Chapter 3 Notes

Overview

> $$\begin{cases} Open\ and\ closed\ subset\ of\ a\ metric\ space \\ Formal\ properties\ of\ open\ subsets \\ Closed\ sets\ by\ convergence\ of\ sequences \\ Convergence/Continuity\ using\ open\ sets \\ Equivalence\ for\ two\ distances \\ Homeomorphism \end{cases}$$

3.1. Open sets and closed sets

Definition for open sets and closed sets

> Definition 3.1.1.
>
> Let (X, d) be a metric space, and let $A \subseteq X$.
>
> Open: we say that A is open in (X, d) if $\forall p \in A$, $\exists \epsilon > 0$, s.t. $B_\epsilon(p) \subseteq A$.
>
> Closed: If $B \subseteq X$ then we say that B is closed in (X, d) if $X \backslash B$ is open.
>
> 一般的开集证明思路: 证明所证集内的任一元素的任一开球内的元素都在所证集中. i.e. 所证集内任一元素的任一开球是所证集的子集.

Formal properties of open sets

> Lemma 3.1.4 — Open sets properties
>
> Let (X, d_X) be a metric space.
>
> 1. The subsets $\emptyset$ and X are open
> 2. An arbitrary union of open sets is open (任意并)
> 3. A finite intersection of open sets is open (有限交)
>
> About closed sets:

> 1. The subsets $\emptyset$ and X are closed
> 2. The arbitrary intersection of closed sets is closed（任意交）
> 3. A finite union of closed sets is closed (有限并)

Closed sets by the convergence of sequences

> Lemma 3.1.8.
>
> Let (X, d) be a metric space, and let $A \subseteq X$. Then,
>
> The subset A is closed $\iff \forall(x_n)$ of A, if (x_n) converges to $l \in X$ then $l \in A$.

3.2. Topology

> Definition 3.2.1.
>
> A topology $\mathcal{U}$ on X is a collection of subsets $\mathcal{U}_i$ of X, called the <u>open subsets</u> of X (开子集族), that satisfies the following properties.
>
> (T_1) The subsets $\emptyset$, $X \in \mathcal{U}$
>
> (T_2) If $\mathcal{U}_i \in \mathcal{U}$ for all $i \in I \implies \bigcup_{i \in I} \mathcal{U}_i \in \mathcal{U}$ (任意并)
>
> (T_3) If $\mathcal{U}_1, \ldots, \mathcal{U}_N \in \mathcal{U} \implies \bigcap_{i=1}^{N} \mathcal{U}_i \in \mathcal{U}$ (有限交)

> Remark 3.2.5.
>
> $$\begin{cases} Discrete\ topology\ \mathscr{P}(X) \\ Trivial\ topology\ \mathcal{U} = \{\emptyset, X\} \end{cases}$$

Convergence/Continuity in topological terms

> Theorem 3.2.8. — <u>Convergence</u>
>
> Let (X, d) be a metric space, (x_n) a sequence of X, and let $l \in X$. Then the following are equivalent:
>
> 1. (x_n) converges to l
> 2. $\forall$ <u>open subset</u> $\mathcal{U} \subseteq X$ with $l \in \mathcal{U}$, $\exists N \in \mathbb{N}$ s.t. $(x_n) \in \mathcal{U} \quad \forall n > N$
>
> Tips: 当收敛时(x_n)只有有限多的项在$\mathcal{U}$外面

> Theorem 3.2.9. — <u>Continuity</u>
>
> Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. Then the following are equivalent.
>
> 1. f is continuous.
> 2. $\forall$ <u>open set</u> $\mathcal{U} \subseteq Y$, the *inverse image* $f^{-1}(\mathcal{U})$ is open in X.
>
> Remark 3.2.10.
>
> $\forall$ open subset $\mathcal{U} \subseteq Y$, $f^{-1}(\mathcal{U})$ is open $\iff \forall$ closed set $C \subseteq Y$, $f^{-1}(C)$ is closed.
>
> Tips: 上述的闭集用开集表示即可.

