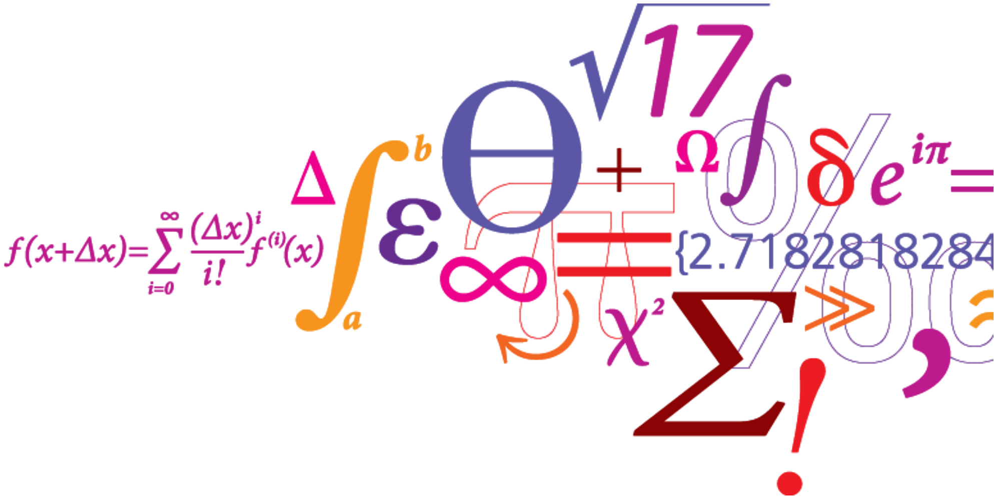
Large-Scale Optimization Problem in Energy Systems: Applications of Decomposition Techniques

Lecture 9: Applications of Benders Decomposition to Stochastic Programming



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Learning Objectives

- At the end of this session the students should be able to:

- Recognize the decomposable structure of a stochastic optimization problem (number of complicating variables and subproblems)

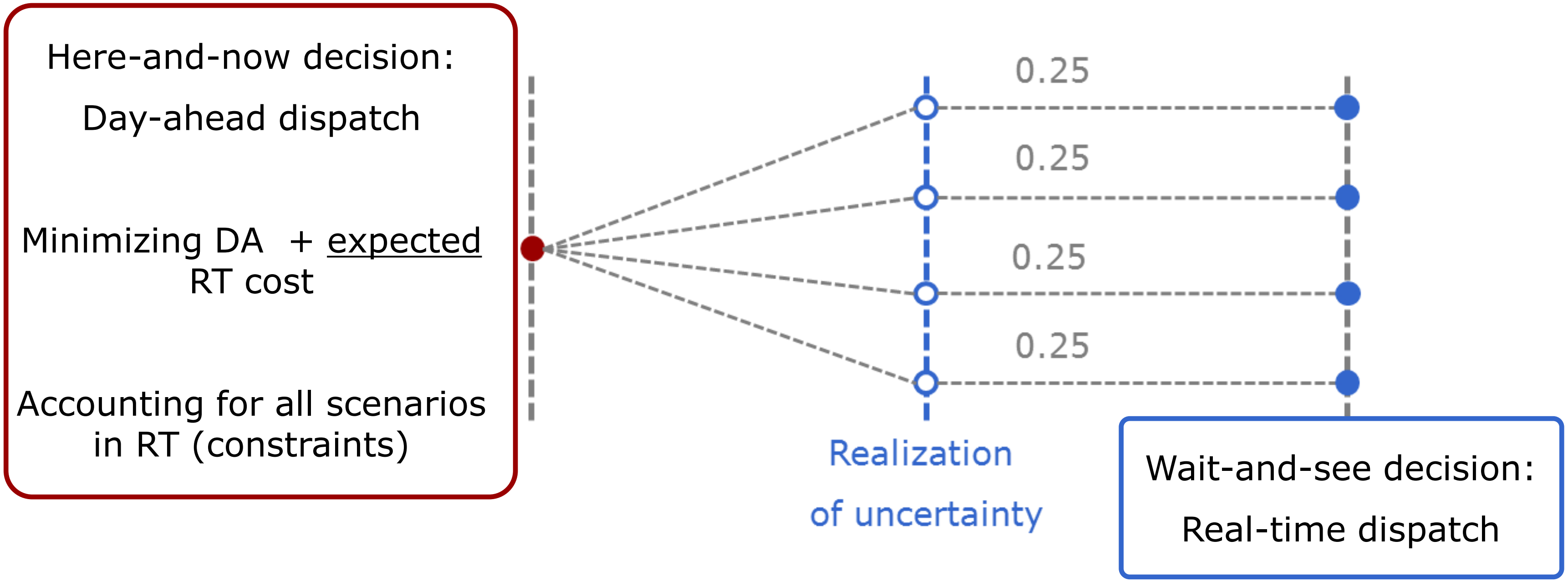
- Write a Benders decomposition algorithm for a general stochastic optimization problem

- Write and solve a Benders decomposition algorithm for a simple example of stochastic market clearing

Stochastic Programming – A Reminder

Example: Stochastic Market Clearing

- considers demand, wind power, etc. as a potential "scenario" in the balancing market
- determines day-ahead dispatch according to all scenarios



Here-and-now decision:

Day-ahead dispatch

Minimizing DA + expected RT cost

Accounting for all scenarios in RT (constraints)

Wait-and-see decision:

Real-time dispatch

Stochastic Programming – A Reminder

Problem: As the number of scenarios increases the complexity of stochastic problems increases!

- We can use decomposition techniques to regain tractability

Stochastic Programming – A Reminder

Example: Stochastic linear optimization problem

$$\min_{x, y_{s_1}, \ldots, y_{s_N}} \quad Ax + \pi_{s_1} A_{s_1} y_{s_1} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}$$

$$\begin{array}{lllll}
\text{s.t.} & B^0 x & & & = d^0 \\
 & C^0 x & & & \leq e^0 \\
 & B^{s_1} x & + F^{\overset{s}{1}} y_{s_1} & & = d^{s_1} \\
 & C^{s_1} x & + G^{\overset{s}{1}} y_{s_1} & & \leq e^{s_1} \\
 & & \ldots & & \\
 & & \ldots & & \\
 & B^{s_N} x & & + F^{\overset{s}{N}} y_{s_N} & = d^{s_N} \\
 & C^{s_N} x & & + G^{\overset{s}{N}} y_{s_N} & \leq e^{s_N}
\end{array}$$

Stochastic Programming – A Reminder

Example: Stochastic linear optimization problem

$$\min_{x, y_{s_1}, \ldots, y_{s_N}} \quad Ax + \pi_{s_1} A_{s_1} y_{s_1} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}$$

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& B^{s_1} x + F^{\overset{s}{1}} y_{s_1} && = d^{s_1} \\
& C^{s_1} x + G^{\overset{s}{1}} y_{s_1} && \leq e^{s_1} \\
& \qquad\qquad \ldots && \\
& \qquad\qquad \ldots && \\
& B^{s_N} x \qquad\qquad + F^{\overset{s}{N}} y_{s_N} && = d^{s_N} \\
& C^{s_N} x \qquad\qquad + G^{\overset{s}{N}} y_{s_N} && \leq e^{s_N}
\end{aligned}$$

x "here-and-now" variables

Stochastic Programming – A Reminder

Example: Stochastic linear optimization problem

$$\min_{x, y_{s_1}, \ldots, y_{s_N}} \quad Ax + \pi_{s_1} A_{s_1} y_{s_1} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}$$

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& C^{s_N} x \qquad + G^{\overset{s}{N}} y_{s_N} && \leq e^{s_N}
\end{aligned}$$

x "here-and-now" variables

y "wait-and-see" variables for scenario $s_1, \ldots, s_N$

Stochastic Programming – A Reminder

Example: Stochastic linear optimization problem

$$\min_{x, y_{s_1}, \ldots, y_{s_N}} \quad Ax + \boxed{\pi_{s_1} A_{s_1} y_{s_1} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}}$$

Expected cost
(weighted sum)

$$\begin{aligned}
\text{s.t. } & B^0 x && = d^0 \\
& C^0 x && \leq e^0 \\
& B^{s_1} x + F^{\overset{s}{1}} y_{s_1} && = d^{s_1} \\
& C^{s_1} x + G^{\overset{s}{1}} y_{s_1} && \leq e^{s_1} \\
& \qquad \ldots \\
& \qquad \ldots \\
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& C^{s_N} x \qquad + G^{\overset{s}{N}} y_{s_N} && \leq e^{s_N}
\end{aligned}$$

x ”here-and-now” variables

y ”wait-and-see”
variables for scenario
$s_1, \ldots, s_N$

Stochastic Programming – A Reminder

Example: Stochastic linear optimization problem

$$\min_{x, y_{s_1}, \dots, y_{s_N}} \quad Ax + \boxed{\pi_{s_1} A_{s_1} y_{s_1} + \dots + \pi_{s_N} A_{s_N} y_{s_N}}$$

Expected cost (weighted sum)

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\text{s.t. } & B^0 x && = d^0 \\
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& \qquad \dots && \\
& \qquad \dots && \\
& B^{s_N} x \qquad + F^{\overset{s}{N}} y_{s_N} && = d^{s_N} \\
& C^{s_N} x \qquad + G^{\overset{s}{N}} y_{s_N} && \leq e^{s_N}
\end{aligned}$$

x "here-and-now" variables

y "wait-and-see" variables for scenario $s_1, \dots, s_N$

Question:

Is this Problem decomposable?

Stochastic Programming – A Reminder

Example: Stochastic linear optimization problem

$$\min_{x, y_{s_1}, \ldots, y_{s_N}} \quad Ax + \boxed{\pi_{s_1} A_{s_1} y_{s_1} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}}$$

Expected cost
(weighted sum)

$$\begin{array}{llllll}
\text{s.t.} & B^0 x & & & = d^0 \\
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& C^{s_N} x & & +G^{\overset{s}{N}} y_{s_N} & \leq e^{s_N}
\end{array}$$

x "here-and-now" variables

y "wait-and-see"
variables for scenario
$s_1, \ldots, s_N$

x ("here-and-now" variables) is the set of complicating variables

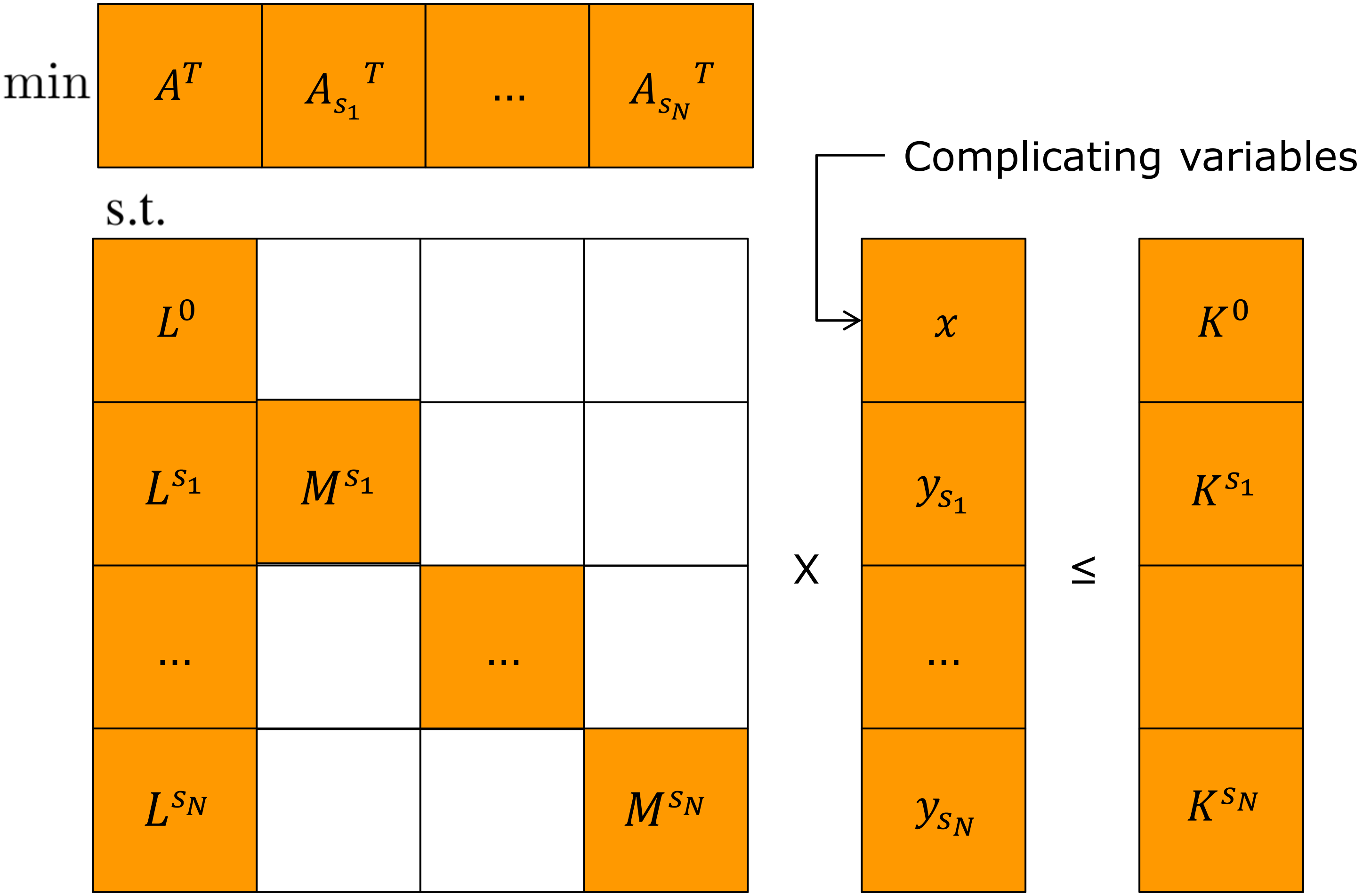
Stochastic Programming – Benders Decomposition

$$\min \begin{bmatrix} A^T & {A_{s_1}}^T & \dots & {A_{s_N}}^T \end{bmatrix}$$

s.t.

$$\begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \text{X} \begin{bmatrix} x \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \leq \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

Stochastic Programming – Benders Decomposition

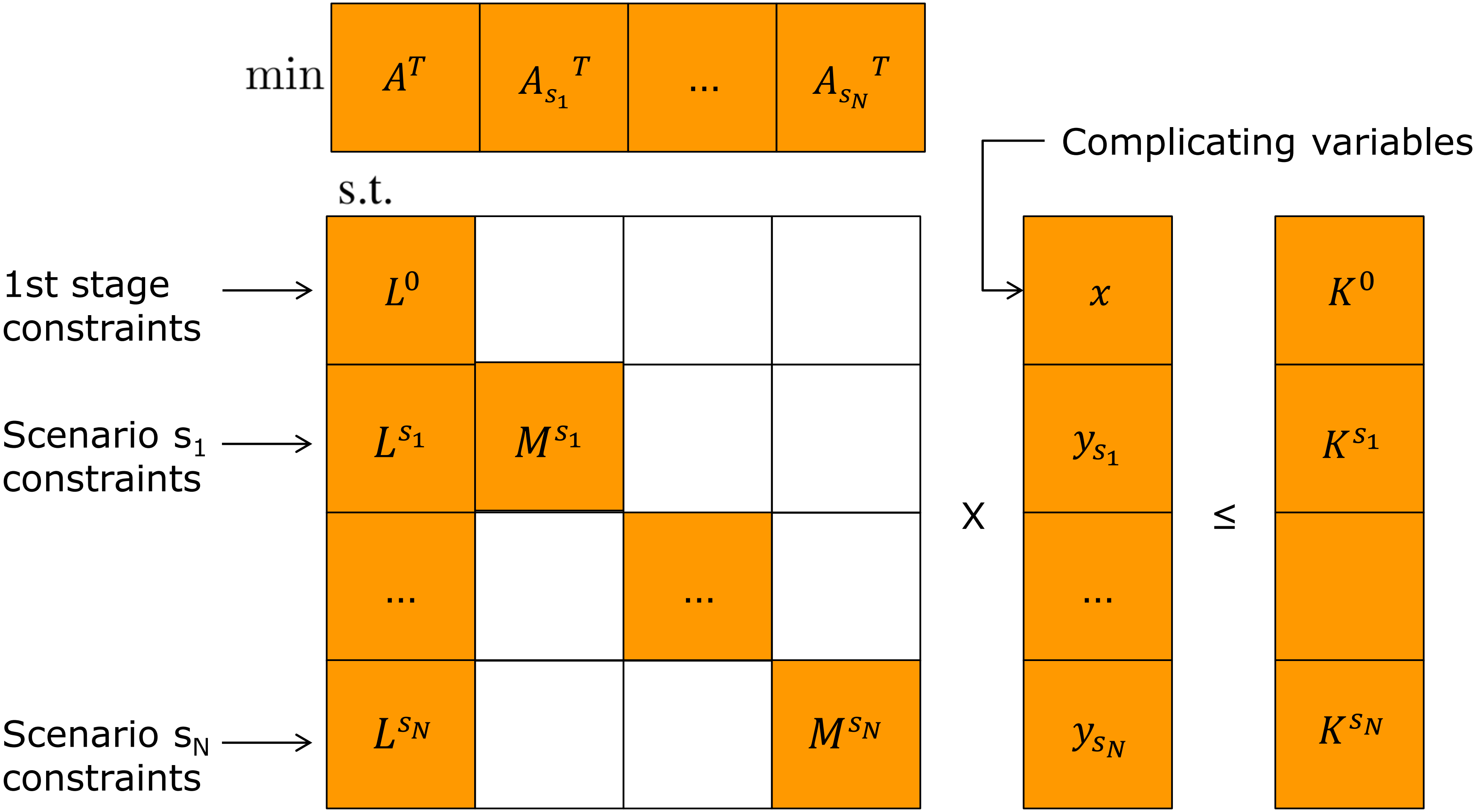


$$\min \begin{bmatrix} A^T & {A_{s_1}}^T & \dots & {A_{s_N}}^T \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \mathrm{X} \begin{bmatrix} x \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \leq \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

Complicating variables

Stochastic Programming – Benders Decomposition



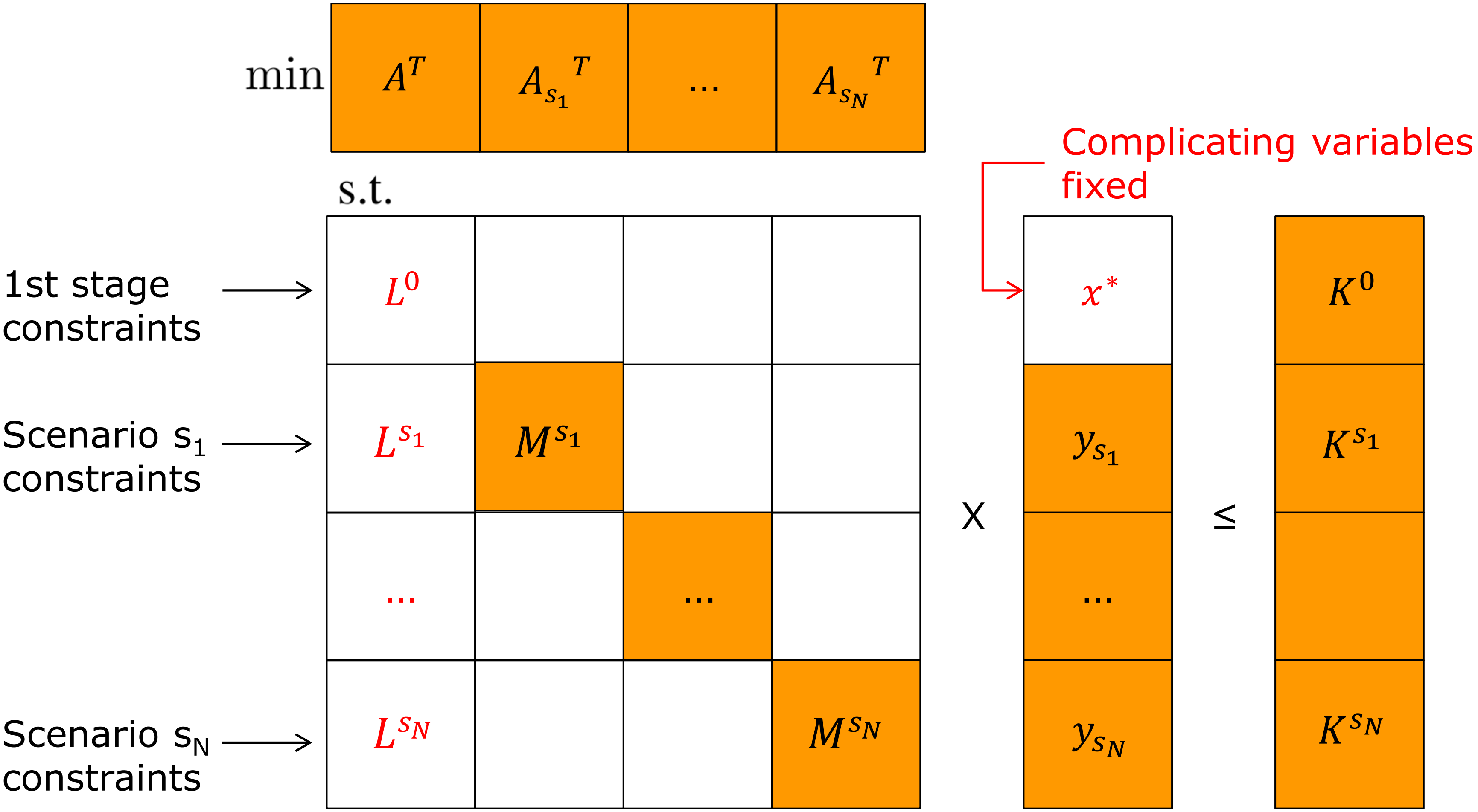
$$\min \begin{bmatrix} A^T & {A_{s_1}}^T & \dots & {A_{s_N}}^T \end{bmatrix}$$

s.t.

