Chapter 1 Notes

1.1. Starting out

> 4 axioms for metric:
>
> (M0): non-negative
>
> (M1): $\forall x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
>
> (M2): Symmetry
>
> (M3): Triangle inequality: $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$

> Lemma 1.1.6
>
> $\max(a+b, c+d) \leq \max(a, c) + \max(b, d)$

> Cauchy-Schwarz Inequality
>
> $\forall a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$, the following inequality holds:
>
> $$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2$$

> d_∞ for $\mathbb{R}^n$:
>
> $$d_\infty(x, y) = \underset{1 \leq i \leq n}{max} |x_i - y_i|$$
>
> d_p for $\mathbb{R}$:
>
> $$d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$$

> *discrete distance*
>
> $$d_{discr}(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

1.2. Subspace distance

> We can just measure distances on the subset using the global distance defined on the larger metric space

1.3. Spaces of real sequences

To extend our definition of distance d_p to spaces whose elements are sequences,

$$d_p(A,B) = (\sum_{n=0}^{\infty}|A_n - B_n|^p)^{\frac{1}{p}} = \lim_{N\to\infty}(\sum_{n=0}^{N}|A_n - B_n|^p)^{\frac{1}{p}}$$

and

$$d_\infty((A_n),(B_n)) = \sup_{n\in\mathbb{N}}|A_n - B_n|$$

By definition, a distance is required to be a non-negative real number, which $+\infty$ is not.

Thus, we define l^p and l^∞ sequence space:

$$l^p = \{(A_n)|\ \sum_{n=0}^{\infty}|A_n|^p < \infty\}$$

$$l^\infty = \{(A_n)|\ (A_n)\ is\ bounded\}$$

We say (A_n) is bounded if $\exists M \in \mathbb{R}$ s.t. $|A_n| \le M,\ \forall n \in \mathbb{N}$

Theorem 1.3.4.

$\forall p \ge 1$ (and also for $p = \infty$) the function d_p defines a distance on the set l^p

Proposition 1.3.5.

For $p \le q$, the inclusion $l^p \subseteq l^q$ holds (and this inclusion also holds when $q = \infty$)

Corollary 1.3.6.

(l^p, d_q) is a metric space (a metric subspace of (l^q, d_q)) whenever $p \le q$

Theorem 1.3.10 — 闵可夫斯基不等式

$$(\sum_{n=0}^{\infty}|A_n + B_n|^p)^{\frac{1}{p}} \le (\sum_{n=0}^{\infty}|A_n|^p)^{\frac{1}{p}} + (\sum_{n=0}^{\infty}|B_n|^p)^{\frac{1}{p}} \qquad (1)$$

where $p \ge 1$

Tips: when p=2, (1) is Cauchy-Schwarz inequality

1.4. Spaces of functions

Definition 1.4.1

$$C[0,1] := \{f : [0,1] \to \mathbb{R} \quad s.t.\ \ f\ is\ continuous\}$$

Metric on $C[0,1]$

Definition 1.4.2.

$$d_{L^1}(f,g) = \int_0^1 |f(x) - g(x)|dx$$

> $$d_{L^\infty}(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$$

1.5. Product metrics

1.6. Isometries

> Definition 1.6.1.
>
> An *isometry* from (X, d_x) to (Y, d_Y) is a function $\phi : X \to Y$ s.t.
>
> (1) $\forall x_1, x_2 \in X, d_Y(\phi(x_1), \phi(x_2)) = d_x(x_1, x_2)$
>
> (2) $\phi \; is \; surjective$

> Lemma 1.6.2
>
> An isometry is *injective*

> Definition 1.6.3.
>
> Two metric spaces $(X, d_x), (Y, d_Y)$ are *isometric* if there exists an isometry
> $\phi : (X, d_x) \to (Y, d_Y)$