Ficha nº 1

1. $\vec{f}(t) = (t, t^2)$, $t \in \mathbb{R}$

a) $\vec{f}(0) = (0,0)$

$\vec{f}(-1) = (-1,1)$

b) $\vec{f}(t) = (f_1(t), f_2(t))$ onde $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto t$

$f_2 : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto t^2$

$D_{\vec{f}} = D_{f_1} \cap D_{f_2} = \mathbb{R} \cap \mathbb{R} =$
$= \mathbb{R}$

2. $\vec{f}(t) = (\cos t, \operatorname{sen} t)$, $t \in \mathbb{R}$

a) $\vec{f}(0) = (\cos 0, \operatorname{sen} 0) = (1,0)$

$\vec{f}(\pi) = (\cos \pi, \operatorname{sen} \pi) = (-1,0)$

b) $\vec{f}(t) = (f_1(t), f_2(t))$ onde $f_1 : \mathbb{R} \to \mathbb{R}$, $t \mapsto \cos t$

$f_2 : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto \operatorname{sen} t$

$D_{\vec{f}} = D_{f_1} \cap D_{f_2} = \mathbb{R}.$

3. $\vec{f}(t) = \left(\frac{1}{t}, \ln t\right)$

a) $\vec{f}(1) = (1, \ln 1) = (1,0)$

$\vec{f}(2) = \left(\frac{1}{2}, \ln 2\right)$

b) $\vec{f}(t) = (f_1(t), f_2(t))$ onde $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto \frac{1}{t}$

$f_2 : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto \ln t$

$D_{\vec{f}} = D_{f_1} \cap D_{f_2} =$
$= \mathbb{R}\setminus\{0\} \cap]0,+\infty[$
$=]0,+\infty[= \mathbb{R}_+$

4 a) $\lim\limits_{t\to 1}\left(t^2+1, \frac{1}{t}\right) = \left(1+1, \frac{1}{1}\right) = (2,1)$

b) $\lim\limits_{t\to 0}\left(2t+1, \ln t\right) = (1, \ln 0) = (1, -\infty)$

5. a) $\vec{R}(t) = \left(\frac{1}{t+1}, \cos t, t^3\right)$

$\vec{R}'(t) = \left(\left(\frac{1}{t+1}\right)', (\cos t)', (t^3)'\right)$

$\vec{R}'(t) = \left(-\frac{1}{(t+1)^2}, -\operatorname{sen} t, 3t^2\right)$

b) $\vec{R}(t) = \left(e^{t^2}, \ln(t+1)\right)$

$\vec{R}'(t) = \left(\left(e^{t^2}\right)', \left(\ln(t+1)\right)'\right)$

$\vec{R}'(t) = \left(2t \cdot e^{t^2}, \frac{1}{t+1}\right)$

6. $\int_0^1 \vec{F}(t)\,dt = \int_0^1 \left(t^2, e^{2t}\right) dt = \left(\int_0^1 t^2 dt, \int_0^1 e^{2t} dt\right)$

$= \left(\left[\frac{t^3}{3}\right]_0^1, \left[\frac{e^{2t}}{2}\right]_0^1\right) = \left(\frac{1}{3}, \frac{e^2}{2} - \frac{e^0}{2}\right) = \left(\frac{1}{3}, \frac{e^2-1}{2}\right).$

7. $\vec{R}(t) = (\operatorname{sen} t, \cos t)$ é contínua em $\frac{\pi}{4}$?

$\vec{R}\left(\frac{\pi}{4}\right) = \left(\operatorname{sen}\frac{\pi}{4}, \cos\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\lim_{t\to\frac{\pi}{4}} \vec{R}(t) = \lim_{t\to\frac{\pi}{4}} (\text{sen}\, t, \cos t) = \left(\text{sen}\frac{\pi}{4}, \cos\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$ ③

Como $\lim_{t\to\frac{\pi}{4}} \vec{R}(t) = \vec{R}\left(\frac{\pi}{4}\right)$, então $\vec{R}(t)$ é contínua em $\frac{\pi}{4}$.

