Phys 311 Lecture 8 16.10.2024

Black Body Radiation

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$$

(Wave Equation)

$f = g(x \pm ct)$ will be a solution.

- How do you solve such an equation?

Important: Is there a conserved quantity for this equation?

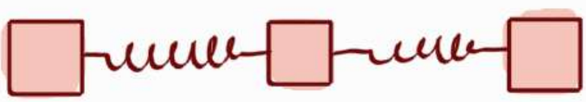
→ Generally, a conservation eqn. is in the form:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0}$$

→ I have an idea of a quadratic object in f so that we have a conserved quantity like this:

$$\left.\begin{aligned} \rho &\equiv \frac{1}{2}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial f}{\partial x}\right)^2 \\ \frac{\partial \rho}{\partial t} &= \frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t^2} + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial t} \end{aligned}\right\} \text{mechanical energy density}$$

$$\left.\begin{aligned} j &\equiv \frac{\partial f}{\partial x}\frac{\partial f}{\partial t} \\ \frac{\partial j}{\partial x} &= -\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial t} \end{aligned}\right\} \text{current density}$$



$\sin(kx - \omega t)$ is a solution.

$$-\omega^2 \sin(kx-\omega t) - (-k^2)\sin(kx-\omega t) = 0$$

iff $\boxed{\omega^2 = k^2}$ $\longrightarrow$ Can easily create oscillatory motion.

(Put a charge in a circular motion)

(sin functions are complete in the sense ... Fourier etc.)

$$\rho = \frac{1}{2}\omega^2 \cos^2(kx-\omega t) + \frac{1}{2}k^2\cos^2(kx-\omega t)$$

$$\boxed{\rho = A^2\omega^2\cos^2(kx-\omega t)}$$

@ fixed time:

direction

$$j = -k(-\omega)A^2\cos^2(kx-\omega t) = \text{sign}(k)\rho$$

$$\boxed{j = \text{sign}(k)\rho}$$

Intensity at $x=0$ $\quad \frac{1}{T}\int_0^T dt\, A^2\omega^2\cos^2(kx_0 - \omega t)$

$\hookrightarrow$ period $T = \frac{1}{f}$ with $\omega = 2\pi f$

$\lambda = cT = \frac{c}{f} \sim 3m$

Standing Waves

$$f(x=0,t) = f(x=L,t) = 0$$

$$A\sin(kx-\omega t) + B\sin(kx-\omega t) = f$$

at $x=0$ $\quad A\sin(-\omega t) + B\sin(-\omega t) = 0 \quad B=-A$

at $x=L$ $\quad A\sin(Lk-\omega t) + B\sin(kL-\omega t) = 0$

expand: $\sin(kL)\cos(\omega t) - \cos(kL)\sin(\omega t)$
$\sin(kL)\cos(\omega t) + \cos(kL)\sin(\omega t)$

$$\boxed{2A\cos(\omega t)\sin(kl)=0}$$

$$\boxed{\begin{array}{l}\sin(kL)=0\\ A\sin(kx)\cos(\omega t)=f\end{array}}$$

$\boxed{k=\frac{\pi n}{L}}$ standing wave pattern.

$$\omega = vk \qquad \omega = 2\pi f \qquad f = \frac{v}{2\pi}k = \frac{vn}{2L} = \sqrt{\frac{T}{\mu}}\frac{n}{2L}$$

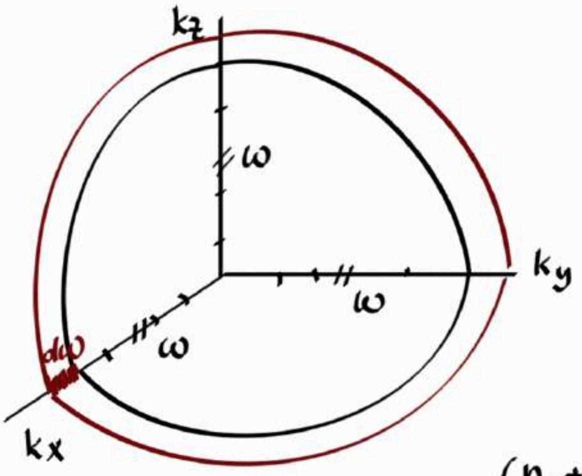
Degeneracies: Configurations with same energies

in 3D :

standing rectangular box

$$\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)$$
$$= c^2\left(\frac{n_x^2\pi^2}{L_x^2} + \frac{n_y^2\pi^2}{L_y^2} + \frac{n_z^2\pi^2}{L_z^2}\right)$$

