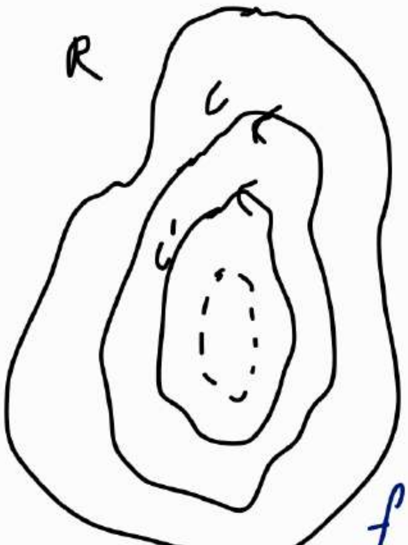
PHYS 325 Lecture 6 15.10.24

Cauchy - Goursat Theorem

$R \rightarrow$ simply connected region

$$\oint_C dz\, f(z) = 0$$

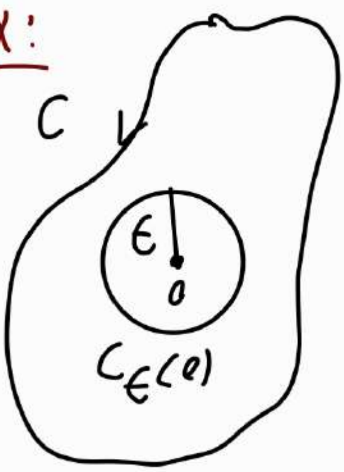
f analytic on R



$$\oint_C dz\, f(z) = \oint_{C'} dz\, f(z)$$

f analytic on R

Ex:

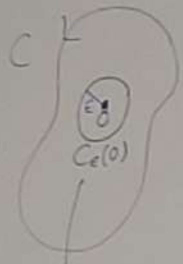


$n \in \mathbb{Z}$

$$\oint_C dz\, z^n$$

$z = re^{i\theta}$, $r = \epsilon$ on $C_\epsilon(0)$

Ex:



circle of radius ϵ and center at 0

$n \in \mathbb{Z}$

$$\oint_C dz\, z^n$$

$$z = r e^{i\theta}, \quad r = \epsilon \text{ on } C_\epsilon(0)$$

$$dz = \epsilon\, d(e^{i\theta}) = \epsilon\, i\, d\theta\, e^{i\theta}$$

$n \in \mathbb{Z}_{\geq 0}$

$$\oint_C dz\, z^n = \oint_{C_\epsilon(0)} dz\, z^n = \int_0^{2\pi} d\theta\, (i\epsilon e^{i\theta})(\epsilon^n e^{in\theta})$$

$$= i\epsilon^{n+1} \int_0^{2\pi} d\theta\, e^{i(n+1)\theta} = i\epsilon^{n+1} \frac{e^{i(n+1)\theta}}{i(n+1)} \Bigg|_0^{2\pi}$$

$$= \frac{\epsilon^{n+1}}{n+1} \left(e^{i\, 2\pi(n+1)} - 1 \right) = 0$$

$$\cos 2\pi(n+1) + i \sin 2\pi(n+1) = 1$$

"We could just use Cauchy-Goursat Theorem"

$n = -1$

z^{-1} is not analytic. (it's not even defined) at 0.

I can't apply the theorem in this case.

$$\oint_C dz \, \frac{1}{z} = 2\pi i$$

Indeed it isn't 0.

very important result (why?)

Writing Polar Coord. outside the origin.

z

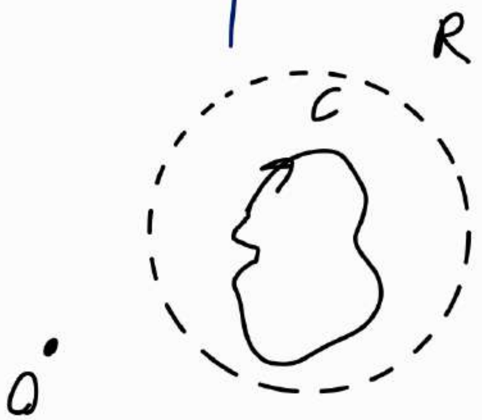
z_0

$\cdot 0$

$z - z_0 = |z - z_0| e^{i\theta}$

Consider this: (important)

$\frac{1}{z}$ is analytic on R



$$\oint \frac{dz}{z} = \begin{cases} 2\pi i & \text{if } C \text{ enclose } 0 \\ 0 & \text{otherwise} \end{cases}$$