8. $$\vec{R}(t) = \begin{cases} (1,0,1) & \text{se } t=0 \\ \left(\frac{\text{sen}\, t}{t}, \frac{1-\cos t}{t}, t+1\right) & \text{se } t\neq 0 \end{cases}$$

$D_{\vec{R}} = \mathbb{R}$

Quando $t\neq 0$, $\vec{R}(t)$ é contínua, pois as funções reais $R_1(t) = \frac{\text{sen}\, t}{t}$, $R_2(t) = \frac{1-\cos t}{t}$, $R_3(t) = t+1$ são contínuas em $\mathbb{R}\setminus\{0\}$ (quocientes de funções contínuas em $\mathbb{R}\setminus\{0\}$).

Quando $t=0$:

$\vec{R}(0) = (1,0,1)$

$$\lim_{t\to 0} \vec{R}(t) = \lim_{t\to 0} \left(\frac{\text{sen}\, t}{t}, \frac{1-\cos t}{t}, t+1\right) =$$

$$= \left(\lim_{t\to 0} \frac{\text{sen}\, t}{t}, \lim_{t\to 0} \frac{1-\cos t}{t}, \lim_{t\to 0} (t+1)\right)$$

$= (1, 0, 1)$.

9. $\vec{f}(t) = \left(\frac{\cos t}{t}, \ln t, \sqrt{t+1}\right)$

a) $\vec{f}(\pi) = \left(\frac{\cos \pi}{\pi}, \ln \pi, \sqrt{\pi+1}\right) = \left(-\frac{1}{\pi}, \ln \pi, \sqrt{\pi+1}\right)$

b) $D_{\vec{f}} = D_{\frac{\cos t}{t}} \cap D_{\ln t} \cap D_{\sqrt{t+1}}$

$D_{\vec{f}} = \mathbb{R}\setminus\{0\} \cap]0,+\infty[\cap [-1,+\infty[$

$D_{\vec{f}} =]0,+\infty[$

c) $\lim_{t\to 1} \vec{f}(t) = \left(\lim_{t\to 1} \frac{\cos t}{t}, \lim_{t\to 1} \ln t, \lim_{t\to 1} \sqrt{t+1}\right) =$

$= \left(\cos 1, 0, \sqrt{2}\right)$

$\lim_{t\to 0} \vec{f}(t) = \left(\lim_{t\to 0} \frac{\cos t}{t}, \lim_{t\to 0} \ln t, \lim_{t\to 0} \sqrt{t+1}\right) =$

$= \left(+\infty, -\infty, 1\right).$

d) $\vec{f}$ é contínua em $]0,+\infty[$ pois as funções componentes são contínuas nesse conjunto.

10. $\vec{R}(t) = \vec{u} + \vec{v}.\cos t + \vec{w}.\operatorname{sen} t$

$\vec{u} = 2\vec{e_1} + \vec{e_2}$, $\vec{v} = \vec{e_2} - \vec{e_3}$ $\vec{w} = \vec{e_2} + \vec{e_3}$

$\vec{R}(t) = 2\vec{e_1} + \vec{e_2} + (\vec{e_2} - \vec{e_3}).\cos t + (\vec{e_2} + \vec{e_3}).\operatorname{sen} t =$

$= 2\vec{e_1} + (1 + \cos t + \operatorname{sen} t)\,\vec{e_2} + (-\cos t + \operatorname{sen} t)\,\vec{e_3}$

$\vec{R}(t) = (2, 1 + \cos t + \operatorname{sen} t, -\cos t + \operatorname{sen} t)$

b) $\vec{R}'(t) = (0, -\operatorname{sen} t + \cos t, \operatorname{sen} t - \cos t)$

c) $\vec{R}''(t) = (0, -\cos t - \operatorname{sen} t, \cos t + \operatorname{sen} t)$.

