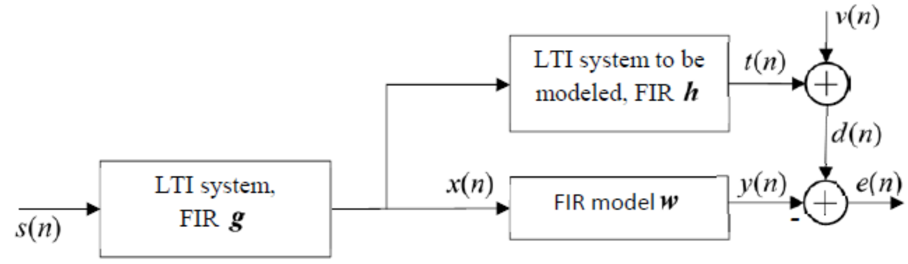
Estimation, system identification, and Wiener solution

Consider the following system:



where:

$s(n)$ and $v(n)$ are sequences from white-noise, zero-mean, mean-ergodic and correlation-ergodic random processes, with power $\sigma_s^2 = 10$ and $\sigma_v^2 = 1$, respectively. $x(n)$ and $d(n)$ are sample sequences observed from $0 \le n \le N-1$.

$\boldsymbol{g} = [1,1]$ i.e., FIR impulse response $g(n) = \delta(n) + \delta(n-1)$

$\boldsymbol{h} = [1, 2, 1]$ i.e., FIR impulse response $h(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$

$\boldsymbol{w} = [w_0, w_1, w_2]$ is a 3-tap FIR filter used to model the LTI system with impulse response $\boldsymbol{h}$

$N = 10000$ samples

Solution from theoretical correlation

1. Knowing the values of $\boldsymbol{g}$, $\boldsymbol{h}$, σ_s^2 and σ_v^2, find the theoretical expression for the auto-correlation $\phi_{xx}(l)$, the cross-correlation $\phi_{dx}(l)$ and the power σ_d^2 (i.e., from $\phi_{dd}(0)$).
2. Using the values of $\phi_{xx}(l)$, $\phi_{dx}(l)$, σ_d^2, compute numerically the minimum mean square error (MMSE, σ_e^2) achievable by the optimal Wiener solution, i.e., use the MMSE form which <u>does not require</u> knowledge of the optimum Wiener filter coefficients.
3. Using the values of $\phi_{xx}(l)$, $\phi_{dx}(l)$, find numerically the coefficients $\boldsymbol{w}$ of the optimal Wiener solution.
4. Using the values of $\phi_{dx}(l)$, σ_d^2, $\boldsymbol{w}$, compute numerically the minimum mean square error (MMSE, σ_e^2) achievable by the optimal Wiener solution, i.e., MMSE form which <u>requires</u> knowledge of the optimum Wiener filter coefficients. Compare the result with step 2, they should be the same.
5. With the same coefficients $\boldsymbol{w}$ found in step 3, generate the error signal $e(n)$ and measure experimentally an estimate of the MMSE i.e., compute $\hat{\sigma}_e^2$. Compare with the true MMSE of step 4.

Solution from unbiased estimates of correlation

6. Next assume that you don't know the values of $\boldsymbol{g}$, $\boldsymbol{h}$, σ_s^2 and σ_v^2. So instead, compute the unbiased estimates of $\hat{\phi}_{xx}(l)$, $\hat{\phi}_{dx}(l)$ from the observed sequences $x(n)$ and $d(n)$.
7. Using the above estimates $\hat{\phi}_{xx}(l)$, $\hat{\phi}_{dx}(l)$, re-compute numerically the optimal coefficients $\boldsymbol{w}$ of the Wiener solution.
8. Repeat step 4 but using the optimal coefficients $\boldsymbol{w}$ found in step 7. This means that for true MMSE computation you will use again the ideal values of $\phi_{dx}(l)$, σ_d^2 (step 1), unlike for the computation of the coefficients $\boldsymbol{w}$ in step 7. Compare with the MMSE found in step 4.

Solution from frequency domain, using Welch estimates of PSDs

9. Next, instead of using the unbiased estimates $\hat{\phi}_{xx}(l)$, $\hat{\phi}_{dx}(l)$ to compute the coefficients $\boldsymbol{w}$ as in step 7, compute the PSD estimates $\hat{\phi}_{xx}(e^{j\omega})$, $\hat{\phi}_{dx}(e^{j\omega})$ using a Welch estimation method and a 3 tap FFT/DFT (size-3 FFT/DFT). Then compute the frequency domain Wiener solution $W(e^{j\omega}) = \hat{\phi}_{dx}(e^{j\omega}) / \hat{\phi}_{xx}(e^{j\omega})$ (in the FFT/DFT domain, $W(k) = \hat{\phi}_{dx}(k) / \hat{\phi}_{xx}(k)$), and finally compute $\boldsymbol{w} = [w_0, w_1, w_2]$ from the inverse FFT/DFT of $W(k)$.

   Note: since the FFT/DFT size that we use here is much less than the number of available time samples ($3 << N$) , make sure that the FFT/DFT function that you use will not discard the signal samples when you specify a FFT size smaller than the signal length N. Alternatively, you can use a FFT/DFT size of N instead of 3, but after you compute $\boldsymbol{w}$ from the IFFT you must keep only the first 3 coefficients out of N values (but in practice this would be a waste of computations).

   Note: for the special case considered here (system to be modeled is a causal FIR system and we use enough coefficients to model it) there is no difference between the solution obtained from the frequency domain and the time domain Wiener solutions. But in general this is not true: "unconstrained" Wiener solution versus FIR causal Wiener solution, more details to be covered later.

10. Repeat (again) step 4 but using the optimal coefficients $\boldsymbol{w}$ found in step 9. This means that for true MMSE computation you will use again the ideal values of $\phi_{dx}(l)$, σ_d^2 (step 1), unlike for the computation of the coefficients $\boldsymbol{w}$ in step 9. Compare with the MMSE found in step 4.