Was late to class... See chapter 5.2 26.09.24

(vi) Kummer's Test

Let $\sum_{k=0}^{\infty} a_k$ be a series of positive terms $(a_k > 0)$

Let $\rho = \lim_{k\to\infty} \left[a_k \left| \frac{u_k}{u_{k+1}} \right| - a_{k+1} \right]$

if $\rho > 0$ abs. convergent

if $\rho < 0$ and Σ is div. $\rightarrow \Sigma u_k$ div.

if ...

(vii) Raabe test)

(viii) Gauss' Test

⋮

Special Case: Assume:

$$\left| \frac{u_k}{u_{k+1}} \right| = \frac{k^2 + a_1 k + b_1}{k^2 + a_2 k + b_2} \quad (k^2 + a_2 k + b_2 \neq 0)$$

$$\begin{array}{l|l} k^2 + a_1 k + a_0 & k^2 + b_1 k + b_0 \\ \hline - \; k^2 + b_1 k + b_0 & 1 + \frac{a_1 - b_1}{k} \\ \hline (a_1 - b_1) k + (a_0 - b_0) & \\ - \; (a_1 - b_1) k + b(a_1 - b_1) + b_0 \left(\frac{a_1 - b_1}{k} \right) & \\ \hline (a_0 - b_0) - b_1 (& \end{array}$$

$$k^2+a_1k+a_0 = (k^2+b_1k+b_0)\left[1+\frac{a_1-b_1}{k}\right]+a_0$$
$$-b_0-b_1(a_1-b_1)-\frac{b_0(a_1-b_1)}{k}$$

$$\frac{k^2 a_1 k+a_0}{k^2+b_1k+b_0} = 1+\frac{a_1-b_1}{k}+\frac{k^2\left(a_0-b_0-b_1(a_1-b_1)-\frac{b_0(a_1-b_1)}{k}\right)}{k^2(k^2+b_1k+b_0)}$$

$$B(k) = \qquad (\cdots)$$

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connected to spherical symmetry

spherical symmetric dist.

Example: Legendre Series

Arises from a power series solution of a DE. (Section 8.5)

represents

- electrostatic potential
- wave funct.

$$\sum_{j=0}^{\infty} a_{2j}x^{2j} = \sum_{k=0}^{\infty} u_k(x) \qquad u_k(x)=a_{2k}x^{2k}$$

$$x\in[-1,1]$$
$$l>0$$

$$\frac{a_{2j+2}}{a_{2j}} = \frac{2j(2j+1)-l(l+1)}{(2j+1)(2j+2)}.$$

$$\frac{u_{k+1}(x)}{u_k(x)} = \frac{a_{2k+2}x^{2k+2}}{a_{2k}x^{2k}} = \frac{2k(2k+1)-l(l+1)}{(2k+1)(2k+2)}x^2$$

Suppose $\ell = 2k_0 \quad k_0 \in \mathbb{Z}_{\geq 0}$

$$\frac{a_{2k_0+2}}{a_{2k_0}} = \frac{2k_0(2k_0+1) - 2k_0(2k_0+1)}{(2k_0+1)(2k_0+2)} = 0$$

$a_{2k_0+2} = 0$

$a_{2k_0+4} = 0$

If ℓ is an even integer, we have a polynomial, "Legendre Polynomials"

Suppose ℓ is not even.

Apply ratio test to see convergence

$$\lim_{k \to \infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| = x^2 \rightarrow$$ the series is convergent if $x^2 < 1 \quad x \in (-1, 1)$

Test is inconclusive at the end points.

Get this to the standard form...

$x=1$ $\qquad \dfrac{(2k+1)(2k+2)}{2k\,(2k+1)-\ell(\ell+1)} = \dfrac{4k^2+6k+2}{4k^2+2k-\ell(\ell+1)}$

$$= \frac{k^2+\frac{3}{2}k+\frac{1}{2}}{k^2+\frac{1}{2}k-\frac{\ell(\ell+1)}{4}}$$

