Ficha nº7

1.a) $\int_0^1 \int_{-1}^2 (2y^2-3x)\,dx\,dy = \int_0^1 \left[2y^2x - \frac{3x^2}{2}\right]_{x=-1}^{x=2} dy =$

$= \int_0^1 \left[2y^2 \times 2 - \frac{3\times 4}{2} - \left(2y^2(-1) - \frac{3\times 1}{2}\right)\right] dy = \int_0^1 \left[4y^2 - 6 + 2y^2 + \frac{3}{2}\right] dy$

$= \int_0^1 \left(6y^2 - \frac{9}{2}\right) dy = \left[\frac{6y^3}{3} - \frac{9}{2}y\right]_{y=0}^{y=1} = \frac{6}{3} - \frac{9}{2} - 0 = -\frac{5}{2}$

1.b) $\int_0^{\pi/2} \int_0^{\pi} (u \operatorname{sen} t + t \operatorname{sen} u)\,dt\,du = \int_0^{\pi/2} \left[-u\cos t + \frac{t^2}{2}\operatorname{sen} u\right]_{t=0}^{t=\pi} du$

$= \int_0^{\pi/2} \left[-u\cos\pi + \frac{\pi^2}{2} \cdot \operatorname{sen} u - \left(-u\cos 0 + \frac{0}{2}\cdot \operatorname{sen} u\right)\right] du =$

$= \int_0^{\pi/2} \left(u + \frac{\pi^2}{2} \cdot \operatorname{sen} u + u\right) du = \int_0^{\pi/2} \left(2u + \frac{\pi^2}{2}\cdot \operatorname{sen} u\right) du =$

$= \left[u^2 - \frac{\pi^2}{2}\cos u\right]_{u=0}^{\pi/2} = \frac{\pi^2}{4} - \frac{\pi^2}{2}\cos\frac{\pi}{2} - 0 + \frac{\pi^2}{2}\cdot\cos 0 = \frac{3\pi^2}{4}$

1.c) $\int_0^1 \int_{\sqrt{x}}^{x^2} 2x^2 y\,dy\,dx = \int_0^1 \left[\cancel{2}x^2\frac{y^2}{\cancel{2}}\right]_{y=\sqrt{x}}^{y=x^2} dx =$

$= \int_0^1 \left[x^2x^4 - x^2\cdot x\right] dx = \int_0^1 (x^6 - x^3)\,dx = \left[\frac{x^7}{7} - \frac{x^4}{4}\right]_0^1 = \frac{1}{7} - \frac{1}{4} = -\frac{3}{28}$

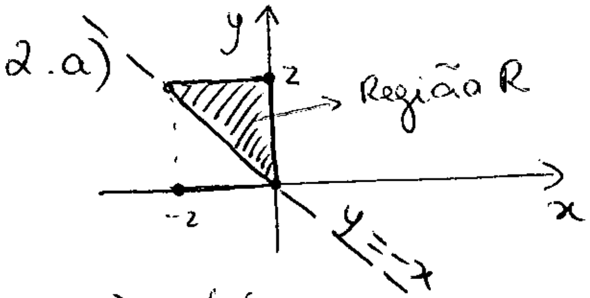
1.d) $\int_0^1 \int_0^u \sqrt{u^2+4}\,dv\,du$

$= \int_0^1 [v]_{v=0}^{v=u} \sqrt{u^2+4}\cdot du = \int_0^1 u\sqrt{u^2+4}\,du = \frac{1}{2}\int_0^1 2u\,(u^2+4)^{1/2} du$

$= \frac{1}{2}\left[\frac{(u^2+4)^{3/2}}{\frac{3}{2}}\right]_{u=0}^{u=1} = \frac{1}{3}\left[(1+4)^{3/2} - 4^{3/2}\right] = \frac{1}{3}\left[5^{3/2} - 4^{3/2}\right] =$

$= \frac{1}{3}\left[5\sqrt{5} - 8\right]$

2.a)



i) $\iint_R dx\,dy \longrightarrow y_{min} \leq y \leq y_{max} \rightarrow$ entre duas constantes y_{min}, y_{max}

Neste caso $\boxed{\begin{array}{l} 2 \leq y \leq 0 \\ -y \leq x \leq 0 \end{array}}$

$$\int_0^2 \int_{-y}^0 dx\,dy$$

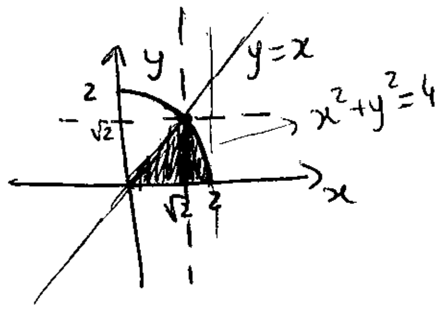
ii) $\iint_R dy\,dx$

$x_{min} \leq x \leq x_{max}$, x_{min}, x_{max} têm que ser constantes

$\boxed{\begin{array}{l} -2 \leq x \leq 0 \\ -x \leq y \leq 2 \end{array}}$

$$\int_{-2}^0 \int_{-x}^2 dy\,dx$$

2.b)



A equação desta secção no 1º quadrante da circunferência $x^2+y^2=4$ pode ser escrita como $y = \sqrt{4-x^2}$, $0 \leq x \leq 2$