$$\begin{array}{l} \\ \text{1st stage constraints} \rightarrow \\ \text{Scenario } s_1 \text{ constraints} \rightarrow \\ \\ \text{Scenario } s_N \text{ constraints} \rightarrow \end{array} \begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \times \begin{bmatrix} x \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \le \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

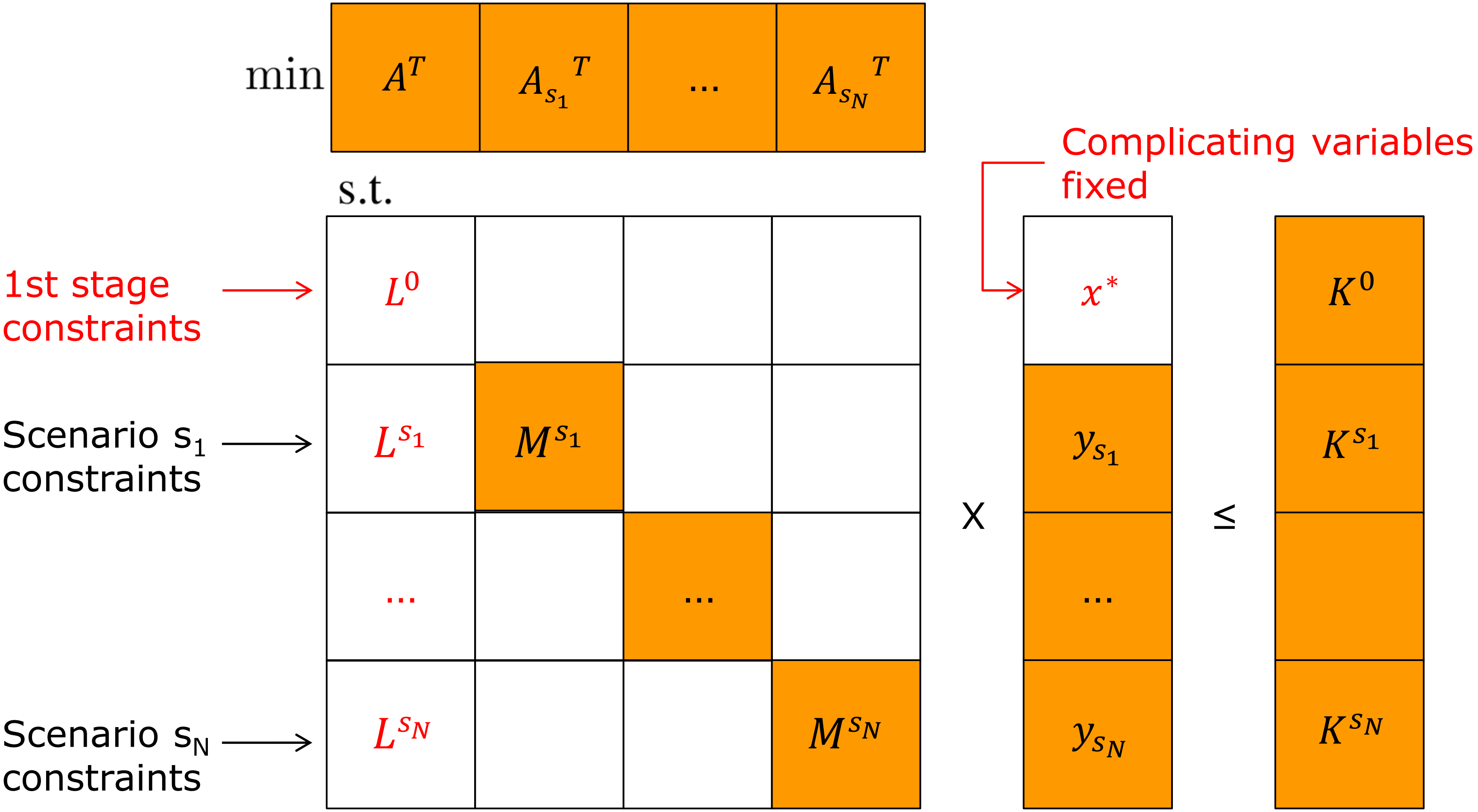
Complicating variables

Stochastic Programming – Benders Decomposition



Question: How many subproblems (at least) are there?

Stochastic Programming – Benders Decomposition



$$\min \begin{bmatrix} A^T & A_{s_1}{}^T & \dots & A_{s_N}{}^T \end{bmatrix}$$

s.t.

$$\begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \text{X} \begin{bmatrix} x^* \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \le \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

1st stage constraints

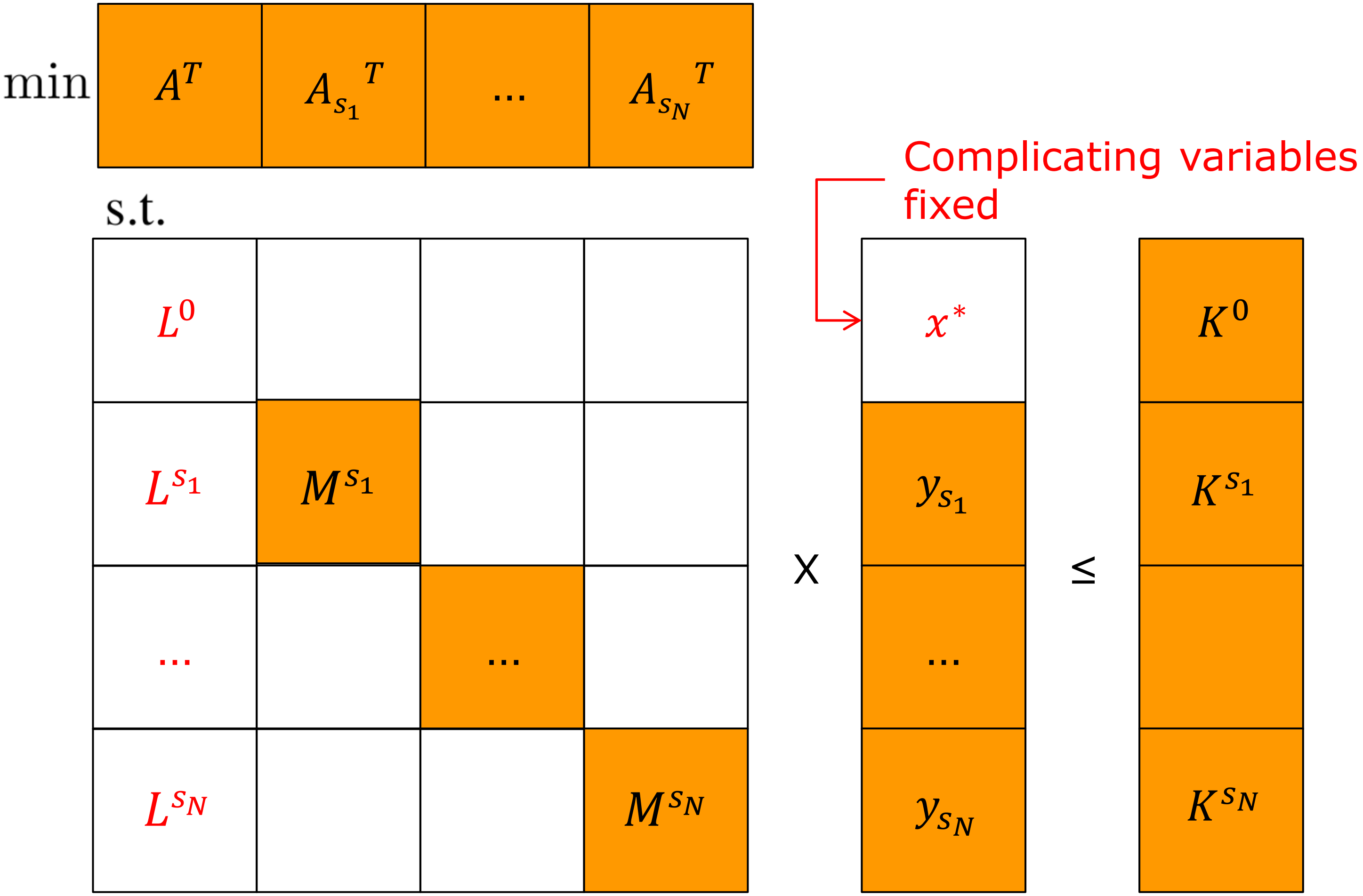
Scenario s_1 constraints

Scenario s_N constraints

Complicating variables fixed

Question: How many subproblems (at least) are there?

Stochastic Programming – Benders Decomposition



$$\min \begin{bmatrix} A^T & A_{s_1}{}^T & \dots & A_{s_N}{}^T \end{bmatrix}$$

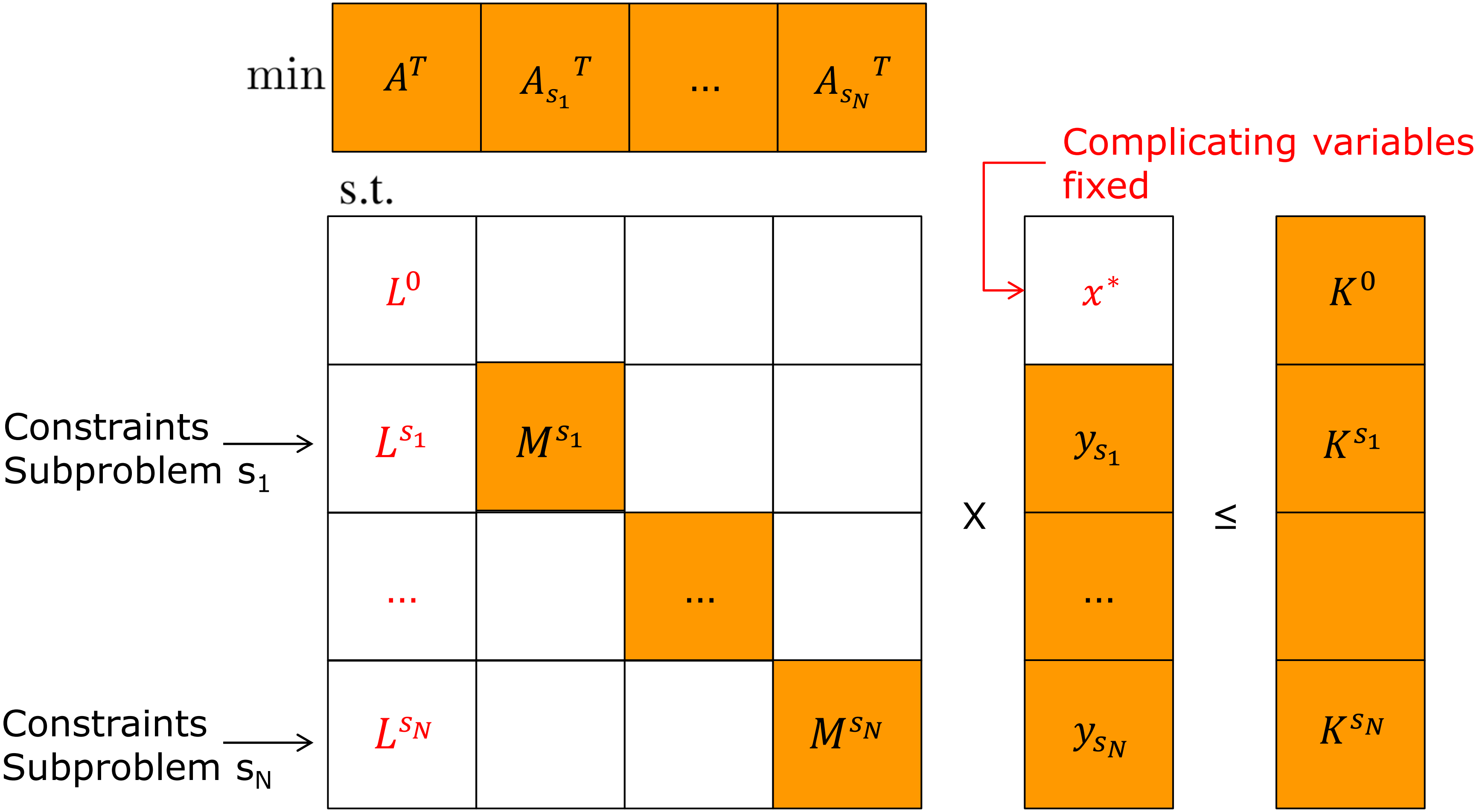
s.t.

$$\begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \times \begin{bmatrix} x^* \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \le \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

Complicating variables fixed

- One subproblem per scenario (at least)!

Stochastic Programming – Benders Decomposition



$$\min \begin{bmatrix} A^T & {A_{s_1}}^T & \dots & {A_{s_N}}^T \end{bmatrix}$$

s.t.

$$\begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \times \begin{bmatrix} x^* \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \leq \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

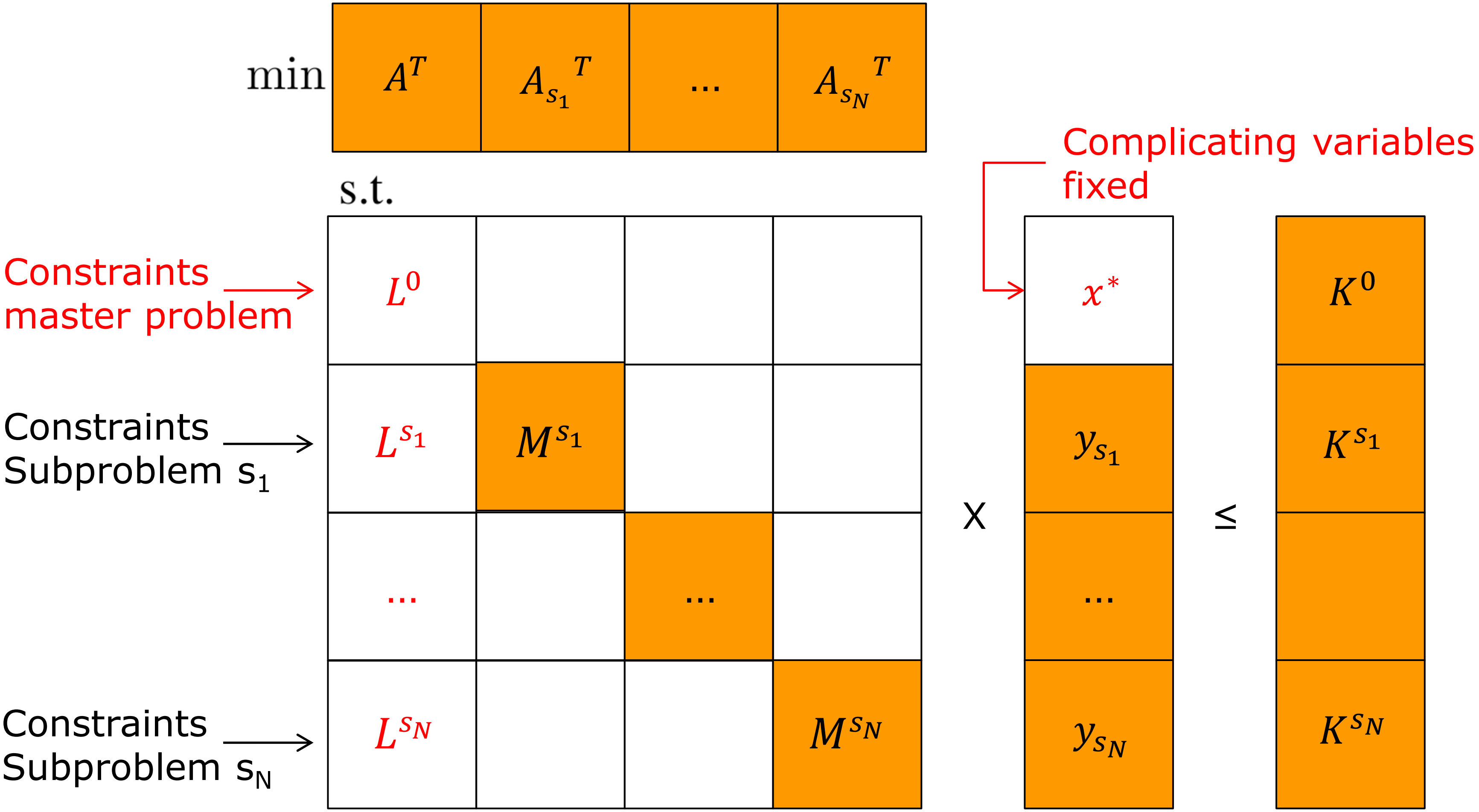
Complicating variables fixed

Constraints Subproblem s_1

Constraints Subproblem s_N

- One subproblem per scenario (at least)!

Stochastic Programming – Benders Decomposition



$$\min \begin{bmatrix} A^T & {A_{s_1}}^T & \dots & {A_{s_N}}^T \end{bmatrix}$$

s.t.

$$\begin{bmatrix} L^0 & & & \\ L^{s_1} & M^{s_1} & & \\ \dots & & \dots & \\ L^{s_N} & & & M^{s_N} \end{bmatrix} \text{X} \begin{bmatrix} x^* \\ y_{s_1} \\ \dots \\ y_{s_N} \end{bmatrix} \leq \begin{bmatrix} K^0 \\ K^{s_1} \\ \\ K^{s_N} \end{bmatrix}$$

Constraints master problem

Constraints Subproblem s_1

Constraints Subproblem s_N

Complicating variables fixed

- One subproblem per scenario (at least)!

Stochastic Programming – Benders Decomposition

This formulation of a stochastic optimization problem:

$$
\begin{aligned}
\min_{x, y_{s_1}, \dots, y_{s_N}} \quad & \mathbb{E} f(x, y_{s_1}, \dots y_{s_N}) \\
\text{s.t.} \quad & h(x) = 0 \\
& g(x) \leq 0 \\
& h^s(x, y_s) = 0, \quad s = \{s_1, \dots, s_N\} \\
& g^s(x, y_s) \leq 0, \quad s = \{s_1, \dots, s_N\}
\end{aligned}
$$

Stochastic Programming – Benders Decomposition

Can be reformulated as follows:

$$\min_{\boxed{x_{s_1},\ldots,x_{s_N}},y_{s_1},\ldots,y_{s_N}} \mathbb{E}f(x_{s_1},\ldots,x_{s_N},y_{s_1},\ldots y_{s_N})$$

1 variable per scenario

$$\begin{aligned}
\text{s.t. } & h(x_s) = 0, \quad s = \{s_1,\ldots,s_N\} \\
& g(x_s) \leq 0, \quad s = \{s_1,\ldots,s_N\} \\
& h^s(x_s, y_s) = 0, \quad s = \{s_1,\ldots,s_N\} \\
& g^s(x_s, y_s) \leq 0, \quad s = \{s_1,\ldots,s_N\} \\
& \boxed{x_s = x_{s'}, \quad s, s' = \{s_1,\ldots,s_N\}}
\end{aligned}$$

Non-anticipativity constraints

Stochastic Programming – Benders Decomposition

Can be reformulated as follows:

$$\min_{\boxed{x_{s_1},\ldots,x_{s_N}},y_{s_1},\ldots,y_{s_N}} \quad \mathbb{E}f(x_{s_1},\ldots,x_{s_N},y_{s_1},\ldots y_{s_N})$$

1 variable per scenario

$$\begin{aligned}
\text{s.t.}\quad & h(x_s)=0, && s=\{s_1,\ldots,s_N\}\\
& g(x_s)\leq 0, && s=\{s_1,\ldots,s_N\}\\
& h^s(x_s,y_s)=0, && s=\{s_1,\ldots,s_N\}\\
& g^s(x_s,y_s)\leq 0, && s=\{s_1,\ldots,s_N\}
\end{aligned}$$

Non-anticipativity constraints

$$\boxed{x_s = x_{s'}, \quad s,s'=\{s_1,\ldots,s_N\}}$$

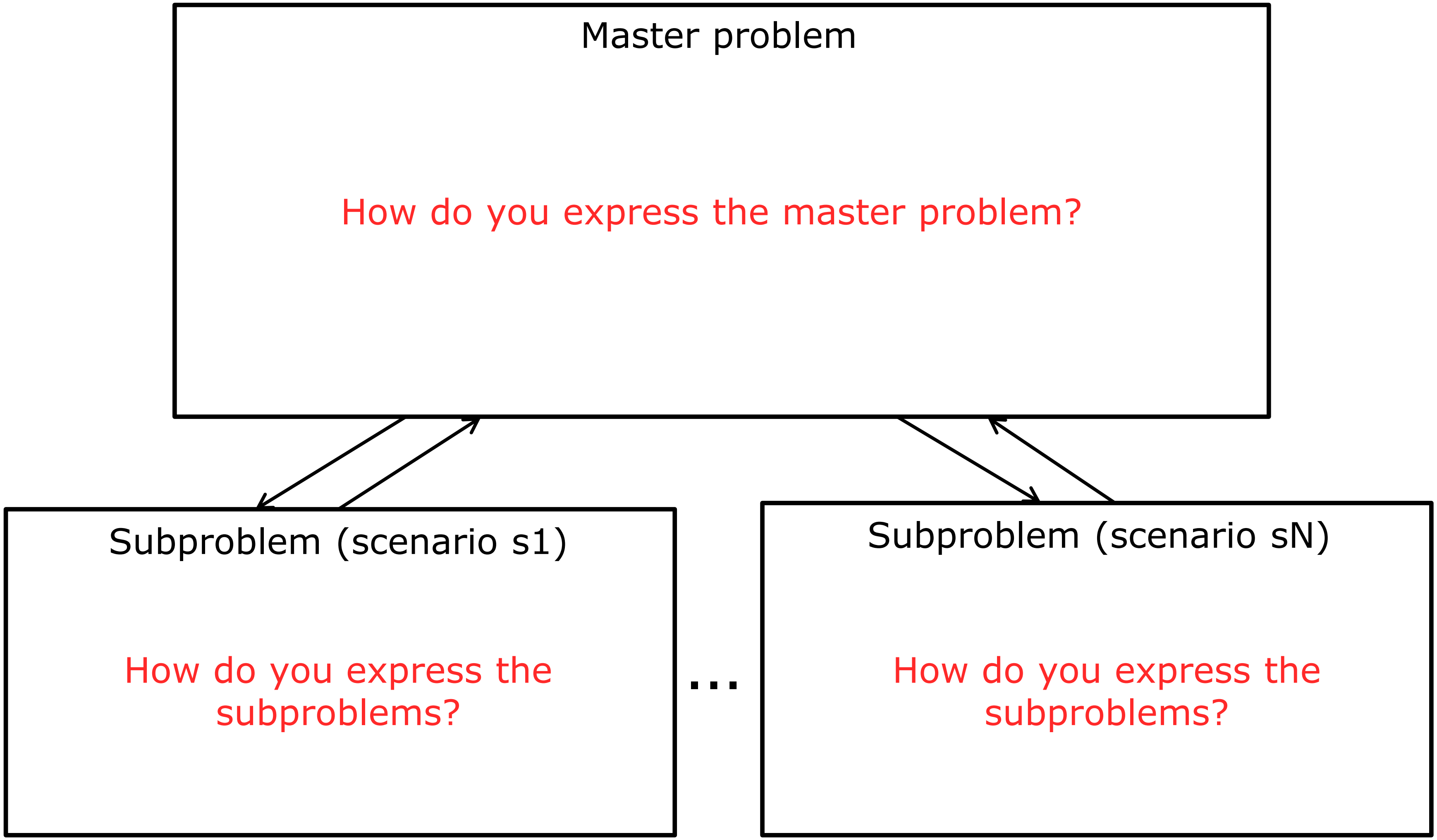
Question: Advantage of using complicating variables Vs. Complicating constraints for stochastic optimization problems?