One has to count the degeneracies



$$\frac{4}{3}\pi(\omega+\Delta\omega)^3 - \frac{4\pi}{3}\omega^3 \approx 4\pi\,\omega^2\,d\omega$$

$$\omega^3 + 3\Delta\omega\,\omega^2 + 3\Delta\omega^2\omega + \Delta\omega^3$$

continuous k_x, k_y, k_z

$$\frac{(n_x+1)^2}{L_x^2} - \frac{n_x^2}{L_x^2} = \frac{2n_x}{L_x^2} \qquad \frac{\Delta\omega^2_{nearest}}{\omega^2}$$

when L's are large

when $\omega \nearrow$ dots $\nearrow$,

it can be considered in continuum.

$$\frac{2n_x}{L_x^2}\cdot\frac{L_x^2}{n_x^2}$$

$$\frac{1}{n_x}$$

Basic Statistical Mechanics

"Partition Function"

- If a system is isolated. All accessible states are equally likely to happen.

- Assume a large system, and a small subsystem is in equilibrium with the large system (Temperature is the same for both)

$$P_r \propto e^{-\beta E_r} \qquad \beta = \frac{1}{kT}$$

(Weinberg's book: Definition of Temperature)

$$\sum_{r=1}^{\infty} P_r = 1 \Rightarrow \boxed{Z = \sum e^{-\beta E_r}} \text{ partition function.}$$

$$P_r = \frac{e^{-\beta E_r}}{Z} \qquad \frac{dZ}{d\beta} = -\sum E_r e^{-\beta E_r} = -Z\sum E_r P_r$$

How does Quantum Mechanichs Enter The Game Here?

Harmonic Oscillator Classical.

$$E = \frac{p^2}{2m} + \frac{1}{2} k x^2 \qquad z = \int dp\, dx\, e^{\frac{-\beta p^2}{2m}}\, e^{\frac{-\beta k x^2}{2}}$$

$$z = \int dp\, e^{-\beta p^2/2m} \int dx\, e^{-\beta k x^2/2}$$

$$z = \int_{-\infty}^{\infty} du\, e^{-\beta u^2} = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} ds\, e^{-s^2} \qquad \frac{dz}{d\beta} = -\frac{1}{2} \frac{1}{\beta^{3/2}} (\qquad)$$

$$s = \sqrt{\beta}\, u \qquad -\frac{1}{z} \frac{dz}{d\beta} = \frac{1}{2\beta} = \frac{kT}{2}$$

$$\bar{E} \equiv \sum_r E_r P_r \Rightarrow \bar{E} = -\frac{1}{z} \frac{dz}{d\beta} = -\frac{d \ln z}{d\beta}$$

Quantum Oscillator

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Quantum mechanical energies are not continuous functions.

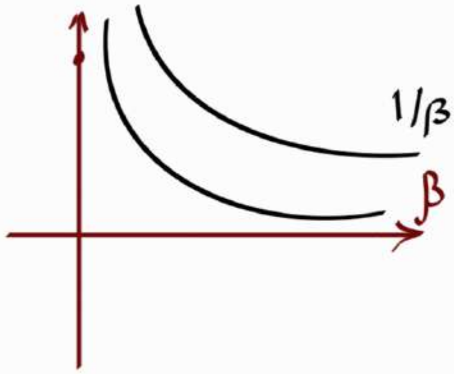
$$E = \hbar \omega (n + 1/2) \qquad n = 0, 1, 2, 3, \ldots$$

$$Z = \sum_{n=0}^{\infty} e^{-\beta \hbar \omega (n+1/2)} = \frac{e^{-\beta \frac{\hbar \omega}{2}}}{1 - e^{-\beta \hbar \omega}}$$

$$\bar{E} = -\frac{dz}{d\beta} \frac{1}{z} = -\left[-\frac{\hbar\omega}{2} \frac{b}{(1-a)} + \frac{b(-\hbar\omega)a}{(1-a)^2} \right]^{a} = \frac{b}{1-a} \left(+\frac{\hbar\omega}{2} + \frac{\hbar\omega a}{1-a} \right)$$

Small β approximation
can be performed

No equipartiton therm.



$$\boxed{\bar{E} - \frac{\hbar\omega}{2} = \frac{\hbar\omega\, e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}}$$