CSM
Fall 2019

Electrical Engineering 16B
CSM test

1. First-Order Differential Equation Practice

(a) Solve the equation $\frac{d}{dt}f(t) = 7f(t)$ given that $f(0) = 3$

Solutions: This is a basic first-order differential eqution practice. We know the solution of the equation $\frac{d}{dt}f(t) = 7f(t)$ is in the form $f(t) = Ae^{7t}$, as we can check by plugging it into the differential equation. We can find the value of A by using $f(0) = 3$. The final solution is $f(t) = 3e^{7t}$.

(b) Solve the equation $\frac{d}{dt}f(t) + 5f(t) = 0$ given that $f(0) = 2$

Solutions: $f(t) = 2e^{-5t}$.

(c) Solve the equation $3\frac{d}{dt}f(t) + 6f(t) = 0$ given that $f(0) = 7$

Solutions: $f(t) = 7e^{-2t}$.

(d) Solve the equation $\frac{d}{dt}f(t) = 2f(t) + 2$ given that $f(0) = 3$

Solutions: To get started on this problem, we can write the equation as $\frac{d}{dt}f(t) + 1 = 2(f(t) + 1)$. Then we solve for $f(t) + 1$ like the previous problems and obtain $f(t) + 1 = 4e^{2t}$. The final solution is $f(t) = 4e^{2t} - 1$.

(e) Solve the equation $5\frac{d}{dt}f(t) + 3f(t) - 10 = 0$ given that $f(0) = 8$

Solutions: $f(t) = \frac{14}{3}e^{\frac{-3}{5}t} + \frac{10}{3}$.

(f) Solve the equation $6\frac{d}{dt}f(t) + 4 = 3f(t)$ given that $f(0) = \frac{1}{3}$

Solutions: $f(t) = -e^{\frac{1}{2}t} + \frac{4}{3}$.

(g) Solve the equation $3f(t) - 6f'(t) = 4$ given that $f(2) = \frac{1}{3}$

Solutions: $f(t) = -e^{\frac{1}{2}t-1} + \frac{4}{3}$

2. RC Circuit

Consider the circuit below, assume that when $t \leq 0$, the capacitor has no charge stored ($V_c(t = 0) = 0$). At $t = 0$, the switch closes. Assume that $V_s = 5\,\text{V}$, $R = 100\,\Omega$, and $C = 10\,\mu\text{F}$.

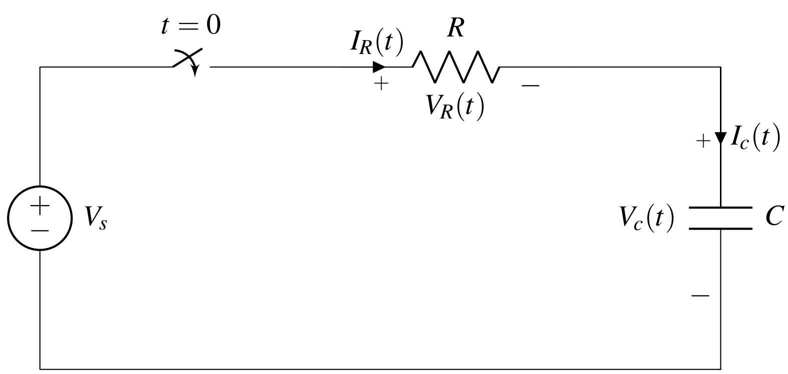

Figure 1: RC Circuit with Voltage Source

(a) Write out the KCL equations associated with the circuit when the switch is open. Then write out the differential equations for the case when the switch is closed.

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Solutions:

When the swtich is open, we denote the voltage to the left of resistor to be y. We obtain the following differential equations. KCL at node between the resistor and capacitor gives

$$\frac{y - V_C}{R} = C\frac{dV_C}{dt}$$

Since the current going into the resistor and the voltage source is zero, KCL gives:

$$\frac{y - V_C}{R} = 0$$

and

$$i_{V_S} = 0$$

When the switch is closed, the following still holds:

$$\frac{y - V_C}{R} = C\frac{dV_C}{dt}$$

Additionally, KCL at the node between the resistor and voltage sources gives:

$$\frac{V_S - V_C}{R} + i_{V_S} = 0$$

(b) Write out the differential equation for $V_c(t)$ after the switch closes.

Solutions:

From the previous problem we know that when the switch is closed,

$$\frac{V_S - V_C}{R} = C\frac{dV_C}{dt}$$

Thus we obtain

$$C\frac{dV_C}{dt} + \frac{V_C}{R} - \frac{V_S}{R} = 0$$

(c) What is the initial condition for $V_c(t)$ (i.e. $V_c(t = 0)$) and what is $V_c(t \to \infty)$?