Stochastic Programming – Benders Decomposition

To sum up…

- Stochastic (convex) optimization problems are decomposable
- Complicating variables: first-stage (here-and-now) decision variables
- Number of subproblems: One per scenario (at-least)

Stochastic Programming – Benders Decomposition



Master problem

How do you express the master problem?

Subproblem (scenario s1)

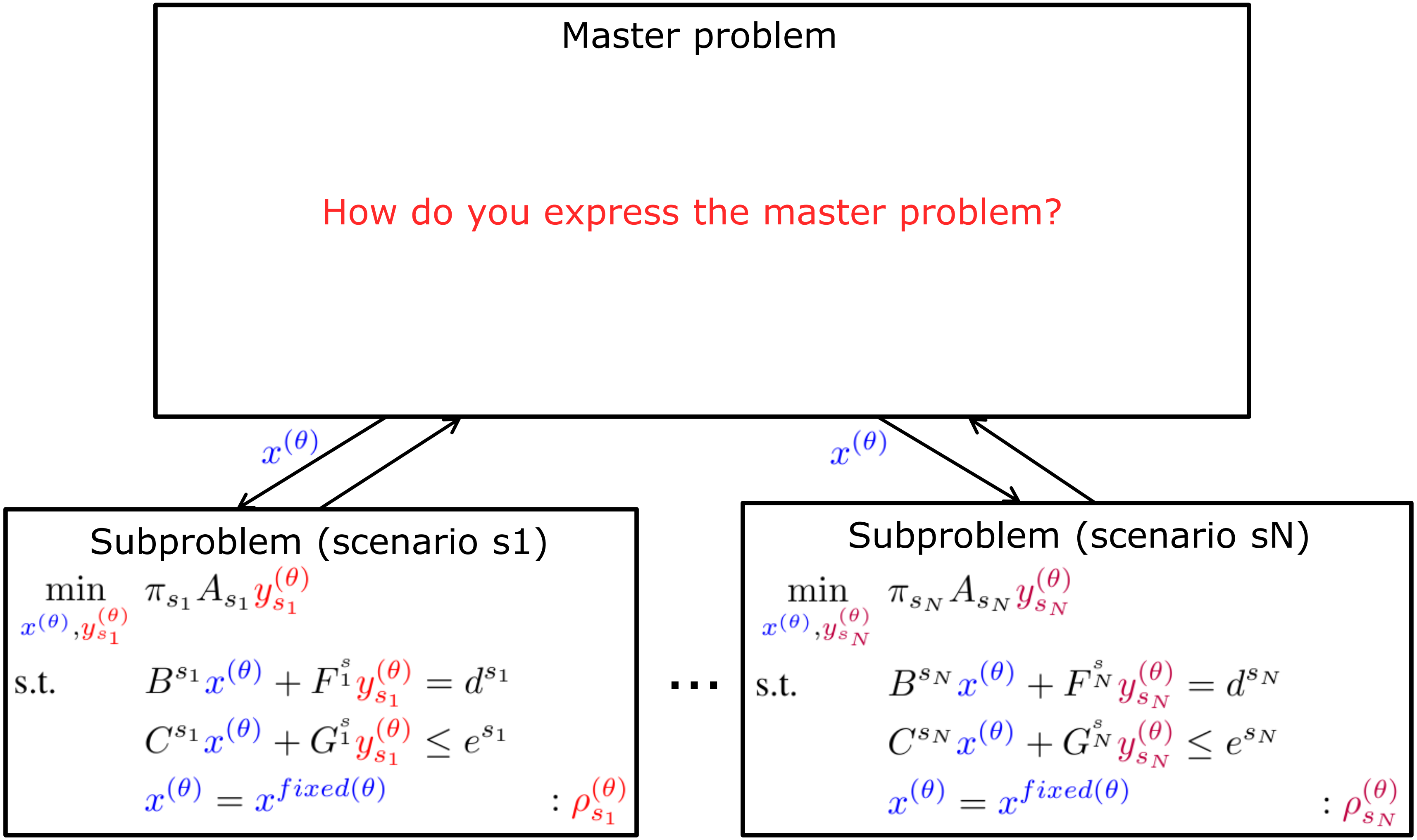
How do you express the subproblems?

…

Subproblem (scenario sN)

How do you express the subproblems?

Stochastic Programming – Benders Decomposition



Master problem

How do you express the master problem?

$x^{(\theta)}$

$x^{(\theta)}$

Subproblem (scenario s1)

$$\begin{aligned} \min_{x^{(\theta)}, y_{s_1}^{(\theta)}} \quad & \pi_{s_1} A_{s_1} y_{s_1}^{(\theta)} \\ \text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{s_1} y_{s_1}^{(\theta)} = d^{s_1} \\ & C^{s_1} x^{(\theta)} + G^{s_1} y_{s_1}^{(\theta)} \leq e^{s_1} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_1}^{(\theta)} \end{aligned}$$

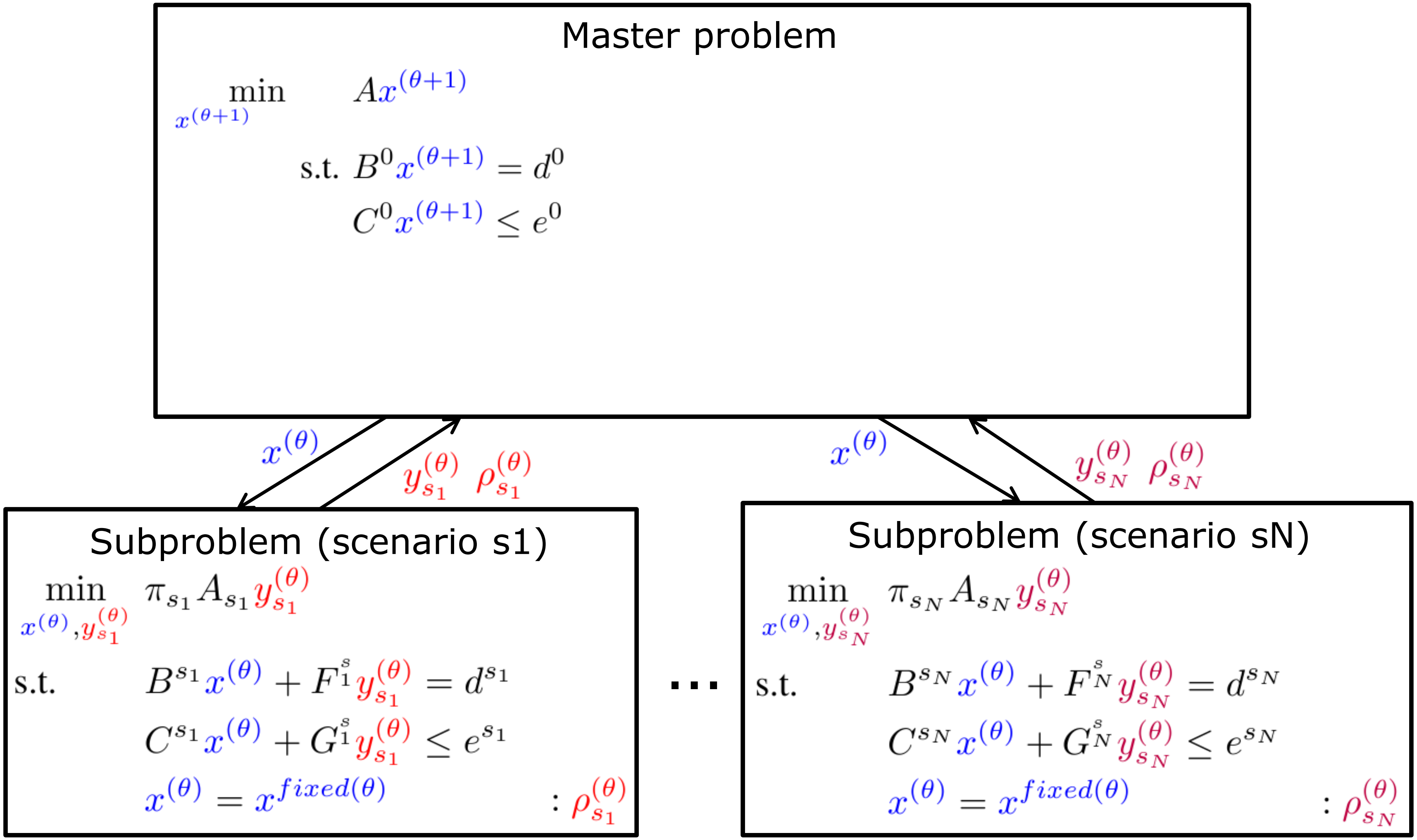
…

Subproblem (scenario sN)

$$\begin{aligned} \min_{x^{(\theta)}, y_{s_N}^{(\theta)}} \quad & \pi_{s_N} A_{s_N} y_{s_N}^{(\theta)} \\ \text{s.t.} \quad & B^{s_N} x^{(\theta)} + F^{s_N} y_{s_N}^{(\theta)} = d^{s_N} \\ & C^{s_N} x^{(\theta)} + G^{s_N} y_{s_N}^{(\theta)} \leq e^{s_N} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_N}^{(\theta)} \end{aligned}$$

θ: current iteration, k= previous iterations

Stochastic Programming – Benders Decomposition



Master problem

$$\begin{aligned} \min_{x^{(\theta+1)}} \quad & Ax^{(\theta+1)} \\ \text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\ & C^0 x^{(\theta+1)} \le e^0 \end{aligned}$$

Subproblem (scenario s1)

$$\begin{aligned} \min_{x^{(\theta)}, y_{s_1}^{(\theta)}} \quad & \pi_{s_1} A_{s_1} y_{s_1}^{(\theta)} \\ \text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{\overset{s}{1}} y_{s_1}^{(\theta)} = d^{s_1} \\ & C^{s_1} x^{(\theta)} + G^{\overset{s}{1}} y_{s_1}^{(\theta)} \le e^{s_1} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_1}^{(\theta)} \end{aligned}$$

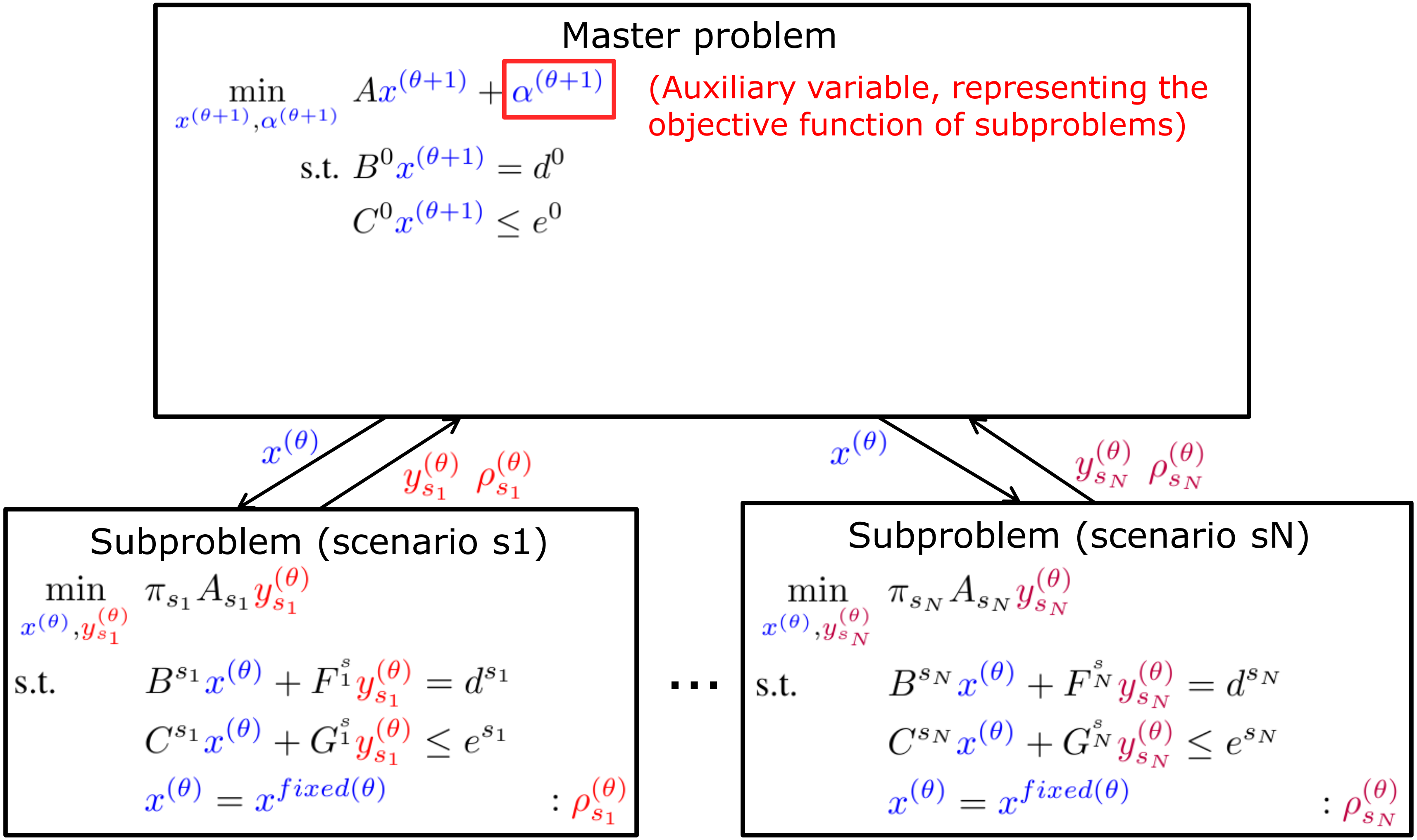
…

Subproblem (scenario sN)

$$\begin{aligned} \min_{x^{(\theta)}, y_{s_N}^{(\theta)}} \quad & \pi_{s_N} A_{s_N} y_{s_N}^{(\theta)} \\ \text{s.t.} \quad & B^{s_N} x^{(\theta)} + F^{\overset{s}{N}} y_{s_N}^{(\theta)} = d^{s_N} \\ & C^{s_N} x^{(\theta)} + G^{\overset{s}{N}} y_{s_N}^{(\theta)} \le e^{s_N} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_N}^{(\theta)} \end{aligned}$$

θ: current iteration, k= previous iterations

Stochastic Programming – Benders Decomposition



Master problem

$$\begin{aligned} \min_{x^{(\theta+1)},\alpha^{(\theta+1)}} \quad & Ax^{(\theta+1)} + \boxed{\alpha^{(\theta+1)}} \\ \text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\ & C^0 x^{(\theta+1)} \leq e^0 \end{aligned}$$

(Auxiliary variable, representing the objective function of subproblems)

$x^{(\theta)}$

$y_{s_1}^{(\theta)}$ $\rho_{s_1}^{(\theta)}$

$x^{(\theta)}$

$y_{s_N}^{(\theta)}$ $\rho_{s_N}^{(\theta)}$

Subproblem (scenario s1)

$$\begin{aligned} \min_{x^{(\theta)},y_{s_1}^{(\theta)}} \quad & \pi_{s_1} A_{s_1} y_{s_1}^{(\theta)} \\ \text{s.t.} \quad & B^{s_1} x^{(\theta)} + F_1^s y_{s_1}^{(\theta)} = d^{s_1} \\ & C^{s_1} x^{(\theta)} + G_1^s y_{s_1}^{(\theta)} \leq e^{s_1} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_1}^{(\theta)} \end{aligned}$$

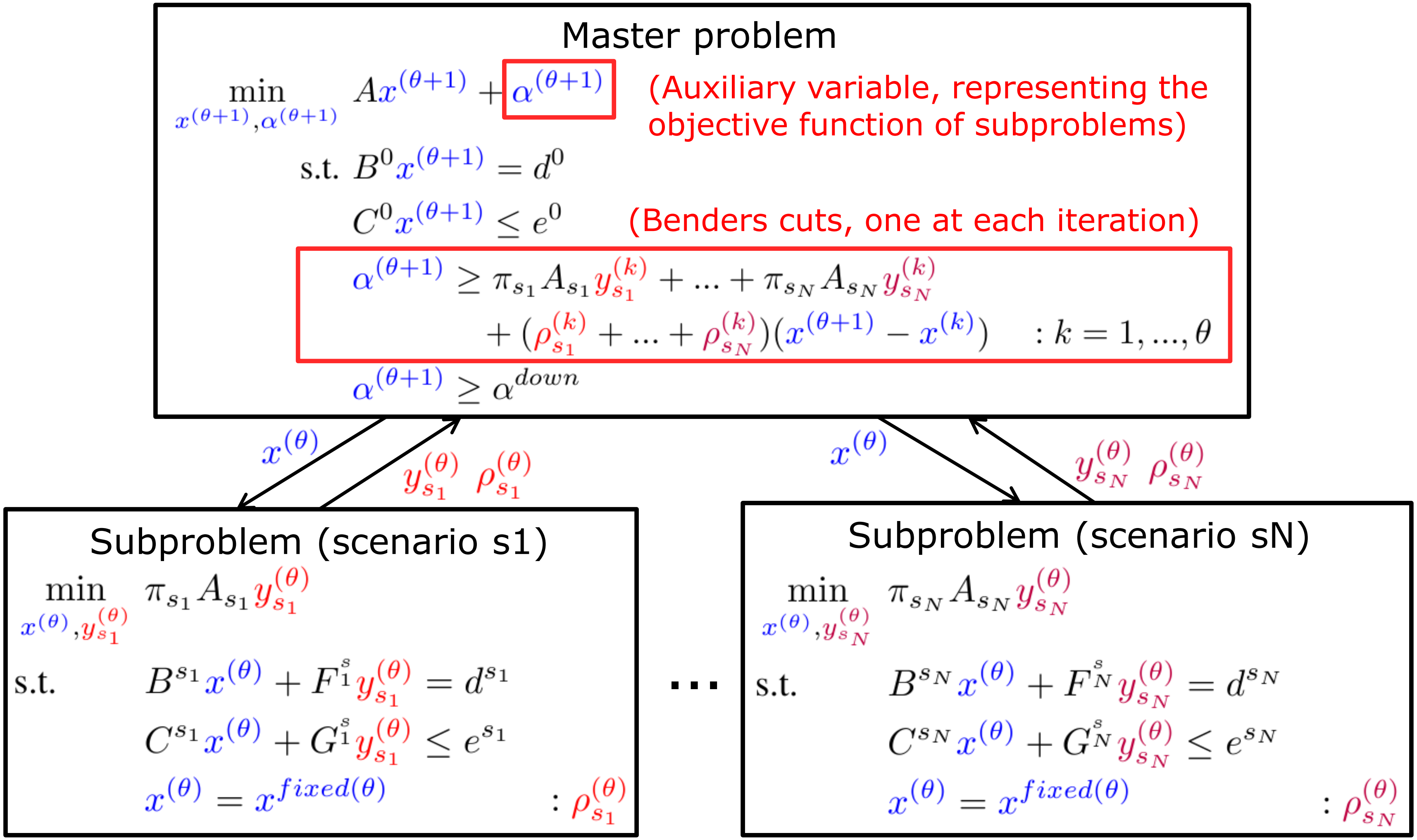
…

Subproblem (scenario sN)

$$\begin{aligned} \min_{x^{(\theta)},y_{s_N}^{(\theta)}} \quad & \pi_{s_N} A_{s_N} y_{s_N}^{(\theta)} \\ \text{s.t.} \quad & B^{s_N} x^{(\theta)} + F_N^s y_{s_N}^{(\theta)} = d^{s_N} \\ & C^{s_N} x^{(\theta)} + G_N^s y_{s_N}^{(\theta)} \leq e^{s_N} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_N}^{(\theta)} \end{aligned}$$