11. $\vec{R}(0) = \vec{e_3}$, $\vec{R}'(0) = \vec{e_1} + \vec{e_2}$, $\vec{R}''(t) = -\vec{e_3}$

$$\vec{R}''(t) = -\vec{e_3} = (0, 0, -1)$$

$\vec{R}'(t) = (C_1, C_2, -t + C_3)$ e $\vec{R}'(0) = \vec{e_1} + \vec{e_2} = (1, 1, 0)$

$\vec{R}'(0) = (C_1, C_2, C_3) = (1, 1, 0)$

Daí $C_1 = C_2 = 1$, $C_3 = 0$

Logo

$$\vec{R}'(t) = (1, 1, -t)$$

E

$\vec{R}(t) = \left(t + D_1, t + D_2, -\frac{t^2}{2} + D_3\right)$ $\vec{R}(0) = (0, 0, 1)$

$\vec{R}(0) = (D_1, D_2, D_3) = (0, 0, 1)$ Daí

$D_1 = D_2 = 0$, $D_3 = 1$

Logo

$$\vec{R}(t) = \left(t, t, -\frac{t^2}{2} + 1\right)$$

Determinar t_0, tal que $\vec{R}(t_0)$ é vector director do plano xoy. Nesse caso, $\vec{R}(t_0)$ tem que ter a 3ª componente nula.

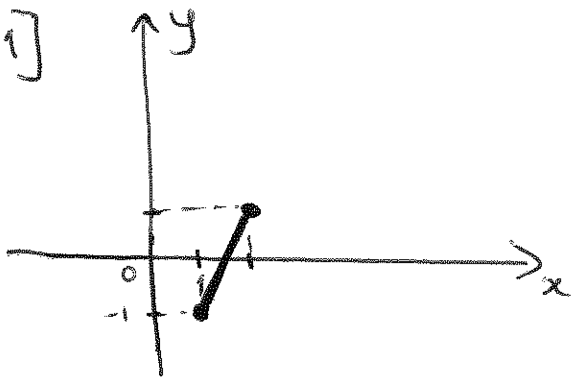
$$-\frac{t_0^2}{2} + 1 = 0 \Leftrightarrow -t_0^2 = -2 \Leftrightarrow \boxed{t_0 = \pm\sqrt{2}}$$

12. $\vec{R}(t) = \vec{u} + t\vec{v}$, $t \in \mathbb{R}$ $\quad \vec{u} = (a, b, c)$
$\vec{v} = (l, m, n)$

a) $\vec{R}(t) = (a, b, c) + t(l, m, n) =$
$\vec{R}(t) = (a + tl, b + tm, c + tn)$.

b) $\vec{R}'(t) = \left((a+tl)', (b+tm)', (c+tn)'\right) = (l, m, n) = \vec{v}$.

13. a) $\begin{cases} x = 1+t \\ y = -1+2t \end{cases}$, $t \in [0,1]$



$t = 0 \Rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$ $(1,-1)$

$t = 1 \Rightarrow \begin{cases} x = 2 \\ y = -1+2 \end{cases}$ $(2,1)$

$\begin{cases} x = 1+t \\ y = -1+2t \end{cases}$ $\begin{cases} t = x-1 \\ 2t = y+1 \end{cases}$ $\begin{cases} t = x-1 \\ t = \frac{y+1}{2} \end{cases}$ $\Rightarrow x - 1 = \frac{y+1}{2}$ → recta.

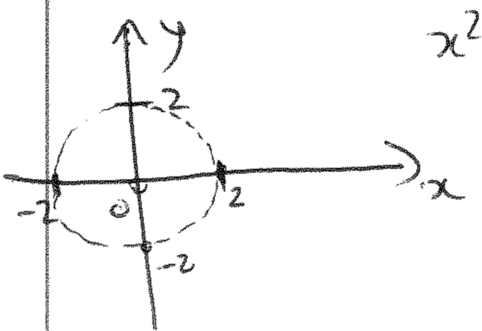
$x - 1 = \frac{y+1}{2}$, $t \in [0,1]$ → segmento de recta entre $(1,-1)$ e $(2,1)$.

b) $\vec{F}(t) = (2\cos t, 2\,\text{sen}\,t)$ $\quad t \in [0, 2\pi]$

$\begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\,t \end{cases}$ Como $(2\cos t)^2 + (2\,\text{sen}\,t)^2 = 4(\cos^2 t + \text{sen}^2 t) =$
$= 4 \times 1 = 4$

$x^2 + y^2 = (2\cos t)^2 + (2\,\text{sen}\,t)^2 = 4$.