Solutions:

No charge is on the capacitor before time $t = 0$. Using $q = VC$, we know that $V_c = 0\,\text{V}$ before $t = 0$.

At $t = 0$, the switch closes. Since voltage across the capacitor cannot change instantaneously,

$$V_c(t = 0) = 0.$$

As t goes to infinity, the capacitor will become fully charged and the current goes to zero. Therefore, the voltage of the capacitor equals the voltage source:

$$V_c(t \to \infty) = V_S.$$

(d) Using the initial conditions found in the previous parts, find an expression for $V_c(t)$ in terms of V_s, R, and C.

Solutions:

The general solution to the equation

$$\frac{dy}{dt} = \lambda y$$

is

$$y(t) = Ke^{\lambda t},$$

where K is a constant and λ is the eigenvalue of the equation. In our case, we know

$$C\frac{dV_C}{dt} + \frac{V_C}{R} - \frac{V_S}{R} = 0,$$

which can be written as

$$\frac{dV_C}{dt} = -\frac{V_C}{RC} + \frac{V_S}{RC}.$$

The solution will be in the form

$$V_c(t) = Ke^{-\frac{t}{RC}} + V_S.$$

To find K, we can plug in our initial condition at $t = 0$:

$$V_c(t = 0) = K + V_S = 0.$$

So our overall equation ends up being

$$V_c(t) = -V_S e^{-\frac{t}{RC}} + V_S.$$

We can also double check our answer by looking at the state for t $\to \infty$:

$$V_c(t \to \infty) = -V_S e^{-\infty} + V_S = V_S,$$

which agrees with our answer from previous part.

(e) On what order of magnitude of time (nanoseconds, milliseconds, 10's of seconds, etc.) does this circuit settle (V_c is $> 95\%$ of its value as $t \to \infty$)?

Solutions:

The time constant τ of an RC circuit is just $\tau = RC$. For our circuit:

$$\tau = RC = 100\,\Omega \cdot 10\,\mu\text{F} = 0.001\,\text{s}$$

After 3 time constants, the voltage will be 95% of its steady state value

$$3\tau = 0.003\,\text{s}$$

The circuit will settle on the order of milliseconds.

(f) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

Solutions:

To reduce settling time, reduce τ. We can achieve this by

i. Lowering the value of R or

ii. Lowering the value of C.

Notice how the value of V_s does not change the settling time.

3. Fun with Inductors

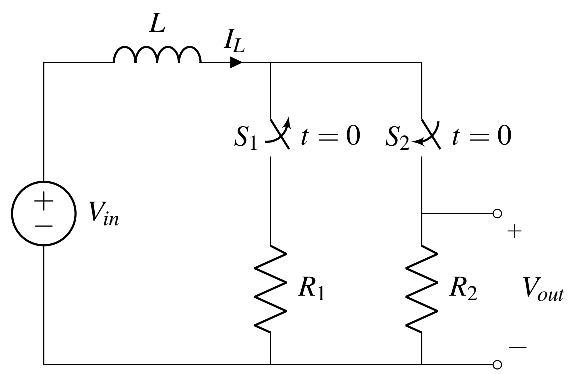

Figure 2: Circuit A

(a) Consider circuit A. Assuming that for $t < 0$, switch S_1 is on and switch S_2 is off (and both switches have been in these states indefinitely), what is $i_L(0)$?

Solutions: When S_1 is on and S_2 is off for a long period of time, $\frac{di_L}{dt} = 0$ because the circuit will have reached a steady state, and the current through R_1 will be equal to i_L. We find

$$V_{in} - V_L - V_{R_1} = 0$$

$$V_{in} - L\frac{di_L}{dt}(0) - i_L(0)R_1 = 0$$

$$V_{in} - i_L(0)R_1 = 0$$

$$i_L(0) = \frac{V_{in}}{R_1}$$

(b) Now let's assume that for $t \geq 0$, S_1 is off and S_2 is on. Solve for $V_{out}(t)$ for $t \geq 0$.