θ: current iteration, k= previous iterations

Stochastic Programming – Benders Decomposition



Master problem

$$\begin{aligned}\min_{x^{(\theta+1)},\alpha^{(\theta+1)}} \quad & Ax^{(\theta+1)} + \boxed{\alpha^{(\theta+1)}} \\ \text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\ & C^0 x^{(\theta+1)} \leq e^0 \\ & \alpha^{(\theta+1)} \geq \pi_{s_1} A_{s_1} y_{s_1}^{(k)} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}^{(k)} \\ & \quad + (\rho_{s_1}^{(k)} + \ldots + \rho_{s_N}^{(k)})(x^{(\theta+1)} - x^{(k)}) \quad : k = 1, \ldots, \theta \\ & \alpha^{(\theta+1)} \geq \alpha^{down}\end{aligned}$$

(Auxiliary variable, representing the objective function of subproblems)

(Benders cuts, one at each iteration)

$x^{(\theta)}$

$y_{s_1}^{(\theta)}$ $\rho_{s_1}^{(\theta)}$

$x^{(\theta)}$

$y_{s_N}^{(\theta)}$ $\rho_{s_N}^{(\theta)}$

Subproblem (scenario s1)

$$\begin{aligned}\min_{x^{(\theta)},y_{s_1}^{(\theta)}} \quad & \pi_{s_1} A_{s_1} y_{s_1}^{(\theta)} \\ \text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{s_1} y_{s_1}^{(\theta)} = d^{s_1} \\ & C^{s_1} x^{(\theta)} + G^{s_1} y_{s_1}^{(\theta)} \leq e^{s_1} \\ & x^{(\theta)} = x^{fixed(\theta)} \quad : \rho_{s_1}^{(\theta)}\end{aligned}$$

…

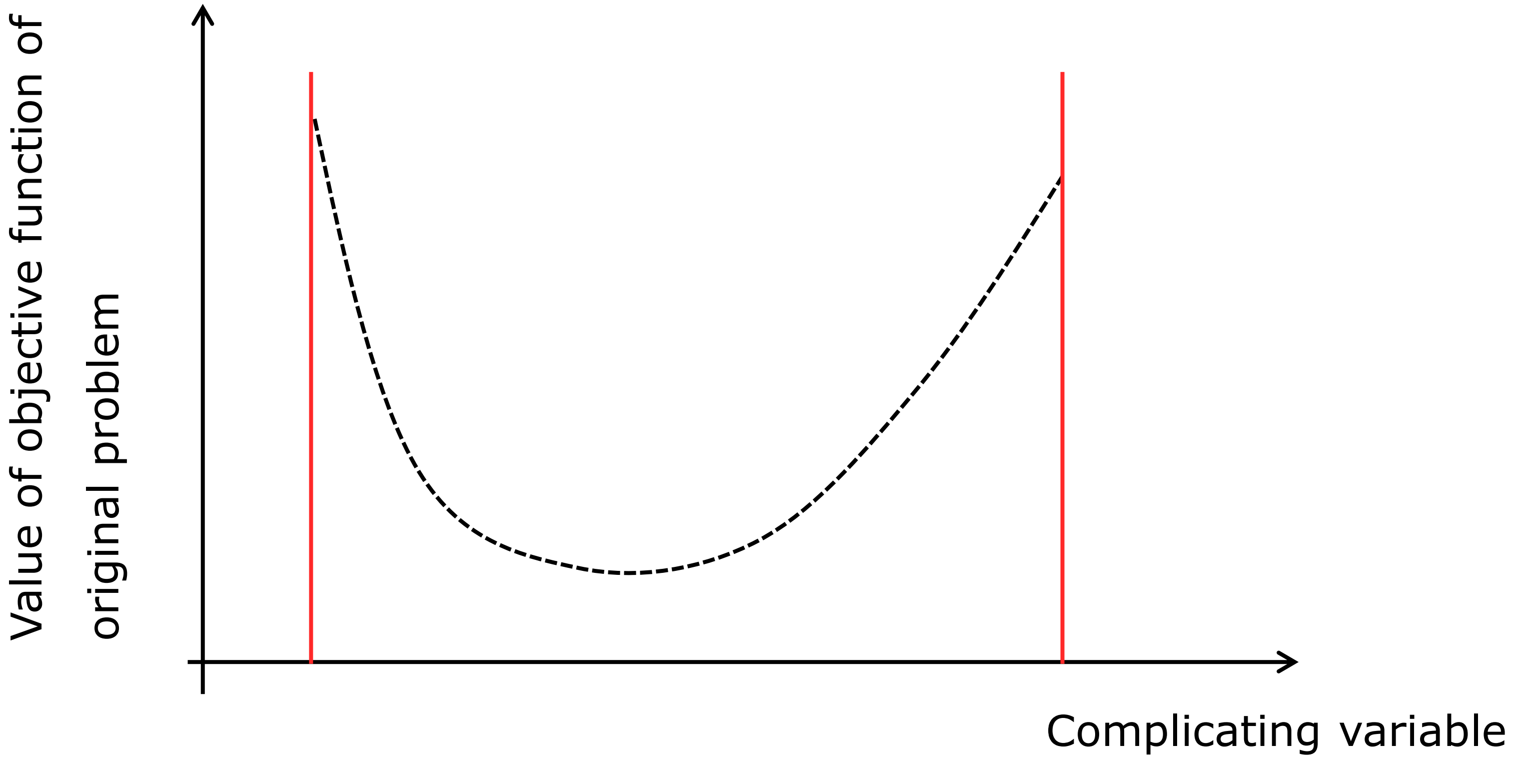
Subproblem (scenario sN)

$$\begin{aligned}\min_{x^{(\theta)},y_{s_N}^{(\theta)}} \quad & \pi_{s_N} A_{s_N} y_{s_N}^{(\theta)} \\ \text{s.t.} \quad & B^{s_N} x^{(\theta)} + F^{s_N} y_{s_N}^{(\theta)} = d^{s_N} \\ & C^{s_N} x^{(\theta)} + G^{s_N} y_{s_N}^{(\theta)} \leq e^{s_N} \\ & x^{(\theta)} = x^{fixed(\theta)} \quad : \rho_{s_N}^{(\theta)}\end{aligned}$$

θ: current iteration, k= previous iterations

Benders Decomposition – Benders Cuts

Question: What do the Benders cuts do?



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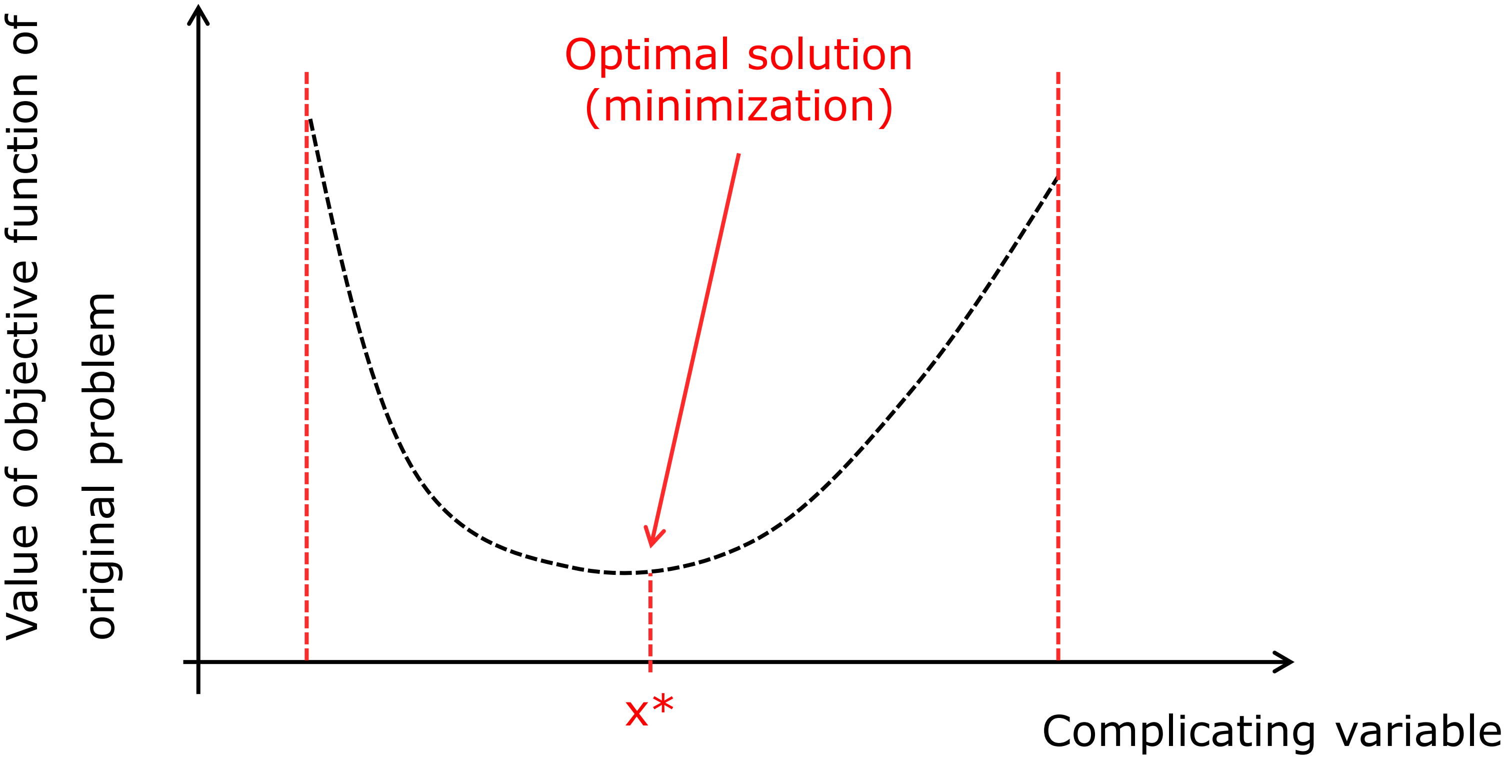

Benders Decomposition – Benders Cuts

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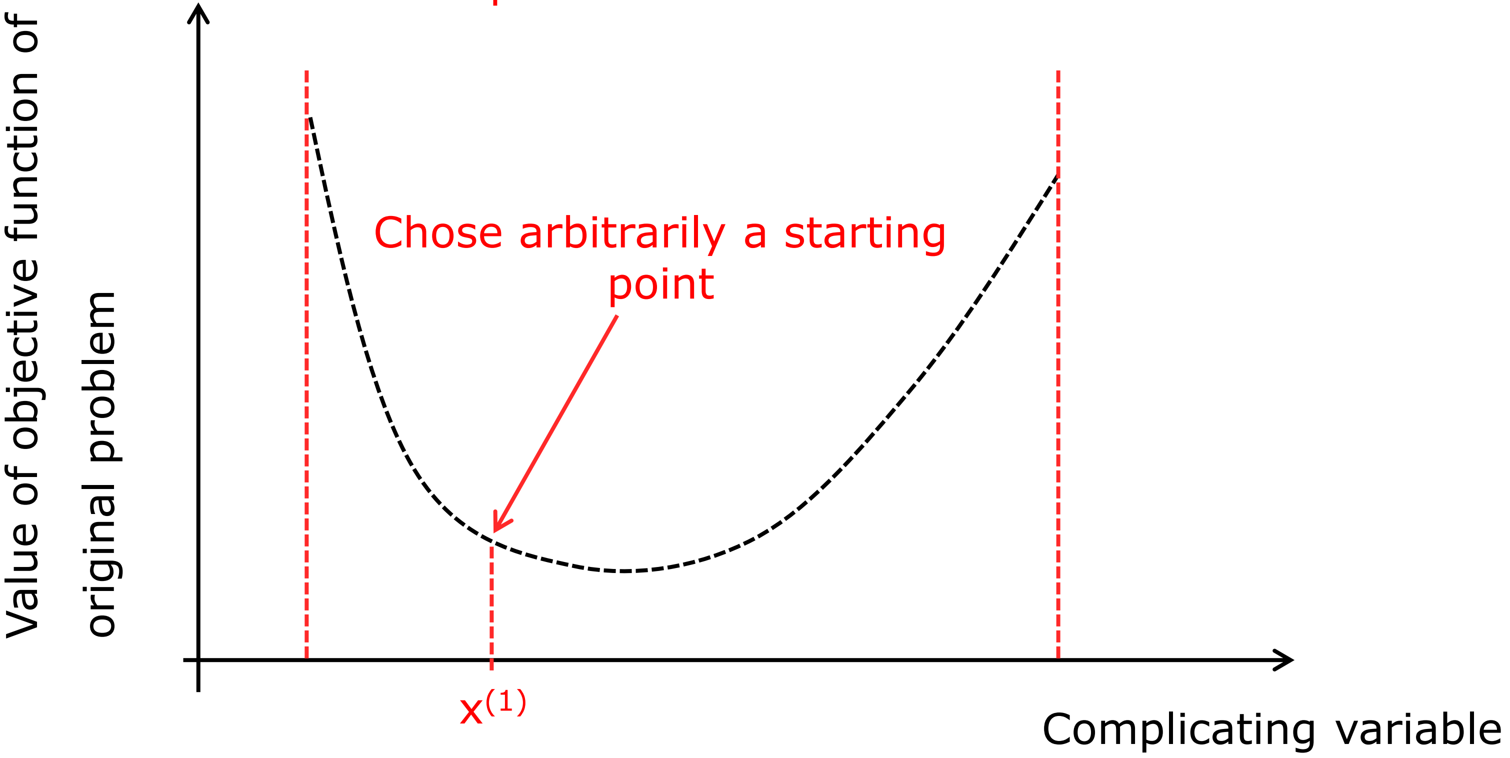
Iterative process: iteration 1



Benders Decomposition – Benders Cuts

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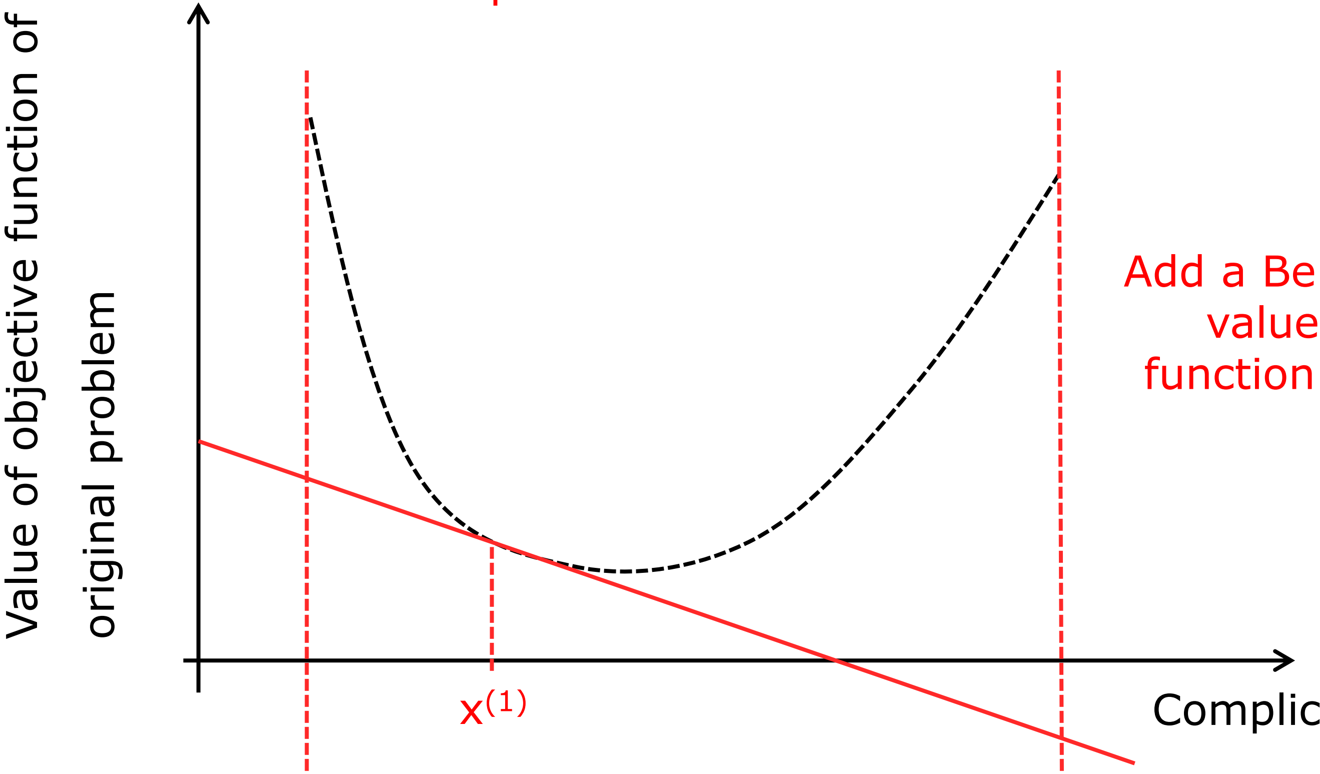
Iterative process: iteration 1



Add a Benders cut: using the value of the objective function and the sensitivies

Benders Decomposition – Benders Cuts

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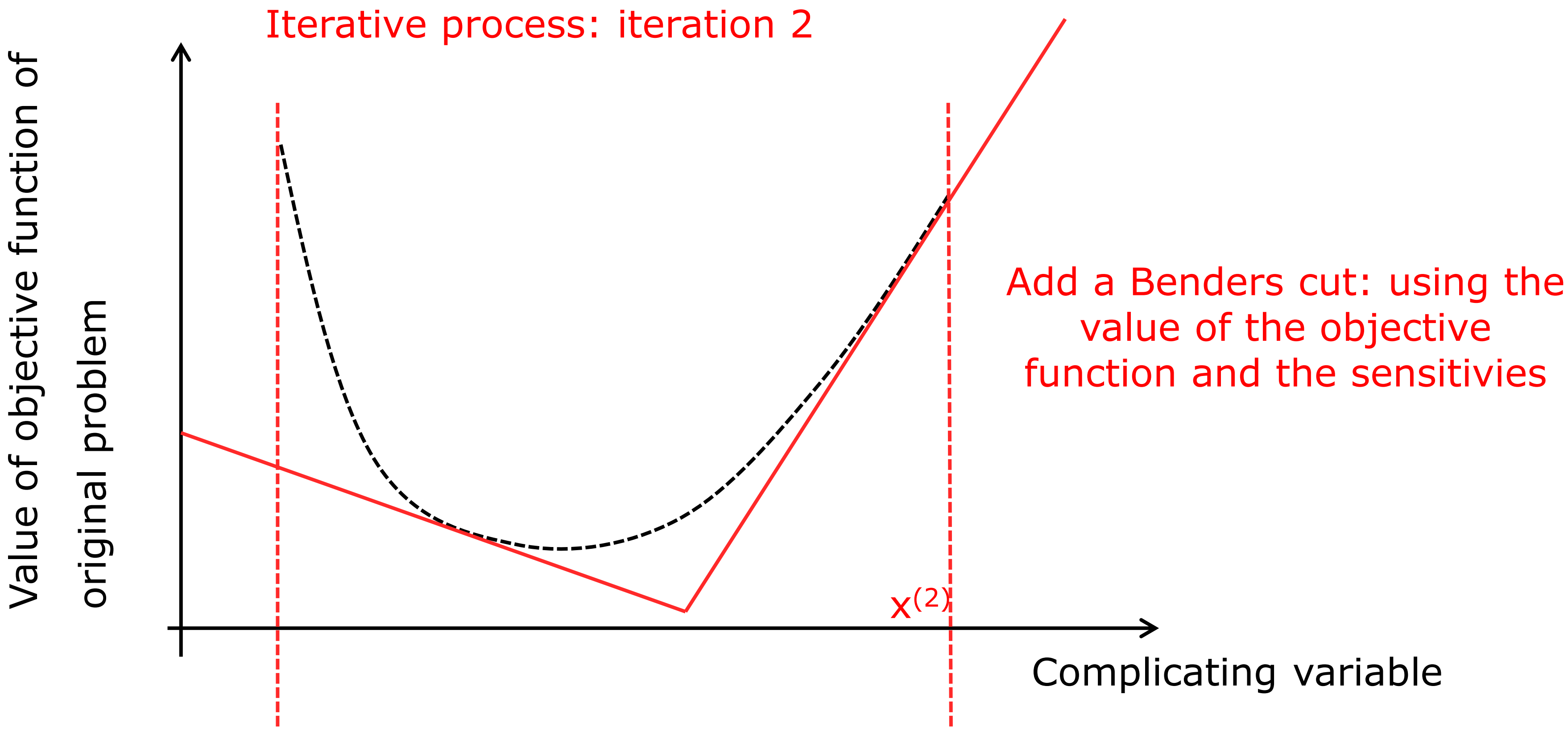

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Benders Decomposition – Benders Cuts

