Ficha nº 1-A
Funções trigonométricas inversas

1. a) $\text{arcsen}\left(-\frac{\sqrt{2}}{2}\right) = y$

y tem que satisfazer duas condições:

$$\begin{cases} y \in D'_{\text{arcsen}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \text{e} \\ \text{sen}\, y = -\frac{\sqrt{2}}{2} \end{cases}$$

. Assim $y = -\frac{\pi}{4}$.

b) $\text{cotg}\left(\text{arcsen}\left(-\frac{4}{5}\right)\right) = \text{cotg}\, \alpha = ?$, representando $\alpha = \text{arcsen}\left(-\frac{4}{5}\right)$.

Se $\alpha = \text{arcsen}\left(-\frac{4}{5}\right) \Leftrightarrow \text{sen}\, \alpha = -\frac{4}{5} \wedge \alpha \in 4^{\circ}$ Quadrante.

Da relação $1 + \text{cotg}^2 \alpha = \frac{1}{\text{sen}^2 \alpha}$ e $\text{sen}\, \alpha = -\frac{4}{5}$, tem-se

$$1 + \text{cotg}^2 \alpha = \frac{25}{16} \Leftrightarrow \text{cotg}^2 \alpha = \frac{9}{16} \Leftrightarrow \text{cotg}\, \alpha = \pm \frac{3}{4}.$$

Como $\alpha \in 4^{\circ}$ Quadrante , $\text{cotg}\, \alpha = -\frac{3}{4}$.

c) $\cos\left[\text{arcsen}\, \frac{1}{2} - \text{arccos}\, \frac{3}{5}\right] = \cos(\alpha - \beta)$, representando

$\alpha = \text{arcsen}\, \frac{1}{2}$ e $\beta = \text{arccos}\, \frac{3}{5}$.

$$\cos(\alpha - \beta) = \cos\alpha . \cos\beta + \text{sen}\, \alpha . \text{sen}\, \beta.$$

Como $\alpha = \text{arcsen}\, \frac{1}{2} \Leftrightarrow \text{sen}\, \alpha = \frac{1}{2}$ e $\alpha \in 1^{\circ}$ Q

$\beta = \text{arccos}\, \frac{3}{5} \Leftrightarrow \cos\beta = \frac{3}{5} \wedge \beta \in 1^{\circ}$ Q

Assim, $\cos(\alpha-\beta) = \frac{3}{5} \cdot \cos\alpha + \frac{1}{2} \cdot \operatorname{sen}\beta$.

Como $\operatorname{sen}\alpha = \frac{1}{2}$, $\cos\alpha = \pm\sqrt{1-\frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$ e $\alpha \in 1^\circ Q$,

logo $\cos\alpha = \frac{\sqrt{3}}{2}$.

Como $\cos\beta = \frac{3}{5}$, $\operatorname{sen}\beta = \pm\sqrt{1-\frac{9}{25}} = \pm\frac{4}{5}$ e $\alpha \in 1^\circ Q$,

logo $\operatorname{sen}\beta = \frac{4}{5}$.

. Assim,

$$\cos\left[\operatorname{arcsen}\frac{1}{2} - \operatorname{arccos}\frac{3}{5}\right] = \frac{3\sqrt{3}+4}{10}.$$

2. a) $R = \operatorname{arcsen}\left(\operatorname{sen}\frac{\pi}{2}\right) + 4\operatorname{arcsen}\left(-\frac{1}{2}\right) + 2\operatorname{arccos}\left(-\frac{\sqrt{2}}{2}\right) =$

$= \frac{\pi}{2} + 4\left(-\frac{\pi}{6}\right) + 2\left(-\frac{\pi}{4}\right) = -\frac{2\pi}{3}.$

b) $R = \cos^2\left(\frac{1}{2}\operatorname{arccos}\frac{1}{3}\right) - \operatorname{sen}^2\left(\frac{1}{2}\operatorname{arccos}\frac{1}{3}\right) = \cos^2\alpha - \operatorname{sen}^2\alpha$,

representando $\alpha = \frac{1}{2}\operatorname{arccos}\frac{1}{3}$.

Assim,

$$R = \cos^2\alpha - \operatorname{sen}^2\alpha = \cos(2\alpha) = \cos\left(\operatorname{arccos}\frac{1}{3}\right) = \frac{1}{3}.$$

c) $R = \operatorname{tg}^2\left(\operatorname{arcsen}\frac{3}{5}\right) - \operatorname{cotg}^2\left(\operatorname{arccos}\frac{4}{5}\right) = \operatorname{tg}^2\alpha - \operatorname{cotg}^2\beta$,

representando $\alpha = \operatorname{arcsen}\frac{3}{5}$ e $\beta = \operatorname{arccos}\frac{4}{5}$.

Fazendo $\alpha = \operatorname{arcsen}\frac{3}{5} \Leftrightarrow \operatorname{sen}\alpha = \frac{3}{5}$ e $\alpha \in 1^\circ Q$.

Daí, $\cos\alpha = \pm\sqrt{1-\frac{9}{25}} = \pm\frac{4}{5}$. Como $\alpha \in 1^\circ Q$, $\cos\alpha = \frac{4}{5}$.

Por outro lado, $\beta = \arccos \frac{4}{5} \Leftrightarrow \cos\beta = \frac{4}{5}$.

Como a função cos é injectiva em $[0,\pi]$, tem-se que $\alpha = \beta$.