ou $x = \sqrt{4-y^2}$, $0 \leq y \leq 2$

i) $\iint_R dx\,dy$

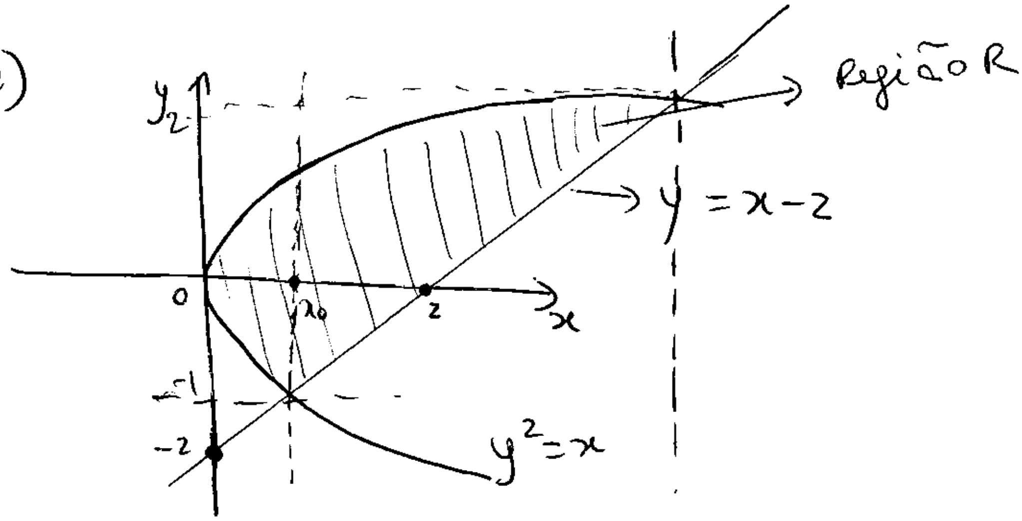
$\boxed{\begin{array}{l} 0 \leq y \leq \sqrt{2} \\ y \leq x \leq \sqrt{4-y^2} \end{array}}$

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} dx\,dy$$

ii) $\iint_R dy\,dx$ $\quad \begin{array}{l} 0 \leq x \leq \sqrt{2} \\ 0 \leq y \leq x \end{array} + \begin{array}{l} \sqrt{2} \leq x \leq 2 \\ 0 \leq y \leq \sqrt{4-x^2} \end{array}$

$$\int_0^{\sqrt{2}}\int_0^{x} dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} dy\,dx$$

c)



i) $\iint_R dy\,dx$

x_0 determina-se, através da intersecção dos curvas $y = x-2$ e $y^2 = x$

$$\begin{cases} y = x-2 \\ y^2 = x \end{cases} \quad y = y^2 - 2 \Leftrightarrow y^2 - y - 2 = 0 \Leftrightarrow y = 2 \vee \underline{\underline{y = -1}}$$

$y = -1 \Rightarrow x = (-1)^2 = 1$

$y = 2 \Rightarrow x = 4$

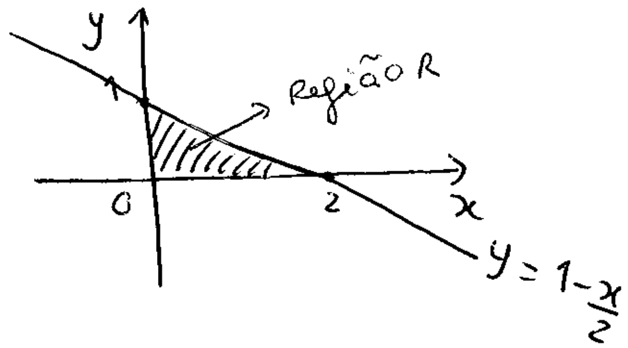
$$\boxed{\begin{array}{l} 0 \leq x \leq 1 \\ -\sqrt{x} \leq y \leq \sqrt{x} \end{array} \quad \cup \quad \begin{array}{l} 1 \leq x \leq 4 \\ x-2 \leq y \leq \sqrt{x} \end{array}}$$

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy\,dx$$

ii) $\boxed{\begin{array}{l} -1 \leq y \leq 2 \\ y^2 \leq x \leq y+2 \end{array}}$

$$\int_{-1}^{2} \int_{y^2}^{y+2} dx\,dy$$

3. a) $\iint_R x \, dA$

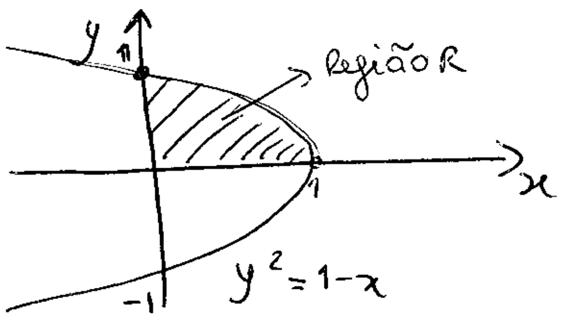


$$\begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq 1 - \frac{x}{2} \end{cases}$$

$$\int_0^2 \int_0^{1-x/2} x \, dy \, dx = \int_0^2 x \left[y \right]_0^{1-\frac{x}{2}} dx = \int_0^2 x\left(1 - \frac{x}{2}\right) dx =$$

$$= \int_0^2 \left(x - \frac{x^2}{2} \right) dx = \left[\frac{x^2}{2} - \frac{x^3}{6} \right]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}$$

b) $\iint_R (2x + y^2) \, dA$



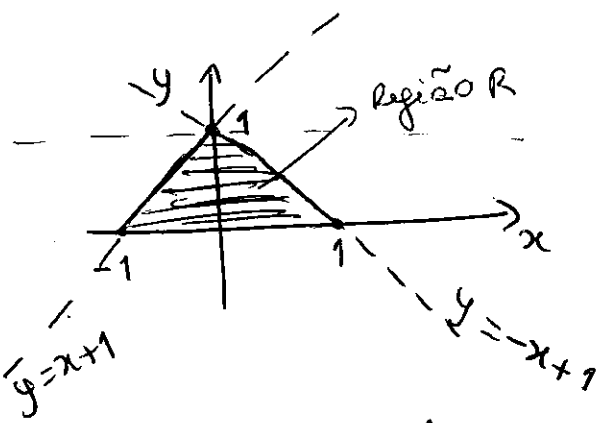
Região R $\begin{cases} 0 \leq y \leq 1 \\ 0 \leq x \leq 1 - y^2 \end{cases}$

$$\int_0^1 \int_0^{1-y^2} (2x + y^2) \, dx \, dy = \int_0^1 \left[x^2 + y^2 x \right]_{0=x}^{x=1-y^2} dy =$$

$$= \int_0^1 \left[(1-y^2)^2 + y^2(1-y^2) - 0 \right] dy = \int_0^1 \left[(1-y^2)(1 - \cancel{y^2} + \cancel{y^2}) \right] dy$$

$$= \int_0^1 (1 - y^2) \, dy = \left[y - \frac{y^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

c) $\iint_R y \, dA$

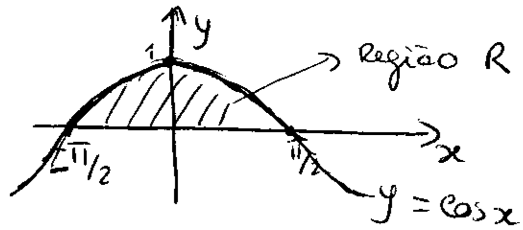


$$\begin{cases} 0 \leq y \leq 1 \\ y - 1 \leq x \leq 1 - y \end{cases}$$

$$\int_0^1 \int_{y-1}^{1-y} y \, dx \, dy = \int_0^1 y \left[x \right]_{y-1}^{1-y} dy = \int_0^1 y \left[1 - y - y + 1 \right] dy =$$

$$= \int_0^1 y(2-2y)\,dy = \int_0^1 (2y-2y^2)\,dy = \left[y^2 - \frac{2y^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

