# Quantitative Research Case Study

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# Q1 and Q3. Pricing European and American call options.

The goal is to find the value of the European call option and American call option knowing the date of maturity $N$, strike price $K$, initial price of the asset at period 0 is $S_0 = 1$ and the asset price at period $j$ is $S_j = (1+v)S_{j-1}$ with risk-neutral probability $p = 1/2$ and $S_j = (1-v)S_{j-1}$ with probability $1/2$. The interest rate in the given model is 0.

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- For each period $j = 1, ..., N$ calculate all possible values of the prices of the asset and put them into the matrix $S$: $S[i][j]$ has to contain the price of asset in period $j$ provided price for this asset moved up $j - i$ times before.

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- Calculate the values of the option at period $N$ using $f(S) = \max(S - K, 0)$ and use the standard formulas for option pricing. Namely for European:

$$V_{k-1}(S) = \frac{1}{1+r}(pV_k(S(1+v)) + (1-p)V_k(S(1-v))) \tag{1}$$

For American:

$$V_{k-1}(S) = \max(\frac{1}{1+r}(pV_k(S(1+v)) + (1-p)V_k(S(1-v))), f(S)) \tag{2}$$

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- Update matrix $S$: column $j$ has to contain values of the option at period $j$ for all possible $j+1$ prices of the asset in this period.

Note: it was assumed that interest rate $r = 0$. However, pricing works even if $r$ is some number from $(0, 1)$. The only difference is that we need to change risk-neutral measure. Using the standard formula, risk-neutral $p$ is:

$$p = \frac{1 + r - (1 - v)}{1 + v - (1 - v)} = \frac{r + v}{2v} = \frac{1}{2} + \frac{r}{2v} \tag{3}$$

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- Update matrix $S$: column $j$ has to contain values of the option at period $j$ for all possible $j+1$ prices of the asset in this period.
- The value of the option will be in $S[0][0]$

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# Q2. Calibration of parameter $v$

The goal is to calibrate parameter $v$ to match the price (denoted $V_0$) of some European call option on the asset $S$ given the date of maturity $N$ and strike price $K$.
For the given model, I can derive the formula for the price of a European option:

$$V_0 = \frac{1}{(1+r)^N} \sum_{j=0}^{N} C_N^j p^j (1-p)^{N-j} f\left((1+v)^j (1-v)^{N-j}\right) \tag{4}$$

where $p$ is risk-neutral probability and $f(S) = \max(S-K, 0)$.

# Q2. Calibration of parameter $v$

It was shown that $p$ depends on $v$. Then

$$V_0 = \frac{1}{(1+r)^N} \sum_{j=0}^{N} C_N^j (\frac{1}{2} + \frac{r}{2v})^j (\frac{1}{2} - \frac{r}{2v})^{N-j} f((1+v)^j (1-v)^{N-j}) \quad (5)$$

And we end up with the function of $v$, provided other parameters are known. Hence, the problem will be solved by finding the root of the function:

$$V(v) = \frac{1}{(1+r)^N} \sum_{j=0}^{N} C_N^j (\frac{1}{2} + \frac{r}{2v})^j (\frac{1}{2} - \frac{r}{2v})^{N-j} f((1+v)^j (1-v)^{N-j}) - V_0 \quad (6)$$

where $V_0$ is given as a price of the European call option.

## Q2. Calibration of parameter $v$

Given $N$,$K$,$r$, and $V_0$, three scenarios are possible:

1. $V(v)$ has unique root $v^*$ on the interval $(r, 1)$ and this is the solution

Example 1. Assume $N = 2$, $K = 0.5$, $r = 0$, $V_0 = 0.5$. Clearly, risk-neutral $p = 0.5$. Then

$$V(v) = (\frac{1}{2})^2(f(1+v)^2) + 2f((1+v)(1-v)) + f((1-v)^2) - 0.5 \quad (7)$$

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1. $V(v)$ has unique root $v^*$ on the interval $(r, 1)$ and this is the solution
2. $V(v)$ doesn't have a root on the interval $(r, 1)$
3. $V(v)$ has infinitely many roots on the interval $(r, 1)$

