1. **Controllable Canonical Form and Eigenvalue Placement**

Consider a linear discrete time system below ($\vec{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $\vec{b} \in \mathbb{R}^n$).

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t)$$

If the system is *controllable*, then there exists a transformation $\vec{z} = T\vec{x}$ (where T is an invertible $n \times n$ matrix) such that in the transformed coordinates, the system is in *controllable canonical form*, which is given by

$$\vec{z}(t+1) = \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}u(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

The characteristic polynomials of the matrices A and $\widetilde{A}$ are the same and given by

$$\det(\lambda I - A) = \det(\lambda I - \widetilde{A}) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_0 \tag{1}$$

(a) Show that $\widetilde{A} = TAT^{-1}$ and $\widetilde{\vec{b}} = T\vec{b}$.

Solution:

Starting from the original system, we have

$$\begin{aligned} \vec{x}(t+1) &= A\vec{x}(t) + \vec{b}u(t) \\ \Rightarrow T\vec{x}(t+1) &= TA\vec{x}(t) + T\vec{b}u(t) \\ \Rightarrow T\vec{x}(t+1) &= TA(T^{-1}T)\vec{x}(t) + T\vec{b}u(t) \\ \Rightarrow \vec{z}(t+1) &= TAT^{-1}\vec{z}(t) + T\vec{b}u(t) \end{aligned}$$

Comparing to the canonical form, we have

$$\widetilde{A} = TAT^{-1}, \text{ and } \widetilde{\vec{b}} = T\vec{b}$$

(b) Show that A and $\widetilde{A}$ have the same eigenvalues (*Hint: let $\vec{v}$ be an eigenvector of A; use $T\vec{v}$ for $\widetilde{A}$*)

Solution:

Let λ and $\vec{v}$ be an eigenvalue and its corresponding eigenvector of A. That is, $A\vec{v} = \lambda\vec{v}$. Next we consider $\widetilde{A}T\vec{v}$:

$$\begin{aligned} \widetilde{A}(T\vec{v}) &= TAT^{-1}T\vec{v} \\ &= TA\vec{v} \\ &= T(\lambda\vec{v}) \\ &= \lambda(T\vec{v}) \end{aligned}$$

We have shown that λ is also an eigenvalue of $\widetilde{A}$ and its corresponding eigenvector is $T\vec{v}$.

(c) Let the controllability matrices C and $\widetilde{C}$ be $C = \begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{n-1}\vec{b} \end{bmatrix}$ and $\widetilde{C} = \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} & \cdots & \widetilde{A}^{n-1}\widetilde{\vec{b}} \end{bmatrix}$, respectively. Show that $\widetilde{C} = TC$, which is equivalent to $T = \widetilde{C}C^{-1}$.

Solution:

$$\begin{aligned}
\widetilde{C} &= \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} & \cdots & \widetilde{A}^{n-1}\widetilde{\vec{b}} \end{bmatrix} \\
&= \begin{bmatrix} T\vec{b} & (TAT^{-1})(T\vec{b}) & \cdots & (TA^{n-1}T^{-1})(T\vec{b}) \end{bmatrix} \\
&= \begin{bmatrix} T\vec{b} & TA\vec{b} & \cdots & TA^{n-1}\vec{b} \end{bmatrix} \\
&= TC
\end{aligned}$$

Now, consider the specific system

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t) = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u(t) \tag{2}$$

(d) Show that the system (2) is controllable.

Solution:

$$C = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & -2 \end{bmatrix} \text{ (full rank)}$$

Since the system is controllable, there exists a transformation $\vec{z} = T\vec{x}$ such that

$$\vec{z}(t+1) = \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}u(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3}$$

(e) Compute the matrix $\widetilde{A}$.

Solution:

$$\det(\lambda I - A) = (\lambda+2)(\lambda+1) = \lambda^2 + 3\lambda + 2 = \lambda^2 + a_1\lambda + a_0$$

By inspection, we have $a_1 = 3$ and $a_0 = 2$. Thus,

$$\widetilde{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

(f) Compute the controllability matrices $C = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix}$ and $\widetilde{C} = \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} \end{bmatrix}$.

Solution:

$$C = \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & -2 \end{bmatrix}$$

$$\widetilde{C} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

(g) Compute the transformation matrix $T = \widetilde{C}C^{-1}$.