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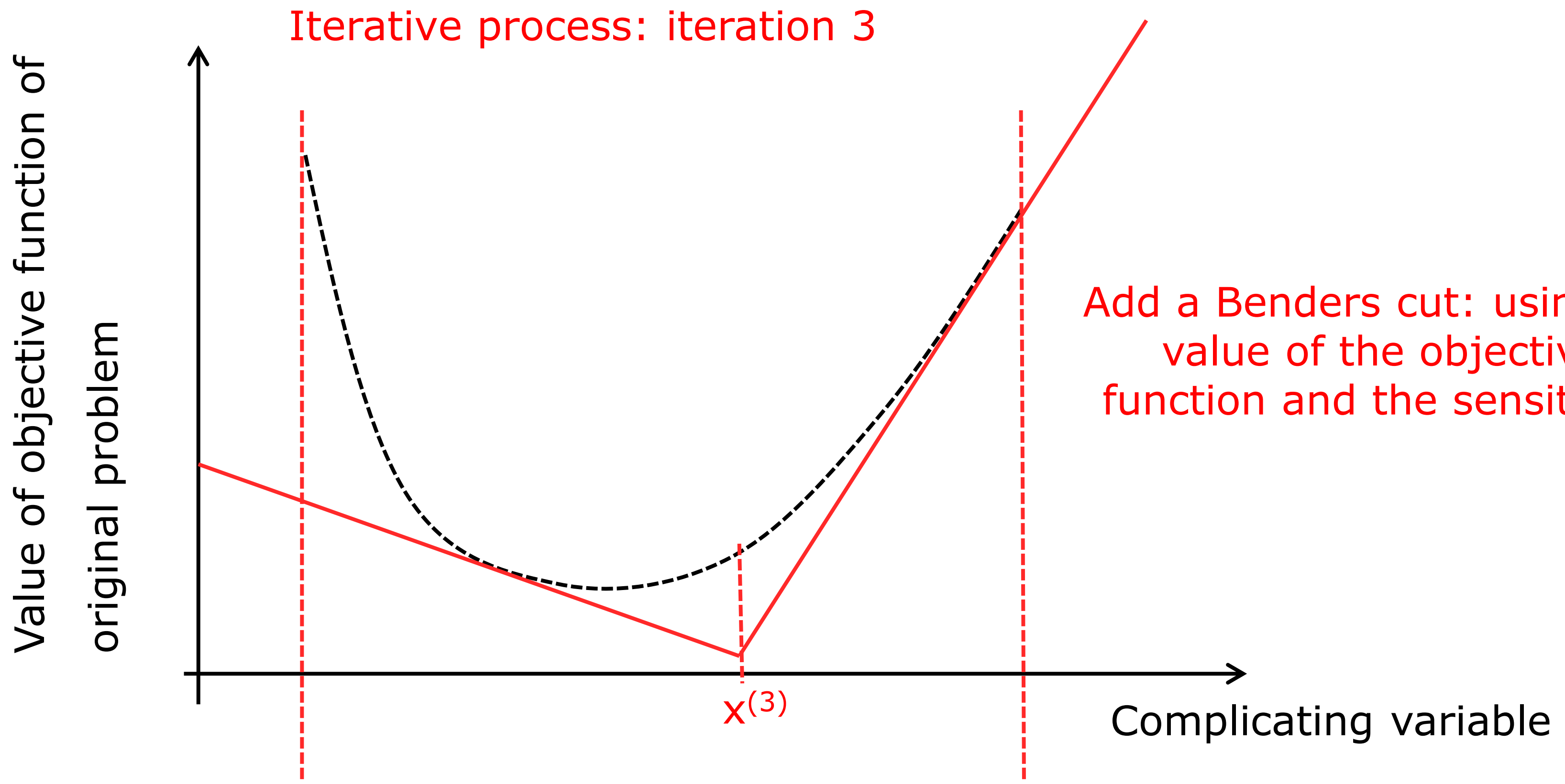

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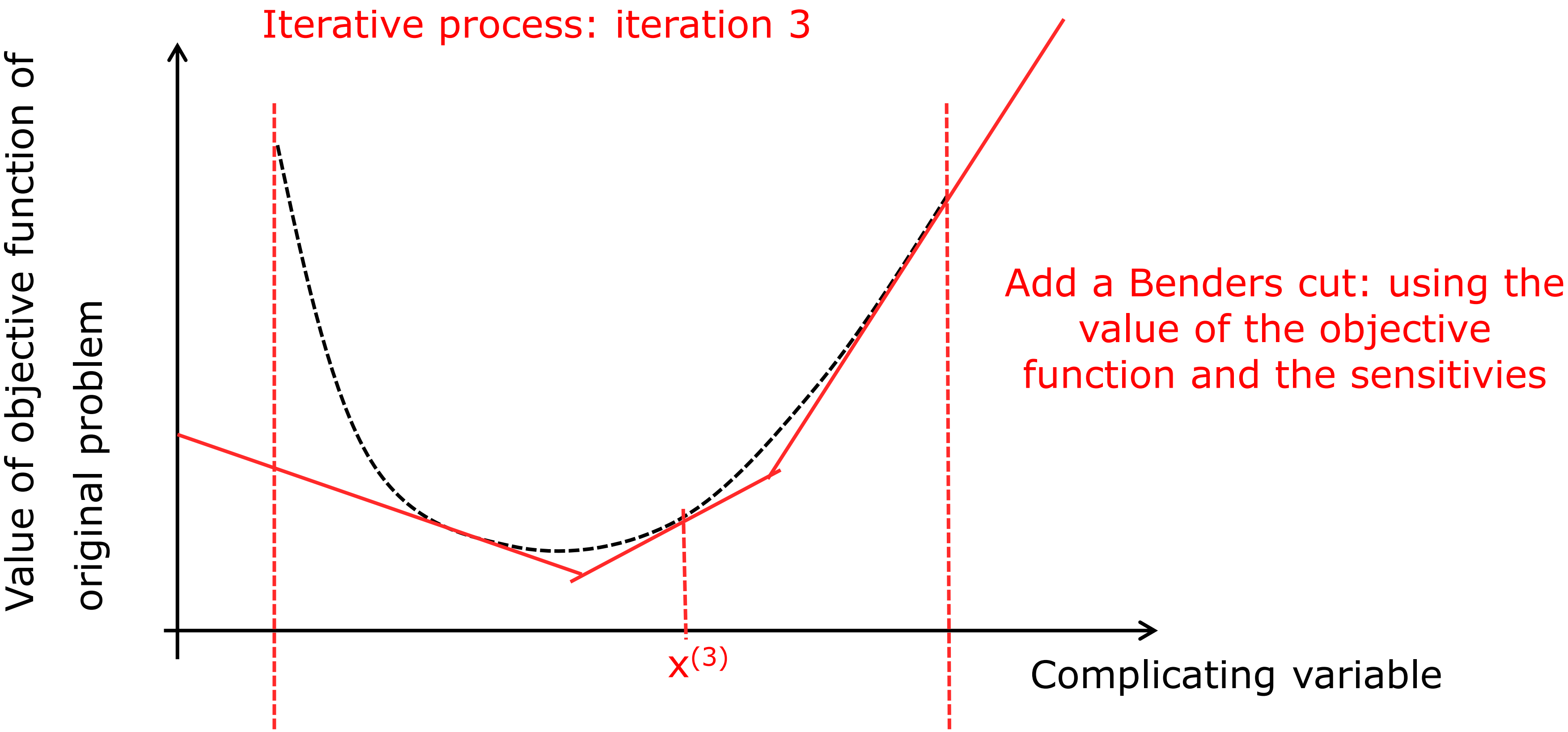

Benders Decomposition – Benders Cuts

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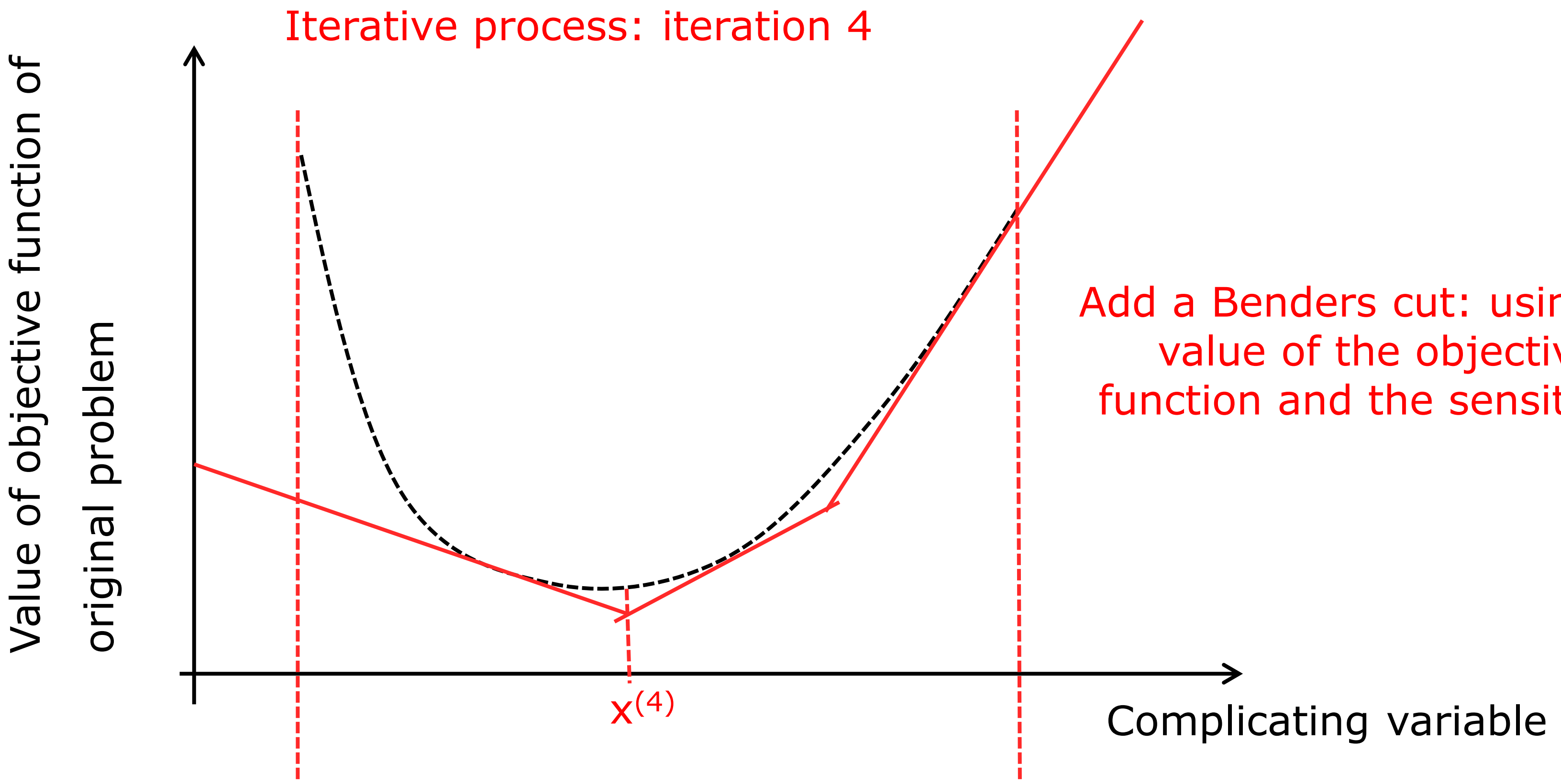

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Benders Decomposition – Benders Cuts

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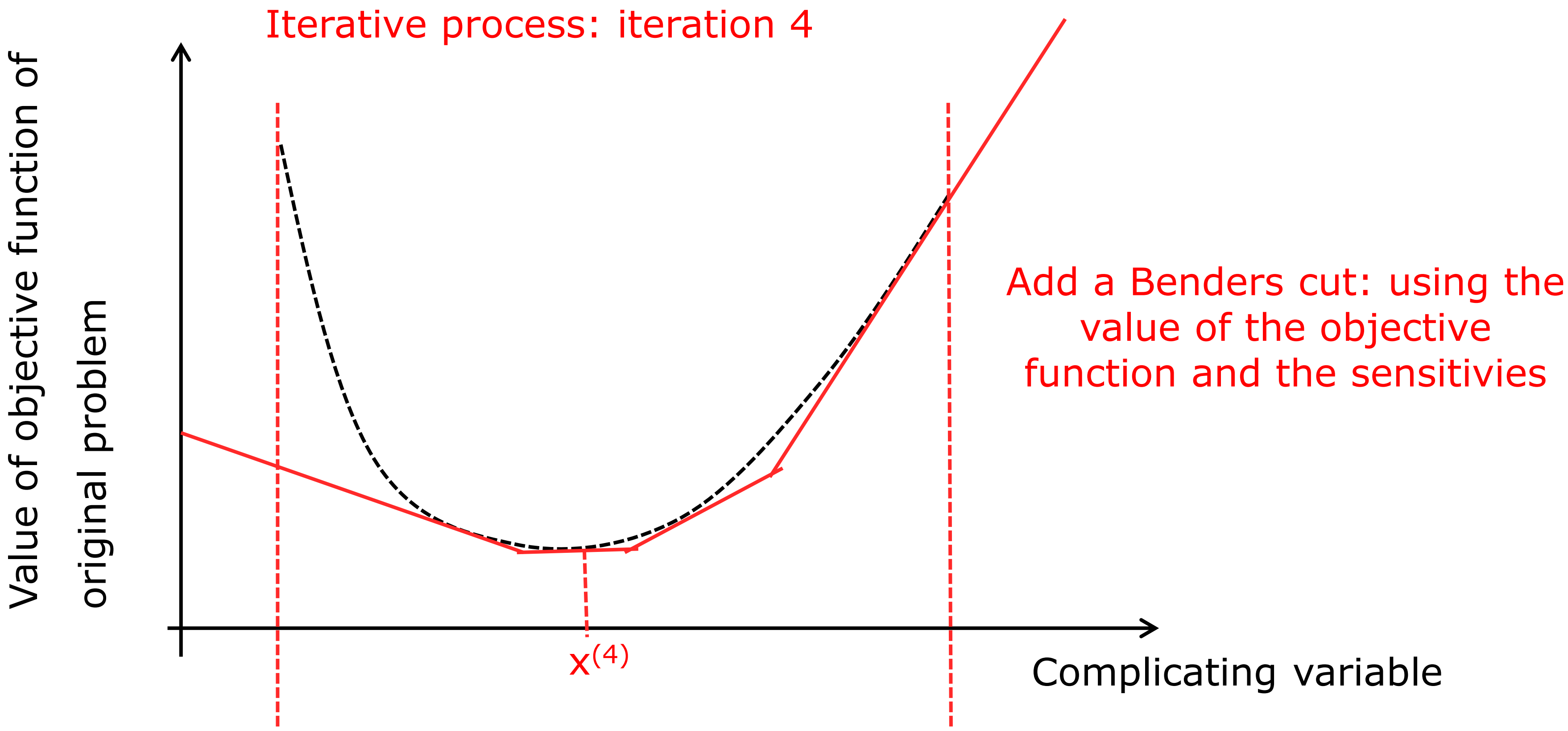


Add a Benders cut: using the value of the objective function and the sensitivies

Benders Decomposition – Benders Cuts

Let's go back to our stochastic optimization problem…

Master problem

$$
\begin{aligned}
\min_{x^{(\theta+1)},\alpha^{(\theta+1)}} \quad & Ax^{(\theta+1)} + \alpha^{(\theta+1)} \\
\text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\
& C^0 x^{(\theta+1)} \leq e^0 \\
& \alpha^{(\theta+1)} \geq \pi_{s_1} A_{s_1} y_{s_1}^{(k)} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}^{(k)} \\
& \qquad + (\rho_{s_1}^{(k)} + \ldots + \rho_{s_N}^{(k)})(x^{(\theta+1)} - x^{(k)}) \qquad : k = 1, \ldots, \theta \\
& \alpha^{(\theta+1)} \geq \alpha^{down}
\end{aligned}
$$

Benders Decomposition – Benders Cuts

Let's go back to our stochastic optimization problem…

Master problem

$$
\begin{aligned}
\min_{x^{(\theta+1)},\alpha^{(\theta+1)}} \quad & Ax^{(\theta+1)} + \alpha^{(\theta+1)} \\
\text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\
& C^0 x^{(\theta+1)} \leq e^0 \\
& \alpha^{(\theta+1)} \geq \pi_{s_1} A_{s_1} y_{s_1}^{(k)} + ... + \pi_{s_N} A_{s_N} y_{s_N}^{(k)} \\
& \quad \boxed{+ (\rho_{s_1}^{(k)} + ... + \rho_{s_N}^{(k)})(x^{(\theta+1)} - x^{(k)})} \quad : k = 1, ..., \theta \\
& \alpha^{(\theta+1)} \geq \alpha^{down}
\end{aligned}
$$

Question: Why doesn't the weight of the scenarios (Π_s) multiply the sensitivities in my expression of the Benders cuts?

Benders Decomposition – Benders Cuts

Let’s go back to our stochastic optimization problem…

Subproblem (scenario s1)

$$\begin{aligned}
\min_{x^{(\theta)},y^{(\theta)}_{s_1}} \quad & \pi_{s_1} A_{s_1} y^{(\theta)}_{s_1} \\
\text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{\overset{s}{1}} y^{(\theta)}_{s_1} = d^{s_1} \\
& C^{s_1} x^{(\theta)} + G^{\overset{s}{1}} y^{(\theta)}_{s_1} \leq e^{s_1} \\
& x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho^{(\theta)}_{s_1}
\end{aligned}$$

Benders Decomposition – Benders Cuts

Let’s go back to our stochastic optimization problem…

Subproblem (scenario s1)

$$\min_{x^{(\theta)},y_{s_1}^{(\theta)}} \quad \pi_{s_1} A_{s_1} y_{s_1}^{(\theta)}$$

$$\begin{aligned} \text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{s_1} y_{s_1}^{(\theta)} = d^{s_1} \\ & C^{s_1} x^{(\theta)} + G^{s_1} y_{s_1}^{(\theta)} \leq e^{s_1} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_1}^{(\theta)} \end{aligned}$$

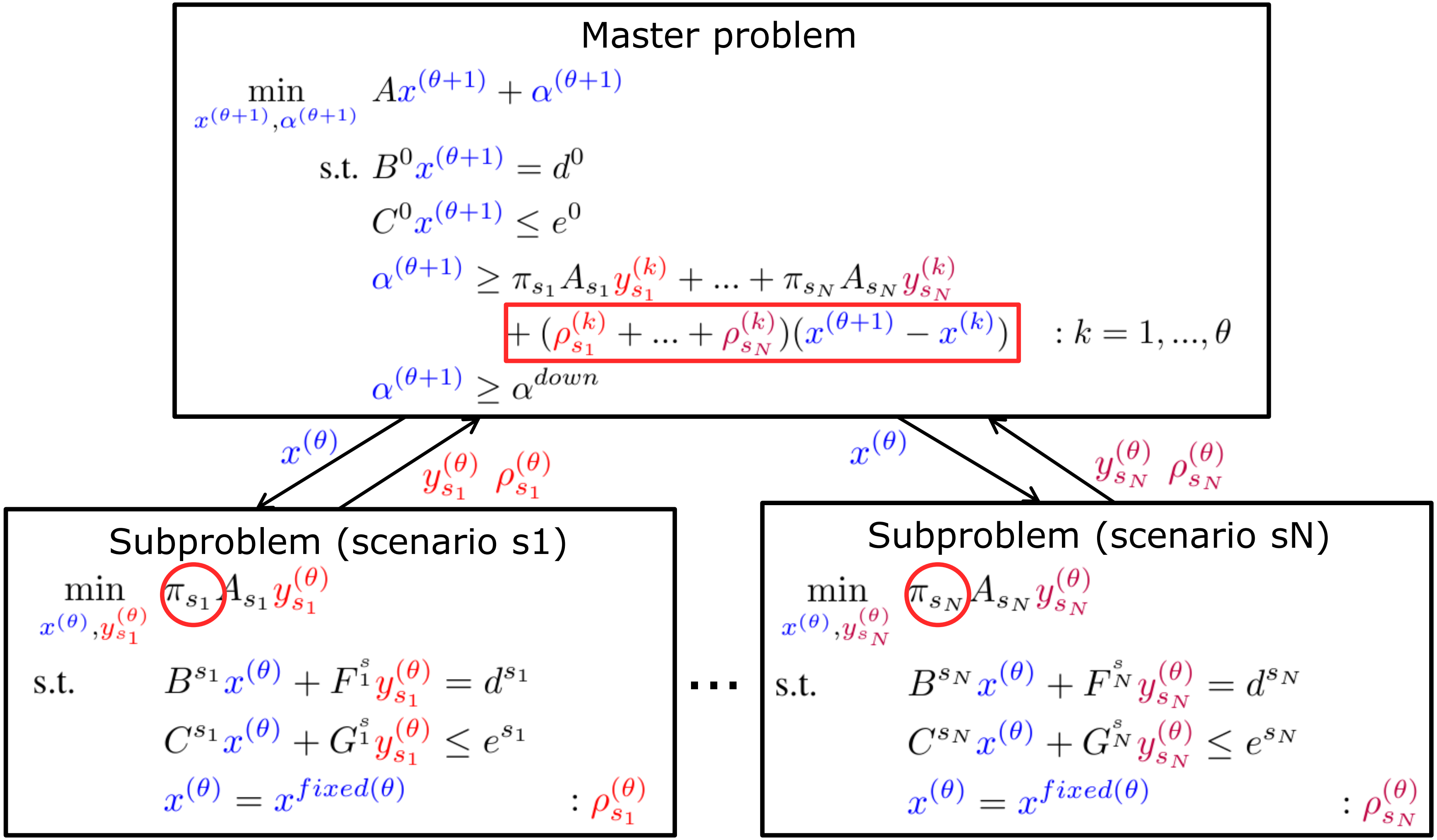
The weight of each scenario is already accounted for in the objective function of the subproblem.

This influences the sensitivities (ρ_s)

Careful not to account for the weights twice!!

