Ficha 2A - Séries de Taylor

Fórmula de Taylor

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \underbrace{f(c)}_{n=0} + \underbrace{f'(c)(x-c)}_{n=1} + \underbrace{\frac{f''(c)}{2!}(x-c)^2}_{n=2} + \underbrace{\frac{f'''(c)}{3!}(x-c)^3}_{n=3} + \cdots$$

① a) $f(x) = x^3 + 4x^2 - x + 1$ para $c = 1 \Rightarrow$ potências de base $x-1$

$f(x) = x^3 + 3x^2 - x + 1$ $\quad f(1) = 5$

$f'(x) = 3x^2 + 8x - 1$ $\quad f'(1) = 10$

$f''(x) = 6x + 8$ $\quad f''(1) = 14$

$f'''(x) = 6$ $\quad f'''(x) = 6$

$f^{iv}(x) = 0$ $\quad f^{iv}(x) = 0$

$$f(x) = 5 + 10(x-1) + \frac{14}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 + 0 + 0 + 0 + 0 + 0 + 0 + \cdots$$

$$f(x) = 5 + 10(x-1) + 7(x-1)^2 + (x-1)^3$$

$x \in \mathbb{R}$

$D_f = \mathbb{R}$

b) $g(x) = \cos x$ para $c = \frac{\pi}{2} \Rightarrow$ potências de base $x - \frac{\pi}{2}$

$g(x) = \cos x$ $\quad g(\frac{\pi}{2}) = 0$

$g'(x) = -\operatorname{sen} x$ $\quad g'(\frac{\pi}{2}) = -1$

$g''(x) = -\cos x$ $\quad g''(\frac{\pi}{2}) = 0$

$g'''(x) = \operatorname{sen} x$ $\quad g'''(\frac{\pi}{2}) = 1$

$g^{iv}(x) = \cos x$ $\quad$ $g^{iv}(x) = 0$

$g^{v}(x) = -\operatorname{sen} x$ $\quad$ $g^{v}(x) = -1$

$g^{vi}(x) = -\cos x$ $\quad$ $g^{vi}(x) = 0$

$g^{vii}(x) = \operatorname{sen} x$ $\quad$ $g^{vii}(x) = 1$

$$f(x) = 0 - (x - \frac{\pi}{2}) + 0 + \frac{1}{3!}(x-\frac{\pi}{2})^3 + 0 - \frac{1}{5!}(x-\frac{\pi}{2})^5 + 0 + \frac{1}{7!}(x-\frac{\pi}{2})^7 + \dots$$

$$f(x) = \cos x = -(x-\frac{\pi}{2}) + \frac{1}{3!}(x-\frac{\pi}{2})^3 - \frac{1}{5!}(x-\frac{\pi}{2})^5 + \frac{1}{7!}(x-\frac{\pi}{2})^7 - \dots$$

$$= \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-1)!}(x-\frac{\pi}{2})^{2n-1}$$

$$\text{Raio } R = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{\frac{1}{(2n-1)!}}{\frac{1}{(2n+2-1)!}} = \lim \frac{(2n+1)!}{(2n-1)!} =$$

$$= \lim \frac{(2n+1)(2n)(2n-1)!}{(2n-1)!} = \lim (2n+1)(2n) = +\infty$$

$x \in \mathbb{R}$

e) $h(x) = x^5$ $\quad$ para $c = -2 \Rightarrow$ potências de base $x+2$

$h(x) = x^5$ $\quad$ $h(-2) = (-2)^5 = -2^5$

$h'(x) = 5x^4$ $\quad$ $h'(-2) = 5(-2)^4 = 5\times 2^4$

$h''(x) = 20x^3$ $\quad$ $h''(-2) = 5\times4(-2)^3 = -5\times 2^5$

$h'''(x) = 60x^2$ $\quad$ $h'''(-2) = 5\times4\times3(-2)^2 = 5\times3\times2^4$

$h^{iv}(x) = 120x$ $\quad$ $h^{iv}(-2) = 5\times4\times3\times2(-2) = -5\times3\times2^4$

$h^{v}(x) = 120$ $\quad$ $h^{v}(-2) = 5\times4\times3\times2 = 5\times3\times2^3$

$h^{vi}(x) = 0$

$$f(x) = -2^5 + 5\times 2^4(x+2) - \frac{5\times 2^5}{2!}(x+2)^2 + \frac{5\times 3\times 2^4}{3!}(x+2)^3 - \frac{5\times 3\times 2^4}{4!}(x+2)^4 + \frac{5\times 3\times 2^3}{5!}(x+2)^5$$

$$f(x) = -2^5 + 5\times 2^4(x+2) - 5\times 2^4(x+2) + 5\times 2^3(x+2)^3 - 5\times 2(x+2)^4 + (x+2)^5$$

$$f(x) = x^5 = -2^5 + 5\times 2^4(x+2) - 5\times 2^4(x+2)^2 + 5\times 2^3(x+2)^3 - 10(x+2)^4 + (x+2)^5$$