4.a) $\iint_R \operatorname{sen}^2 x \, dA$

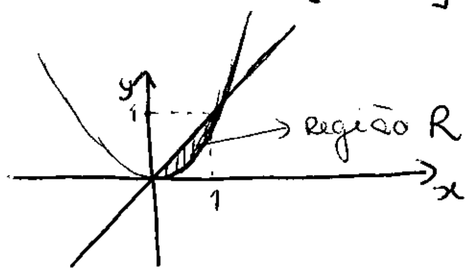


Região R $\rightarrow \begin{cases} -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 \leq y \leq \cos x \end{cases}$

$$\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} \operatorname{sen}^2 x \, dy\, dx = \int_{-\pi/2}^{\pi/2} \left[y \operatorname{sen}^2 x\right]_{y=0}^{y=\cos x} dx =$$

$$= \int_{-\pi/2}^{\pi/2} \cos x \cdot \operatorname{sen}^2 x \, dx = \frac{1}{3}\left[\operatorname{sen}^3 x\right]_{-\pi/2}^{\pi/2} = \frac{1}{3}\left[1-(-1)\right] = \frac{2}{3}$$

4.b) $\iint_R xy \, dA$



$0 \leq x \leq 1$

$x^2 \leq y \leq x$

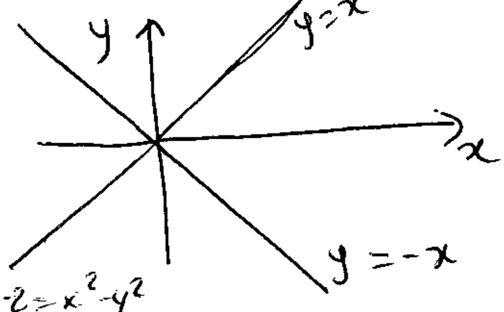
$$\int_0^1 \int_{x^2}^{x} xy \, dy\, dx = \int_0^1 \left[\frac{1}{2} x y^2\right]_{y=x^2}^{y=x} dx = \int_0^1 \frac{1}{2} x (x^2 - x^4)\, dx =$$

$$= \frac{1}{2}\int_0^1 (x^3 - x^5)\, dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{24}$$

c) $\iint_R (x^2 - y^2)\, dA$

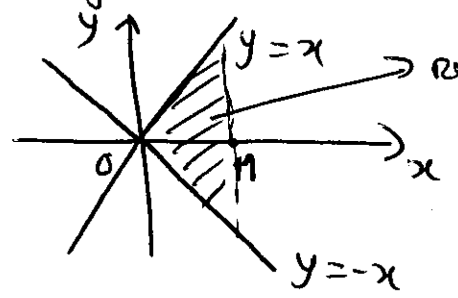
~~A intersecção~~ Para determinar a região R, é útil fazer a intersecção da superfície $z = x^2 - y^2$ com o plano XOY:

$\begin{cases} z = x^2 - y^2 \\ z = 0 \end{cases}$ $\left\{ x^2 - y^2 = 0 \Leftrightarrow y = \pm x \right.$



Retas $y = x$ e $y = -x$ são a intersecção do plano XOY com a superfície $z = x^2 - y^2$

No plano xOy, a região R está limitada entre $x=0$ e $x=1$



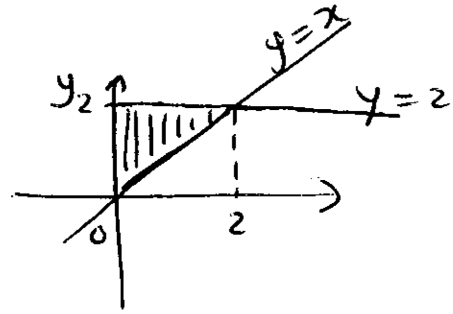
Região R:
$$\begin{cases} 0 \leq x \leq 1 \\ -x \leq y \leq x \end{cases}$$

$$\iint_R x^2-y^2\, dA = \int_0^1 \int_{-x}^{x} (x^2-y^2)\, dy\, dx = \int_0^1 \left[x^2 y - \frac{y^3}{3}\right]_{-x}^{x} dx =$$

$$= \int_0^1 \frac{4}{3}x^3\, dx = \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1 = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}.$$

5. Note que, nos exercícios indicados, com a ordem de integração dada, não é possível primitivar as funções. Assim, em todos eles, é conveniente trocar a ordem de integração.

a) $\displaystyle\int_0^2 \int_x^2 \exp(-y^2)\, dy\, dx$



Região R: $\boxed{\begin{array}{l} x \leq y \leq 2 \\ 0 \leq x \leq 2 \end{array}}$

ou pode ser descrita por

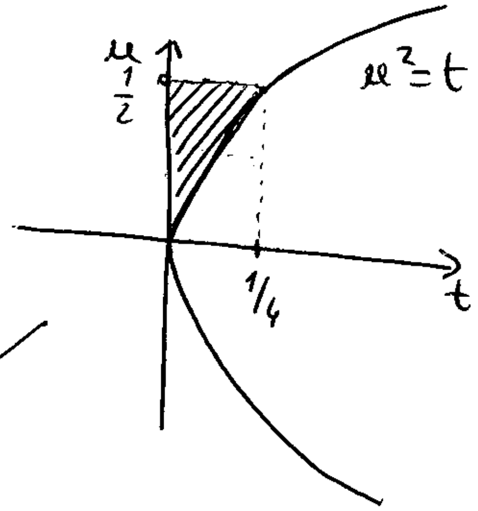
$\boxed{\begin{array}{l} 0 \leq y \leq 2 \\ 0 \leq x \leq y \end{array}}$

$$\int_0^2 \int_0^y e^{-y^2}\, dx\, dy = \int_0^2 y\, e^{-y^2} dx = -\frac{1}{2}\left[e^{-y^2}\right]_0^2 = \frac{1}{2}\left(1-e^{-4}\right)$$

