Chapter 4 Notes

> Overview
>
> $$\begin{cases} Cauchy\ sequence \\ Completeness \\ CMT \\ Comapact\ metric\ space \\ Compactness\ properties \\ Compact\ subsets \\ Compact\ implies\ complete \end{cases}$$

4.0.最小上界定理

> Theorem 4.0.1. — LUB定理
>
> 实数非空子集有上界，则它有最小上界 $\Longrightarrow$ 实数完备性
>
> 这个定理对于度量空间的推广并不可行，所以用柯西收敛来定义完备.

4.1. Cauchy convergence and completeness

Cauchy sequence and convergence

> 为了用不依赖极限值 $l \in X$ 的表示法来定义收敛性，遂引入柯西收敛这一只依赖于(x_n)序列中元素的表示法.
>
> Definition 4.1.1.
>
> Let (X, d) be a metric space and let (x_n) be a sequence of X. We say that it is *Cauchy convergent* (or just *Cauchy*) if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon\ \forall n, m > N$
>
> Lemma 4.1.2.
>
> $$(x_n)\ converges \implies (x_n)\ is\ Cauchy$$
>
> 然而，反之却不一定成立.

Tips: 前面提到，收敛值 l 必须满足 $l \in X$ (收敛值要在集内). 所以柯西收敛有时不一定收敛. 但是，若收敛，则无论收敛值是否在X内，一定柯西收敛.

Cauchy convergence and completeness

Definition 4.1.4. — 完备性的定义

A metric space (X, d) is *complete* if every Cauchy sequence *converges*.

Tips: 提到收敛，收敛值必须在X中.

Theorem 4.1.6 — Completeness and closed

Let (X, d) be a metric space, and $A \subseteq X$ be a subset, and let d_A be the distance induced by d on the subset A.

1. If (A, d_A) is *complete*, then A is *closed* in (X, d_X)
2. If (X, d) is *complete* and A is *closed* in (X, d_X), then (A, d_A) is also *complete*.

(X, d_X) is complete and $Y \subset X$, then (Y, d_X) is complete $\iff Y \subset X$ is closed

Remark 4.1.7.

同胚的两个度量空间，其中一个是完备的不一定意味着另一个也是完备的.

例如：$((-\pi/2, \pi/2), d_1)$ 与 $(\mathbb{R}, d_1)$ 同胚，但后者完备，前者不完备.

4.2. Completeness of $\mathbb{R}^N$

In order to prove $(\mathbb{R}^N, d_p)$ is a complete metric space ($p = \infty$ is accepted).

4.3. The contraction mapping theorem (CMT)

Definition 4.3.1. — 不动点

Let X be a set, $f: X \to X$ a function and let $p \in X$. We say that p is a fixed point if $f(p) = p$.

Definition 4.3.3. — 压缩映射

Let (X, d) be a metric space. Then $f: X \to X$ is a *contraction* when $\exists\, constant\ L \in [0, 1)$ s.t. $d(f(x), f(y)) \leq L \cdot d(x, y)\ \forall x, y \in X$.

Theorem 4.3.4. — CMT

Suppose (X, d) is a *complete* metric space. If $f: (X, d) \to (X, d)$ is a *contraction*, then f has a *unique fixed point*.

Lemma 4.3.7.

压缩映射 f 是连续的.

Tips: CMT证明过程大致是：

1. 用反证法来证明不动点的唯一性.
2. 为了证明不动点的存在，任取$x_0 \in X$, 递归定义$x_{n+1} = f(x_n), n = 0, 1, 2, \cdots$. 后证 $d(x_{n+1}, x_n) \leq L^n d(x_1, x_0)$, 并推导出 $(x_n)_{n=0}^{\infty}$ 是柯西列，且该柯西列极限为 f 的不动点.

3. Remark 4.3.9.
证明过程中，有，

$$d(x_n, x_m) \leq d(x_1, x_0) \cdot \frac{L^m}{1-L}$$

So, take $n \to \infty$,

$$d(l, x_m) \leq d(x_1, x_0) \cdot \frac{L^m}{1-L}$$

$\forall x \in X$, 总有充分大的m, s.t. $\forall \varepsilon > 0$,

$$d(f(x), x) \cdot \frac{L^m}{1-L} < \varepsilon$$

这阐述了 $f^m(x)$ 序列向极限值不动点 l 逼近的性质.