Solutions:

$$V_{in} - V_L - V_{out} = 0$$

$$V_{in} - L\frac{di_L}{dt} - i_L R_2 = 0$$

$$\frac{di_L}{dt} + \frac{R_2}{L}i_L = \frac{V_{in}}{L}$$

This is a non-homogenous first order differential equation in i_L. We can solve for $i_L(t)$ and then use Ohm's law to find $V_{out}(t)$ after this has been solved.

$$\frac{di_L}{dt} + \frac{R_2}{L}(i_L - \frac{V_{in}}{R_2}) = 0$$

Let $\tilde{i_L} = i_L - \frac{V_{in}}{R_2}$. We now have:

$$\frac{d\tilde{i_L}}{dt} + \frac{R_2}{L}\tilde{i}_L = 0$$

The general solution is given by:

$$\tilde{i}_L(t) = c_1 e^{-\frac{R_2}{L}t}$$

Resubstituting back i_L, we have:

$$i_L(t) = \frac{V_{in}}{R_2} + c_1 e^{-\frac{R_2}{L}t}$$

Applying initial conditions, we know:

$$i_L(0) = \frac{V_{in}}{R_2} + c_1 = \frac{V_{in}}{R_1}$$

$$c_1 = \frac{V_{in}}{R_1} - \frac{V_{in}}{R_2}$$

Our solution for $i_L(t)$ thus becomes:

$$i_L(t) = \frac{V_{in}}{R_2} + (\frac{V_{in}}{R_1} - \frac{V_{in}}{R_2})e^{-\frac{R_2}{L}t}$$

Since $V_{out}(t) = i_L(t)R_2$,

$$V_{out}(t) = V_{in}(1 + (\frac{R_2}{R_1} - 1)e^{-\frac{R_2}{L}t})$$

(c) If $V_{in} = 1V$, $L = 1nH$, $R_1 = 1k\Omega$, and $R_2 = 10k\Omega$, what is the maximum value of $V_{out}(t)$ for $t \geq 0$?

Solutions: Since the coefficient in front of our time-varying component $e^{-\frac{R_2}{L}t}$, given by $\frac{R_2}{R_1} - 1 = 9$, is positive, $V_{out}(t)$ undergoes decay over time. Therefore, the maximum value is achieved at $t = 0$:

$$\max V_{out}(t) = V_{out}(0) = \frac{R_2}{R_1}V_{in} = 10V$$

(d) In general, if we want $\max V_{out}(t)$ to be greater than V_{in}, what relationship needs to be maintained between the values of R_1 and R_2?

Solutions: As long as the coefficient on our exponential term, given by $\frac{R_2}{R_1} - 1$, is greater than 0 (i.e. when $\frac{R_2}{R_1} > 1$) then the maximum value of $V_{out}(t)$ will be achieved at $t = 0$ and will have a value of $\frac{R_2}{R_1}V_{in} > V_{in}$. Otherwise, if $\frac{R_2}{R_1} \leq 1$, the maximum value of $V_{out}(t)$ is reached at $t = \infty$, where $V_{out} = V_{in}$ regardless of R_2 and R_1. Therefore, our necessary condition for the maximum of V_{out} to be greater than V_{in} is:

$$R_2 > R_1$$

(e) Now assume that at time $t = t_1$, switch S_2 was turned off, and switch S_1 was turned back on. Solve for $i_L(t)$ for $t > t_1$. If $R_2 > R_1$, how does this $i_L(t)$ for $t > t_1$ compare with the initial condition $i_L(0)$ you found in part (a)?

Solutions: Our new initial condition for $t > t_1$ is given by plugging in $t = t_1$ into the equation for $i_L(t)$ we found in part (b). Thus, $i_L(t_1) = \frac{V_{in}}{R_2} + (\frac{V_{in}}{R_1} - \frac{V_{in}}{R_2})e^{-\frac{R_2}{L}t_1}$.
We can write the relationship between the current through the inductor and the current through R_1:

$$i_L = i_{R_1}$$

$$i_L = \frac{V_{R_1}}{R_1}$$

$$i_L = \frac{V_{in} - V_L}{R_1}$$

$$i_L = \frac{V_{in}}{R_1} - \frac{L\frac{di_L}{dt}}{R_1}$$

$$\frac{di_L}{dt} + \frac{R_1}{L}i_L = \frac{V_{in}}{L}$$

This is a first order non-homogeneous differential equation similar to that found in part (b), except with R_1 in place of R_2. Following those steps in part (b), we find the general solution:

$$i_L(t) = \frac{V_{in}}{R_1} + c_1 e^{-\frac{R_1}{L}t}$$

To find c_1 we apply our initial condition:

$$i_L(t_1) = \frac{V_{in}}{R_1} + c_1 e^{-\frac{R_1}{L}t_1} = \frac{V_{in}}{R_2} + (\frac{V_{in}}{R_1} - \frac{V_{in}}{R_2})e^{-\frac{R_2}{L}t_1}$$

$$c_1 e^{-\frac{R_1}{L}t_1} = (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(1 - e^{-\frac{R_2}{L}t_1})$$

$$c_1 = \frac{(\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(1 - e^{-\frac{R_2}{L}t_1})}{e^{-\frac{R_1}{L}t_1}}$$

