$\boxed{+5}$

1. $f(x,y) = xy$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$

2. $f(x,y) = x^2 + y^2(1+\sin x)$ $\quad (a,b) = (\pi, 2)$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2x + y^2 \cos x,\ 2y(1+\sin x)\right)$$

$$\vec{\nabla} f(\pi, 2) = (2\pi - 4,\ 4)$$

3. $f(x,y,z) = \left(\frac{x}{y}\right)^z$ $\quad \vec{u} = 2\vec{e}_1 + 2\vec{e}_2 - 2\vec{e}_3$

$$\frac{\partial f}{\partial x} = \frac{z}{y}\left(\frac{x}{y}\right)^{z-1} \qquad \frac{\partial f}{\partial y} = -\frac{zx}{y^2}\left(\frac{x}{y}\right)^{z-1}$$

$$\frac{\partial f}{\partial z} = \ln\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^z \qquad \boxed{\frac{d}{dx}(a^u) = \ln a \cdot a^u \cdot \frac{du}{dx}}$$

$$\nabla f(1,1,1) = \left(\frac{\partial f}{\partial x}(1,1,1), \frac{\partial f}{\partial y}(1,1,1), \frac{\partial f}{\partial z}(1,1,1)\right)$$

$$= (1, -1, 0)$$

$$\vec{u} = (2, 2, -2) \qquad \|\vec{u}\| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$$

$$D_{\vec{v}} f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$$

$$= (1, -1, 0)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = 0$$

④ $f(x,y,z) = \operatorname{sen}\left(\frac{xz}{x^2+y^2}\right)$

a) $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\frac{\partial f}{\partial x} = \frac{z(x^2+y^2) - 2x^2 z}{(x^2+y^2)^2} \cos\left(\frac{xz}{x^2+y^2}\right) = \frac{zx^2 - zy^2 - 2zx^2}{(x^2+y^2)^2} \cos\left(\frac{xz}{x^2+y^2}\right)$$

$$\frac{\partial f}{\partial x} = \frac{-zy^2 - zx^2}{(x^2+y^2)^2} \cos\left(\frac{xz}{x^2+y^2}\right)$$

$$\frac{\partial f}{\partial y} = -\frac{2yxz}{(x^2+y^2)^2} \cos\left(\frac{xz}{x^2+y^2}\right)$$

$$\frac{\partial f}{\partial z} = \frac{x\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}} \cos\left(\frac{xz}{x^2+y^2}\right) = \frac{x}{x^2+y^2} \cos\left(\frac{xz}{x^2+y^2}\right)$$

b) $\vec{\nabla} f(2,1,0) = \left(0, 0, \frac{2}{5}\cos 0\right) = (0, 0, 2/5)$

c) $D_{(1,1,1)} f(2,1,0) = \vec{\nabla} f(2,1,0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = (0,0,2/5) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right.$

$D_{(1,1,1)} f(2,1,0) = 0 + 0 + 2/5\sqrt{3} = 2/5\sqrt{3}$

(5) $D_{\vec{u}} f(a,b) = \vec{u} \cdot \vec{\nabla} f(a,b) = \|\vec{\nabla} f(a,b)\| \cdot \cos\phi$

a)

direcção segundo a qual f tem menor taxa de variação ?

$D_{\vec{u}} f(a,b)$ é máxima quando $\cos\phi = 1 \Rightarrow \phi = 0$

ou seja $\vec{u}$ tem a direcção do $\vec{\nabla} f$.

$\therefore D_{\vec{u}} f(a,b) = \|\vec{\nabla} f(a,b)\|$

b)

Menor taxa de variação $\Rightarrow \cos\phi = -1 \Rightarrow \phi = 180^\circ$

$\vec{\nabla} f(a,b)$ sentido oposto a $\vec{u}$.

$\therefore D_{\vec{u}} f(a,b) = -\|\vec{\nabla} f(a,b)\|$

c)

taxa de variação nula $\Rightarrow \cos\phi = 0 \Rightarrow$

$\Rightarrow \vec{\nabla} f(a,b) \perp \vec{u}$

(6) $f(x,y,z) = (x^2 + \cos z)\exp(-x+y)$

$$\vec{\nabla} f(1,1,\pi) = \left(\frac{\partial f}{\partial x}(1,1,\pi), \frac{\partial f}{\partial y}(1,1,\pi), \frac{\partial f}{\partial z}(1,1,\pi) \right) = (1,0,0)$$