(2)

$$f(x) = \frac{e^{x^2} - 1}{x} \qquad x \neq 0$$

Sabemos que $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots \qquad u \in \mathbb{R}$

então

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

$$f(x) = \frac{1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots - 1}{x}$$

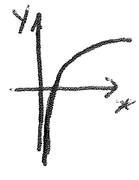
$$f(x) = x + \frac{x^3}{2!} + \frac{x^5}{3!} + \frac{x^7}{4!} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{n!}(x)^{2n+1}$$

$$\text{Raio } R = \lim \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim \frac{(n+1)!}{n!} = \lim (n+1) = +\infty$$

$x \in \mathbb{R}$, mas $x \neq 0$ (denominador)

$\therefore \ x \in \mathbb{R} \setminus \{0\}$

③ a) $n(x) = \ln x$ $\qquad c = 0$



$n(x) = \ln x$ $\qquad n(0) = \ln(0)$ não existe em $\mathbb{R}$

$n'(x) = \frac{1}{x}$ $\qquad n'(0) = \frac{1}{0}$ não existe em $\mathbb{R}$

zero no denominador

$n''(x) = -\frac{1}{x^2}$

Não conseguimos aplicar a fórmula de Taylor $\left(\sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \right)$, pois n e n' não estão definidos no ponto 0.

b) $n(x) = x^{3/2}$ $\qquad c = 0$

$n(0) = 0^{3/2} = 0$

$n'(x) = \frac{3}{2} x^{1/2}$ $\qquad n'(0) = \frac{3}{2} \times \sqrt{0} = 0$

$n''(x) = \frac{3}{4} x^{-1/2} = \frac{3}{4} \times \frac{1}{\sqrt{x}}$ $\qquad n''(0) = \frac{3}{0}$ não existe em $\mathbb{R}$

Também não conseguimos aplicar a fórmula de Taylor, pois a partir da 2ª derivada, inclusive, as derivadas não estão definidas no ponto 0.

④ $f(x) = \frac{e^x - e^{-x}}{2}$

Sabemos que $e^n = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \frac{n^4}{4!} + \frac{n^5}{5!} + \frac{n^6}{6!} + \cdots \quad n \in \mathbb{R}$

então $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$

$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots$

$e^x - e^{-x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots - 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \cdots$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \dots$$

logo

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)!} x^{2n-1}$$

Nota: $\frac{e^x - e^{-x}}{2} = \operatorname{sh} x$

5) $f(x) = \ln x$ com $c = e$ e $n = 3$

$f(x) = \ln x$ $\qquad f(e) = \ln e = 1$

$f'(x) = \frac{1}{x}$ $\qquad f'(e) = \frac{1}{e}$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(e) = -\frac{1}{e^2}$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(x) = \frac{2}{e^3}$

$$f(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{e^2 2!}(x-e)^2 + \frac{2}{e^3 3!}(x-e)^3 + R_3(x)$$

$$f(x) = \ln x = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + R_3(x)$$

<u>ou</u>

Sabemos que $\ln(u+1) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$

então

$$\ln x = \ln\left[(x-e)+e\right] = \ln\left[\frac{(x-e)+e}{e} \times e\right] = \ln\left[\frac{(x-e)+e}{e}\right] + \ln e$$

$$\ln x = 1 + \ln\left[\frac{x-e}{e}+1\right]$$

seja $u = \frac{x-e}{e}$

Tem-se que

$$\ln(\cancel{x}+1) = \frac{x-e}{e} - \frac{\left(\frac{x-e}{e}\right)^2}{2} + \frac{\left(\frac{x-e}{e}\right)^3}{3} + R_3(x)$$

Então

$$\ln x = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + R_3(x)$$

6

a) Sabemos que $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \quad x \in \mathbb{R}$

então

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\dots\right)}{x^2} =$$

$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \dots}{x^2} = \lim_{x\to 0}\left(\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^6 - \dots\right) = \frac{1}{2}$$

b) Sabemos que $\operatorname{cotg} u = \frac{1}{u} - \frac{u}{3} - \frac{u^3}{45} - \frac{2u^5}{945} - \dots \quad u \in \left]0,\pi\right[$

