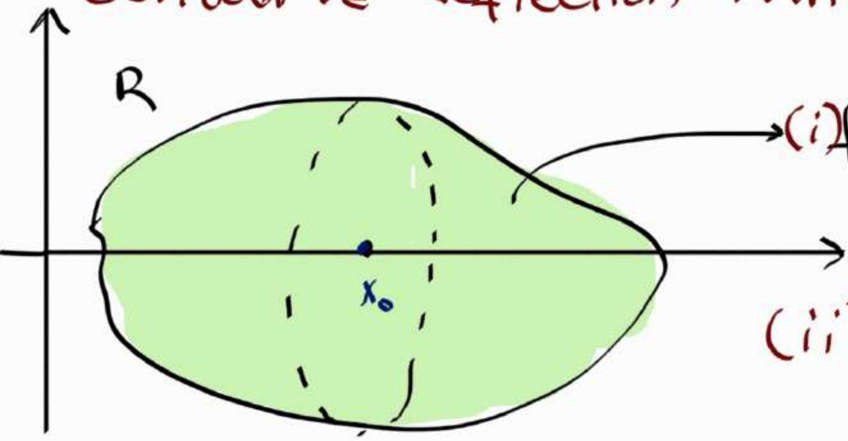
Phys 325 Lecture 7 17.10.2024

An Application of Taylor Exponsion:

Schwortz Reflection Principle



(i) f is analytic in R.

(ii) f is real on the real axis.

<u>EX:</u> e^z (satisfies both)

Pick a point x_0, both in R & real axis.

→ Let's Taylor expand $f(z)$ around x_0.

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{x_0} (z - x_0)^n$$

↓ This derivatives are all real

- $f(z)$ is real on the real axis.

$$f'(x_0) = \lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(x_0 + \Delta z) - f(x_0)}{\Delta z} \in \mathbb{R}$$

the way I approach x_0 doesn't matter. Δz can be real.

- $z = re^{i\theta}$

- $(z^n)^* = (r^n e^{in\theta})^*$

$$= r^n e^{-in\theta} = (re^{-i\theta})^n$$

$$(z^n)^* = (z^*)^n$$

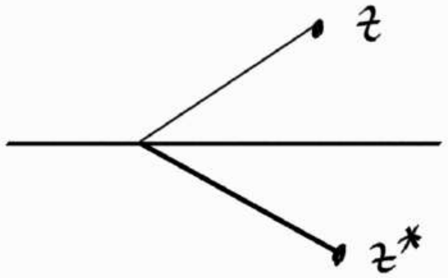
- Let's apply this to $[(z-x_0)^n]^*$:

$$[(z-x_0)^n]^* = [(z-x_0)^*]^n = (z^* - x_0)^n$$

$$\Rightarrow \boxed{[f(z)]^* = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f(z)}{dz^n}\right|_{z=x_0} (z^* - x_0)^n = f(z^*)}$$

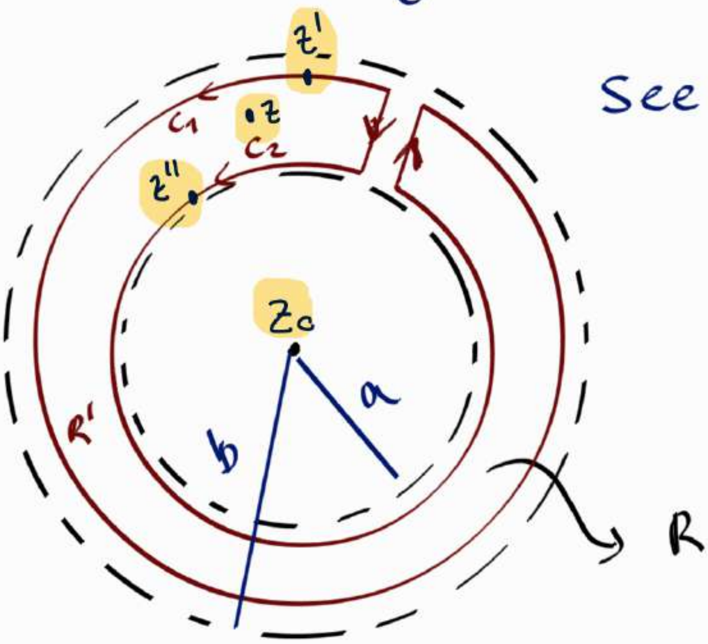
- "z, z^* are reflections of each other along the real axis."