Stochastic Programming – Benders Decomposition

This formulation of the master problem and subproblems:



Master problem

$$
\begin{aligned}
\min_{x^{(\theta+1)},\alpha^{(\theta+1)}} \quad & Ax^{(\theta+1)} + \alpha^{(\theta+1)} \\
\text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\
& C^0 x^{(\theta+1)} \leq e^0 \\
& \alpha^{(\theta+1)} \geq \pi_{s_1} A_{s_1} y_{s_1}^{(k)} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}^{(k)} \\
& \qquad \boxed{+ (\rho_{s_1}^{(k)} + \ldots + \rho_{s_N}^{(k)})(x^{(\theta+1)} - x^{(k)})} \quad : k = 1, \ldots, \theta \\
& \alpha^{(\theta+1)} \geq \alpha^{down}
\end{aligned}
$$

Subproblem (scenario s1)

$$
\begin{aligned}
\min_{x^{(\theta)},y_{s_1}^{(\theta)}} \quad & \boxed{\pi_{s_1}} A_{s_1} y_{s_1}^{(\theta)} \\
\text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{s_1} y_{s_1}^{(\theta)} = d^{s_1} \\
& C^{s_1} x^{(\theta)} + G^{s_1} y_{s_1}^{(\theta)} \leq e^{s_1} \\
& x^{(\theta)} = x^{fixed(\theta)} \quad : \rho_{s_1}^{(\theta)}
\end{aligned}
$$

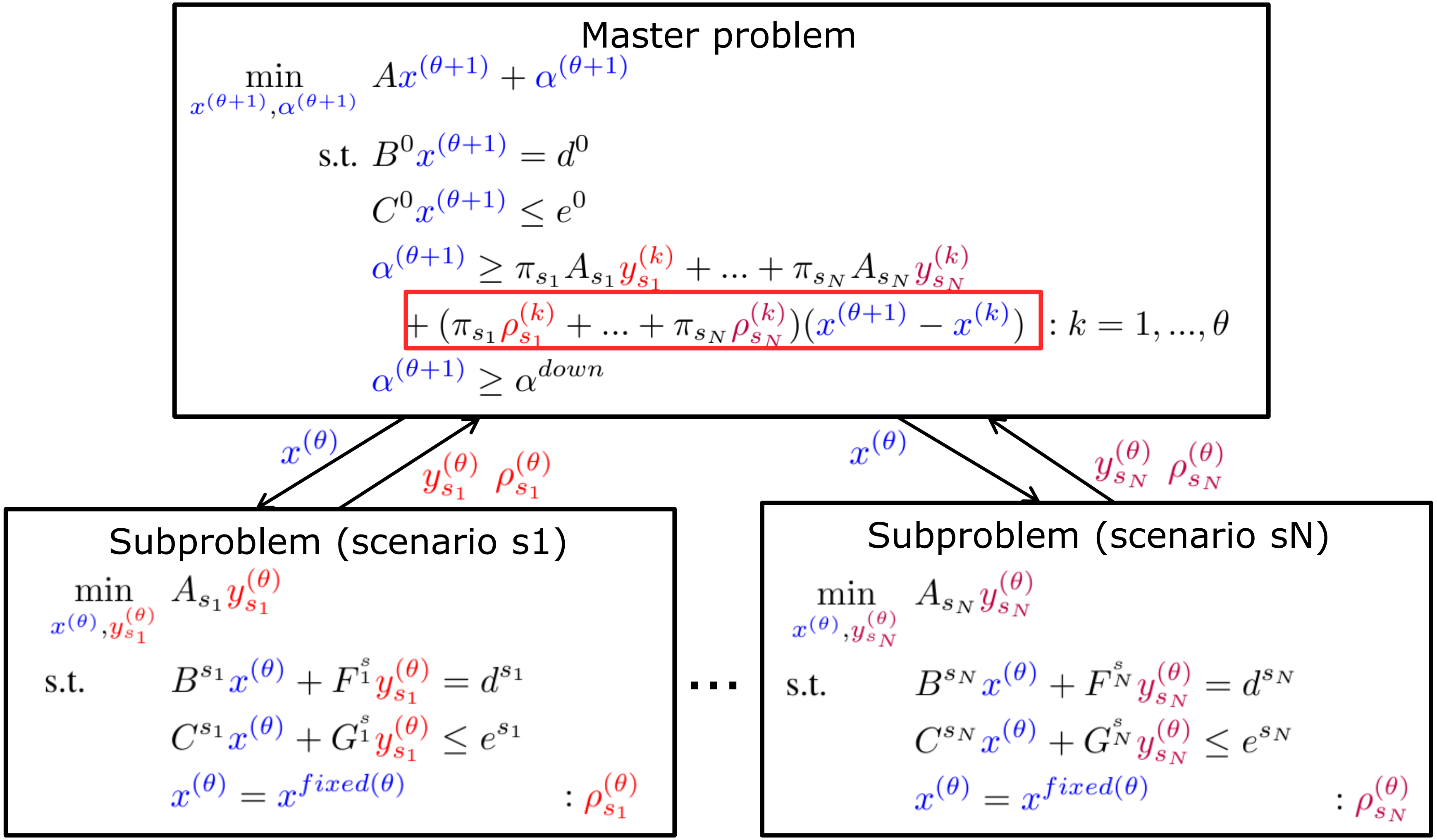
...

Subproblem (scenario sN)

$$
\begin{aligned}
\min_{x^{(\theta)},y_{s_N}^{(\theta)}} \quad & \boxed{\pi_{s_N}} A_{s_N} y_{s_N}^{(\theta)} \\
\text{s.t.} \quad & B^{s_N} x^{(\theta)} + F^{s_N} y_{s_N}^{(\theta)} = d^{s_N} \\
& C^{s_N} x^{(\theta)} + G^{s_N} y_{s_N}^{(\theta)} \leq e^{s_N} \\
& x^{(\theta)} = x^{fixed(\theta)} \quad : \rho_{s_N}^{(\theta)}
\end{aligned}
$$

Stochastic Programming – Benders Decomposition

is equivalent to the following formulation:



Master problem

$$\begin{aligned} \min_{x^{(\theta+1)},\alpha^{(\theta+1)}} \quad & Ax^{(\theta+1)} + \alpha^{(\theta+1)} \\ \text{s.t.} \quad & B^0 x^{(\theta+1)} = d^0 \\ & C^0 x^{(\theta+1)} \leq e^0 \\ & \alpha^{(\theta+1)} \geq \pi_{s_1} A_{s_1} y_{s_1}^{(k)} + \ldots + \pi_{s_N} A_{s_N} y_{s_N}^{(k)} \\ & \boxed{+ (\pi_{s_1} \rho_{s_1}^{(k)} + \ldots + \pi_{s_N} \rho_{s_N}^{(k)})(x^{(\theta+1)} - x^{(k)})} : k = 1, \ldots, \theta \\ & \alpha^{(\theta+1)} \geq \alpha^{down} \end{aligned}$$

$x^{(\theta)}$

$y_{s_1}^{(\theta)} \quad \rho_{s_1}^{(\theta)}$

$x^{(\theta)}$

$y_{s_N}^{(\theta)} \quad \rho_{s_N}^{(\theta)}$

Subproblem (scenario s1)

$$\begin{aligned} \min_{x^{(\theta)},y_{s_1}^{(\theta)}} \quad & A_{s_1} y_{s_1}^{(\theta)} \\ \text{s.t.} \quad & B^{s_1} x^{(\theta)} + F^{s}_{1} y_{s_1}^{(\theta)} = d^{s_1} \\ & C^{s_1} x^{(\theta)} + G^{s}_{1} y_{s_1}^{(\theta)} \leq e^{s_1} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_1}^{(\theta)} \end{aligned}$$

…

Subproblem (scenario sN)

$$\begin{aligned} \min_{x^{(\theta)},y_{s_N}^{(\theta)}} \quad & A_{s_N} y_{s_N}^{(\theta)} \\ \text{s.t.} \quad & B^{s_N} x^{(\theta)} + F^{s}_{N} y_{s_N}^{(\theta)} = d^{s_N} \\ & C^{s_N} x^{(\theta)} + G^{s}_{N} y_{s_N}^{(\theta)} \leq e^{s_N} \\ & x^{(\theta)} = x^{fixed(\theta)} \qquad : \rho_{s_N}^{(\theta)} \end{aligned}$$

Benders Decomposition – Stopping criterion

Question: When can we consider that the algorithm has converged?

Benders Decomposition – Stopping criterion

Question: When can we consider that the algorithm has converged?

- When the value of the objective function of the ”original problem” in the subproblems and in the master problem have converged

We define, at each iteration:

$$LB^{(\theta)} = Ax^{(\theta)} + \alpha^{(\theta)}$$

$$UB^{(\theta)} = Ax^{(\theta)} + \pi_{s_1} A_{s_1} y_{s_1}^{(\theta)} + ... + \pi_{s_N} A_{s_N} y_{s_N}^{(\theta)}$$

When $|LB^{(\theta)} - UB^{(\theta)}| \leq \varepsilon$, the algorithm has converged.

Benders Decomposition Algorithm

Step 0: Initialization

Set $\theta=1$, $x^{fixed(1)} = x^{initial}$, and $LB^{(1)} = -\infty$

Step 1: Solve subproblems

For each subproblem obtain the value of primal ($y_s^{(\theta)}$) and dual ($\rho_s^{(\theta)}$), variables, and objective function $\Pi_s A^s y_s^{(\theta)}$. Compute the upper-bound $UB^{(\theta)}$.

Step 2: check convergence

If $|LB^{(\theta)} - UB^{(\theta)}| \leq \varepsilon$, the algorithm has converged.

Otherwise, θ <- (θ +1) and go to step 3.

Step 3: Solve master problem

Obtain the values of $x^{(\theta)}$ and $\alpha^{(\theta)}$. Compute the lower bound $LB^{(\theta)}$.

Go back to step 1.

Applications: Stochastic Market Clearing

To sum up…

- We have expressed the Benders algorithm (compact form) for a generic (linear) stochastic optimization problem
- For a convex optimization problem, Benders decomposition guarantees convergence to a global optimum

➢ Let’s look at an example: Stochastic market clearing

Stochastic Market Clearing – Benders Decomposition

Recall this example (lecture 4):

- Capacity: 100MW
- Production cost: 10$/MW
- Inflexible in RT

- Capacity: 30MW
- Production cost: 20$/MW
- Fully flexible in RT

- Capacity: 70MW
- Production cost: 0$/MW
- Uncertain production in DA
- 4 equiprobable scenarios: 30MW, 60MW, 70MW, 10MW

- Inelastic load: 120MW
- Curtailment cost: 80$/MW

Stochastic Market Clearing – Benders Decomposition

Stochastic market clearing formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},p_{s_1}^{G_2,RT},p_{s_2}^{G_2,RT},p_{s_3}^{G_2,RT},p_{s_4}^{G_2,RT},p_{s_1}^{spill},p_{s_2}^{spill},p_{s_3}^{spill},p_{s_4}^{spill},p_{s_1}^{shed},p_{s_2}^{shed},p_{s_3}^{shed},p_{s_4}^{shed}}$$

$$\left[10p^{G1,DA} + 20p^{G_2,DA}\right] + 0.25\big[20p_{s_1}^{G_2,RT} + 20p_{s_2}^{G_2,RT} + 20p_{s_3}^{G_2,RT} + 20p_{s_4}^{G_2,RT}$$
$$+ 80p_{s_1}^{shed} + 80p_{s_2}^{shed} + 80p_{s_3}^{shed} + 80p_{s_4}^{shed}\big]$$

$$\text{s.t. } 0 \le p^{G1,DA} \le 100$$
$$0 \le p^{G2,DA} \le 30$$
$$0 \le p^{W,DA} \le 70$$
$$p^{G1,DA} + p^{G2,DA} + p^{W,DA} = 120 \ : \lambda^{DA}$$
$$0 \le p^{G2,DA} + p_{s_1}^{G2,RT} \le 30$$
$$0 \le p_{s_1}^{spill} \le 30$$
$$0 \le p_{s_1}^{shed} \le 120$$
$$p_{s_1}^{G2,RT} + \left(30 - p^{W,DA} - p_{s_1}^{spill}\right) + p_{s_1}^{shed} = 0 \ : \lambda_{s_1}^{RT}$$

Stochastic Market Clearing – Benders Decomposition

Stochastic market clearing formulation:

DA dispatch (first stage decision variables)

$$\text{Minimize}_{\boxed{p^{G_1,DA},p^{G_2,DA},p^{W,DA}},p_{s_1}^{G_2,RT},p_{s_2}^{G_2,RT},p_{s_3}^{G_2,RT},p_{s_4}^{G_2,RT},p_{s_1}^{spill},p_{s_2}^{spill},p_{s_3}^{spill},p_{s_4}^{spill},p_{s_1}^{shed},p_{s_2}^{shed},p_{s_3}^{shed},p_{s_4}^{shed}}$$

$$\boxed{\left[10p^{G_1,DA}+20p^{G_2,DA}\right]}+0.25\left[20p_{s_1}^{G_2,RT}+20p_{s_2}^{G_2,RT}+20p_{s_3}^{G_2,RT}+20p_{s_4}^{G_2,RT}\right.$$

DA dispatch cost

$$\left.+80p_{s_1}^{shed}+80p_{s_2}^{shed}+80p_{s_3}^{shed}+80p_{s_4}^{shed}\right]$$

$$\text{s.t.}\ \boxed{\begin{aligned}&0\leq p^{G1,DA}\leq 100\\&0\leq p^{G2,DA}\leq 30\\&0\leq p^{W,DA}\leq 70\\&p^{G1,DA}+p^{G2,DA}+p^{W,DA}=120\ :\lambda^{DA}\end{aligned}}$$

DA dispatch constraints

$$\begin{aligned}&0\leq p^{G2,DA}+p_{s_1}^{G_2,RT}\leq 30\\&0\leq p_{s_1}^{spill}\leq 30\\&0\leq p_{s_1}^{shed}\leq 120\\&\quad p_{s_1}^{G_2,RT}+\left(30-p^{W,DA}-p_{s_1}^{spill}\right)+p_{s_1}^{shed}=0\ :\lambda_{s_1}^{RT}\end{aligned}$$

Stochastic Market Clearing – Benders Decomposition

Stochastic market clearing formulation:

RT redispatch per scenario (second stage decision variables)

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA}\boxed{p_{s_1}^{G_2,RT},p_{s_2}^{G_2,RT},p_{s_3}^{G_2,RT},p_{s_4}^{G_2,RT},p_{s_1}^{spill},p_{s_2}^{spill},p_{s_3}^{spill},p_{s_4}^{spill},p_{s_1}^{shed},p_{s_2}^{shed},p_{s_3}^{shed},p_{s_4}^{shed}}}$$

$$\left[10p^{G_1,DA} + 20p^{G_2,DA}\right] + \boxed{\begin{aligned}0.25\big[&20p_{s_1}^{G_2,RT} + 20p_{s_2}^{G_2,RT} + 20p_{s_3}^{G_2,RT} + 20p_{s_4}^{G_2,RT} \\ &+ 80p_{s_1}^{shed} + 80p_{s_2}^{shed} + 80p_{s_3}^{shed} + 80p_{s_4}^{shed}\big]\end{aligned}}$$

<u>expected</u> RT redispatch cost

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA} \leq 100 \\
& 0 \leq p^{G2,DA} \leq 30 \\
& 0 \leq p^{W,DA} \leq 70 \\
& p^{G1,DA} + p^{G2,DA} + p^{W,DA} = 120 \ : \lambda^{DA}
\end{aligned}$$

$$\boxed{\begin{aligned}
& 0 \leq p^{G2,DA} + p_{s_1}^{G_2,RT} \leq 30 \\
& 0 \leq p_{s_1}^{spill} \leq 30 \\
& 0 \leq p_{s_1}^{shed} \leq 120 \\
& \quad p_{s_1}^{G_2,RT} + \left(30 - p^{W,DA} - p_{s_1}^{spill}\right) + p_{s_1}^{shed} = 0 \ : \lambda_{s_1}^{RT}
\end{aligned}}$$

RT dispatch constraints for scenario s_1

(identical set of constraints for each scenario s_1 to s_4)

Stochastic Market Clearing – Benders Decomposition

Stochastic market clearing formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},p_{s_1}^{G_2,RT},p_{s_2}^{G_2,RT},p_{s_3}^{G_2,RT},p_{s_4}^{G_2,RT},p_{s_1}^{spill},p_{s_2}^{spill},p_{s_3}^{spill},p_{s_4}^{spill},p_{s_1}^{shed},p_{s_2}^{shed},p_{s_3}^{shed},p_{s_4}^{shed}}$$

$$\left[10p^{G1,DA} + 20p^{G_2,DA}\right] + 0.25\left[20p_{s_1}^{G_2,RT} + 20p_{s_2}^{G_2,RT} + 20p_{s_3}^{G_2,RT} + 20p_{s_4}^{G_2,RT}\right.$$
$$\left. + 80p_{s_1}^{shed} + 80p_{s_2}^{shed} + 80p_{s_3}^{shed} + 80p_{s_4}^{shed}\right]$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA} \leq 100 \\
& 0 \leq p^{G2,DA} \leq 30 \\
& 0 \leq p^{W,DA} \leq 70 \\
& p^{G1,DA} + p^{G2,DA} + p^{W,DA} = 120 \ : \lambda^{DA} \\
& 0 \leq p^{G2,DA} + p_{s_1}^{G2,RT} \leq 30 \\
& 0 \leq p_{s_1}^{spill} \leq 30 \\
& 0 \leq p_{s_1}^{shed} \leq 120 \\
& \quad p_{s_1}^{G2,RT} + \left(30 - p^{W,DA} - p_{s_1}^{spill}\right) + p_{s_1}^{shed} = 0 \ : \lambda_{s_1}^{RT}
\end{aligned}$$

Question:

Is this problem decomposable?

Stochastic Market Clearing – Benders Decomposition

Stochastic market clearing formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},p_{s_1}^{G_2,RT},p_{s_2}^{G_2,RT},p_{s_3}^{G_2,RT},p_{s_4}^{G_2,RT},p_{s_1}^{spill},p_{s_2}^{spill},p_{s_3}^{spill},p_{s_4}^{spill},p_{s_1}^{shed},p_{s_2}^{shed},p_{s_3}^{shed},p_{s_4}^{shed}}$$

$$\left[10p^{G_1,DA} + 20p^{G_2,DA}\right] + 0.25\big[20p_{s_1}^{G_2,RT} + 20p_{s_2}^{G_2,RT} + 20p_{s_3}^{G_2,RT} + 20p_{s_4}^{G_2,RT}$$
$$+ 80p_{s_1}^{shed} + 80p_{s_2}^{shed} + 80p_{s_3}^{shed} + 80p_{s_4}^{shed}\big]$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA} \leq 100 \\
& 0 \leq p^{G2,DA} \leq 30 \\
& 0 \leq p^{W,DA} \leq 70 \\
& p^{G1,DA} + p^{G2,DA} + p^{W,DA} = 120 \ : \lambda^{DA} \\
& 0 \leq p^{G2,DA} + p_{s_1}^{G2,RT} \leq 30 \\
& 0 \leq p_{s_1}^{spill} \leq 30 \\
& 0 \leq p_{s_1}^{shed} \leq 120 \\
& \quad p_{s_1}^{G2,RT} + (30 - p^{W,DA} - p_{s_1}^{spill}) + p_{s_1}^{shed} = 0 \ : \lambda_{s_1}^{RT}
\end{aligned}$$

- Complicating variables: DA dispatch variables
- 4 subproblems (one per scenario)

Stochastic Market Clearing – Benders Decomposition

Subproblems formulation (compact):

$$\text{Minimize}_{p^{G_1,DA(\theta)},p^{G_2,DA(\theta)},p^{W,DA(\theta)},p^{G_2,RT(\theta)}_{s_1},p^{spill(\theta)}_{s_1},p^{shed(\theta)}_{s_1}}$$

$$0.25\left[20p^{G_2,RT(\theta)}_{s_1}+80p^{shed(\theta)}_{s_1}\right]$$

$$\begin{aligned}
\text{s.t.}\, & 0 \leq p^{G2,DA(\theta)} + p^{G2,RT(\theta)}_{s_1} \leq 30 \\
& 0 \leq p^{spill(\theta)}_{s_1} \leq 30 \\
& 0 \leq p^{shed(\theta)}_{s_1} \leq 120 \\
& p^{G2,RT(\theta)}_{s_1} + \left(30 - p^{W,DA(\theta)} - p^{spill(\theta)}_{s_1}\right) + p^{shed(\theta)}_{s_1} = 0 \ : \lambda^{RT}_{s_1} \\
& p^{G1,DA(\theta)} = p^{G1,fixed(\theta)} : \rho^{G1(\theta)}_{s_1} \\
& p^{G2,DA(\theta)} = p^{G2,fixed(\theta)} : \rho^{G2(\theta)}_{s_1} \\
& p^{W,DA(\theta)} = p^{W,fixed(\theta)} : \rho^{W(\theta)}_{s_1}
\end{aligned}$$

Sensitivies of subproblem s_1, with respect to all complicating variables

(Identical formulation for scenarios s_1 to s_4)

Stochastic Market Clearing – Benders Decomposition

Master problem formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},\alpha^{(\theta)}} \left[10p^{G1,DA(\theta)} + 20p^{G_2,DA(\theta)}\right] + \alpha^{(\theta)}$$

$$\text{s.t. } 0 \leq p^{G1,DA(\theta)} \leq 100$$

$$0 \leq p^{G2,DA(\theta)} \leq 30$$

$$0 \leq p^{W,DA(\theta)} \leq 70$$

$$p^{G1,DA(\theta)} + p^{G2,DA(\theta)} + p^{W,DA(\theta)} = 120 \ : \lambda^{DA}$$

Benders cuts?