7

b) $\displaystyle\int_0^{1/4}\int_{\sqrt{t}}^{1/2} \frac{e^u}{u}\,du\,dt$

Região R $\begin{cases} 0 \leq t \leq \frac{1}{4} \\ \sqrt{t} \leq u \leq \frac{1}{2} \end{cases}$ $\longrightarrow u = \sqrt{t} \Rightarrow u^2 = t \longrightarrow$



ou pode ser descrito na forma:

Região R. $\begin{cases} 0 \leq u \leq \frac{1}{2} \\ 0 \leq t \leq u^2 \end{cases}$

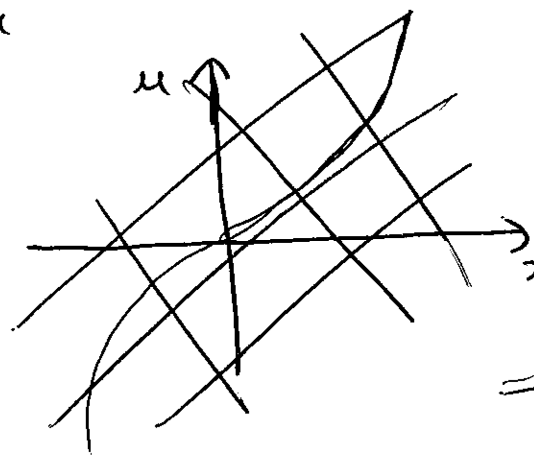
$$\int_0^{1/2}\int_0^{u^2} \frac{e^u}{u}\,dt\,du = \int_0^{1/2} \frac{e^u}{u}\cdot\left[t\right]_0^{u^2} du = \int_0^{1/2} u\cdot e^u\,du =$$

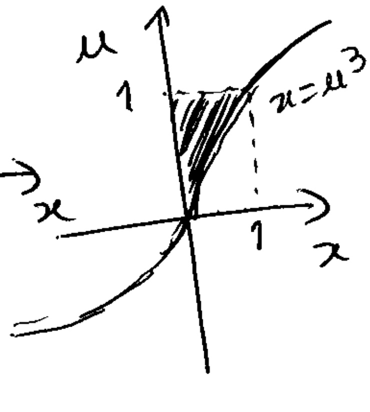
$$= \left[(u-1)e^u\right]_0^{1/2} = 1 - \frac{\sqrt{e}}{2}.$$

c) $\displaystyle\int_0^{1}\int_{x^{1/3}}^{1} \frac{1}{1+u^4}\,du\,dx$

Região R:

$\begin{cases} 0 \leq x \leq 1 \\ x^{1/3} \leq u \leq 1 \end{cases}$ $\longrightarrow$





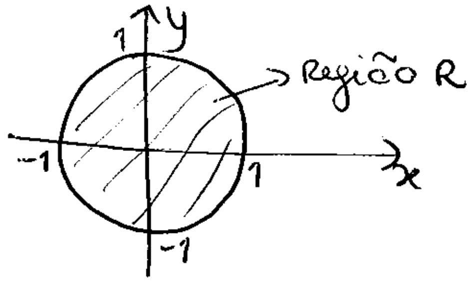
ou pode ser descrito na forma:

Região R $\begin{cases} 0 \leq u \leq 1 \\ 0 \leq x \leq u^3 \end{cases}$

$$\int_0^{1}\int_0^{u^3} \frac{1}{1+u^4}\cdot dx\,du = \int_0^1 \frac{1}{1+u^4}\left[x\right]_0^{u^3} du = \int_0^1 \frac{u^3}{1+u^4}\,du =$$

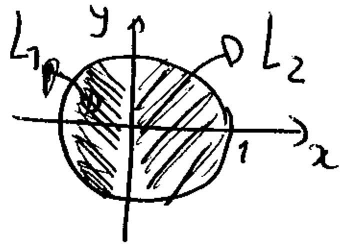
$$= \frac{1}{4}\left[\ln\left(1+u^4\right)\right]_0^1 = \frac{\ln 2}{4}.$$

6. i)



$\iint_R x \, dA = 0$ pois

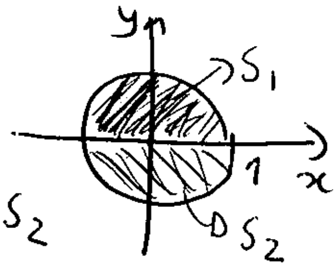
$\iint_{L_1} x \, dA = -\iint_{L_2} x \, dA$ onde



pois a função $f(x,y) = x$ é simétrica nas regiões indicadas e $R = L_1 \cup L_2$

$\iint_R x^2 y \, dA = 0$ pois

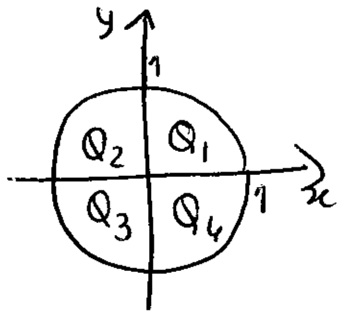
$\iint_{S_2} x^2 y \, dA = -\iint_{S_1} x^2 y \, dA$



e $x^2 \geq 0$ em todo R. e $R = S_1 \cup S_2$

$\iint_R xy \, dA = 0$ pois

$\iint_{Q_1 \cup Q_3} xy \, dA = -\iint_{Q_2 \cup Q_4} xy \, dA$



ii) $\iint_R e^x \, dA = 2\iint_{S_1} e^x \, dA$, pois $z = e^x$ depende de x mas não de y

$\iint_R x^2 \, dA = 4\iint_{Q_1} x^2 \, dA$, $z = x^2$, depende de x, n̄ depende de y e para x positivo e negativo, o valor é o mesmo.

$\iint_R (x^2 + y) \, dA = \iint_R x^2 \, dA + \iint_R y \, dA = \iint_R x^2 \, dA = 4\iint_{Q_1} x^2 \, dA.$

7. a) Área de R $= \iint_R dA$

Justificar que $\iint_R \frac{1}{1+x^4+y^4} dA \leq \iint_R dA$

Basta mostrar que $\frac{1}{1+x^4+y^4} \leq 1$.