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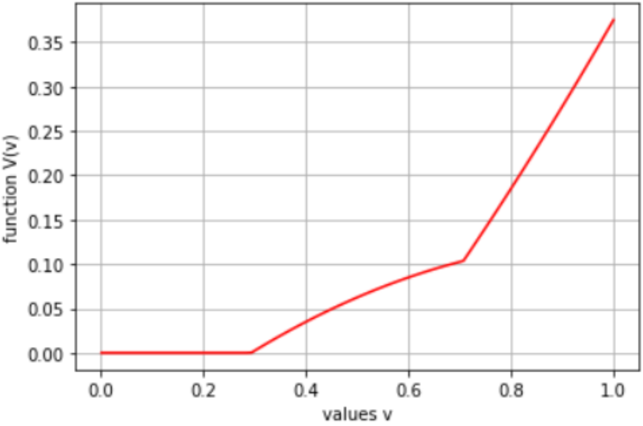
# Q2. Calibration of parameter $v$

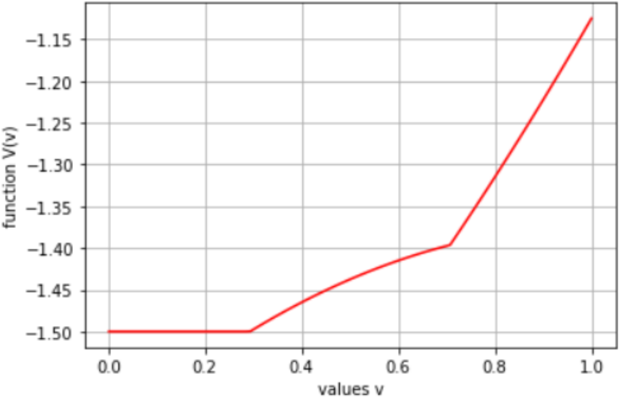
Illustration to case 3 (infinitely many roots)




In this case calibration of a unique parameter $v$ is not possible

# Q2. Calibration of parameter $v$

Example 2. Assume $N = 2$, $K = 0.5$, $r = 0$, $V_0 = 2$. Function $V(v)$ doesn't have roots at the interval $(r, 1)$




Therefore, no $v$ matches the given value $V_0$ of the option.

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1. "Brute force" method simply check possible $v$ one by one looking for the root.
2. "Binary section" method calculates the values $V(v)$ at the ends of the given interval (starting from $(r, 1)$) and in the middle of the interval. If signs of the function $V(v)$ at the left and at the middle point are different, then the root is in $(r, (r+1)/2)$, otherwise it is in $((r+1)/2, 1)$. Repeat this until approximate root is detected.

# Q4. Estimation of $\max_{1 \leq j \leq N} S_j$

The goal is to estimate the expectation of $\max_{1 \leq j \leq N} S_j$ given the parameter $v$.

To do this I use Monte-Carlo method in the following steps:

- Knowing parameter $v$ (and corresponding risk-neutral $p$) and $S_0 = 1$ generate random path for prices $S$ of the asset $(1, S_1, S_2, ..., S_N)$ and find the maximum the price $S$ reaches along the path. Denote the price $S_M^1$

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- Repeat path generation $I$ times and obtain the sample of $I$ maximal prices $(S_M^1, S_M^2, ..., S_M^I)$
- Find the sample average of $(S_M^1, S_M^2, ..., S_M^I)$ and by the Law of Large Numbers average $\bar{S}^M$ is an approximation of the mathematical expectation of $\max_{1 \leq j \leq N} S_j$

# Q5. Calibration of $(v_1, v_2, ..., v_N)$

The goal is to calibrate parameters $(v_1, v_2, ..., v_N)$ knowing the prices of $N$ European call options (denoted by $V_1, V_2, ..., V_N$ and strike price $K$ (here I allow interest rate $r$ to be $> 0$).
To begin with, I write the code that will price a European option with the given strike price $K$, $r$ and known parameters $(v_1, v_2, ..., v_N)$. The steps of the solution are as follows:

- Calculate risk-neutral probabilities for each period: $p_i = \frac{1}{2} + \frac{r}{2v_i}$ (if $r = 0$, then for all $i$ $p_i = 0.5$)

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- Calculate risk-neutral probabilities for each period: $p_i = \frac{1}{2} + \frac{r}{2v_i}$ (if $r = 0$, then for all $i$ $p_i = 0.5$)
- $S_0 = 1$ with probability 1. Assuming that possible prices $S_{j-1}$ (and corresponding probabilities) are known, I calculate $S_j = (1 + v_j)S_{j-1}$ and $S_j = (1 - v_j)S_{j-1}$. In the period $j$ price $S_j$ can take $2^j$ values (with possibly different probabilities if $r > 0$).

# Q5. Calibration of $(v_1, v_2, ..., v_N)$

**Example 3.** Assume $N = 2$ and $(v_1, v_2)$ are given. Then risk neutral probability measure is $p_1 = \frac{1}{2} + \frac{r}{2v_1}$ and $p_2 = \frac{1}{2} + \frac{r}{2v_2}$ correspondingly.
Hence $S_1 = (1 + v_1)$ with probability $p_1$ and $S_1 = (1 - v_1)$ with probability $(1 - p_1)$.
Next,

$$S_2 = \begin{cases} (1+v_1)(1+v_2) & \text{with probability } p_1 p_2 \\ (1+v_1)(1-v_2) & \text{with probability } p_1(1-p_2) \\ (1-v_1)(1+v_2) & \text{with probability } (1-p_1)p_2 \\ (1-v_1)(1-v_2) & \text{with probability } (1-p_1)(1-p_2) \end{cases} \tag{8}$$

# Q5. Calibration of $(v_1, v_2, ..., v_N)$

To store all the paths for $S$ and corresponding probabilities together I will use Trees. Namely, one node on the level $j$ keeps one possible value for the price $S_j$ with the probability, and it is linked to three other nodes.

**Example 3 (cont.)** An example of a node is $[1 + v_1, p_1]$. It is linked to the parental node $[1, 1]$ as $S_1$ was obtained from $S_0$ and to two children nodes $[(1 + v_1)(1 + v_2), p_1 p_2]$ and $[(1 + v_1)(1 - v_2), p_1(1 - p_2)]$.

# Q5. Calibration of $(v_1, v_2, ..., v_N)$

Assume now that prices of $N$ European call options with different dates of maturity are known: $(V_1, V_2, ..., V_N)$ and I have to calibrate parameters $(v_1, ..., v_N)$. Clearly, the value of the option that expires in period 1 is

$$V_1 = \frac{1}{1+r}(pf(1+v_1) + (1-p)f(1-v_1)) \tag{9}$$

where $p$ is risk-neutral probability that depends on $v$. Consider the function:

$$V_1(v) = \frac{1}{1+r}(p(v)f(1+v) + (1-p(v))f(1-v) - V_1. \tag{10}$$

Root of this function $v*$ is calibrated value $v_1$. Can be found by "brute force" search or binary section.

# Q5. Calibration of $(v_1, v_2, ..., v_N)$

Assume that $v_1^*, v_2^*, .., v_{j-1}^*$ are calibrated. Then I take price of the European call option with maturity date $j$. The formula for the price is:

$$V_j = \frac{1}{(1+r)^j} \sum f((1 \pm v_1^*)...(1 \pm v_j))(\frac{1}{2} \pm \frac{r}{2v_1^*})...(\frac{1}{2} \pm \frac{r}{2v_j}) \tag{11}$$

All the parameters $v_1^*, v_2^*, .., v_{j-1}^*$ are known, hence, the only unknown variable is $v_j$. Write down the function:

$$V_j(v) = \frac{1}{(1+r)^j} \sum f((1 \pm v_1^*)...(1 \pm v))(\frac{1}{2} \pm \frac{r}{2v_1^*})...(\frac{1}{2} \pm \frac{r}{2v}) - V_j \tag{12}$$

Then the root of this function is calibrated $v_j$. Therefore, I calibrate parameters one by one.

Note: issues with the existence or uniqueness of the root of some function described in Q2 are present in this model as well.