$x^2 + y^2 = 4$ → circunf. de raio 2

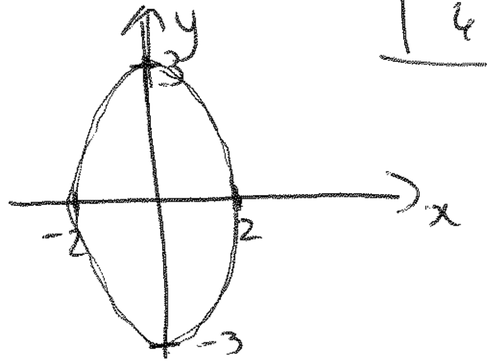


c) $\vec{r}(t) = (2\cos t, 3\,\text{sen}\,t) \quad t \in [0, 2\pi]$

$$\begin{cases} x = 2\cos t \\ y = 3\,\text{sen}\,t \end{cases} \Leftrightarrow \begin{cases} \frac{x}{2} = \cos t \\ \frac{y}{3} = \text{sen}\,t \end{cases}$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \text{sen}^2 t = 1.$$

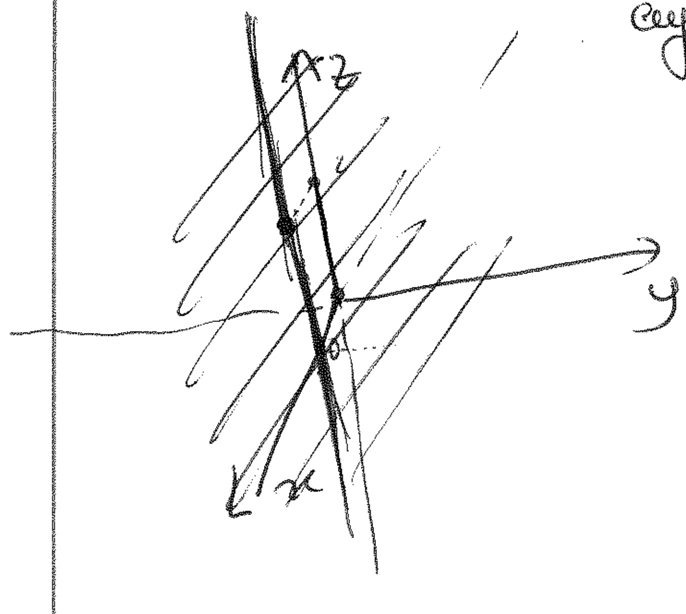
$$\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1} \rightarrow \text{elipse}$$

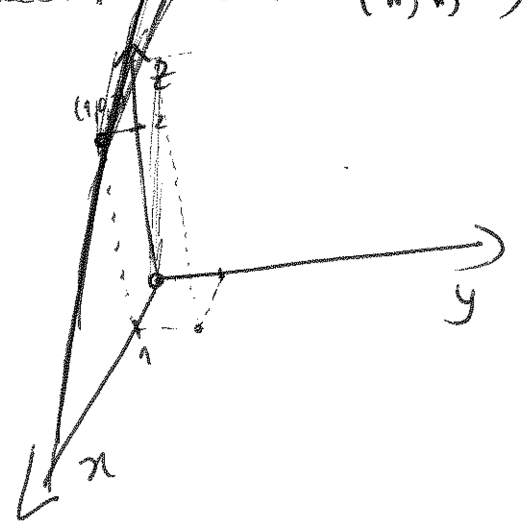


d) $\vec{r}(t) = (1+t, t, 2+3t) \longrightarrow \begin{cases} x = 1+t \\ y = t \\ z = 2+3t \end{cases}, \quad t \in \mathbb{R}$.

$$\Leftrightarrow \begin{cases} x - 1 = t \\ y = t \\ \frac{z-2}{3} = t \end{cases}, t \in \mathbb{R} \Leftrightarrow x - 1 = y = \frac{z-2}{3}$$

$\rightarrow$ Recta que passa em $(1,0,2)$ cujo vector director é $(1,1,3)$.