Como $x^4+y^4 \geq 0 \Leftrightarrow 1+x^4+y^4 \geq 1 \Leftrightarrow \frac{1}{1+x^4+y^4} \leq 1$ c.q.m.

b) $1+x^2+y^2 \geq 1+x^2 \quad , \forall (x,y) \in R$

$\frac{1}{1+x^2+y^2} \leq \frac{1}{1+x^2} \implies \frac{x}{1+x^2+y^2} \leq \frac{x}{1+x^2}$

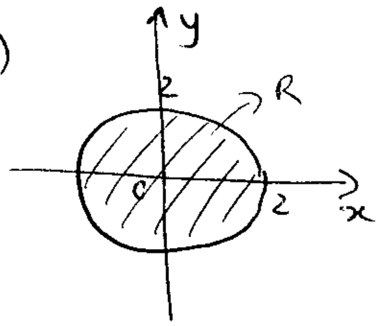
$\hookrightarrow$ pois $x > 0$.

Assim $\iint_R \frac{x}{1+x^2+y^2} dA \leq \iint_R \frac{x}{1+x^2} dA = \int_0^1 \int_0^1 \frac{x}{1+x^2} dx\, dy =$

$= \int_0^1 \left[\frac{1}{2} \ln(1+x^2)\right]_0^1 dy = \int_0^1 \frac{\ln 2}{2} \cdot dy = \frac{\ln 2}{2} [y]_0^1 = \frac{\ln 2}{2}$.

Ficha 7 — Integrais duplos em coordenadas polares

1 a)



$x^2+y^2=4 \Rightarrow R=2$

$\downarrow$

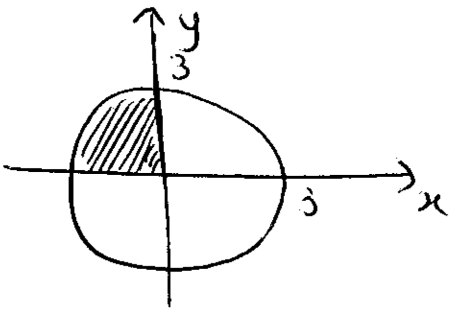
$x = R\cos\theta$

$y = R\,\text{sen}\,\theta$

$x^2+y^2 \leq 4 \Rightarrow 0 \leq R \leq 2 \ \wedge\ 0 < \theta \leq 2\pi$

$$\int_0^2\int_0^{2\pi} f(R\cos\theta, R\,\text{sen}\,\theta)\, R \, d\theta\, dR$$

b)



$x^2+y^2 \leq 9 \Rightarrow 0 \leq R \leq 3$

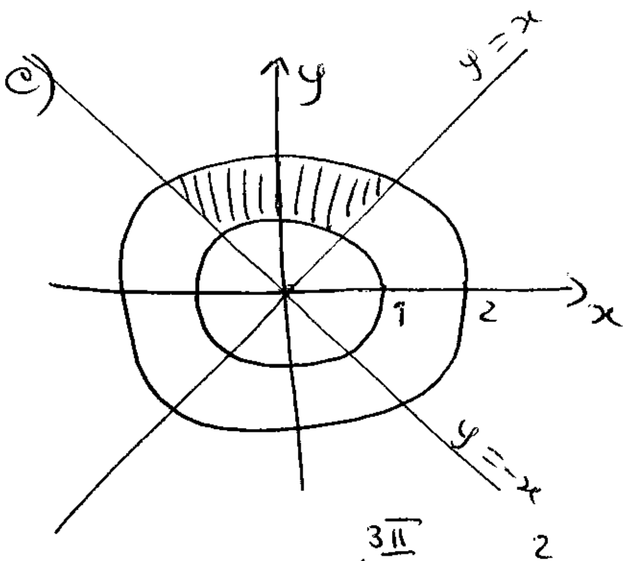
No 2º quadrante, entre

$x=0 \wedge y \geq 0 \Rightarrow \theta = \frac{\pi}{2}$

$y=0 \wedge x \leq 0 \Rightarrow \theta = \pi$

$$\int_0^3\int_{\frac{\pi}{2}}^{\pi} f(R\cos\theta, R\,\text{sen}\,\theta)\, R\, d\theta\, dR$$

c)



$1 \leq x^2+y^2 \leq 4 \Rightarrow 1 \leq R \leq 2$

$y \geq x \wedge x \geq 0 \Rightarrow \frac{\pi}{2} \geq \theta \geq \frac{\pi}{4}$

$y \geq -x \wedge x \leq 0 \Rightarrow \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

$1 \leq R \leq 2$

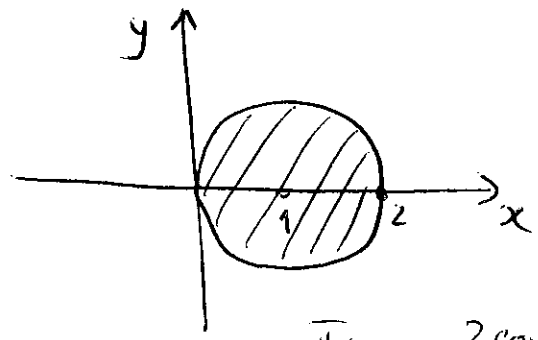
$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

$$\int_{\pi/4}^{\frac{3\pi}{4}}\int_1^2 f(R\cos\theta, R\,\text{sen}\,\theta)\, R\, dR\, d\theta$$

Int. duplos em coord. polares

2

d)



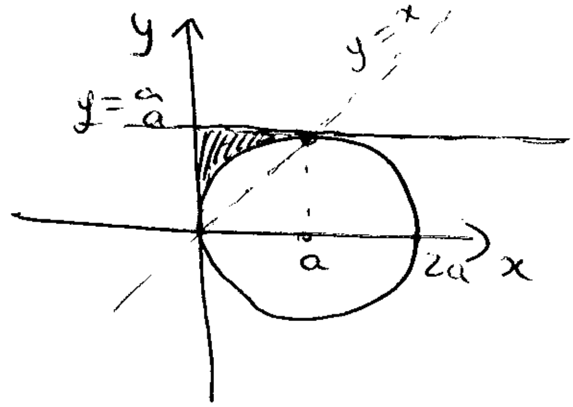
$(x-1)^2 + y^2 = 1 \Rightarrow R = 2\cos\theta$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$0 \leq R \leq 2\cos\theta$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\cos\theta} f(R\cos\theta, R\,\mathrm{sen}\,\theta) R \, dR \, d\theta$$

e)