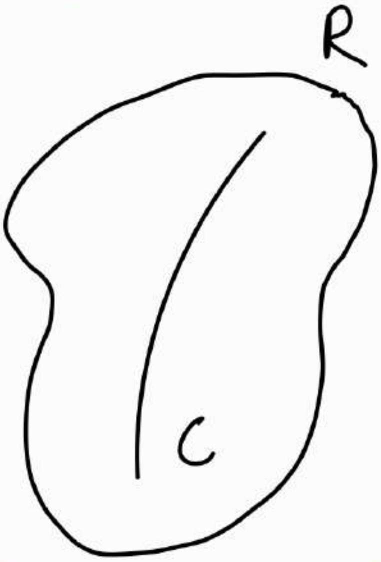
($\oint \frac{dz}{z} = 0$ here)

$n < -1$

$$= \frac{\epsilon^{n+1}}{n+1}\left(e^{i\,2\pi(n+1)} - 1\right) = 0$$

$$\cos 2\pi(n+1) + i \sin 2\pi(n+1) = 1$$

Change of Variable in Contour Integration



$$\int_C dz\, f(z)$$

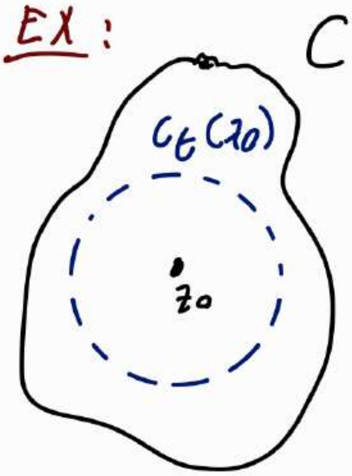
- $z = g(z')$ g is analytic on R w/ analytic inverse g^{-1}.

f analytic on R

- Let $C' = g^{-1}(C)$

$$* \quad \int_C dz\, f(z) = \int_{C} dz' \underbrace{\boxed{\frac{dg(z')}{dz'}}}_{\text{Jacobian Factor}} f(g(z'))$$

EX:



$$\oint_C dz\,(z-z_0)^n$$

Remember:

$$\boxed{\oint dz\, z^n} = \begin{cases} 2\pi i & n=-1 \\ 0 & \text{otherwise} \end{cases}$$

we deform the contour and make it a circle

$$(*) \quad \oint_C dz\,(z-z_0)^n = \oint_{C_\epsilon(z_0)} dz\,(z-z_0)^n$$

Change of variable.

$$z = z' + z_0 = g(z') \qquad \frac{dg(z')}{dz'} = 1$$

$$(*) = \boxed{\int_{C_\epsilon(0)} dz' \cdot 1 \cdot (z')^n}$$

=

so, same result.

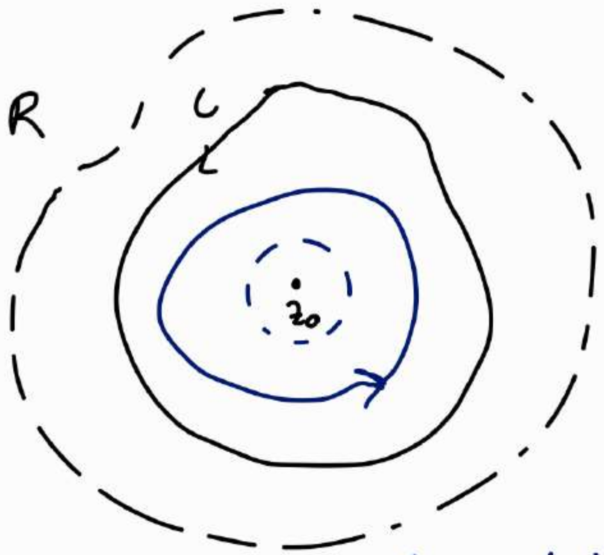
$$C_\epsilon(z_0) : |z - z_0| = \epsilon$$

$$C_\epsilon(0) : |z'| = \epsilon$$

P.S: If C enclose z_0 : $\oint_C dz\,(z-z_0)^n = \begin{cases} 2\pi i & n=-1 \\ 0 & \text{otherwise} \end{cases}$

if not $\oint_C dz\,(z-z_0)^n = 0$

Cauchey Integral Formula



$$\oint dz \, \frac{f(z)}{z - z_0}$$

f analyt' on R

⊗ For any $\epsilon > 0$ $\exists \, \delta(\epsilon)$ s.t. $|f(z) - f(z_0)| < \epsilon$ for $|z - z_0| < \delta$ This follows from continuity of f at z_0

$$\oint_C dz \, \frac{f(z)}{z - z_0} = \oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \, \frac{f(z)}{z - z_0}$$

$$= \oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \, \frac{f(z) - f(z_0)}{z - z_0} + f(z_0) \underbrace{\oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \, \frac{1}{z - z_0}}_{2\pi i}$$

