PHYS 301 LECTURE 5

07.10.24

Simple Rigid Body Problem

For any system we showed the angular m. is:

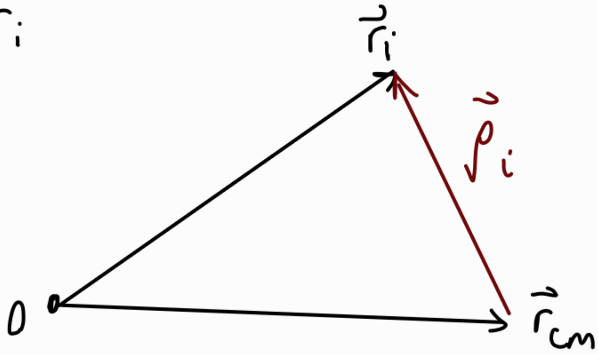
$$\vec{L}_{tot} = \sum_{i=1}^{N} m_i \vec{r}_i \times \vec{v}_i$$

Rigid Body Constraints:

Being on the same axis isn't enough (solo system)

→ $|r_i - r_j|$ are constants $= c_{ij}$ defines the shape.

Center of mass: $\vec{r}_{cm} = \underbrace{\sum_{i=1}^{N} m_i}_{M_{Tot.}} \vec{r}_i$



Define a new coordinate $\vec{\rho}_i$:

$$\vec{r}_i = \vec{r}_{cm} + \vec{\rho}_i \Rightarrow \sum_{i=1}^{N} m_i \rho_i = 0 \quad (*)$$

$$\vec{V}_i = \vec{V}_{cm} + \frac{d\vec{\rho}_i}{dt} \Rightarrow \sum_{i=1}^{N} m_i \frac{d\rho_i}{dt} = 0$$

Pluggin' these back: $\vec{L}_{Tot} = \sum_{i-}^{N} m_i (\vec{r}_{cm} + \vec{\rho}_i) \times \left(\vec{V}_{cm} + \frac{d\rho_i}{dt}\right)$

$$\vec{L}_{Tot} = \underbrace{m_{Tot}\, \vec{r}_{cm} \times \vec{V}_{cm}}_{\text{CM acts like a point. Angular momentum due to that}} + \vec{r}_{cm} \times \cancel{\sum_{i=1}^{N} m_i \frac{d\rho_i}{dt}}^{\,0} + \cancel{\sum_{i=1}^{N} m_i \rho_i}^{\,0} \times V_{cm}$$

$$\boxed{+ \sum_{i=1}^{N} m_i \rho_i \times \frac{d\rho_i}{dt}}$$

→ So the motion can be separated into two parts. rotation around CM and motion of CM as a point

So, $$\vec{L}^{(CM)}_{Tot} \equiv m_{Tot}\, \vec{r}_{cm} \times \vec{v}_{cm} + \sum_{i=1}^{N} m_i \rho_i \times \frac{d\rho_i}{dt}$$

$$\vec{\tau}_{Tot} = \sum_i \vec{r}_i \times F_i = \sum_i (\vec{r}_{cm} + \vec{\rho}_i) \times \vec{F}_i = \vec{r}_{cm} \times \overbrace{\sum_i F_i}^{F_{net}} + \sum_i \vec{\rho}_i \times \vec{F}_i$$

$$\frac{d\vec{L}^{(CM)}_{Tot}}{dt} = \sum \vec{\rho}_i \times \vec{F}_i$$

Last time we defined angular velocity:

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$|\vec{r}_i - \vec{r}_j| \equiv c_{ij}$ is constant

$|\vec{\rho}_i - \vec{\rho}_j| \equiv c_{ij} \longrightarrow \vec{\omega}_i \equiv \vec{\omega}$

This is why solar system isn't a rigid body.

$|\vec{\rho}_i|$ is also constant: $\vec{\rho}_i \cdot \vec{\rho}_i$ is constant.