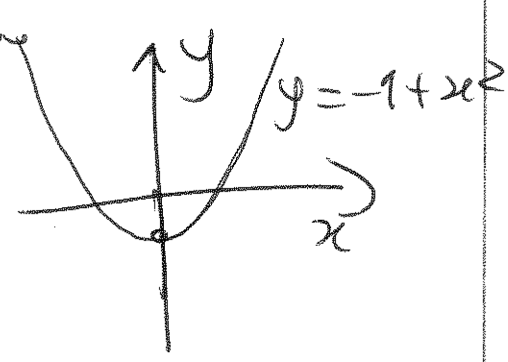




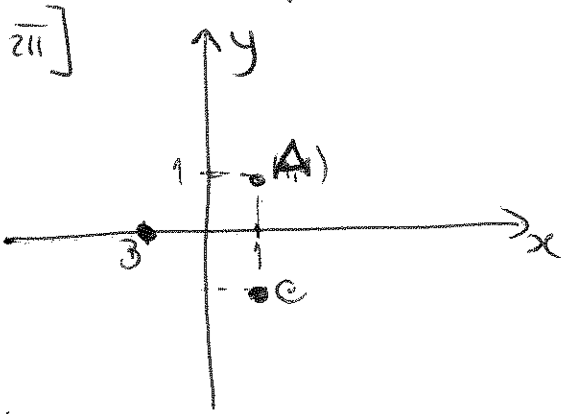
e) $\begin{cases} x = t \\ y = -1 + t^2 \end{cases} , t \in \mathbb{R}$

substituindo t na 2ª eq. cartesiana

$y = -1 + x^2 \rightarrow$ parábola



14. $\vec{R}(\theta) = \begin{cases} x = \cos(2\theta) \\ y = \cos\theta \end{cases} , \theta \in [0, 2\pi]$



$\vec{R}(0) = (\cos 0, \cos 0) = (1,1) \rightarrow A$

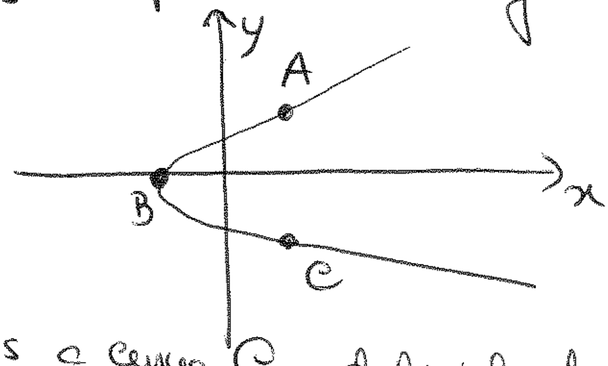
$\vec{R}\left(\frac{\pi}{2}\right) = \left(\cos\pi, \cos\frac{\pi}{2}\right) = (-1, 0) \rightarrow B$

$\vec{R}(\pi) = (\cos 2\pi, \cos\pi) = (1, -1) \rightarrow C$

$\vec{R}(2\pi) = (\cos 4\pi, \cos 2\pi) = (1, 1) \rightarrow A$

b) $\cos(2\theta) = 2\cos^2\theta - 1 \Rightarrow x = 2y^2 - 1 \rightarrow$ parábola.

c) $\frac{x+1}{2} = y^2 \rightarrow$ parábola ao longo do eixo $\vec{Ox}$



d) Não. Pois a curva C, definida por $\vec{R}(\theta)$ não contempla ~~toda~~ toda a parábola, só entre o ponto A e C, passando por B.

15. a) $\vec{R}(t) = (6t, -t^3, 3t^2)$

$\vec{v}(t) = \dfrac{d\vec{R}(t)}{dt} = (6, -3t^2, 6t)$

b) $\vec{R}(t) = \left(\frac{1}{t}, e^{t^2}, \ln(2t)\right), t>0$

$\vec{v}(t) = \dfrac{d\vec{R}(t)}{dt} = \left(-\frac{1}{t^2}, 2te^{t^2}, \frac{1}{t}\right), t>0.$

16 a) $\vec{R}(t) = \left(\cos t, -e^2, \frac{1}{t+1}\right)$

Vector tangente $\vec{R}'(t) = \left(-\text{sen}\, t, 0, -\frac{1}{(t+1)^2}\right)$

b) é igual à alínea b) do ex. 15

17. $\vec{e}(t) = (t, t^2, e^t)$

Vector tangente $(1, 2t, e^t)$

Vector tangente à curva C no ~~ponto~~ quando $t=0$

$(1,0,1)$.

