15 лекция

14 декабря 2022 г. 13:43

Применение преобразования Фурье

$$\begin{cases} u_t - a^2 u_{xx} = 0, & x\in\mathbb{R},\ t>0 \\ u(x,0)=\varphi(x); & x\in\mathbb{R}\end{cases}$$

$$U(\lambda,t) = \widehat{u(x,t)} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda x} u(x,t)\,dx$$

$$\Phi(\lambda) = \widehat{\varphi(x)} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda x}\varphi(x)\,dx$$

$$\begin{cases} U_t(\lambda,t) + a^2\lambda^2 U(\lambda,t) = 0, \\ U(\lambda,0) = \Phi(\lambda)\end{cases}$$

Решение: $U(\lambda,t) = C(\lambda)e^{-a^2\lambda^2 t}$, $\lambda\in\mathbb{R}$

$U(\lambda,t) = \Phi(\lambda)e^{-a^2\lambda^2 t}$

Без источников тепла

$$\int_0^{+\infty} e^{-p^2u^2}\cos(qu)\,du = I(q)$$

$$\frac{d}{dq}I(q) = -\int_0^{+\infty} u e^{-p^2u^2}\sin(qu)\,du = -\frac{1}{2p^2}\left(e^{-p^2u^2}\sin(qu)\right)\Big|_0^{+\infty} + q\int_0^{+\infty} e^{-p^2u^2}\cos(qu)\,du\Big) = -\frac{q}{2p^2}I(q)$$

$$ue^{-p^2u^2} = \frac{1}{-2p^2}\frac{d}{du}\left(e^{-p^2u^2}\right)$$

$$I'(q) + \frac{q}{2p^2}I(q) = 0$$

$$I(q) = C(p)\,e^{-\frac{q^2}{4p^2}}$$

$$C(p) = I(0) = \int_0^{+\infty} e^{-p^2u^2}du = \frac{1}{p}\int_0^{+\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2p}$$

$$\int_0^{+\infty} e^{-p^2u^2}\cos(qu)\,du = \frac{\sqrt{\pi}}{2p}e^{-\frac{q^2}{4p^2}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\lambda x}U(\lambda,t)\,d\lambda = \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(s)\left(\int_{\mathbb{R}} e^{i\lambda(x-s)}e^{-a^2\lambda^2 t}d\lambda\right)ds =$$

$$= \frac{2}{2\pi}\int_{\mathbb{R}}\varphi(s)\left[\int_0^{+\infty}\cos(\lambda(x-s))\,e^{-a^2\lambda^2 t}d\lambda\right]ds =$$

$$= \frac{1}{\pi}\int_{\mathbb{R}}\varphi(s)\frac{\sqrt{\pi}}{2a\sqrt{t}}e^{-\frac{(x-s)^2}{4a^2t}}ds$$

Ответ: $u(x,t) = \dfrac{1}{2a\sqrt{\pi t}}\displaystyle\int_{\mathbb{R}}\varphi(s)\,e^{-\frac{(x-s)^2}{4a^2t}}ds$

$$\begin{cases} u_t - a^2u_{xx} = f(x,t), & x\in\mathbb{R} \\ u(x,0) = 0 & t>0\end{cases}$$

$$U(\lambda,t) = \frac{1}{\sqrt{2\pi}}\int e^{-i\lambda x}u(x,t)\,dx$$

$$F(\lambda,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda x}f(x,t)\,dx$$

С источникам тепла

$$\begin{cases} U_t + a^2\lambda^2 U = F(\lambda,t) \\ U(\lambda,0) = 0\end{cases},\quad U(\lambda,t) = C(\lambda)e^{-a^2\lambda^2 t}$$

$$C_t = F(\lambda,t)e^{a^2\lambda^2 t}$$

$$C(\lambda,t) = \int_0^t F(\lambda,\tau)e^{a^2\lambda^2\tau}d\tau + C_1(\lambda)$$

$$U_{н.о.} = C_1(\lambda)e^{-a^2\lambda^2 t} + \int_0^t F(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}d\tau$$

неоднородное

$$U(\lambda,0) = 0 \Rightarrow C_1(\lambda) = 0$$

$$U_{рно} = \int_0^t F(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}d\tau$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\lambda x}U(\lambda,t)\,d\lambda = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x}\left[\int_0^t\left(\int_{\mathbb{R}} e^{-i\lambda s}f(s,\tau)\,ds\right)e^{-a^2\lambda^2(t-\tau)}d\tau\right]d\lambda =$$

$$= \frac{1}{2\pi}\int_0^t d\tau\int_{\mathbb{R}} f(s,\tau)\,ds\int_{\mathbb{R}} e^{i\lambda(x-s)-a^2\lambda^2(t-\tau)}d\lambda = \frac{2}{2\pi}\int_0^t d\tau\int_{\mathbb{R}} f(s,\tau)\,ds\int_0^{+\infty}\cos(\lambda(x-s))\,e^{-a^2\lambda^2(t-\tau)}d\lambda = \frac{1}{2a\sqrt{\pi}}\int_0^t\frac{d\tau}{\sqrt{t-\tau}}\int_{\mathbb{R}} f(s,\tau)\,e^{-\frac{(x-s)^2}{4a^2(t-\tau)}}ds$$

Формула Пуассона

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-\lambda)^2}{4a^2t}}\varphi(\lambda)\,d\lambda + \int_0^t\int_{\mathbb{R}}\frac{e^{-\frac{(x-\lambda)^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi(t-\tau)}}f(\lambda,\tau)\,d\tau\,d\lambda$$

$$\sigma(f): \{\omega : \hat{f}(\omega)\neq 0\}$$

$$f(t) = \begin{cases} 1, & t\in[-T,T] \\ 0, & t\notin[-T,T]\end{cases};$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{T} e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-i\omega}\right)e^{-i\omega t}\Big|_{-T}^{T} = \frac{1}{\sqrt{2\pi}}\,\frac{2\left(e^{-i\omega T} - e^{i\omega T}\right)}{(-i\omega)2} = \sqrt{\frac{2}{\pi}}\,\frac{\sin(\omega T)}{\omega}$$

$$\Delta\omega = \frac{2\pi}{T}$$
$$\Delta T = 2T \quad\Rightarrow \Delta T\cdot\Delta\omega = 4\pi$$

$$x_0 = \frac{1}{\|f\|^2_{L_2}}\int_{-\infty}^{\infty} x\,|f(x)|^2dx$$

$$(\Delta f) = \frac{1}{\|f\|^2_{L_2}}\int_{\mathbb{R}}(x-x_0)^2|f(x)|^2dx$$