$\boxed{\text{F.4}}$

(1) $z = \cos(x^2 y) \; ; \; x = s^3 t^2 \; ; \; y = s^2 + \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial s} \qquad z < \begin{matrix} x < \begin{matrix} s \\ t \end{matrix} \\ y < \begin{matrix} s \\ t \end{matrix} \end{matrix}$$

$$= -2xy \sin(x^2 y) \times (3s^2t^2) + (-x^2 \sin(x^2 y)) \times (2s)$$

$$= -2(s^3t^2)\left(s^2+\frac{1}{t}\right) \sin\left(s^6t^4\left(s^2+\frac{1}{t}\right)\right) \times (3s^2t^2) +$$
$$+ \left(-s^6t^4 \sin\left(s^6t^4\left(s^2+\frac{1}{t}\right)\right) \times (2s)\right.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial t}$$

$$= -2xy \sin(x^2 y) \times (2s^3t) - x^2 \sin(x^2 y)\left(-\frac{1}{t^2}\right)$$

$$= -2(s^3t^2)\left(s^2+\frac{1}{t}\right) \sin\left(s^6t^4\left(s^2+\frac{1}{t}\right)\right)(2s^3t) +$$
$$+ \frac{s^6t^4}{t^2} \sin\left(s^6t^4\left(s^2+\frac{1}{t}\right)\right)$$

(2) $z = txy^2 \; ; \; x = t + \ln(y+t^2) \; ; \; y = e^t$

$$z < \begin{matrix} t \\ x < \begin{matrix} y - t \\ t \end{matrix} \\ y - t \end{matrix}$$

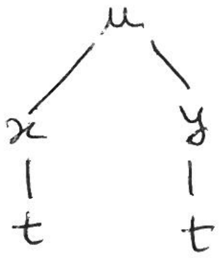
$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{dy}{dt} + \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= xy^2 + ty^2\left(\frac{1}{y+t^2} \cdot e^t\right) + ty^2\left(1 + \frac{2t}{y+t^2}\right) + 2txy\,(e^t)$$

$$= (t + \ln(e^t+t^2))\, e^{2t} + te^{2t}\left[\frac{e^t}{e^t+t^2} + 1 + \frac{2t}{e^t+t^2}\right] + 2t(t+\ln(e^t+t^2))e^{2t}$$

③ $\dfrac{d^2u}{dt^2} = ?$ $\qquad u = e^{x-2y}$ $\qquad x = \sin t$

$y = t^3$



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = e^{x-2y}\cos t + (-2)e^{x-2y}\, 3t^2$$

$$\frac{du}{dt} = e^{x-2y}\cos t - 6t^2 e^{x-2y}$$

$$\frac{du}{dt} = e^{\sin t - 2t^3}\cos t - 6t^2 e^{\sin t - 2t^3}$$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(e^{\sin t - 2t^3}\cos t - 6t^2 e^{\sin t - 2t^3}\right)$$

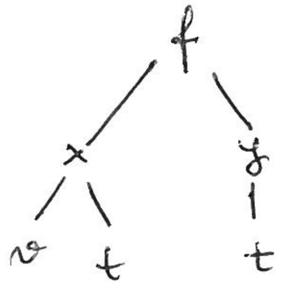
$$\frac{d^2u}{dt^2} = (\cos t - 6t^2)\, e^{\sin t - 2t^3}\cos t - e^{\sin t - 2t^3}\sin t$$

$$-12t e^{\sin t - 2t^3} - 6t^2(\cos t - 6t^2)\, e^{\sin t - 2t}$$

④ $z = f(x,y)$ $\qquad \dfrac{\partial^2 z}{\partial v^2} = ?$ $\qquad \dfrac{\partial^2 z}{\partial v \partial t} = ?$ $\qquad \dfrac{\partial^2 z}{\partial t^2} = ?$

$x = 2v + \ln t$

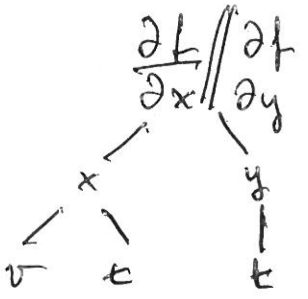
$y = 1/t$



$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} = \frac{\partial f}{\partial x}\cdot 2$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(2\frac{\partial f}{\partial x}\right) = 2\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right)$$

$$= 2\cdot\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial v} = 2\frac{\partial^2 f}{\partial x^2}\cdot 2 = 4\frac{\partial^2 f}{\partial x^2}$$