- $\rho_i \cdot \dfrac{d\rho_i}{dt} = 0 \Rightarrow \vec{\rho}_i \perp \dfrac{d\vec{\rho}_i}{dt}$

- $\dfrac{d\vec{\rho}_i}{dt} = \vec{\omega}_i \times \vec{\rho}_i$

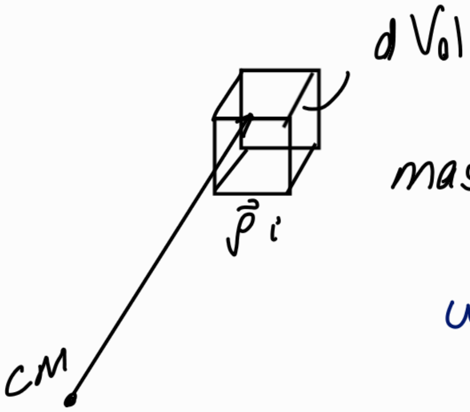
Finding $\vec{L}^{cm}_{Tot}$ For Rigid Bodies

$$\vec{L}^{cm}_{\substack{Tot\\rigid}} = \sum_{i=1}^{N} m \vec{\rho}_i \times (\vec{\omega} \times \vec{\rho}_i) \not\Rightarrow \vec{L}^{cm}_{\substack{Tot\\rigid}} = I\vec{\omega} \text{ ?}$$

generally L, ω don't have to be parallel.

if continuous body:

$$m_i \longrightarrow dm_i \qquad \sum_i \longrightarrow \int dV_{ol}\, \mu(\vec{\rho})\, \vec{\rho} \times (\vec{\omega} \times \vec{\rho})$$

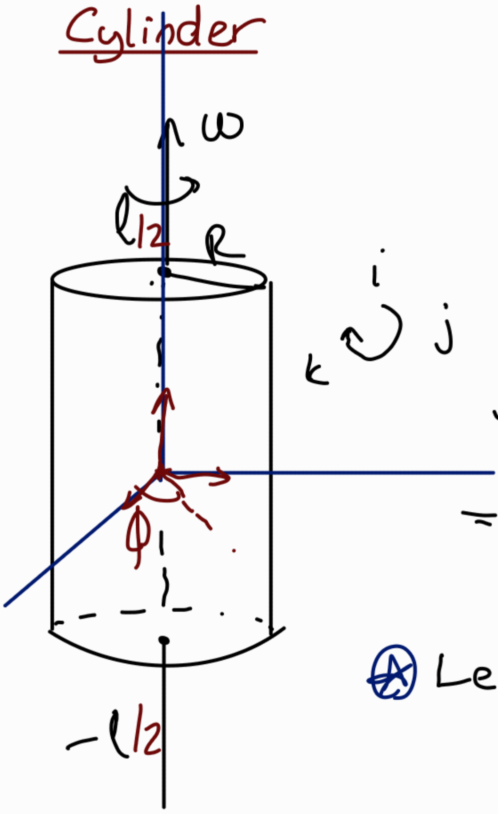


mass density $\mu(\vec{\rho})$

Why use symmetrical objects?

It's just easier.

Cylinder



$$\vec{\rho} = z\hat{k} + x\hat{i} + y\hat{j}$$
$$\vec{\omega} = \omega\hat{k}$$

$$\vec{\omega} \times \vec{\rho} = \omega(x\hat{j} - y\hat{i})$$

$$\vec{\rho} \times (\vec{\omega} \times \vec{\rho}) = \omega(z\hat{k} + x\hat{i} + y\hat{j}) \times (x\hat{j} - y\hat{i})$$

$$= \omega(-zx\hat{i} - zy\hat{j} + x^2\hat{k} + y^2\hat{k})$$

Let it be a homogeneous rigid body!

$$dm = \mu\, dx\, dy\, dz$$

$$\int_{body} \omega\mu\, dx\, dy\, dz\, (\underbrace{-zx\hat{i} - zy\hat{j}}_{\text{cancel due to symmetry}} + (x^2+y^2)\hat{k})$$