Solution:

First, we compute

$$C^{-1} = \frac{1}{\frac{1}{2}(-2) - \frac{1}{2}(-1)} \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$$

Then we have

$$T = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(h) Show that the system (2) is *unstable* in open-loop.

Solution:

From the characteristic polynomial $\lambda^2 + 3\lambda + 2$, we know it is unstable since the eigenvalues are $\lambda = -1$ and -2.

Now, we want to make the system *stable* by applying state feedback for the system in the canonical form (3). That is, let $u(t)$ be $u(t) = -\vec{k}^T\vec{z}(t) = \begin{bmatrix} -k_0 & -k_1 \end{bmatrix} \vec{z}(t)$. After applying state feedback, the systems (2) and (3) have the form

$$\vec{x}(t+1) = A_{cl}\vec{x}(t)$$
$$\vec{z}(t+1) = \widetilde{A}_{cl}\vec{z}(t)$$

(i) Compute A_{cl} and $\widetilde{A}_{cl}$ in terms of k_0 and k_1.

Solution:

For the original system, we have (note that $u(t) = -\vec{k}^T\vec{z}(t) = -\vec{k}^T T\vec{x}(t)$)

$$\begin{aligned} \vec{x}(t+1) &= A\vec{x}(t) + \vec{b}(-\vec{k}^T T\vec{x}(t)) \\ &= (A - \vec{b}\vec{k}^T T)\vec{x}(t) \end{aligned}$$

Thus,

$$A_{cl} = A + \vec{b}(-\vec{k}^T)T = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -k_0 - k_1 & k_0 - k_1 \\ -k_0 - k_1 & k_0 - k_1 \end{bmatrix} = \begin{bmatrix} -2 - \frac{1}{2}k_0 - \frac{1}{2}k_1 & 0 + \frac{1}{2}k_0 - \frac{1}{2}k_1 \\ -3 - \frac{1}{2}k_0 - \frac{1}{2}k_1 & -1 + \frac{1}{2}k_0 - \frac{1}{2}k_1 \end{bmatrix}$$

For the canonical form, we have

$$\begin{aligned} \vec{z}(t+1) &= \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}(-\vec{k}^T\vec{z}(t)) \\ &= (\widetilde{A} - \widetilde{\vec{b}}\vec{k}^T)\vec{z}(t) \end{aligned}$$

Thus,

$$\widetilde{A}_{cl} = \widetilde{A} + \widetilde{\vec{b}}(-\vec{k}^T) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -k_0 & -k_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 - k_0 & -3 - k_1 \end{bmatrix}$$

(j) Compute $\vec{k}$ so that $\widetilde{A}_{cl}$ has eigenvalues $\lambda = \pm\frac{1}{2}$ (*Hint: use Formula (1)*).

Solution:

First, for the given eigenvalues we have

$$(\lambda + \frac{1}{2})(\lambda - \frac{1}{2}) = \lambda^2 - \frac{1}{4}$$

Second, by comparison

$$-a_1 = 0 = -3 - k_1$$
$$-a_0 = -(-\frac{1}{4}) = -2 - k_0$$

We have

$$k_1 = -3$$
$$k_0 = -\frac{9}{4}$$

(k) Using the $\vec{k}$ you derived in the previous part, show that A_{cl} also has eigenvalues $\lambda = \pm\frac{1}{2}$ by explicit calculation.

Solution:

$$A_{cl} = \begin{bmatrix} -2 - \frac{1}{2}k_0 - \frac{1}{2}k_1 & 0 + \frac{1}{2}k_0 - \frac{1}{2}k_1 \\ -3 - \frac{1}{2}k_0 - \frac{1}{2}k_1 & -1 + \frac{1}{2}k_0 - \frac{1}{2}k_1 \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{-3}{8} & \frac{-5}{8} \end{bmatrix}$$

The characteristic polynomial is given by

$$(\frac{5}{8} - \lambda)(\frac{-5}{8} - \lambda) + \frac{9}{64} = \lambda^2 - \frac{16}{64} = \lambda^2 - \frac{1}{4}$$

Thus, $\lambda = \pm\frac{1}{2}$.