Triangle Inequality

$$\rightarrow \left| \oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \, \frac{|f(z) - f(z_0)|}{|z - z_0|} = \frac{2}{\delta(\epsilon)} \oint_{C_{\delta(\epsilon)}(z_0)} dz \, |f(z) - f(z_0)|$$

(circumference) (Darbet theorem)

$$\le \frac{\cancel{2}}{\cancel{\delta(\epsilon)}}\, \epsilon\, 2\pi \frac{\cancel{\delta(\epsilon)}}{\cancel{2}} = 2\pi\epsilon$$

I can take the limit.

$$\lim_{\epsilon\to 0} \oint_C dz \frac{f(z)}{z-z_0} = \lim_{\epsilon\to 0} \oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \frac{f(z)}{z-z_0}$$

$$= \oint_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \frac{f(z)-f(z_0)}{z-z_0} + f(z_0) \underbrace{\int_{C_{\frac{\delta(\epsilon)}{2}}(z_0)} dz \frac{1}{z-z_0}}_{2\pi i}$$

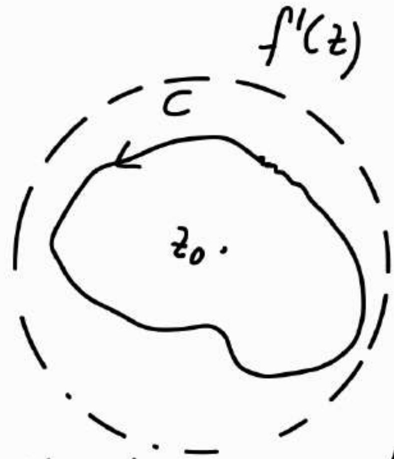
$$= \boxed{2\pi i\, f(z_0)}$$

Derivatives

The derivatives of all orders of an analytic function exist and are themselves analytic.

Moreover:

$$\left.\frac{d^n f(z)}{dz^n}\right|_{z_0} = \frac{n!}{2\pi i} \oint dz \frac{f(z)}{(z-z_0)^{n+1}}$$

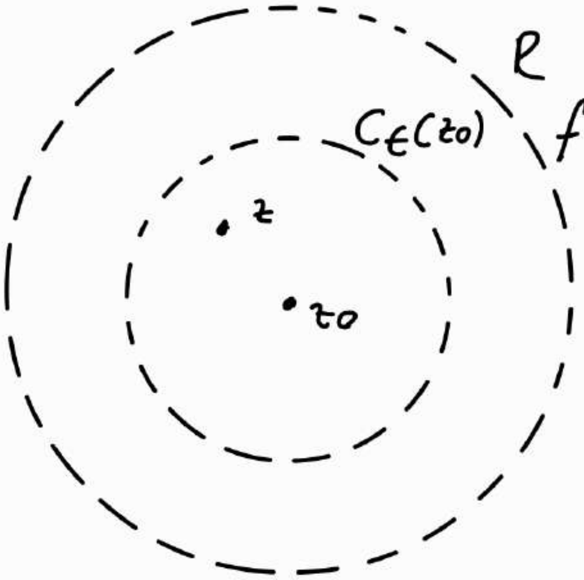


R simply connected

(✱) There is no analouge of this formula for real variables.

Proof: Denney & Krzywicki or Arfken

Taylor Expansion



$$f(z) = \frac{1}{2\pi i}\oint_{C_\epsilon(z_0)} dz' \frac{f(z')}{z'-z} = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i}\oint_C dz' \frac{1}{z'-z_0} \frac{f(z')}{1-\frac{z-z_0}{z'-z_0}}$$

f analytic on R

follows from the geometry of circle ← $\boxed{|z-z_0| < |z'-z_0|}$

$$\left|\frac{z-z_0}{z'-z_0}\right| = \frac{|z-z_0|}{|z'-z_0|} < 1$$

$$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(z'-z_0)^n}$$

(Series is convergent by the ratio test)

$$f(z) = \frac{1}{2\pi i}\oint_C dz' \underbrace{\sum_{n=0}^{\infty} \frac{f(z')}{z'-z_0}\frac{(z-z_0)^n}{(z'-z_0)^n}}_{\text{uniformly convergent}} = \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i}\int dz \frac{f(z')}{(z'-z_0)^{n+1}}$$

- You can show it's uniformly convergent with **M-Test**

$$= \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{n!} \frac{d^n f(z)}{dz^n}\bigg|_{z_0}$$

for $\left|\frac{z-z_0}{z'-z_0}\right| < 1$ $\therefore$ integration and summation can be interchanged.

Let f be not analytic at z_1, in other words z_1 is the singularity of f nearest to z_0 (Taylor expansion is valid until there)

Radius of convergence $= |z_1 - z_0|$