→ "useful in quantum mechanical scattering theory"



Laurent Expansion

We have a function f, analytic in annular region R. (annuler = between 2 circles)



See: annuler region is multiply connected!

$$f(z) = \underbrace{\frac{1}{2\pi i}\oint dz' \frac{f(z')}{z'-z}}_{S_1} - \underbrace{\frac{1}{2\pi i}\oint dz'' \frac{f(z'')}{z''-z}}_{S_2}$$

- $z' - z = z' - z_0 - (z - z_0)$ $\qquad |z' - z_0| > |z - z_0|$

- $z'' - z = z'' - z_0 - (z - z_0)$ $\qquad |z'' - z_0| < |z - z_0|$

Putting in the parenthesis:

$$S_1 = \frac{1}{2\pi i}\oint_{C_1} dz' \frac{f(z')}{(z' - z_0)} \frac{1}{1 - \frac{z - z_0}{z' - z_0}}$$

$$= \frac{1}{2\pi i}\oint dz' \frac{f(z')}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \oint_{C_1} dz' \frac{f(z')}{(z'-z_0)^{n+1}} \right]}_{\frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z_0}} (z - z_0)^n$$

$$f(z) = S_1 - S_2$$

Now focusing on S_2:

$$S_2 = \frac{1}{2\pi i} \oint_{C_2} dz'' \frac{f(z'')}{z'' - z} = -\frac{1}{2\pi i} \oint_{C_2} dz'' \frac{f(z'')}{-(z''-z_0)+(z-z_0)}$$

$$= -\frac{1}{2\pi i} \oint_{C_2} dz'' \frac{f(z'')}{\quad} \qquad (\dots)$$

$$-S_2 = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{C_2} dz'' \, f(z'')(z''-z_0)^n \right] \frac{1}{(z-z_0)^{n+1}}$$

$$S_1 = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \oint_{C_1} dz' \frac{f(z')}{(z'-z_0)^{n+1}} \right]}_{\frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z_0}} (z-z_0)^n$$

$$f(z) = S_1 - S_2$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n \begin{cases} \dfrac{1}{2\pi i} \displaystyle\oint_{C_1} dz' \frac{f(z')}{(z'-z_0)^{n+1}} = \dfrac{1}{n!} \dfrac{d^n f(z)}{dz^n}\Big|_{z_0} & n \geq 0 \\ \dfrac{1}{2\pi i} \displaystyle\oint_{C_2} dz'' \frac{f(z'')}{(z''-z_0)^{n+1}} & n < 0 \end{cases}$$

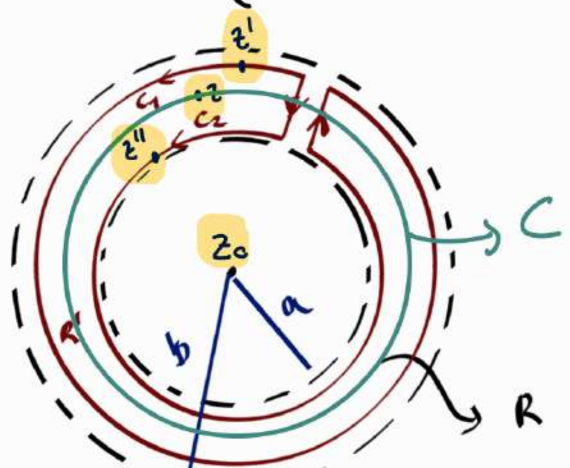
$$-S_2 = \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \oint_{C_2} dz'' \, f(z'') \, (z''-z_0)^{n-1} \right] \frac{1}{(z-z_0)^n} = \sum_{n=-\infty}^{-1} \left[\frac{1}{2\pi i} \oint_{C_2} dz'' \frac{f(z'')}{(z''-z_0)^{n+1}} \right] (z-z_0)^n$$

$$S_1 = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \oint_{C_1} dz' \frac{f(z')}{(z'-z_0)^{n+1}} \right]}_{= \frac{1}{n!} \frac{d^n f(z)}{dz^n}\big|_{z_0}} (z-z_0)^n$$