$$\alpha^{(\theta)} \geq \alpha^{down}$$

Stochastic Market Clearing – Benders Decomposition

Master problem formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},\alpha^{(\theta)}}$$

$$\left[10p^{G1,DA(\theta)} + 20p^{G_2,DA(\theta)}\right] + \alpha^{(\theta)}$$

$$\text{s.t. } 0 \le p^{G1,DA(\theta)} \le 100$$

$$0 \le p^{G2,DA(\theta)} \le 30$$

$$0 \le p^{W,DA(\theta)} \le 70$$

$$p^{G1,DA(\theta)} + p^{G2,DA(\theta)} + p^{W,DA(\theta)} = 120 \;\; : \lambda^{DA}$$

Benders cuts: one per iteration

$$\begin{aligned}\alpha^{(\theta)} \ge 0.25\big[&20p_{s_1}^{G_2,RT(k)} + 20p_{s_2}^{G_2,RT(k)} + 20p_{s_3}^{G_2,RT(k)} + 20p_{s_4}^{G_2,RT(k)} \\ &+ 80p_{s_1}^{shed(k)} + 80p_{s_2}^{shed(k)} + 80p_{s_3}^{shed(k)} + 80p_{s_4}^{shed(k)}\big] \\ &+ (\rho_{s_1}^{G1(k)} + \rho_{s_2}^{G1(k)} + \rho_{s_3}^{G1(k)} + \rho_{s_4}^{G1(k)})(p^{G1,DA(\theta)} - p^{G1,DA(k)}) \\ &+ (\rho_{s_1}^{G2(k)} + \rho_{s_2}^{G2(k)} + \rho_{s_3}^{G2(k)} + \rho_{s_4}^{G2(k)})(p^{G2,DA(\theta)} - p^{G2,DA(k)}) \\ &+ (\rho_{s_1}^{W(k)} + \rho_{s_2}^{W(k)} + \rho_{s_3}^{W(k)} + \rho_{s_4}^{W(k)})(p^{W,DA(\theta)} - p^{W,DA(k)}) \;\; : k = 1, ..., \theta - 1\end{aligned}$$

$$\alpha^{(\theta)} \ge \alpha^{down}$$

Stochastic Market Clearing – Benders Decomposition

Master problem formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},\alpha^{(\theta)}}$$

$$\left[10p^{G1,DA(\theta)} + 20p^{G_2,DA(\theta)}\right] + \alpha^{(\theta)}$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA(\theta)} \leq 100 \\
& 0 \leq p^{G2,DA(\theta)} \leq 30 \\
& 0 \leq p^{W,DA(\theta)} \leq 70 \\
& p^{G1,DA(\theta)} + p^{G2,DA(\theta)} + p^{W,DA(\theta)} = 120 \;\; : \lambda^{DA} \\
& \alpha^{(\theta)} \geq \boxed{\begin{aligned} & 0.25\big[20p_{s_1}^{G_2,RT(k)} + 20p_{s_2}^{G_2,RT(k)} + 20p_{s_3}^{G_2,RT(k)} + 20p_{s_4}^{G_2,RT(k)} \\ & \qquad + 80p_{s_1}^{shed(k)} + 80p_{s_2}^{shed(k)} + 80p_{s_3}^{shed(k)} + 80p_{s_4}^{shed(k)}\big] \end{aligned}} \\
& \qquad + (\rho_{s_1}^{G1(k)} + \rho_{s_2}^{G1(k)} + \rho_{s_3}^{G1(k)} + \rho_{s_4}^{G1(k)})(p^{G1,DA(\theta)} - p^{G1,DA(k)}) \\
& \qquad + (\rho_{s_1}^{G2(k)} + \rho_{s_2}^{G2(k)} + \rho_{s_3}^{G2(k)} + \rho_{s_4}^{G2(k)})(p^{G2,DA(\theta)} - p^{G2,DA(k)}) \\
& \qquad + (\rho_{s_1}^{W(k)} + \rho_{s_2}^{W(k)} + \rho_{s_3}^{W(k)} + \rho_{s_4}^{W(k)})(p^{W,DA(\theta)} - p^{W,DA(k)}) \;\; : k = 1, ..., \theta - 1 \\
& \alpha^{(\theta)} \geq \alpha^{down}
\end{aligned}$$

Objective function of subproblems
(previous iterations

Stochastic Market Clearing – Benders Decomposition

Master problem formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},\alpha^{(\theta)}}$$
$$\left[10p^{G1,DA(\theta)} + 20p^{G_2,DA(\theta)}\right] + \alpha^{(\theta)}$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA(\theta)} \leq 100 \\
& 0 \leq p^{G2,DA(\theta)} \leq 30 \\
& 0 \leq p^{W,DA(\theta)} \leq 70 \\
& p^{G1,DA(\theta)} + p^{G2,DA(\theta)} + p^{W,DA(\theta)} = 120 \ \ : \lambda^{DA} \\
& \alpha^{(\theta)} \geq 0.25\left[20p_{s1}^{G_2,RT(k)} + 20p_{s2}^{G_2,RT(k)} + 20p_{s3}^{G_2,RT(k)} + 20p_{s4}^{G_2,RT(k)} \right. \\
& \qquad \left. + 80p_{s_1}^{shed(k)} + 80p_{s_2}^{shed(k)} + 80p_{s_3}^{shed(k)} + 80p_{s_4}^{shed(k)}\right] \\
& \qquad \boxed{+ (\rho_{s_1}^{G1(k)} + \rho_{s_2}^{G1(k)} + \rho_{s_3}^{G1(k)} + \rho_{s_4}^{G1(k)})(p^{G1,DA(\theta)} - p^{G1,DA(k)})} \\
& \qquad + (\rho_{s_1}^{G2(k)} + \rho_{s_2}^{G2(k)} + \rho_{s_3}^{G2(k)} + \rho_{s_4}^{G2(k)})(p^{G2,DA(\theta)} - p^{G2,DA(k)}) \\
& \qquad + (\rho_{s_1}^{W(k)} + \rho_{s_2}^{W(k)} + \rho_{s_3}^{W(k)} + \rho_{s_4}^{W(k)})(p^{W,DA(\theta)} - p^{W,DA(k)}) \ \ : k = 1, ..., \theta - 1 \\
& \alpha^{(\theta)} \geq \alpha^{down}
\end{aligned}$$

Sensitivies of subproblems, with respect to complicating variable $p^{G1,DA}$

Stochastic Market Clearing – Benders Decomposition

Master problem formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},\alpha^{(\theta)}}$$

$$\left[10p^{G1,DA(\theta)} + 20p^{G_2,DA(\theta)}\right] + \alpha^{(\theta)}$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA(\theta)} \leq 100 \\
& 0 \leq p^{G2,DA(\theta)} \leq 30 \\
& 0 \leq p^{W,DA(\theta)} \leq 70 \\
& p^{G1,DA(\theta)} + p^{G2,DA(\theta)} + p^{W,DA(\theta)} = 120 \ \ : \lambda^{DA} \\
& \alpha^{(\theta)} \geq 0.25\big[20p_{s1}^{G_2,RT(k)} + 20p_{s2}^{G_2,RT(k)} + 20p_{s3}^{G_2,RT(k)} + 20p_{s4}^{G_2,RT(k)} \\
& \qquad + 80p_{s_1}^{shed(k)} + 80p_{s_2}^{shed(k)} + 80p_{s_3}^{shed(k)} + 80p_{s_4}^{shed(k)}\big] \\
& \qquad + (\rho_{s_1}^{G1(k)} + \rho_{s_2}^{G1(k)} + \rho_{s_3}^{G1(k)} + \rho_{s_4}^{G1(k)})(p^{G1,DA(\theta)} - p^{G1,DA(k)}) \\
& \qquad \boxed{+ (\rho_{s_1}^{G2(k)} + \rho_{s_2}^{G2(k)} + \rho_{s_3}^{G2(k)} + \rho_{s_4}^{G2(k)})(p^{G2,DA(\theta)} - p^{G2,DA(k)})} \\
& \qquad + (\rho_{s_1}^{W(k)} + \rho_{s_2}^{W(k)} + \rho_{s_3}^{W(k)} + \rho_{s_4}^{W(k)})(p^{W,DA(\theta)} - p^{W,DA(k)}) \ \ : k = 1, ..., \theta - 1 \\
& \alpha^{(\theta)} \geq \alpha^{down}
\end{aligned}$$

Sensitivies of subproblems, with respect to complicating variable $p^{G2,DA}$

Stochastic Market Clearing – Benders Decomposition

Master problem formulation:

$$\text{Minimize}_{p^{G_1,DA},p^{G_2,DA},p^{W,DA},\alpha^{(\theta)}}$$

$$\left[10p^{G1,DA(\theta)} + 20p^{G_2,DA(\theta)}\right] + \alpha^{(\theta)}$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p^{G1,DA(\theta)} \leq 100 \\
& 0 \leq p^{G2,DA(\theta)} \leq 30 \\
& 0 \leq p^{W,DA(\theta)} \leq 70 \\
& p^{G1,DA(\theta)} + p^{G2,DA(\theta)} + p^{W,DA(\theta)} = 120 \quad : \lambda^{DA} \\
& \alpha^{(\theta)} \geq 0.25\left[20p_{s1}^{G_2,RT(k)} + 20p_{s2}^{G_2,RT(k)} + 20p_{s3}^{G_2,RT(k)} + 20p_{s4}^{G_2,RT(k)}\right. \\
& \qquad \left. + 80p_{s_1}^{shed(k)} + 80p_{s_2}^{shed(k)} + 80p_{s_3}^{shed(k)} + 80p_{s_4}^{shed(k)}\right] \\
& \qquad + (\rho_{s_1}^{G1(k)} + \rho_{s_2}^{G1(k)} + \rho_{s_3}^{G1(k)} + \rho_{s_4}^{G1(k)})(p^{G1,DA(\theta)} - p^{G1,DA(k)}) \\
& \qquad + (\rho_{s_1}^{G2(k)} + \rho_{s_2}^{G2(k)} + \rho_{s_3}^{G2(k)} + \rho_{s_4}^{G2(k)})(p^{G2,DA(\theta)} - p^{G2,DA(k)}) \\
& \qquad \boxed{+ (\rho_{s_1}^{W(k)} + \rho_{s_2}^{W(k)} + \rho_{s_3}^{W(k)} + \rho_{s_4}^{W(k)})(p^{W,DA(\theta)} - p^{W,DA(k)})} : k = 1, ..., \theta - 1 \\
& \alpha^{(\theta)} \geq \alpha^{down}
\end{aligned}$$

Sensitivies of subproblems, with respect to complicating variable $p^{W,DA}$

Stochastic Market Clearing – Benders Decomposition

For this afternoon:

- Solve this stochastic market clearing using a Benders decomposition algorithm (compact and non-compact)
- Compare the results and computational time with the non-decomposed problem
- Think about the extension for a multi time step market clearing: How can it be decomposed? (# of complicating variables and subproblems)
- Think about the extension for a Unit Commitment problem (binaries in DA or RT dispatch): Can you apply Benders decomposition? Is convergence guaranteed?

Stochastic Market Clearing – Results

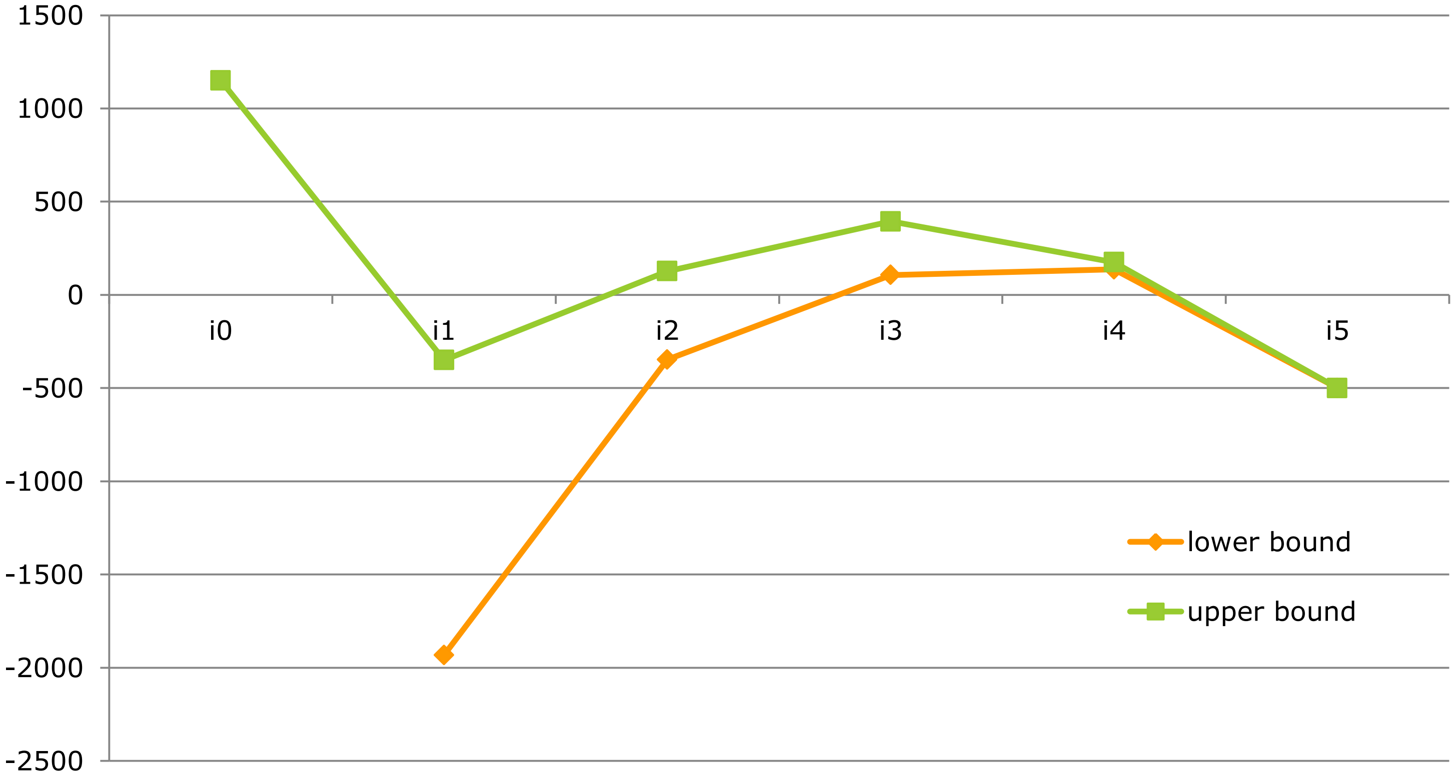
Results:

	DA	RT (w1)	RT (w2)	RT (w3)	RT (w4)
G1 [MW]	90	-	-	-	-
G2 [MW]	30	-30	-30	-30	-10
WP [MW]	0	+30	+60 (30 spilled)	+70 (40 spilled)	+10
Load [MW]	120	-	-	-	-
Market price [$/MWh]	10	20	0	0	20
System cost [$]	1500	-600	-600	-600	-200

- Total expected system cost = 1500 – 0.25*[600+600+600+200] = $1000

Stochastic Market Clearing – Results

Results: Benders algorithm convergence



$$\begin{aligned} UB^{(k)} = 0.25\big[&20p_{s_1}^{G_2,RT(k)} + 20p_{s_2}^{G_2,RT(k)} + 20p_{s_3}^{G_2,RT(k)} + 20p_{s_4}^{G_2,RT(k)} \\ &+ 80p_{s_1}^{shed(k)} + 80p_{s_2}^{shed(k)} + 80p_{s_3}^{shed(k)} + 80p_{s_4}^{shed(k)}\big] \\ LB^{(k)} = \alpha^{(k)}& \end{aligned}$$

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA},p_t^{G_2,DA},p_t^{W,DA},p_{s_1,t}^{G_2,RT},p_{s_2,t}^{G_2,RT},p_{s_3,t}^{G_2,RT},p_{s_4,t}^{G_2,RT},p_{s_1,t}^{spill},p_{s_2,t}^{spill},p_{s_3,t}^{spill},p_{s_4,t}^{spill},p_{s_1,t}^{shed},p_{s_2,t}^{shed},p_{s_3,t}^{shed},p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$

$$+ 80p_{s_1,t}^{shed} + 80p_{s_2,t}^{shed} + 80p_{s_3,t}^{shed} + 80p_{s_4,t}^{shed}\big]$$

$$\begin{aligned}
\text{s.t. } & 0 \leq p_t^{G1,DA} \leq 100,\ \ t = t_1, \ldots, t_{24} \\
& 0 \leq p_t^{G2,DA} \leq 30,\ \ t = t_1, \ldots, t_{24} \\
& 0 \leq p_t^{W,DA} \leq 70,\ \ t = t_1, \ldots, t_{24} \\
& p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \ \ : \lambda_t^{DA},\ \ t = t_1, \ldots, t_{24} \\
& 0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G_2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30 \\
& 0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real} \\
& 0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1} \\
& \quad p_{s_1,t_1}^{G_2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \ \ : \lambda_{s_1,t_1}^{RT}
\end{aligned}$$

Extension: Multi Time Step Market

DA dispatch (first stage decision variables), indexed by time step

$$\text{Minimize}_{\boxed{p_t^{G_1,DA},p_t^{G_2,DA},p_t^{W,DA}},p_{s_1,t}^{G_2,RT},p_{s_2,t}^{G_2,RT},p_{s_3,t}^{G_2,RT},p_{s_4,t}^{G_2,RT},p_{s_1,t}^{spill},p_{s_2,t}^{spill},p_{s_3,t}^{spill},p_{s_4,t}^{spill},p_{s_1,t}^{shed},p_{s_2,t}^{shed},p_{s_3,t}^{shed},p_{s_4,t}^{shed}}$$

$$\boxed{\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right]} + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$

$$+ 80p_{s_1,t}^{shed} + 80p_{s_2,t}^{shed} + 80p_{s_3,t}^{shed} + 80p_{s_4,t}^{shed}\big]$$

DA dispatch cost

$$\text{s.t.}\ \boxed{\begin{aligned} &0 \leq p_t^{G1,DA} \leq 100,\ \ t = t_1, ..., t_{24} \\ &0 \leq p_t^{G2,DA} \leq 30,\ \ t = t_1, ..., t_{24} \\ &0 \leq p_t^{W,DA} \leq 70,\ \ t = t_1, ..., t_{24} \\ &p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \ \ : \lambda_t^{DA},\ \ t = t_1, ..., t_{24} \end{aligned}}$$

DA dispatch constraints, for all time steps

$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G_2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$

$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$

$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$

$$p_{s_1,t_1}^{G_2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \ \ : \lambda_{s_1,t_1}^{RT}$$

Extension: Multi Time Step Market

RT redispatch per scenario and time step (second stage decision variables)

$$\text{Minimize}_{p_t^{G_1,DA},p_t^{G_2,DA},p_t^{W,DA},p_{s_1,t}^{G_2,RT},p_{s_2,t}^{G_2,RT},p_{s_3,t}^{G_2,RT},p_{s_4,t}^{G_2,RT},p_{s_1,t}^{spill},p_{s_2,t}^{spill},p_{s_3,t}^{spill},p_{s_4,t}^{spill},p_{s_1,t}^{shed},p_{s_2,t}^{shed},p_{s_3,t}^{shed},p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$
$$+ 80p_{s_1,t}^{shed} + 80p_{s_2,t}^{shed} + 80p_{s_3,t}^{shed} + 80p_{s_4,t}^{shed}\big]$$

Expected RT redispatch cost

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100, \;\; t = t_1, ..., t_{24}$$
$$0 \leq p_t^{G2,DA} \leq 30, \;\; t = t_1, ..., t_{24}$$
$$0 \leq p_t^{W,DA} \leq 70, \;\; t = t_1, ..., t_{24}$$
$$p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \;\; : \lambda_t^{DA}, \;\; t = t_1, ..., t_{24}$$
$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G_2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$
$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$
$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$
$$p_{s_1,t_1}^{G_2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \;\; : \lambda_{s_1,t_1}^{RT}$$

RT dispatch constraints for scenario s_1 and time step t_1

(identical set of constraints for each scenario and time step)

Extension: Multi Time Step Market

RT redispatch per scenario and time step (second stage decision variables)

$$\text{Minimize}_{p_t^{G_1,DA}, p_t^{G_2,DA}, p_t^{W,DA}, p_{s_1,t}^{G_2,RT}, p_{s_2,t}^{G_2,RT}, p_{s_3,t}^{G_2,RT}, p_{s_4,t}^{G_2,RT}, p_{s_1,t}^{spill}, p_{s_2,t}^{spill}, p_{s_3,t}^{spill}, p_{s_4,t}^{spill}, p_{s_1,t}^{shed}, p_{s_2,t}^{shed}, p_{s_3,t}^{shed}, p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$
$$+ 80p_{s_1,t}^{shed} + 80p_{s_2,t}^{shed} + 80p_{s_3,t}^{shed} + 80p_{s_4,t}^{shed}\big]$$

Expected RT redispatch cost

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100, \ \ t = t_1, ..., t_{24}$$
$$0 \leq p_t^{G2,DA} \leq 30, \ \ t = t_1, ..., t_{24}$$
$$0 \leq p_t^{W,DA} \leq 70, \ \ t = t_1, ..., t_{24}$$
$$p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \ : \lambda_t^{DA}, \ \ t = t_1, ..., t_{24}$$

$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G_2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$
$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$
$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$
$$p_{s_1,t_1}^{G_2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \ : \lambda_{s_1,t_1}^{RT}$$

RT dispatch constraints for scenario s_1 and time step t_1

(identical set of constraints for each scenario and time step)

Question: How can this problem be decomposed?