2. Controllable Canonical Form

Consider the linear continuous-time system below.

$$\frac{d}{dt}\vec{s}(t) = A_r\vec{s}(t) + \vec{b}_r u(t) = \begin{bmatrix} 7 & -14 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{s}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

(a) There exists a transformation $\vec{z}(t) = T\vec{s}(t)$ such that the resulting system is in controllable canonical form:

$$\frac{d}{dt}\vec{z}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Find this transformation matrix T.

HINT: The column vectors of T are just the standard basis vectors $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ *arranged in some order.*

Solution:

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) For the system given by $(A_r, \vec{b}_r, \vec{s}, u)$, compute a state feedback law $u(t) = \vec{f}^T\vec{s}(t) = \begin{bmatrix} f_0 & f_1 & f_2 \end{bmatrix}\vec{s}(t)$ such that the resulting closed-loop system has eigenvalues $\lambda = -1, -2, -4$.
What is the vector $\vec{f}$?

Solution:

$$\vec{f} = \begin{bmatrix} -14 \\ 0 \\ -16 \end{bmatrix}$$

(c) Consider another system below:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) = \begin{bmatrix} -3 & -11 & 4 \\ 4 & 10 & -4 \\ 1 & 1 & 0 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}u(t)$$

Compute a state feedback law $u(t) = \vec{k}^T\vec{x}(t) = \begin{bmatrix} k_0 & k_1 & k_2 \end{bmatrix}\vec{x}(t)$ such that the resulting closed-loop system has eigenvalues $\lambda = -1, -2, -4$. **What is the vector $\vec{k}$?**
HINT: the matrices A and A_r are related by

$$\underbrace{\begin{bmatrix} 7 & -14 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A_r} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -3 & -11 & 4 \\ 4 & 10 & -4 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$\vec{k}^T = \vec{f}^T T = \begin{bmatrix} -14 & 0 & -16 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 14 & -16 \end{bmatrix}$$

3. **Controllable Canonical Form (20 pts)**

When we are trying to stabilize a robot, it is sometimes useful to put the dynamics into a standard form that lets us more easily adjust its behavior.

Consider the linear continuous-time system below.

$$\frac{d}{dt}\vec{s}(t) = A_r\vec{s}(t) + \vec{b}_r u(t) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \vec{s}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

(a) (10 pts) There exists a transformation $\vec{z}(t) = T\vec{s}(t)$ such that the resulting system is in controllable canonical form:

$$\frac{d}{dt}\vec{z}(t) = \begin{bmatrix} 0 & 1 \\ \alpha_0 & \alpha_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Find this transformation matrix T and the resulting α_0 and α_1 in controllable canonical form.

HINT: The column vectors of T are just the standard basis vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ arranged in some order.

Solution: From the hint and we know T cannot be the identity matrix, so

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

With T, we can derive the canonical form matrix A_c by

$$A_c = TA_rT^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

Thus, $\alpha_0 = -1$ and $\alpha_1 = 2$.

(b) (10 pts) For the system given by (A_r, $\vec{b}_r$, $\vec{s}$, u), **use the controllable canonical form from the previous part** to obtain a state feedback law $u(t) = \vec{f}^T\vec{z}(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \vec{z}(t)$ such that the resulting closed-loop system has eigenvalues $\lambda = -1, -2$ and then use the transformation T to get $u(t) = \vec{g}^T\vec{s}(t) = \begin{bmatrix} g_0 & g_1 \end{bmatrix} \vec{s}(t)$ a control law in terms of the original state variable $\vec{s}(t)$.

What are the vectors $\vec{f}$ and $\vec{g}$?

You will get full credit if you correctly use the properties of controllable canonical form to do this, but you may check your answer by another method if you so desire.