3.3. Equivalent distances

Equivalence for two distances

> Two distances are equivalent if the notions of convergence and continuity defined using one are the same as those defined using the other. (上面使用了开集来定义度量空间内收敛性与连续性)

> Definition 3.3.1.
>
> Let X be a set. Let d, d' be two different distances on X. Then we say that d and d' are equivalent whenever the open sets of (X, d) coincide with those of (X, d') . We write $d \sim d'$ to denote that two distances are equivalent.
>
> Tips: 即开集表示一致.

> Corollary 3.3.2. — Convergence
>
> Let (X, d) and (X, d') be metric spaces with $d \sim d'$, and let (x_n) be sequence of X and $l \in X$. Then we have
>
> $$(x_n) \xrightarrow{d} l \iff (x_n) \xrightarrow{d'} l$$
>
> Corollary 3.3.3. — Continuity
>
> Let (X, d_X), (X, d'_X) and (Y, d_Y), (Y, d'_Y) be metric spaces where $d_X \sim d'_X$ and $d_Y \sim d'_Y$. Then
>
> $$\underbrace{f:\ (X, d_X) \to (Y, d_Y)}_{is\ continuous} \iff \underbrace{f:\ (X, d'_X) \to (Y, d'_Y)}_{is\ continuous}$$

> Lemma 3.3.5.
>
> Let (X, d), (X, d') be metric spaces on the same underlying set. Suppose $\forall x, y$, $\exists C > 0$, s.t.
>
> $$d(x, y) \leq C \cdot d'(x, y)$$
>
> Then,
>
> $$\mathcal{U} \subseteq (X, d)\ open \implies \mathcal{U} \subseteq (X, d')\ open$$
>
> Corollary 3.3.6. — Give an easy sufficient condition for two distances to be equivalent
>
> According to the lemma above, if $\forall x, y \in X, \exists C, C' > 0$, s.t.
>
> $$d(x, y) \leq C \cdot d'(x, y) \quad and \quad d'(x, y) \leq C' \cdot d(x, y)$$
>
> Then according to definition 3.3.1., d is equivalent to d'.

> From Chapter 1, if $p \neq q$, then $(\mathbb{R}^n, d_p)$ and $(\mathbb{R}, d_q)$ are *not isometric*.
>
> Corollary 3.3.7. — equivalence for d_p and d_q
>
> On $\mathbb{R}^n$, the distances d_p and d_q are equivalent $\forall p, q \geq 1$ including $p, q = \infty$.

> The case of the space of functions $C[0, 1]$ and two distances d_{L^1} and d_{L^∞}:
>
> Lemma 3.3.8.

The inequality $d_{L^1}(f,g) \le d_{L^\infty}(f,g)$ holds $\forall f, g \in C[0,1]$.

Remark 3.3.9.

The distance d_{L^1} and d_{L^∞} are not equivalent.

Remark 3.3.12.

For $p \ge 1$, the space of sequences l^p can be endowed with distances d_q and $d_{q'}$ $\forall p \le q < q'$. These two distances are not equivalent.

3.4. Homeomorphisms

Definition 3.4.1. — 集到集的双连续映射

Let (X, d_X), (Y, d_Y) be metric spaces. We say that $f: X \to Y$ is a homeomorphism when

1. f is bijective
2. Both f and f^{-1} are continuous

We say that the two metric spaces are homeomorphic when there exists such a f.

f可以将(X, d_X)中的开集通过映射:

$$\mathcal{U} \to f(\mathcal{U}), \quad \mathcal{V} \to f^{-1}(\mathcal{V})$$

来得到(Y, d_Y)中的开集.

Example 3.4.2.

Let $f: (X, d_X) \to (Y, d_Y)$ be an isometry, then f is a homeomorphism.