Homework #3

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1 Problem B

Consider a negative binomial distribution with r = 3 and p = 0.4. Find the skewness of this distribution.

Solution. The skewness of a random variable is:

$$\begin{aligned} Skewness(X) &= E[(X - EX)^3/\sigma^3] \\ &= (E(X^3) - 3(EX)E(X^2) + 3(EX)^2EX - (EX)^3)/\sigma^3 \\ &= (E(X^3) - 3EX((E(X^2) + (EX)^2)) - (EX)^3)/\sigma^3 \\ &= (E(X^3) - 3EX\sigma^2 - (EX)^3)/\sigma^3 \end{aligned} \tag{1.1}$$

Using the mailing tube on (4.6) we find:

$$\sum_{i=0}^{\infty} t^i = \frac{1}{1-t} \tag{1.2}$$

Then if we take the derivative we get:

$$\sum_{i=1}^{\infty} i \cdot t^{i-1} = \frac{1}{(1-t)^2} \tag{1.3}$$

And again:

$$\sum_{i=2}^{\infty} i(i-1)(t^{i-2}) = \frac{2}{(1-t)^3} \tag{1.4}$$

And one last time:

$$\sum_{i=3}^{\infty} i(i-1)(i-2)(t^{i-3}) = \frac{6}{(1-t)^4} \tag{1.5}$$

Using mailing tube for the expected value of a random variable with negative binomial distribution (4.38) and mailing tube for variance of a random variable with negative binomial distribution (4.39), we already know the values of EX and Var(X). So we must solve for $E(X^3)$.

$$\begin{aligned} E(X^3) &= \sum_{i=3}^{\infty} i^3 \binom{i-1}{2} (1-p)^{i-3} p^3 \\ &= \sum_{i=3}^{\infty} i^3 \frac{(i-1)(i-2)}{2} (1-p)^{i-3} p^3 \\ &= \frac{p^3}{2} \sum_{i=3}^{\infty} i^3 (i-1)(i-2)(1-p)^{i-3} \\ &= 720 \end{aligned} \tag{1.6}$$

Using our result for (1.5), we can solve for $E(X^3)$ using t = (1-p). Now we substitute our results into (1.1) to solve for skewness.

$$EX = \frac{r}{p} \tag{1.7}$$

$$Var(X) = \sigma^2 = \frac{r - rp}{p^2} \tag{1.8}$$

$$(E(X^3) - 3EX\sigma^2 - (EX)^3)/\sigma^3 = 1.19257 \tag{1.9}$$

2 Problem C

Consider our simple board game example without bonus rolls. Let Xi be the square you land on after your ith turn. These random variables satisfy the Markov property. You will win each time you reach or pass 0, with the amount of your winnings being i+1 dollars if your win involves landing on square i. Find the following:

(1) Find the long-run average number of rolls between wins.

Solution. Draw the matrix first,

$$P = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \end{bmatrix}$$

Calling the function $findpi1()$ from the textbook, we get:

$\pi = (0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125)$

So suppose we roll 8000 rolls, the number of wins is $0 \cdot 8000 \cdot 0.125 + 1 \cdot 8000 \cdot 0.125 + 2 \cdot 8000 \cdot 0.125 + 3 \cdot 8000 \cdot 0.125 + 4 \cdot 8000 \cdot 0.125 + 5 \cdot 8000 \cdot 0.125 + 6 \cdot 8000 \cdot 0.125 + 7 \cdot 8000 \cdot 0.125 = 3500$

So the long-run average number of rolls between wins = 8000/3500 = 2.2857

(2) Find the long-run value of total winnings per turn.

Solution. Let W denotes the winnings of each turn, then E(W) is the long-run value of total winnings per turn.

$E(W) = 1 \cdot P(W = 1) + 2 \cdot P(W = 2) + 3 \cdot P(W = 3) + 4 \cdot P(W = 4) + 5 \cdot P(W = 5) + 6 \cdot P(W = 6) + 7 \cdot P(W = 7) + 8 \cdot P(W = 8) = \frac{1}{8} \cdot 1 \cdot 6 \cdot \frac{1}{6} + \frac{1}{8} \cdot 2 \cdot 5 \cdot \frac{1}{6} + \frac{1}{8} \cdot 3 \cdot 4 \cdot \frac{1}{6} + \frac{1}{8} \cdot 4 \cdot 3 \cdot \frac{1}{6} + \frac{1}{8} \cdot 5 \cdot 2 \cdot \frac{1}{6} + \frac{1}{8} \cdot 6 \cdot 1 \cdot \frac{1}{6} = \frac{7}{6} = 1.1667$

(3) Find ETj, where Tj is the number of rolls needed to win if one starts at square j.

Solution. For all ET_j, we use (3.13) to find the expected values.

