Chapter 2 Notes

Overview

$$\begin{cases} Convergence\ for\ sequence\ elements \\ Continuity\ for\ function \\ Continuity\ relates\ with\ Convergence \end{cases}$$

2.1. Convergence in metric space

> Definition 2.1.1. — Convergence: ϵ language
>
> $(X_n)_{n\in\mathbb{N}}$ converges to $l \in \mathbb{R}$ if $\forall\epsilon > 0$,$\exists N \in \mathbb{N}$, s.t. $|l - x_n| < \epsilon\ \forall n > \mathbb{N}$

> Definition 2.1.3.
>
> A sequence of elements of a set X is a function $x:\ \mathbb{N} \to X$ wrote as $x = (x_n)$

> Definition 2.1.4. — Convergence: metric language
>
> (X_n) converges to l in X if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, l) < \epsilon\ \forall n > \mathbb{N}$

> Definition 2.1.7. — Ball
>
> Let (X, d) be a metric space, $p \in X$, $R \in \mathbb{R}_{>0}$
>
> Open ball:
>
> $$B_R^d(p) = \{x \in X:\ d(x,p) < R\}$$
>
> Closed ball:
>
> $$\bar{B}_R^d(p) = \{x \in X:\ d(x,p) \leq R\}$$
>
> Tips: $x \in X$ in both open ball and closed ball

> Definition 2.1.13. — Convergence: Ball language
>
> (X_n) converges to $l \in X$ if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $x_n \in B_\epsilon^d(l)\ \forall N > \mathbb{N}$

> Definition 2.1.15
>
> A sequence (x_n) in X is *eventually constant* if $\exists l \in X$ and $\exists N \in \mathbb{N}$ s.t. $x_n = l\ \forall n > N$

2.2. Continuity in metric spaces

> Definition 2.2.1. — Continuity: $\epsilon - \delta$ language
>
> Let $f:\ \mathbb{R} \to \mathbb{R}$. f is continuous at x_0 if $\forall\epsilon > 0\ \exists\delta > 0$ s.t.

$$|f(x) - f(x_0)| = d_1(f(x), f(x_0)) < \epsilon$$

Whenever $|x - x_0| = d_1(x, x_0) < \delta$

We say that f is continuous if f is continuous if f is continuous at every $x_0 \in \mathbb{R}$

Using lim,

If $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$|f(x) - l| = d_1(f(x), l) < \epsilon$$

Whenever $0 < |x - x_0| < \delta$, we say that

$$\lim_{x \to x_0} f(x) = l$$

Now extend these definitions to the general case of functions between metric spaces

Definition 2.2.2. — Continuity: functions between metric spaces

Let (X, d_X), (Y, d_Y) be metric spaces. Let $f: X \to Y$ be a function, and let $x_0 \in X$. Then we say that f is continuous at x_0 if $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$d_Y(f(x), f(x_0)) < \epsilon$$

whenever $d_X(x, x_0) < \delta$. We say that f is continuous if it is continuous $\forall x_0 \in X$

Similarly using lim,

If $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$d_Y(f(x), l) < \epsilon$$

whenever $0 < d_X(x, x_0) < \delta$, we say that

$$\lim_{x \to x_0} f(x) = l$$

f is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$

Remark 2.2.3. — Direct images and Inverse images of a function notation

We have $f: X \to Y$ and $f(x) \in Y$

Define that for a subset $A \subseteq X$, its direct image (像集):

$$f(A) := \{y \in Y : \exists x \in A \quad s.t. \; f(x) = y\}$$

Similarly for a subset $B \subseteq Y$ we define its inverse image/preimage (原像集):

$$f^{-1}(B) := \{x \in X : \quad s.t. \; f(x) \in B\}$$

由此我们可以用ball语言来重写连续性的定义

f is continuous at $x_0 \iff \forall \epsilon > 0 \; \exists \delta > 0$ s.t. $f(B_\delta^{d_X}(x_0)) \subseteq B_\epsilon^{d_Y}(f(x_0))$

Where the latter is $d_x(X, X_0) < \delta \to d_Y(f(x), f(x_0)) < \epsilon$

Corollary 2.2.6.

If f and g are both continuous, then so is $g \circ f$

Tips: 先f再g

Recast continuity in terms of convergence(class test 1考过)

Lemma 2.2.7.

Let (X, d_X) and (Y, d_Y) be metric spaces, $f: \ X \to Y$ a function and $p \in X$. The following are equivalent

$$\begin{cases} f \ is \ continuous \ at \ p \\ \forall (x_n) \subseteq X \ s.t. \ x_n \xrightarrow{d_X} p, \ f(x_n) \xrightarrow{d_Y} f(p) \end{cases}$$

Remark: 这将收敛性与连续性联系了起来

Remark 2.2.8.

If f is continuous at p, by the above we have

$$f(\underset{n}{lim} \ x_n) = \underset{n}{lim} f(x_n)$$