$y = a \Rightarrow a = R\,\mathrm{sen}\,\theta \Leftrightarrow R = \frac{a}{\mathrm{sen}\,\theta}$

$y = a \Rightarrow R = a.\mathrm{cosec}\,\theta$

$(x-a)^2 + y^2 = a^2 \Rightarrow R = 2a\cos\theta$

$y = x \Rightarrow \theta = \frac{\pi}{4}$

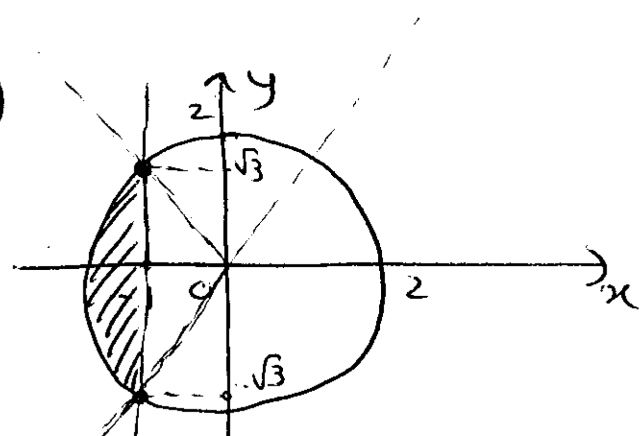
$x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$2a\cos\theta \leq R \leq a\,\mathrm{cosec}\,\theta$

$$\int_{\frac{\pi}{4}}^{\pi/2} \int_{2a\cos\theta}^{a.\mathrm{cosec}\,\theta} f(R\cos\theta, R\,\mathrm{sen}\,\theta) R \, dR \, d\theta$$

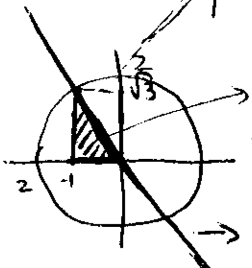
f)



$x^2 + y^2 \leq 4 \Rightarrow R \leq 2$

$x = -1 \Leftrightarrow -1 = R\cos\theta \Leftrightarrow R = -\frac{1}{\cos\theta} \Leftrightarrow R = -\sec\theta$

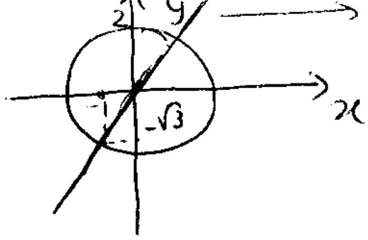
$-\sec\theta \leq R \leq 2$



triângulo rectângulo de hipotenusa 2 e um cateto 1.

$\rightarrow y = -\sqrt{3}x \Leftrightarrow \frac{y}{x} = -\sqrt{3} \Leftrightarrow \mathrm{tg}\,\theta = -\sqrt{3} \Leftrightarrow \frac{2\pi}{3} = \theta$

3



$y=\sqrt{3}x \Rightarrow \frac{y}{x}=\sqrt{3} \Rightarrow tg\theta=\sqrt{3} \Rightarrow$

$\frac{4\pi}{3}=\theta$

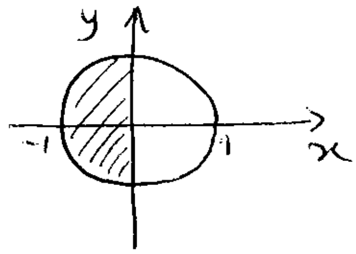
$$\frac{2\pi}{3}\le\theta\le\frac{4\pi}{3}$$

$\frac{2\pi}{3}\le\theta\le\frac{4\pi}{3}$

$-\sec\theta\le R\le 2$

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\int_{-\sec\theta}^{2} f(R\cos\theta, R\,sen\theta)\,R\,dR\,d\theta$$

2.a) $\iint_R x\,dx\,dy$



$dx\,dy = R\,dR\,d\theta$

$\begin{cases} x=R\cos\theta \\ y=R\,sen\theta \end{cases}$

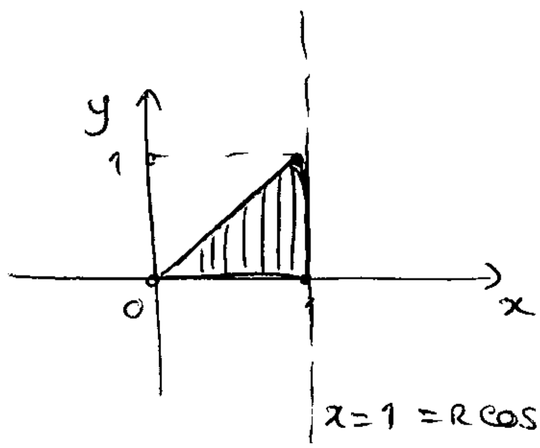
$0\le R\le 1$

$\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$

$$\int_{\pi/2}^{3\pi/2}\int_0^1 R\cos\theta\cdot R\,dR\,d\theta = \int_{\pi/2}^{3\pi/2}\int_0^1 R^2\cos\theta\,dR\,d\theta =$$

$$\int_{\pi/2}^{3\pi/2}\left[\frac{R^3}{3}\cdot\cos\theta\right]_{R=0}^{R=1} d\theta = \int_{\pi/2}^{3\pi/2}\cos\theta\,d\theta = [sen\theta]_{\pi/2}^{3\pi/2} = -1-1=-2$$

b) $\iint_R \tan^2\theta\,dA$



$x=1=R\cos\theta$

$R=\frac{1}{\cos\theta}=\sec\theta$

$0\le\theta\le\frac{\pi}{4}$

$0\le R\le\sec\theta$

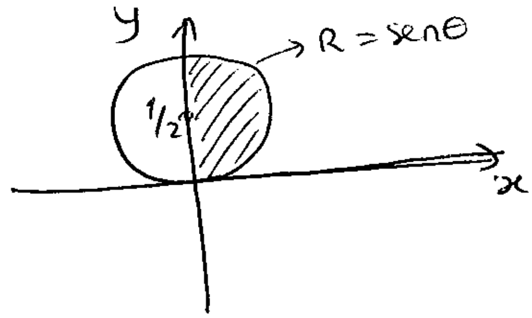
$$\int_0^{\pi/4}\int_0^{\sec\theta}\tan^2\theta\,R\,dR\,d\theta = \int_0^{\pi/4}\left[\frac{R^2}{2}\right]_0^{\sec\theta}\tan^2\theta\,.d\theta =$$

$$\frac{1}{2}\int_0^{\pi/4} \underbrace{\sec^2\theta}_{f'} \cdot \underbrace{\tan^2\theta}_{f^2} \, d\theta = \frac{1}{2}\left[\frac{\tan^3\theta}{3}\right]_0^{\pi/4} = \frac{1}{2}\left[\frac{\tan^3 \pi/4}{3} - \frac{\tan^3 0}{3}\right] =$$