Solution:

First we find a feedback law for $\vec{f}$ and $\vec{z}$. With the feedback, we have

$$\frac{d}{dt}\vec{z}(t) = \left(A_c + \begin{bmatrix} 0 & 0 \\ f_0 & f_1 \end{bmatrix}\right)\vec{z}(t) = \begin{bmatrix} 0 & 1 \\ -1+f_0 & 2+f_1 \end{bmatrix}\vec{z}(t)$$

The characteristic polynomial of the closed-loop system is given by

$$\lambda^2 - (2+f_1)\lambda - (-1+f_0)$$

Our goal is to place eigenvalues at -1 and -2, which is characterized by the polynomial

$$(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$$

By comparison of coefficients, we have

$$\begin{aligned} -2 - f_1 &= 3 \\ 1 - f_0 &= 2 \end{aligned}$$

Thus,

$$\vec{f} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

Then, using the property of the controllable canonical form, we have

$$\vec{g}^T = \vec{f}^T T = \begin{bmatrix} -1 & -5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -1 \end{bmatrix}$$

4. Controllable Canonical Form - Eigenvalues Placement

Consider the following linear discrete time system

$$\vec{x}(t+1) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

(a) Is this system controllable?

Solution: We calculate

$$\mathscr{C} = [B, AB, A^2B] = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that $\mathscr{C}$ matrix is full rank and hence our system is controllable. **Answer:** We calculate

$$\mathscr{C} = [B, AB, A^2B] = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that $\mathscr{C}$ matrix is full rank and hence our system is controllable.

(b) Is the linear discrete time system stable?

Solution: We have to calculate the eigenvalues of matrix A. Thus,

$$0 = \det(A - \lambda I) = \lambda^3 + 4\lambda^2 + 3\lambda = \lambda(\lambda+3)(\lambda+1)$$

Since the eigenvalue at -3 is outside the unit circle, this system is unstable. **Answer:** We have to calculate the eigenvalues of matrix A. Thus,

$$0 = \det(A - \lambda I) = \lambda^3 + 4\lambda^2 + 3\lambda = \lambda(\lambda+3)(\lambda+1)$$

Since the eigenvalue at -3 is outside the unit circle, this system is unstable.

(c) Bring the system to the controllable canonical form

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

using transformation $\vec{z}(t) = T\vec{x}(t)$

Solution: The characteristic polynomial for this system is $\lambda^3 + 4\lambda^2 + 3\lambda$. So putting this system in control canonical form gives

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

To compute the transformation T, we need the matrices

$$R_n = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{R}_n = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

Then T is given by

$$T = \tilde{R}_n R_n^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

. **Answer:** The characteristic polynomial for this system is $\lambda^3 + 4\lambda^2 + 3\lambda$. So putting this system in control canonical form gives

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

To compute the transformation T, we need the matrices

$$R_n = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{R}_n = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

Then T is given by

$$T = \tilde{R}_n R_n^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

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(d) Using state feedback $u(t) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \vec{z}(t)$ place the eigenvalues at $0, 1/2, -1/2$.

Solution: The closed loop system in z coordinates is given by

$$\widetilde{A} + \widetilde{B}\widetilde{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 - 3 & k_3 - 4 \end{bmatrix}$$

which has characteristic polynomial $\lambda^3 + (4 - k_3)\lambda^2 + (3 - k_2)\lambda - k_1$. To place the eigenvalues at 0, 1/2, -1/2, the desired characteristic polynomial is $\lambda(\lambda - 1/2)(\lambda + 1/2) = \lambda^3 - 1/4\ \lambda$. So we should choose $k_1 = 0, k_2 = 13/4, k_3 = 4$. If we want the feedback controller in terms of $x(t)$, we write $u(t) = K\vec{x}(t)$ where

$$K = \widetilde{K}T = \begin{bmatrix} 0 & 13/4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -35/4 & 19/4 \end{bmatrix}.$$

Answer: The closed loop system in z coordinates is given by

$$\widetilde{A} + \widetilde{B}\widetilde{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 - 3 & k_3 - 4 \end{bmatrix}$$

which has characteristic polynomial $\lambda^3 + (4 - k_3)\lambda^2 + (3 - k_2)\lambda - k_1$. To place the eigenvalues at 0, 1/2, -1/2, the desired characteristic polynomial is $\lambda(\lambda - 1/2)(\lambda + 1/2) = \lambda^3 - 1/4\ \lambda$. So we should choose $k_1 = 0, k_2 = 13/4, k_3 = 4$. If we want the feedback controller in terms of $x(t)$, we write $u(t) = K\vec{x}(t)$ where

$$K = \widetilde{K}T = \begin{bmatrix} 0 & 13/4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -35/4 & 19/4 \end{bmatrix}.$$

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