PFPL
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Chapter 2 Exercise

Questions

2.1 Give an inductive definition of the judgment $\max(m; n; p)$ where m nat, n nat, p nat, with the meaning that p is the larger of m and n. Prove that every m and n are related to a unique p by this judgment.

$$\frac{}{\max(m; \mathrm{zero}; m)} \qquad \frac{}{\max(\mathrm{zero}; \mathrm{succ}(n); \mathrm{succ}(n))} \qquad \frac{\max(m; n; p)}{\max(\mathrm{succ}(m); \mathrm{succ}(n); \mathrm{succ}(p))}$$

Proof. *If $n = 0$, then we have a unique $p = m$. Similarly, if $m = 0$, then $p = n$. Otherwise we need to prove that if every m and n are related to a unique p, then every every succ(m) and succ(n) are related to a unique q, and by definition we know that $q = succ(p)$. Thus, by rule induction, every m and n are related to a unique p.* ☐

2.5 Give an inductive definition of the *binary natural numbers*, which are either zero, twice a binary number, or one more than twice a binary number. The size of such a representation is logarithmic, rather than linear, in the natural number it represents.

$$\frac{}{\mathrm{zero\ binary}} \qquad \frac{x \text{ binary}}{\mathrm{twice}(x) \text{ binary}} \qquad \frac{x \text{ binary}}{\mathrm{twice{+}1}(x) \text{ binary}}$$

2.6 Give an inductive definition of addition of binary natural numbers as defined in Exercise **2.5**.
Hint: Proceed by analyzing both arguments to the addition, and make use of an auxiliary function to compute the successor of a binary number. Hint: Alternatively, define both the sum and the sum-plus-one of two binary numbers mutually recursively.

$$\frac{}{\mathrm{succ}(\mathrm{zero}; \mathrm{twice{+}1}(\mathrm{zero}))} \qquad \frac{}{\mathrm{succ}(\mathrm{twice}(n); \mathrm{twice{+}1}(n))}$$

$$\frac{\mathrm{succ}(n; m)}{\mathrm{succ}(\mathrm{twice{+}1}(n); \mathrm{twice}(m))}$$

$$\frac{}{\mathrm{add}(n; \mathrm{zero}; n)} \qquad \frac{}{\mathrm{add}(\mathrm{zero}; n; n)}$$

$$\frac{\mathrm{add}(m; n; p)}{\mathrm{add}(\mathrm{twice}(m); \mathrm{twice}(n); \mathrm{twice}(p))}$$

$$\frac{\mathrm{add}(m; n; p)}{\mathrm{add}(\mathrm{twice}(m); \mathrm{twice{+}1}(n); \mathrm{twice{+}1}(p))}$$

$$\frac{\mathrm{add}(m; n; p)}{\mathrm{add}(\mathrm{twice{+}1}(m); \mathrm{twice}(n); \mathrm{twice{+}1}(p))}$$

$$\frac{\mathrm{add}(m; n; p) \quad \mathrm{succ}(p; q)}{\mathrm{add}(\mathrm{twice{+}1}(m); \mathrm{twice{+}1}(n); \mathrm{twice}(q))}$$