4

$$= \frac{1}{6}.$$

2.c) $$\iint_R \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$$



$$0 \leq \theta \leq \frac{\pi}{2}$$

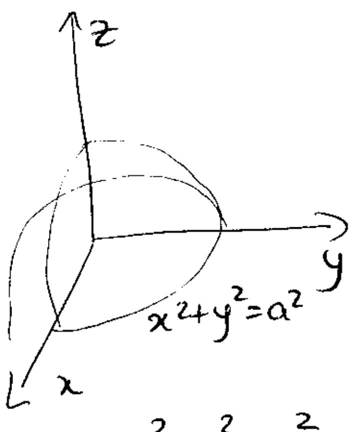
$$0 \leq R \leq \operatorname{sen}\theta$$

$$\int_0^{\pi/2}\int_0^{\operatorname{sen}\theta} \frac{1}{\sqrt{1-R^2}}\,R\,dR\,d\theta = \int_0^{\pi/2}\left(-\frac{1}{2}\right)\int_0^{\operatorname{sen}\theta} \underbrace{-2R}_{f'}\,\underbrace{(1-R^2)^{-1/2}}_{f^{-1/2}}\,dR\,d\theta$$

$$= -\frac{1}{\cancel{2}}\int_0^{\pi/2}\left[\frac{(1-R^2)^{1/2}}{\cancel{1}/2}\right]_{R=0}^{R=\operatorname{sen}\theta} d\theta = -\int_0^{\pi/2}\left(\sqrt{1-\operatorname{sen}^2\theta} - 1\right)d\theta =$$

$$= \int_0^{\pi/2}(1-\cos\theta)\,d\theta = \left[\theta - \operatorname{sen}\theta\right]_0^{\pi/2} = \frac{\pi}{2} - \operatorname{sen}\frac{\pi}{2} = \frac{\pi}{2} - 1.$$

3.a)



Esfera : $x^2+y^2+z^2=a^2$

$$z^2 = a^2 - x^2 - y^2$$

$$z = \pm\sqrt{a^2-x^2-y^2}$$

$$\iint_R \sqrt{a^2-x^2+y^2}\,dx\,dy$$

$x = R\cos\theta$

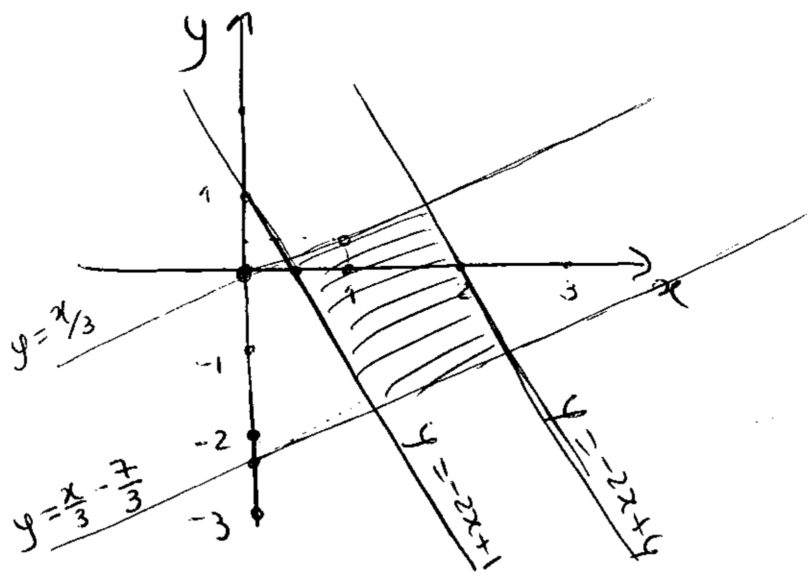
$y = R\operatorname{sen}\theta$

$0 \leq R \leq a$

$0 \leq \theta \leq 2\pi$

$$\int_0^a \int_0^{2\pi} R\sqrt{a^2-R^2}\,d\theta\,dR =$$

1. $\displaystyle\iint_R \frac{x-3y}{2x+y}\,dx\,dy$



$$\begin{cases} u = x-3y \\ v = 2x+y \end{cases}$$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

e $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7$$

Assim $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{7}$

e $dx\,dy = \frac{1}{7}\,du\,dv$

$y = \frac{x}{3} \Rightarrow u = x - 3\times\frac{x}{3} = 0$

$y = \frac{x}{3} - \frac{7}{3} \Rightarrow u = x - 3\left(\frac{x}{3} - \frac{7}{3}\right) = 7$

$0 \leq u \leq 7$

$y = -2x+4 \Rightarrow v = 2x - 2x + 4 = 4$

$y = -2x+1 \Rightarrow v = 2x - 2x + 1 = 1$

$1 \leq v \leq 4$

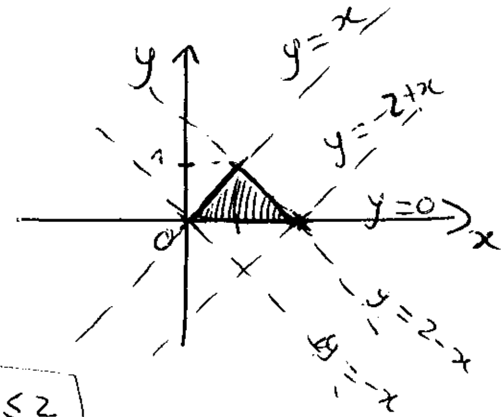
$$\int_0^7 \int_1^4 \frac{u}{v}\cdot\frac{1}{7}\,dv\,du = \frac{1}{7}\int_0^7\int_1^4 u\cdot\frac{1}{v}\,dv\,du$$