então

$$\lim_{x\to 0} \frac{1}{x}\left(\operatorname{cotg} x - \frac{1}{x}\right) = \lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots - \frac{1}{x}\right)\right] =$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\left(-\frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots\right)\right] = \lim_{x\to 0}\left(-\frac{1}{3} - \frac{x^2}{45} - \frac{2x^5}{945} - \dots\right) =$$

$$= -1/3$$

c) Sabemos que $e^{u} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \quad u \in \mathbb{R}$

então

$$e^{-t^2} = 1 + (-t^2) + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots$$

$$= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$$

logo

$$\int_0^x e^{-t^2} dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right) dt$$

$$= \left[t - \frac{t^3}{3} + \frac{t^5}{5\times 2!} - \frac{t^7}{7\times 3!} + \cdots\right]_0^x = x - \frac{x^3}{3} + \frac{x^5}{5\times 2!} - \frac{x^7}{7\times 3!} + \cdots =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)(n-1)!}$$

(7) Vamos determinar o desenvolvimento em série de Taylor numa vizinhança de $\frac{\pi}{4}$ da função $\operatorname{sen} x$.

$f(x) = \operatorname{sen} x \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\operatorname{sen} x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{IV}(x) = \operatorname{sen} x \qquad f^{IV}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$f(x) = \operatorname{sen} x = \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots\right]$$

$$43^\circ = 45^\circ - 2^\circ \Leftrightarrow \frac{43}{180}\pi = \frac{\pi}{4} - \frac{2}{180}\pi \Leftrightarrow \frac{43}{180}\pi = \frac{\pi}{4} - \frac{\pi}{90}$$

$$\operatorname{sen} 43^\circ = \operatorname{sen}\left(\frac{\pi}{4} - \frac{\pi}{90}\right)$$

Se $x = \frac{\pi}{4} - \frac{\pi}{90}$

$$\operatorname{sen}\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \frac{\sqrt{2}}{2}\left[1 + \left(\left(\frac{\pi}{4} - \frac{\pi}{90}\right) - \frac{\pi}{4}\right) - \frac{1}{2!}\left(\left(\frac{\pi}{4} - \frac{\pi}{90}\right) - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(\left(\frac{\pi}{4} - \frac{\pi}{90}\right) - \frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(\left(\frac{\pi}{4} - \frac{\pi}{90}\right) - \frac{\pi}{4}\right)^4 +\right.$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(-\frac{\pi}{90}\right) - \frac{1}{2!}\left(-\frac{\pi}{90}\right)^2 - \frac{1}{3!}\left(-\frac{\pi}{90}\right)^3 + \frac{1}{4!}\left(-\frac{\pi}{90}\right)^4 + \cdots\right]$$

então

$$\operatorname{sen} 43^\circ = \frac{\sqrt{2}}{2}\left[1 - \frac{\pi}{90} - \frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 + \frac{1}{4!}\left(\frac{\pi}{90}\right)^4 + \cdots\right]$$

<u>outra maneira</u>

Sabemos que $\operatorname{sen}(a \mp b) = \operatorname{sen} a \cos b \mp \operatorname{sen} b \cos a$, $\forall a, b \in \mathbb{R}$ e

$$\operatorname{sen} 43^\circ = \operatorname{sen}\left(\frac{\pi}{4} - \frac{\pi}{90}\right)$$

então

$$\operatorname{sen}\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \operatorname{sen}\frac{\pi}{4}\cos\frac{\pi}{90} - \operatorname{sen}\frac{\pi}{90}\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cos\frac{\pi}{90} - \frac{\sqrt{2}}{2}\operatorname{sen}\frac{\pi}{90} =$$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{90} - \operatorname{sen}\frac{\pi}{90}\right)$$

Sabemos que:

$$\operatorname{sen} u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots \quad u \in \mathbb{R}$$

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots \quad u \in \mathbb{R}$$

então

$$\cos\frac{\pi}{90} - \operatorname{sen}\frac{\pi}{90} = 1 - \frac{\left(\frac{\pi}{90}\right)^2}{2!} + \frac{\left(\frac{\pi}{90}\right)^4}{4!} - \frac{\left(\frac{\pi}{90}\right)^6}{6!} + \cdots - \left(\frac{\pi}{90} - \frac{\left(\frac{\pi}{90}\right)^3}{3!} + \frac{\left(\frac{\pi}{90}\right)^5}{5!} - \frac{\left(\frac{\pi}{90}\right)^7}{7!} + \cdots\right)$$

$$\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{90} - \operatorname{sen}\frac{\pi}{90}\right) = \frac{\sqrt{2}}{2}\left[1 - \frac{\pi}{90} - \frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 + \frac{1}{4!}\left(\frac{\pi}{90}\right)^4 - \frac{1}{5!}\left(\frac{\pi}{90}\right)^5 - \cdots\right]$$