Stability and Controllability

1. Basic Practice

(a) Consider the single variable system:

$$\frac{d}{dt}x(t) = 5x(t) + u(t)$$

Is this system stable?

Solution: The real part of the eigenvalue for the continuous time system is not negative. SO ths sytem is not stable.

(b) Design an input $u(t)$ to stabilize the system.

Solution: To make the real part of the eigenvalue smaller than zero, we can choose $u(t) = -6x(t) as an example.$

2. Continuous Time System

(a) Consider the following continuous time system:

$$\frac{d^2}{dt^2}x(t) = -x(t)$$

We convert this system to state space form with $x_1(t) = x(t)$ and $x_2(t) = \frac{d}{dt}x(t)$. Our representation becomes:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

What is the correct values in A?

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(b) Which best describes the behavior of the above system?

Solution:

Unstable since the eigenvalues are i and $-i$. Noise can cause oscillations that never die out

(c) We want to change the behavior of the system using a feedback control model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

We set $u(t) = K\vec{x}(t)$, where $K = [k_1\ k_2]$. What values of k_1 and k_2 will keep the oscillatory behavior of the system but stabilize it? Does $k_1 = -1$ and $k_2 = -2$ work? What about $k_1 = 3$ and $k_2 = 2$?

Solution:

$k_1 = -1$ and $k_2 = -2$ will keep the oscillatory behavior and make the system stable, while $k_1 = 3$ and $k_2 = 2$ wouldn't. We can see this by computing the eigenvalues of the system after adding control.

(d) What values of k_1 and k_2 will remove the oscillatory behavior completely and still stabilize the system?

Solution:

As an example, using feedback values $k_1 = -1$ and $k_2 = -3$ gives system eigenvalues of -1 and -2, thus the system is not oscilllatory since they are both real and remains stable because they are negative.

3. Eigenvalue Placement

(a) Consider the following 2 variable system in controller canonical form:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

What is the characteristic polynomial of A?

Solution: $\lambda^2 - 4\lambda + 5$

(b) Is this system stable?

Solution: No, it isn't stable, because one of the eigenvalues have positive real components.

(c) We wish the put the system in feedback, i.e set $u(t) = K\vec{x}(t)$, then our system becomes:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}K\vec{x}(t)$$

Where $K = [k_1\ k_2]$. What should k_1 and k_2 be if we want our system to have eigenvalues $\{-1, -2\}$?

Solution: $k_1 = 3$ and $k_2 = -7$

4. Discrete Time Feedback

(a) Consider the scalar system: $x(i+1) = 1.5x(i) + u(i)$. Given the controller $u(i) = kx(i)$, for what value of k can we have the system to behave like:$x(i+1) = \lambda x(i)$ where $\lambda = 0.7$?

Solution: To make the system's eigenvalue be 0.7, we can choose $k = -0.8$.

(b) Given the system $\vec{x}(i+1) = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\vec{x}(i) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u_1(i) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_2(i)$. Let $u_1(i) = k_1x_1(i)$ and $u_2(i) = k_2x_2(i)$. What value of k_1 and k_2 would make the system stable with eigenvalues $\lambda_1 = \lambda_2 = \frac{1}{2}$?

Solution: We can choose $k_1 = -2.5$ and $k_2 = -4.5$.

(c) Given the matrix $\begin{bmatrix} 2 & 1 \\ -3+2k_1 & 4+2k_2 \end{bmatrix}$, what should k_1 and k_2 be for the matrix to have eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{-1}{3}$?

Solution: Coefficient match to find $k_1 = \frac{-1}{4}$ and $k_2 = \frac{-35}{12}$

(d) Given the matrix $\begin{bmatrix} 2+k_1 & 7+k_2 \\ 3 & -1 \end{bmatrix}$, what should k_1 and k_2 be for the matrix to have eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$?

Solution: Coefficient match to find $k_1 = 6$ and $k_2 = -13$

(e) Given the system $\vec{x}(t+1) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$. Is the system stable?

Solution: For a discrete system to be stable, $|\lambda_i| < 1$ Characteristic polynomial:

$$(2-\lambda)^2 + 1 = 0$$
$$\lambda^2 - 4\lambda + 5 = 0$$
$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm j$$
$$|\lambda_1| = |\lambda_2| = \sqrt{5} > 1$$

The magnitude of the eigenvalues are greater than 1, so the system is unstable.

(f) Given the feedback controller $u(t) = K\vec{x}(t)$ for the previous system, where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. What should k_1 and k_2 be for the system to reach $\vec{x}(t) = 0$ from any states in 2 time steps?

Solution:

The system can be written as:

$$\vec{x}(t+1) = (A+BK)\vec{x}(t)$$

For this system to converge in 2 steps, we need:

$$\lambda_1 = \lambda_2 = 0$$

Where λ_1 and λ_2 are the eigenvalues of $(A+BK)$

$$A+BK = \begin{bmatrix} 2 & -1 \\ 1+k_1 & 2+k_2 \end{bmatrix}$$

Characteristic polynomial:

$$(2-\lambda)(2+k_2-\lambda) + 1 + k_1 = \lambda^2 + \lambda(-4-k_2) + 5 + k_1 + 2k_2$$

Coefficient match to:

$$(\lambda+0)(\lambda+0) = \lambda^2$$
$$-4-k_2 = 0 \Rightarrow k_2 = -4$$
$$5 + k_1 + 2k_2 = 5 + k_1 - 8 = 0 \Rightarrow k_1 = 3$$

(g) Given the system $\vec{x}(t+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$. Is the system controllable? What should k_1 and k_2 be to put the eigenvalues of the system at $\lambda = -1 \pm j$

Solution: The controllability matrix

$$\mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has full rank. So the system is controllable.
A state feedback controller has the form

$$u(t) = K\vec{x}(t) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

With this control the closed loop system is

$$\vec{x}(t+1) = (A+BK)\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \vec{x}(t).$$

The characteristic equation of the closed loop system is

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ k_1 & k_2 - \lambda \end{bmatrix} = \lambda^2 - k_2\lambda - k_1.$$

The desired closed loop characteristic equation is

$$0 = (\lambda - (-1+j))(\lambda - (-1-j)) = \lambda^2 + 2\lambda + 2$$

So we should choose $k_1 = k_2 = -2$.

(h) Given the system $\vec{z}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 11 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$. Design state feedback so that the system has eigenvalue $0, 1/2, -1/2$.

Solution: The closed loop system in z coordinates is given by

$$\widetilde{A} + \widetilde{B}\widetilde{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2+5 & k_3+11 \end{bmatrix}$$

which has characteristic polynomial $\lambda^3 + (-11-k_3)\lambda^2 + (-5-k_2)\lambda - k_1$. To place the eigenvalues at 0, 1/2, -1/2, the desired characteristic polynomial is $\lambda(\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = \lambda^3 - \frac{1}{4}\lambda$. So we should choose $k_1 = 0, k_2 = \frac{-19}{4}, k_3 = -11$.

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