$h \leq 1$ divergence

$h > 1$ convergence

Remember

$$\left|\frac{u_k}{u_{k+1}}\right| = \frac{k^2+a_1k+a_0}{k^2+b_1k+b_0}$$

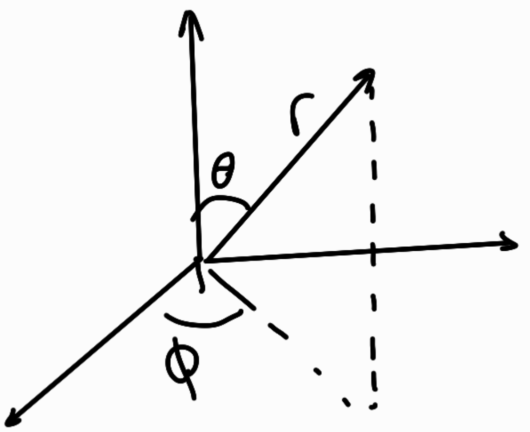
$(k^2+a_2k+b_2 \neq 0)$

$a_1 = 3/2, \quad b_1 = 1/2$

$h = a_1 - b_1 = 1$

Physical / Geometrical meaning of the X:

(not coordinate)



$x = \cos\theta$

$x = \pm 1$

$\theta = 0, \pi$

- You don't want a wave function to be inf. in the z axis.

What's the way out of this z axis?

Remember what happens when $\underbrace{\ell \text{ is even.}}$ → polynomial.

(Quantization of angular momentum)

Many quantization solutions arise this way.

Sequences of Functions

$\{f_n(x)\}_{n=0}^{\infty}$ $f_n(x)$ is a real valued function on (a,b) $\forall n \in \mathbb{Z}_{\geq 0}$

Pick $x \in (a,b)$ then we have the numerical sequence

$$f_0(x),\ f_1(x),\ f_2(x),\ \dots$$

$\lim_{n\to\infty} f_n(x)$ suppose the seq. is convergent

for $\forall x \in (a,b)$

$$\lim_{n\to\infty} f_n(x) = f(x)$$ (Pointwise limit)

Ex:

$$f_n(x) = \frac{n|x|}{1+n|x|}$$ $f_n(0)=0$

$$\lim_{n\to\infty} f_n(x) = \begin{Bmatrix} 0 & x=0 \\ 1 & x\neq 0 \end{Bmatrix} = f(x)$$ discontinuous at $x=0$

Uniform Convergence:

Suppose $\lim_{n\to\infty} f_n(x) = f(x)$

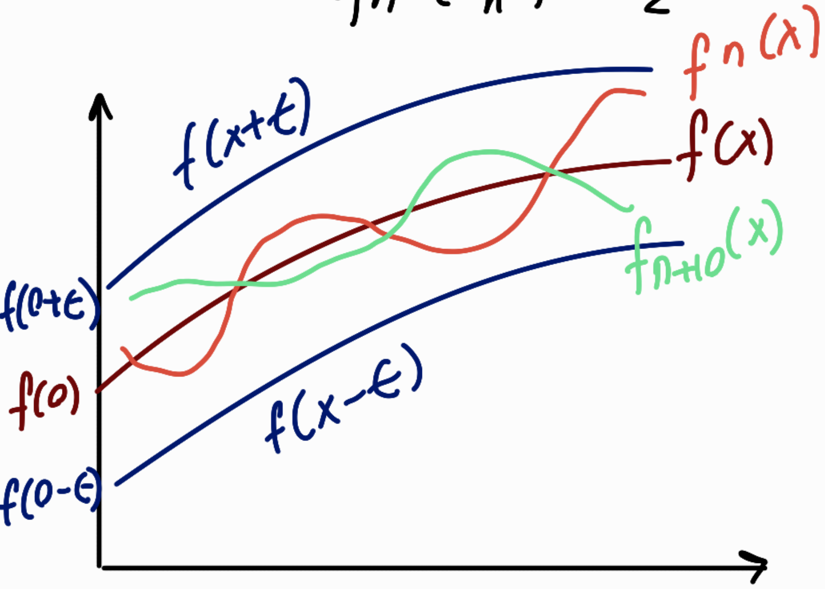
$$\forall \epsilon>0\ \underbrace{\exists N\in\mathbb{N}}_{N=N(\epsilon,x)}\ \text{s.t. } |f_n(x)-f(x)|<\epsilon \quad \forall n>N$$

Because for different x's,
I have different sequences.

If $N(\epsilon, x)$ is independent of x then convergence is called uniform.

Return to example:

$$f_n\left(\frac{1}{n}\right) = \frac{1}{2}$$



Uniform convergence

$$\lim_{n\to\infty} f_n(x) = f(x)$$

(uniform)