$$f(z) = S_1 - S_2$$

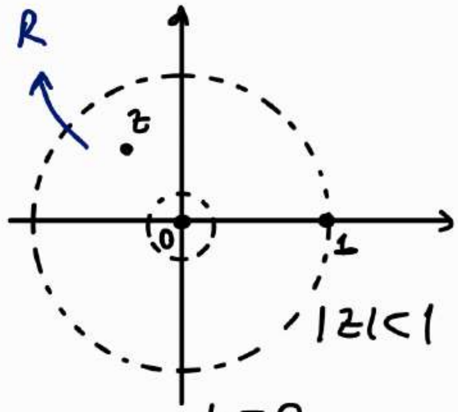
- You can deform C_1 and C_2 to another curve

$$a_n \left\{ \cdots = \frac{1}{2\pi i} \oint_{C} dz' \frac{f(z')}{(z'-z)^{n+1}} \right.$$



Example: $f(z) = \dfrac{1}{z(z-1)}$

singularities: $z=0$, $z=1$
R doesn't include 0 or 1



$b=0$

$a \to 1$

Find the Laurent expansion around $z_0 = 0$

pole of order 1

(This example is also in the book)

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

write in the form:

$$= -\frac{1}{z} - \frac{1}{1-z}$$

See this is the sum of a geometric series:

$$\frac{1}{1-z} = 1 + z + z^2 \dots = \sum_{n=0}^{\infty} z^n$$

$$a_n = \begin{cases} -1 & n \geq -1 \\ 0 & n < -1 \end{cases}$$

$$-\frac{1}{z} - 1 - z - z^2 \dots = -\sum_{n=-1}^{\infty} -z^n$$

coefficients $a_0 = -1$, $a_1 = -1$

$$(*) \quad \boxed{f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n}$$

$a_{-N} \neq 0$

then z_0 is called a pole of order N.

→ pole of order 1 is called a simple pole

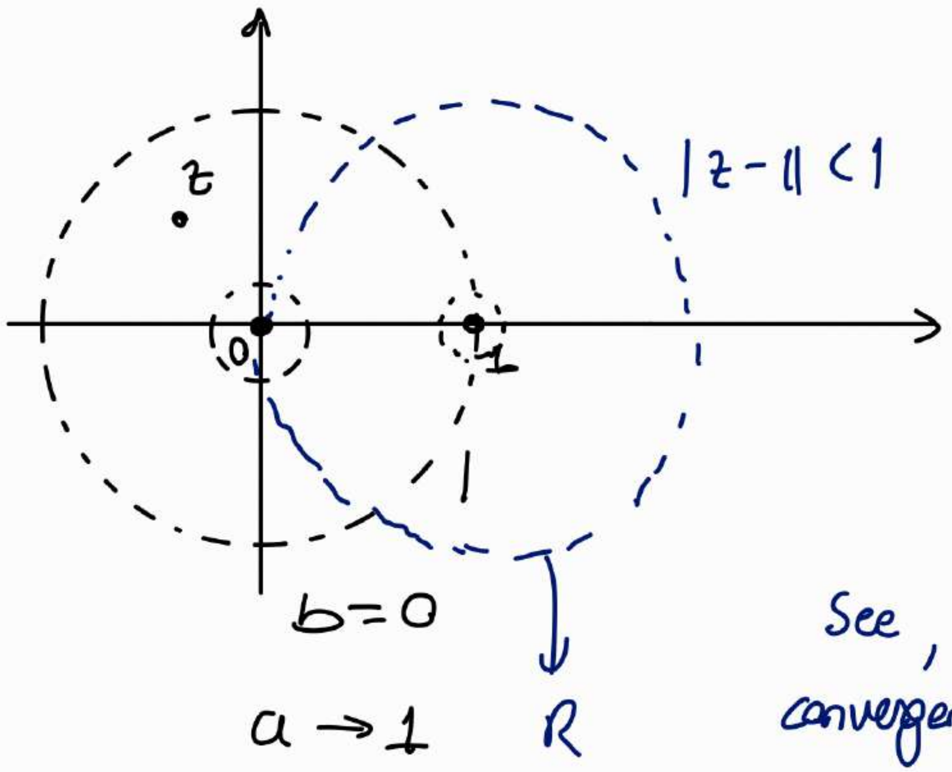
Let's look at the other singularity.