$E(T_7) = 1 \cdot P(T_7 = 1) = 1 \cdot 1 = 1$

$E(T_6) = 1 \cdot P(T_6 = 1) + 2 \cdot P(T_6 = 2) = 1 \cdot \frac{5}{6} + 2 \cdot \frac{1}{6} = \frac{7}{6} = 1.1667$

$E(T_5) = 1 \cdot P(T_5 = 1) + 2 \cdot P(T_5 = 2) + 3 \cdot P(T_5 = 3) = 1 \cdot \frac{4}{6} + 2 \cdot \frac{11}{36} + 3 \cdot \frac{1}{36} = \frac{49}{36} = 1.3611$

$E(T_4) = 1 \cdot P(T_4 = 1) + 2 \cdot P(T_4 = 2) + 3 \cdot P(T_4 = 3) + 4 \cdot P(T_4 = 4)$
$= 1 \cdot \frac{3}{6} + 2 \cdot \frac{15}{36} + 3 \cdot \frac{17}{126} + 4 \cdot \frac{1}{216} = \frac{343}{216} = 1.5879$

$E(T_3) = 1 \cdot P(T_3 = 1) + 2 \cdot P(T_3 = 2) + 3 \cdot P(T_3 = 3) + 4 \cdot P(T_3 = 4) + 5 \cdot P(T_3 = 5)$
$= 1 \cdot \frac{2}{6} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{4}{27} + 4 \cdot \frac{23}{1296} + 5 \cdot \frac{1}{1296} = \frac{2401}{1296} = 1.8526$

$E(T_2) = 1 \cdot P(T_2 = 1) + 2 \cdot P(T_2 = 2) + 3 \cdot P(T_2 = 3) + 4 \cdot P(T_2 = 4) + 5 \cdot P(T_2 = 5) + 6 \cdot P(T_2 = 6) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{5}{9} + 3 \cdot \frac{25}{108} + 4 \cdot \frac{55}{1296} + 5 \cdot \frac{29}{7776} + 6 \cdot \frac{1}{7776} = \frac{16807}{7776} = 2.1614$

$E(T_1) = 2 \cdot P(T_2 = 2) + 3 \cdot P(T_2 = 3) + 4 \cdot P(T_2 = 4) + 5 \cdot P(T_2 = 5) + 6 \cdot P(T_2 = 6) + 7 \cdot P(T_2 = 7) = 2 \cdot \frac{7}{12} + 3 \cdot \frac{35}{108} + 4 \cdot \frac{37}{432} + 5 \cdot \frac{1}{108} + 6 \cdot \frac{35}{46656} + 7 \cdot \frac{1}{46656} = \frac{118153}{46656} = 2.5324$

$E(T_0) = 2 \cdot P(T_0 = 2) + 3 \cdot P(T_0 = 3) + 4 \cdot P(T_0 = 4) + 5 \cdot P(T_0 = 5) + 6 \cdot P(T_0 = 6) + 7 \cdot P(T_0 = 7) + 8 \cdot P(T_0 = 8)$
$= 2 \cdot \frac{5}{12} + 3 \cdot \frac{91}{216} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{155}{7776} + 6 \cdot \frac{199}{46656} + 7 \cdot \frac{41}{279936} + 8 \cdot \frac{1}{279936} = \frac{775087}{279936} = 2.7688$

3 Problem D

On p.72, we use properties of infinite series to derive the fact that the mean of a geometric random variable with success probability p is 1/p. But one can derive this quickly and simply, without using properties of series, using (4.68) instead.
Take V to be the geometric random variable, and let U be the indicator variable for the event V = 1. Use the "memoryless" property here; if we don't get a success on the first trial, the mean remaining time is the same as if we are starting fresh.

Solution. The mailing tube in (4.68) says:

$$E(V) = \sum_c P(U = c)E(V|U = c)$$

where c ranges through the support of U. In our case the support of U is 0,1 because U is the indicator variable for the success of V=1 with probability p. We know that if we don't get a success on the first trial, the mean remaining time remains the same as if we haven't encountered a failure. Therefore we can split the summation into components of the support:

$$E(V) = P(U = 1)E(V|U = 1) + P(U = 0)E(V|U = 0)$$

Since we know the probability of the success of V, we know the probability where U=1 and U=0. Also, the expected value of V when U=1 is 1 since U=1 when V=1. So,

$$E(V) = p(1) + (1 - p)E(V|U = 0)$$

The expected value of V given U=0 can be written as E(V+1) since value V was 0 on the first trial and an additional trial needs to be done in order to obtain the first success. In terms of lines in the notebook, if we don't receive a success on the first trial, the number of trials is incremented by 1. Then we can simplify our equation:

$$\begin{aligned} E(V) &= p(1) + (1 - p)E(V|U = 0) \\ &= p + (1 - p)E(V + 1) \end{aligned}$$

We can also use the existing mailing tube (3.24) to move the constant out:

$$\begin{aligned} E(V) &= p + (1 - p)E(V + 1) \\ &= p + (1 - p)(E(V) + 1) \\ &= p + (E(V) + 1 - pE(V) - p) \\ &= E(V) + 1 - pE(V) \\ E(V) &= E(V) + 1 - pE(V) \\ 0 &= 1 - pE(V) \\ pE(V) &= 1 \\ E(V) &= \frac{1}{p} \end{aligned}$$

Thus we have used (4.68) to derive the fact that the mean of a geometric random variable is $\frac{1}{p}$.