Tips: 4.3.3. 中，是不能有 $L = 1$ 的 (有些教材就 $L = 1$ 的情况称f为压缩映射，就 $L \in [0, 1)$ 的情况称f为严格压缩映射）.

Lemma 4.3.6. — 单变量实值函数是压缩映射的准则

Let $f : [a, b] \to [a, b]$ be *differentiable* with $|f'(x)| \leq L < 1$ $\forall x \in [a, b]$. Then f is a *contraction* when $[a, b]$ is endowed with the distance d_1.

证明由中值定理所得.

4.4. Compactness

Definition 4.4.1. — subsequence 子列的概念

Let (X_n) be a sequence in X. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. We say that $(x_{n_k})_{k\in\mathbb{N}}$ is a *subsequence* of $(x_n)_{n\in\mathbb{N}}$.

如果(X, d)中的序列(x_n)收敛于l，那么其子列也同样收敛于l.

Compactness Definition

Definition 4.4.2. — 紧致性的定义

We say that a metric space (X, d) is *compact* if every sequence admits a convergent subsequence. (当且仅当(X, d)中的每一个序列都至少有一个收敛子列.)

Compactness Properties

Proposition 4.4.5.

If $X \subseteq (\mathbb{R}, d_1)$ is s.t. (X, d_1) is *compact*, then X has a *maximum* and a *minimum*.

Tips:

1. 先证有界.
2. 再利用紧致性证明上确界在 X 中.

Corollary 4.4.8.

> Suppose (X, d_X) is *compact*, and that $f : X \to \mathbb{R}$ is *continuous* (with the distance d_1 on $\mathbb{R}$). Then f has a *minimum* and a maximum.

> Theorem 4.4.6. Key Result — 连续映射保持紧致性
>
> If $f : (X, d_X) \to (Y, d_Y)$ is a *continuous* function, if (X, d_X) is *compact*, then so is $(f(X), d_Y)$.
>
> 证明: $f(x_{n_k}) \to f(l)$, 再用紧致性定义叙述即可.

> Corollary 4.4.7. 紧致性是一种拓扑性质
>
> If (X, d_X) is *homeomorphic* to (Y, d_Y) then,
>
> $$(X, d_X) \textit{ is compact } \iff (Y, d_Y) \textit{ is compact}$$
>
> 如果这里的"*紧致*" 换成 "*完备*"，则不成立.

Compactness for subsets of $\mathbb{R}^N$

> Question: 什么样的$(\mathbb{R}, d_1)$ 的度量子空间是紧致的?
>
> Lemma 4.4.10
>
> Let (X, d) be a metric space. Suppose that $C \subseteq X$ and (C, d) is *compact*. Then C is a *closed subset* of (X, d).
>
> 证明: 若不是闭子集，则存在序列不收敛于C（序列收敛于L, 则其每一子列也收敛于L), 则其不紧致，与紧致前提矛盾.
>
> Theorem 4.4.11 — Bolzano-Weierstrass Theorem
>
> Let $X \subseteq (\mathbb{R}, d_1)$. Then,
>
> $$(X, d_1) \textit{ is compact } \iff X \textit{ is closed and bounded}$$
>
> Proof: 同名定理：每一有界序列都有收敛子列.
>
> Theorem 4.4.12. 上述定理的多维推广
>
> $(\mathbb{R}^N, d_p)$ 的紧致子集是有界闭子集.（if and only if ）
>
> Definition 4.4.13. $\mathbb{R}^N$上有界的定义
>
> $X \subseteq \mathbb{R}^N$ is bounded if there exists an $R > 0$ s.t. $X \subseteq B_R(0)$.

Compactness and completeness

> Lemma 4.4.17.
>
> Let (X, d) be a metric space. Suppose (x_n) is a *Cauchy sequence* and a *subsequence* (x_{n_k}) *converges* to l . Then (x_n) *converges* to l .
>
> Corollary 4.4.18.
>
> A compact metric space is complete.
>
> 相反叙述是错误的.