18. $\mathcal{C}: \vec{R}(t) = (\cos t, \text{sen}\, t, t)$

$\vec{R}\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

$\vec{R}'(t) = (-\text{sen}\, t, \cos t, 1) \quad \vec{R}'\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$

b) Recta $s: (x,y,z) = P + t\vec{v}$, P ponto da recta $\left(\vec{R}\left(\frac{\pi}{4}\right)\right)$

$\vec{v}$ vector director da recta $\left(\vec{R}'\left(\frac{\pi}{4}\right)\right)$

$(x,y,z) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) + t\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$

$$\begin{cases} x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t \\ y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t \\ z = \frac{\pi}{4} + t. \end{cases}, t \in \mathbb{R}.$$

19. $\vec{R}(t) = (2t,\ 8-3t^2,\ 3t+4)$

$\vec{v}(t) = \vec{R}'(t) = (2, -6t, 3)$

$\vec{a}(t) = \vec{R}''(t) = (0, -6, 0)$

20. $\vec{R}(t) = (2t - 2\,\text{sen}\,t,\ 2 - 2\cos t)$

Posição $\vec{R}(0) = (0, 2-2) = (0,0)$

Velocidade $\vec{v}(t) = \vec{R}'(t) = (2 - 2\cos t,\ +2\,\text{sen}\,t)$

$\vec{R}'(0) = (0,0)$

aceleração $\vec{a}(t) = \vec{R}''(t) = (2\,\text{sen}\,t,\ 2\cos t)$

$\vec{a}(0) = (0, 2)$

Quando $t = \frac{3\pi}{2}$

$\vec{R}\left(\frac{3\pi}{2}\right) = \left(3\pi - 2\,\text{sen}\,\frac{3\pi}{2},\ 2 - 2\cos\frac{3\pi}{2}\right) =$

$= (3\pi - 2(-1),\ 2) = (3\pi + 2, 2)$

$\vec{v}\left(\frac{3\pi}{2}\right) = \left(2 - 2\cos\frac{3\pi}{2},\ 2\,\text{sen}\,\frac{3\pi}{2}\right) = (2, -2)$

$\vec{a}\left(\frac{3\pi}{2}\right) = \left(2\,\text{sen}\,\frac{3\pi}{2},\ 2\cos\frac{3\pi}{2}\right) = (-2, 0)$

b) reta tangente s:

$(x,y,z) = \vec{R}\left(\frac{3\pi}{2}\right) + t\,\vec{R}'\left(\frac{3\pi}{2}\right)$

$= (3\pi+2, 2) + t(2,-2)$, $t \in \mathbb{R}$

$$\boxed{(x,y,z) = (3\pi + 2 + 2t,\ 2 - 2t).}$$

21. $\begin{cases} x = t^2 - 1 \\ y = t^3 - t \end{cases}, \quad t \in \mathbb{R}.$

a) Recta tangente à curva é horizontal se a 2ª componente ~~~~ do vector director é nula.

O vector director determina-se pela derivada de $\vec{R}(t)$

$$\vec{R}'(t) = (2t, 3t^2 - 1)$$

A 2ª componente nula $3t^2 - 1 = 0 \Leftrightarrow t^2 = \frac{1}{3} \Leftrightarrow \boxed{t = \pm\frac{\sqrt{3}}{3}}$

$\begin{cases} x = \frac{1}{3} - 1 \\ y = \frac{\sqrt{3}}{9} - \frac{\sqrt{3}}{3} \end{cases}$ $\left(-\frac{2}{3}, -\frac{2\sqrt{3}}{9}\right)$ e $\begin{cases} x = \frac{1}{3} - 1 \\ y = -\frac{\sqrt{3}}{9} + \frac{\sqrt{3}}{3} \end{cases}$ $\left(-\frac{2}{3}, \frac{2\sqrt{3}}{9}\right)$

b) A recta tangente à curva é vertical se a 1ª componente do vector director é nula.

$$\vec{R}'(t) = (2t, 3t^2 - 1) \qquad 2t = 0 \Leftrightarrow t = 0.$$

$t = 0$

$\begin{cases} x = -1 \\ y = 0 \end{cases}$ $(-1, 0) \to$ ponto onde a recta tangente à curva é vertical.