$$= \omega \mu \, l \, \hat{k} \int dx\,dy \underbrace{(x^2+y^2)}_{}$$

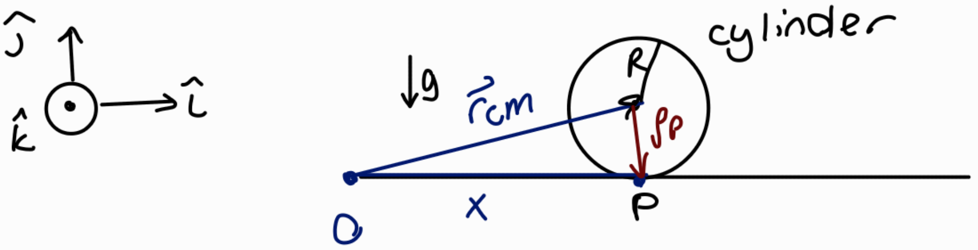
$\perp$ distance from particle to $\hat{k}$

$$\vec{L}_{Tot} = I \vec{\omega} \qquad I = \mu \, l \int dx\,dy \, (x^2+y^2)$$

★ $\vec{L}$ and $\vec{\omega}$ are parallel only when the rotation axis is on symmetric axis and rotation axis is stable!

Rolling without Slipping is a vector condition:

Contact point has zero relative velocity



Rolls without slipping towards the right

$\vec{\omega} = -\omega_0 \hat{k}$

$\vec{V}_{cm} = V_0 \hat{\imath}$

$$\vec{V}_P = \vec{V}_{cm} + \vec{\omega} \times \vec{\rho}_P \qquad \vec{\rho}_P = -R\hat{\jmath}$$

$$V_P = V_0 \hat{\imath} + \omega_0 R \underbrace{(+\hat{k}) \times (+\hat{\jmath})}_{-\hat{\imath}}$$

By definition:

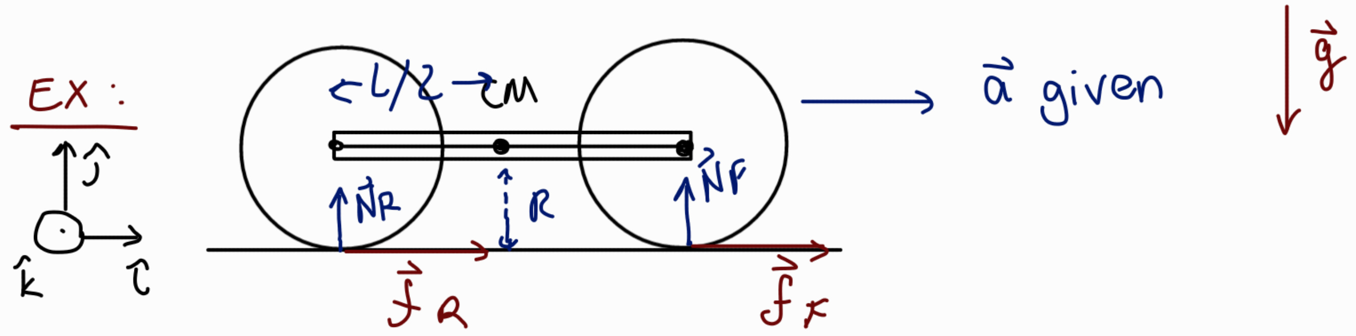
$$\Rightarrow V_P = 0 \quad \Rightarrow \quad V_0 = \omega_0 R$$

generic condition: $\vec{V}_P = 0 \Rightarrow \vec{V}_{cm} = -\vec{\omega} \times \vec{\rho}_P = \vec{\rho}_P \times \vec{\omega}$

* Torque causes car to move by inducing friction.
Let's apply something similar here

There will be a static friction force
(Because P is static)

EX:



$$\vec{F}^{TOT}_{CM} = m_{Total}\vec{a} = \vec{f}_R + \vec{f}_F$$

$$\vec{\tau}^{TOT}_{CM} = 0 = \hat{k}\left(\frac{L}{2}\vec{N}_F - \frac{L}{2}N_R + Rf_F + Rf_R\right) = 0$$

$$\vec{N}_R - \vec{N}_F = \frac{2}{L}Rm_{Tot}a \qquad \vec{N}_R + \vec{N}_F = m_{Tot}\vec{g}$$

- $\vec{N}_R = \left(\frac{2R}{L}a + \vec{g}\right)\frac{m_{Tot}}{2}$
- $\vec{N}_F = \left(\vec{g} - \frac{2Ra}{L}\right)\frac{m_{Tot}}{2}$

$$\vec{N}_R > \vec{N}_F$$