Find the Laurent expansion around

$z_0 = 1 \longrightarrow$ simple pole

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$= -\frac{1}{1+(z-1)} + \frac{1}{z-1}$$



$$\boxed{= \sum_{n=0}^{\infty} (-1)^{n+1} (z-1)^n + \frac{1}{z-1}}$$

$a_{-1} = 1$

$a_n = (-1)^{n+1}$

$n \geq 0$

See, this series is convergent only on R

Behaviour at infinity:

Suppose a $f(z)$ (analytic in some region...)

→ What happens when $z \to \infty$.

→ We do a change of variable:

$$z = \frac{1}{w} \qquad f(z) = f\left(\frac{1}{w}\right) = \tilde{f}(w)$$

$z \to \infty$

$w \to 0$. ⊛ if $\tilde{f}(w)$ is analytic at $w=0$
then $f(z)$ is called analytic at $z=\infty$

⊛ Similarly, if $\tilde{f}(w)$ has a pole at $w=0$,
$f(z)$ has a pole at $z=\infty$

$$f(z) = \frac{1}{z(z-1)} = \frac{w}{\frac{1}{w}-1} = \underbrace{\frac{w^2}{1-w}}_{=\tilde{f}(w)}$$

$|w|<1$ $|z|>1$

$$\tilde{f}(w) = w^2(1 + w + w^2 + \cdots)$$

$$= \sum_{n=2}^{\infty} w^n = \sum_{n=2}^{\infty} \frac{1}{z^n} = \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

Fully analytic. → Taylor. No singularities.

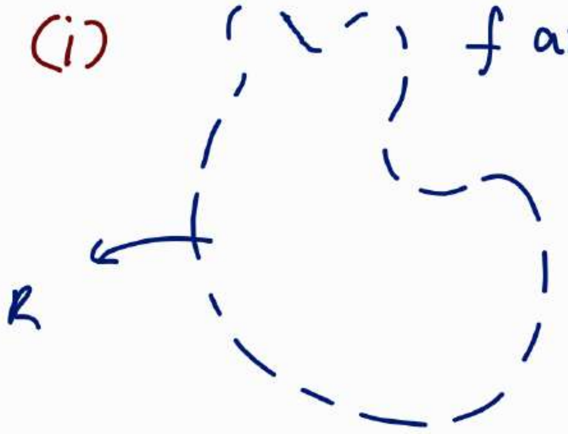
→ A function $f(z)$ which is analitic everywhere on $\mathbb{C}$ (complex plane doesn't include infinity) is called an entire function

E.g.: $f(z) = e^z$

(Proofs on Denery & Kritzwicky)

Some Properties, etc

(i)



f analytic on R

$|f|$ cannot have a local maximum on R

(no analogue on real numbers)

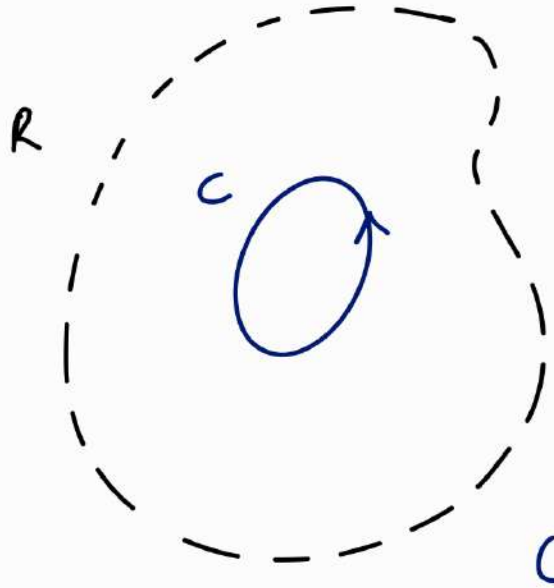
(<u>Proof:</u> pg 41 of DK)

(ii) Cauchy Lioville Theorem

A bounded entire function must be a constant.

(<u>Proof:</u> pg 42 of DK)

(iii) Morera's Theorem



assume f is continuous on R. and for any closed path C.

$$\oint_C dz\, f(z) = 0$$

Then f is analitic on R.

(Proof: page 43 of DK)