22. $\|\vec{R}(t)\| = k \Leftrightarrow \vec{R}(t) \cdot \vec{R}(t) = k^2$ ~~~~ Derivando,

$\vec{R}'(t) \cdot \vec{R}(t) + \vec{R}(t) \cdot \vec{R}'(t) = 0 \Leftrightarrow \vec{R}(t) \cdot \vec{R}'(t) = 0.$

Significa que quando a norma do vector posição é constante, $\forall t \in \mathbb{R}$ então o vector tangente é perpendicular à curva determinada por $\vec{R}(t)$.

23. $t = 0 \quad \Rightarrow \vec{R}(0) = (3, 6, 5)$

$\vec{v}(0) = (1, -1, 0)$

Recta s: $(x, y, z) = (3, 6, 5) + t(1, -1, 0), \quad t \in \mathbb{R}$

$= (3 + t, 6 - t, 5).$

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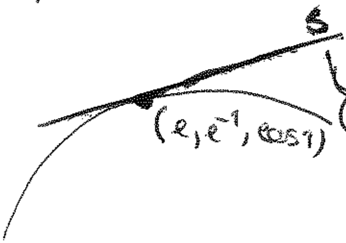
24. $\vec{R}(t) = \left(e^{t}, e^{-t}, \cos t\right)$

$\vec{R}'(t) = \left(e^{t}, -e^{-t}, -\mathrm{sen}\, t\right)$

Vector director da recta tangente à curva quando $t = 1$.

$\vec{R}'(1) = \left(e, -e^{-1}, -\mathrm{sen}\, 1\right)$



Ponto $\vec{R}(1) = \left(e, e^{-1}, \cos 1\right)$

Recta tangente à curva:

$$(x,y,z) = \left(e, e^{-1}, \cos 1\right) + t\left(e, -e^{-1}, -\mathrm{sen}\, 1\right), \quad t \in \mathbb{R}$$

$$= \left(e + te, e^{-1} - e^{-1}t, \cos 1 - t\,\mathrm{sen}\, 1\right)$$

Quando $t = 3$, a partícula está na recta.

$$(x,y,z) = \left(e + 3e, e^{-1} - 3e^{-1}, \cos 1 - 3\,\mathrm{sen}\, 1\right)$$

$$= \left(4e, -2e^{-1}, \cos 1 - 3\,\mathrm{sen}\, 1\right) \rightarrow \text{no ponto.}$$

25. $\vec{R}(1) = (0,0,e)$ $\quad \vec{R}'(1) = (3,4,e)$

$\vec{R}''(t) = \vec{a}(t) = \left(t, t^2, e^t\right)$

$\vec{R}'(t) = \vec{v}(t) = \left(\frac{t^2}{2} + C_1, \frac{t^3}{3} + C_2, e^t + C_3\right)$

$$\vec{R}'(1) = \left(\frac{1}{2} + C_1, \frac{1}{3} + C_2, e + C_3\right) = (3, 4, e)$$

$C_1 = 3 - 1/2$, $\quad C_2 = 4 - 1/3$, $\quad C_3 = 0$

$C_1 = \frac{5}{2}$, $\quad C_2 = \frac{11}{3}$, $\quad C_3 = 0$

$\vec{R}'(t) = \vec{v}(t) = \left(\frac{t^2}{2} + \frac{5}{2}, \frac{t^3}{3} + \frac{11}{3}, e^t\right)$

$$\vec{R}(t) = \left(\frac{t^3}{6} + \frac{5}{2}t + D_1, \frac{t^4}{12} + \frac{11}{3}t + D_2, e^t + D_3\right)$$

$$\vec{R}(1) = \left(\frac{1}{6} + \frac{5}{2} + D_1, \frac{1}{12} + \frac{11}{3} + D_2, e + D_3\right) = (0, 0, e)$$

$D_1 = -\frac{8}{3}$, $\quad D_2 = -\frac{45}{12} = -\frac{15}{4}$, $\quad D_3 = 0$

$$\vec{R}(t) = \left(\frac{t^3}{6} + \frac{5}{2}t - \frac{8}{3}, \frac{t^4}{12} + \frac{11t}{3} - \frac{15}{4}, e^t \right)$$

$$\vec{R}(0) = \left(-\frac{8}{3}, -\frac{15}{4}, 1 \right).$$