(2)

$$= \frac{1}{7}\int_0^7 u\left[\ln|v|\right]_{v=1}^{4} du = \frac{1}{7}\int_0^7 u\,(\ln 4)\,du = \frac{\ln 4}{7}\int_0^7 u\,du =$$

$$= \frac{\ln 4}{7}\left[\frac{u^2}{2}\right]_0^7 = \frac{\ln 4}{7}\times\frac{7^2}{2} = \frac{7\ln 4}{2}$$

2. $\displaystyle\iint_R \cos\frac{x-y}{x+y}\,dx\,dy$



$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$

$y=-x \Rightarrow u=0$

$y=2-x \Rightarrow u=2$ $\qquad \boxed{0 \le u \le 2}$

$y=x \Rightarrow v=0$

~~$y=-2+x \Rightarrow$~~

$y=0 \Rightarrow \begin{cases} u=x \\ v=x \end{cases} \Rightarrow u=v \qquad \boxed{0 \le v \le u}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \quad e \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

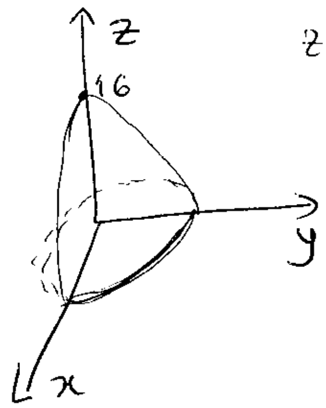
$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$\int_0^2\int_0^u \cos\frac{v}{u}\cdot\frac{1}{2}\,dv\,du =$$

$$= \frac{1}{2}\int_0^2 u\cdot\left[\operatorname{sen}\frac{v}{u}\right]_{v=0}^{v=u} du = \frac{1}{2}\int_0^2 u\left(\operatorname{sen}1 - \operatorname{sen}0\right)du = \frac{1}{2}\int_0^2 u\cdot\operatorname{sen}1\cdot du =$$

$$= \frac{1}{2}\cdot\operatorname{sen}1\left[\frac{u^2}{2}\right]_0^2 = \frac{1}{2}\operatorname{sen}1\left[\frac{4}{2}-0\right] = \operatorname{sen}1$$

3. $\begin{cases} u = x \\ v = 2y \end{cases}$ $\qquad \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$



$z = 16 - x^2 - 4y^2 \to$ parabolóide ao longo do eixo Oz com a concavidade virada para baixo.

A região R determina-se pela intersecção do parabolóide com o plano xOy.

$\begin{cases} z = 16 - x^2 - 4y^2 \\ z = 0 \end{cases} \quad \Big\} \ x^2 + 4y^2 = 16$

$$\iint_R (16 - x^2 - 4y^2)\,dx\,dy$$

Com a mudança de variável $u = x$, $v = 2y$, fica

$x^2 + 4y^2 = 16 \Rightarrow u^2 + v^2 = 16 \to$ circunferência centrada na origem e raio 4.

$$\iint_R (16 - x^2 - 4y^2)\,dx\,dy = \iint_U (16 - u^2 - v^2) \times \frac{1}{2}\,du\,dv$$

Como a região U é o interior da circunferência centrada na origem e raio 4, convém mudar as variáveis (u,v) para as coordenadas polares (R,θ).

$\begin{cases} u = R\cos\theta \\ v = R\,\text{sen}\,\theta \end{cases}$ $\qquad 0 \leq R \leq 4 \quad 0 \leq \theta \leq 2\pi$

$$\frac{\partial(u,v)}{\partial(R,\theta)} = R$$

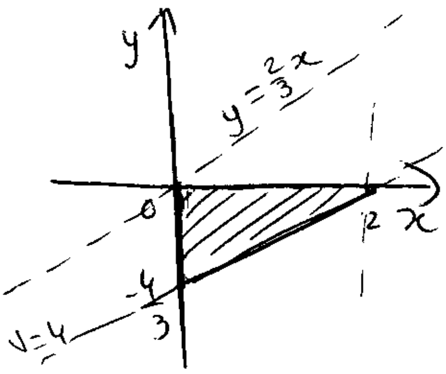
$$\iint_U (16 - u^2 - v^2) \cdot \frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^4 \int_0^{2\pi} (16 - R^2) R \,d\theta\, dR =$$

$$= \frac{1}{2}\int_0^4 2\pi\,(16R - R^3)\,dR = \pi\left[\frac{16R^2}{2} - \frac{R^4}{4}\right]_0^4 = \pi\left[\frac{16 \times 4^2}{2} - \frac{4^4}{4}\right] = 64\pi$$

(4)

4. $\iint_R (2x-3y)^2(x+y)^2\,dx\,dy$

$$\begin{cases} u = x+y \\ v = 2x-3y \end{cases} \qquad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$



$2x-3y=4 \Leftrightarrow y = \frac{2}{3}x - \frac{4}{3}$

$2x-3y=4$

$$x=0 \Rightarrow \begin{cases} u = y \\ v = -3y \end{cases} \Rightarrow v = -3u$$

$$y=0 \Rightarrow \begin{cases} u = x \\ v = 2x \end{cases} \Rightarrow v = 2u$$

$2x-3y=4 \Rightarrow v=4$

$y=\frac{2}{3}x \Rightarrow v=0$

$$\boxed{\begin{array}{c} 0 \le v \le 4 \\ -\frac{v}{3} \le u \le \frac{v}{2} \end{array}}$$

$$\int_0^4 \int_{-v/3}^{v/2} v^2 u^2 \frac{1}{5}\,du\,dv = \frac{1}{5}\int_0^4 v^2 \left[\frac{u^3}{3}\right]_{-v/3}^{v/2} dv =$$

$$= \frac{1}{5}\int_0^4 \frac{v^2}{3}\left[\frac{v^3}{2^3} + \frac{v^3}{3^3}\right] dv = \frac{1}{15}\left(\frac{1}{8}+\frac{1}{27}\right)\int_0^4 v^2 \cdot v^3\,dv =$$

$$= \frac{1}{15}\left(\frac{1}{8}+\frac{1}{27}\right)\left[\frac{v^6}{6}\right]_0^4 = \frac{1}{5}\left(\frac{1}{8}+\frac{1}{27}\right)\left(\frac{4^6}{6}\right).$$