(7) $f(x,y) = \ln \|\vec{r}\|$; $\vec{r} = (x,y)$; $\|\vec{r}\| = \sqrt{x^2+y^2}$

$$\Updownarrow$$

$$f(x,y) = \ln \sqrt{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$

$$\vec{\nabla} f(x,y) = \frac{1}{x^2+y^2}(x,y) = \frac{1}{\|\vec{r}\|}(x,y) = \frac{\vec{r}}{\|\vec{r}\|}$$

(8) a) $P \rightarrow (2,1,3)$; $M \rightarrow (5,5,15)$

vector $\vec{PM} \rightarrow (3,4,12) = M - P = (5,5,15) - (2,1,3)$

$$D_{\vec{PM}} f(2,1,3) = \vec{\nabla} f(2,1,3) \cdot \vec{PM}$$

$$= (4,5,3) \cdot (3,4,12) = 12 + 20 + 36 = 68$$

b) A direcção $\vec{u}$, tal que $\measuredangle(\vec{u}, \vec{OX}) = 60°$

$$\vec{u} \rightarrow (\cos 60°, \sin 60°) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u}$$

$$= (0,-9) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = -\frac{9\sqrt{3}}{2}$$

(9) $$\begin{cases} D_{\left(\frac{\sqrt2}{2},\frac{\sqrt2}{2}\right)} f(a,b) = \vec\nabla f(a,b)\cdot\left(\frac{\sqrt2}{2},\frac{\sqrt2}{2}\right) = 3\sqrt2 \\ D_{\left(\frac{3}{5},-\frac{4}{5}\right)} f(a,b) = \vec\nabla f(a,b)\cdot\left(\frac{3}{5},-\frac{4}{5}\right) = 5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{\partial f}{\partial x}(a,b)\cdot\frac{\sqrt2}{2} + \frac{\partial f}{\partial y}(a,b)\cdot\frac{\sqrt2}{2} = 3\sqrt2 \\ \frac{\partial f}{\partial x}(a,b)\cdot\frac{3}{5} - \frac{\partial f}{\partial y}(a,b)\,\frac{4}{5} = 5 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial f}{\partial x}(a,b) = 7 \\ \frac{\partial f}{\partial y}(a,b) = -1 \end{cases}$$

(10) $\vec{u} = ? = (u_1, u_2)$; $D_{\vec u} f(2,0) = -1$; $f(x,y) = xy$

$$\left(\frac{\partial f}{\partial x}(2,0), \frac{\partial f}{\partial y}(2,0)\right)\cdot(u_1,u_2) = -1 \Leftrightarrow (0,2)(u_1,u_2) = -1 \Leftrightarrow$$

$\Leftrightarrow \quad 2u_2 = -1$

$\Leftrightarrow \quad u_2 = -\frac{1}{2}$

$\therefore\ \vec u = \left(u_1, -\frac{1}{2}\right)$ //

(11) $f(x,y,z) = x^2+y^2-z^2$

A taxa de variação máxima no pto (a,b,c) é $\|\vec\nabla f(a,b,c)\|$

$\vec\nabla f(a,b,c) = (2a, 2b, -2c) = \sqrt{4a^2+4b^2+4c^2} = 2\sqrt{a^2+b^2+c^2}$,

a sua metade é $\frac{\|\vec\nabla f(a,b,c)\|}{2} = \frac{2\sqrt{a^2+b^2+c^2}}{2} = \sqrt{a^2+b^2+c^2}$

Pretende-se determinar $\vec u$ tal que:

$D_{\vec u} f(a,b,c) = \sqrt{a^2+b^2+c^2} \Leftrightarrow \vec\nabla f(a,b,c)\cdot(u_1,u_2,u_3) = \sqrt{a^2+b^2+c^2}$

$\Leftrightarrow (2a,2b,-2c)\cdot(u_1,u_2,u_3) = \sqrt{a^2+b^2+c^2}$

$\Leftrightarrow 2a u_1 + 2b u_2 - 2c u_3 = \sqrt{a^2+b^2+c^2} \Leftrightarrow u_3 = \frac{\sqrt{a^2+b^2+c^2}}{-2c} + \frac{a}{c}u_1 + \frac{b}{c}u_2$, $c \neq 0$

$\therefore\ \vec u = \left(u_1, u_2, \frac{\sqrt{a^2+b^2+c^2}}{-2c} + \frac{a}{c}u_1 + \frac{b}{c}u_2\right)$, $c\neq 0$.