter $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\ dx = \int_0^{\pi/4} \frac{1}{(1+\operatorname{tg}^2 t)^2}\,\frac{1}{\cos^2 t}\, dt = \int_0^{\pi/4} cos^2 t\, dt = \frac{\pi}{8} + \frac{1}{4}.$