$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{t}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{t^2}\right)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{1}{t} - \frac{1}{t^2}\frac{\partial f}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right)\left(-\frac{1}{t^2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{t^2}\right)'_t$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial t}\ \frac{1}{t} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt}\ \frac{1}{t} - \frac{1}{t^2}\frac{\partial f}{\partial x}$$

$$- \frac{1}{t^2}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}\right] + \frac{\partial f}{\partial y}\left(\frac{2t}{t^4}\right)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}\cdot\left(\frac{1}{t}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\left(-\frac{1}{t^3}\right) - \frac{1}{t^2}\frac{\partial f}{\partial x} - \frac{1}{t^2}\left(\frac{\partial^2 f}{\partial x \partial y}\frac{1}{t} + \frac{\partial^2 f}{\partial y^2}\left(-\frac{1}{t^2}\right)\right)$$

$$+ \frac{2}{t^3}\frac{\partial f}{\partial y}$$

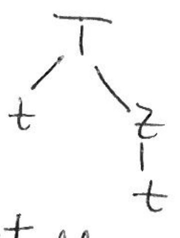
$$\frac{\partial^2 z}{\partial t \partial v} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\cdot\frac{1}{t} + \frac{\partial f}{\partial y}\left(-\frac{1}{t^2}\right)\right)$$

$$= \frac{1}{t}\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right) - \frac{1}{t^2}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{t}\cdot\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial v} - \frac{1}{t^2}\cdot\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial v}$$

$$= \frac{1}{t}\frac{\partial^2 f}{\partial x^2}\cdot 2 - \frac{1}{t^2}\frac{\partial^2 f}{\partial x \partial y}\cdot 2$$

(5) $T = e^{-t} z \qquad z = f(t)$



a) $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial t} + \dfrac{\partial T}{\partial z}\dfrac{dz}{dt} = -e^{-t} f(t) + e^{-t}\dfrac{df}{dt}$

b) $f(t) = e^t$

$$\frac{dT}{dt} = -e^{-t}e^{t} + e^{-t}e^{t} = -1 + 1 = 0$$

(F4) Funções implícitas

1. $1+y=x^2-\ln y \rightarrow F(x,y)=1+y-x^2+\ln y=0$

a)
- $F(\sqrt{2},1)=1+1-2+\ln 1=0$ ✓
- $\frac{\partial F}{\partial x}=-2x$, $\frac{\partial F}{\partial y}=1+\frac{1}{y}$. As funções $\frac{\partial F}{\partial x}$ e $\frac{\partial F}{\partial y}$ são contínuas numa vizinhança do ponto $(\sqrt{2},1)$
- $\frac{\partial F}{\partial y}(\sqrt{2},1)=1+1=2\neq 0$ ✓

$\Rightarrow$ A equação $F(x,y)=0$ define implicitamente y com função de x numa vizinhança do ponto $(\sqrt{2},1)$.

b) $$\frac{dy(\sqrt{2})}{dx}=-\frac{\frac{\partial F}{\partial x}(\sqrt{2},1)}{\frac{\partial F}{\partial y}(\sqrt{2},1)}=-\frac{-2\sqrt{2}}{1+1}=\sqrt{2}$$

$$\frac{d^2y}{dx^2}(\sqrt{2})=\frac{d}{dx}\left(\frac{dy(\sqrt{2})}{dx}\right)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{-2x}{1+\frac{1}{y}}\right)=\frac{d}{dx}\left(\frac{2x}{1+\frac{1}{y}}\right)=$$
$$=\frac{2\left(1+\frac{1}{y}\right)-2x\,(-y^{-2})\frac{dy}{dx}}{\left(1+\frac{1}{y}\right)^2}$$

$$\frac{d^2y}{dx^2}(\sqrt{2})=\frac{2\,(1+1)-2\sqrt{2}\,(-1)\,\frac{dy}{dx}(\sqrt{2})}{2^2}=\frac{4+2\sqrt{2}\cdot\sqrt{2}}{4}=2$$

c) equação da recta tangente no ponto de abcissa

$y=mx+b$ $\quad m=\frac{dy}{dx}(\sqrt{2})\overset{b)}{=}\sqrt{2}$

$$\left.\begin{array}{l} y=\sqrt{2}x+b \\ x=\sqrt{2} \\ y=1 \end{array}\right\} \Rightarrow \begin{array}{l} 1=\sqrt{2}\sqrt{2}+b \\ b=-1 \end{array}$$

(2.) $x^3+y^3=6xy$ $\qquad F(x,y)=x^3+y^3-6xy=0$

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{3x^2-6y}{3y^2-6x}$$

$$=-\frac{x^2-2y}{y^2-2x}$$

$F(x,y(x))=0$

$$\frac{dF}{dx}=\underbrace{\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}}=0$$

$$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$

(3.) $x^2y^2+x-2y^3=0$ $\qquad F(x,y)=x^2y^2+x-2y^3$

$$\frac{y}{x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{2xy^2+1}{2x^2y-6y^2}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{2xy^2+1}{2x^2y-6y^2}\right)=$$

$$=-\frac{\left(2y^2+4xy\frac{dy}{dx}\right)\left(2x^2y-6y^2\right)-\left(2xy^2+1\right)\left(4xy+2x^2\frac{dy}{dx}-12\right.}{\left(2x^2y-6y^2\right)^2}$$