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA}, p_t^{G_2,DA}, p_t^{W,DA}, p_{s_1,t}^{G_2,RT}, p_{s_2,t}^{G_2,RT}, p_{s_3,t}^{G_2,RT}, p_{s_4,t}^{G_2,RT}, p_{s_1,t}^{spill}, p_{s_2,t}^{spill}, p_{s_3,t}^{spill}, p_{s_4,t}^{spill}, p_{s_1,t}^{shed}, p_{s_2,t}^{shed}, p_{s_3,t}^{shed}, p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10 p_t^{G_1,DA} + 20 p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20 p_{s_1,t}^{G_2,RT} + 20 p_{s_2,t}^{G_2,RT} + 20 p_{s_3,t}^{G_2,RT} + 20 p_{s_4,t}^{G_2,RT}$$
$$+ 80 p_{s_1,t}^{shed} + 80 p_{s_2,t}^{shed} + 80 p_{s_3,t}^{shed} + 80 p_{s_4,t}^{shed}\big]$$

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100, \ t = t_1, ..., t_{24}$$
$$0 \leq p_t^{G2,DA} \leq 30, \ t = t_1, ..., t_{24}$$
$$0 \leq p_t^{W,DA} \leq 70, \ t = t_1, ..., t_{24}$$
$$p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \ : \lambda_t^{DA}, \ t = t_1, ..., t_{24}$$
$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G_2,RT} \leq 30, \quad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$
$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$
$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$
$$p_{s_1,t_1}^{G_2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \ : \lambda_{s_1,t_1}^{RT}$$

Question: How can this problem be decomposed?

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA},p_t^{G_2,DA},p_t^{W,DA},p_{s_1,t}^{G_2,RT},p_{s_2,t}^{G_2,RT},p_{s_3,t}^{G_2,RT},p_{s_4,t}^{G_2,RT},p_{s_1,t}^{spill},p_{s_2,t}^{spill},p_{s_3,t}^{spill},p_{s_4,t}^{spill},p_{s_1,t}^{shed},p_{s_2,t}^{shed},p_{s_3,t}^{shed},p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$
$$+ 80p_{s_1,t}^{shed} + 80p_{s_2,t}^{shed} + 80p_{s_3,t}^{shed} + 80p_{s_4,t}^{shed}\big]$$

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100,\ \ t = t_1, ..., t_{24}$$
$$0 \leq p_t^{G2,DA} \leq 30,\ \ t = t_1, ..., t_{24}$$
$$0 \leq p_t^{W,DA} \leq 70,\ \ t = t_1, ..., t_{24}$$
$$p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t\ \ : \lambda_t^{DA},\ \ t = t_1, ..., t_{24}$$
$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$
$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$
$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$
$$p_{s_1,t_1}^{G2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0\ \ : \lambda_{s_1,t_1}^{RT}$$

- Complicating variables: day-ahead dispatch variables for all time steps
- One subproblem per time step and scenario

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA}, p_t^{G_2,DA}, p_t^{W,DA}, p_{s_1,t}^{G_2,RT}, p_{s_2,t}^{G_2,RT}, p_{s_3,t}^{G_2,RT}, p_{s_4,t}^{G_2,RT}, p_{s_1,t}^{spill}, p_{s_2,t}^{spill}, p_{s_3,t}^{spill}, p_{s_4,t}^{spill}, p_{s_1,t}^{shed}, p_{s_2,t}^{shed}, p_{s_3,t}^{shed}, p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10 p_t^{G_1,DA} + 20 p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25 \big[20 p_{s_1,t}^{G_2,RT} + 20 p_{s_2,t}^{G_2,RT} + 20 p_{s_3,t}^{G_2,RT} + 20 p_{s_4,t}^{G_2,RT}$$

$$+ 80 p_{s_1,t}^{shed} + 80 p_{s_2,t}^{shed} + 80 p_{s_3,t}^{shed} + 80 p_{s_4,t}^{shed}\big]$$

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100, \;\; t = t_1, ..., t_{24}$$

$$0 \leq p_t^{G2,DA} \leq 30, \;\; t = t_1, ..., t_{24}$$

$$0 \leq p_t^{W,DA} \leq 70, \;\; t = t_1, ..., t_{24}$$

$$p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \;\; : \lambda_t^{DA}, \;\; t = t_1, ..., t_{24}$$

$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$

$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$

$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$

$$p_{s_1,t_1}^{G2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \;\; : \lambda_{s_1,t_1}^{RT}$$

Question: What happens if we add ramping constraints in the DA or RT dispatch?

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA}, p_t^{G_2,DA}, p_t^{W,DA}, p_{s_1,t}^{G_2,RT}, p_{s_2,t}^{G_2,RT}, p_{s_3,t}^{G_2,RT}, p_{s_4,t}^{G_2,RT}, p_{s_1,t}^{spill}, p_{s_2,t}^{spill}, p_{s_3,t}^{spill}, p_{s_4,t}^{spill}, p_{s_1,t}^{shed}, p_{s_2,t}^{shed}, p_{s_3,t}^{shed}, p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10 p_t^{G_1,DA} + 20 p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25 \big[20 p_{s_1,t}^{G_2,RT} + 20 p_{s_2,t}^{G_2,RT} + 20 p_{s_3,t}^{G_2,RT} + 20 p_{s_4,t}^{G_2,RT}$$

$$+ 80 p_{s_1,t}^{shed} + 80 p_{s_2,t}^{shed} + 80 p_{s_3,t}^{shed} + 80 p_{s_4,t}^{shed}\big]$$

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100, \; t = t_1, ..., t_{24}$$

$$0 \leq p_t^{G2,DA} \leq 30, \; t = t_1, ..., t_{24}$$

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$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$

$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$

$$0 \leq p_{s_1,t_1}^{shed} \leq L_{t_1}$$

$$p_{s_1,t_1}^{G2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \; : \lambda_{s_1,t_1}^{RT}$$

$$\boxed{\begin{aligned} &-R^{G1,D} \leq p_{t+1}^{G1,DA} - p_t^{G1,DA} \leq R^{G1,U}, \; t = t_1, ..., t_{23} \\ &-R^{G2,D} \leq p_{t+1}^{G2,DA} - p_t^{G2,DA} \leq R^{G2,U}, \; t = t_1, ..., t_{23} \end{aligned}}$$

DA ramping constraints
(inter-temporal)

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA}, p_t^{G_2,DA}, p_t^{W,DA}, p_{s_1,t}^{G_2,RT}, p_{s_2,t}^{G_2,RT}, p_{s_3,t}^{G_2,RT}, p_{s_4,t}^{G_2,RT}, p_{s_1,t}^{spill}, p_{s_2,t}^{spill}, p_{s_3,t}^{spill}, p_{s_4,t}^{spill}, p_{s_1,t}^{shed}, p_{s_2,t}^{shed}, p_{s_3,t}^{shed}, p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$

$$+ 80p_{s_1,t}^{shed} + 80p_{s_2,t}^{shed} + 80p_{s_3,t}^{shed} + 80p_{s_4,t}^{shed}\big]$$

$$\text{s.t. } 0 \leq p_t^{G1,DA} \leq 100, \; t = t_1, ..., t_{24}$$

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$$p_t^{G1,DA} + p_t^{G2,DA} + p_t^{W,DA} = L_t \; : \lambda_t^{DA}, \; t = t_1, ..., t_{24}$$

$$0 \leq p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT} \leq 30, \qquad -30 \leq p_{s_1,t_1}^{G2,RT} \leq 30$$

$$0 \leq p_{s_1,t_1}^{spill} \leq W_{s_1,t_1}^{real}$$

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$$p_{s_1,t_1}^{G2,RT} + \left(W_{s_1,t_1}^{real} - p_{t_1}^{W,DA} - p_{s_1,t_1}^{spill}\right) + p_{s_1,t_1}^{shed} = 0 \; : \lambda_{s_1,t_1}^{RT}$$

$$\boxed{\begin{aligned} -R^{G1,D} &\leq p_{t+1}^{G1,DA} - p_t^{G1,DA} \leq R^{G1,U}, \; t = t_1, ..., t_{23} \\ -R^{G2,D} &\leq p_{t+1}^{G2,DA} - p_t^{G2,DA} \leq R^{G2,U}, \; t = t_1, ..., t_{23} \end{aligned}}$$

DA ramping constraints
(inter-temporal)

- ➢ The problem is still decomposable per scenario and per time step

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA}, p_t^{G_2,DA}, p_t^{W,DA}, p_{s_1,t}^{G_2,RT}, p_{s_2,t}^{G_2,RT}, p_{s_3,t}^{G_2,RT}, p_{s_4,t}^{G_2,RT}, p_{s_1,t}^{spill}, p_{s_2,t}^{spill}, p_{s_3,t}^{spill}, p_{s_4,t}^{spill}, p_{s_1,t}^{shed}, p_{s_2,t}^{shed}, p_{s_3,t}^{shed}, p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$

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DA ramping constraints
(inter-temporal)

$$-R^{G2,D} \leq (p_{t_1+1}^{G2,DA} + p_{s_1,t_1+1}^{G2,RT}) - (p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT})$$

$$(p_{t_1+1}^{G2,DA} + p_{s_1,t_1+1}^{G2,RT}) - (p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT}) \leq R^{G2,U}$$

RT ramping constraints
(inter-temporal)

Extension: Multi Time Step Market

$$\text{Minimize}_{p_t^{G_1,DA},p_t^{G_2,DA},p_t^{W,DA},p_{s_1,t}^{G_2,RT},p_{s_2,t}^{G_2,RT},p_{s_3,t}^{G_2,RT},p_{s_4,t}^{G_2,RT},p_{s_1,t}^{spill},p_{s_2,t}^{spill},p_{s_3,t}^{spill},p_{s_4,t}^{spill},p_{s_1,t}^{shed},p_{s_2,t}^{shed},p_{s_3,t}^{shed},p_{s_4,t}^{shed}}$$

$$\sum_{t=t_1}^{t_{24}} \left[10p_t^{G_1,DA} + 20p_t^{G_2,DA}\right] + \sum_{t=t_1}^{t_{24}} 0.25\big[20p_{s_1,t}^{G_2,RT} + 20p_{s_2,t}^{G_2,RT} + 20p_{s_3,t}^{G_2,RT} + 20p_{s_4,t}^{G_2,RT}$$

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DA ramping constraints
(inter-temporal)

$$-R^{G2,D} \leq (p_{t_1+1}^{G2,DA} + p_{s_1,t_1+1}^{G2,RT}) - (p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT})$$

$$(p_{t_1+1}^{G2,DA} + p_{s_1,t_1+1}^{G2,RT}) - (p_{t_1}^{G2,DA} + p_{s_1,t_1}^{G2,RT}) \leq R^{G2,U}$$

RT ramping constraints
(inter-temporal)

- The problem is only decomposable per scenario

Extension: Unit-Commitment Problem

- How can you formulate the Benders algorithm for a unit commitment problem (binary variables in the day-ahead dispatch)?

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  - Careful: In the compact form the master problem variables are still variables of the subproblems… BUT there are not binaries in the subproblems
  - However, convergence is not guaranteed: non-convexity with respect to the complicating variables

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- What about binary variables in the real time dispatch?
    - Can not generate Benders cuts directly (sensitivities)
    - Alternatives: binaries as complicating variables (large number?), primal-cut Benders

Stochastic Bilevel Optimization Problem: Strategic Offering

Upper-level: strategic producer

Maximize **Expected** profit $(x,y_{s1},\ldots,y_{sN},\lambda_{s1},\ldots,\lambda_{sN})$

Subject to:

market clearing (scenario 1)

Maximize social welfare (x,y_{s1})

Subject to:

- Balance equations at nodes $(y_{s1}) : \lambda_{s1}$
- Transmission constraints $(y_{s1}) : \sigma_{s1}$
- Production bounds $(x,y_{s1}) : \mu_{s1}$

…

market clearing (scenario N)

Maximize social welfare (x,y_{sN})

Subject to:

- Balance equations at nodes $(y_{sN}) : \lambda_{sN}$
- Transmission constraints $(y_{sN}) : \sigma_{sN}$
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Stochastic Bilevel Optimization Problem: Strategic Offering

Upper-level: strategic producer

Maximize **Expected** profit $(x, y_{s1}, \ldots, y_{sN}, \lambda_{s1}, \ldots, \lambda_{sN})$

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KKT conditions

$(x,\ y_{s1}\ ,\ \lambda_{s1},\ \sigma_{s1},\ \mu_{s1})$

…

market clearing (scenario N)

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Fortuny-Amat linearization

$(x,\ y_{s1}\ ,\ \lambda_{s1},\ \sigma_{s1},\ \mu_{s1}\ ,\ u_{s1})$

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market clearing (scenario N)

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…

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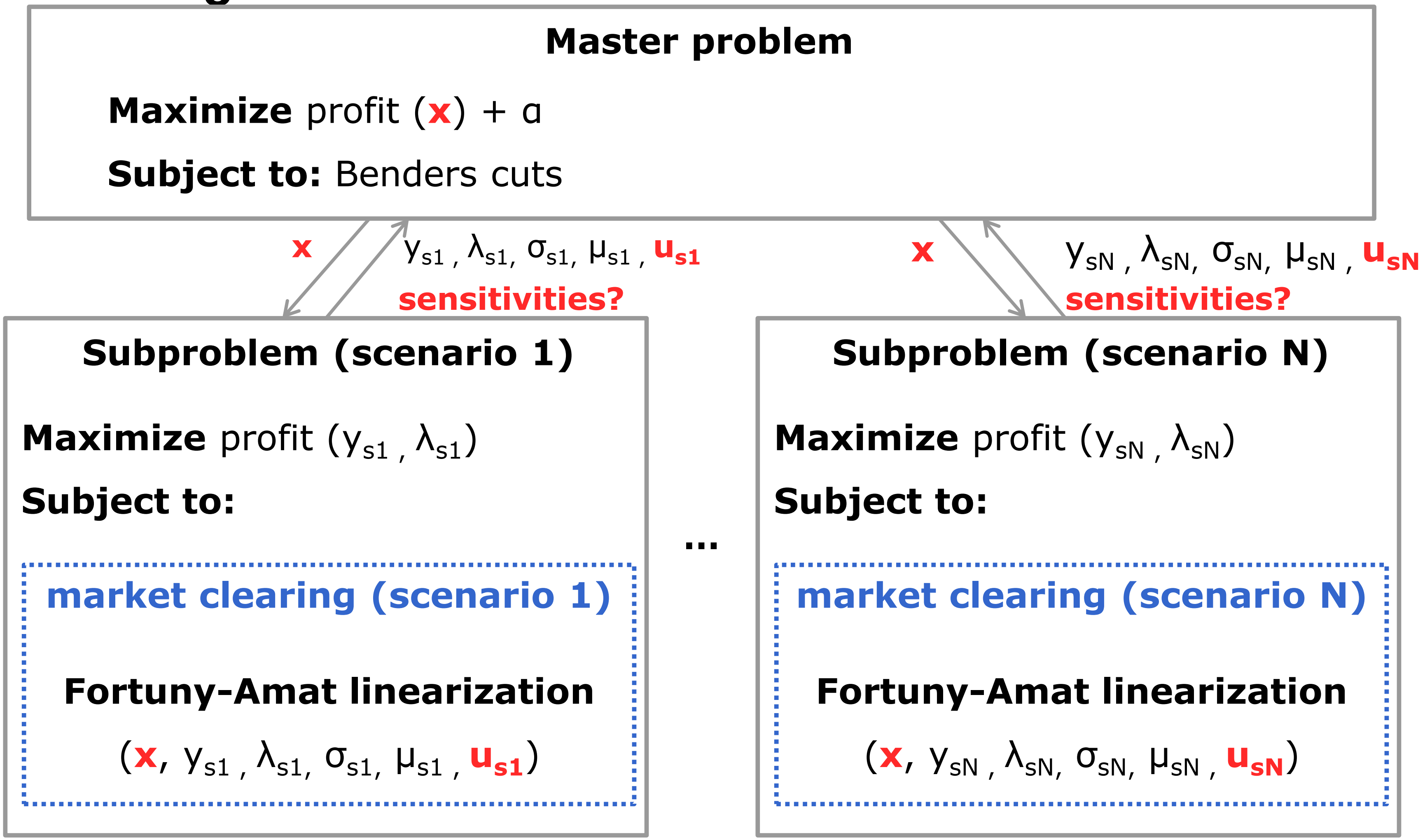
x (upper-level & here-and-now decision variable) is a complicating variable

Stochastic Bilevel Optimization Problem: Strategic Offering



Master problem

Maximize profit (x) + α

Subject to: Benders cuts

x

y_{s1} , λ_{s1}, σ_{s1}, μ_{s1} , u_{s1}

sensitivities?

x

y_{sN} , λ_{sN}, σ_{sN}, μ_{sN} , u_{sN}

sensitivities?

Subproblem (scenario 1)

Maximize profit (y_{s1} , λ_{s1})

Subject to:

market clearing (scenario 1)

Fortuny-Amat linearization

(x, y_{s1} , λ_{s1}, σ_{s1}, μ_{s1} , u_{s1})

…

Subproblem (scenario N)

Maximize profit (y_{sN} , λ_{sN})

Subject to:

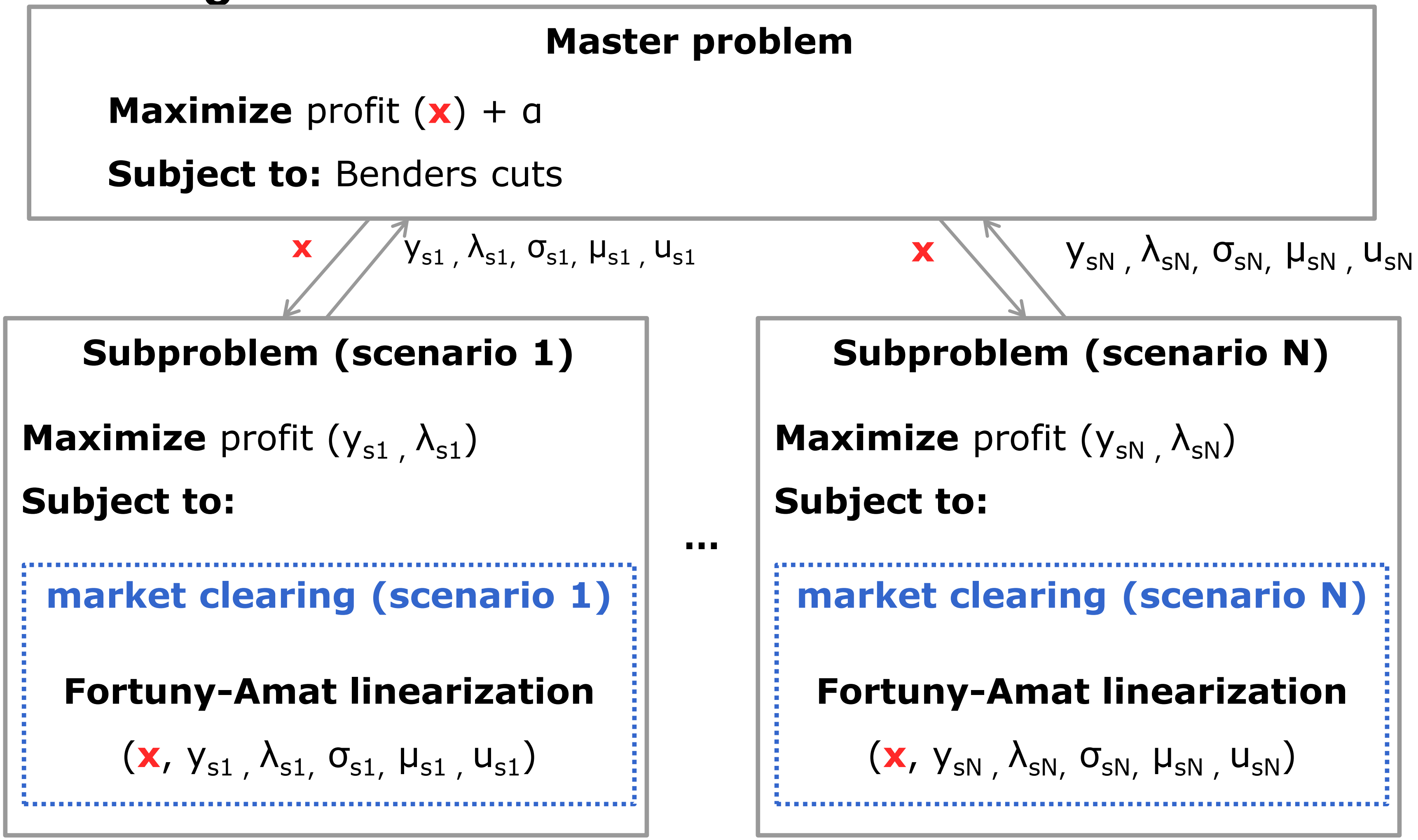
market clearing (scenario N)

Fortuny-Amat linearization

(x, y_{sN} , λ_{sN}, σ_{sN}, μ_{sN} , u_{sN})

Binary variables (Fortuny-Amat)! Alternatives?

Stochastic Bilevel Optimization Problem: Strategic Offering



Master problem

Maximize profit (x) + α

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x

y_{s1} , λ_{s1}, σ_{s1}, μ_{s1} , u_{s1}

x

y_{sN} , λ_{sN}, σ_{sN}, μ_{sN} , u_{sN}

Subproblem (scenario 1)

Maximize profit (y_{s1} , λ_{s1})

Subject to:

market clearing (scenario 1)

Fortuny-Amat linearization

(x, y_{s1} , λ_{s1}, σ_{s1}, μ_{s1} , u_{s1})

…

Subproblem (scenario N)

Maximize profit (y_{sN} , λ_{sN})

Subject to:

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Fortuny-Amat linearization

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Thank you for your attention!