Assim,

$$R = tg^2\left(\arcsen \frac{3}{5}\right) - cotg^2\left(\arccos \frac{4}{5}\right) = tg^2\alpha - cotg^2\alpha =$$

$$= \frac{sen^2\alpha}{\cos^2\alpha} - \frac{\cos^2\alpha}{sen^2\alpha} = \frac{\frac{9}{25}}{\frac{16}{25}} - \frac{\frac{16}{25}}{9/25} = \frac{9}{16} - \frac{16}{9} = -\frac{175}{144}.$$

3.a) $f(-1) - f\left(-\frac{3}{2}\right) = \frac{\pi}{3} - 2\arccos(-1+1) - \frac{\pi}{3} + 2\arccos\left(-\frac{3}{2}+1\right)$

$$= -2\arccos 0 + 2\arccos\left(-\frac{1}{2}\right) = -2\times\frac{\pi}{2} + 2\left(\pi - \frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

b) $D_f = \{x \in \mathbb{R} : x+1 \in D_{\arccos}\}$

$= \{x \in \mathbb{R} : x+1 \in [-1,1]\}$

$= \{x \in \mathbb{R} : -1 \leq x+1 \leq 1\} = \{x \in \mathbb{R} : -2 \leq x \leq 0\} = [-2,0]$

$D'_f = \{y \in \mathbb{R} : y = f(x), x \in D_f\}$

Tem-se $\quad \arccos(x+1) \in D'_{\arccos}$

$\arccos(x+1) \in [0,\pi]$

$0 \leq \arccos(x+1) \leq \pi$

$0 \geq -2\arccos(x+1) \geq -2\pi$

$\frac{\pi}{3} \geq \frac{\pi}{3} - 2\arccos(x+1) \geq -\frac{5\pi}{3}$

$D'_f = \left[-\frac{5\pi}{3}, \frac{\pi}{3}\right]$

c) $f(x)=0 \Leftrightarrow \frac{\pi}{3} - 2\arccos(x+1) = 0 \Leftrightarrow$

$\Leftrightarrow -2\arccos(x+1) = -\frac{\pi}{3} \Leftrightarrow \arccos(x+1) = \frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow x+1 = \cos\frac{\pi}{6} \Leftrightarrow x = \frac{\sqrt{3}}{2} - 1 \in D_f$

d) $y = f(x) \Leftrightarrow x = f^{-1}(y)$

$y = \frac{\pi}{3} - 2\arccos(x+1) \Leftrightarrow y - \frac{\pi}{3} = -2\arccos(x+1) \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{6} - \frac{y}{2} = \arccos(x+1) \Leftrightarrow$

$\Leftrightarrow \cos\left(\frac{\pi}{6} - \frac{y}{2}\right) = x+1 \Leftrightarrow$

$\Leftrightarrow x = \cos\left(\frac{\pi}{6} - \frac{y}{2}\right) - 1$

$$f^{-1} : \left[-\frac{5\pi}{3}, \frac{\pi}{3}\right] \longrightarrow [-2, 0]$$

$$y \rightsquigarrow f^{-1}(y) = \cos\left(\frac{\pi}{6} - \frac{y}{2}\right) - 1.$$

e) $f(x) \leq -\frac{\pi}{3} \Leftrightarrow \frac{\pi}{3} - 2\arccos(x+1) \leq -\frac{\pi}{3} \Leftrightarrow -2\arccos(x+1) \leq -\frac{2\pi}{3}$

$\Leftrightarrow \arccos(x+1) \geq \frac{\pi}{3} \Leftrightarrow x+1 \leq \cos\frac{\pi}{3} \Leftrightarrow$

A troca de sentido de desigualdade deve-se ao facto de cos ser uma função decrescente.

$\Leftrightarrow x \leq \frac{1}{2} - 1 \Leftrightarrow x \leq -\frac{1}{2}$

$$S = \left]-\infty, -\frac{1}{2}\right] \cap D_f = \left]-\infty, -\frac{1}{2}\right] \cap [-2, 0] = \left[-2, -\frac{1}{2}\right]$$

4. a) $g'(t) = \frac{dg}{dt}(t) = \left(3t \cdot \text{arcsen}\sqrt{t^2-1}\right)' =$

$$= 3\,\text{arcsen}\sqrt{t^2-1} + 3t\left(\text{arcsen}\sqrt{t^2-1}\right)'$$

$$= 3\,\text{arcsen}\sqrt{t^2-1} + 3t\,\frac{\left(\sqrt{t^2-1}\right)'}{\sqrt{1-(t^2-1)}} =$$

$(\text{arcsen}\,u)' = \frac{u'}{\sqrt{1-u^2}}$

$$= 3\,\text{arcsen}\sqrt{t^2-1} + \frac{3t^2}{\sqrt{2-t^2}\,\sqrt{t^2-1}}$$

b) $f'(y) = \frac{df}{dy}(y) = \left(\frac{1}{\cos y}\right)' - \left(\text{arctg}\left(\frac{y}{2}\right)\right)'$

$$= -\frac{1}{\cos^2 y} - \frac{\frac{1}{2}}{1+\frac{y^2}{4}} =$$

$$= -\frac{1}{\cos^2 y} - \frac{2}{y^2+4}\,.$$

$(\text{arctg}\,u)' = \frac{u'}{1+u^2}\,.$