$$c_1 = (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1 - R_2}{L}t_1})$$

Thus, we have

$$i_L(t) = \frac{V_{in}}{R_1} + (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1 - R_2}{L}t_1})e^{-\frac{R_1}{L}t}$$

for $t > t_1$. We also see that as $t \to \infty$, $i_L(t)$ for $t > t_1$ becomes:

$$i_L(t = \infty) = \frac{V_{in}}{R_1} + (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1 - R_2}{L}t_1})e^{-\frac{R_1}{L}\infty}$$

$$i_L(t = \infty) = \frac{V_{in}}{R_1} + (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1 - R_2}{L}t_1})(0)$$

$$i_L(t = \infty) = \frac{V_{in}}{R_1} = i_L(0)$$

Thus, if we turn S_2 back off and S_1 back on as was described in this part, we will eventually revert back to the initial state from which we started! Specifically, if $R_2 > R_1$, $i_L(t)$ at $t = t_1$ will be less than our initial condition $i_L(0)$, and $i_L(t)$ will rise to $i_L(0)$ over time.

4. **Shrinking Transistor Size [OPTIONAL]**

Moore's Law is a 1965 observation by Fairchild Semiconductor Research and Development Lab's director, Gordon Moore, that the number of transistors on an integrated circuit chip doubles every 1.5-2 years. This observation has dominated the computer industry into modern day, where we now have sophisticated processes to create transistors that have a smallest feature size that is 10 nm across.

(a) Given that silicon atoms in a lattice are separated by ≈ 0.543nm, how many silicon atoms thick is a 10nm feature?

Solutions: We have a total length of 10 nm that is composed of atoms that are separated by 0.543 nm. The number of atoms $n = \frac{total\ length}{distance\ per\ atom} = \frac{10\ nm}{0.543\ nm} = 18.4162$. A 10 nm feature is approximately 18 atoms thick.

(b) Here is a transmission electron microscope (TEM) image of a FINFET transistor from Intel, where you can see the silicon atoms in an ordered lattice.

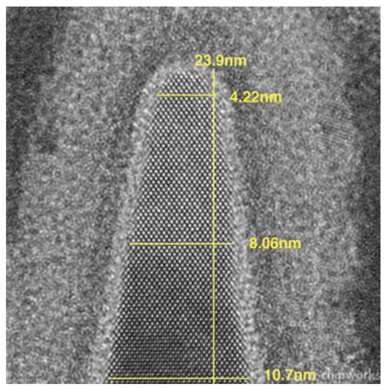

Figure 3: TEM Image of Intel FINFET Transistor

How many atoms across is the feature where it is 4.22nm wide? **Solutions:** There are 12 atoms (white dots) across the feature.

5. If the most recent 10 nm technology was released in 2017, by applying Moore's Law, when would we expect features that are 1 silicon atom wide? Assume that the feature size is reduced by half every time the number of transistors doubles.

Solutions: We can write out the equation for the number of transistors on a chip in year y_n relative to previous year y_0 as

$$N_{y_n} = N_{y_0} * 2^{(y_n - y_0)/t_{double}}, \tag{1}$$

where t_{double} is the number of years it takes for the number of transistors to double.

If the feature width W is halved every time the number of transistors doubles, then we can write the equation for the width in future year y_n as

$$W_{y_n} = \frac{W_{y_0}}{2^{(y_n - y_0)/t_{double}}}. \tag{2}$$

Plugging in for $W_{y_0} = 10nm$, $y_0 = 2017$, and $W_{y_n} = 0.543nm$, and rearranging, we get:

$$2^{(y_n - y_0)/t_{double}} = \frac{W_{y_0}}{W_{y_n}} \tag{3}$$

$$2^{(y_n - 2017)/t_{double}} = \frac{10nm}{0.543nm} = 18.4162 \tag{4}$$

$$\frac{y_n - 2017}{t_{double}} \log(2) = \log(18.4162) \tag{5}$$

$$\frac{y_n - 2017}{t_{double}} = \log(18.4162)/\log(2) = 4.2029 \tag{6}$$

$$y_n = 4.2029 * t_{double} + 2017 \tag{7}$$

If $t_{double} = 1.5$, $y_n = 2023$ and if $t_{double} = 2$, $y_n = 2025$. It will take between 6-8 years to reach a feature that is only 1 silicon atom wide if we halve the transistor feature width every 1.5-2 years.

6. Do you think Moore's Law can continue forever? Why or why not?

Solutions: No, Moore's Law cannot continue forever. Among other reasons, we cannot create a reliable process that consistently creates features that are 1 atom wide. Also, our use of the materials assumes that the features have some thickness, and that assumption breaks down when there are such few atoms across a device.

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